Quadrature - II

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1 Introduction

This report discusses Gauss-Legendre quadrature, the algorithm to be employed, and describes Legendre polynomials. In addition, it discusses the accuracy and computational time required by the algorithm when applied to the function:

$$I = \int_{-1}^{1} f(x)dx \tag{1}$$

where,

$$f(x) = e^{-x}\sin^2(4x) \tag{2}$$

This method of quadrature has also been compared to the Midpoint method of quadrature using a Riemann sum.

2 Gaussian Quadrature

- **Quadrature rule:** An approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration.
- **n-point Gaussian quadrature rule:** A quadrature rule constructed to yield an exact result for polynomials of degree $2n-1$ or less by a suitable choice of the nodes x_i and weights w_i for $i = 1, ..., n$.
- The most common domain of integration for such a rule is taken as $[-1, 1]$, so the rule is stated as

  $$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \tag{3}$$

  which is exact for polynomials of degree $2n-1$ or less. Here,

  1. n is the number of sample points used.
  2. w_i are quadrature weights.
  3. x_i are the quadrature nodes.

3 Legendre Polynomials

Legendre polynomials: A system of complete and orthogonal polynomials, with a vast number of mathematical properties, and numerous applications. Denoted by $P_n(x)$.
Definition by construction as an orthogonal system: In this approach, the polynomials

are defined as an orthogonal system with respect to the weight function $w(x) = 1$ over the interval $[-1, 1]$. That is, $P_n(x)$ is a polynomial of degree n, such that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0 \quad \text{if } n \neq m \tag{4}$$

With the additional standardization condition $P_n(1) = 1$, all the polynomials can be uniquely determined.

Definition via generating function: The Legendre polynomials can also be defined as the coefficients in a formal expansion in powers of t of the generating function:

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{5}$$

Definition by Legendre's differential equation:

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{6}$$

Definition by Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n \tag{7}$$

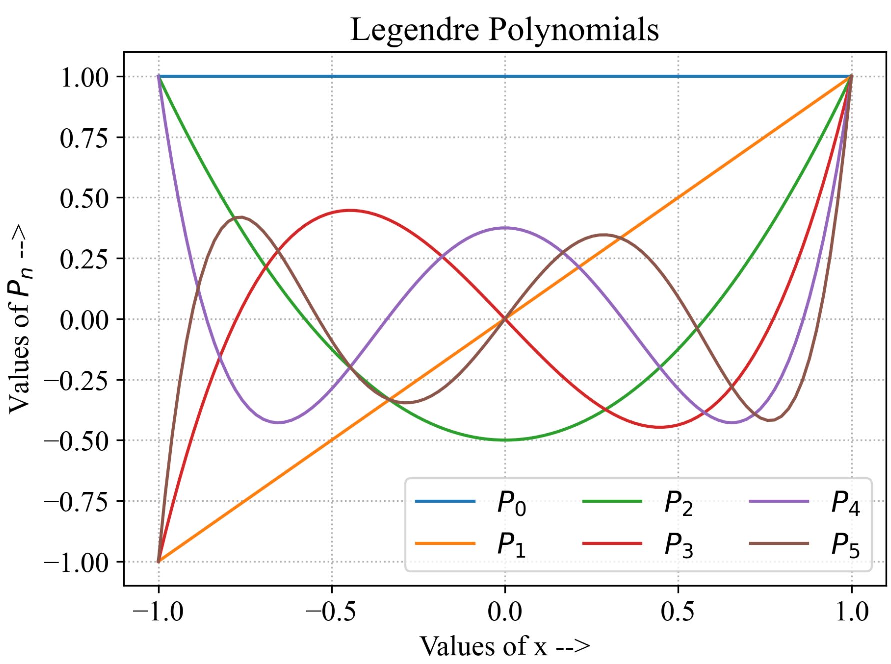

Figure 1: The first 5 Legendre polynomials.

The first five Legendre polynomials are as follows:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
- $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

4 Gauss-Legendre Quadrature

Gauss–Legendre quadrature is a form of Gaussian quadrature for approximating the definite integral of a function.
For integrating over the interval $[-1, 1]$, the associated orthogonal polynomials are Legendre polynomials denoted by $P_n(x)$ and the rule takes the form:

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \tag{8}$$

where,

1. n is the number of sample points used.
2. w_i are quadrature weights. The weights are given by the formula:

$$w_i = \frac{2}{(1 - x_i^2)\left[P'_n(x_i)\right]^2} \tag{9}$$

3. x_i are the roots of the n^{th} Legendre polynomial.

This choice of quadrature weights w_i and quadrature nodes x_i is the unique choice that allows the quadrature rule to integrate degree $2n - 1$ polynomials exactly.

5 Midpoint method:

In this method, we start by defining a co-ordinate system, and plotting the function f. This gives us a figure and an area under it.
We then subdivide our figure into smaller rectangles of base Δx. We add up all of these small areas to obtain an approximate value for the total area. This is the Riemann sum.
For this method, we approximate the function f at the midpoint for each interval.

This gives multiple rectangles with base Δx and height $f(a + \frac{(2i+1)}{2}\Delta x)$. Doing this for $i = 0, 1, \ldots, n-1$ and adding up the resulting areas gives

$$A_{MP} = \Delta x(f(a + \frac{\Delta x}{2}) + f(a + \frac{3}{2}\Delta x) + ... + f(b - \frac{\Delta x}{2})) \tag{10}$$

$$A_{MP} = \sum_{i=0}^{n-1} f(a + \frac{(2i+1)}{2}\Delta x)\Delta x \tag{11}$$

The error E_{MP} is given by:

$$E_{MP} = |A_t - A_{MP}| = |\int_a^b f(x)dx - A_{MP}| \tag{12}$$

where A_t is the true area of the figure which is the value of the definite integral.

6 Analysis

In this section, the variation in error as we change the number of terms N will be discussed. The differences between the Gauss-Legendre method and the right endpoint method will also be explained with reference to number of terms for a target error and computational cost. The entire analysis is with reference to the integral:

$$I = \int_{-1}^{1} f(x)dx \tag{13}$$

where,

$$f(x) = e^{-x}\sin^2(4x) \tag{14}$$

6.1 Error variation with number of terms:

The following figure shows the variation in error as we increase the number of sample points N.
We see that the error drops rapidly from about 1 to 10^{-15} as we increase N beyond 20.
The error is not monotonically decreasing. This is because we change our points of evaluation and weights with every N. This gives rise to a new set of terms everytime. Hence, while the error does drop overall, the decrease is not monotonic.

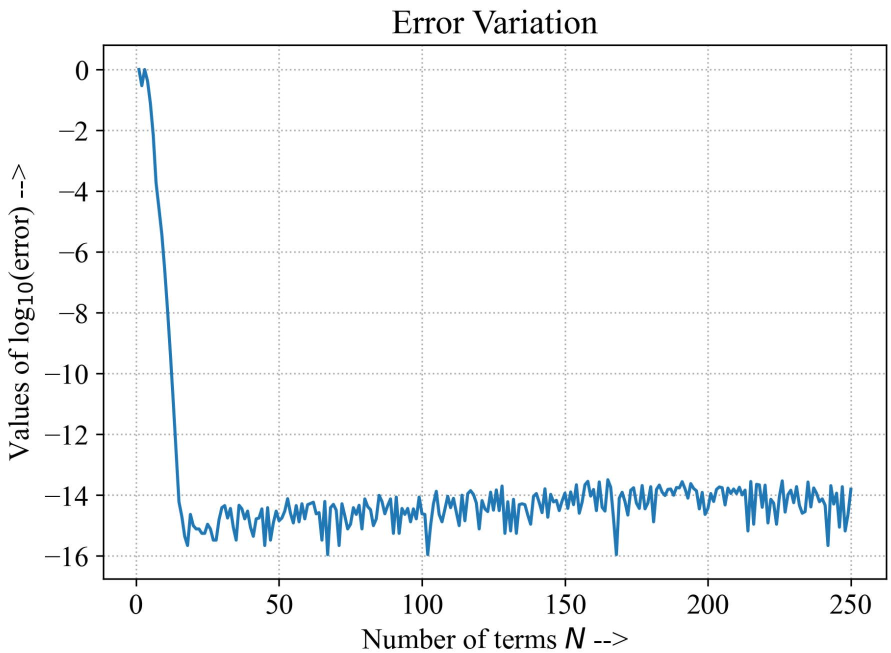

Figure 2: Variation in error with N.

6.2 Variation in number of terms for a given target error:

In this subsection, we compare the number of terms for a given target error between the Gauss-Legendre method and the best method of quadrature using Riemann sums.
Both midpoint method and trapezoidal method given results upto about the same accuracy for a particular N, hence here we consider midpoint method to be the best.
The number of segments required to get an error of $10^{-}8$ is over 15000, hence we stop plotting at $error = 10^{-7}$.
From the figure, we can infer that the Gauss-Legendre quadrature requires significantly fewer terms than the midpoint method. Hence, it is the more efficient method.

