Quadrature - I

Reva Dhillon

September 20, 2022

1 Introduction

This report discusses the following four different methods of Riemann summation for performing **Quadrature**:

- Left endpoint
- Right endpoint
- Mid point
- Trapezoidal

In addition, it elucidates the errors encountered using each method as we vary the values of Δx.
It also elaborates on the computational cost of each method using the integral:

$$I = \int_{0}^{\frac{\pi}{2}} \sin(x)dx \tag{1}$$

2 Quadrature

In mathematics, **quadrature** is a historical term which means the process of determining area.
With the invention of integral calculus, we obtained a method of finding areas of figures using definite integrals.
Therefore quadrature, also known as **numerical quadrature**, is analogous to numerical integration which refers to calculating the numerical values of definite integrals.
To calculate the area of a complex figure defined by function f, we generally follow these steps:

- Define a co-ordinate system.
- Within the chosen co-ordinate system, subdivide the figure into smaller segments whose areas are known.
- Add up all of these small areas to obtain an approximate value for the total area. This is the Riemann sum S. In mathematics, a Riemann sum is a certain kind of approximation of an integral by a finite sum.
- This approximation can be improved by increasing the number of segments and varying their shape appropriately.
- If we let the number of segments n tend to ∞, we obtain the true area of the figure:

  $$A_t = \int_{a}^{b} f(x)dx \tag{2}$$

  where $f(x)$ gives the shape of the figure and A_t gives the true area between the $x = a$ and $x = b$.

In the following subsections, the four methods of Riemann summation listed below shall be discussed to perform quadrature.

1. Left endpoint
2. Right endpoint
3. Mid point
4. Trapezoidal

These four methods of Riemann summation are usually best approached with partitions of equal size. The interval $[a, b]$ is therefore divided into n segments(subintervals), each of length

$$\Delta x = \frac{b-a}{n} \tag{3}$$

The points in the interval will then be $\{a, a+\Delta x, a+2\Delta x, ..., a+(n-2)\Delta x, a+(n-1)\Delta x, b\}$.

2.1 Left endpoint method:

For the left Riemann sum, approximating the function by its value at the left endpoint gives multiple rectangles with base Δx and height $f(a + i\Delta x)$. Doing this for $i = 0, 1, \ldots, n-1$ and adding up the resulting areas gives

$$A_{LE} = \Delta x(f(a) + f(a + \Delta x) + f(a + 2\Delta x) + ... + f(a + (n-1)\Delta x)) \tag{4}$$

$$A_{LE} = \sum_{i=0}^{n-1} f(a + i\Delta x)\Delta x \tag{5}$$

The left Riemann sum amounts to an overestimation if f is monotonically decreasing on this interval, and an underestimation if it is monotonically increasing.
The error E_{LE} is given by:

$$E_{LE} = |A_t - A_{LE}| = |\int_a^b f(x)dx - A_{LE}| \tag{6}$$

2.2 Right endpoint method:

For the right Riemann sum, approximating the function by its value at the right endpoint gives multiple rectangles with base Δx and height $f(a + i\Delta x)$. Doing this for $i = 1, 2, \ldots, n$ and adding up the resulting areas gives

$$A_{RE} = \Delta x(f(a + \Delta x) + f(a + 2\Delta x) + ... + f(b)) \tag{7}$$

$$A_{RE} = \sum_{i=1}^{n} f(a + i\Delta x)\Delta x \tag{8}$$

The right Riemann sum amounts to an underestimation if f is monotonically decreasing, and an overestimation if it is monotonically increasing.
The error E_{RE} is given by:

$$E_{RE} = |A_t - A_{RE}| = |\int_a^b f(x)dx - A_{RE}| \tag{9}$$

2.3 Mid point method:

For this method, we approximate the function f at the midpoint for each interval.
This gives multiple rectangles with base Δx and height $f(a + \frac{(2i+1)}{2}\Delta x)$. Doing this for $i = 0, 1, \dots, n-1$ and adding up the resulting areas gives

