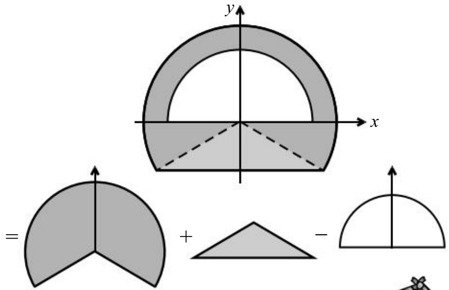
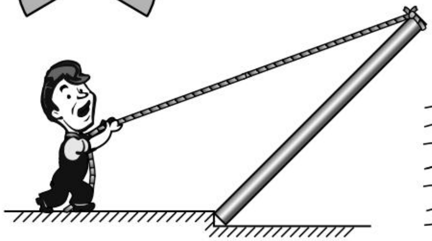
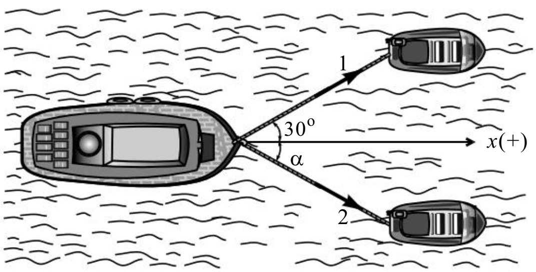


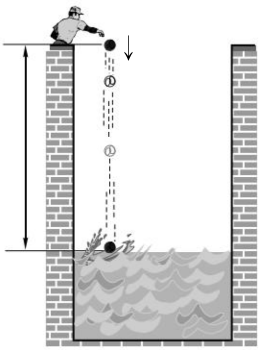


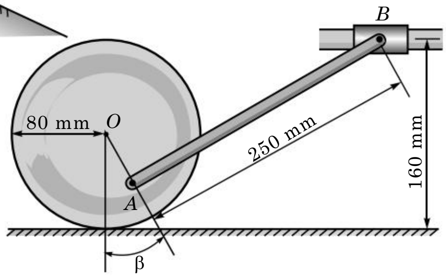
ENGINEERING MECHANICS

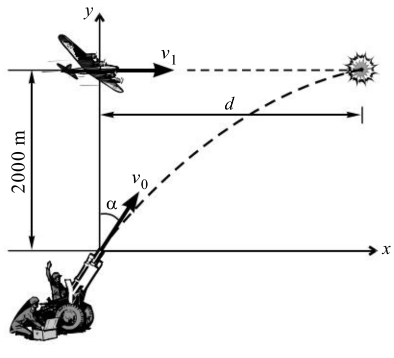
STATICS AND DYNAMICS

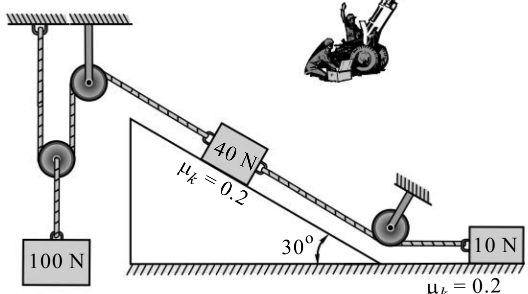
by N. H. Dubey

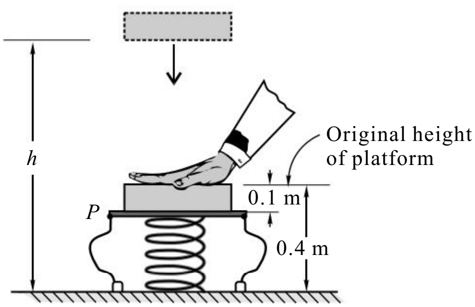


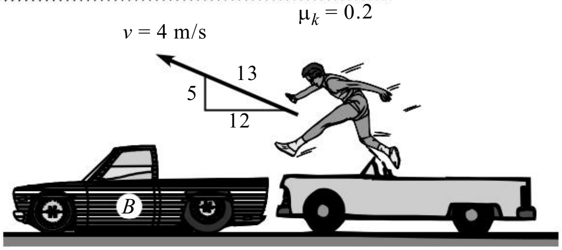












About the Author

Prof. N. H. Dubey is currently Visiting Professor at D. J. Sanghvi College of Engineering, Mumbai. He obtained his bachelor's degree in Production Engineering and postgraduate in Advertising and Public Relations. He has honed his computer skills at the Garware Institute. He has an industrial experience of one year with Premier Automobile Ltd. As of today, he has more that 26 years of teaching experience and is also associated with the Indian Society for Technical Education, New Delhi.

He is also the author of the following textbooks:

1. Engineering Drawing (Degree)

2. Machine Drawing (Solid Geometry)

3. Engineering Drawing (Diploma)

4. Computer Aided Engineering Drawing

5. Engineering Graphics

6. Technical Drawing Application

7. Question Bank in Engineering Drawing

His text are widely acknowledged by the teachers as well as student community alike.

Engineering Mechanics

Statics and Dynamics

N. H. Dubey

Visiting Professor
D. J. Sanghvi College of Engineering,
Mumbai.

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*❋ **Ek Tuhi Nirankar** ❋*

This book is dedicated to

Sadguru Baba Hardev Singhji Maharaj
of Sant Nirankari Mission

and

My Loving Parents

Late Shri. Hansraj M. Dubey

and

Late Smt. Kamaladevi H. Dubey

Preface

Engineering Mechanics is the branch of Physics which deals with the study of the effect of force system acting on a particle or a rigid body which may be at rest or in motion.

Engineering Mechanics is considered a basic subject for engineering students irrespective of branches, study of this subject helps develop the thinking, analytical ability and imaginative skill of the student.

Engineering Mechanics is the basic subject which supports many other subjects like Strength of Materials, Theory of Machines, Kinetics of Machines, Dynamics of Machines, Fluid Mechanics, Fluid Machines, Machine Design, Tool Design, etc. to apply the engineering concept for manufacturing of various products and projects such as automobile, aircrafts, electric motors, robots, construction of roadways, railways, bridges, dams, power transmission towers, projectile of missiles, satellites and many more.

Many years of teaching experience and interaction with the students has given me a clear picture of the difficulties faced by the students. I have made a dedicated attempt, while writing this book, to overcome all types of problems faced by students. A sincere effort has been made to develop an interest within the student and to motivate and develop analytical ability. The content of the book is presented very clearly and systematically.

Constructive suggestions and comments for improvement of the book are most welcome and will be appreciated.

N. H. Dubey

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About The Book

This book caters to the need of first-year engineering students desiring to achieve a firm foundation on the subject of Engineering Mechanics. It aims to support the learning of Statics and Dynamics with theoretical material, applications and a sufficient number of solved problems which have been selected from examination question papers and set in a sequential order. This text is a sincere attempt to make the subject simple and easy to understand.

Features

- An ideal offering for the complete course on Engineering Mechanics
- Unique chapter structure for self-study of students
- Exam-oriented pedagogy which is similar to examinations of various Universities:
  - 327 Solved Problems
  - 93 Review Questions
  - 101 Multiple Choice Questions
  - 365 Exercise Problems
  - 79 Fill in the blanks

The online learning center of this book can be accessed at **http: //www.mhhe.com/dubey/em1**. The site contains Solution Manual and PowerPoint Slides for Instructors.

Feedback

Constructive suggestions and comments for improvement of the book are most welcome and will be appreciated.

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Publisher's Note

Constructive suggestions and criticism always go a long way in enhancing any endeavour. We request all readers to email us their valuable comments/views/feedback for the betterment of the book at **info.india@mheducation.com** mentioning the title and author name in the subject line. Also, please feel free to report any piracy of the bok spotted by you.

Acknowledgement

I am thankful to my colleagues and many other friends who have directly/indirectly help me in preparing this book. I specially thank to my friend Prof. Manoj Jadhav for subject interaction.

I sincerely thank to all my engineering students for their co-operation, which has strengthen my passion in teaching for last twenty four years.

I owe sincere thanks to McGraw Hill Education (India), for taking deep interest and producing this book in the best possible form.

I am grateful to my wife, Sangeeta, and sons, Anand and Vishwas, for constant support and cheerful ambience. I acknowledge my deep gratitude to all the members of Dubey family for their warm encouragement.

I express my gratitude to village Manikpur, near Varanasi, who cradled me in their love and imbibed willpower and passion in me.

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  CMR Engineering College.

Content

Roadmap to the Syllabus

(As per the Latest Syllabus of R13)

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY

HYDERABAD

I-Year B.Tech.

Unit - I

- **INTRODUCTION TO ENGINEERING MECHANICS :** Basic Concepts, Resultants of Force System : Parallelogram Law, Forces and Components, Resultant of Coplanar Concurrent Forces, Components of Forces in Space, Moment of Force, Principle of Moments, Coplanar Applications, Couples, Resultant of Any Force System, Equilibrium of Force Systems, Free Body Diagrams, Equations of Equilibrium, Equilibrium of Planar Systems, Equilibrium of Spatial Systems.

GOTO - Chapters 1, 2, 3, and 4

Unit - II

- **FRICTION :** Introduction, Theory of Friction, Angle of Friction, Laws of Friction, Static and Dynamic Frictions, Motion of Bodies, Wedge, Screw, Screw-jack, and Differential Screw-jack.
- **TRANSMISSION OF POWER :** Flat Belt Drives, Types of Flat Belt Drives, Length of Belt, Tensions, Tight Side, Slack Side, Initial and Centrifugal, Power Transmitted and Condition for Maximum Power.

GOTO - Chapters 8 and 9

Unit - III

- **CENTROIDS AND CENTERS OF GRAVITY :** Introduction, Centroids and Centre of Gravity of Simple Figures (from Basic Principles), Centroids of Composite Figures, Theorem of Pappus, Center of Gravity of Bodies and Centroids of Volumes.
- **MOMENTS OF INERTIA :** Definition, Polar Moment of Inertia, Radius of Gyration, Transfer Formula for Moment of Inertia, Moment of Inertia for Composite Areas, Products of Inertia, Transfer Formula for Product of Inertia. **Mass Moment of Inertia :** Moment of Inertia of Masses, Transfer Formula for Mass Moments of Inertia, Mass Moment of Inertia of Composite Bodies.

GOTO - Chapters 5 and 6

Unit - IV

- **KINEMATICS OF A PARTICLE :** Motion of a Particle, Rectilinear Motion, Motion Curves, Rectangular Components of Curvilinear Motion, Kinematics of Rigid Body, Types of Rigid Body Motion, Angular Motion, Fixed Axis Rotation.
- **KINETICS OF PARTICLES :** Translation, Analysis as a Particle and Analysis as a Rigid Body in Translation, Equations of Plane Motion, Angular Motion, Fixed Axis Rotation, Rolling Bodies.

GOTO - Chapters 10, 11, 12, 13, 14 and 17

Unit - V

- **WORK - ENERGY METHOD :** Work - Energy Equations for Translation, Work-Energy Applications to Particle Motion, Work Energy Applied to Connected Systems, Work Energy Applied to Fixed Axis Rotation and Plane Motion, Impulse and Momentum.
- **MECHANICAL VIBRATIONS :** Definitions and Concepts, Simple Harmonic Motion, Free Vibrations, Simple and Compound Pendulums, Torsion Pendulum, Free Vibrations Without Damping, General Cases.

GOTO - Chapters 15, 16 and 18

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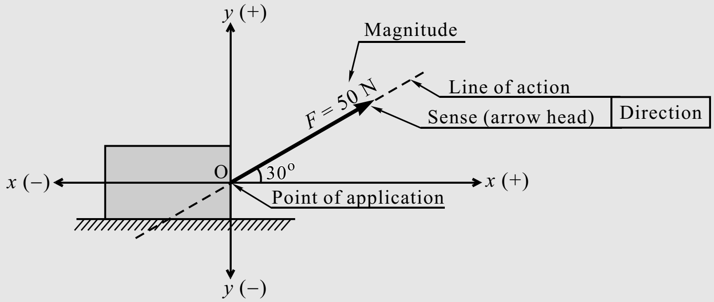


INTRODUCTION TO ENGINEERING MECHANICS

1.1 Introduction

- **Science :** *A systematic study of the structure and behaviour of natural and physical world based on facts that one can prove. Science is the study of living and non-living things*. There are various branches of science such as natural science, physical science, earth science, life science, etc. Physics, chemistry, mathematics, biology, etc., are a few subjects of science.
- **Physics :** It is *sub-branch of science which studies the property of matter and energy.* Deals with the study of mechanics, thermodynamics, electricity, magnetism, sound, light, nuclear physics, electronics, etc., among others.
- **Mechanics :** It is *the branch of physics which deals with the study of effect of force system acting on a particle or a rigid body which may be at rest or in motion.* The purpose of mechanics is to explain and predict physical phenomena and thus to lay the foundation for engineering application.

1.2 Applications of Engineering Mechanics

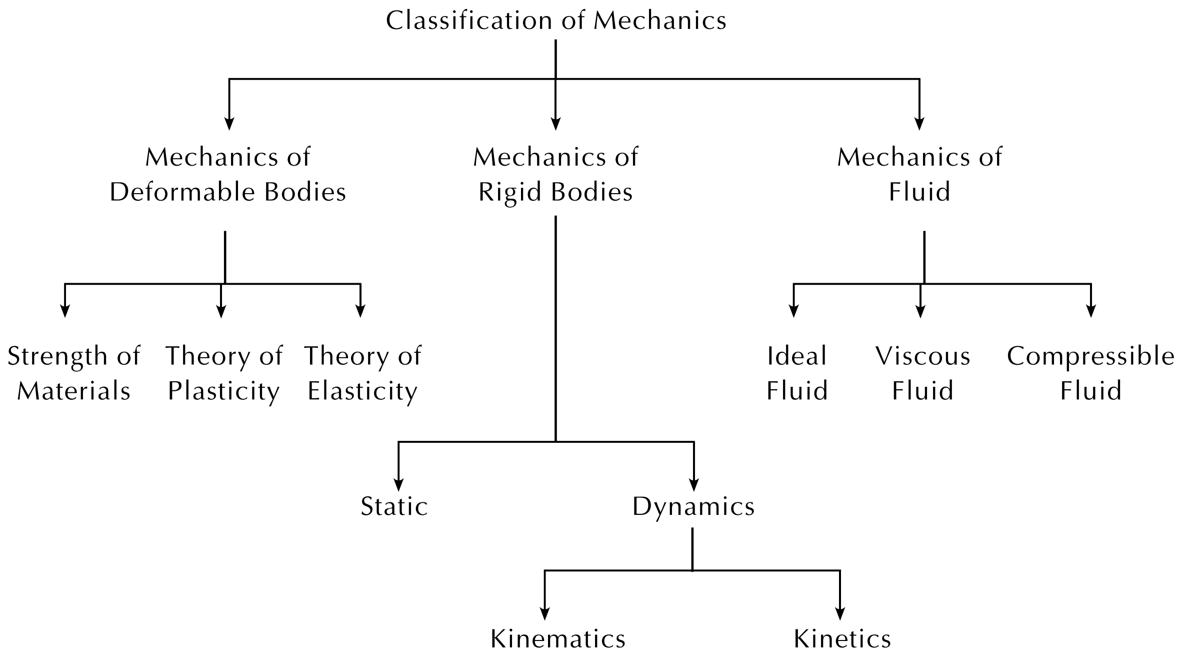
Engineering Mechanics : It is considered as one of the basic subjects for engineering students, irrespective of branches as it develops the thinking and imaginative skill of students. It supports many other subjects in manufacturing of various products and projects. Some of the applications of engineering mechanics are shown in Figs.1.2-i and ii.

Engineering : *It is the application of scientific knowledge which is used by an engineer to design and manufacture the product that serve the human society*.

Apart from what nature has given, man has produced many artificial goods which may range from a small pin to a huge multistorey building. The main role of an engineer is to apply the engineering concepts for manufacturing various products and work on projects such as automobiles, aircrafts, electric motors, robots, television, mobile, satellite, construction of roadways, railways, bridges, dams, power transmission towers, skyscrapers, projectile of missiles, launching of rockets, radar communication structure, trusses, lifting machines like crane, hoist, screw jack, elevators, conveyor belt, cargo ship, submarine, etc.

1.3 Classification of Mechanics

Mechanics can be classified broadly into the following :



- **Statics :** It is the study of the effect of force system acting on a particle or rigid body which is at rest.
- **Dynamics :** It is the study of the effect of force system acting on a particle or a rigid body which is in motion. It can also be stated as the study of *geometry of motion* with or without reference to the *cause of motion*.

  It must be noted that

  1. the study of geometry of motion means relationship between displacement, velocity, acceleration and time.

  2. with reference to *cause of motion* means mass and the force causing the motion are considered.

  The sub-branches of dynamics are kinematics and kinetics.
- **Kinematics :** It is the study of geometry of motion without reference to the cause of motion (i.e. mass and force causing motion are not considered).
- **Kinetics :** It is the study of geometry of motion with reference to the cause of motion (i.e. mass and force causing motion are considered).

1.4 Basic Concepts

- **Space :** It is *the region which extends in all directions and contains everything in it*. The concept of space is associated with the notion of the position of point P. The position of P can be defined by fundamental quantity length measured from certain reference point, called origin, in three given directions. These lengths are known as *coordinates of point P* (x,y,z).

- **Time :** It is *the measure of duration between two successive events.* Time is the basic quantity involved in the analysis of dynamics but not in statics.

- **Matter :** It is that *which occupies space and can be perceived by our senses.*

- **Mass :** It is *the quantity of matter contained in a body.* These quantities do not change on account of the position occupied by the body. The force of attraction exerted by the Earth on two different bodies with equal mass will be in same manner. Mass is *the property of body which measures its resistance to a change of motion*. Its S.I. unit is kg.

- **Scalar :** A physical quantity which requires only *magnitude* for its complete description is known as *scalar*. For example, distance, area, volume, mass, work, power, energy, time, density, speed, etc. Scalar quantities are added and subtracted by simple arithmetic methods.

- **Vector :** A physical quantity which requires both *magnitude* and *direction* for its complete description is known as *vector*. For example, force, displacement, velocity, acceleration, momentum, moment, couple, torque, impulse, weight, etc.

1.5 Idealisation in Mechanics

While studying the effect of force system acting on a particle or a rigid body which may be at rest or in motion, in mechanics some assumptions (idealisations) are made to simplify the problem without affecting the actual results.

- **Particle :** It is defined as an entity having considerable mass but negligible dimension.
  A body whose shape and size are not considered in analysis of problem and all the forces acting on a given body are assumed to act at a single point is considered to be a particle.

  Meaning of Particle in Engineering Mechanics : While studying the problem in engineering mechanics, dimensions of a particular body can be neglected and it can be assumed as particles for simplicity of solution. For example,

  1. An artificial satellite though large in size is assumed as a particle while studying its orbital motion around Earth. Here the size of satellite is negligible as compared to the Earth and the size of its orbit.

  2. A train moving from one station to another is under observation for kinematic quantity such as displacement, velocity and acceleration w.r.t. time. Here the size of train is negligible as compared to its distance between two stations which may be in kilometres, so the train is treated as a particle. It means the point (train) is moving from one position to another.

- **Rigid Body :** A solid body having considerable mass as well as dimension.

  Combination of large number of particles form a *body* and is defined as *the matter limited in all directions and has dimensions*.

  It is defined as *a body in which the particle does not change the relative position whatever large force may be applied.* In other words, *the body which is capable of withstanding changes in its shape and size and does not deform under the action of forces is termed as a rigid body.*

  For example, beams and columns of building structures do deform under the action of loads they carry, but the actual deformation that has taken place in structures is negligible and therefore, the body is assumed to be a rigid body.

Note : It is a hypothetical concept. No body is perfectly rigid in the universe. Assumption of a rigid body is made, in most of the cases, when the actual deformation that has taken place in structures are negligible and such assumptions of a rigid body helps for analysis of problem. Therefore, in engineering mechanics we are going to assume all given bodies as rigid bodies.

1.6 Laws of Mechanics (Fundamental Principles)

The study of mechanics depends upon few fundamental principles which are based on experimental evidence.

1. **Newton's First Law of Motion :** Every body continues in its state of rest or of uniform motion in a straight line unless an external force acts on it. Newton's first law defines the principle of the equilibrium of forces, which is the main topic of concern in *statics*.

2. **Newton's Second Law of Motion :** The rate of change of momentum of a body is directly proportional to the force acting on it and takes place in the direction of applied force.

$$\text{Thus , } F = \frac{mv - mu}{t} = m\,\frac{(v-u)}{t}$$

$$\therefore\ F = ma$$

   where F is the resultant force acting on a body of mass m moving with acceleration a.

   As per Newton's second law, the acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force ($F = ma$). This law forms the basis for most of the analysis in *dynamics*.

3. **Newton's Third Law of Motion :** To every action, there is an equal and opposite reaction. The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction and collinear. It means that forces always occur in pairs of equal and opposite forces.

4. **Newton's Law of Gravitation :** The force of attraction between any two bodies in the universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. For example, if m_1 and m_2 are the masses of two bodies and r is the distance between them, then the force of attraction F between them is given by

$$F \propto \frac{m_1 m_2}{r^2} \qquad \therefore\ F = \frac{G m_1 m_2}{r^2}$$

   where G is the universal gravitational constant.

5. **Principle of Transmissibility of Force :** It states that *the condition of equilibrium or uniform motion of rigid body will remain unchanged if the point of application of a force acting on a rigid body is transmitted to act at any other point along its line of action.*

Fig. 1.6-i

Refer to Fig. 1.6-i. A force F acting on the rigid body at point A can be replaced by the same force F at the point B provided points A and B lie on the same line of action of the force. Though the nature changes, as shown from PUSH to PULL, but the external effect remains unchanged due to the principle of transmissibility of force.

6. **Law of Parallelogram of Force :** If two forces acting simultaneously on a body at a point are represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of a parallelogram which passes through the point of intersection of the two sides representing the forces. Refer to Fig. 1.6-ii.

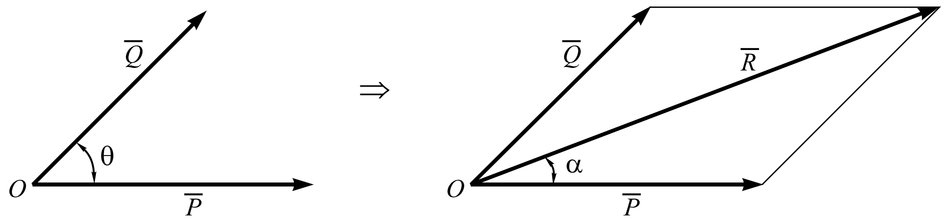


Fig. 1.6-ii

$$R = \sqrt{P^2 + Q^2 + 2\,PQ \cos\theta} \qquad \text{...(1.1)}$$

$$\tan\alpha = \frac{Q \sin\theta}{P + Q\cos\theta} \qquad \text{...(1.2)}$$

1.7 Concept of Force

Every body at rest has a tendency to remain at rest. Similarly, a body in motion has a tendency to remain in motion. This is known as the *property of inertia*. The state of the body changes only if an external agency acts on it.

1. **Force :** An external agency which changes or tends to change the state of rest or of uniform motion of a body upon which it acts is known as *force*.

2. **One Newton Force :** It is *a force required to produce an acceleration of* 1 m/s^2 *in a body of mass* 1 kg.

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

$$F = m \times a \qquad \text{...(1.3)}$$

$$\therefore\ 1\ \text{N} = 1\ \text{kg} \times 1\ \text{m/s}^2$$

3. **Characteristics of Force :**
   (a) Magnitude
   (b) Direction (line of action and sense)
   (c) Point of application

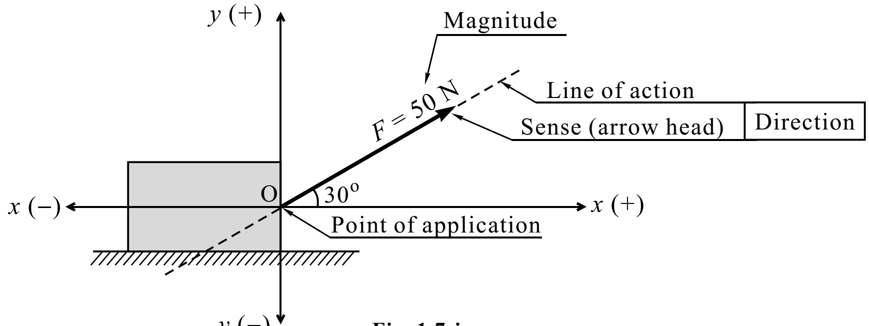


Fig. 1.7-i

4. **Graphical Representation of Force :** A force is represented graphically by drawing a *straight line parallel to the line of action of the force*. Taking some suitable scale for the magnitude of the force its *magnitude is represented by the length of this line drawn to scale* as shown in Fig. 1.7-ii. The direction is given by *the angle substended by the straight line with the reference axis* and sense by means of *an arrow head placed at the end of the line*.

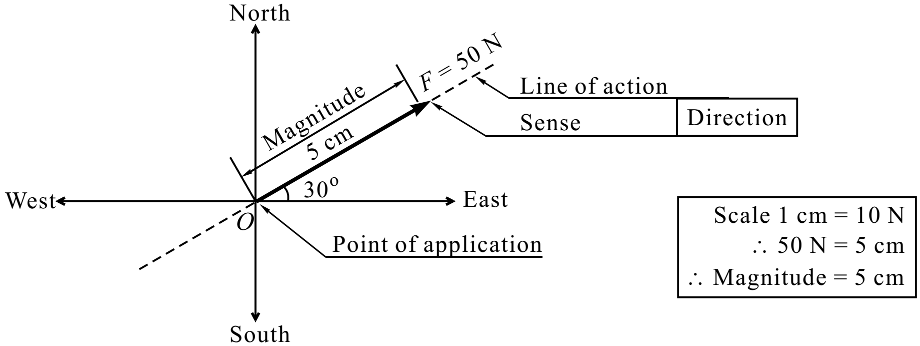


Fig. 1.7-ii

1.8 Classification of Force System

A force system may be classified as follows :

Force System
- Coplanar
  - Concurrent
  - Parallel
    - Like
    - Unlike
  - Non-concurrent and Non-parallel
- Non-coplanar
  - Concurrent
  - Parallel
    - Like
    - Unlike
  - Non-concurrent and Non-parallel

Force System : When a number of forces act simultaneously on a body then they are said to form a *force system*.

Depending upon whether the line of action of all the forces acting on the body lies in the same plane or in different plane, the force system may be classified as follows :

(i) ***Coplanar Force System* :** If the line of action of all the forces in the system lies on the same plane then it is called a *coplanar force system*.

(ii) ***Non-coplanar Force System* :** If the line of action of all the forces in the system do not lie on the same plane then it is called a *non-coplanar force system*.

These two force systems can be subclassified into three groups:

(a) ***Concurrent Force System* :** If the line of action of all the forces in the system passes through single point then it is called a *concurrent force system*. It can be further subclassified into coplanar concurrent force ans non-coplanar concurrent force (Fig. 1.8-i).

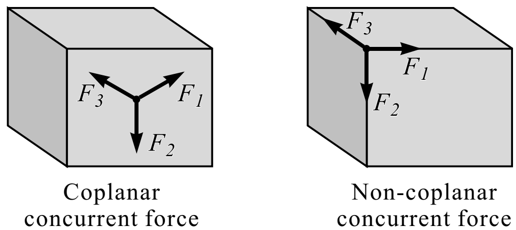


Fig. 1.8-i

(b) ***Parallel Force System* :** If the line of action of all the forces in the system are parallel to each other then it is called a *parallel force system*. Parallel force system can be further subclassified into two groups, *like* and *unlike*.

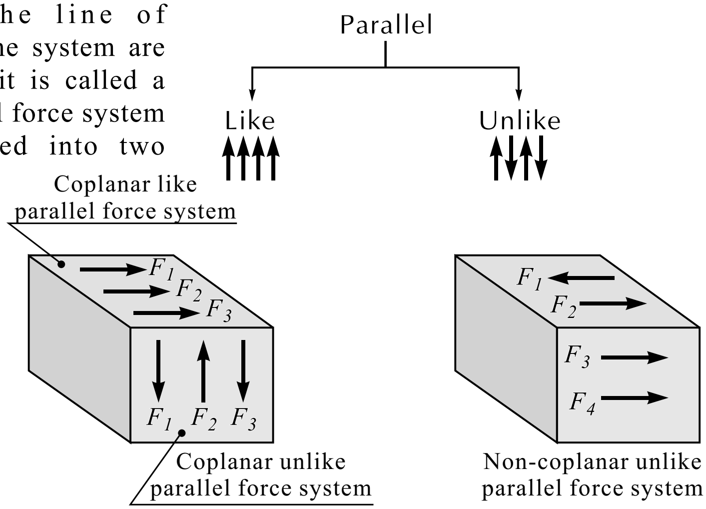


Fig. 1.8-ii

(c) ***General Force System* :** If the line of action of all the forces in the system are neither concurrent nor parallel, then it is known as *non-concurrent and non-parallel force system*.

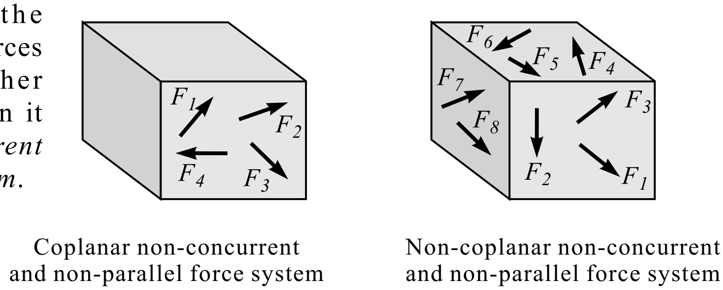
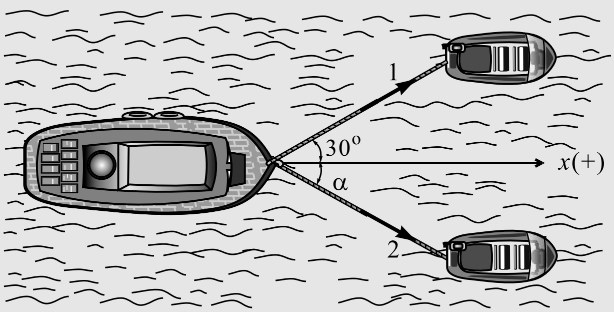


Fig. 1.8-iii

2

Resultant of Force System



2.1 Introduction

Resultant Force is *a single force which replaces the given force system having the same effect.*

In this chapter, we shall learn to find the resultant of force system for

1. concurrent force system,
2. parallel force system and
3. general force system.

The process of finding the resultant of any number of forces is called the *composition of forces*.

Composition of Forces

Forces may be combined (added) to obtain a single force which produces the same effect as the original system of forces. This single force is known as *resultant force*. The process of finding the resultant of forces is called *composition of forces*.

Force is a vector quantity. The method of addition of forces (vectors) is based on the parallelogram law.

Parallelogram Law

The resultant of any two non-collinear concurrent forces may be found by this law which states that "*If two forces acting simultaneously on a body at a point are represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of parallelogram which passes through the point of intersection of the two sides representing the forces*".

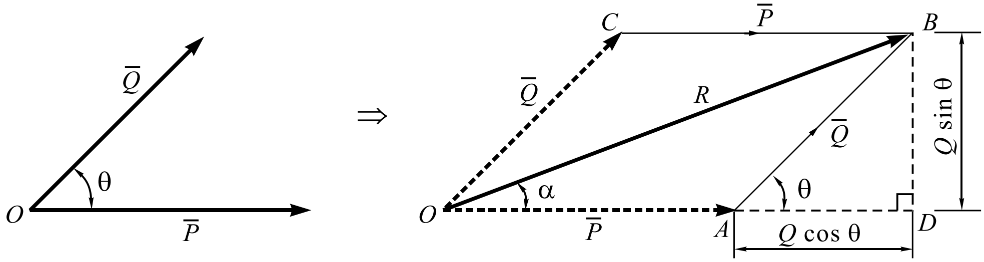


Fig. 2.1-i

Let P and Q be the two concurrent forces with an included angle θ acting at and away from the point O as shown in Fig. 2.1-i. They are represented by two adjacent sides OA and OC of the parallelogram $OABC$.

Draw a perpendicular from point B on OA extended, meeting at point D. As OC is parallel to AB and $OC = AB = Q$, $OA = P$ and $OB = R$.

In ΔODB,

$$OB^2 = OD^2 + BD^2$$

$$OB^2 = (OA + AD)^2 + BD^2$$

$$R^2 = (P + Q \cos \theta)^2 + (Q \sin \theta)^2$$

$$R^2 = P^2 + 2\,PQ \cos^2 \theta + Q^2 \cos^2 \theta + Q^2 \sin^2 \theta$$

$$R^2 = P^2 + Q^2 + 2\,PQ \cos \theta$$

Magnitude of resultant $R = \sqrt{P^2 + Q^2 + 2\,PQ \cos \theta}$ ...(2.1)

In ΔOBD : Let α be the angle made by R with P,

$$\tan \alpha = \frac{BD}{OD} = \frac{BD}{OA + AB}$$

$$\tan \alpha = \frac{Q \sin \theta}{P + Q \cos \theta} \qquad ...(2.2)$$

Triangle Law of Force (Corollary of Parallelogram Law)

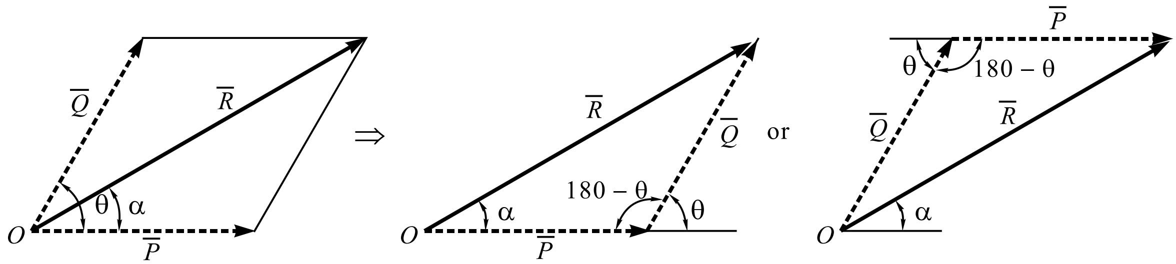


Fig. 2.1-ii

If two forces are represented by their force vector placed tip to tail; their resultant is the vector directed from the tail of first vector to the tip of the second vector. Refer to Fig. 2.1-ii.

