Stat 3470: Introduction to Probability and Statistics for Engineers
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2.3 Counting Techniques

For equally likely outcomes computing probabilities reduces to counting.

If N denotes the number of outcomes in a sample space and $N(A)$ denotes the number of outcomes contained in an event A, then

$$P(A) = \frac{N(A)}{N}$$

Some rules that are useful in counting techiques:

The Product Rule for Ordered Pairs

Proposition

If the first element or object of an ordered pair (O_1, O_2) can be selected in n_1 ways, and for each of these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is $\boldsymbol{n_1 n_2}$**.**

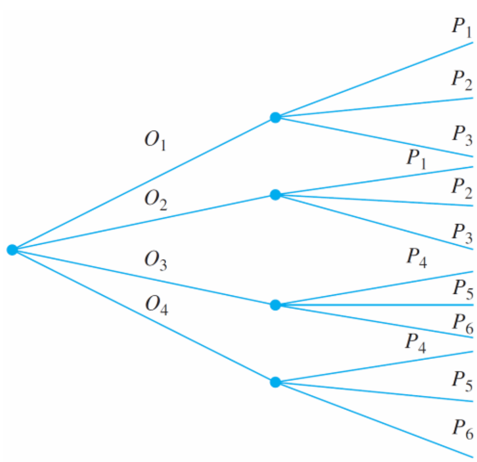
Examples: Throw a die twice.

A family has just moved to a new city and requires the services of both an obstetrician and a pediatrician. There are two easily accessible medical clinics, each having two obstetricians and three pediatricians.

The family will obtain maximum health insurance benefits by joining a clinic and selecting both doctors from that clinic. In how many ways can this be done?

Denote the obstetricians by O_1, O_2, O_3, and O_4 and the pediatricians by $P_1, \ldots, P_6$.

Then we wish the number of pairs (O_i, P_j) for which O_i and P_j are associated with the same clinic.



A More General Product Rule

Example:

If a six-sided die is tossed five times in succession rather than just twice, then each possible outcome is an ordered collection of five numbers such as (1, 3, 1, 2, 4) or (6, 5, 2, 2, 2).

Definition

An ordered collection of k objects is called a k-*tuple*.
(so a pair is a 2-tuple and a triple is a 3-tuple etc.)

Product Rule for *k*-Tuples

Suppose a set consists of ordered collections of k elements (k-tuples) and that there are n_1 possible choices for the first element; for each choice of the first element, there are n_2 possible choices of the second element; . . . ; for each possible choice of the first k – 1 elements, there are n_k choices of the kth element. Then there are $\boldsymbol{n_1 n_2 \bullet \cdot \cdot \cdot \bullet n_k}$ possible k-tuples.

Example:

Suppose that a home remodeling job involves first purchasing several kitchen appliances then hiring a plumber and then and electrican. The kitchen appliances will all be purchased from the same dealer, and there are five dealers, 12 plumbers, and 9 electricians in the area.

With the dealers denoted by D_i , i=1..5, the plumbers denoted by P_i, i=12, and the electriciams are denoted by E_i I=1..9 there are

$$N = n_1 n_2 n_3 = (5)(12)(9) = 540$$

3-tuples of the form (D_i, P_j, Q_k), so there are 540 ways to choose first an appliance dealer, then a plumbing contractor, and finally an electrical contractor.

Permutations and Combinations

Example:

Suppose, for example, that a college of engineering has seven departments, which we denote by *a*, *b*, *c*, *d*, *e*, *f*, and *g*. Each department has one representative on the college's student council. From these seven representatives, one is to be chosen chair, another is to be selected vice-chair, and a third will be secretary.

How many ways are there to select the three officers?

Definition

An ordered subset is called a **permutation**.

The number of permutations of size k that can be formed from the n distinct individuals or objects in a group will be denoted by $\boldsymbol{P_{k,n}}$.

Factorial notation: $m! = m(m-1)(m-2)\bullet \cdot \cdot \cdot \cdot \bullet(2)(1.)$

This gives 1! = 1, and we also define 0! = 1.

Proposition

$$P_{k,n} = \frac{n!}{(n-k)!} = n(n-1)(n-2)\bullet \cdot \cdot \cdot \cdot \bullet(n-(k-2))(n-(k-1))$$

Apply to above example:

$$P_{3,7} = (7)(6)(5) = \frac{(7)(6)(5)(4!)}{4!} = \frac{7!}{4!}$$

Example:

Again refer to the student council scenario, and suppose that three of the seven representatives are to be selected to attend a statewide convention.

The order of selection is not important; all that matters is which three get selected.

How many ways the three can be selected?

If the three individuals are *a*, *c*, *g*, they can be ordered in 3! = 6 ways to produce permutations: *a*, *c*, *g* *a*, *g*, *c* *c*, *a*, *g* *c*, *g*, *a* *g*, *a*, *c* *g*, *c*, *a*

Similarly, there are 3! = 6 ways to order the combination
b, *c*, *e* to produce permutations, and in fact 3! ways to order any particular combination of size 3 to produce permutations.

So computing this from the above we get:

Definition

An unordered subset is called a combination.

The number of combinations is denoted as $C_{k,n}$, or as, $\binom{n}{k}$ is the number of ways are there to select a subset of size *k* from a set of size *n*.

Proposition:

$$C_{k,n} = \binom{n}{k} \frac{n!}{k! \cdot (n-k)!} = \frac{n(n-1)(n-2)\cdot \ldots \cdot (n-(k-2))(n-(k-1))}{k\cdot(k-1)\cdot \ldots \cdot 2\cdot 1}$$

Apply to above example:

$$C_{3,7} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{7!}{3!4!} = \binom{7}{3}$$

The relationship between the number of combinations and the number of permutations:

Example:

A particular iPod playlist contains 100 songs, 10 of which are by the Beatles.

What is the probability that the first Beatles song heard is the fifth song played if the songs are played in random order?

In order for this event to occur, it must be the case that the first four songs played are not Beatles' songs (NBs) and that the fifth song is by the Beatles (B).

The number of ways to select the first five songs is

The number of ways to select these five songs so that the first four are NBs and the next is a B is

The random shuffle assumption implies that each outcome is equally likely.

Therefore the desired probability is the ratio of the number of outcomes for which the event of interest occurs to the number of possible outcomes: