Hyperbolic and Trigonometric Functions

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1 Introduction

This report discusses the graphs, properties and applications of the hyperbolic functions $\sinh x$ and $\cosh x$ and trigonometric functions.The report also lists various formulae associated with these functions.

2 Hyperbolic Functions

The **hyperbolic trigonometric functions** are functions that extend the notion of the parametric equations for a unit circle ($x = \cos t$ and $y = \sin t$) to the parametric equations for a hyperbola, which yield the following two fundamental hyperbolic equations:

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2} \qquad \cosh x = \frac{\exp(x) + \exp(-x)}{2} \tag{1}$$

The graphs of these two functions can be seen in Figure 1.
Hyperbolic functions have several applications.
For instance, they are related to the curve one traces out when chasing an object that is moving linearly.
They also define the shape of a chain being held by its endpoints and are used to design arches that will provide stability to structures. This shape, defined as the graph of the function $y = \lambda \cosh \frac{y}{\lambda}$, is also referred to as a **catenary**.
Hyperbolas, which are closely related to the hyperbolic functions, also define the shape of the path a spaceship takes when it uses the **gravitational slingshot** effect to alter its course via a planet's gravitational pull propelling it away from that planet at high velocity.

2.1 Properties of $\sinh x$:

- Exponential definition:

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2} \tag{2}$$

- Complex trigonometric definition:

$$\sinh x = -i \sin(ix) \tag{3}$$

- The range of $\sinh(x)$: $\mathit{Range} = (-\infty, \infty)$.
  This is depicted in Figue 1.

- Odd function:

$$\sinh(-x) = -\sinh(x) \tag{4}$$

- Sums of arguments:

$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \tag{5}$$

- Subtraction formula:

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y) \tag{6}$$

- Derivative:

$$\frac{d\sinh(x)}{dx} = \cosh(x) \tag{7}$$

- Taylor Series:

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{8}$$

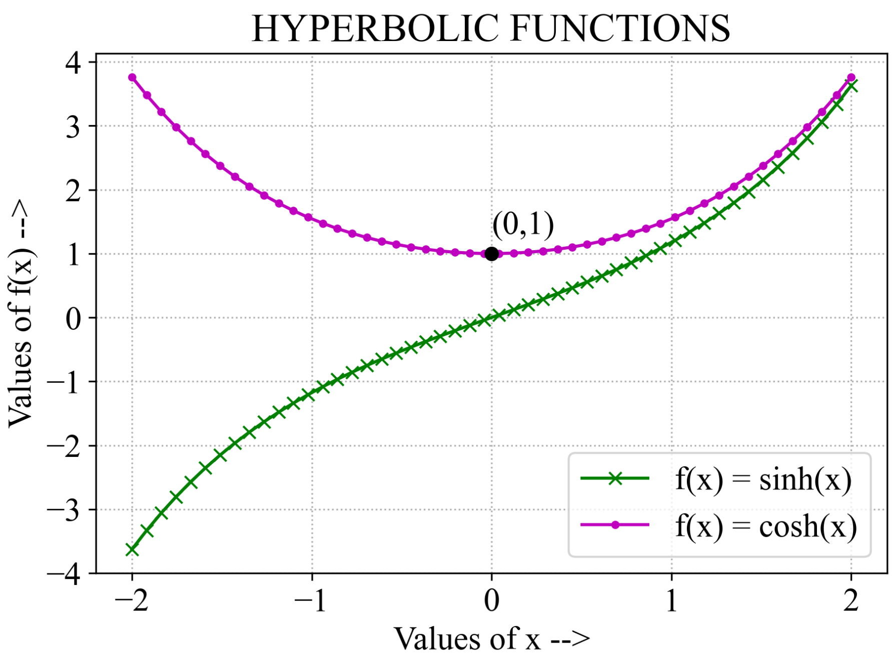

Figure 1: Hyperbolic sine and cosine functions from −2 to 2.

2.2 Properties of $\cosh x$:

- Exponential definition:

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2} \tag{9}$$

- Complex trigonometric definition:

$$\cosh x = \cos(ix) \tag{10}$$

- The $\cosh(x)$ function is always positive. $Range = [\,1, \infty)$.
  This has been shown in Figure 1 along with the minima point in cartesian co-ordinates $(0, 1)$.

- Even function:
$$\cosh(-x) = \cosh(x) \tag{11}$$

- Sums of arguments:
$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \tag{12}$$

- Subtraction formula:
$$\cosh(x - y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y) \tag{13}$$

- Derivative:
$$\frac{d\cosh(x)}{dx} = \sinh(x) \tag{14}$$

- Taylor Series:
$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \tag{15}$$

- Hyperbolic sine and cosine satisfy:
$$\cosh^2(x) - \sinh^2(x) = 1 \tag{16}$$

3 Trigonometric Functions

In mathematics, the trigonometric functions, also called **circular functions**, **angle functions** or **goniometric functions** are real functions which relate an angle of a right-angled triangle to ratios of two side lengths.
The six trigonometric function are: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$, $\sec(x)$, $\csc(x)$.
A few of the applications of trigonometric ratios are as follows:

- **Astronomy**: Trigonometric tables were created over decades ago for computations in this field. It helps in determining the distance between the stars and planets. The tables help in locating the position of a sphere and this kind of trigonometry is called **spherical trigonometry**.
- **Aviation**: It helps in determining the speed, direction, and distance of an airplane along with keeping in mind the speed and direction of the wind.
- **Measurements**: It is used to calculate the angle of elevation, angle of depression and lengths along different directions.
- **Construction**: When constructing a building, aspects like roof inclination, rook slopes, perpendicular and parallel walls, light angles, sun shading, etc, require trigonometry.

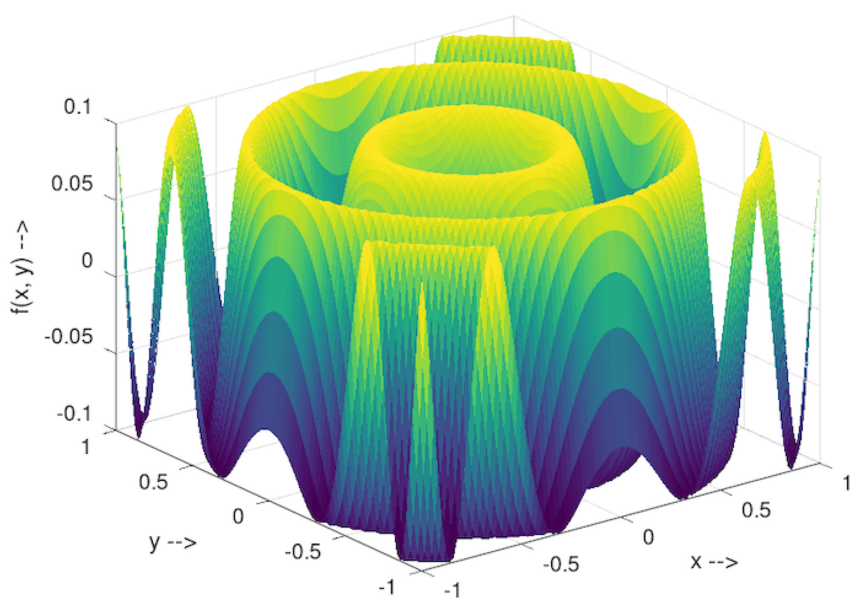

Figure 2: $f(x,y) = \sin(10(x^2 + y^2))/10$

3.1 Definition and properties of $\sin(x)$ and $\cos(x)$:

- Right-angled triangle definition:

$$sin(\theta) = \frac{opposite}{hypotenuse} \qquad cos(\theta) = \frac{base/adjacent}{hypotenuse} \tag{17}$$

- $Domain = (-\infty, \infty)$
  $Range = [-1, 1]$ (This is graphically depicted in Figure 2.)
  $Period = 2\pi$

- Unit-circle definition: For a unit circle mapped on the Euclidean plane using the cartesian co-ordinate system, for the point (x, y) lying on the circle: $x = \cos(\theta)$ and $y = \sin(\theta)$.

- First order derivatives:

$$\frac{d\sin(x)}{dx} = \cos(x) \qquad \frac{d\cos(x)}{dx} = -\sin(x) \tag{18}$$

- Taylor series:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{19}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{20}$$

- Odd function: $\sin(-\theta) = -\sin(\theta)$
  Even function: $\cos(-\theta) = \cos(\theta)$

4 Conclusion

The aim of this report is to enumerate the various properties and formulae associated with hyperbolic and trigonometric functions and to show their graphical representation. It also discusses the applications of these functions in practical situations.