By cosine rule, we have

$$\therefore\ R^2 = P^2 + Q^2 - 2\,PQ \cos (180 - \theta)$$

$$\therefore\ R^2 = P^2 + Q^2 + 2\,PQ \cos \theta$$

$$\therefore\ R = \sqrt{P^2 + Q^2 + 2\,PQ \cos \theta}$$

2.1.1 Solved Problems on Composition of Forces by Parallelogram and Triangle Law

Problem 1

Find the resultant of the given forces.

(i) By Parallelogram law

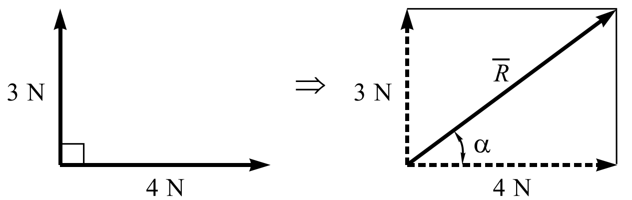


Fig. 2.1(a) : Given

$$\therefore\ R = \sqrt{4^2 + 3^2 + 2 \times 3 \times 4 \cos 90^o}$$

$R = 5$ N ***Ans.***

$$\tan \alpha = \frac{3 \sin 90^o}{4 + 3 \cos 90^o}$$

$$\therefore\ \alpha = 36.87^o$$

(ii) By Triangle law

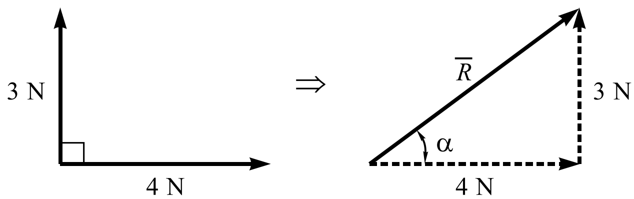


Fig. 2.1(b) : Given

By cosine rule

$$\therefore\ R = \sqrt{4^2 + 3^2 - 2 \times 3 \times 4 \cos 90^o}$$

$R = 5$ N ***Ans.***

By sine rule

$$\frac{R}{\sin 90^o} = \frac{3}{\sin \alpha}$$

$$\sin \alpha = \frac{3}{5}$$

$$\therefore\ \alpha = 36.87^o$$

Problem 2

Find the resultant of the given forces.

(i) By Parallelogram law

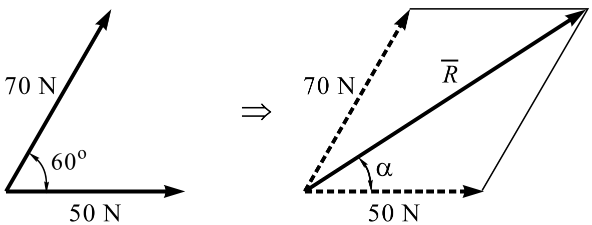


Fig. 2.2(a) : Given

$$R = \sqrt{50^2 + 70^2 + 2 \times 50 \times 70 \cos 60^o}$$

$R = 104.4$ N ***Ans.***

$$\tan \alpha = \frac{70 \sin 60^o}{50 + 70 \cos 60^o}$$

$\tan \alpha = 0.7132$ $\therefore\ \alpha = 35.5^o$

(ii) By Triangle law

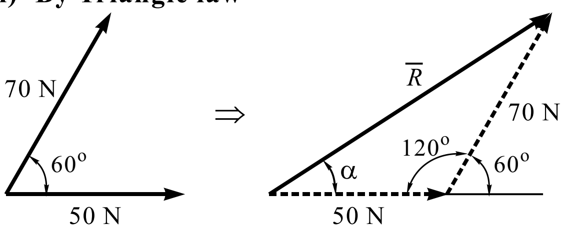


Fig. 2.2(b) : Given

By cosine rule

$$R = \sqrt{50^2 + 70^2 - 2 \times 50 \times 70 \cos 120^o}$$

$R = 104.4$ N ***Ans.***

By sine rule

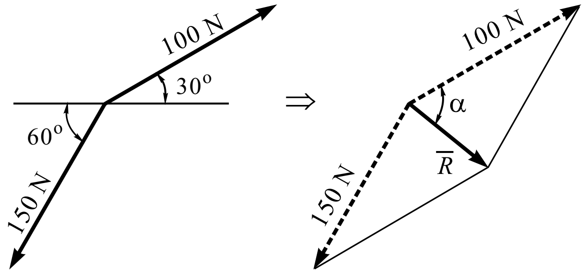
$$\frac{R}{\sin 120^o} = \frac{70}{\sin \alpha}$$

$$\sin \alpha = \frac{70 \sin 120^o}{104.4} \qquad \therefore\ \alpha = 35.5^o$$

Problem 3

Find the resultant of the given forces.

(i) By Parallelogram law



$$R = \sqrt{100^2 + 150^2 + 2 \times 100 \times 150 \cos 150^\circ}$$

$$R = 80.74 \text{ N} \quad \textit{Ans.}$$

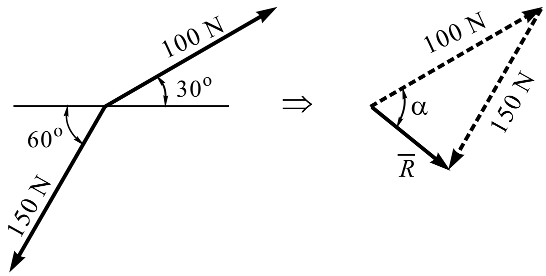
$$\tan\alpha = \frac{150 \sin 150^\circ}{100 + 150 \cos 150^\circ}$$

$$\tan\alpha = |-2.51|$$

$$\therefore \alpha = 68.26^\circ$$

Fig. 2.3(a) : Given

(ii) By Triangle law



By cosine rule

$$R = \sqrt{100^2 + 150^2 - 2 \times 100 \times 150 \cos 30^\circ}$$

$$R = 80.74 \text{ N} \quad \textit{Ans.}$$

By sine rule

$$\frac{R}{\sin 30^\circ} = \frac{150}{\sin\alpha}$$

$$\sin\alpha = \frac{150 \sin 30^\circ}{80.74} = 0.9289$$

$$\therefore \alpha = 68.26^\circ$$

Fig. 2.3(b) : Given

Problem 4

Two forces of 400 N and 600 N act at an angle 60° to each other. Determine the resultant in magnitude and direction if **(i)** the forces have same sense, and **(ii)** the forces have different senses.

Solution

Case (i) : Forces have same sense

Method I : Parallelogram Law

Let both forces be pull forces.

Magnitude of resultant R is given by

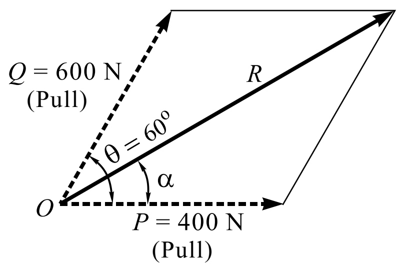


Fig. 2.4(a)

$$R = \sqrt{P^2 + Q^2 + 2\,PQ \cos\theta}$$

$$\therefore R = \sqrt{(400)^2 + (600)^2 + 2 \times 400 \times 600 \cos 60^\circ}$$

$$\therefore R = 871.78 \text{ N}$$

Direction of the resultant

$$\tan\alpha = \frac{Q \sin\theta}{P + Q\cos\theta}$$

$$\therefore \tan\alpha = \frac{600 \sin 60^\circ}{400 + 600 \cos 60^\circ} = 0.742$$

$$\therefore \alpha = \tan^{-1}(0.742) = 36.59^\circ$$

Method II : Triangle Law

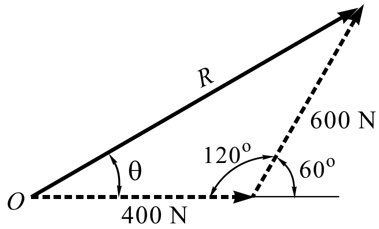


Fig. 2.4(b)

By cosine rule

$$R = \sqrt{(400)^2 + (600)^2 - 2 \times 400 \times 600 \cos 120^o}$$

$$\therefore \; R = 871.78 \text{ N}$$

By sine rule

$$\frac{600}{\sin \theta} = \frac{871.78}{\sin 120^o}$$

$$\therefore \; \theta = 36.59^o$$

$$\therefore \text{ resultant } R = 871.78 \text{ N} \quad (\measuredangle 36.59^o) \quad \textbf{\textit{Ans.}}$$

Method III : Resolution of Force

$$\Sigma F_x = 400 + 600 \cos 60^o = 700 \text{ N } (\rightarrow)$$

$$\Sigma F_y = 600 \sin 60^o = 519.62 \text{ N } (\uparrow)$$

Magnitude of resultant R

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$R = \sqrt{(700)^2 + (519.62)^2}$$

$$R = 871.78 \text{ N}$$

Inclination of resultant θ

$$\tan \theta = \frac{\Sigma F_y}{\Sigma F_x} \quad \therefore \; \theta = \tan^{-1}\left(\frac{519.62}{700}\right)$$

$$\therefore \; \theta = 36.59^o$$

$$\therefore \text{ resultant } R = 871.78 \text{ N} \quad (\measuredangle 36.59^o) \quad \textbf{\textit{Ans.}}$$

Case (ii) : When both forces have different senses

Method I : Parallelogram Law

Let P be a pull force and Q be push force.

Convert Q as a pull force by extending its line of action.
($\because$ Principle of transmissibility of force)

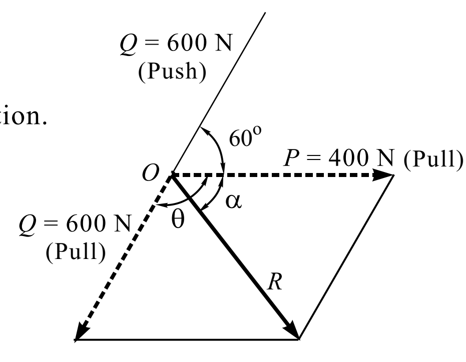


Fig. 2.4(c)

$$\therefore \; \theta = 180^o - 60^o = 120^o$$

Magnitude of the resultant

$$R = \sqrt{P^2 + Q^2 + 2\,PQ \cos \theta}$$

$$\therefore \; R = \sqrt{400^2 + 600^2 + 2 \times 400 \times 600 \cos 120^o}$$

$$\therefore \; R = 529.15 \text{ N}$$

Direction of the resultant

$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta} = \frac{600\sin 120^o}{400 + 600\cos 120^o} = 5.196$$

$$\therefore\ \alpha = \tan^{-1}(5.196) = 79.11^o$$

Method II : Triangle Law

By cosine rule

$$R = \sqrt{(400)^2 + (600)^2 - 2\times 400\times 600\cos 60^o}$$

$$\therefore\ R = 529.15\text{ N}$$

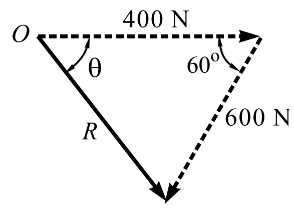


Fig. 2.4(d)

By sine rule

$$\frac{600}{\sin\theta} = \frac{529.15}{\sin 60^o}$$

$$\therefore\ \sin\theta = 6.982$$

$$\therefore\ \theta = 79.11^o$$

$\therefore$ resultant $R = 529.15$ N $\left(\searrow 79.11^o\right)$ ***Ans.***

Method III : Resolution of Force

$$\Sigma F_x = 400 - 600\cos 60^o$$

$$\therefore\ \Sigma F_x = 100\text{ N }(\rightarrow)$$

$$\Sigma F_y = -600\sin 60^o = -519.62$$

$$\therefore\ \Sigma F_y = 519.62\text{ N }(\downarrow)$$

Magnitude of resultant R

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$R = \sqrt{(100)^2 + (519.62)^2}$$

$$R = 529.15\text{ N}$$

Inclination of resultant angle θ

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x}$$

$$\therefore\ \theta = \tan^{-1}\left(\frac{519.62}{100}\right)$$

$$\therefore\ \theta = 79.11^o$$

$\therefore$ resultant $R = 529.15$ N $\left(\searrow 79.11^o\right)$ ***Ans.***

Problem 5

A car is pulled by means of two ropes as shown in Fig. 2.5(a). The tension in one rope is P = 2.6 kN. If the resultant of two forces applied at O is directed along the x-axis of the car. Find the tension in the other rope and magnitude of the resultant.

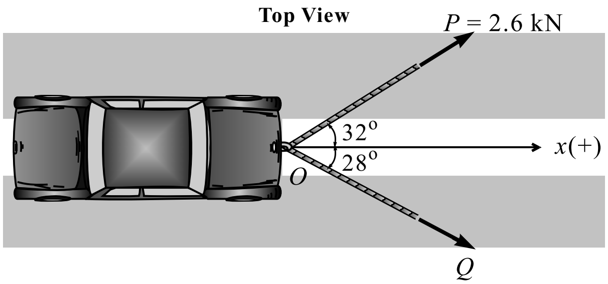


Fig. 2.5(a)

Solution

(i) *Method I : Parallelogram Law*

By parallelogram law, we have

$$\tan 28^\circ = \frac{2.6 \sin 60^\circ}{Q + 2.6 \cos 60^\circ}$$

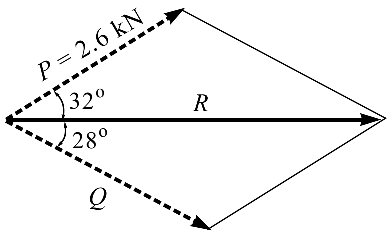


Fig. 2.5(b)

$$Q \tan 28^\circ + 2.6 \cos 60^\circ \times \tan 28^\circ = 2.6 \sin 60^\circ$$

$\therefore$ $Q = 2.934$ kN ***Ans.***

$$\therefore\ R = \sqrt{(2.6)^2 + (2.934)^2 + 2 \times 2.6 \times 2.934 \cos 60^\circ}$$

$\therefore$ $R = 4.8$ kN $(\rightarrow)$ ***Ans.***

(ii) *Method II : Triangle Law*

By sine rule

$$\frac{2.6}{\sin 28^\circ} = \frac{Q}{\sin 32^\circ} = \frac{R}{\sin 120^\circ}$$

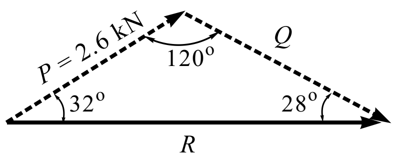


Fig. 2.5(c)

$$Q = \frac{2.6 \sin 32^\circ}{\sin 28^\circ} \qquad R = \frac{2.6 \sin 120^\circ}{\sin 28^\circ}$$

$Q = 2.934$ kN $\left(\searrow 28^\circ\right)$ $\qquad R = 4.796$ kN $(\rightarrow)$ ***Ans.***

(iii) *Method III : Resolution of Force*

$\because$ Resultant is horizontal

$\therefore$ $\Sigma F_y = 0$

$\therefore$ $2.6 \sin 32^\circ - Q \sin 28^\circ = 0$

$\therefore$ $Q = 2.934$ kN $\left(\searrow 28^\circ\right)$ ***Ans.***

$\because$ Resultant is horizontal

$\therefore$ $R = \Sigma F_x$

$R = 2.6 \cos 32^\circ + 2.934 \cos 28^\circ$

$R = 4.796$ kN $(\rightarrow)$ ***Ans.***

Problem 6

Find the magnitude of forces F_1 and F_2 if they act at right angle, their resultant is $\sqrt{34}$ N. If they act at an angle 60^o; their resultant is 7 N.

Solution

(i) By parallelogram law, we have

$$\text{Resultant } R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos\theta}$$

$$\sqrt{34} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos 90^o}$$

$$34 = F_1^2 + F_2^2 \quad \text{... (I)}$$

$$7 = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos 60^o}$$

$$49 = F_1^2 + F_2^2 + 2F_1F_2 \cos 60^o \quad \text{... (II)}$$

(ii) Putting the value of Eq. (I) in Eq. (II)

$$49 = 34 + 2F_1F_2 \cos 60^o$$

$$49 = 34 + F_1F_2$$

$$F_1F_2 = 15$$

$$\therefore F_1 = \frac{15}{F_2} \quad \text{... (III)}$$

Putting the value in Eq. (I), we get

$$34 = \left(\frac{15}{F_2}\right)^2 + F_2^2$$

$$34F_2^2 = F_2^4 + 225$$

$$F_2^4 - 34F_2^2 + 225 = 0$$

Solving quadratic equation, we get

$F_2 = 5$ N and $F_2 = 3$ N

From Eq. (III), we get

If $F_2 = 5$ N then $F_1 = 3$ N

If $F_2 = 3$ N then $F_1 = 5$ N ***Ans.***

Problem 7

To move a boat uniformly along the river at a given speed, a resultant force R = 520 N is required. Two men pull with force P and Q, by means of ropes, to do this. The ropes makes an angle of 30° and 40° respectively with the sides of the river as shown in Fig. 2.7. **(i)** Determine the force P and Q, **(ii)** If θ_1 = 30°, find the value of θ_2 such that the force in the rope Q is minimum. What is the minimum force Q ?

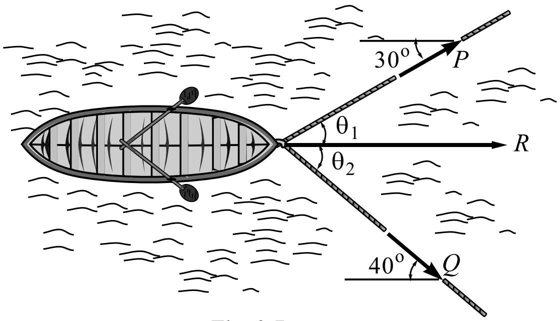


Fig. 2.7

Solution

(i) $R_x = \Sigma F_x$

$520 = P \cos 30^\circ + Q \cos 40^\circ$

$520 = 0.866\, P + 0.766\, Q$ ... (I)

$\Sigma F_y = 0$

$0 = P \sin 30^\circ - Q \sin 40^\circ$

$0 = 0.5\, P - 0.6428\, Q$ ... (II)

Solving Eqs. (I) and (II), we get

$P = 355.7$ N and $Q = 276.68$ N ***Ans.***

(ii) For the force Q to be minimum, force P and Q should be perpendicular to each other.

$\theta_1 + \theta_2 = 90^\circ$ $(\theta_1 = 30^\circ$ given)

$\therefore \theta_2 = 60^\circ$

$R_x = \Sigma F_x$

$520 = P \cos 30^\circ + Q_{min} \cos 60^\circ$

$520 = 0.866\, P + 0.5\, Q_{min}$ ... (III)

$\Sigma F_y = 0$

$0 = P \sin 30^\circ - Q_{min} \sin 60^\circ$

$0 = 0.5\, P - 0.866\, Q_{min}$ ... (IV)

Solving Eqs.(III) and (IV), we get

$Q_{min} = 260$ N ***Ans.***

2.2 Resolution of Force

The process of breaking the force into a number of components, which are equivalent to the given forces is called *resolution of force*.

The law of parallelogram shows how to combine two forces into a resultant force whereas resolution of force is an inverse operation in which a given force is replaced by two components which are equivalent to the given force.

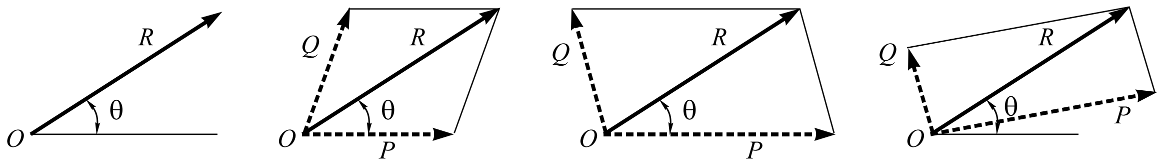


Fig. 2.2-i

Single force R is replaced by two components of force P and Q.

Figure 2.2-i shows that we can have infinite number of pair of oblique components of single force R such that R is the diagonal of various parallelogram.

Note : Even component of force can be further splitted into subcomponents which means resolution of single force can have infinite number of components too.

2.2.1 Solved Problems on Resolution of Force into Oblique Components of Force

Problem 8

Resolve the 100 N force acting a 30° to horizontal into two components, one along horizontal and other along 120° to horizontal.

Solution

(i) *Method I : By Parallelogram law*

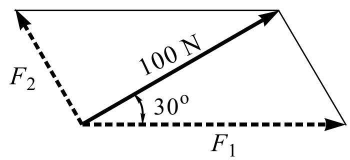


Fig. 2.8(a)

$$100 = \sqrt{F_1^2 + F_2^2 + 2\,F_1F_2 \cos 120^\circ}$$

$$10000 = F_1^2 + F_2^2 - F_1F_2 \qquad \text{..... (I)}$$

$$\tan 30^\circ = \frac{F_2 \sin 120^\circ}{F_1 + F_2 \cos 120^\circ}$$

$$0.5774\,F_1 - 0.2887\,F_2 = 0.866\,F_2$$

$$F_1 = 2\,F_2 \qquad \text{..... (II)}$$

Solving Eqs. (I) and (II)

$F_1 = 115.47$ N (→) $F_2 = 57.76$ N (60°)

(ii) *Method II : By Triangle law*

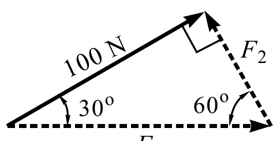


Fig. 2.8(b)

By sine rule

$$\frac{100}{\sin 60^\circ} = \frac{F_1}{\sin 90^\circ} = \frac{F_2}{\sin 30^\circ}$$

$F_1 = 115.47$ N (→) $F_2 = 57.76$ N (60°) ***Ans.***

2.2.2 Resolution of Force into Rectangular Components of Force

Usually we require rectangular components of force. The process of breaking the force into mutually perpendicular components which are equivalent to the given force is called *rectangular components of force*.

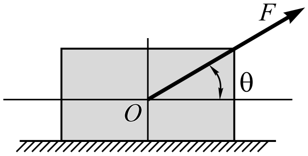


Fig. 2.2.2-i

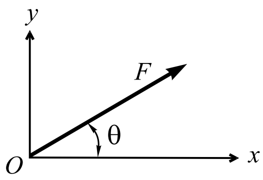


Fig. 2.2.2-ii

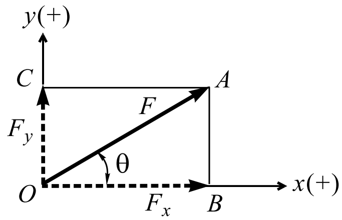


Fig. 2.2.2-iii

Consider a force of magnitude F acting at an angle θ with horizontal (Fig. 2.2.2-i), taking O as origin draw x-axis and y-axis (Fig. 2.2.2-ii). Let the force F be represented by line OA drawn to the scale. Draw perpendicular from point A on the x-axis to mark B and on y-axis to mark C. OB (F_x) and OC (F_y) are the mutually perpendicular components of force F (Fig. 2.2.2-iii).

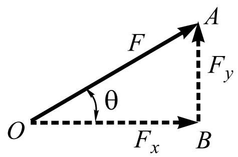


Fig. 2.2.2-iv : By Triangle Law of Force

By trigonometric convention, we have the relation of components F_x and F_y with F and θ.

$$\sin\theta = \frac{F_y}{F} \quad \text{and} \quad \cos\theta = \frac{F_x}{F}$$

$$\therefore \; F_y = F\sin\theta \quad \text{and} \quad F_x = F\cos\theta$$

$$F = \sqrt{F_x^{\,2} + F_y^{\,2}} \text{ and } \theta = \tan^{-1}\left(\frac{F_y}{F_x}\right).$$

Sign Conventions

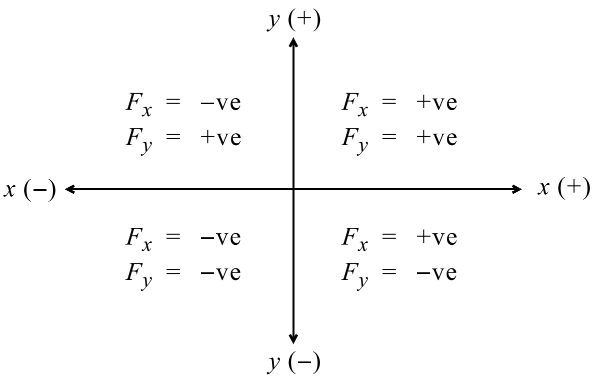


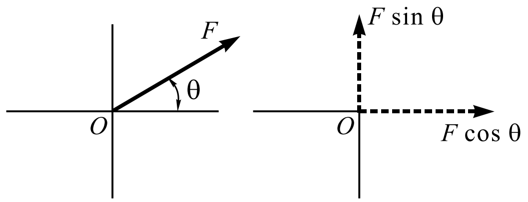
Fig. 2.2.2-v

1. Forces acting horizontally towards right are +ve and left are −ve.
2. Forces acting vertically upward are +ve and downward are −ve.

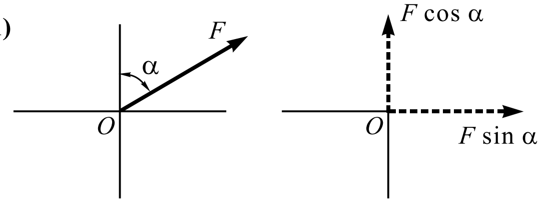
Example 1

Resolve the given force F into horizontal and vertical components.

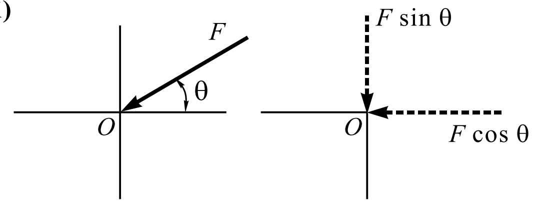
(i)



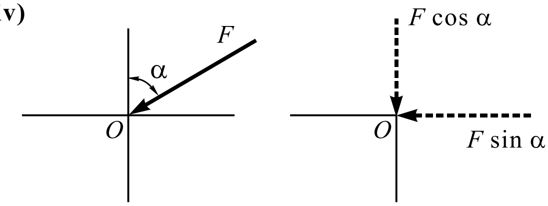
(ii)



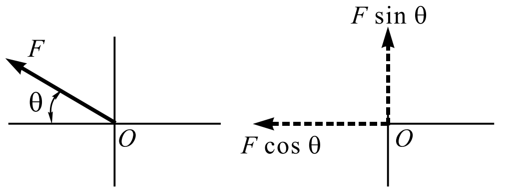
(iii)



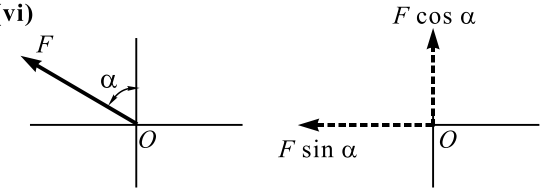
(iv)



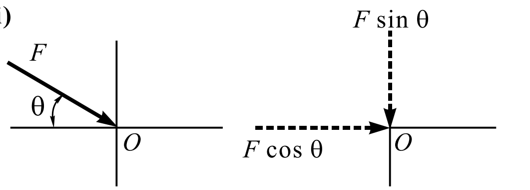
(v)



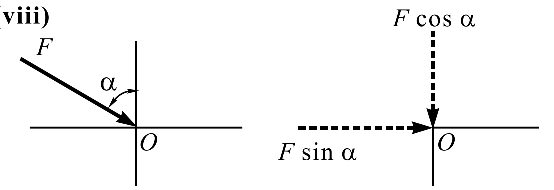
(vi)



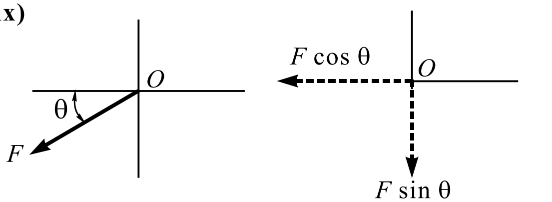
(vii)



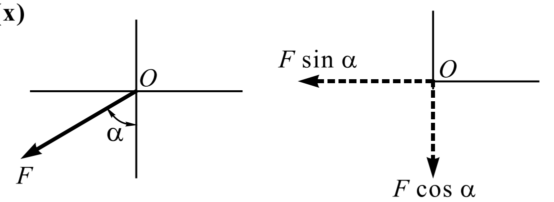
(viii)



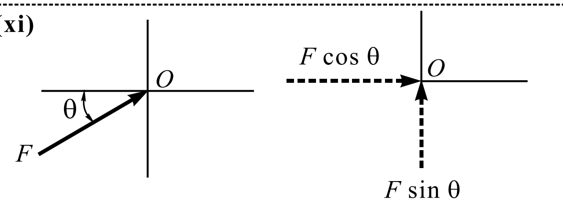
(ix)



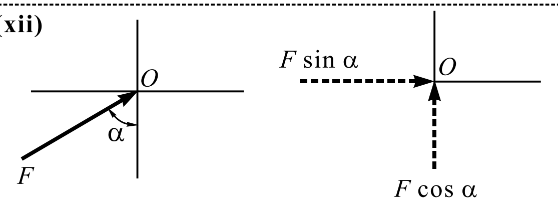
(x)



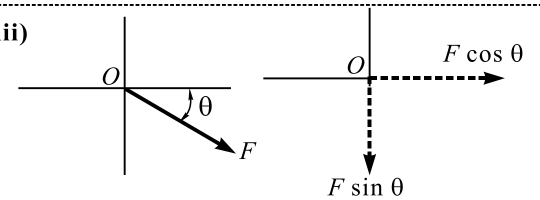
(xi)



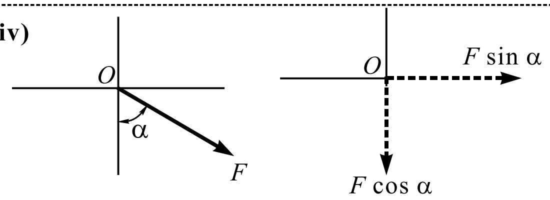
(xii)



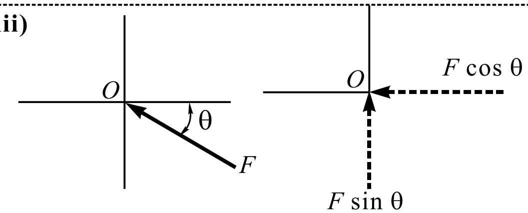
(xiii)



(xiv)



(xiii)



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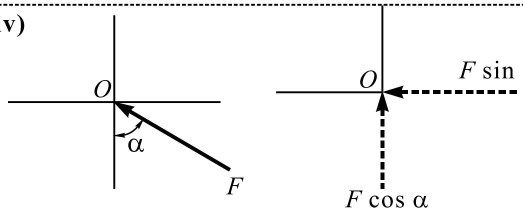
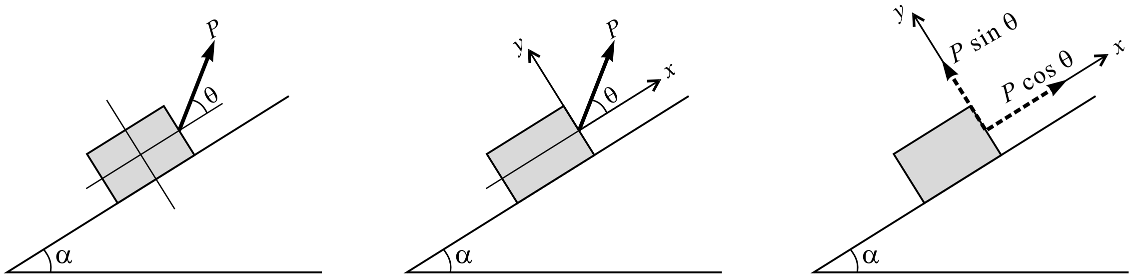


Example 2

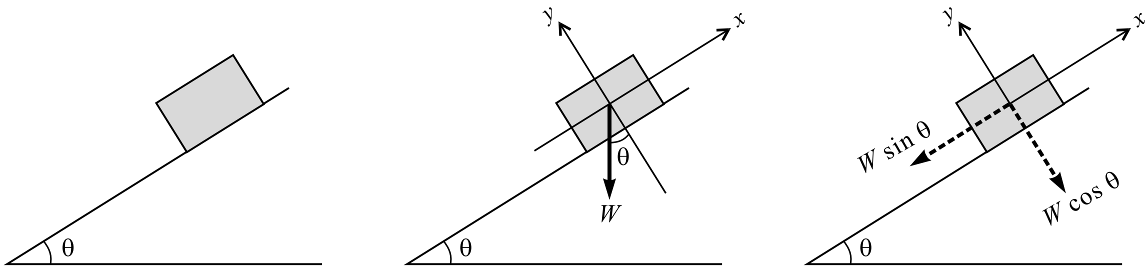
The orientation of *x*-axis and *y*-axis need not be always horizontal and vertical. Resolve the force *P* along the *x*-axis and *y*-axis which are parallel and perpendicular to the inclined plane.



Given **Solution**

Example 3

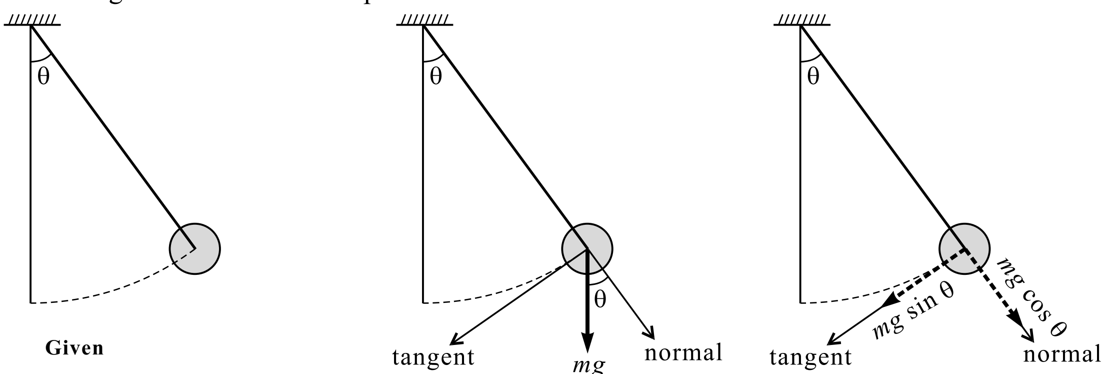
Resolve the weight *W* of a block parallel and perpendicular to the inclined plane.