$$A_{MP} = \Delta x(f(a + \frac{\Delta x}{2}) + f(a + \frac{3}{2}\Delta x) + ... + f(b - \frac{\Delta x}{2})) \tag{10}$$

$$A_{MP} = \sum_{i=0}^{n-1} f(a + \frac{(2i+1)}{2}\Delta x)\Delta x \tag{11}$$

The error E_{MP} is given by:

$$E_{MP} = |A_t - A_{MP}| = |\int_a^b f(x)dx - A_{MP}| \tag{12}$$

2.4 Trapezoidal method:

For this method, instead of rectangles we form trapeziums for each interval by drawing a line through $f(a + i\Delta x)$ and $f(a + (i+1)\Delta x)$. The area of each such trapezium is:

$$\Delta A_i = \frac{1}{2}\Delta x(f(a + i\Delta x) + f(a + (i+1)\Delta x)) \tag{13}$$

Doing this for $i = 0, 1, \dots, n-1$ and adding up the resulting areas gives

$$A_{TZ} = \sum_{i=0}^{n-1} \Delta A_i \tag{14}$$

$$A_{TZ} = \frac{1}{2}\Delta x(f(a) + 2f(a + \Delta x) + 2f(a + 2\Delta x) + ... + f(b)) \tag{15}$$

The error E_{TZ} is given by:

$$E_{TZ} = |A_t - A_{TZ}| = |\int_a^b f(x)dx - A_{TZ}| \tag{16}$$

The approximation obtained with the trapezoid rule for a function is the same as the average of the left and right Riemann sums of that function.

3 Comparative Study

In this section, the errors and variations with Δx for all four methods shall be discussed. The integral used for these studies is:

$$I = \int_0^{\frac{\pi}{2}} \sin(x)dx \tag{17}$$

This integral gives the area enclosed by the sin curve, the x-axis and the lines $x = 0$, $x = \frac{\pi}{2}$. The value of I is the true area A_t:

$$A_t = I = \int_0^{\frac{\pi}{2}} \sin(x)dx = 1 \tag{18}$$

3.1 Absolute error vs Δx:

The figures below show the variations in absolute error with Δx for each method on linear axis and log-log scale. The least value of Δx graphed is $\Delta x = 0.0157$(Linear).

3.1.1 Left Endpoint Method:

- We can observe that for the Left Endpoint method, the absolute error increases with increase in Δx that is, with decrease in number of subintervals n.
- The regression model is a straight line which approximately passes through origin. This indicates that as n tends to ∞, that is, Δx tends to 0, absolute error tends to 0. This means that the left Riemann sum will be equal to the true area. $Slope \approx 0.6$.
- Maximum error = 1.
- The regression model on the log-log scale for this method shows a decline in absolute error as Δx decreases.Absolute error $< 10^{-2}$ for $\Delta x = 0.0157$.

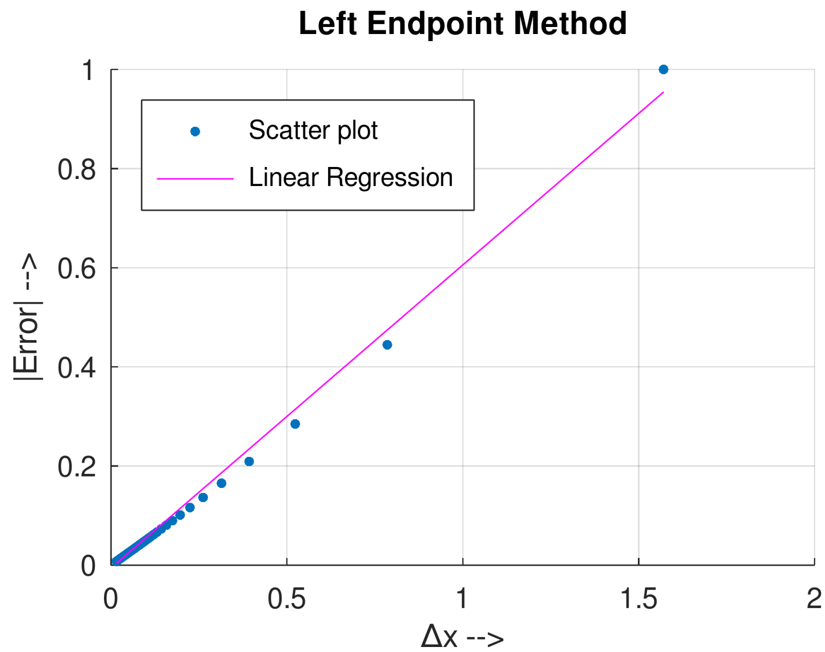

Figure 1: Absolute error vs Δx (Linear scale)

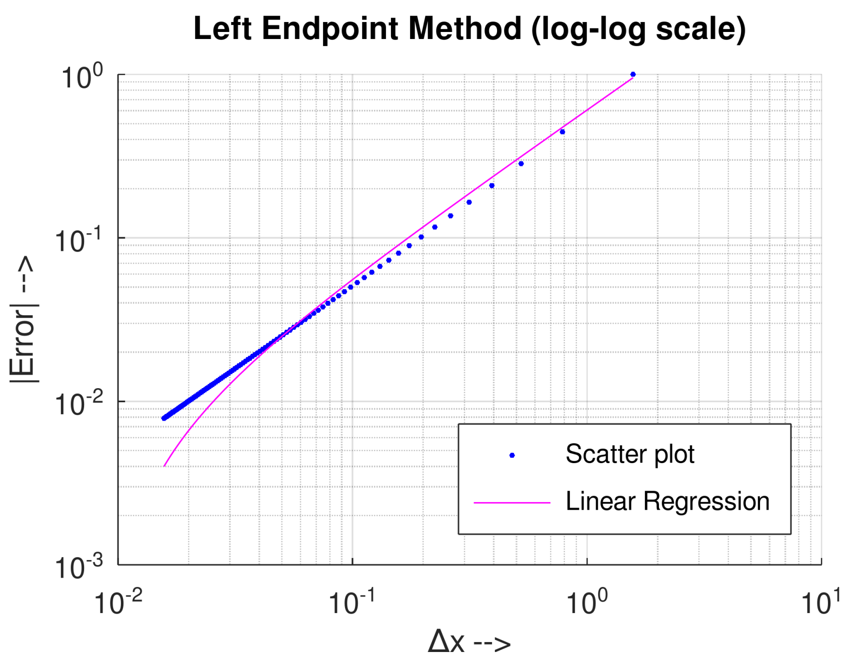

Figure 2: Absolute error vs Δx (Log-log scale)

3.1.2 Right Endpoint Method:

- We can observe that for the Right Endpoint method, the absolute error increases with increase in Δx that is, with decrease in number of subintervals n.
- The regression model is a straight line which approximately passes through origin similar to the left endpoint method. However the slope of the best-fit line is smaller in the model for the right endpoint method. $Slope \approx 0.4$.
- In this method as well, as n tends to ∞, that is, Δx tends to 0, absolute error tends to 0. This means that the right Riemann sum will be equal to the true area.
- Maximum error < 0.6.
- The regression model on the log-log scale for this method shows a net decline in absolute error as Δx decreases. However, the value given by the regression model in this method for the minimum plotted value of Δx is higher than that given by the left-endpoint method. Absolute error$> 10^{-2}$ for $\Delta x = 0.0157$.

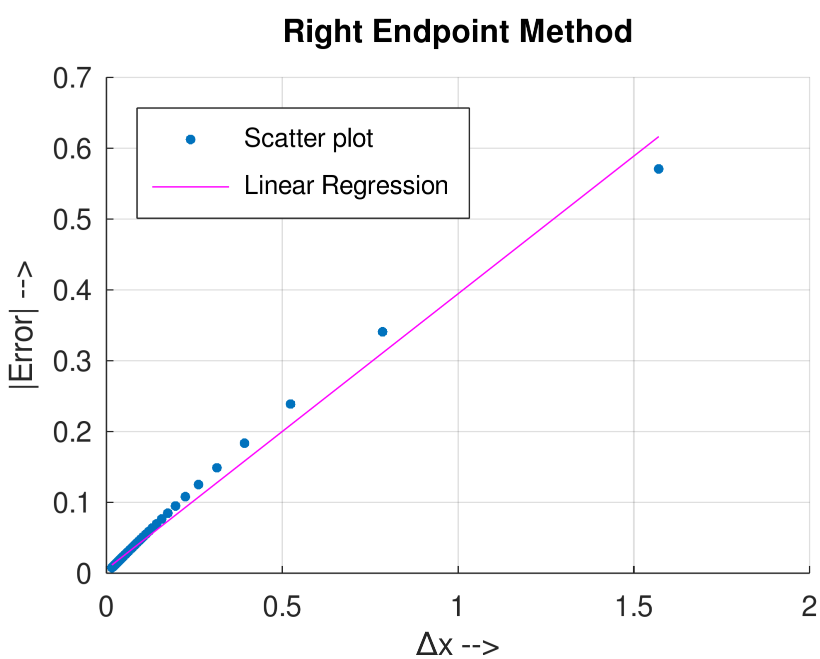

Figure 3: Absolute error vs Δx (Linear scale)

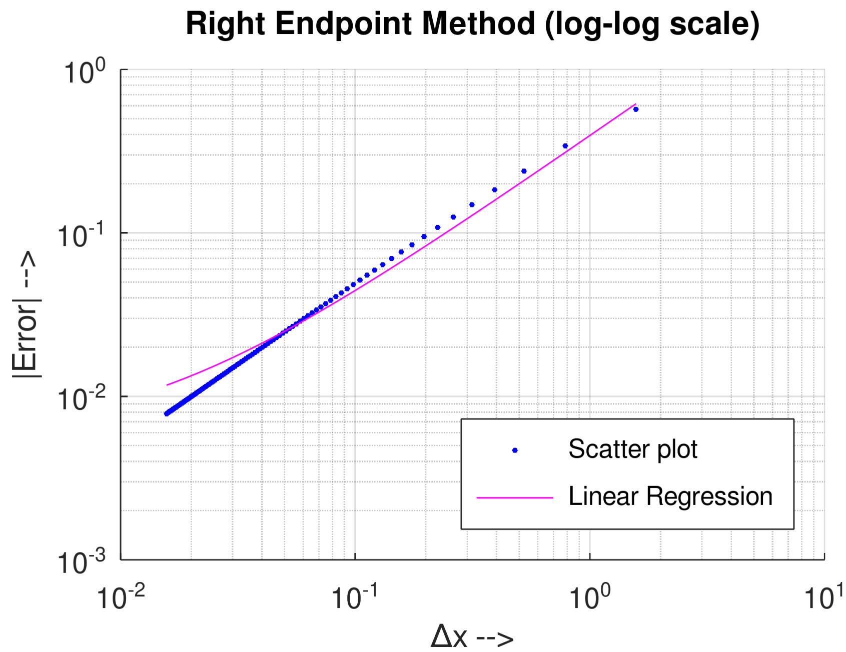

Figure 4: Absolute error vs Δx (Log-log scale)

3.1.3 Midpoint Method:

- We can observe that for the Midpoint method, the absolute error increases with increase in Δx that is, with decrease in number of subintervals n.
- The regression model is a straight line which approximately passes through origin similar to the previous methods. However the slope of the best-fit line is smaller in this model as compared to the other two. *Slope* ≈ 0.053.
- In this method as well, as n tends to ∞, that is, Δx tends to 0, absolute error tends to 0. This means that the Riemann sum will be equal to the true area.
- Maximum error < 0.12.
- The regression model on the log-log scale for this method shows a steep decline in absolute error as Δx decreases.
- For this method, the best-fit line has negative values and zero even though the scatter plot for absolute error values does not. To plot this on the log-log scale the magnitude of the most negative value$+10^{-6}$ is taken as the correction term. So all values will be shifted on the log-log scale. Correction term $= 1.9924 \times 10^{-3} + 10^{-6}$.

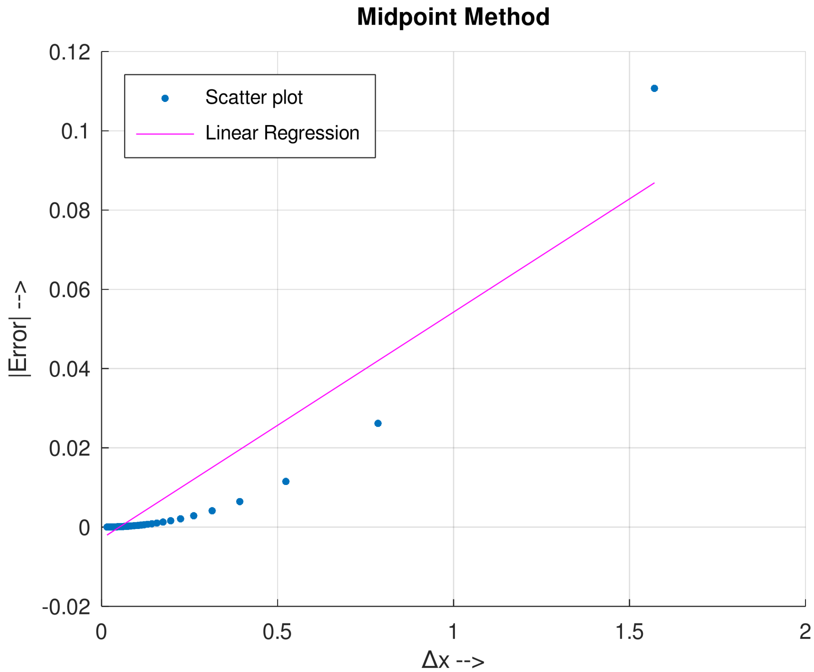

Figure 5: Absolute error vs Δx (Linear scale)

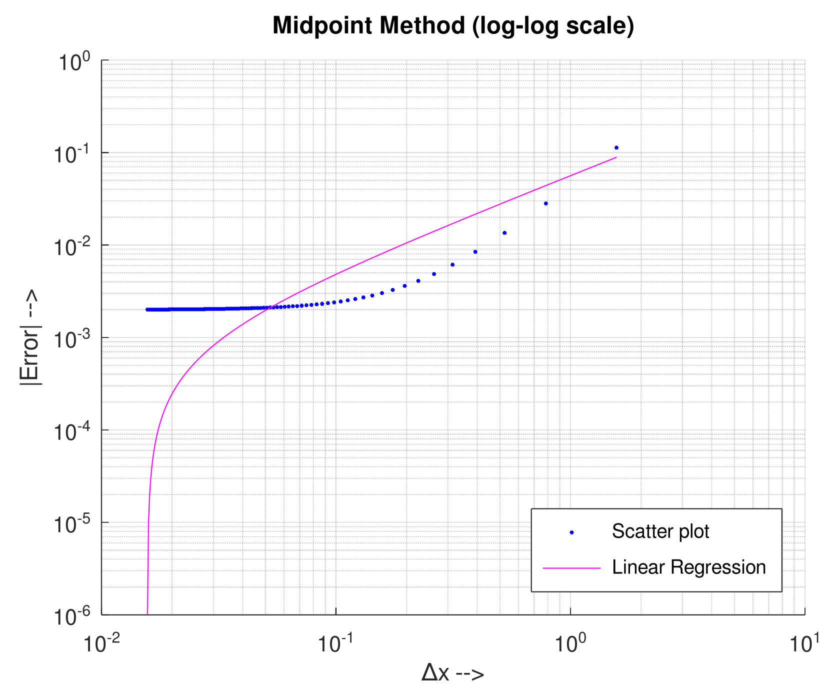

Figure 6: Absolute error vs Δx (Log-log scale)

3.1.4 Trapezoidal Method:

- We can observe that for the Trapezoidal method, the absolute error increases with increase in Δx that is, with decrease in number of subintervals n.

- The regression model is a straight line which approximately passes through origin similar to the previous methods. However the slope of the best-fit line is slightly greater in this model as compared to that of the midpoint method. $Slope \approx 0.1$.

- In this method as well, as n tends to ∞, that is, Δx tends to 0, absolute error tends to 0. This means that the Riemann sum will be equal to the true area.

- Maximum error < 0.25.

- The regression model on the log-log scale for this method shows a steep decline in absolute error as Δx decreases.

- For this method, the best-fit line has negative values and zero even though the scatter plot for absolute error values does not. To plot this on the log-log scale the magnitude of the most negative value $+10^{-6}$ is taken as the correction term. So all values will be shifted on the log-log scale. Correction term $= 3.8533 \times 10^{-3} + 10^{-6}$.

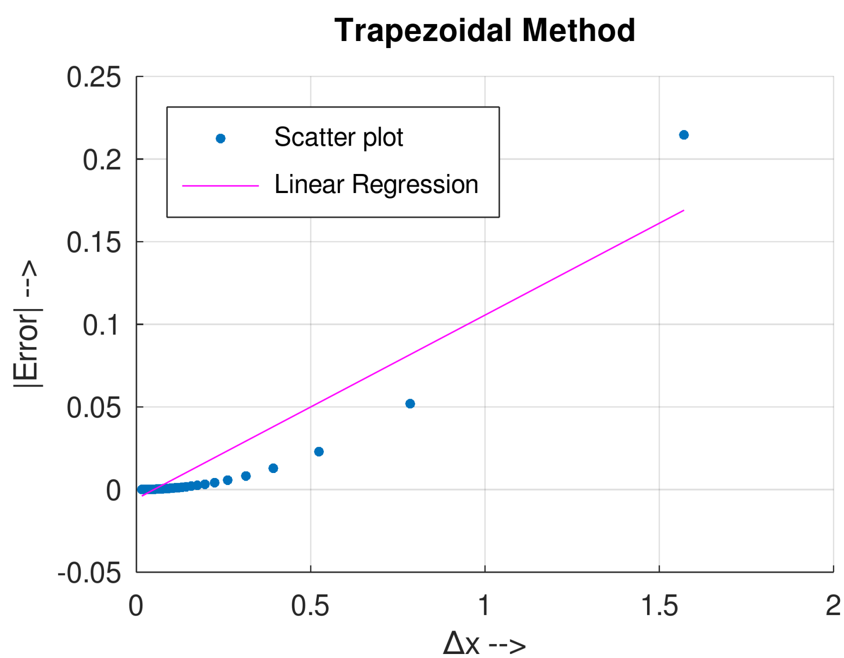

Figure 7: Absolute error vs Δx (Linear scale)

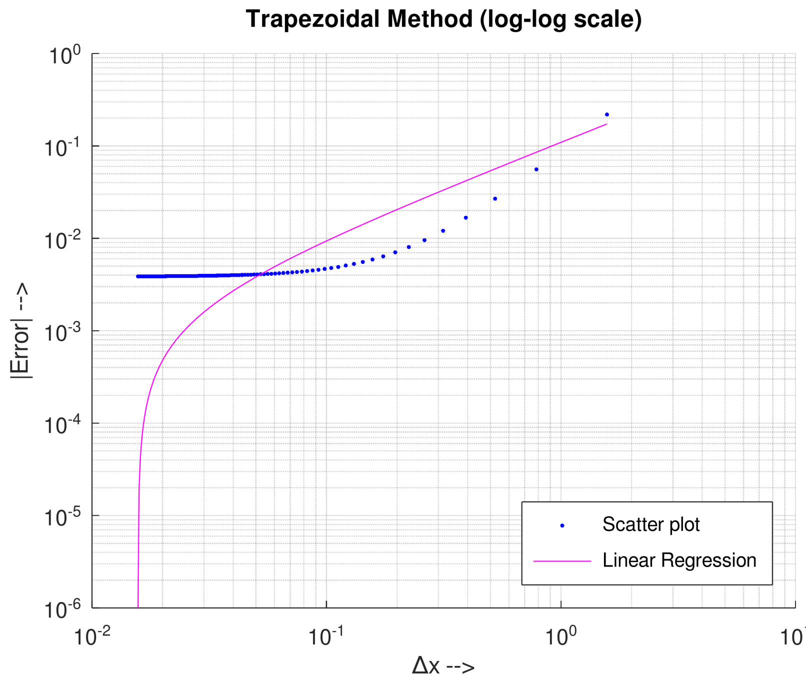

Figure 8: Absolute error vs Δx (Log-log scale)

3.2 I_{LE} and I_{RE} as functions of Δx:

The following plots show the values of I_{LE} and I_{RE} as we change Δx.

- $I_{LE} = 0$ for one segment$(n = 1)$, that is $\Delta x = 1.57$.
- I_{LE} value increases as we decrease Δx.
- As the number of segments n tend to ∞, I_{LE} tends to the true value of the integral $= 1$.
- $I_{RE} \approx 1.57$ for one sgment$(n = 1)$, that is $\Delta x = 1.57$.
- I_{RE} value decreases as we decrease Δx.
- As the number of segments n tend to ∞, I_{RE} tends to the true value of the integral $= 1$.

