MATHEMATICS

SECTION A

January 29, 2024

1 Vector:

1. For any two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, prove that

$$\left(\overrightarrow{a} \times \overrightarrow{b}\right)^2 = \overrightarrow{a}^2 \ \overrightarrow{b}^2 - \left(\overrightarrow{a} \cdot \overrightarrow{b}\right)^2$$

2. Find the equation of planes passing through the intersection of the planes $\overrightarrow{r} \cdot \left(2\hat{i} + 6\hat{j}\right) + 12 = 0$ and $\overrightarrow{r} \cdot \left(3\hat{i} - \hat{j} + 4\hat{k}\right) = 0$ and are at a unit distance from origin.

3. Find the vector equation of the line passing through $(2, 1, -1)$ and parallel to the line $\overrightarrow{r} = \left(\hat{i} + \hat{j}\right) + \lambda\left(2\hat{i} - \hat{j} + \hat{k}\right)$. Also, find the distance between these two lines.

2 Probability:

4. The probability of two A and B coming to school on time are $2/7$ and $4/7$, respectively. Assuming that the events 'A coming on time' and 'B coming on time' are independent, find the probability of only one of them coming to school on time.

5. If $P(A) = 0.6$, $P(B) = 0.5$ and $P(B \mid A) = 0.4$, find $P(A \cup B)$ and $P(A \mid B)$.

6. A bag contains 5 red and 3 black balls and another bag contains 2 red and 6 black balls. Two balls are drawn at random (*without replacement*) from one of the bags and both are found to be red. Find the probability that balls are drawn from the first bag.

3 Matrix:

7. $A = \begin{pmatrix} 8 & 2 \\ 3 & 2 \end{pmatrix}$,then find $\left|\text{adj } A\right|$.

8. Using the properties of determinants, prove the following :

$$\begin{vmatrix} a^2 & bc & ac + c^2 \\ a^2 & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

4 Differential equation:

9. If $y = 2\sqrt{\sec\left(e^{2x}\right)}$, then find $\dfrac{dy}{dx}$.
10. Find the differential equation of the y of curves represented by $y^2 = a\left(b^2 - x^2\right)$.
11. Find the particular solution of the differential equation: $\left(1 + e^{2x}\right)dy + \left(1 + y^2\right)e^x dx = 0$, given that $y(0) = 1$.
12. If $X^p y^q = (x + y)^{p+q}$, prove that $\dfrac{dy}{dx} = \frac{y}{x}$ and $\dfrac{d^2y}{dx^2} = 0$.
13. If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y), \left|x\right| < 1, \left|y\right| < 1$,show that $\dfrac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

5 Integration:

14. Find:

$$\int e^x (\cos x - \sin x) \operatorname{cosec}^2 x dx$$

15. Find:

$$\int (\sin x \cdot \sin 2x \cdot \sin 3x)\, dx$$

16. Evaluate:

$$\int_{-1}^{2} \left|x^3 - x\right| dx$$

6 Linear Form:

17. Using integration, find the area of the following region:

$$\left\{(x, y) : x^2 + y^2 \leq 16a^2 and y^2 \leq 6ax\right\}$$

7 Algebra:

18. Let an operation $*$ on the set of natural numbers N be defined by $a * b = a^b$. Find
    (a) whether $*$ is a binary or not, and
    (b) if it is a binary, then is it commutative or not.

8 Functions:

19. Find whether the function $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$; is increasing or decreasing in the interval $\frac{3\pi}{8} < x < \frac{5\pi}{8}$.

9 optimization:

20. The sum of the perimeters of circle and a square is **K**, where **K** is some constant. Prove that the sum of their areas is least when the side of the square is twice the radius of the circle.