Given **Solution**

Example 4

A simple pendulum bob of mass *m* is hanging as shown in the figure below. Resolve the weight of bob into tangent and normal components.



Given **Solution**

2.3 Resultant of Concurrent Force System Using Method of Resolution

If the number of forces is more than two, then its resultant can be found out conveniently by the *method of resolution*.

Procedure

Step 1 : Find $R_x = \Sigma F_x$.

Resolve all the forces along the horizontal x-axis and take the algebraic sum of force components considering proper sign convention (+ve $\rightarrow$).

Step 2 : Find $R_y = \Sigma F_y$.

Resolve all the forces along the vertical y-axis and take the algebraic sum of force components considering proper sign convention (+ve $\uparrow$).

Step 3 : Find R.

Magnitude of resultant force is given by $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

Step 4 : Find θ.

Inclination of the line of action of resultant force with horizontal x-axis is given by

$$\tan\theta = \left|\frac{\Sigma F_y}{\Sigma F_x}\right| \quad \therefore \quad \theta = \tan^{-1}\left(\frac{\Sigma F_y}{\Sigma F_x}\right)$$

Note : Use only positive value of ΣF_x and ΣF_y for finding θ with horizontal x-axis so the value of θ will be always acute angle.

Step 5 : Position of resultant

Resultant may lie in any four quadrant depending on the signs of ΣF_x and ΣF_y

(i) ΣF_x (+ve) and ΣF_y (+ve) Ist Quadrant

(ii) ΣF_x (−ve) and ΣF_y (+ve) IInd Quadrant

(iii) ΣF_x (−ve) and ΣF_y (−ve) IIIrd Quadrant

(iv) ΣF_x (+ve) and ΣF_y (−ve) IVth Quadrant

The point of application of the resultant for concurrent force system is the point of concurrency.

Note :
- If resultant is horizontal then $\Sigma F_y = 0$ and $\Sigma F_x = R$.
- If resultant is vertical then $\Sigma F_x = 0$ and $\Sigma F_y = R$.
- If resultant is zero then $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

Hint : For finding resultant of two concurrent forces, one can prefer parallelogram law or triangle law as discussed in previous chapter.

Problem 9

An eye bolt is being pulled from ground by three forces as shown in Fig. 2.9(a). Determine the equilibrants on the eye bolt which resist to come out.

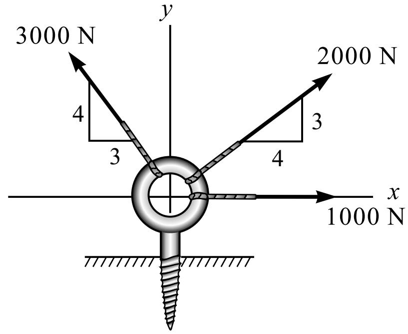
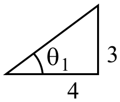


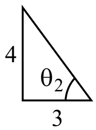
Fig. 2.9(a)

Note : The direction of force is defined by its slope.

Solution

(i) 

$\tan\theta_1 = \frac{3}{4} \quad \therefore \theta_1 = 36.87^o$

(ii) 

$\tan\theta_2 = \frac{4}{3} \quad \therefore \theta_2 = 53.13^o$

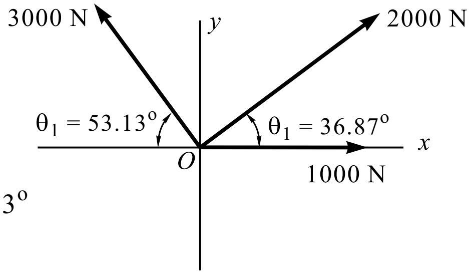


Fig. 2.9(b)

(iii) $\Sigma F_x = 1000 + 2000 \cos 36.87^o - 3000 \cos 53.13^o$

$\therefore \Sigma F_x = 799.99 \approx 800 \text{ N } (\rightarrow)$ ***Ans.***

(iv) $\Sigma F_y = 2000 \sin 36.87^o + 3000 \sin 53.13^o$

$\therefore \Sigma F_y = 3600 \text{ N } (\uparrow)$ ***Ans.***

(v) Magnitude of resultant R

$$R = \sqrt{(800)^2 + (3600)^2}$$

$R = 3687.82$ N ***Ans.***

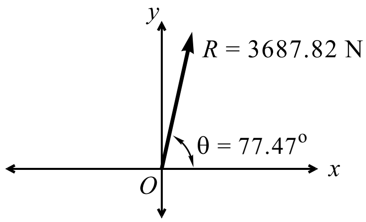


Fig. 2.9(c)

(vi) Inclination of resultant θ

$$\theta = \tan^{-1}\left(\frac{3600}{800}\right)$$

$\therefore \theta = 77.47^o$ ***Ans.***

(vii) Position of the resultant [Refer to Fig. 2.9(c)]

$\because \Sigma F_x$ is +ve $(\rightarrow)$ and ΣF_y is +ve $(\uparrow)$

$\therefore$ resultant force R lies in the first quadrant.

(viii) Since equilibrant is equal in magnitude, opposite in direction and collinear to that of resultant force.

$\therefore$ Equilibrant E lies in the third quadrant. [Refer to Fig. 2.9(d)]

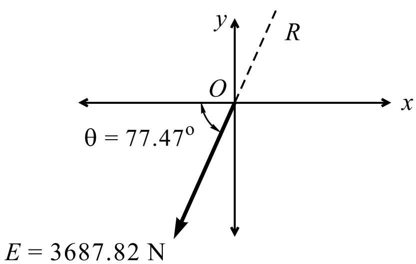


Fig. 2.9(d)

Ans.

Problem 10

Forces 7 kN, 10 kN, 10 kN and 3 kN, respectively act at one of the angular point of regular pentagon toward the other four points taken in order. Find their resultant completely.

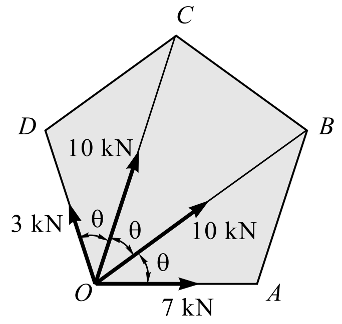


Fig. 2.10(a)

Solution

Regular pentagon is a polygon having five sides of equal length. The point of intersection of two sides is called an angular point.
Therefore, pentagon has five angular points.
Refer to Fig. 2.10(a).

Included angles of any regular polygon

$$= 180 - \frac{360}{\text{Number of sides}}$$

For pentagon included angle

$$= 180 - \frac{360}{5} = 108^o$$

$$\therefore \ \theta = \frac{108}{3} = 36^o$$

Represent the concurrent forces as shown in Fig. 2.10(b).

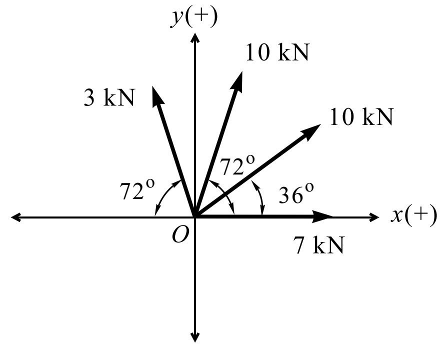


Fig. 2.10(b)

(i) $\Sigma F_x = 7 + 10 \cos 36^o + 10 \cos 72^o - 3 \cos 72^o$

$\Sigma F_x = 17.25 \text{ kN } (\rightarrow)$

(ii) $\Sigma F_y = 10 \sin 36^o + 10 \sin 72^o + 3 \sin 72^o$

$\Sigma F_y = 18.24 \text{ kN } (\uparrow)$

(iii) Magnitude of resultant R

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$R = \sqrt{(17.25)^2 + (18.24)^2}$$

$R = 25.10$ kN ***Ans.***

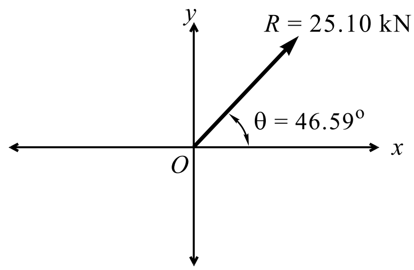


Fig. 2.10(c)

(iv) Inclination of the resultant θ

$$\theta = \tan^{-1}\left(\frac{\Sigma F_y}{\Sigma F_x}\right) \quad \therefore \theta = \tan^{-1}\left(\frac{18.24}{17.25}\right)$$

$\therefore \ \theta = 46.59^o$ ***Ans.***

(v) Position of the resultant is shown in Fig. 2.10(c).

Problem 11

The top end of vertical boot is connected by two cables having tension T_1 = 500 N and T_2 = 1500 N, as shown in Fig. 2.11(a). The third cable AB is used as a guy wire. Determine the tension in cable AB if the resultant of the three concurrent forces acting at A is vertical. Also find the resultant.

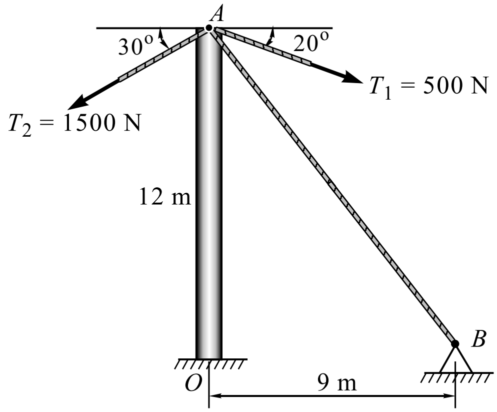


Fig. 2.11(a)

Solution

Fig. 2.11(b) shows the concurrent force system. The tension in cable AB (T_3) will try to exert a pull at A.

It is given that the resultant of the force system is vertical. It means that algebraic sum of all the force components in horizontal x-axis direction must be zero.

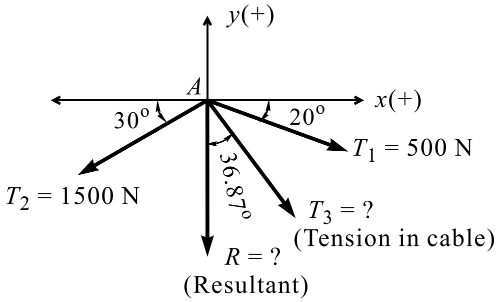


Fig. 2.11(b)

(i) $\therefore \Sigma F_x = 0$

$\therefore 500 \cos 20^\circ - 1500 \cos 30^\circ + T_3 \sin 36.87^\circ = 0$

$\therefore T_3 = 1381.98$ N ***Ans.***

(ii) $\because$ Resultant is vertical

$\therefore R = \Sigma F_y$

$\therefore R = -500 \sin 20^\circ - 1500 \sin 30^\circ - 1381.98 \cos 36.87^\circ$

$\therefore R = -2026.59$ N

$\therefore R = 2026.59$ N $(\downarrow)$ ***Ans.***

Problem 12

Determine the magnitude and direction of forces F_1 and F_2, shown in Fig. 2.12 when the resultant of the given force system is found to be 800 N along positive x-axis.

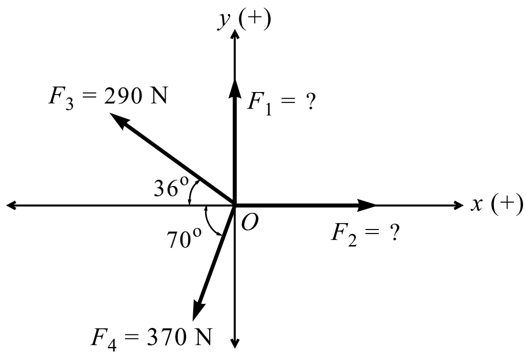


Fig. 2.12

Solution

(i) $\Sigma F_x = 800$

$F_2 - 290 \cos 36^\circ - 370 \cos 70^\circ = 800$

$F_2 = 1161.2$ N $(\rightarrow)$ ***Ans.***

(ii) $\Sigma F_y = 0$

$F_1 + 290 \sin 36^\circ - 370 \sin 70^\circ = 0$

$F_1 = 177.23$ N $(\uparrow)$ ***Ans.***

Problem 13

A force $R = 25$ kN, acting at O, has three components F_A, F_B and F_C as shown in Fig. 2.13(a). If $F_C = 20$ kN, find F_A and F_B.

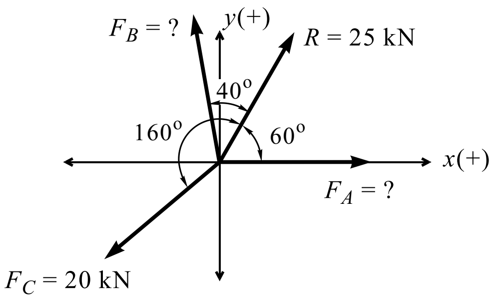


Fig. 2.13(a)

Solution

Refer to Fig. 2.13(b).

(i) To find F_B

$$R_y = \Sigma F_y$$

$$\therefore\ 25 \sin 60^o = F_B \sin 80^o - 20 \sin 40^o$$

$$\therefore\ F_B = 35.04 \text{ kN} \quad \textbf{\textit{Ans.}}$$

(ii) To find F_A

$$R_x = \Sigma F_x$$

$$\therefore\ 25 \cos 60^o = F_A - 35.04 \cos 80^o - 20 \cos 40^o$$

$$\therefore\ F_A = 33.91 \text{ kN} \quad \textbf{\textit{Ans.}}$$

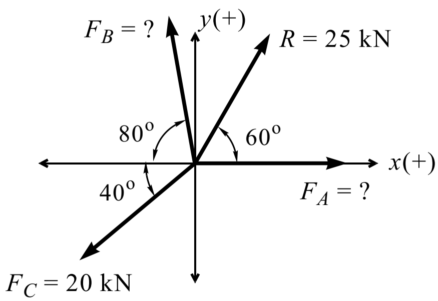


Fig. 2.13(b)

Problem 14

Find force F_4 completely, so as to give the resultant of the system of forces as shown in Fig. 2.14(a).

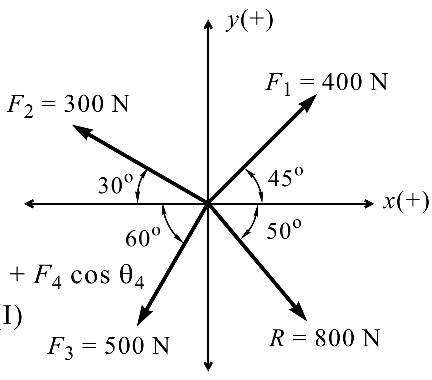


Fig. 2.14(a)

Solution

Assume F_4 in Ist quadrant making angle θ_4 with x-axis.

(i) $R_x = \Sigma F_x$

$$\therefore\ 800 \cos 50^o = 400 \cos 45^o - 300 \cos 30^o - 500 \cos 60^o + F_4 \cos \theta_4$$

$$\therefore\ F_4 \cos \theta_4 = 741.19 \text{ N } (\rightarrow) \qquad \ldots \text{(I)}$$

(ii) $R_y = \Sigma F_y$

$$-800 \sin 50^o = 400 \sin 45^o + 300 \sin 30^o - 500 \sin 60^o + F_4 \sin \theta_4$$

$$\therefore\ F_4 \sin \theta_4 = -612.66 \text{ N}$$

$$\therefore\ F_4 \sin \theta_4 = 612.66 \text{ N } (\downarrow) \qquad \ldots \text{(II)}$$

(iii) Dividing Eq. (II) by Eq. (I)

$$\therefore\ \tan \theta_4 = \frac{612.66}{741.19} \quad \therefore\ \theta_4 = 39.58^o \quad \textbf{\textit{Ans.}}$$

(iv) From Eq. (I), we get

$$F_4 = \frac{741.19}{\cos 39.58^o} \quad \therefore\ F_4 = 961.67 \text{ N} \quad \textbf{\textit{Ans.}}$$

(v) Position of F_4 [Refer to Fig. 2.14(b)].

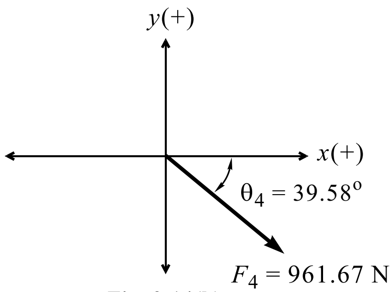


Fig. 2.14(b)

Problem 15

The striker of a carom board laying on the board is being pulled by four players as shown in Fig. 2.15(a). The players are sitting exactly at the centre of the four sides. Determine the resultant of forces in magnitude and direction.

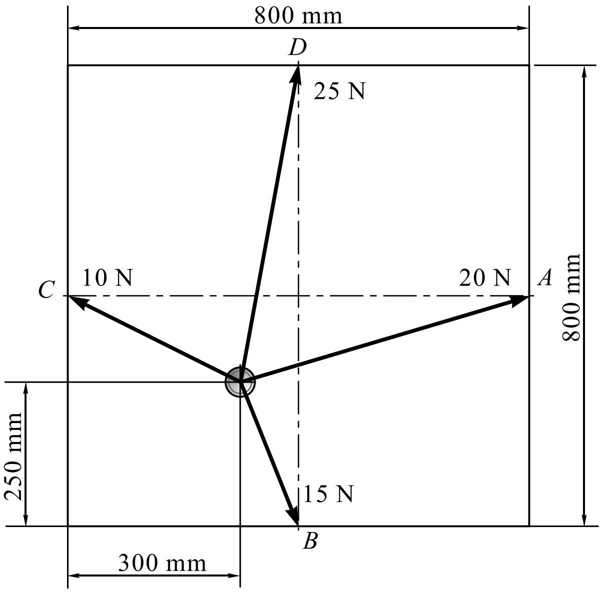


Fig. 2.15(a)

Solution

(i) $\tan \theta_1 = \dfrac{150}{500}$ $\therefore \theta_1 = 16.7^o$

$\tan \theta_2 = \dfrac{550}{100}$ $\therefore \theta_2 = 79.7^o$

$\tan \theta_3 = \dfrac{150}{300}$ $\therefore \theta_2 = 26.56^o$

$\tan \theta_4 = \dfrac{250}{100}$ $\therefore \theta_2 = 68.2^o$

(ii) $\Sigma F_x = 20 \cos \theta_1 + 25 \cos \theta_2 - 10 \cos \theta_3 + 15 \cos \theta_4$

$\Sigma F_x = 20.25 \text{ N } (\rightarrow)$

(iii) $\Sigma F_y = 20 \sin \theta_1 + 25 \sin \theta_2 + 10 \sin \theta_3 - 15 \sin \theta_4$

$\therefore \Sigma F_y = 20.89 \text{ N } (\uparrow)$

(iv) $R = \sqrt{(20.25)^2 + (20.89)^2}$

$\therefore R = 29.09$ N ***Ans.***

(v) $\theta = \tan^{-1}\left(\dfrac{20.89}{20.25}\right)$

$\therefore \theta = 45.89^o$ ***Ans.***

(vi) Position of resultant R is shown in Fig. 2.15(c).

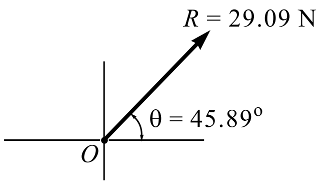


Fig. 2.15(c)

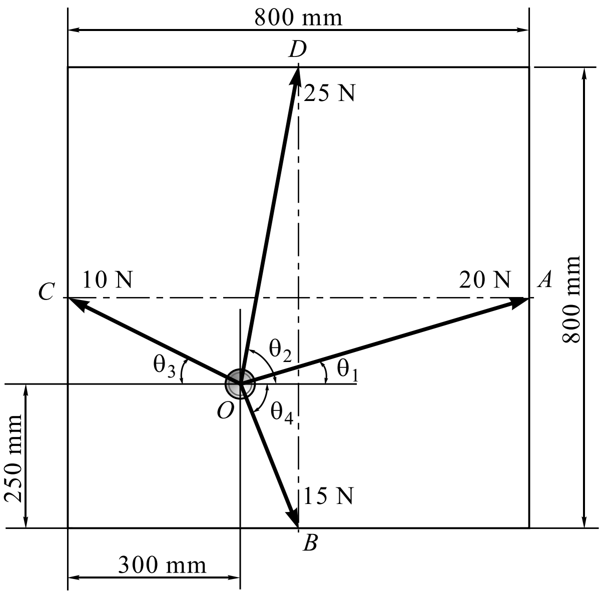


Fig. 2.15(b)

Problem 16

Determine the resultant of the three forces originating at point (3, –3) and passing through the point indicated : 126 N through (8, 6), 183 N through (2, –5) and 269 N through (–6, 3).

Solution

To find θ_1, θ_2 and θ_3.

$$\tan\theta = \left|\frac{y_2 - y_1}{x_2 - x_1}\right| \quad \text{(slope)}$$

$$\theta_1 = \tan^{-1}\left|\frac{9}{5}\right| \qquad \therefore\ \theta_1 = 60.95^\circ$$

$$\theta_2 = \tan^{-1}\left|\frac{-2}{-1}\right| \qquad \therefore\ \theta_2 = 63.44^\circ$$

$$\theta_3 = \tan^{-1}\left|\frac{6}{-9}\right| \qquad \therefore\ \theta_3 = 33.69^\circ$$

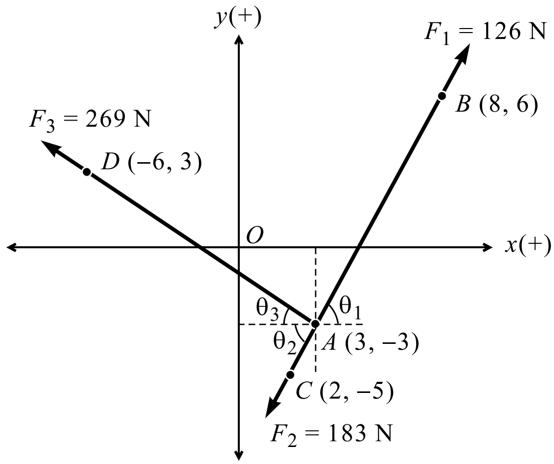


Fig. 2.16(a)

(i) $\Sigma F_x = 126\cos 60.95^\circ - 183\cos 63.44^\circ - 269\cos 33.69^\circ$

$$\Sigma F_x = -244.47 \text{ N}$$

$$\therefore\ \Sigma F_x = 244.47 \text{ N } (\leftarrow)$$

(ii) $\Sigma F_y = 126\sin 60.95^\circ - 183\sin 63.44^\circ + 269\sin 33.69^\circ$

$$\Sigma F_y = 95.68 \text{ N } (\uparrow)$$

(iii) Magnitude of resultant R

$$R = \sqrt{(244.47)^2 + (95.68)^2}$$

$$\therefore\ R = 262.53 \text{ N} \qquad \textit{Ans.}$$

(iv) Inclination of the resultant θ

$$\theta = \tan^{-1}\left(\frac{95.68}{244.47}\right)$$

$$\therefore\ \theta = 21.37^\circ \qquad \textit{Ans.}$$

(v) Position of the resultant w.r.t. point A is as shown in Fig. 2.16(b).

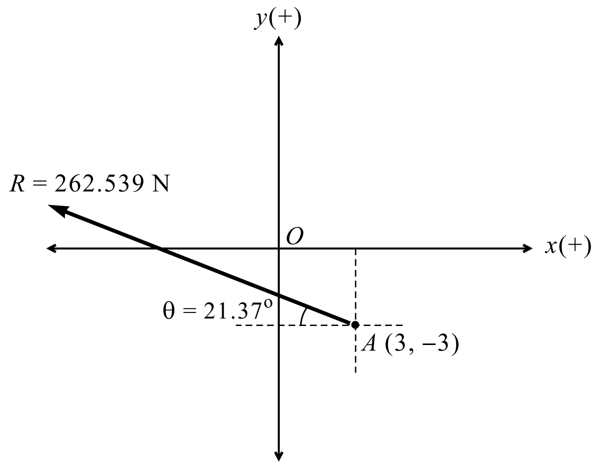


Fig. 2.16(b)

Problem 17

A body is acted upon by forces as below. Find the resultant of these forces.

(i) 50 N acting due East

(ii) 100 N 50^o North of East

(iii) 75 N 20^o West of North

(iv) 120 N acting 30^o South of West

(v) 90 N acting 25^o West of South

(vi) 80 N acting 40^o South of East

All forces are acting from the point *O*.

Solution

Draw the sketch of all forces acting at point *O*.

Show geographical axis North (*N*), South (*S*), East (*E*) and West (*W*). [Refer to Fig. 2.17(a)]

Note : 50^o North of East means measure an angle of 50^o from East toward North. Similar guideline is to be followed for all forces.

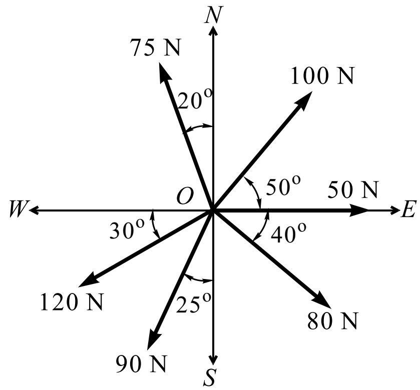


Fig. 2.17(a)

(i) $\Sigma F_x = 50 + 100 \cos 50^o - 75 \sin 20^o - 120 \cos 30^o - 90 \sin 25^o + 80 \cos 40^o$

$\therefore \Sigma F_x = 7.95 \text{ N} (\rightarrow)$ ***Ans.***

(ii) $\Sigma F_y = 100 \sin 50^o + 75 \cos 20^o - 120 \sin 30^o - 90 \cos 25^o - 80 \sin 40^o$

$\Sigma F_y = -45.91 \text{ N}$

$\therefore \Sigma F_y = 45.91 \text{ N} (\downarrow)$ ***Ans.***

(iii) $R = \sqrt{(7.95)^2 + (45.91)^2}$

$R = 46.59 \text{ N}$ ***Ans.***

(iv) $\theta = \tan^{-1}\left(\frac{45.91}{7.95}\right)$

$\therefore \theta = 80.18^o$ ***Ans.***

(v) Position of the resultant w.r.t point *O* is as shown in Fig. 2.17(b).

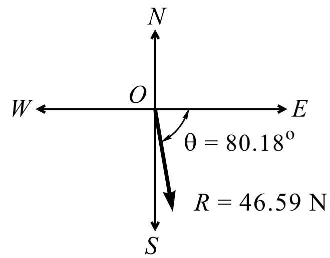


Fig. 2.17(b)

Problem 18

For the system shown (Fig. 2.18), determine

(i) the required value of α if resultant of three forces is to be vertical and

(ii) the corresponding magnitude of resultant.

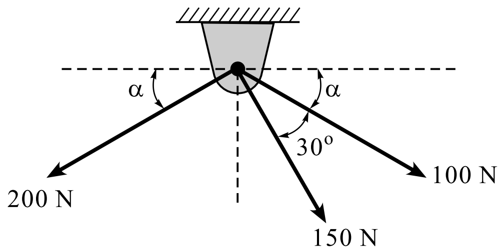


Fig. 2.18

Solution

(i) Since resultant is vertical

$\therefore \Sigma F_x = 0$

$100 \cos \alpha + 150 \cos (\alpha + 30) - 200 \cos \alpha = 0$

$150(\cos \alpha \cos 30^o - \sin \alpha \sin 30^o) - 100 \cos \alpha = 0$

$130 \cos \alpha - 75 \sin \alpha - 100 \cos \alpha = 0$

$30 \cos \alpha - 75 \sin \alpha = 0$

$75 \sin \alpha = 30 \cos \alpha$

$$\frac{\sin \alpha}{\cos \alpha} = \frac{30}{75} = \tan \alpha$$

$\therefore \alpha = 21.8^o$ ***Ans.***

(ii) $\because$ Resultant is vertical

$\therefore R = \Sigma F_y$

$\therefore R = -100 \sin \alpha - 150 \sin (\alpha + 30) - 200 \sin \alpha = 0$

$R = -100 \sin 21.8^o - 150 \sin (21.8 + 30)^o - 200 \sin 21.8^o = 0$

$R = -229.29 \text{ N}$

$\therefore R = 229.29 \text{ N } (\downarrow)$ ***Ans.***

2.4 Moment of Force

The rotational effect produced by force is known as *moment of the force*. In other words, the tendency of a force to rotate a rigid body about an axis is measured by the moment of the force about that axis.

Consider a rigid body which is acted upon by a force F as shown in Fig. 2.4-i. Let an axis perpendicular to the plane of the paper pass through point O. The force F will have a tendency to rotate the body about the axis through O.

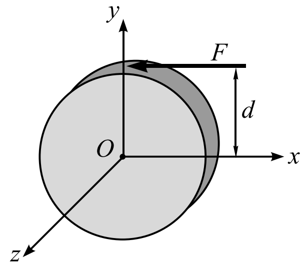


Fig. 2.4-i

The moment of the force about an axis through O is M_O and is given as the product of magnitude of the force F and perpendicular distance d from O to the line of action of the force F.

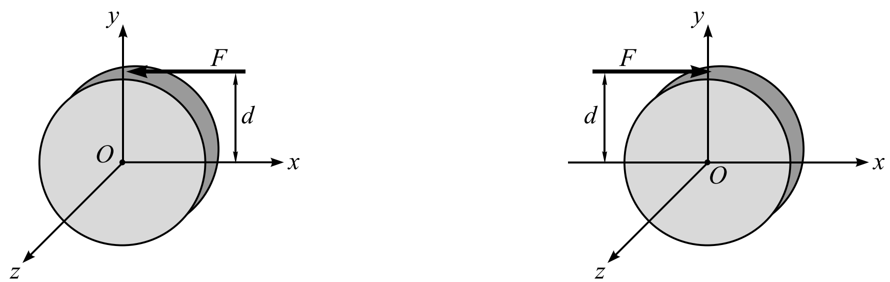
$$M_O = F \times d \quad \text{...(2.3)}$$

Here point O is called the *moment centre* and d is called the *moment arm*. If F is in Newton and d is in meter then S.I. unit of moment of force is N-m.

It is important to note that the moment will be zero if $d = 0$, i.e., the line of action of the force intersects the axis about which the moment is considered. Thus, force produces zero moment about any axis or reference point which intersect the line of action of the force.

Sign Conventions

The moment of a force has not only magnitude but also a direction. This direction is perpendicular to the plane of the paper. The force will try to rotate the body in anticlockwise or clockwise depending upon the relative position of the force and its axis.



Anticlockwise sense (↺) of rotation   Clockwise sense (↻) of rotation

Fig. 2.4-ii

As per the right-hand-thumb rule, we obtain anticlockwise moment as positive and clockwise moment as negative.

Note :
- Moment of force can be added algebraically as scalar quantities with proper sign convention.
- For coplanar force system, moment of force is taken about a point instead of an axis.

Problem 19

A 75 N vertical force is applied to the end of a force 3 m long which is attached to a shaft at O as shown in Fig. 2.19(a). Determine

(i) the moment of the 75 N force about O,

(ii) the magnitude of the horizontal force applied at A which creates the same moment about O and

(iii) the smallest force applied at A which creates the same moment about O.

(iv) How far from the shaft at O a 200 N vertical force must act to create the same moment about O ?

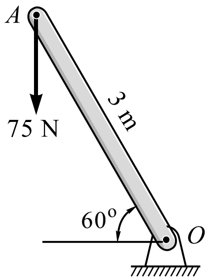


Fig. 2.19(a)

Solution

(i) The perpendicular distance from O to the line of action [Fig. 2.19(b)] of 75 N force is

$OB = 3 \cos 60^\circ = 1.5$ m

The magnitude of moment 75 N force about O is

$$M_O = 75 \times 3 \cos 60^\circ = 75 \times 1.5$$

$$M_O = 112.5 \text{ N-m } (\circlearrowleft)$$ ***Ans.***

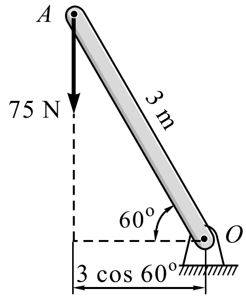


Fig. 2.19(b)

(ii) The perpendicular distance from O to the line of action of force F is $AB = 3 \sin 60^\circ$ [Fig. 2.19(c)].

F should be directed toward left to create anticlockwise moment about O.

$$M_O = F \times d$$

$$112.5 = F \times 3 \sin 60^\circ$$

$$\therefore F = 43.3 \text{ N } (\leftarrow)$$ ***Ans.***

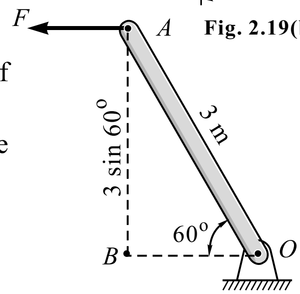


Fig. 2.19(c)

(iii) For the smallest force applied at A which creates the same moment about O, i.e., 112.5 N-m ($\circlearrowleft$) the perpendicular distance from O to the line of action of force d should be maximum [Fig. 2.19(d)].

Select a force perpendicular to lever OA so maximum distance $d = 3$ m.

$$M_O = F \times d$$

$$112.5 = F \times 3$$

$$\therefore F = 37.5 \text{ N } \left(30^\circ \swarrow\right)$$ ***Ans.***

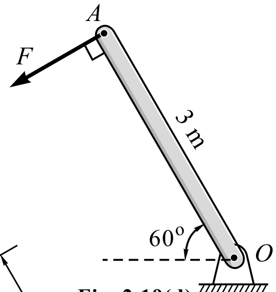


Fig. 2.19(d)

(iv) Let l be the distance at which force 200 N is acting vertically down [Fig. 2.19(d)].

Perpendicular distance from O to the line of action of 200 N force is $l \cos 60^\circ$.

$$M_O = F \times l \cos 60^\circ$$

$$112.5 = 200 \times l \cos 60^\circ$$

$$\therefore l = 1.125 \text{ m}$$ ***Ans.***

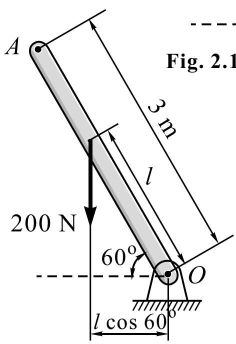


Fig. 2.19(e)

2.5 Couple

Two non-collinear parallel forces of equal magnitude but acting in opposite direction form a *couple*.

It is a special case of parallel forces which produces the *rotary effect* on a rigid body.

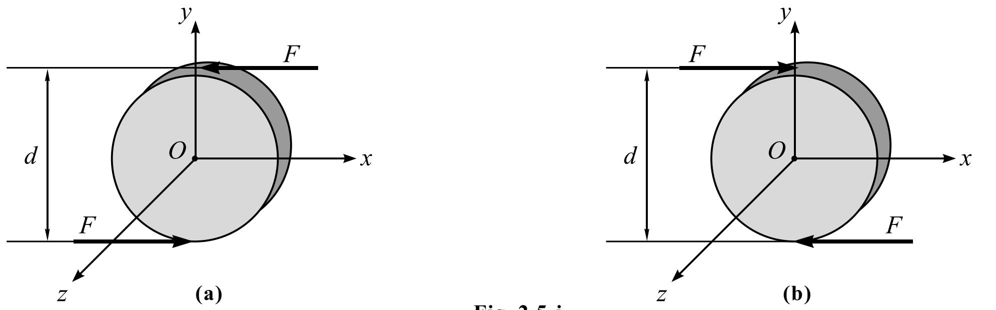


Fig. 2.5-i

Moment of Couple

The magnitude of rotation, known as the *moment of couple*, is the product of common magnitude of the two force F and the perpendicular distance d (arm of couple) between the lines of action.

Refer to Fig. 2.5-i(a) Refer to Fig. 2.5-i(b)

$$M = F \times d \;(\curvearrowright) \qquad M = F \times d \;(\circlearrowleft) \qquad \ldots(2.4)$$

Sign Convention

The couple has not only a magnitude but also a direction, which is perpendicular to the plane of paper. The couple will try to rotate the body in clockwise or anticlockwise which can be identified by simple observation.

As per right-hand-thumb rule, we obtain *anticlockwise sense as +ve and clockwise sense as −ve*.

Properties of Couple

1. Moment of couple is equal to the product of one of the force and the arm of couple.
2. The tendency of couple is to rotate the body about an axis perpendicular to the plane containing the two parallel forces.
3. The resultant force of a couple system is zero.
4. A couple can only rotate the body but cannot translate the body.
5. Moment of couple can be added algebraically as scalar quantity with proper sign convention.
6. A couple can be replaced by a couple only and not by a single force.
7. A couple is a pure turning moment which is always constant. It may be moved anywhere in its own plane on a body without any change of its effect on the body. Thus, a couple acting on a rigid body is known as a *free vector*.
8. A system of parallel forces whose resultant is a couple can attain equilibrium only by another couple of same magnitude but opposite direction.
9. A couple does not have moment centre, like moment of force.

Why is Couple a Free Vector ?

Case (i)

Magnitude of moment of couple M is equal to the product of the common magnitude of the two force F and of the perpendicular distance between the lines of action d. The sense of couple M is anticlockwise by observation.

$$\text{Moment of couple } M = F \times d \ (\circlearrowleft) \qquad \text{...(I)}$$

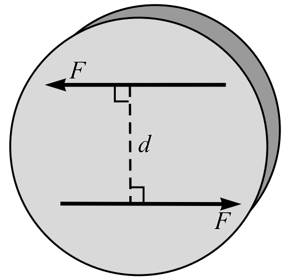


Fig. 2.5-ii(a)

Case (ii)

Take moment of two forces about a point O. As per the sign convention, anticlockwise direction of rotation is positive.

From Fig. 2.5-ii(b), we have

$$M = F \times \frac{d}{2} + F \times \frac{d}{2}$$

$$M = F\left(\frac{d}{2} + \frac{d}{2}\right)$$

$$M = F \times d \ (\circlearrowleft) \qquad \text{...(II)}$$

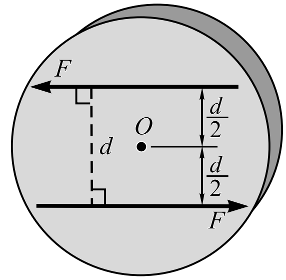


Fig. 2.5-ii(b)

Case (iii)

Take moment of two forces about a point A.

From Fig. 2.5-ii(c), we have

$$M = F \times d_2 - F \times d_1$$

$$M = F\,(d_2 - d_1)$$

$$M = F \times d \ (\circlearrowleft) \qquad \text{...(III)}$$

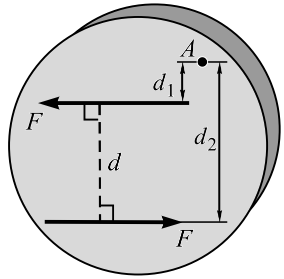


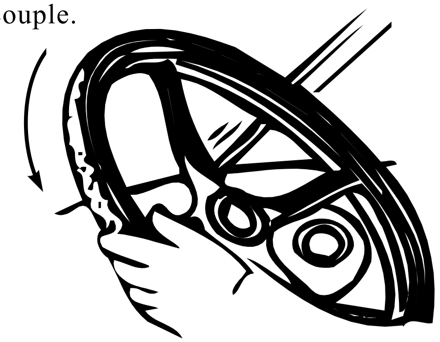
Fig. 2.5-ii(c)

It may be seen that Eqs. (I), (II) and (III) have the same result which shows that moment of couple is constant and independent of any point.

Therefore, '*couple is a free vector*'.

Example 1

The stearing wheel of car is the moment of couple.



Example 2

Rotation of a key to lock or unlock is the moment of couple.

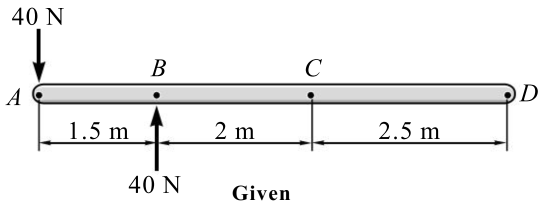
Example 3

Opening or closing of a tap is the moment of couple.

Opening

Closing

Example 4



Given

Solution

From the given figure, we have

(i) Moment of couple $M = 40 \times 1.5 = 60$ N-m $(\circlearrowleft)$

(ii) Moment of forces about point A

$$M_A = 40 \times 1.5 = 60 \text{ N-m } (\circlearrowleft)$$

(iii) Moment of forces about point B

$$M_B = 40 \times 1.5 = 60 \text{ N-m } (\circlearrowleft)$$

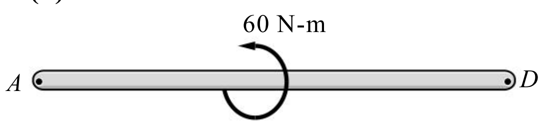
(iv) Moment of forces about point C

$$M_C = 40 \times 3.5 - 40 \times 2 = 40(3.5 - 2)$$
$$= 40 \times 1.5 = 60 \text{ N-m } (\circlearrowleft)$$

(v) Moment of force about point D

$$M_D = 40 \times 6 - 40 \times 4.5 = 40(6 - 4.5)$$
$$= 40 \times 1.5 = 60 \text{ N-m } (\circlearrowleft)$$

From **(i)**, **(ii)**, **(iii)**, **(iv)** and **(v)**

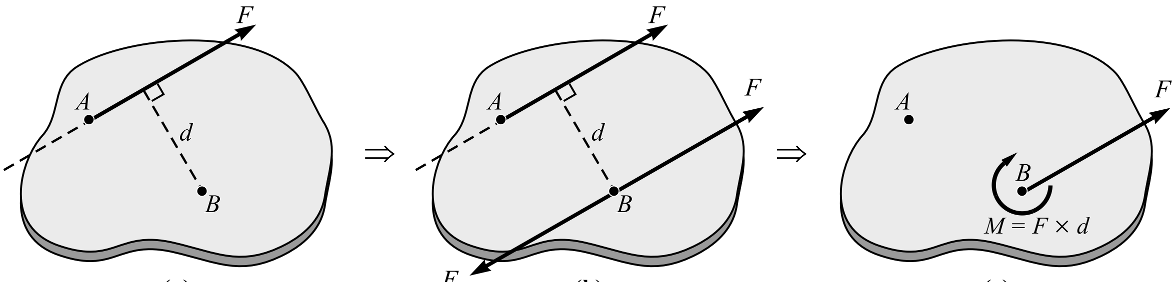


Ans.

Note : The above example shows that *moment of couple is a constant*. Hence, couple is treated as free vector which can be represented anywhere on a rigid body.

2.6 Transfer of Force to the Parallel Position

Resolution of a Force into a Force-Couple System



(a) (b) (c)

Fig. 2.6-i

Refer to Fig. 2.6-i(a). Consider a force acting at point A on a rigid body. This force is to be replaced by a force and couple at point B.

Refer to Fig. 2.6-i(b). Apply two forces, equal in magnitude and opposite in direction, parallel to force F at point B. This addition of forces does not change the original effect on rigid body. Observing carefully we see, out of three forces two forces are acting in opposite direction at A and B form a couple.

$$\text{Moment of couple } M = F \times d \ (\circlearrowright).$$

Refer to Fig. 2.6-i(c). Thus, to shift a force to a new parallel position, a couple is required to be added to the system. Here we can see that moment of couple (M) is equal to moment of force about point B [Refer to Fig. 2.6-i(a), $M_B = F \times d\ (\circlearrowright)$].

Problem 20

Resolve the force $F = 900$ N acting at B into a couple and a force at O. Refer to Fig. 2.20(a).

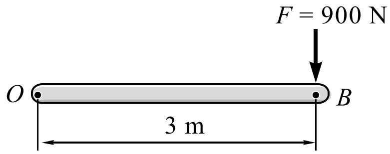


Fig. 2.20(a)

Solution

When force is shifted from point B to O it is shifted as it is along with couple of moment.

$$M = F \times d = 900 \times 3$$

$$M = 2700 \text{ N-m } (\circlearrowright) \quad \textbf{\textit{Ans.}}$$

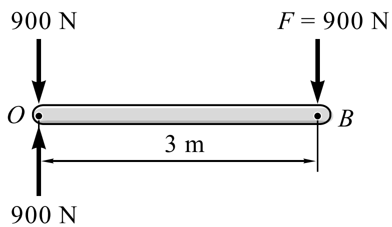


Fig. 2.20(b)

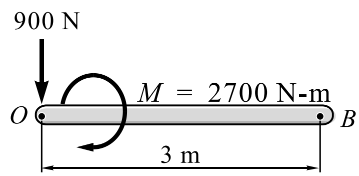


Fig. 2.20(c)

Problem 21

Replace the force (600 N) from point A by equivalent force couple at B. Refer to Fig. 2.21(a).

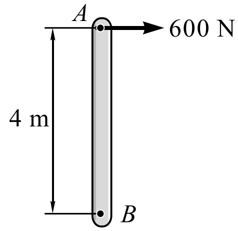


Fig. 2.21(a)

Solution

Refer to Figs. 2.21(b) and (c).

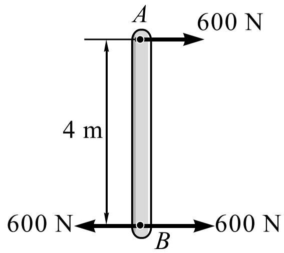


Fig. 2.21(b)

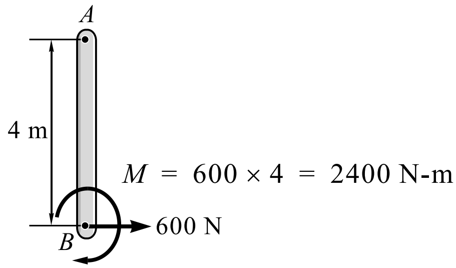


Fig. 2.21(c)

Force (600 N) acting at point A can be replaced at B by a force (600 N) as it is (i.e., Horizontal) and a couple of moment

$$M = 600 \times 4 = 2400 \text{ N-m } (\circlearrowright) \quad \textbf{Ans.}$$

Problem 22

Replace the force (3000 N) from point A by equivalent force couple at B. Refer to Fig. 2.22(a).

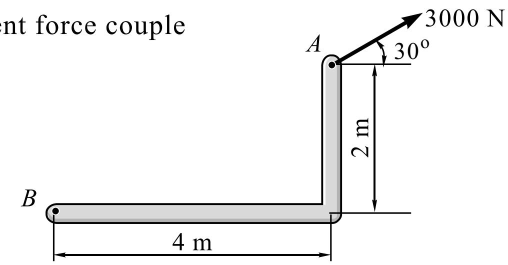


Fig. 2.22(a)

Solution

Refer to Fig. 2.22(b).

$$\text{Couple} = \text{Moment of forces about } B$$

$$\Sigma M_B = 3000 \sin 30^{\circ} \times 4 - 3000 \cos 30^{\circ} \times 2$$

$$\Sigma M_B = 803.85 \text{ N-m } (\circlearrowleft)$$

$$\text{Couple} = 803.85 \text{ N-m } (\circlearrowleft) \quad \textbf{Ans.}$$

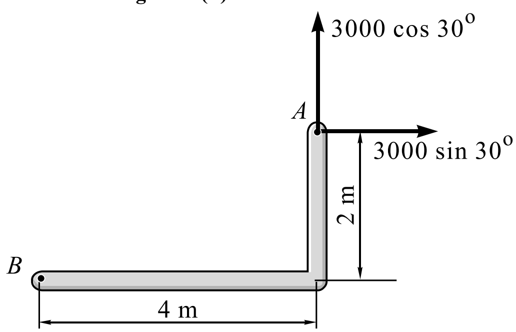


Fig. 2.22(b)

Equivalent force couple at B is shown in Fig. 2.22(c).

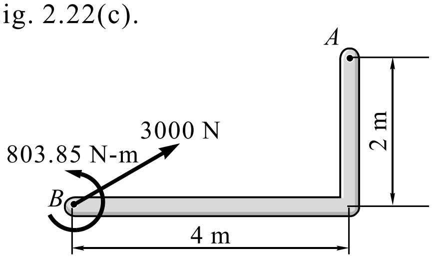


Fig. 2.22(c)

2.7 Varignon's Theorem

It states that *the moment of resultant of all the forces in a plane about any point is equal to the algebraic sum of moment of all the forces about the same point.*

Case : The moment of a force in a plane about any point is equal to the sum of the moments of components of the force about the same point.

$$R \times d = P \times d_1 + Q \times d_2$$

Proof : Consider a force R acting at a point O. Say P and Q are the resolved components of force R (as shown in Fig. 2.7-i). Let R, P and Q form an angle θ, θ_1, θ_2 respectively with x-axis.

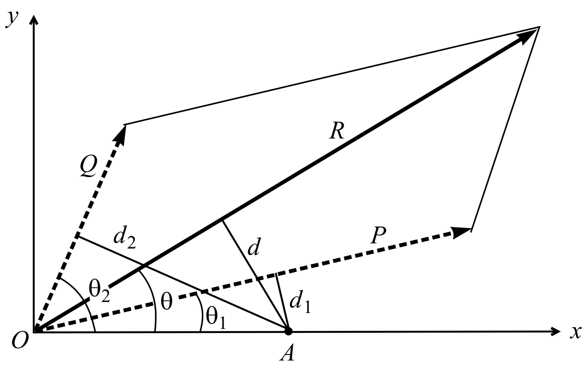


Fig. 2.7-i

The moment of R about an arbitrary point A is $R \times d$ where d is perpendicular distance from A to the line of action of R. Similarly, the moment of P and Q about point A are $P \times d_1 + Q \times d_2$ respectively, where d_1 and d_2 are perpendicular distances from A to the line of action of P and Q respectively.

Since R is the resultant of P and Q, it follows that the sum $P_y + Q_y$ of the y component of two forces P and Q is equal to the y component R_y of their resultant R.

$$\text{i.e.,}\quad R_y = P_y + Q_y$$

$$R_y = R \sin\theta, \quad P_y = P \sin\theta_1 \quad \text{and} \quad Q_y = Q \sin\theta_2$$

$$\therefore\ R \sin\theta = P \sin\theta_1 + Q \sin\theta_2$$

Multiplying both sides by length OA

$$R \times OA \sin\theta = P \times OA \sin\theta_1 + Q \times OA \sin\theta_2$$

$$\text{But}\ \ OA \sin\theta = d,\ \ OA \sin\theta_1 = d_1 \ \text{and}\ \ OA \sin\theta_2 = d_2$$

$$\therefore\ R \times d = P \times d_1 + Q \times d_2$$ ***Hence Proved.***

Note : Varignon's theorem is used for determining the *position of resultant of parallel and general force system*.

2.8 Resultant of Parallel Force System

Procedure

Step 1 : Find resultant $R = \Sigma F$.

Take the algebraic sum of all the parallel forces considering proper sign convention (+ve$\uparrow$ / –ve $\downarrow$).

Step 2 : Find ΣM_O.

Take the algebraic sum moment of forces about a point (say O) considering proper sign convention (+ve $\circlearrowleft$ / –ve $\circlearrowright$).

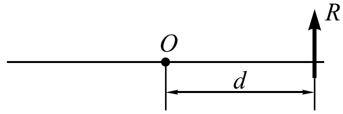
Step 3 : Apply Varignon's theorem,

$\Sigma M_O = R \times d$ (where d is the perpendicular distance between line of action of R and reference point O).

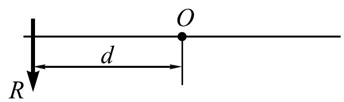
Step 4 : Position of resultant w.r.t. point O.

Resultant may lie to the right or left of the reference point O at a distance d, depending on the sign of ΣF and ΣM_O.

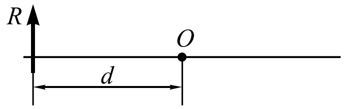
(i) $\Sigma F \uparrow$ and ΣM_O (+ve $\circlearrowleft$)



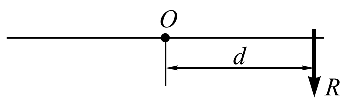
(ii) $\Sigma F \downarrow$ and ΣM_O (+ve $\circlearrowleft$)



(iii) $\Sigma F \uparrow$ and ΣM_O (–ve $\circlearrowright$)



(iv) $\Sigma F \downarrow$ and ΣM_O (–ve $\circlearrowright$)



Note : Resultant of parallel force system may be

- only a single resultant force R (Translational motion),
- only a single resultant couple M (Rotational motion) or
- both a single resultant force R and a single resultant couple M (Translational and rotational motion together)

2.8.1 Solved Problems

Problem 23

Find the resultant of following force system and also find the equivalent force and couple at point A of the same force system shown in Fig. 2.23(a).

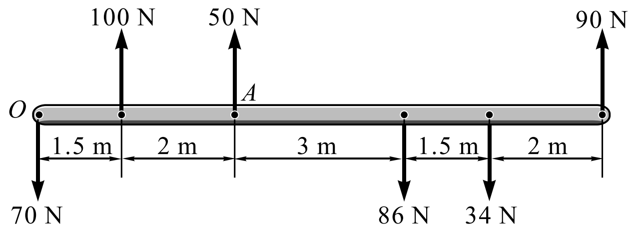


Fig. 2.23(a)

Solution

Case (I)

(i) $R = \Sigma F = -70 + 100 + 50 - 86 - 34 + 90$

$\therefore$ $R = 50$ N ($\uparrow$) ***Ans.***

(ii) $\Sigma M_O = 100 \times 1.5 + 50 \times 3.5 - 86 \times 6.5 - 34 \times 8 + 90 \times 10$

$\therefore$ $\Sigma M_O = 394$ N-m ($\circlearrowleft$) ***Ans.***

(iii) Applying Varignon's theorem, we have

$\Sigma M_O = R \times d$

$$\therefore d = \frac{\Sigma M_O}{R} = \frac{394}{50}$$

$\therefore$ $d = 7.88$ m ***Ans.***

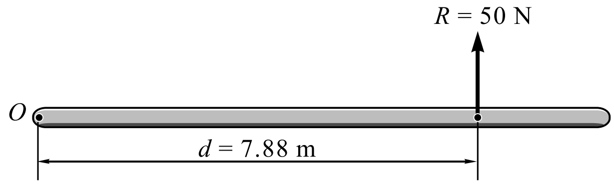


Fig. 2.23(b)

(v) Position of the resultant w.r.t. point O is as shown in Fig. 2.23(b).

Case (II)

(i) $R = \Sigma F = -70 + 100 + 50 - 86 - 34 + 90$

$\therefore$ $R = 50$ N ($\uparrow$) ***Ans.***

(ii) $\Sigma M_A = 70 \times 3.5 - 100 \times 2 - 86 \times 3 - 34 \times 4.5 + 90 \times 6.5$

$\therefore$ $\Sigma M_A = 219$ N-m ($\circlearrowleft$) ***Ans.***

(iii) Equivalent force and couple at point A of the force system is a single force $R = 50$ N ($\uparrow$) and a couple $C = \Sigma M_A = 219$ N-m ($\circlearrowleft$) shown in Fig. 2.15(c).

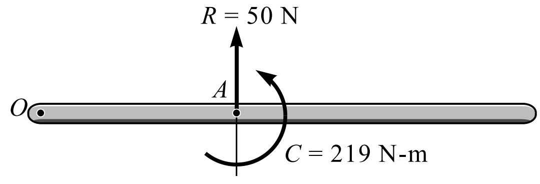
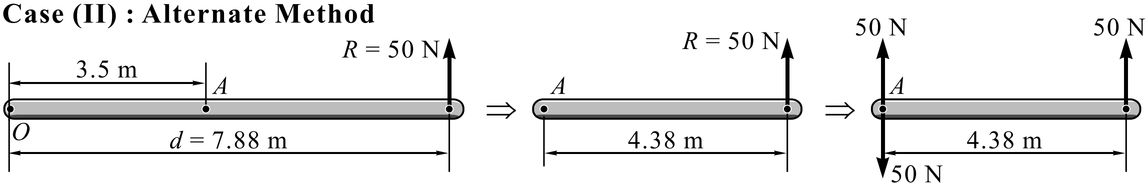


Fig. 2.23(c) ***Ans.***

Case (II) : Alternate Method



(From Case I) **Fig. 2.23(d)**

$\therefore$ Equivalent force and couple at point A =
($C = 50 \times 4.38 = 219$ N-m)

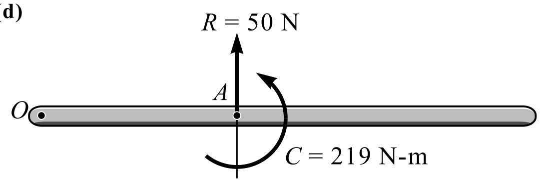


Fig. 2.23(e) ***Ans.***

Problem 24

Replace the force system acting on a bar as shown in Fig. 2.24(a) by a single force.

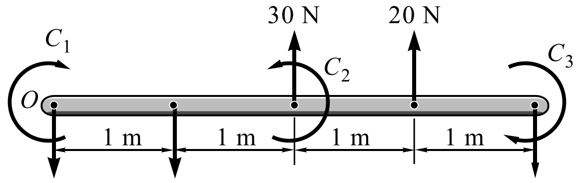


Fig. 2.24(a)

C_1 = 85 N-m
C_2 = 65 N-m
C_3 = 90 N-m

Solution

(i) $R = \Sigma F = -50 - 40 + 30 + 20 - 40$

$R = -80$ N

$R = 80$ N $(\downarrow)$ ***Ans.***

(ii) $\Sigma M_O = -40 \times 1 + 30 \times 2 + 20 \times 3 - 40 \times 4 - 85 + 65 - 90$

$\Sigma M_O = -190$ N-m $\therefore$ $\Sigma M_O = 190$ N-m $(\circlearrowright)$ ***Ans.***

(iii) Applying Varignon's theorem

$\Sigma M_O = R \times d$

$$\therefore \; d = \frac{\Sigma M_O}{R} = \frac{190}{80}$$

$\therefore$ $d = 2.375$ m ***Ans.***

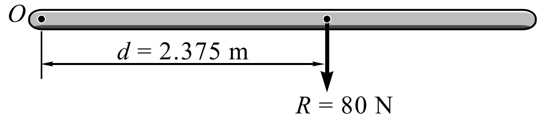


Fig. 2.24(b)

(iv) Position of resultant w.r.t. point O is shown in Fig. 2.24(b).

Problem 25

Find the resultant of given active forces [Fig. 2.25(a)] w.r.t. point B.

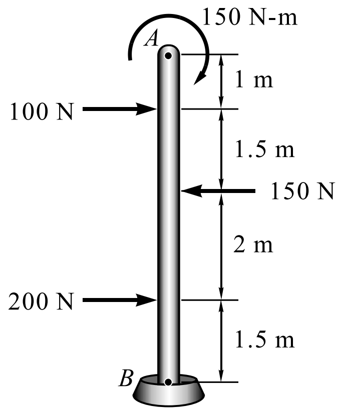


Fig. 2.25(a)

Solution

(i) $R = \Sigma F = 100 - 150 + 200$

$\therefore$ $R = 150$ N $(\rightarrow)$ ***Ans.***

(ii) $\Sigma M_B = -150 - 100 \times 5 + 150 \times 3.5 - 200 \times 1.5$

$\Sigma M_B = -425$ N-m

$\Sigma M_B = 425$ N-m $(\circlearrowright)$ ***Ans.***

(iii) By Varignon's theorem, we have

$\Sigma M_B = R \times h$

$$\therefore \; h = \frac{425}{150}$$

$\therefore$ $h = 2.83$ m ***Ans.***

Position of resultant w.r.t. point B is as shown in Fig. 2.25(b).

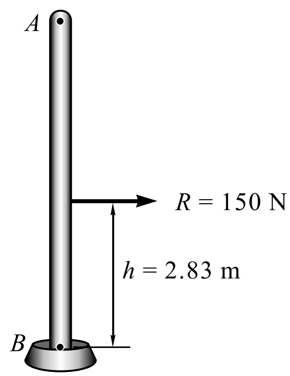


Fig. 2.25(b)

Problem 26

Find the resultant of the force systems shown in Fig. 2.26.

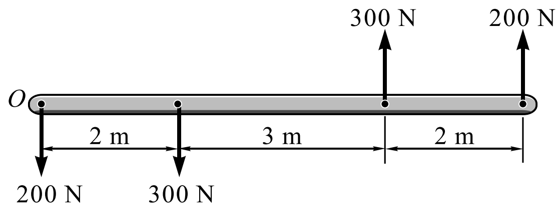


Fig. 2.26

Solution

(i) $R = \Sigma F = -200 - 300 + 300 + 200$

$\therefore$ $R = 0$

$\therefore$ The resultant force $R = 0$ ***Ans.***

$\because$ The resultant force is zero, the resultant may be a couple.

(ii) To find the value of couple, take moment of all forces about point O.

$\Sigma M_O = -300 \times 2 + 300 \times 5 + 200 \times 7$

$\therefore$ $\Sigma M_O = 2300$ N-m $(\circlearrowleft)$ ***Ans.***

(iii) Here the resultant of given force system is a couple of 2300 N-m $(\circlearrowleft)$ and resultant force is zero.

Problem 27

Resolve the force $F = 900$ N acting at B as shown in Fig. 2.27(a) into parallel components at O and A.

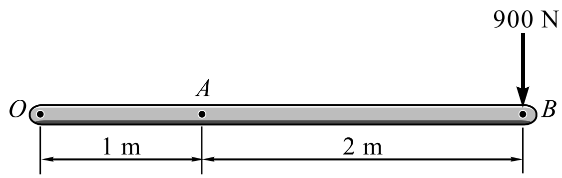


Fig. 2.27(a)

Solution

(i) Assume F_1 and F_2 as the parallel components of forces acting vertical at O and A respectively as shown in Fig. 2.27(b).

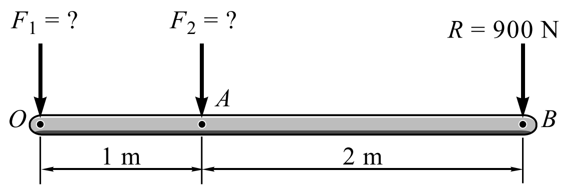


Fig. 2.27(b)

(ii) Taking moment about O and applying Varignon's theorem, we have

$-900 \times 3 = F_1 \times 0 - F_2 \times 1$

$F_2 = 2700$ N$(\downarrow)$

(iii) Taking moment about A and applying Varignon's theorem, we have

$-900 \times 2 = F_1 \times 1 + F_2 \times 0$

$F_1 = -1800$ N (–ve sign indicates wrong assumed direction)

$F_1 = 1800$ N $(\uparrow)$

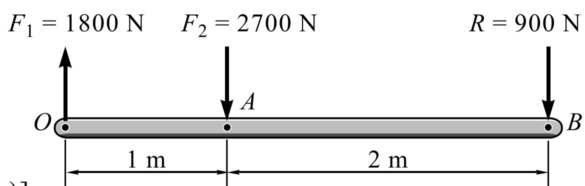


Fig. 2.27(c)

(iv) Position of F_1 and F_2. [Refer to Fig. 2.27(c)].

Problem 28

Determine the resultant of the parallel force shown in Fig. 2.28(a) and locate it w.r.t. O, radius is 1 m.

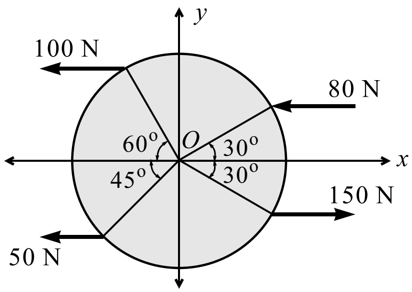


Fig. 2.28(a)

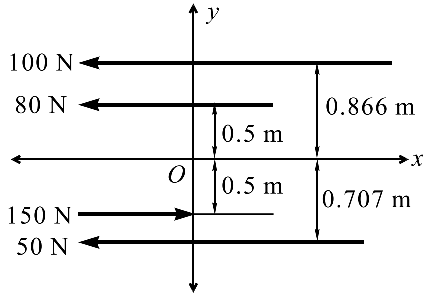


Fig. 2.28(b)

Solution

(i) Magnitude of resultant R

$$R = -100 - 80 + 150 - 50$$

$$\therefore \; R = -80 \text{ N} = 80 \text{ N} \; (\leftarrow) \qquad \textbf{\textit{Ans.}}$$

(ii) $\Sigma M_O = 100 \times 0.866 + 80 \times 0.5 + 150 \times 0.5 - 50 \times 0.707$

$\Sigma M_O = 166.25$ N-m ($\circlearrowleft$)

(iii) Applying Varignon's theorem

$$d = \frac{\Sigma M_O}{R} = \frac{166.25}{80}$$

$$\therefore \; d = 2.078 \text{ m}$$

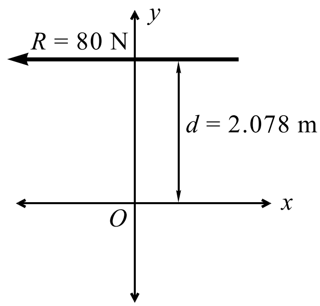


Fig. 2.28(c)

(iv) Position of resultant w.r.t origin O is as shown in Fig. 2.28(c). ***Ans.***

2.9 Resultant of General Force System

Procedure

Step 1 : Find ΣF_x.

Resolve all the forces along the horizontal x-axis and take the algebraic sum of force components considering proper sign convention (+ve →).

Step 2 : Find ΣF_y.

Resolve all the forces along the vertical y-axis and take the algebraic sum of force components considering proper sign convention (+ve ↑).

Step 3 : Find R.

Magnitude of resultant is given by

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

Step 4 : Find θ.

Inclination of the line of action of resultant force with horizontal x-axis is given by

$$\tan\theta = \left|\frac{\Sigma F_y}{\Sigma F_x}\right| \quad \therefore \quad \theta = \tan^{-1}\left(\frac{\Sigma F_y}{\Sigma F_x}\right)$$

Note : Use only positive value of ΣF_x and ΣF_y to find θ with horizontal x-axis so the value of θ will be always acute angle.

Step 5 : ΣM_O

Take the algebraic sum of moment of force about a point (say O) considering proper sign conventions (+ve ↺ / –ve ↻)

Step 6 : Applying Varignon's theorem.

$\Sigma M_O = R \times d$ (where d is the perpendicular distance between the line of action of R and reference point O)

or

$\Sigma M_O = \Sigma F_y \times x$ (where x is the distance between point O and the intersection of line of action of resultant R with x-axis)

or

$\Sigma M_O = \Sigma F_x \times y$ (where y is the distance between point O and the intersection of line of action of resultant R with y-axis)

Step 7 : Position of resultant w.r.t. point O.

Note : It is very easy to select the correct position of R among eight by following the guidelines:

- First observe the direction of ΣF_x and ΣF_y which gives the reference position of R is any of the quadrant passing through origin.
- Since ΣM_O is not zero, it means resultant R should not pass through O. So R is suppose to be at some distance d from O.
- Observe the direction of ΣM_O and shift R on the required side to satisfy the direction of rotation of ΣM_O.

Problem 29

Replace the force system shown in Fig. 2.29(a) by a single force.

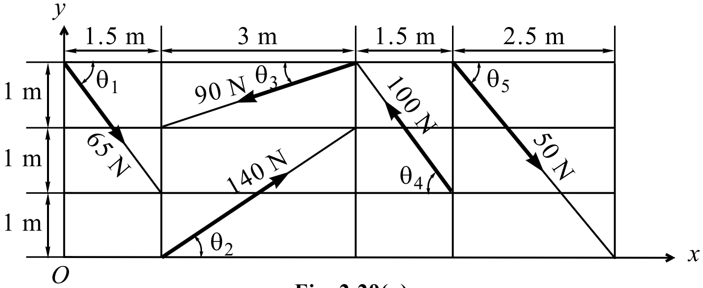


Fig. 2.29(a)

Solution

(i) $\tan\theta_1 = \dfrac{2}{1.5}$ $\therefore$ $\theta_1 = 53.13^o$ $\qquad$ $\tan\theta_4 = \dfrac{2}{1.5}$ $\therefore$ $\theta_1 = 53.13^o$

$\tan\theta_2 = \dfrac{2}{3}$ $\therefore$ $\theta_2 = 33.69^o$ $\qquad$ $\tan\theta_5 = \dfrac{3}{2.5}$ $\therefore$ $\theta_5 = 50.19^o$

$\tan\theta_3 = \dfrac{1}{3}$ $\therefore$ $\theta_3 = 18.44^o$

(ii) $\Sigma F_x = 65\cos 53.13^o + 140\cos 33.69^o - 90\cos 18.44^o - 100\cos 53.13^o + 50\cos 50.19^o$

$\therefore \Sigma F_x = 42.12$ N $(\rightarrow)$

(iii) $\Sigma F_y = -65\sin 53.13^o + 140\sin 33.69^o - 90\sin 18.44^o + 100\sin 53.13^o - 50\sin 50.19^o$

$\therefore \Sigma F_y = 38.78$ N $(\uparrow)$

(iv) $R = \sqrt{(42.12)^2 + (38.78)^2}$

$\therefore R = 57.25$ N ***Ans.***

(v) $\theta = \tan^{-1}\left(\dfrac{38.78}{42.12}\right)$

$\therefore \theta = 42.64^o$ ***Ans.***

(vi) $\Sigma M_O = -65\cos 53.13^o \times 3 + 140\sin 33.69^o \times 1.5 + 90\cos 18.44^o \times 3 - 90\sin 18.44^o \times 4.5$
$+ 100\cos 53.13^o \times 1 + 100\sin 53.13^o \times 6 - 50\cos 50.19^o \times 3 - 50\sin 50.19^o \times 6$

$\therefore \Sigma M_O = 341.03$ N-m $(\circlearrowleft)$

(vii) By Varignon's theorem

$$x = \frac{\Sigma M_O}{\Sigma F_y} = \frac{341.03}{38.78}$$

$\therefore x = 8.79$ m ***Ans.***

(viii) Position of resultant w.r.t. O
[Refer to Fig. 2.29(b)].

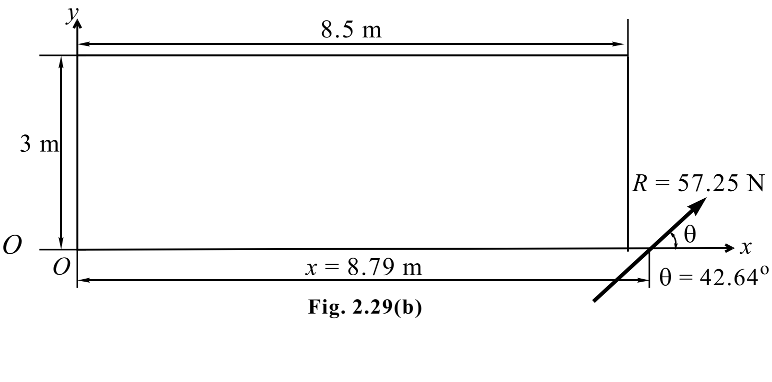


Fig. 2.29(b)

Problem 30

(i) Find the resultant of the force system shown in Fig. 2.30(a).

(ii) Replace the given force and couple by a single force and couple system at *A*.

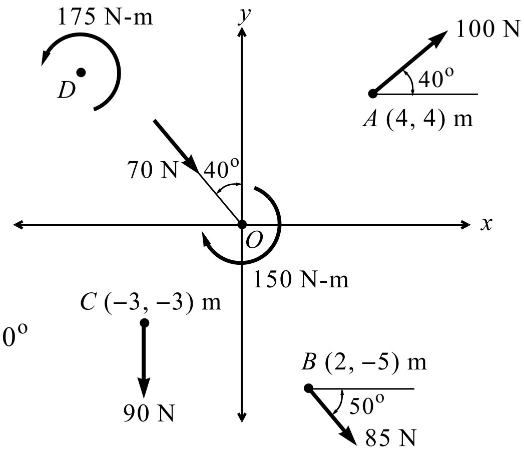


Fig. 2.30(a)

Solution

(i)

(a) $\Sigma F_x = 100 \cos 40^\circ + 85 \cos 50^\circ + 70 \sin 40^\circ$

$\Sigma F_x = 176.24 \text{ N } (\rightarrow)$

(b) $\Sigma F_y = 100 \sin 40^\circ - 85 \sin 50^\circ - 90 - 70 \cos 40^\circ$

$\Sigma F_y = -144.46 \text{ N}$

$\therefore \Sigma F_y = 144.46 \text{ N } (\downarrow)$

(c) Magnitude of resultant *R*

$$R = \sqrt{(176.24)^2 + (144.46)^2}$$

$\therefore R = 227.88 \text{ N}$ ***Ans.***

(d) Inclination of the resultant θ

$$\theta = \tan^{-1}\left(\frac{144.46}{176.24}\right) \quad \therefore \theta = 39.34^\circ$$

(e) $\Sigma M_O = -100 \cos 40^\circ \times 40 + 100 \sin 40^\circ \times 4 + 85 \cos 50^\circ \times 5 - 85 \sin 50^\circ \times 2 + 90 \times 3 + 175 - 150$

$\Sigma M_O = 2369.10 \text{ N-m } (\circlearrowright)$

(f) Applying Varignon's theorem, we have

$\Sigma M_O = R \times d$

$$\therefore d = \frac{2369.10}{227.88}$$

$\therefore d = 10.4 \text{ m}$ ***Ans.***

(g) Position of resultant *R* w.r.t. *O* [Refer to Fig. 2.30(b)].

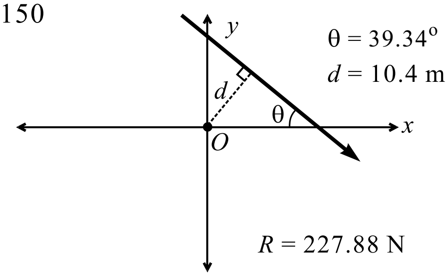


Fig. 2.30(b)

(ii)

Results **(a)** to **(d)** as above in **(i)**

(h) $\Sigma M_A = 85 \cos 50^\circ \times 9 + 85 \sin 50^\circ \times 2 + 90 \times 7 + 70 \sin 40^\circ \times 4 + 70 \cos 40^\circ \times 4 + 175 - 150$

$\Sigma M_A = 1671.43 \text{ N-m } (\circlearrowleft)$

(i) Resultant of a given force system of point *A* is a force and couple as shown in Fig. 2.30(c).

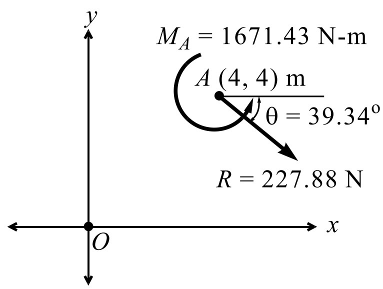


Fig. 2.30(c)

Problem 31

Find the resultant of the force system shown in Fig. 2.31(a). Radius = 2.5 m

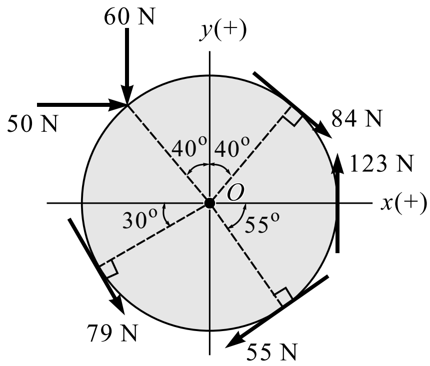


Fig. 2.31(a)

Solution

Refer to Fig. 2.31(b).

(i) $\Sigma F_x = 84 \cos 40^\circ - 55 \cos 35^\circ + 79 \cos 60^\circ + 50$

$\Sigma F_x = 108.79 \text{ N } (\rightarrow)$

(ii) $\Sigma F_y = -84 \sin 40^\circ - 55 \sin 35^\circ - 79 \sin 60^\circ + 123 - 60$

$\Sigma F_y = -90.96 \text{ N}$

$\therefore \Sigma F_y = 90.96 \text{ N } (\downarrow)$

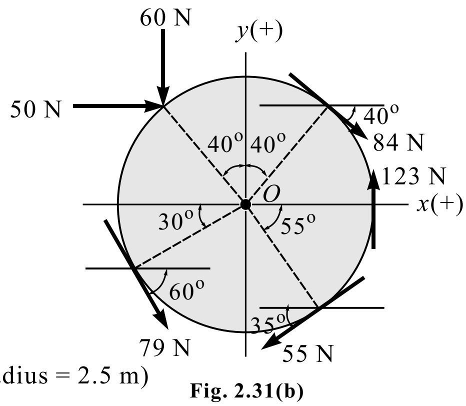


Fig. 2.31(b)

(iii) $R = \sqrt{(108.79)^2 + (90.96)^2}$

$\therefore R = 141.80 \text{ N}$ ***Ans.***

(iv) $\theta = \tan^{-1}\left(\dfrac{90.96}{108.79}\right)$

$\therefore \theta = 39.89^\circ$

(v) $\Sigma M_O = -84 \times 2.5 + 123 \times 2.5 - 55 \times 2.5 + 79 \times 2.5 - 50 \times 2.5 \cos 40^\circ + 60 \times 2.5 \sin 40^\circ$

$\therefore \Sigma M_O = 158.16 \text{ N-m } (\circlearrowleft)$

(vi) By Varignon's theorem

$\Sigma M_O = R \times d$

$\therefore d = \dfrac{158.16}{141.80}$

$\therefore d = 1.12 \text{ m}$ ***Ans.***

(vii) Position of resultant R [Refer to Fig. 2.31(c)].

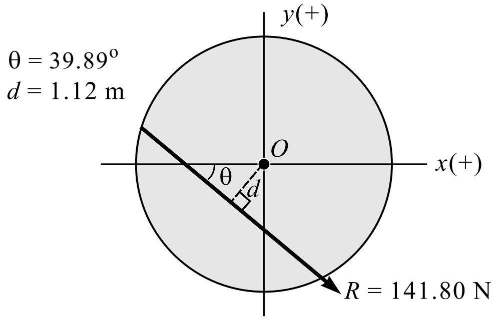


Fig. 2.31(c)

Problem 32

A triangular plate ABC is subjected to four coplanar forces as shown in Fig. 2.32(a). Find the resultant completely and locate its position with respect to point A.

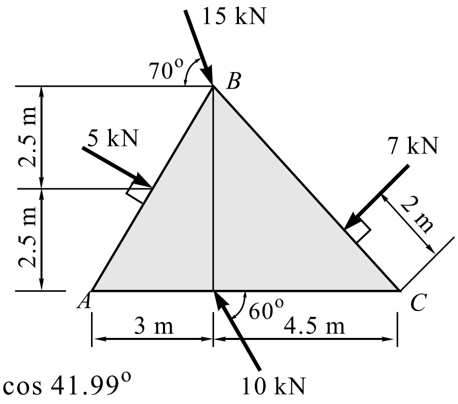


Fig. 2.32(a)

Solution

Refer to Fig. 2.32(b) for geometrical angles.

(i) $\Sigma F_x = 5 \cos 30.96^\circ + 15 \cos 70^\circ - 10 \cos 60^\circ - 7 \cos 41.99^\circ$

$\therefore \Sigma F_x = -0.78 \text{ kN} = 0.78 \text{ kN } (\leftarrow)$

(ii) $\Sigma F_y = -5 \sin 30.96^\circ - 15 \sin 70^\circ + 10 \sin 60^\circ - 7 \sin 41.99^\circ$

$\therefore \Sigma F_y = -12.69 \text{ kN} = 12.69 \text{ kN } (\downarrow)$

(iii) Resultant R

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$R = \sqrt{(0.78)^2 + (12.69)^2}$$

$\therefore R = 12.71$ kN ***Ans.***

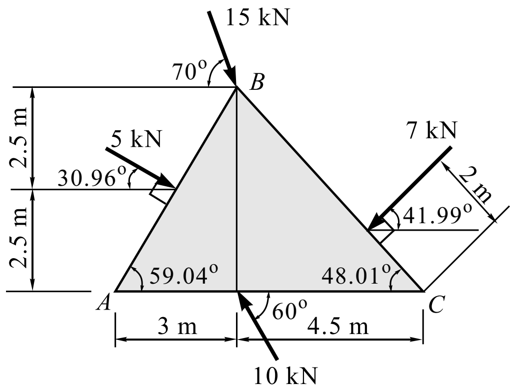


Fig. 2.32(b)

(iv) Direction of the resultant

$$\theta = \tan^{-1}\left(\frac{\Sigma F_y}{\Sigma F_x}\right)$$

$$\therefore \theta = \tan^{-1}\left(\frac{12.69}{0.78}\right)$$

$\therefore \theta = 86.48^\circ$ ***Ans.***

(v) $\Sigma M_A = -5 \cos 30.96^\circ \times 2.5 - 5 \sin 30.96^\circ \times 1.5 - 15 \cos 70^\circ \times 5 - 15 \sin 70^\circ \times 3$
$+ 10 \sin 60^\circ \times 3 + 7 \cos 41.99^\circ \times 2 \sin 48.01^\circ - 7 \sin 41.99^\circ \times (7.5 - 2 \cos 48.01^\circ)$

$\Sigma M_A = -77.66$ kN-m

$\therefore \Sigma M_A = 77.66$ kN-m $(\circlearrowright)$

(vi) By Varignon's theorem, we have

$$x = \frac{\Sigma M_A}{\Sigma F_y} = \frac{77.66}{12.69}$$

$\therefore x = 6.12$ m ***Ans.***

(vii) Position of resultant w.r.t. A
[Refer to Fig. 2.32(c)].

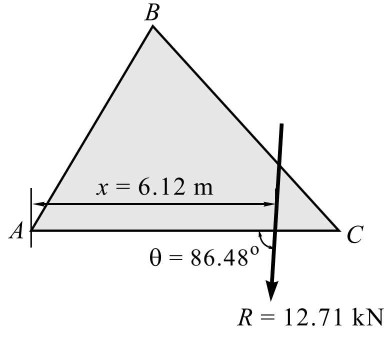


Fig. 2.32(c)

Exercises

[I] Problems

1. Find the resultant of given force system as shown in Fig. 2.E1.

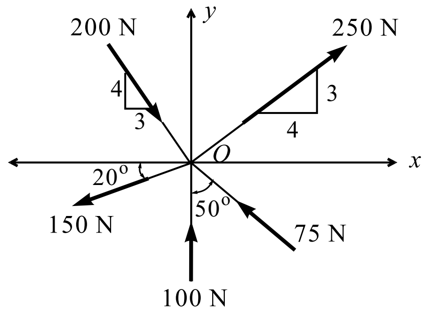
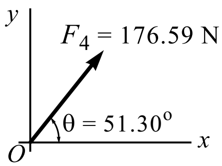


Fig. 2.E1

2. Find fourth force (F_4) completely so as to give the resultant of the system of force as shown in Fig. 2.E2.

[**Ans.**



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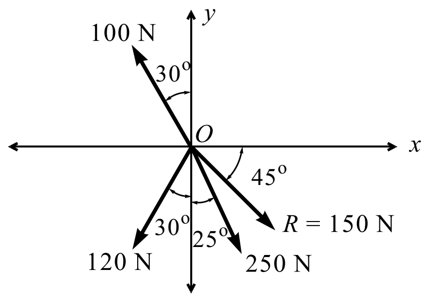
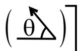


Fig. 2.E2

3. Find the resultant of the force acting on a particle P shown in Fig. 2.E3.

[**Ans.** $R = 500.09$ N 
and $\theta = 88.8°$.]

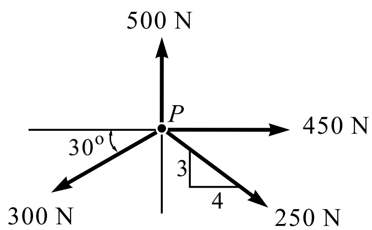


Fig. 2.E3

4. The resultant of the three pulls applied through the three chains attached to bracket is θ as shown in Fig. 2.E4. Determine the magnitude and angle θ of the resultant (R).

[**Ans.** $R = 623.24$ N and $\theta = 75.4°$.]

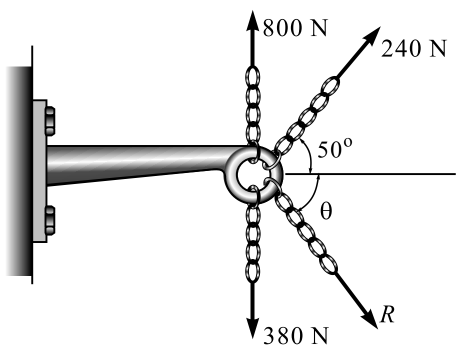


Fig. 2.E4

5. Four forces act on an eye bolt as shown in Fig. 2.E5.
Determine their resultant.

[***Ans.*** $R = 286$ N and $\theta = 88.81^o$.]

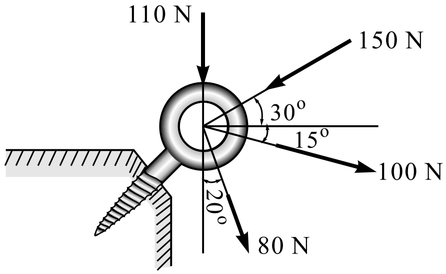


Fig. 2.E5

6. A gusset plate of roof truss is subjected to forces as shown in Fig. 2.E6.
Determine the magnitude of the resultant force and its orientation measured counter clockwise from the positive x-axis.

[***Ans.*** $R = 34.14$ N and $\theta = 180^o\,42'$.]

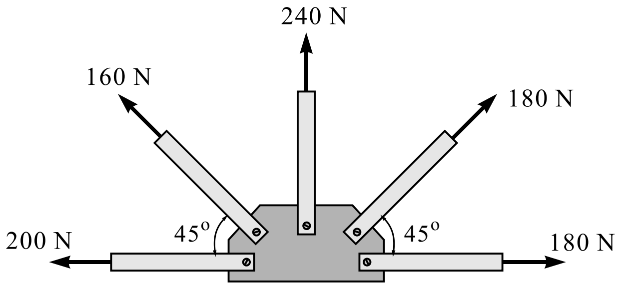


Fig. 2.E6

7. A force R = 25 kN acting at O has three components $F_{A,}$ F_B and F_C as shown in Fig. 2.E7 if F_C = 20 kN, find F_A and F_B.

[***Ans.*** F_A = 33.91 kN and
F_B = 35.04 kN.]

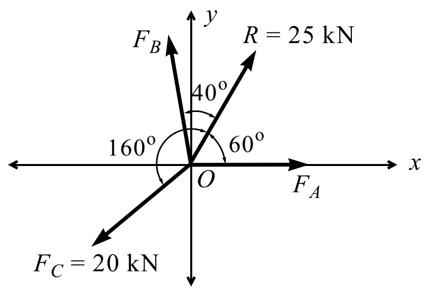


Fig. 2.E7

8. Three force acting at a point are shown in Fig. 2.E8. The direction of the 300 N forces my vary but the angle between them is always 40°. Determine the value of θ for which the resultant of the three forces is directed parallel to x-x along the inclined. Find the result.

[***Ans.*** $\theta = 6.35^o$ and $R = 938.25$ N $(\measuredangle 30^o)$.]

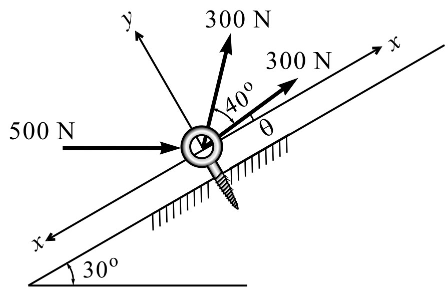


Fig. 2.E8

9. A car is made to move by applying resultant force R = 2000 N along the x-axis. This resultant is developed due to two pulling forces F_1 and F_2 on two ropes, as shown in Fig. 2.E9. Determine the tension in individual ropes.

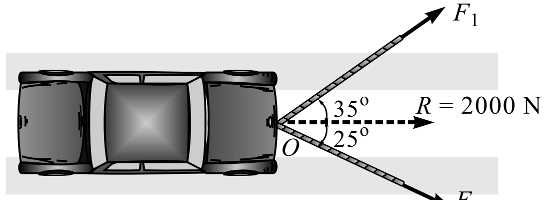


Fig. 2.E9

10. Two locomotive on opposite banks of a canal pull a vessel moving parallel to the banks by means of two horizontal ropes. The tension in these ropes are 2000 N and 2400 N while the angle between them is 60°. Find the resultant pull on the vessel and the angle between each of the ropes and the sides of the canal.

[***Ans.*** $\theta = 33°$, $\alpha = 27°$ and R = 3815.75 N (→).]

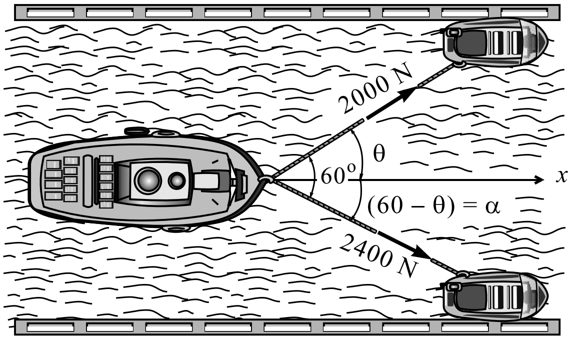


Fig. 2.E10

11. The resultant of four vertical forces is a couple of 300 N.m counterclockwise. Three of the forces are shown in Fig. 2.E11. Determine the fourth.

[***Ans.*** 33 N at 4.47 m to the right of O.]

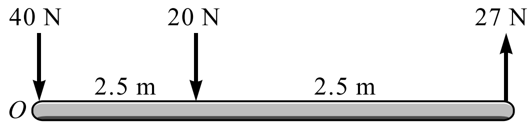
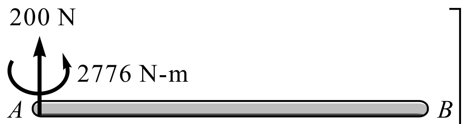


Fig. 2.E11

12. Find the resultant of the system of parallel forces shown in Fig. 2.E12 at point A.

[***Ans.***



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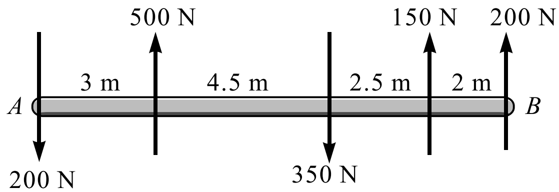


Fig. 2.E12

13. The resultant of the three forces shown in Fig. 2.E13 and other two forces P and Q acting at A and B is a couple of magnitude 120 kN-m clockwise. Determine the forces P and Q.

[***Ans.*** At A force P = 29 kN (↑) and at B force Q = 16 kN (↓).]

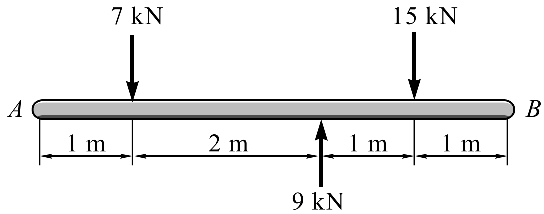


Fig. 2.E13

14. The parallel force system of five forces 12 kN, 15 kN, 24 kN, 30 kN and 20 kN is shown in Fig. 2.E14. Reduce the system to a force and a couple at point P.

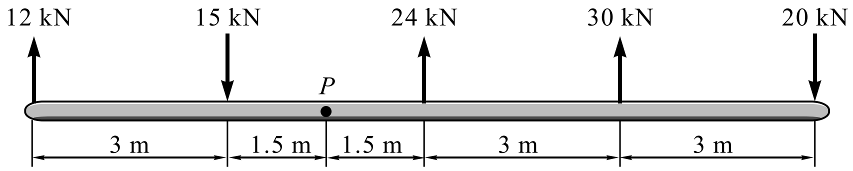
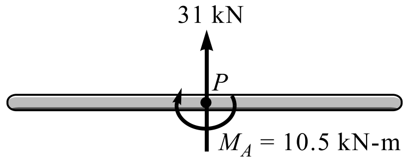


Fig. 2.E14

[***Ans.***



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15. A 150 × 300 mm plate is subjected to four loads as shown in Fig. 2.E15. Find the resultant of the four loads and the two points at which the line of action of the resultant intersects the edges of the plate.

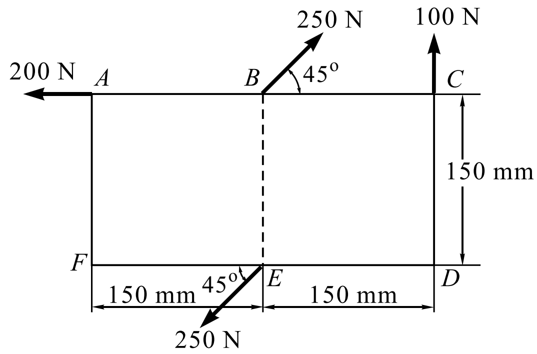


Fig. 2.E15

[***Ans.*** $R = 224$ N, $\theta = 26.6°$,
35 mm to the right of A
and 132.7 mm below C.]

16. Determine the resultant of the non-concurrent non-parallel system of forces. F is in Newton and the coordinates are in metres.

F	20	30	50	10
θ_x	45°	120°	190°	270°
Point of application	(1,3)	(4,–5)	(5,2)	(–2,–5)

[***Ans.*** $R = 54.5$ N, $\theta = 23.2°$ and intersection on x-axis = 3.52 m.]

17. Determine completely the resultant of the four forces shown in Fig. 2.E17. Each force makes a 15° angle with the vertical except the 2000 N force which is vertical.

[***Ans.*** $R = 6830$ N (↓) and
at 16.79 m to the right of A.]

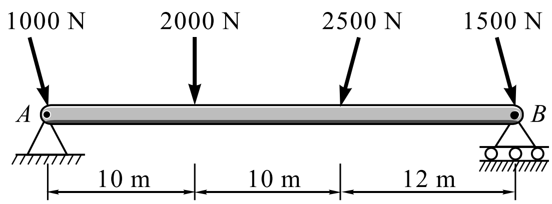


Fig. 2.E17

18. Solve for the resultant of the six loads on the truss shown in Fig. 2.E18. Three loads are vertical. The wind loads are perpendicular to the side. The truss is

[**Ans.** $R = 10.65$ kN (θ), $\theta = 79.2^{\circ}$
and intersection on $AB = 15.41$ m to the right of A.]

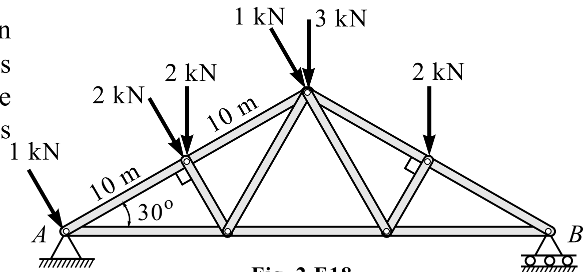


Fig. 2.E18

19. Determine the resultant of the forces acting on the bell crank shown in Fig. 2.E19.

[**Ans.** $R = 1109$ N (θ), $\theta = 79.6^{\circ}$
and intersection on $AO = 157.1$ mm to the left of O.]

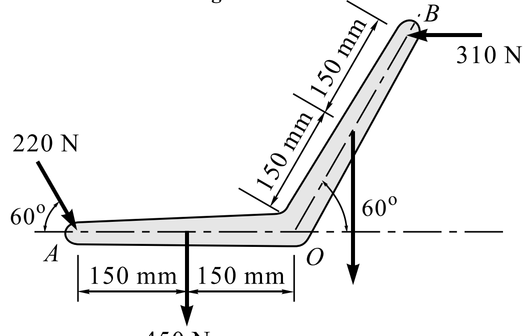


Fig. 2.E19

20. Determine the resultant of the force system in the Fig. 2.E20 and locate it with respect to point A.

[**Ans.** $R = 305$ N (θ), $\theta = 31.6^{\circ}$
and distance from A to the line of action of R is 249 mm.]

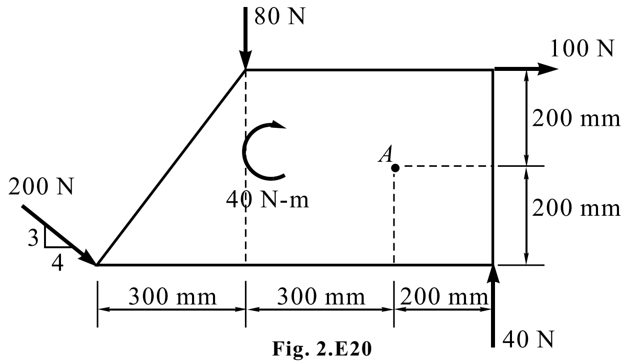


Fig. 2.E20

21. The 200 N force is the resultant of the couple and three forces, two of which are shown in Fig. 2.E21. Determine the third force and locate it with respect to point O.

[**Ans.** $F = 269$ N (θ), $\theta = 48^{\circ}$
and distance from O to the line of action of F is 4.24 m.]

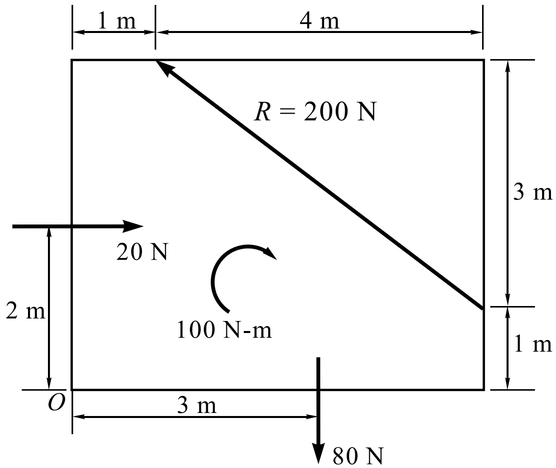


Fig. 2.E21

[II] Review Questions

1. Write the S.I. unit of the following quantities :

(a) Length **(b)** Mass **(c)** Time **(d)** Force

(e) Area **(f)** Volume **(g)** Work **(h)** Power

(i) Energy **(j)** Displacement **(k)** Velocity **(l)** Speed

(m) Acceleration **(n)** Moment of force

2. Group the above quantities in its vector and scalar.

3. Explain the following terms in short :

(a) Mechanics **(b)** Statics **(c)** Dynamics **(d)** Kinematics

(e) Kinetics **(f)** Particle **(g)** Rigid body

4. State the following laws :

(a) Newton's first law of motion **(b)** Newton's second law of motion

(c) Newton's third law of motion **(d)** Newton's law of gravitation

(e) Law of parallelogram of force **(f)** Principle of transmisibility of force

5. What is force ? State its description.

6. Classify the force system.

7. What is meant by :

(a) Resolution of force ?

(b) Composition of force ?

(c) Moment of force ?

(d) Moment of couple ?

8. What is a resultant ?

9. What is meant by equilibrant ?

10. State and prove **(a)** Varignon's theorem, and **(b)** Law of Parallelogram of Forces.

11. Justify, why Triangle Law of Forces is a corollary of Parallelogram Law of Forces ?

12. The resultant of system of parallel forces is zero. What does it signify ?

13. Describe the procedure to find the

(a) resultant of concurrent force system,

(b) resultant of parallel force system and

(c) resultant of general force system.

[III] Fill in the Blanks

1. _________ is the region which extends in all directions and contains everything in it.
2. _________ is measure of duration between successive event.
3. _________ is the quantity of matter contained in a body.
4. A physical quantity which requires only magnitude for its complete description is known as ________.
5. ________ is a matter having considerable mass but negligible dimension.
6. ________ and _______ are equal in magnitude, opposite in direction and collinear in action.
7. Coplanar parallel forces are further subgrouped into _______ parallel forces and _______ parallel forces.
8. As per the right-hand-thumb rule sign convention, anticlockwise moment is taken as ________ and clockwise moment is taken as _______.
9. If the resultant of number of parallel forces is not zero, then the system can be reduced to a ________ force.
10. If the resultant of number of parallel forces is zero, then the system may have a resultant ________ or may be in ________.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. Mechanics is the branch of _________.

   (a) Physics **(b)** Chemistry **(c)** Biology **(d)** Sociology

2. Mechanics of rigid body is the study of _________.

   (a) Strength of material **(b)** Theory of plasticity
   (c) Theory of elasticity **(d)** Statics and dynamics

3. Study of effect of force system acting on particle or rigid body at rest is called _______.

   (a) Dynamics **(b)** Kinematics **(c)** Statics **(d)** Kinetics

4. Study of geometry of motion with reference to the cause of motion is known as ________.

   (a) Statics **(b)** Kinematics **(c)** Kinetics **(d)** Strength of material

5. The relation $F = ma$ is based on ________.

   (a) Newton's 1st law **(b)** Newton's 2nd law **(c)** Newton's 3rd law **(d)** D'Alembert principle

6. Any object whose dimension if not involved in analysis then it is assumed as ________.

   (a) rigid body **(b)** deformable body **(c)** plastic body **(d)** particle

7. Principle of transmisibility of force if used ________ effect remains unchanged.

   (a) internal **(b)** external **(c)** internal and external **(d)** None of these

8. The process of splitting the force into components having the same effect is called as ______.

(a) Composition of force **(b)** Vector addition of force

(c) Resolution of force **(d)** Decomposition of force

9. A turning effect produced by force about a point is called ________.

(a) Moment of couple **(b)** Moment of torque **(c)** Moment of force **(d)** Moment of inertia

10. Couple is a ________ vector.

(a) sliding **(b)** fixed **(c)** null **(d)** free

11. A single force which replaces the given force system having same effect is called _________.

(a) equilibrant **(b)** resultant **(c)** resistant **(d)** equivalent

12. Concurrent force system means _________.

(a) many forces passing through single point

(b) many forces passing through many points

(c) single force passing through single point

(d) None of the above

13. Force is completely defined by its _______.

(a) magnitude **(b)** direction **(c)** point of application **(d)** All of these

14. In concurrent force system if resultant is horizontal then ________.

(a) $\Sigma F_x = 0$ **(b)** $\Sigma F_y = 0$ **(c)** $\Sigma M = 0$ **(d)** $R = 0$

15. For finding resultant of two concurrent forces, one can use ________.

(a) Lami's theorem **(b)** Varignon's theorem **(c)** parallelogram law **(d)** Newton's law

16. "The moment of resultant of all the forces in a plane about any point is equal to the algebraic sum of moment of all the forces about the same point" is called the ________.

(a) Parallelogram law **(b)** Triangle law **(c)** Lami's theorem **(d)** Varignon's theorem

17. Position of resultant force in parallel force system is possible to determine by ________.

(a) Varignon's theorem **(b)** Lami's theorem **(c)** Newton's law **(d)** Polygon law

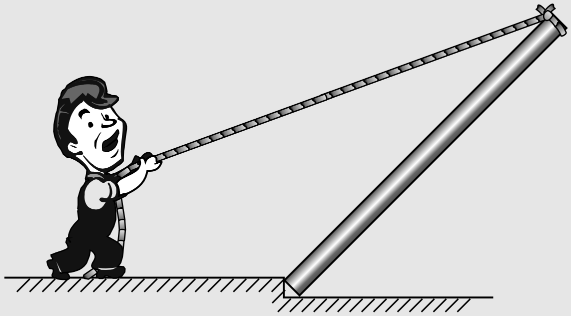
18. If the resultant of parallel force system is both a single resultant force (R) and a single resultant couple (M) then the system is said to perform ________.

(a) translation motion **(b)** rotational motion **(c)** plane motion **(d)** None of these

19. For finding resultant of general force system we use ________ equation/s.

(a) one **(b)** two **(c)** three **(d)** four

3



EQUILIBRIUM OF SYSTEM OF COPLANAR FORCES

3.1 Introduction

In the previous chapter, we have discussed the different types of force system (i.e., *concurrent*, *parallel* and *general*) which were easily identified and their resultants were calculated. In this chapter, we shall discuss the *equilibrium analysis of engineering problem* for which we must originate the force system.

Here we shall introduce the *free body diagram* which is perhaps the most important physical concept in this text. This is always the initial step in solving a problem and often the most critical step. The matter of this chapter is very important for dealing with the subject. Here we shall discuss many basic concepts, such as *two-force body*, *three-force body*, *Lami's theorem*, *types of supports*, *types of loads*, *analysis of simple body*, *analysis of composite bodies*, *frames,* etc.

3.2 Equilibrant

A *force*, which is *equal, opposite* and *collinear to the resultant of a concurrent force system* is known as the *equilibrant of the concurrent force system*.

Equilibrant is the force which, when applied to a body acted by the concurrent force system, keeps the body in equilibrium.

A *single force* which brings the *system to equilibrium*, thus equilibrant is *equal in magnitude*, *opposite in direction* and *collinear to resultant force*.

The force that cancels the effect of the force system acting on the body is also known as *equilibrant*.

3.3 Free Body Diagram (FBD)

*The **Free Body Diagram (FBD)** is a sketch of the body showing all active and reactive forces that acts on it after removing all supports with consideration of geometrical angles and distance given.*

To investigate the equilibrium of a body, we remove the supports and replace them by the reactions which they exert on the body.

The first step in equilibrium analysis is to identify all the forces that act on the body, which is represented by a free body diagram. Therefore, the free body diagram is the most important step in the solution of problems in mechanics.

Importance of FBD

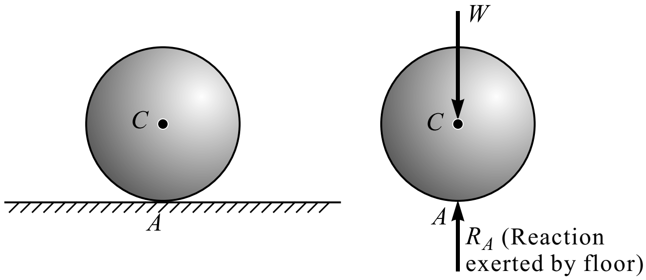
1. The sketch of FBD is the key step that translates a physical problem into a form that can be analysed mathematically.
2. The FBD is the sketch of a body, a portion of a body or two or more connected bodies completely isolated or free from all other bodies, showing the force exerted by all other bodies on the one being considered.
3. FBD represents all active (applied) forces and reactive (reactions) forces. Forces acting on the body that are not provided by the supports are called *active force* (weight of the body and applied forces). *Reactive forces* are those that are exerted on a body by the supports to which it is attached.
4. FBD helps in identifying known and unknown forces acting on a body.
5. FBD helps in identifying which type of force system is acting on the body so by applying appropriate condition of equilibrium, the required unknowns are calculated.

Procedure for Drawing a FBD

1. Draw a neat sketch of the body assuming that all supports are removed.
2. FBD may consist of an entire assembled structure or any combination or part of it.
3. Show all the relevant dimensions and angles on the sketch.
4. Show all the active forces on corresponding point of application and insert their magnitude and direction, if known.
5. Show all the reactive forces due to each support.
6. The FBD should be legible and neatly drawn, and of sufficient size, to show dimensions, since this may be needed in computation of moments of forces.
7. If the sense of reaction is unknown, it should be assumed. The solution will determine the correct sense. A positive result indicates that the assumed sense is correct, whereas a negative result means the assumed sense is incorrect, so the correct sense is opposite to the assumed sense.
8. Use principle of transmissibility wherever convenient.

Example 1

A sphere having weight W is resting on the horizontal floor. It is free to move along the horizontal plane but cannot move vertically downward. A sphere exerts a vertical push against the horizontal surface at the point of contact A. As per Newton's third law, action and reaction are equal and opposite. In FBD, we remove the supporting horizontal plane and replace it by reactive force R_A. Weight W is the active force.

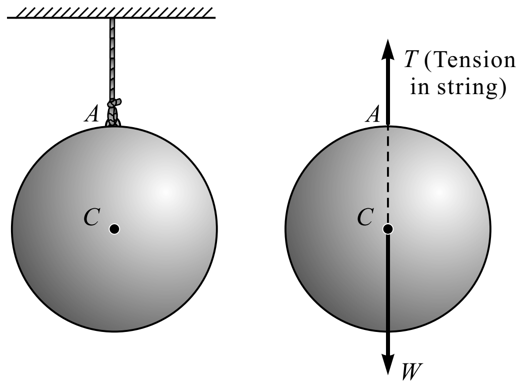


FBD

Note : In this chapter, all contact surface are assumed to be a frictionless. Therefore no FBD is represented with frictional force.

Example 2

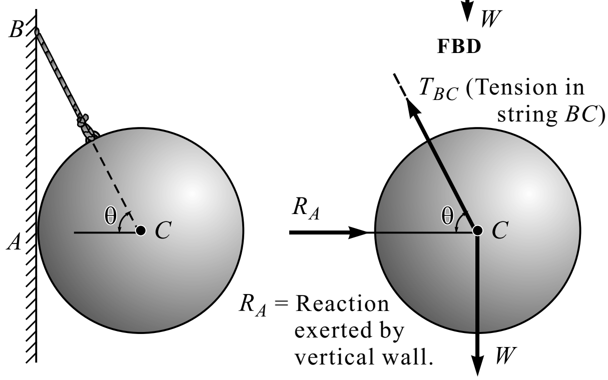
A sphere having weight W is freely suspended by string connected to ceiling. Here sphere can be swing as pendulum but cannot move vertically down as attached by string, thus the sphere exerts a downward pull on the end of the supporting string. In FBD, we remove the string and replace it by reactive force, i.e., the tension T in the string. Weight W is the active force.



FBD

Example 3

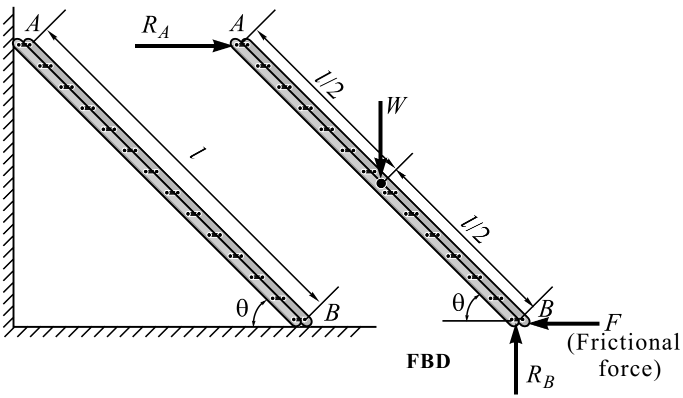
A sphere having weight W is suspended by a string but rest against the vertical wall. Here the sphere is constrained to move downward by string and towards left due to vertical wall. The sphere not only pulls down on string BC but also pushes to the left against the vertical wall at A. In FBD, we remove the string and vertical wall and replace by tension T and reaction R_A. Weight W is the active force.



FBD

Example 4

Ladder having weight W is resting against the rough horizontal floor and smooth vertical wall. R_A is reactive force exerted by vertical wall and R_B is the reactive force exerted by horizontal floor on ladder. F is the reactive frictional force between horizontal floor and ladder, weight W is the active force of ladder.

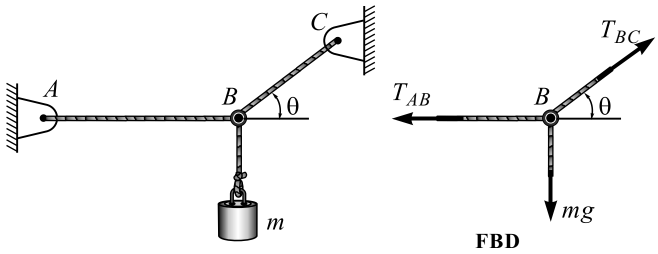


Example 5

A block of mass m kg is suspended by ropes, as shown in the figure.

$T_{AB} \Rightarrow$ Tension in rope AB

$T_{BC} \Rightarrow$ Tension in rope BC

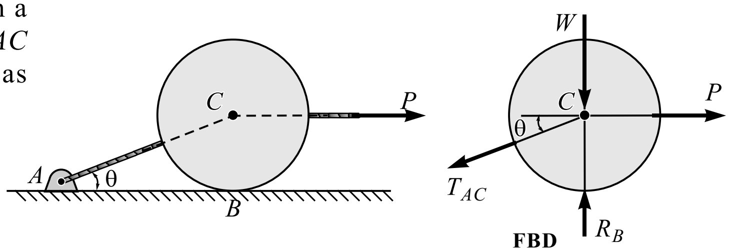


Example 6

A cylinder of weight W supported on a smooth horizontal plane by a cord AC and pulled by applied force P as shown in the figure.

$T_{AC} \Rightarrow$ Tension in cord AC

$R_B \Rightarrow$ Reaction exerted by the floor on cylinder at B

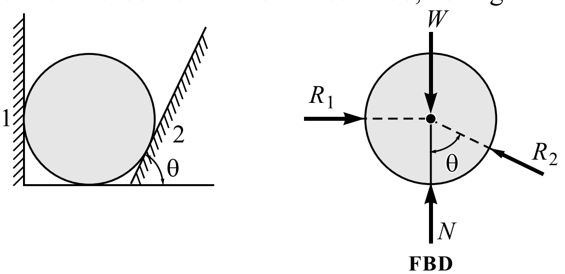


Note : String, rope, cord, cable, wire, thread, chain always experiences tension which is shown by drawing an arrow away from joint or body in FBD.

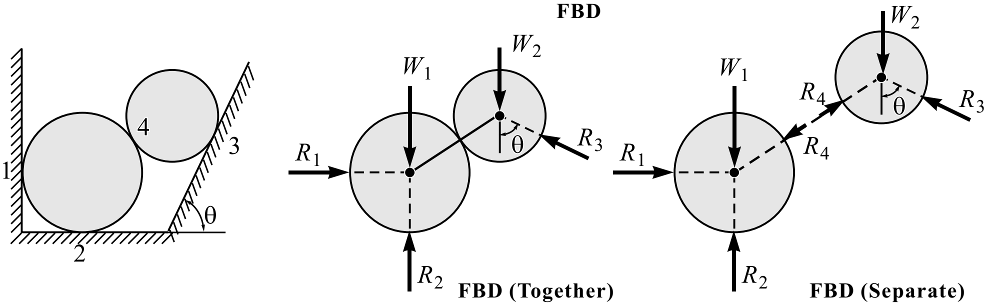
Example 7

Smooth Surface Contact : When a body is in contact with a smooth (frictionless) surface at only one point, the reaction is a force normal to the surface, acting at the point of contact.

(i)



(ii)



3.4 Types of Supports

While drawing FBD the most important step to master is the determination of the support reactions. The structure in the field may be various types such as beam, truss, frame, levers ladder, etc. They are supported with specific arrangements.

Generally, the support offers reactions. Different types of supports and their reactions are classified as follows.

1. **Roller Support :** A roller support is equivalent to a frictionless surface. It can only exert a force that is perpendicular to the supporting surface. The magnitude of the force is then the only unknown force introduced in a FBD when the support is removed. The roller support is free to roll along the surface on which it rests. Since the linear displacement in normal direction to surface of roller is restricted, it offers a reaction in normal direction to surface of roller. For example, a sliding door slides smoothly with the help of a roller support, whereas a conveyer belt can move smoothly on a roller support.

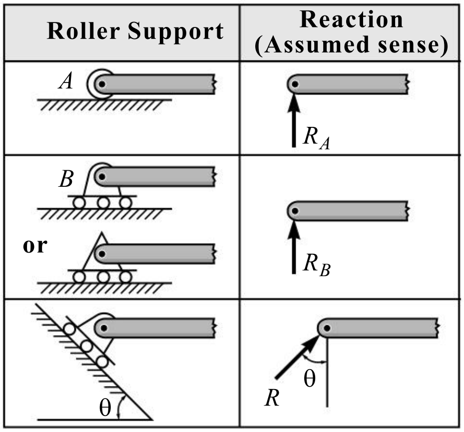


Fig. 3.4-(1)

2. **Hinge (Pin) Support :** The hinge support allows free rotation about the pin end but it does not allows linear displacement of that end. Since linear displacements are restricted in horizontal and vertical directions, the reaction offered at hinge support (say R_A at θ) is resolved into two components, i.e., H_A and V_A. The direction of these two components are uncertain. Therefore, they are initially assumed in FBD

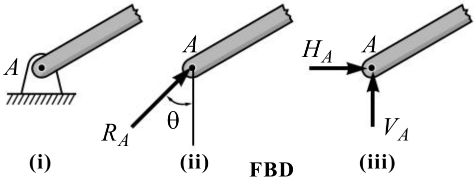


Fig. 3.4(2)

A pin is a cylinder that is slightly smaller than the hole into which it is inserted [refer to Fig. 3.4(3)-i]. Neglecting friction, the pin can only exert a force that is normal to the contact surface (say at point A) shown as R_A [refer to Fig. 3.4(3)-ii]. A pin support thus introduces two unknown, the magnitude of R_A and the angle θ that specifies the direction of R_A. This reaction R_A at θ can be resolved into two components, i.e., horizontal components (H_A) and vertical component (V_A). One should identify the figures given in Fig. 3.4(3)-iv as pin support.

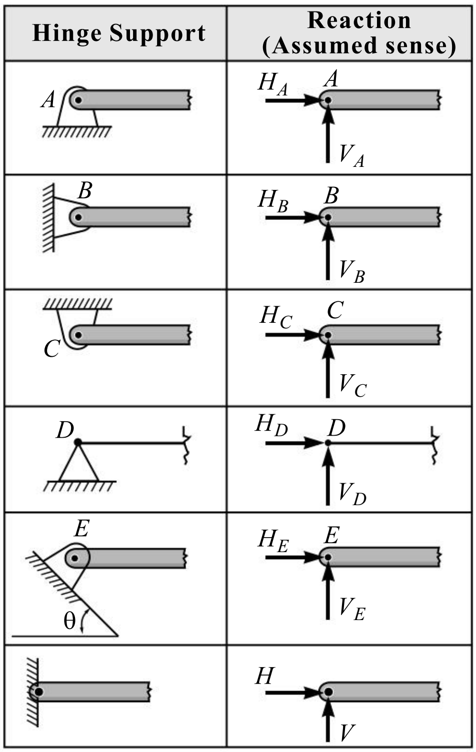
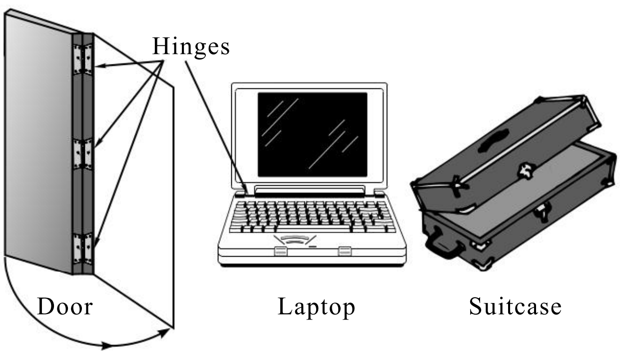


Fig. 3.4(3)-iv

For example, opening and closing of a door is possible by hinges. Same principle applies in the opening and closing of a suitcase or a laptop too.



3. **Fixed (Built in) Support :** When the end of a beam is fixed (built in) then that support is said to be fixed support. Fixed support neither allows linear displacement nor rotation of a beam. Due to these restrictions, the components reaction offered at fixed supports are horizontal component H_A, vertical component V_A and couple component M_A. These components are shown in assumed direction. Refer to Fig. 3.4(3).

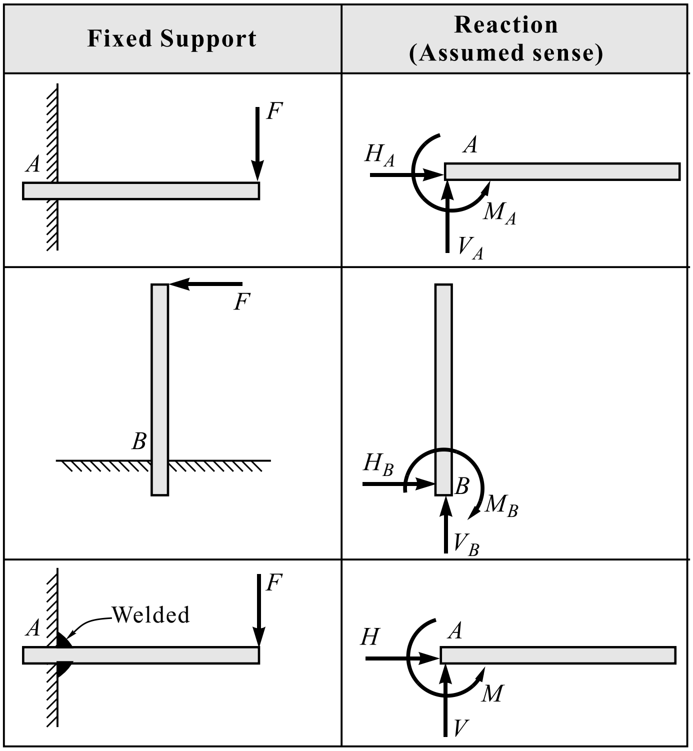


Fig. 3.4(3)

4. **Freely Sliding Guide :** Collar or slider free to move along smooth guides can support force normal to guide only.

   For example, a slider is free to move along a horizontal slot, whereas a collar is free to move along a vertical rod (guide).

   Refer to Fig. 3.4(4).

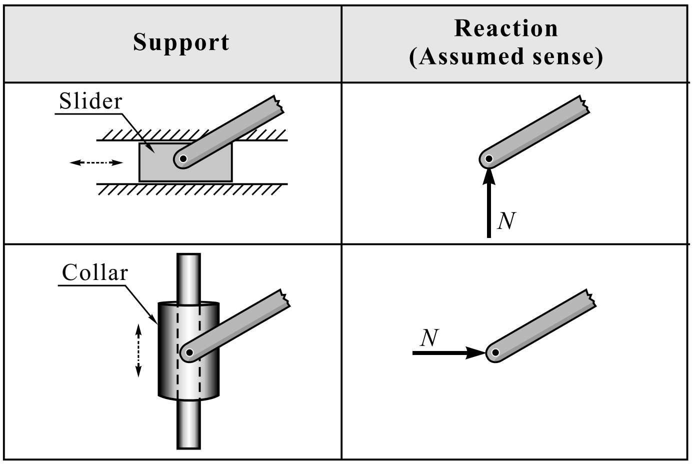


Fig. 3.4(4)

5. **Gravitational Attraction :** The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts towards the Earth through the centre of mass G.

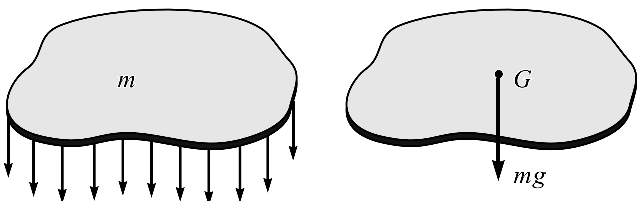


Fig. 3.4(5)

6. **Spring Force :** Spring force is given by the relation $F = kx$ where k is the spring constant and x is the deformation of the spring. Deformation may be due to tension if spring is stretched and compression, if compressed.

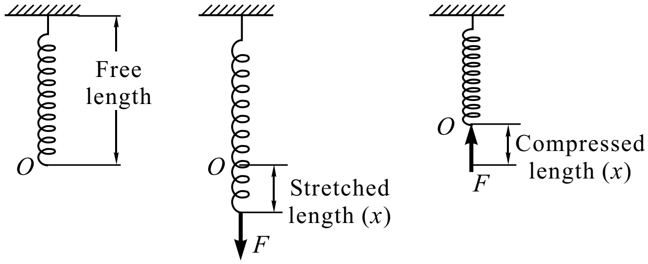


Fig. 3.4(6)

7. **Inextensible String, Cable, Belt Rope, Cord, Chain or Wire :** The force developed in rope is always a tension away from the body in the direction of rope. When one end of a rope is connected to a body, then the rope is not to be considered as a part of the system and it is replaced by tension in FBD as shown in Fig. 3.4(7).

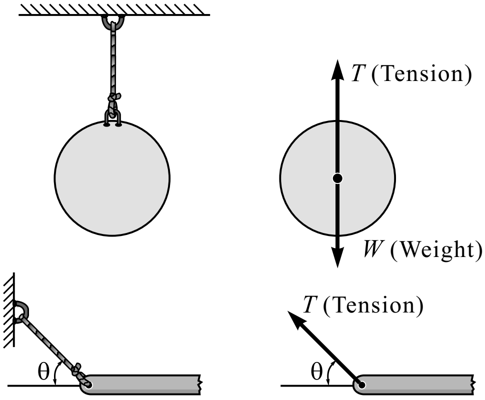


Fig. 3.4(7)

8. **Rope and Frictionless Pulley Arrangement :** When a rope is passing over a frictionless pulley, then the tension on both sides of the rope is same as shown in Fig. 3.4(8)-i, ii and iii.

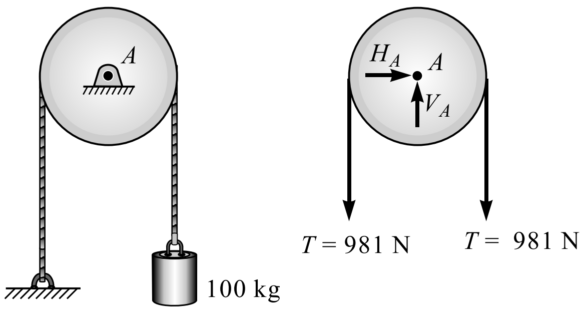


Fig. 3.4(8)-i

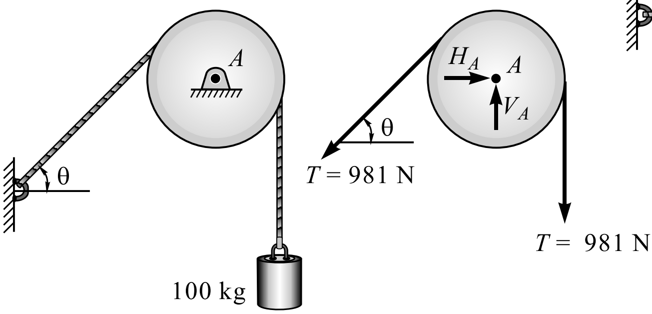


Fig. 3.4(8)-ii

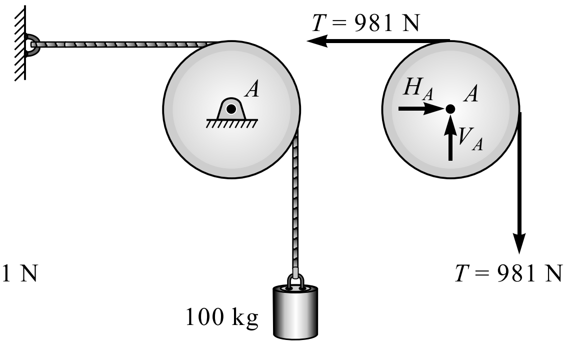


Fig. 3.4(8)-iii

9. **Transfer of Tension of Rope on Frictionless Pulley from Circumference to the Centre of Pulley :** For frictionless pulley, tension on both sides of rope is equal. Therefore, in FBD of pulley, the tension on two sides can be shown at the centre of pulley.

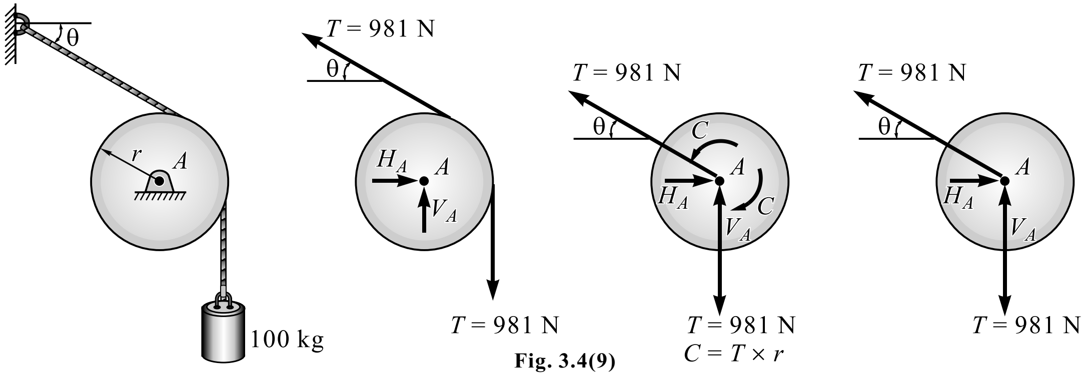


Fig. 3.4(9)

Reason **:** We know that the force can be transferred from one point to the other on the same rigid body by moving at the new point with parallel line of action and adding a couple whose magnitude is equal to the moment of force about new transferred point, here it is centre A. Since both the forces (tension of rope) are tangential to the circle, they are at perpendicular distance, equal to radius of pulley. So the two forces of the same magnitude when transferred to centre A, they also carry the couple of equal magnitude $C = T \times r$ but opposite in sense. Hence they cancel each other.

10. **Straight Rod Supported by Knife Edge (Fulcrum) :**
A straight rod having weight W rests against a horizontal smooth surface and knife edge support at B. It is an exceptional case where the reaction exerted by knife edge is normal to straight rod.
In Fig. 3.4(10), A is the smooth surface and we know reaction R_A will be perpendicular to the smooth surface. Here our emphasis is on the knife edge support which is at B and it exerts the normal reaction R_B perpendicular to the straight rod.

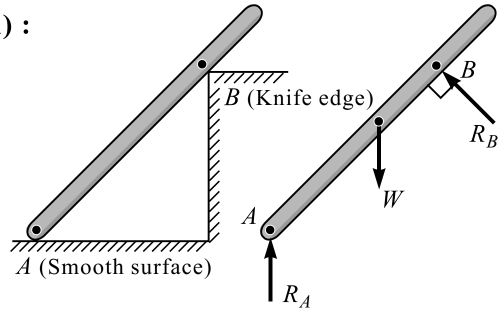


Fig. 3.4(10)

11. **Rigid Body Supported by Knife Edge :** A cylinder having weight W resting against a rectangular block is pulled by force P which is just enough to roll the cylinder over the rectangular block. Since the cylinder is just about to roll over the rectangular block, the reaction at contact B will become zero. The cylinder is subjected to two active forces, i.e., self-weight W and applied force P and one reactive force R due to knife edge support by rectangular block. We know by 'Three-force principle', three non-parallel forces must form concurrent force system for equilibrium condition. Therefore W, P and R must pass through the point of concurrence, as shown in Fig. 3.4(11).

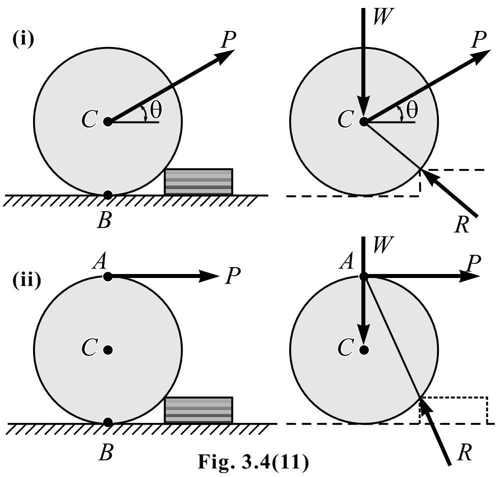


Fig. 3.4(11)

3.5 Lami's Theorem

If three concurrent coplanar forces acting on a body having same nature (i.e., pulling or pushing) are in equilibrium, then each force is proportional to the sine of angle included between the other two forces.

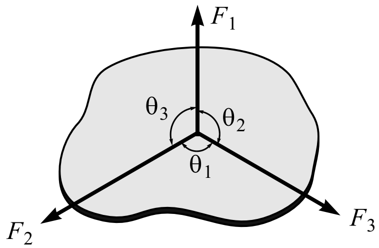


Fig. 3.5-i

By Lami's theorem, we have

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}$$

Proof

By three-force triangle theorem, we get a closed triangle. Refer to Fig. 3.5-ii.

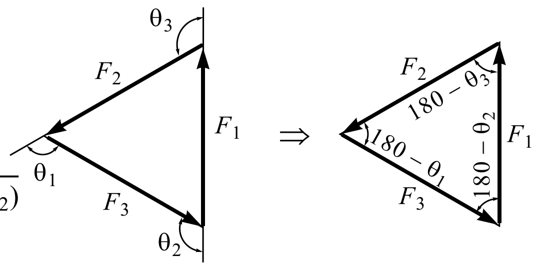


Fig. 3.5-ii

By sine rule, we have

$$\frac{F_1}{\sin(180^\circ - \theta_1)} = \frac{F_2}{\sin(180^\circ - \theta_2)} = \frac{F_2}{\sin(180^\circ - \theta_2)}$$

$$\therefore \frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}$$

Hence proved.

Limitations of Lami's Theorem

1. It is applicable to three non-parallel coplanar concurrent forces only.
2. Nature of three forces must be same (i.e., pulling or pushing)

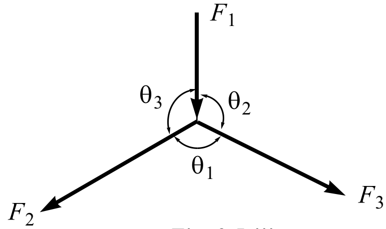


Fig. 3.5-iii

In Fig. 3.5-iii, F_2 and F_3 are pulling forces while F_1 is a pushing force. By principle of transmissibility, one can transmit F_1 on the other side of point of concurrency to make three forces of the same nature (i.e., pulling) and can apply Lami's theorem with due consideration of geometrical change in angles.

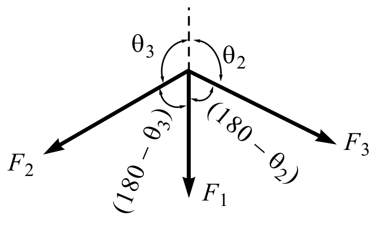


Fig. 3.5-iv

Easy Alternative Vector Approach

Since force is a vector quantity, negative sign (if placed), indicates opposite sense. So, one can prefer to change the force F_1 by introducing negative sign and then applying Lami's theorem as shown in Fig. 3.5-v.

$$\frac{-F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}$$

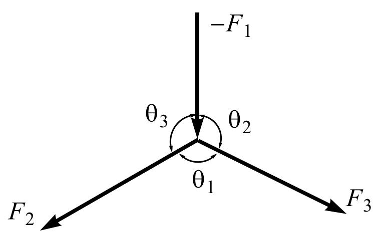


Fig. 3.5-v

3.6 Solved Problems

Problem 1

Find the tension in each rope in Fig. 3.1(a).

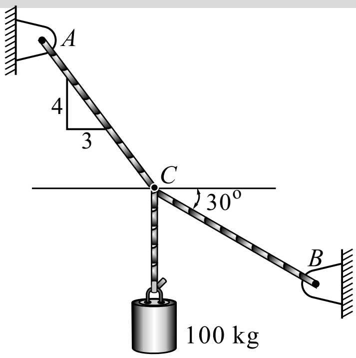


Fig. 3.1(a)

Solution

(i) Consider the FBD of Point C.

(ii) By Lami's theorem,

$$\frac{981}{\sin 156.87^\circ} = \frac{T_{AC}}{\sin 60^\circ} = \frac{T_{BC}}{\sin 143.13^\circ}$$

$$T_{AC} = 2162.76 \text{ N}$$

$$T_{BC} = 1498.41 \text{ N}$$ ***Ans.***

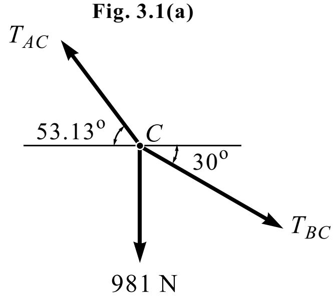


Fig. 3.1(b) : FBD

Problem 2

Block P 5 kg and block Q of mass m kg are suspended through the chord which is in the equilibrium position, as shown in Fig. 3.2(a). Determine the mass of block Q.

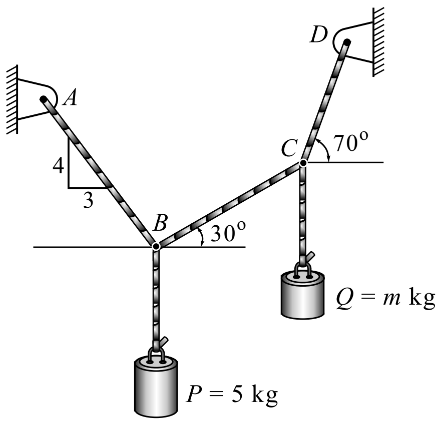


Fig. 3.2(a)

Solution

(i) Consider the FBD of Point B.