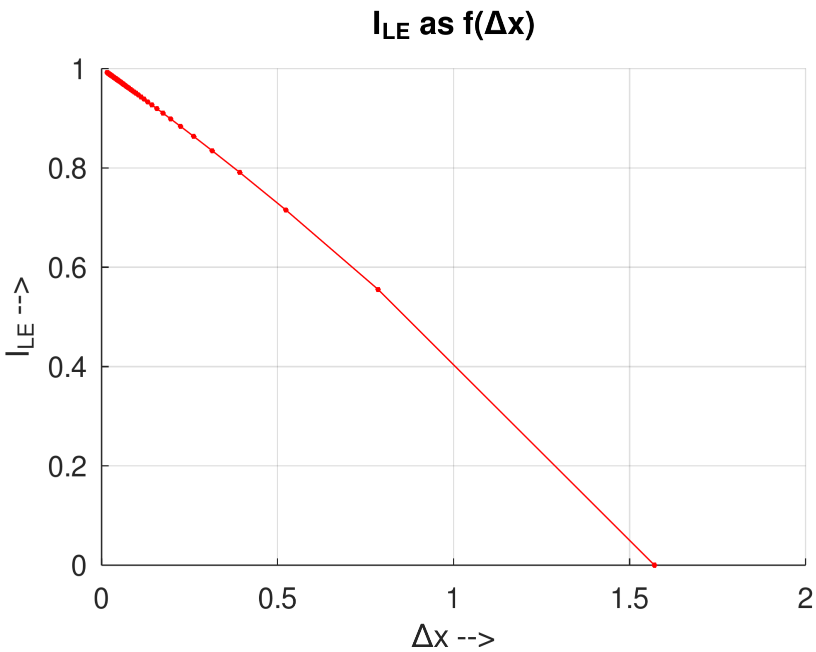

Figure 9: I_{LE} vs Δx

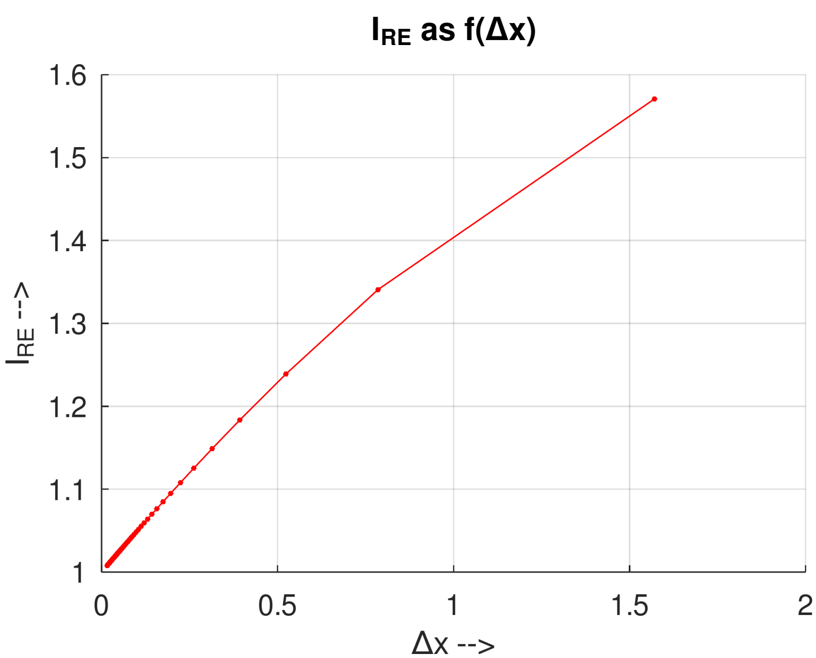

Figure 10: I_{RE} vs Δx

3.3 Computational Cost:

The following figure shows the computational cost for each of these methods.
We see that for larger values of Δx the midpoint method takes the least time, while for

smaller values of Δx the left endpoint method and the right endpoint method take the least time.
In general, the trapezoidal method has the highest computational cost, the midpoint method has a lower computational cost while the left and right endpoint methods have the least computational cost.

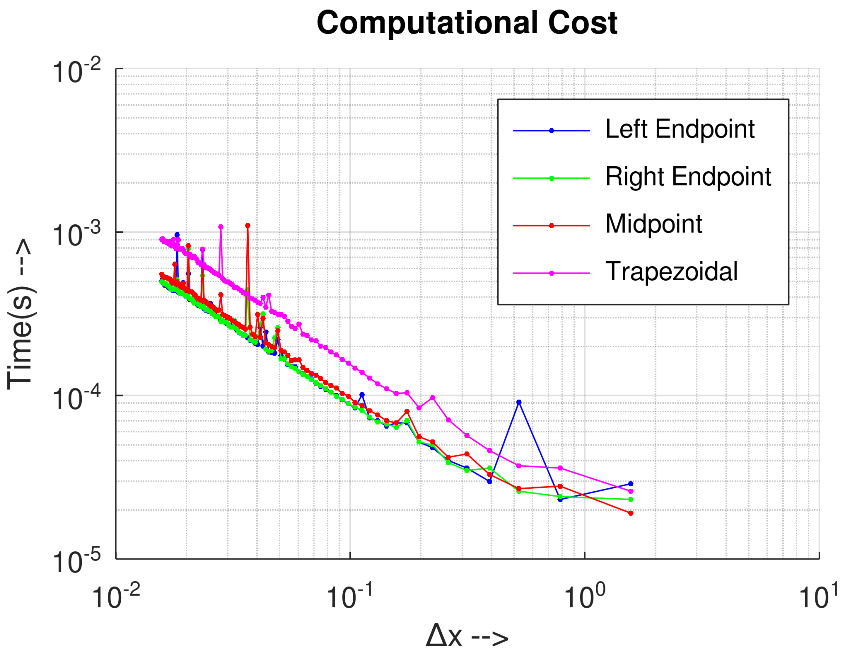

Figure 11: Time($seconds$) vs Δx

4 Conclusion

When selecting the best method of quadrature, we have to check both the errors and the computational cost.
The **slope** of the linear regression model gives us an indication of the magnitude of the error associated with a particular value of Δx.
If the slope is lesser, the error associated with a particular value of Δx is lesser.
Therefore, we would prefer that the slope be minimum.
Hence, using this criteria, for this integral, the midpoint method is the best with $Slope \approx 0.053$.
We also require that the computational cost be minimum so that we save time.
Hence, using this criteria, for this integral, the left or right endpoint methods are the best.
The choice of method is dependant on the requirements of the user.
Hence, I conclude that all of these methods of quadrature are extremely useful in computation of areas.