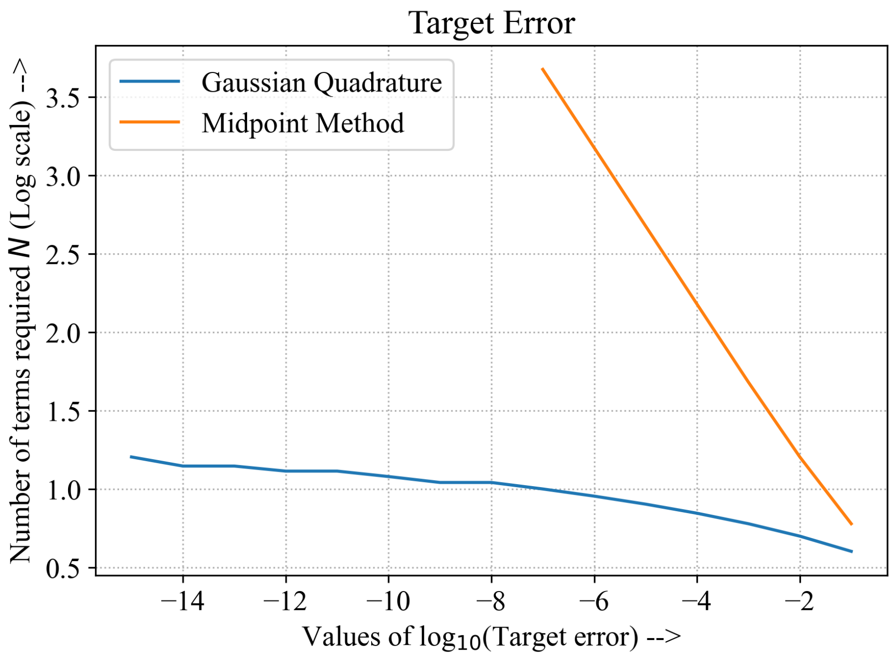

Figure 3: Variation in N for a target error.

6.3 Variation in computation time for a given target error:

In this subsection, we compare the time taken, for a given target error, by the Gauss-Legendre method and the midpoint method.
With for higher values of error the midpoint method is faster, but overall the Gauss-Legendre method is faster.
The slope for the midpoint method is steeper which implies that as we increase the precision required, the time taken by the midpoint method is more than that taken by the Gauss-Legendre method.
Hence, the Gauss-Legendre method is the more efficient method.

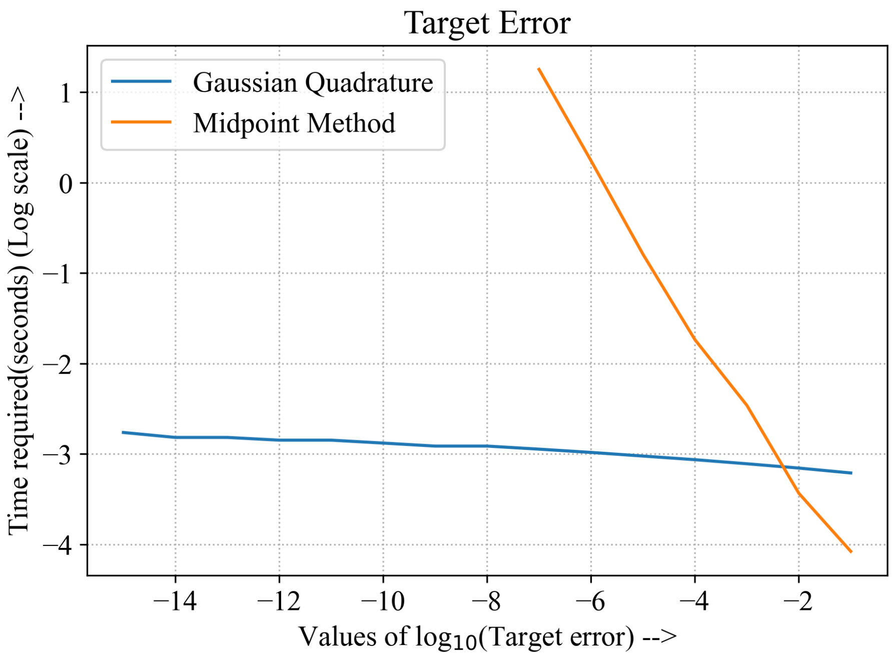

Figure 4: Variation in time for a target error.

7 Conclusion

The Gauss-Legendre method of quadrature is extremely useful and highly precise. Even for an infinite series the error drops rapidly which makes it very accurate. It is used in mathematical and engineering applications and is very advantageous.