(ii) By Lami's theorem,

$$\frac{5 \times 9.81}{\sin 96.87^\circ} = \frac{T_{AB}}{\sin 120^\circ} = \frac{T_{BC}}{\sin 143.13^\circ}$$

$$\therefore T_{AB} = 42.79 \text{ N}$$

$$T_{BC} = 29.64 \text{ N}$$

(iii) Consider the FBD of Point C.

(iv) By Lami's theorem,

$$\frac{m \times 9.81}{\sin 140^\circ} = \frac{29.64}{\sin 160^\circ}$$

$$\therefore m = 5.678 \text{ kg}$$ ***Ans.***

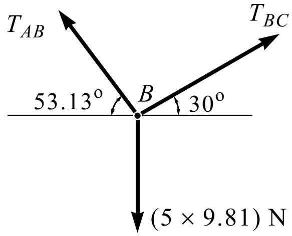


Fig. 3.2(b) : FBD of *B*

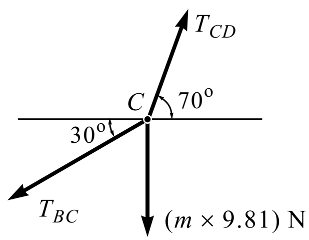


Fig. 3.2(c) : FBD of *C*

Problem 3

Find force transmitted by cable BC as shown in Fig. 3.3(a); E is a frictionless pulley, while B and D are weightless rings.

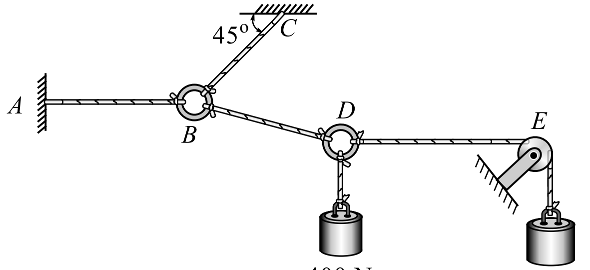


Fig. 3.3(a)

Solution

Let T_{BA} be the tension in the string BA

and T_{BC} be the tension in the string BC.

(i) Consider the FBD of portion *BD*.

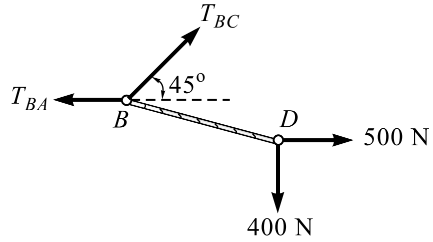


Fig. 3.3(b) : FBD of String BD

From equilibrium condition,

$$\Sigma F_y = 0$$

$$T_{BC} \sin 45^\circ = 400$$

$$\therefore T_{BC} = 565.69 \text{ N } (\measuredangle 45^\circ) \quad \textit{Ans.}$$

Problem 4

A circular roller of weight 1000 N and radius 20 cm hangs by a tie rod AB = 40 cm and rests against a smooth vertical wall at C as shown in Fig. 3.4(a). Determine the tension in the rod and reaction at point C.

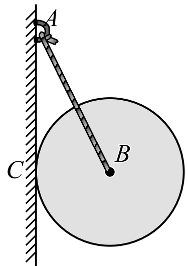


Fig. 3.4(a)

Solution

(i) Draw the FBD of the roller

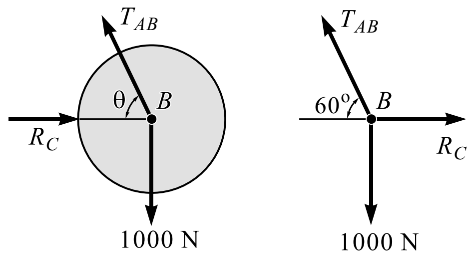


Fig. 3.4(b)

$$\cos \theta = \frac{20}{40}$$

$$\therefore \theta = 60^\circ$$

(ii) By Lami's theorem, we have

$$\frac{1000}{\sin 120^\circ} = \frac{T_{AB}}{\sin 90^\circ} = \frac{R_C}{\sin 150^\circ}$$

$$\therefore T_{AB} = 1154.7 \text{ N } (60^\circ \nwarrow) \quad \textit{Ans.}$$

$$\therefore R_C = 577.35 \text{ N } (\rightarrow) \quad \textit{Ans.}$$

Problem 5

A roller of weight W = 1000 N rests on a smooth inclined plane. It is kept from rolling down the plane by string AC as shown in Fig. 3.5(a). Find the tension in the string and reaction at the point of contact D.

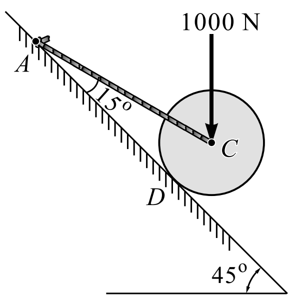


Fig. 3.5(a)

Solution

(i) Draw the FBD of the roller.

(ii) By Lami's theorem,

$$\frac{1000}{\sin 75^o} = \frac{R_D}{\sin 60^o} = \frac{-T_{AC}}{\sin 225^o}$$

$\therefore R_D = 896.58$ N $\left(\angle 45^o\right)$ ***Ans.***

$\therefore T_{AC} = 732$ N ***Ans.***

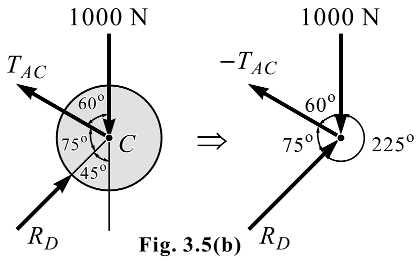


Fig. 3.5(b)

Problem 6

A cylinder of mass 50 kg is resting on a smooth surface, which is inclined at 30^o and 60^o to horizontal as shown in Fig. 3.6(a). Determine the reaction at contact A and B.

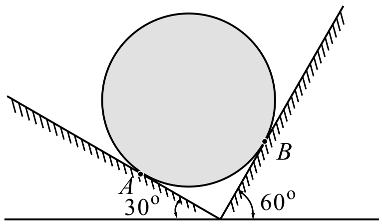


Fig. 3.6(a)

Solution

(i) Consider the FBD of the cylinder.

(ii) By Lami's theorem, we have

$$\frac{50 \times 9.81}{\sin 90^o} = \frac{R_A}{\sin 120^o} = \frac{R_B}{\sin 150^o}$$

$R_A = 424.79$ N

$R_B = 245.25$ N ***Ans.***

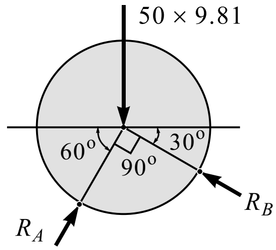


Fig. 3.6(b) : FBD of Cylinder

Problem 7

A 30 kg collar may slide on frictionless vertical rod and is connected to a 34 kg counter weight. Find the value of h for which the system is in equilibrium.

Refer to Fig. 3.7(a) for details.

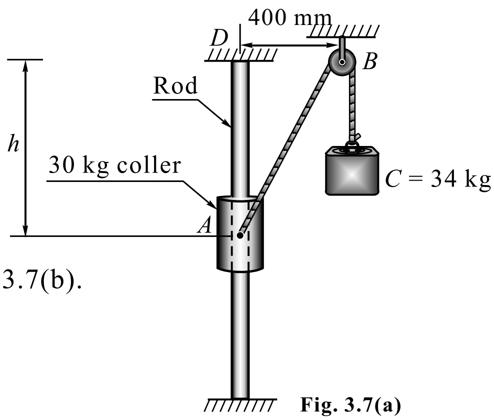


Fig. 3.7(a)

Solution

(i) Consider the FBD of the collar shown in Fig. 3.7(b).

(ii) By Lami's theorem,

$$\frac{T}{\sin 90^o} = \frac{30 \times 9.81}{\sin (180^o - \theta)}$$

$$\sin \theta = \frac{30 \times 9.81}{34 \times 9.81} \times \sin 90^o$$

$\therefore \theta = 61.93^o$

(iii) Refer to Fig. 3.7(c).

$$\tan 61.93^o = \frac{h}{400}$$

$h = 750$ mm ***Ans.***

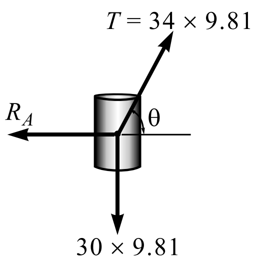


Fig. 3.7(b) : FBD of Collar

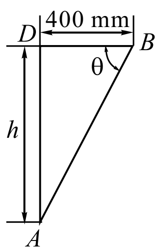


Fig. 3.7(c)

Problem 8

Determine the force P applied at 30^o to the horizontal just necessary to start a roller having radius 50 cm over an obstruction 12 cm high, if the roller is of mass 100 kg, shown in Fig. 3.8(a). Also find the magnitude and direction of P when it is minimum.

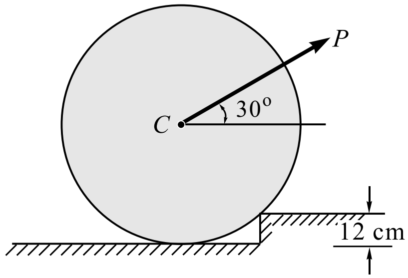


Fig. 3.8(a)

Solution

(i) Consider the FBD of the roller

As per the given condition, P is just sufficient to start the roller. At this instant, the roller will not have any pressure on the horizontal surface. Therefore, the surface will not offer any reaction. We can identify that this body is subjected to three forces, viz., 100×9.81, P and R. Since $100 \times 9.81$and P are passing through C therefore the third force, i.e., the reaction R must also pass through same point C, as per three-force principle.

Let us draw concurrent forces for simplicity, refer to Fig. 3.8(b).

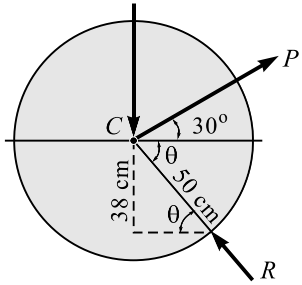


Fig. 3.8(b) : FBD of Roller

$$\sin\theta = \frac{38}{50} \quad \therefore \theta = 49.46^o$$

(ii) Before applying Lami's theorem, we should overcome its limitation, i.e., nature of three forces must be the same. Here P is directed away from point C. Since we know that force is a vector quantity, by placing negative sign we can satisfy the Lami's theorem requirement. Now, nature of three forces is pushing towards point C.

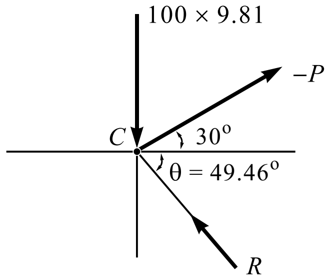


Fig. 3.8(c)

(iii) By Lami's theorem,

$$\frac{981}{\sin 79.46^o} = \frac{-P}{\sin 220.54^o}$$

$$P = \frac{-981 \times \sin 220.54^o}{\sin 79.46^o}$$

$$\therefore P = 648.57 \text{ N} \quad \textbf{\textit{Ans.}}$$

(iv) Method I : To find P_{min}

By Lami's theorem,

$$\frac{-P_{min}}{\sin 220.54^o} = \frac{981}{\sin(\alpha + 49.46^o)}$$

$$P_{min} = \frac{-981 \times \sin 220.54^o}{\sin(\alpha + 49.46^o)} \quad \dots \text{(I)}$$

For P_{min} the denominator should be maximum, i.e.,

$$\sin(\alpha + 49.46^o) = 1$$

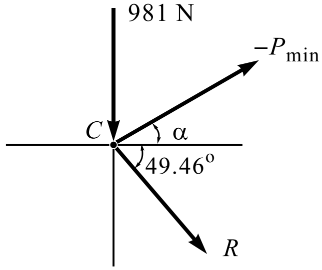


Fig. 3.8(d)

$\therefore\ \alpha + 49.46 = 90 \qquad \therefore\ \alpha = 40.54^\circ$

From Eq. (I)

$$P_{min} = \frac{-981 \times \sin 220.54^\circ}{\sin (40.54^\circ + 49.46^\circ)}$$

$\therefore\ P_{min} = 637.63\ \text{N}\ (\measuredangle \alpha)$ ***Ans.***

(iv) Method II : To find P_{min}

To start the roller over the obstruction, P should balance the anticlockwise moment of W about point A with an equal clockwise moment.

The maximum distance between the point A and line of action of P is AC. Therefore, to create a given moment about A, the force P will be minimum when it acts at right angle to AC as shown in Fig. 3.8(e). Then the P_{min} will make an angle $\alpha = 40.54^\circ$.

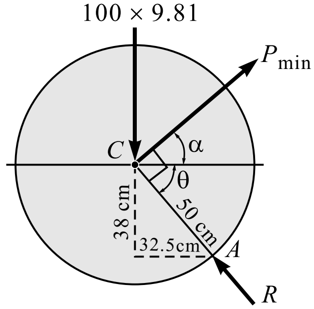


Fig. 3.8(e) : FBD of Roller

$\alpha + \theta = 90^\circ$; $\alpha + 49.49^\circ = 90^\circ \therefore\ \alpha = 40.54^\circ$

$\Sigma M_A = 0$

$100 \times 9.81 \times 32.5 - P_{min} \times 50 = 0$

$\therefore\ P_{min} = 637.63\ \text{N}\ (\measuredangle \alpha)$ ***Ans.***

Problem 9

Two identical rollers each of mass 50 kg are supported by an inclined plane and a vertical wall as shown in Fig. 3.9(a). Assuming smooth surfaces, find the reactions induced at the point of support A, B and C.

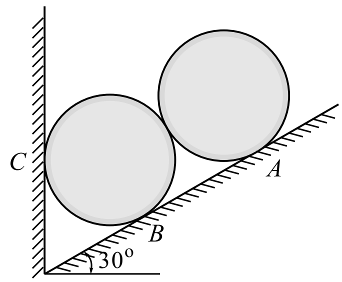


Fig. 3.9(a)

Solution

(i) Consider FBD of both rollers together and let R be the radius of rollers.

(ii) $\Sigma M_O = 0$

$R_A \times 2R - 50 \times 9.81 \cos 30^\circ \times 2R = 0$

$R_A = 424.79\ \text{N}\ (60^\circ \nwarrow)$ ***Ans.***

(iii) $\Sigma F_y = 0$

$R_B \cos 30^\circ + R_A \cos 30^\circ - 50 \times 9.81 - 50 \times 9.81 = 0$

$R_B = 707.97\ \text{N}\ (60^\circ \nwarrow)$ ***Ans.***

(iv) $\Sigma F_x = 0$

$R_C - R_A \sin 30^\circ - R_B \sin 30^\circ = 0$

$R_C = 424.79 \sin 30^\circ + 707.97 \sin 30^\circ$

$R_C = 566.38\ \text{N}\ (\rightarrow)$ ***Ans.***

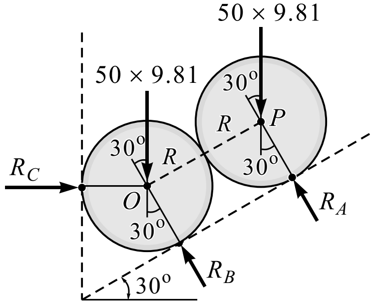


Fig. 3.9(b)

Problem 10

A uniform wheel of 60 cm diameter and weighing 1000 N rest against a rectangular block 15 cm high lying on a horizontal plane as shown in Fig. 3.10(a). It is to be pulled over the block by a horizontal force P applied to the end of a string wound round the circumference of the wheel. Find force P when the wheel is just about to roll over the block.

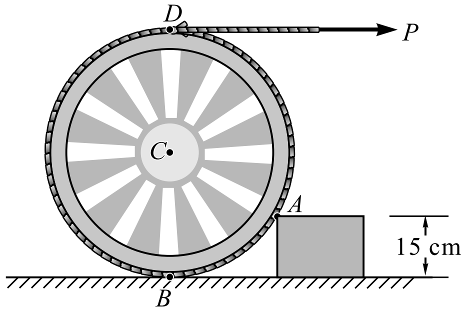


Fig. 3.10(a)

Solution

Since the wheel is just about to roll over the rectangular block, the reaction at contact B will become zero. The wheel is subjected to two active forces, i.e., self-weight 1000 N and horizontal applied force P and one reactive force due to knife edge support by rectangular block. Therefore, by three force principle, all three forces must be concurrent and should pass through point D.

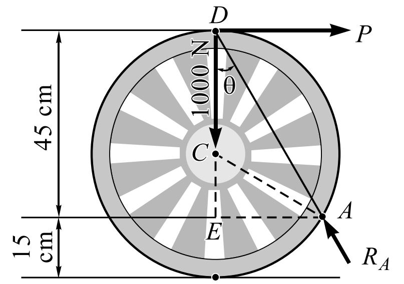


Fig. 3.10(b)

(i) Draw the FBD of wheel corresponding to above discussion. Refer to Fig. 3.10(b).

In $\Delta\, ACE$

$$AE = \sqrt{AC^2 - CE^2} = \sqrt{30^2 - 15^2}$$

$$AE = 25.98 \text{ cm}$$

$$\therefore \tan\theta = \frac{AE}{DE} = \frac{25.98}{45} \quad \therefore \theta = 30^\circ$$

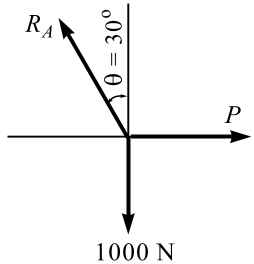


Fig. 3.10(c)

(ii) By Lami's theorem, we have

$$\frac{1000}{\sin 120^\circ} = \frac{P}{\sin 150^\circ}$$

$$\therefore P = 577.35 \text{ N}$$ ***Ans.***

Problem 11

Two cylinders each of diameter 100 mm and each weighing 200 N are placed as shown in Fig. 3.11(a). Assuming that all the contact surfaces are smooth, find the reactions at A, B and C.

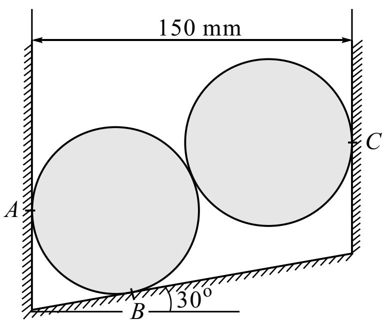


Fig. 3.11(a)

Solution

Note : Assuming the base line inclined at 30° to horizontal.

(i) Consider the FBD of both the rollers together as shown in Fig. 3.11(b).

(ii) From the FBD of lower cylinder.

$\Sigma F_y = 0$

$R_B \cos 30^\circ - 200 - R \sin 60^\circ = 0$

$R_B = 461.89$ N (60°) ***Ans.***

$\Sigma F_x = 0$

$R_A - R\cos 60^\circ - R_B \sin 30^\circ = 0$

$R_A = 346.42$ N $(\rightarrow)$ ***Ans.***

(iii) From the FBD of upper cylinder.

By Lami's theorem,

$$\frac{200}{\sin 120^\circ} = \frac{R_C}{\sin 150^\circ} = \frac{R}{\sin 90^\circ}$$

$R_C = 115.47$ N $(\leftarrow)$

$R = 230.94$ N (60°) ***Ans.***

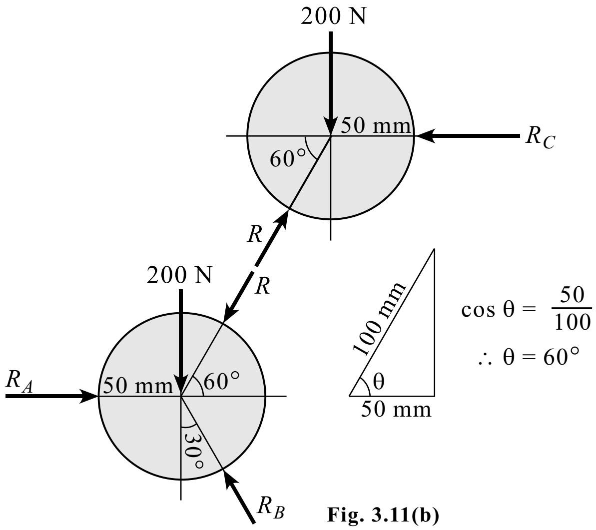


Fig. 3.11(b)

Problem 12

Two spheres, A and B, are resting in a smooth through as shown in Fig. 3.12(a). Draw the free body diagrams of A and B showing all the forces acting on them, both in magnitude and direction. Radius of spheres A and B are 250 mm and 200 mm, respectively.

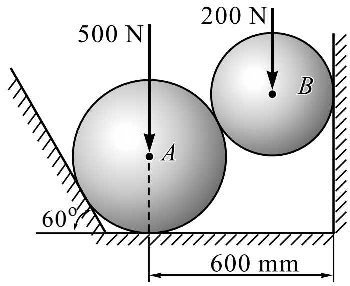


Fig. 3.12(a)

Solution

(i) From Fig. 3.12(b). $AB = 450$ mm and $AC = 400$ mm

$$\cos\theta = \frac{AC}{AB} = \frac{400}{450} \quad \therefore \theta = 27.27^\circ$$

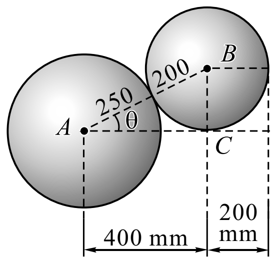


Fig. 3.12(b)

(ii) Consider the FBD of Sphere *B* [Fig. 3.12(c)]

By Lami's theorem,

$$\frac{200}{\sin 152.73^\circ} = \frac{R_1}{\sin 117.27^\circ} = \frac{R_2}{\sin 90^\circ}$$

$\therefore R_1 = 388$ N $(\leftarrow)$ and

$R_2 = 436.51$ N (27.27°) ***Ans.***

(iii) Consider the FBD of Sphere *A* [Fig. 3.12(d)]

$\Sigma F_x = 0$

$R_4 \cos 30^\circ - 436.51 \cos 27.27^\circ = 0$

$R_4 = 448$ N (30°) ***Ans.***

$\Sigma F_y = 0$

$-500 + R_3 - 436.51 \sin 27.27^\circ + 448 \sin 30^\circ = 0$

$R_3 = 476$ N $(\uparrow)$ ***Ans.***

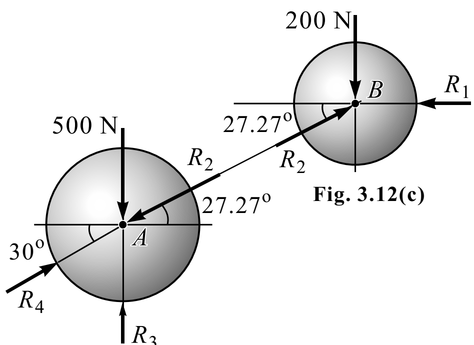


Fig. 3.12(c)

Fig. 3.12(d)

Problem 13

Two spheres A and B of weight 1000 N and 750 N, respectively are kept as shown in the Fig. 3.13(a). Determine the reactions at all contact points 1, 2, 3 and 4. Radius of A = 400 mm and Radius of B = 300 mm.

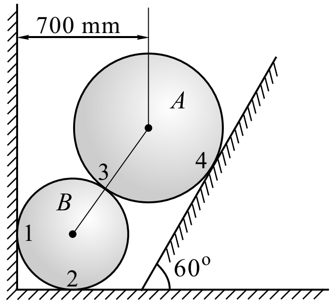


Fig. 3.13(a)

Solution

(i) Consider the FBD of Sphere A [Fig. 3.13(b)]

By Lami's theorem, we have

$$\frac{1000}{\sin(180-30-55.15)^{\circ}} = \frac{R_3}{\sin(90+30)^{\circ}} = \frac{R_4}{\sin(90+55.15)^{\circ}}$$

$$R_3 = 869.14 \text{ N } (55.15^{\circ})$$

$$R_4 = 573.48 \text{ N } (30^{\circ}) \quad \textit{Ans.}$$

(ii) Consider the FBD of Sphere B

$$\Sigma F_x = 0$$

$$R_1 - R_3 \cos 55.15^{\circ} = 0$$

$$R_1 = 496.65 \text{ N } (\rightarrow) \quad \textit{Ans.}$$

$$\Sigma F_y = 0$$

$$R_2 - 750 - R_3 \sin 55.15^{\circ} = 0$$

$$R_2 = 1463.26 \text{ N } (\uparrow) \quad \textit{Ans.}$$

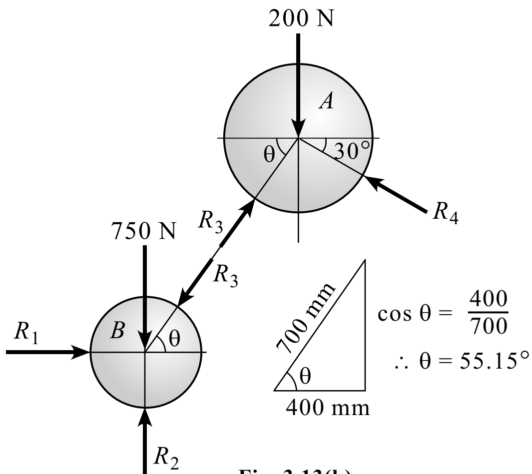


Fig. 3.13(b)

Problem 14

A right-circular cylinder of diameter 40 cm, open at both ends, rests on a smooth horizontal plane. Inside the cylinder, there are two sphere having weights and radii as given [Fig. 3.14(a)]. Find the minimum weight of the cylinder for which it will not tip over.

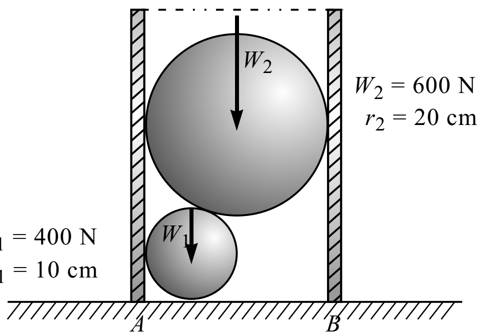


Fig. 3.14(a)

Solution

(i) Consider the FBD of both the spheres together

Let P and Q be the centre points of the spheres.

$$PQ = 30 \text{ cm}$$

$$30^2 = 10^2 + h^2$$

$$h = 28.28 \text{ cm}$$

$$\Sigma M_P = 0$$

$$R_2 \times h - W_2 \times 10 = 0$$

$$R_2 = \frac{600 \times 10}{28.28}$$

$$\therefore R_2 = 212.16 \text{ N } (\leftarrow) \quad \textit{\textbf{Ans.}}$$

$$\Sigma F_x = 0$$

$$R_1 - R_2 = 0$$

$$\therefore R_1 = 212.16 \text{ N } (\rightarrow) \quad \textit{\textbf{Ans.}}$$

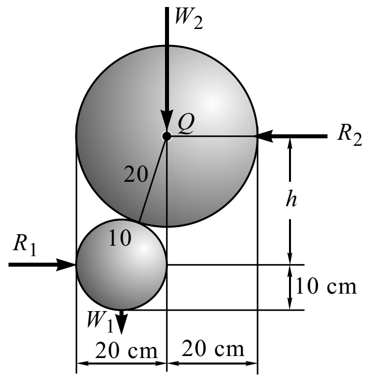


Fig. 3.14(b) : FBD of Both the Spheres Together

(ii) Consider the FBD of the cylinder

If the weight of the cylinder is negligible, the cylinder will tip about point B.
So, to avoid tipping $W_{minimum}$ is required.

$$\Sigma M_B = 0$$

$$W \times 20 + 212.16 \times 10 - 212.16 \times 38.28 = 0$$

$$W = 300 \text{ N } (\downarrow) \quad \textit{\textbf{Ans.}}$$

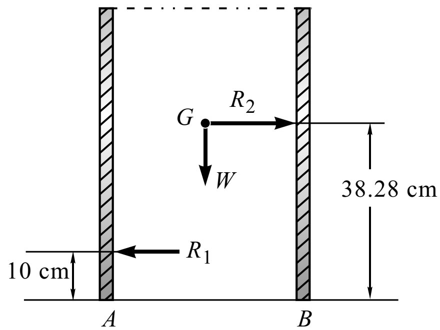
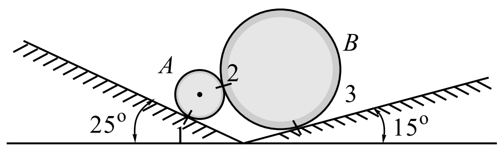


Fig. 3.14(c) : FBD of Cylinder

Problem 15

Determine the reactions at points of contact 1, 2 and 3. Assume smooth surfaces.



$W_A = 1$ kg $r_A = 1$ cm,

$W_B = 4$ kg $r_B = 4$ cm.

Fig. 3.15(a)

Solution

(i) Consider the FBD of both the cylinders together

$$\Sigma F_x = 0$$

$$R_1 \cos 65^\circ - R_3 \cos 75^\circ = 0$$

$$\therefore R_1 = 0.6124\, R_3$$

$$\Sigma F_y = 0$$

$$R_1 \sin 65^\circ + R_3 \sin 75^\circ + 1 \times 9.81 + 4 \times 9.81 = 0$$

$$0.6124\, R_3 \sin 65^\circ + R_3 \sin 75^\circ + 5 \times 9.81 = 0$$

$$R_3 = 32.216 \text{ N } \left(\angle 75^\circ\right) \text{ and } R_1 = 19.729 \text{ N } \left(\angle 65^\circ\right) \quad \textit{\textbf{Ans.}}$$

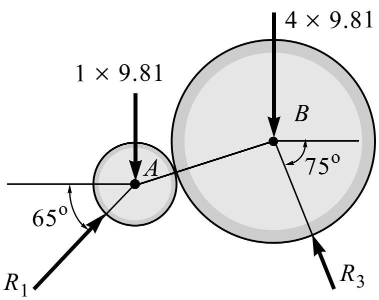


Fig. 3.15(b) : FBD of *A* and *B* Together as a Single Body

(ii) Consider the FBD of 1 kg cylinder *A*

$$\Sigma F_y = 0$$

$$19.729 \sin 65^\circ - 1 \times 9.81 - R_2 \sin \alpha = 0$$

$$R_2 \sin \alpha = 8.071 \qquad \text{......(I)}$$

$$\Sigma F_x = 0$$

$$19.729 \cos 65^\circ - R_2 \cos \alpha = 0$$

$$R_2 \cos \alpha = 8.338 \qquad \text{......(II)}$$

Dividing Eq.(I) by Eq. (II), we get

$$\alpha = 44.07^\circ$$

From Eq. (I), we get

$R_2 = 11.604$ N $(\measuredangle 44.07^\circ)$ ***Ans.***

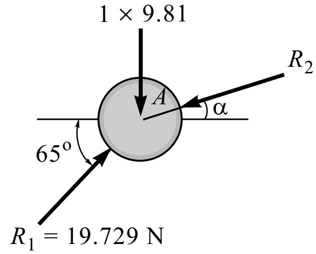


Fig. 3.15(c) : FBD of *A*

Problem 16

Three identical tubes of weights 8 kN each are placed as shown in Fig. 3.16(a). Determine the forces exerted by the tubes on the smooth walls and floor.

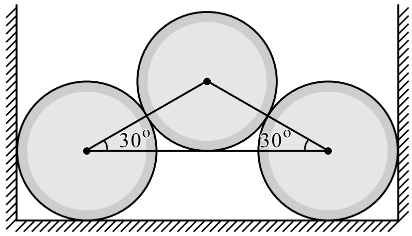


Fig. 3.16(a)

Solution

(i) Consider the FBD of upper tube shown in Fig. 3.16(b). Since tubes are identical and placed symmetrically, reactions R at contact will be same.

(ii) By Lami's theorem

$$\frac{8}{\sin 120^\circ} = \frac{R}{\sin 120^\circ} \quad \therefore R = 8 \text{ kN} \qquad \textit{\textbf{Ans.}}$$

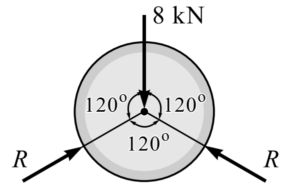


Fig. 3.16(b)

(iii) Consider the FBD of any lower tube (say left) Fig. 3.16(c).

(iv) By Lami's theorem,

$$\frac{8}{\sin 90^\circ} = \frac{R_W}{\sin 120^\circ} = \frac{R_F}{\sin 150^\circ}$$

(v) $R_W = 6.928$ kN (Force exerted by the tubes on the smooth wall) ***Ans.***

$R_F = 4$ kN (Force exerted by the tubes on the smooth floor) ***Ans.***

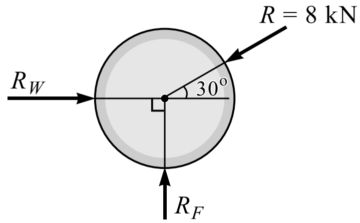


Fig. 3.16(c)

Problem 17

Two smooth circular cylinder is of weight $W = 500$ N each and radius $r = 150$ mm are connected at their centre by a string of length $l = 400$ mm and rest upon a horizontal plane supporting above them a third cylinder of weight 1000 N and radius $r = 150$ mm as shown in Fig. 3.17(a). Find the tension in the string and pressure at the points of contact D and E.

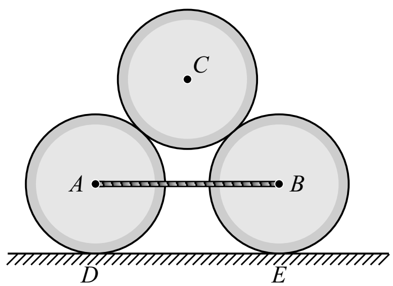


Fig. 3.17(a)

Solution

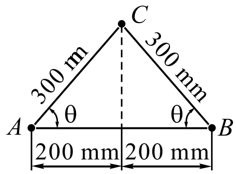


Fig. 3.17(b)

(i) Consider the Δ ABC and simplify its geometric length and angle, as shown in Fig. 3.17(b).

$$\cos\theta = \frac{200}{300} \qquad \therefore\ \theta = 48.19^\circ$$

(ii) Draw the FBD of the upper cylinder C [Fig. 3.20(c)].

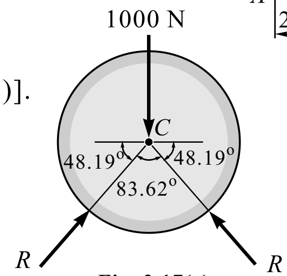


Fig. 3.17(c)

(iii) By Lami's theorem

$$\frac{1000}{\sin 83.62^\circ} = \frac{R}{\sin 138.19^\circ}$$

$\therefore\ R = 670.82$ N ***Ans.***

(iv) Draw the FBD of the lower cylinder A as shown in Fig. 3.17(d).

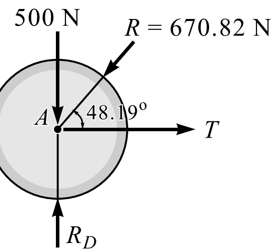


Fig. 3.17(d)

$$\Sigma F_x = 0$$

$$T - 670.82 \cos 48.19^\circ = 0$$

$T = 447.21$ N ***Ans.***

$$\Sigma F_y = 0$$

$$R_D - 500 - 670.82 \sin 48.19^\circ = 0$$

$$R_D = 1000 \text{ N}$$

$R_D = R_E = 1000$ N ***Ans.*** (Since the loading is symmetric, therefore, reaction will be equal.)

Problem 18

Three cylinders are piled up in a rectangular channel, as shown in Fig. 3.18(a). Determine the reactions at point 6 between the cylinder A and the vertical wall of the channel.

(Cylinder A : radius = 4 cm, m = 15 kg,
Cylinder B : radius = 6 cm, m = 40 kg,
Cylinder C : radius = 5 cm, m = 20 kg).

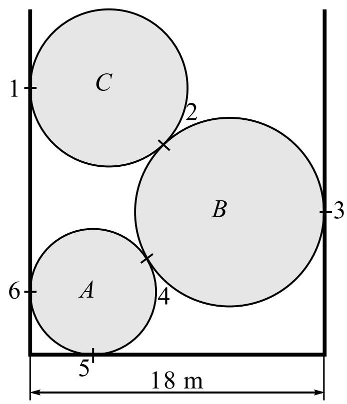


Fig. 3.18(a)

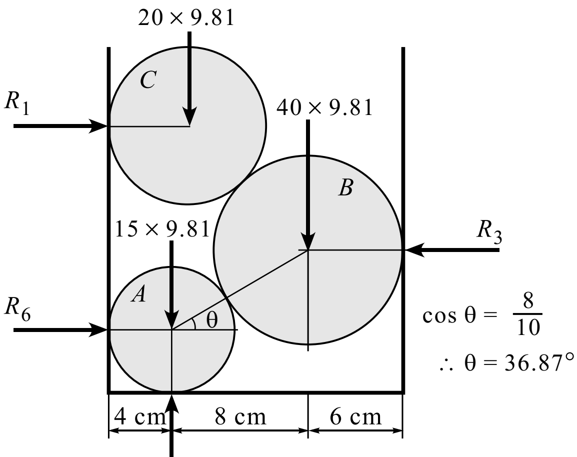


Fig. 3.18(b)

Solution

(i) Consider FBD of entire system, as shown in Fig. 3.18(b).

$\Sigma F_y = 0$

$R_5 - 20 \times 9.81 - 40 \times 9.81 - 15 \times 9.81 = 0$

$R_5 = 735.75$ N ***Ans.***

(ii) Consider the FBD of cylinder *A* [Refer to Fig. 3.18(c)].

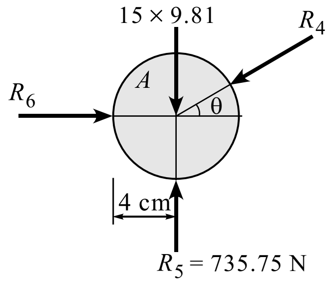


Fig. 3.18(c)

$\Sigma F_y = 0$

$735.75 - 15 \times 9.81 - R_4 \sin 36.87^\circ = 0$

$R_4 = 981$ N ***Ans.***

$\Sigma F_x = 0$

$R_6 - R_4 \cos 36.87^\circ = 0$

$R_6 = 784.8$ N $(\rightarrow)$ ***Ans.***

Problem 19

Three identical spheres *P*, *Q*, *R* of weight *W* are arranged on smooth inclined surface, as shown in Fig. 3.19(a). Determine the angle α which will prevent the arrangement from collapsing.

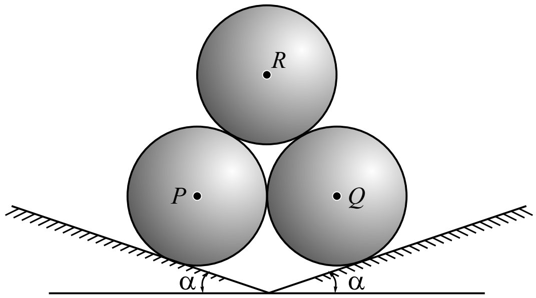


Fig. 3.19(a)

Solution

(i) Consider the FBD of upper sphere *R*

Since spheres are identical, therefore, due to symmetry, reaction at contact point will be same (*R*).

Δ *PQR* is forming equilateral triangle. Thus, included angle between reactions *R* is 60°.

By Lami's theorem,

$$\frac{W}{\sin 60^\circ} = \frac{R}{\sin 150^\circ} \quad \therefore R = 0.577\, W$$

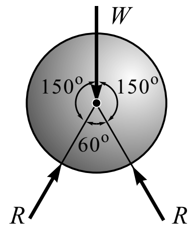


Fig. 3.19(b) : FBD of Sphere *R*

(ii) Consider the FBD of any one lower sphere (say *P*)

The reaction at contact between two lower spheres will be zero because at the required angle α the arrangement is about to collapse.

$\therefore R_2 = 0$

$\Sigma F_x = 0$

$R_1 \sin \alpha = 0.577\, W \cos 60^\circ$ ......(I)

$\Sigma F_y = 0$

$R_1 \cos \alpha = W + 0.577\, W \sin 60^\circ$

Dividing Eq. (I) by Eq. (II)

$$\tan \alpha = \frac{0.577\, W \cos 60^\circ}{W + 0.577\, W \sin 60^\circ} \quad \therefore \alpha = 10.89^\circ \text{ ***Ans.***}$$

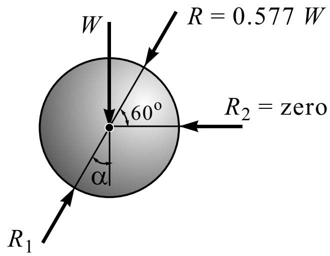


Fig. 3.19(c) : FBD of Sphere *P*

Problem 20

A mass raises a 10 kg joist of length 4 m by pulling on a rope. Find the tension in the rope and reaction at A. Refer Fig. 3.20(a).

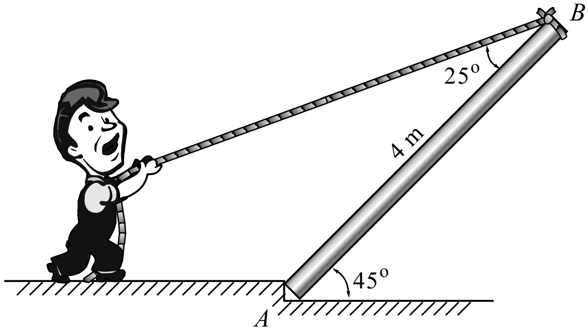


Fig. 3.20(a)

Solution

Method I

(i) Consider the FBD of the joist

By three-force principle in equilibrium R_A, T and 10×9.81 N must pass through a common point say D.

In $\Delta\, BCD$, by sine rule

$$\frac{CD}{\sin 25^\circ} = \frac{2}{\sin 110^\circ}$$

$$CD = 0.9 \text{ m}$$

(ii) In $\Delta\, AEC$,

$$AE = CE = \sqrt{2}$$

$$\tan\theta = \frac{DE}{AE} = \frac{CD + CE}{AE}$$

$$\tan\theta = \frac{0.9 + \sqrt{2}}{\sqrt{2}} \qquad \therefore\ \theta = 58.57^\circ$$

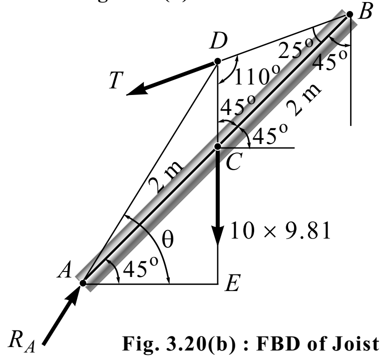


Fig. 3.20(b) : FBD of Joist

(iii) Considering three concurrent forces at point D

By Lami's theorem, we have

$$\frac{98.1}{\sin 141.43^\circ} = \frac{T}{\sin 148.57^\circ} = \frac{R_A}{\sin 70^\circ}$$

$$T = 82.05 \text{ N}$$

$$R_A = 147.86 \text{ N} \ (\angle\theta = 58.57^\circ) \quad \textit{Ans.}$$

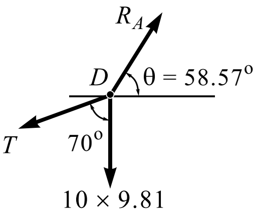


Fig. 3.20(c)

Method II

(i) $\Sigma M_A = 0$

$$T \sin 25^\circ \times 4 - 10 \times 9.81 \times 2 \cos 45^\circ = 0 \ \therefore\ T = 82.07 \text{ N}$$

(ii) $\Sigma F_x = 0$

$$H_A - T\cos 20^\circ = 0 \ \therefore\ H_A = 77.12 \text{ N}$$

(iii) $\Sigma F_y = 0$

$$V_A - 10 \times 9.81 - T \sin 20^\circ = 0 \ \therefore\ V_A = 126.17 \text{ N}$$

(iv) $\tan\theta = \dfrac{V_A}{H_A} = \dfrac{126.17}{77.12} \quad \therefore\ \theta = 58.57^\circ$

(v) $R_A = \sqrt{H_A^2 + V_A^2} = \sqrt{77.12^2 + 126.17^2}$

$$R_A = 147.87 \text{ N} \ (\angle\theta = 58.57) \quad \textit{Ans.}$$

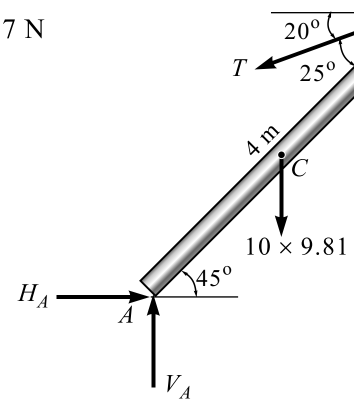


Fig. 3.20(d) : FBD of Joist

Problem 21

A lever AB is hinged at C and attached to a control cable at A, as shown in Fig. 3.21(a). If the lever is subjected to a 75 N vertical force at B, determine the tension in the cable and reaction at C.

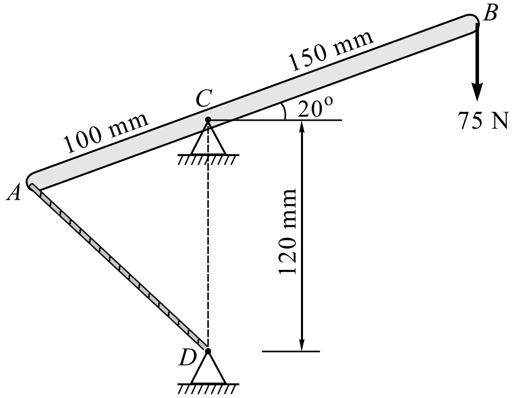


Fig. 3.21(a)

Solution

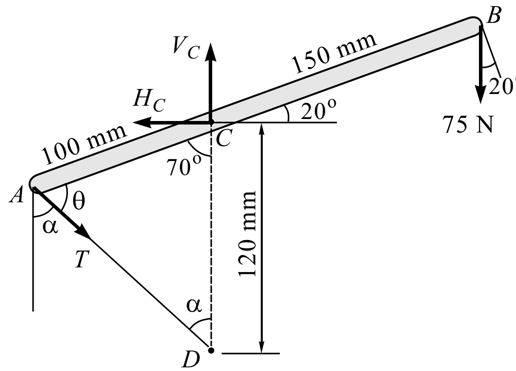


Fig. 3.21(b)

(i) Consider the FBD of Lever AB in Fig. 3.21(b).

(ii) In ΔACD, by cosine rule

$$AD = \sqrt{100^2 + 120^2 - 2 \times 100 \times 120 \cos 70^\circ}$$

$$AD = 127.25 \text{ mm}$$

By sine rule,

$$\frac{127.25}{\sin 70^\circ} = \frac{120}{\sin \theta} \qquad \therefore\ \theta = 62.39^\circ$$

$$\alpha = 180 - 70 - \theta \qquad \therefore\ \alpha = 47.61^\circ$$

(iii) $\Sigma M_C = 0$

$$T \sin 62.39^\circ \times 100 - 75 \cos 20^\circ \times 150 = 0$$

$$T = 119.3 \text{ N}$$

(iv) $\Sigma F_x = 0$

$$-H_C + 119.3 \sin 47.61^\circ = 0$$

$H_C = 88.11$ N $(\leftarrow)$ ***Ans.***

(v) $\Sigma F_y = 0$

$$V_C - 75 - 119.3 \cos 47.61^\circ = 0$$

$V_C = 155.43$ N $(\uparrow)$ ***Ans.***

Problem 22

Two cylinders, having weight W_A = 2000 N and W_B = 1000 N are resting on smooth inclined planes having inclination 60° and 45° with the horizontal respectively, as shown in Fig. 3.22(a). They are connected by a weightless bar *AB* with hinge connections. The bar *AB* makes 15° angle with the horizontal. Find the magnitude of the force *P* required to hold the system in equilibrium.

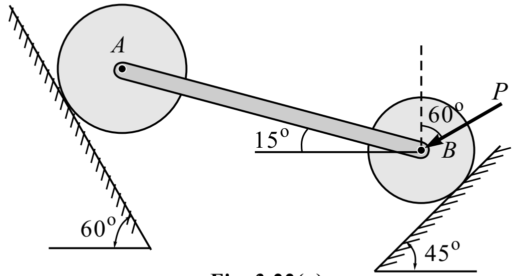


Fig. 3.22(a)

Solution

(i) In a given system of rigid bodies (two cylinders and one bar), the bar *AB* is connected at its extreme ends by frictionless pin. So, we can identify bar *AB* is a two-force member which can be indicated by F_{AB}.

(ii) Consider the FBD of cylinder W_A = 2000 [Fig. 3.22(b)].

By Lami's theorem,

$$\frac{2000}{\sin 135^\circ} = \frac{F_{AB}}{\sin 120^\circ} \quad \therefore\ F_{AB} = 2449.49 \text{ N}$$

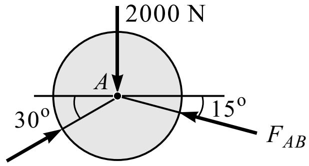


Fig. 3.22(b)

(iii) Consider the FBD of cylinder W_B = 1000 N [Fig. 3.22(c)].

$\Sigma F_x = 0$

$2449.49 \cos 15^\circ - R \cos 45^\circ - P \sin 60^\circ = 0$

$R \cos 45^\circ = 2449.49 \cos 15^\circ - P \sin 60^\circ$ ... (I)

$\Sigma F_y = 0$

$R \cos 45^\circ - P \cos 60^\circ - 2449.49 \sin 15^\circ - 1000 = 0$

$R \cos 45^\circ = P \cos 60^\circ + 2449.49 \sin 15^\circ + 1000$... (II)

Now, solving Eqs. (I) and (II), we get

$P = 535.9$ N ***Ans.***

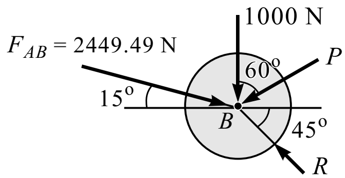


Fig. 3.22(c)

Problem 23

A uniform rod *AB* of length 3*R* and weight *W* rests inside a hemispherical bowl of radius *R* as shown in Fig. 3.23(a). Neglecting friction, determine angle θ corresponding to equilibrium.

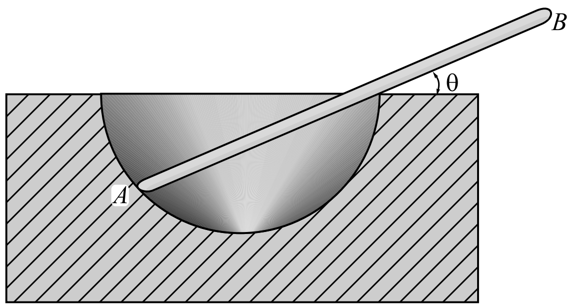


Fig. 3.23(a)

Solution

(i) Refer to the FBD of rod *AB* as shown in Fig. 3.23(b).

The one active force is weight of rod W acting vertically down through the centre of gravity of rod AB and two reactive forces reaction R_A acting along the normal to hemisphere at A and passing through centre O and reaction R_D acting along the normal to the rod AB at D (knife edge support).

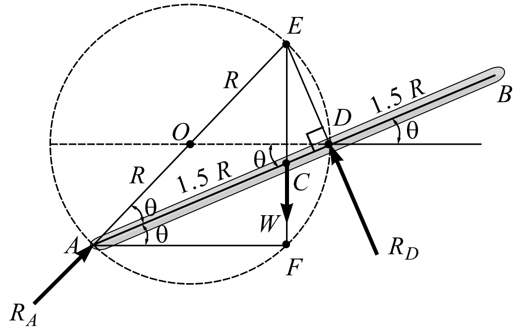


Fig. 3.23(b) : FBD of Rod AB

(ii) By three-force principle in equilibrium these three forces must pass through a common point (say E). This point E must lie on the circle as shown because $\angle ADE = 90^\circ$ (Angle subtended by a diameter at any point lying on the circumference of the circle is a right angle).

(iii) By the geometry of the figure

$\angle\theta = \angle ODA = \angle OAD = \angle DAF$

In Δ EAF and Δ CAF

$EA = 2R$ and $CA = 1.5R$

$\angle EAF = 2\theta$ and $\angle CAF = \theta$

$AF = 2R\cos 2\theta$ and $AF = 1.5R\cos\theta$

(iv) $2R\cos 2\theta = 1.5R\cos\theta$

$2\cos 2\theta = 1.5\cos\theta$

$2(2\cos^2\theta - 1) = 1.5\cos\theta$

$4\cos^2\theta - 1.5\cos\theta - 2 = 0$

Solving the quadratic equation, we get $\theta = 23.2^\circ$ ***Ans.***

3.7 Types of Beam

In a structure, horizontal member which takes transverse load in addition to other loading is called ***beam***. Transverse load means *load perpendicular to the length of the beam*.

In engineering structures like bridges, beam is one of the important structural member. In trusses and frames, pin-joined members take only tensile or compressive load. Beam is capable to take all types of load, i.e., transverse load, tensile load, compressive load, twisting load, etc.

Further beams may carry different types of transverse load such as point load, uniformly distributed load, uniformly varying load, etc.

Classification of Beams : Beams are classified depending upon the type of support, as shown below.

1. **Simply Supported Beam :** As the name indicates, it is the simplest of all beams which is supported by a hinge at one end and a roller at the other end.

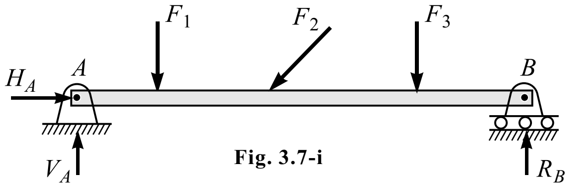


Fig. 3.7-i

2. **Simply Supported Beam with Overhang:** Here, one end or both the ends of simply supported beam is projected beyond the supports, which means that the portion of beam extends beyond the hinge and roller supports.

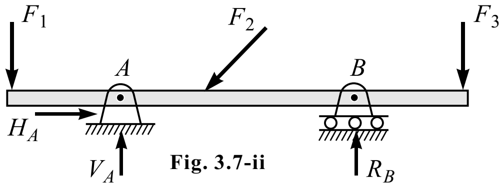


Fig. 3.7-ii

3. **Cantilever Beam :** A beam which is fixed at one end and free at the other end is called a *cantilever beam*. The fixed end is also known as built-in support. The common example is wall bracket, projected balconies, etc. One end of the beam is cast in concrete and is nailed, bolted, riveted or welded. The fixed end does not allow horizontal linear movement, vertical linear movement or rotational movement.

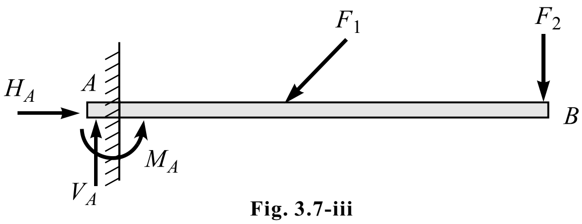


Fig. 3.7-iii

4. **Continuous Beam :** A beam which has more than two support is said to be a *continuous beam*. The extreme left and right supports are the end supports of the beam. Two intermediate supports are shown. Such beams are also called *statically indeterminate beams* because the reactions cannot be obtained by the equation of equilibrium.

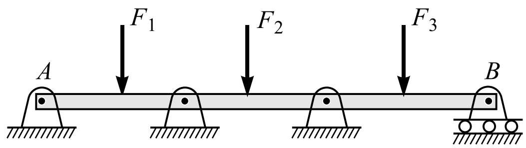


Fig. 3.7-iv

5. **Beams Linked with Internal Hinges :** Here two or more beams are connected to each other by pin joint and continuous beam is formed. Such a joint are called *internal hinges*. Internal hinges allow us to draw FBD of beam at its joint, if required.

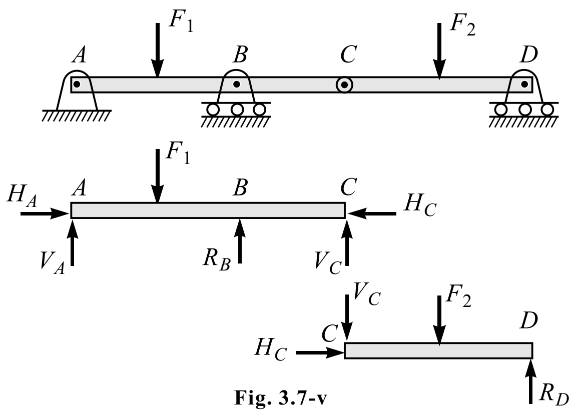


Fig. 3.7-v

3.8 Types of Load

There are two types of load : Point load and distributed load.

1. **Point Load :** If the whole intensity of load is assumed to be concentrated at a point then it is known as *point load*.

   Refer to Fig. 3.8.

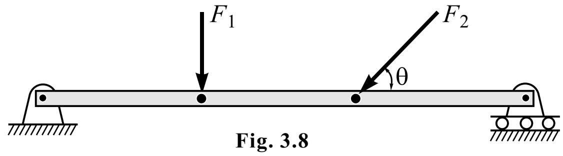


Fig. 3.8

2. **Distributed Load :** The concept of a centroid of an area may be used to solve problem dealing with a beam supporting a distributed load. This load may consist of the weight of materials supported directly or indirectly by the beam or it may be caused by wind or hydraulic pressure. The distributed load may be represented by *plotting the load intensity W supported per unit length*. The load intensity is expressed in N/m or kN/m.

(a) ***Uniformly Distributed Load (UDL)*** **:** If the whole intensity of load is distributed uniformly along the length of loading then it is called Uniformly Distributed Load (UDL). For example a truck loaded with sand of equal height and slab of a building flooring.

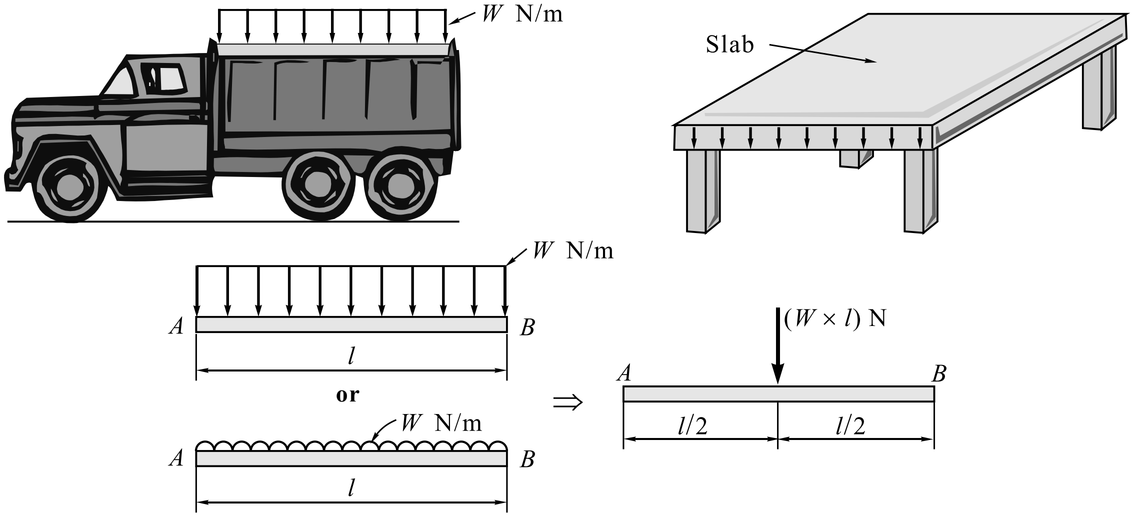
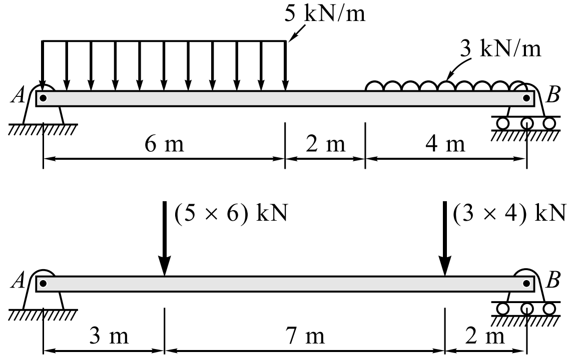


Fig. 3.8-i

Example

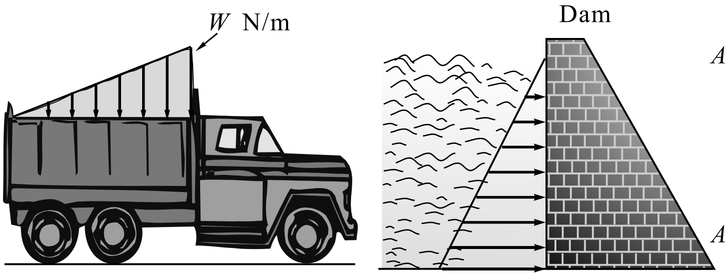
A uniformly distributed load on a beam can be replaced by a concentrated point load. The magnitude of this equivalent point load is equal to the area under loading diagram and it acts through the centroid. The area under the loading diagram is calculated by multiplying the load intensity with length of loading.

Refer to the adjacent figure.



(ii) ***Uniformly Varying Load (UVL)*** **:** If the whole intensity of load is distributed uniformly at varying rate along the length of loading then, it is known as Uniformly Varying Load (UVL). For example a truck loaded with sand, hydraulic pressure varies linearly with the depth.

Refer to Fig. 3.8-ii.



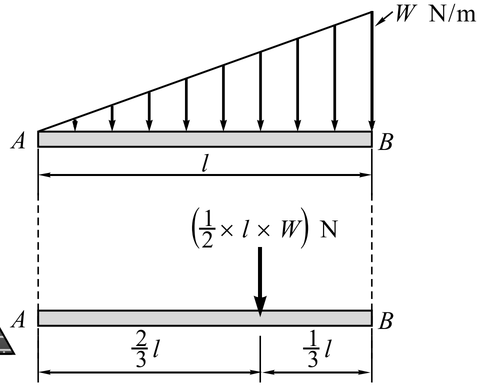
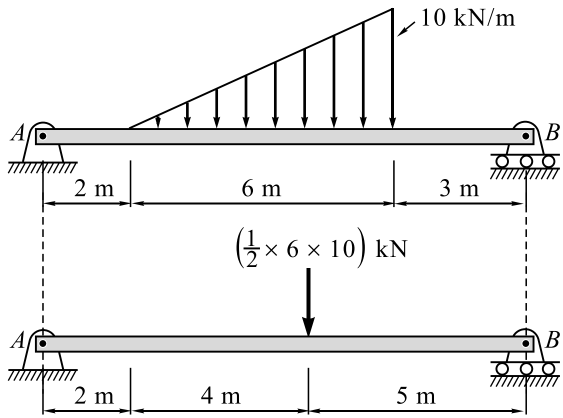


Fig. 3.8-ii

Example

A uniformly varying load on a beam can be replaced by a concentrated point load. The magnitude of this equivalent point load is equal to the area under loading diagram and it acts through the centroid. Generally, UVL is represented by right angled triangle. Area under loading diagram is the area of triangle, i.e.,

$$\frac{1}{2} \times \text{Length of loading} \times \text{Load intensity}$$



(iii) ***UDL and UVL Combined***

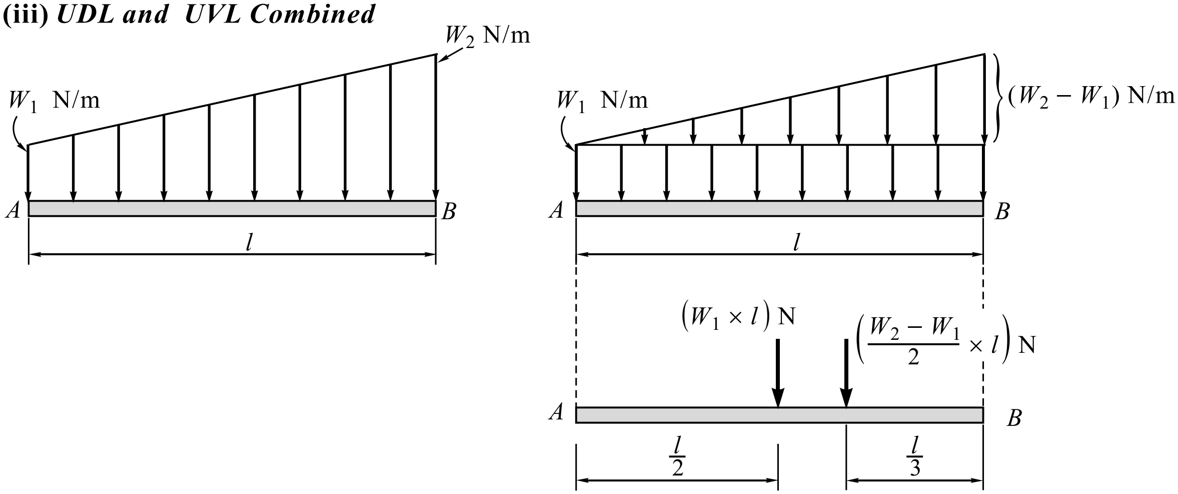
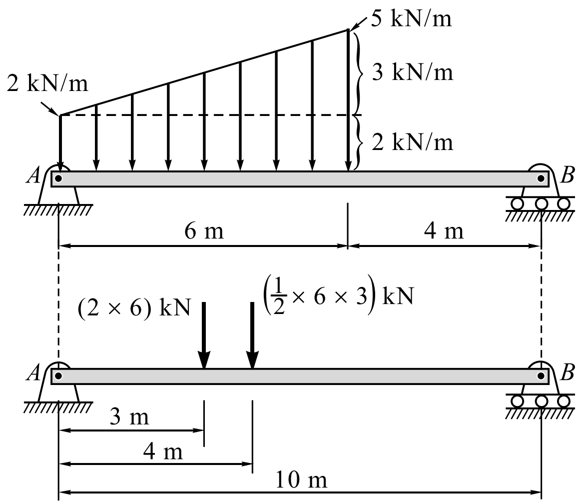


Fig. 3.8-iii

Example



(iv) ***Varying Load with Some Relation*** **:** The varying load is given by some relation, say parabolic nature. It can be replaced by concentrated point load. The magnitude of the equivalent point load is *equal to the area under loading diagram and it acts through the centroid*.

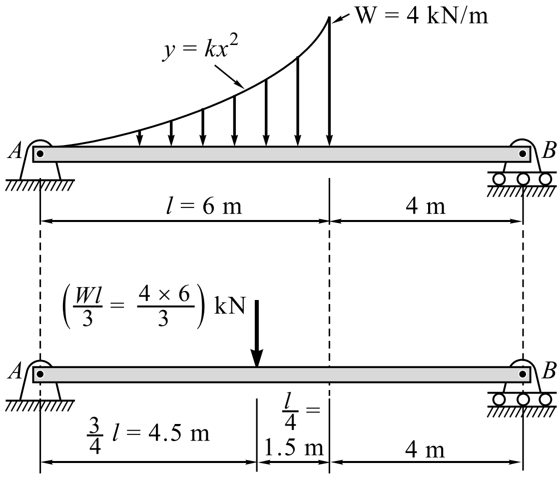


Fig. 3.8-iv

(v) ***Couple***

Example

Though on beam, couples C_1 = 5 kN-m and C_2 = 7 kN-m are shown at specific positions but we know that couple is a free vector, so it can act anywhere along the beam AB. In other words, distance of couples C_1 and C_2 from point A (given 2 m and 6 m) respectively, has no significance as far as position is concerned. As per the requirement of solution say $\Sigma\, M_A$, we can consider given couples C_1 = 5 kN-m (↻) and C_2 = 7 kN-m (↺). Refer to Fig. 3.8-v.

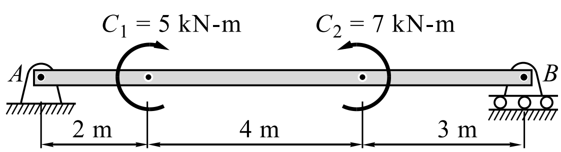


Fig. 3.8-v

3.9 Solved Problems on Support Reactions of Beams

Problem 24

Calculate the support reactions for the beam shown in Fig. 3.24(a).

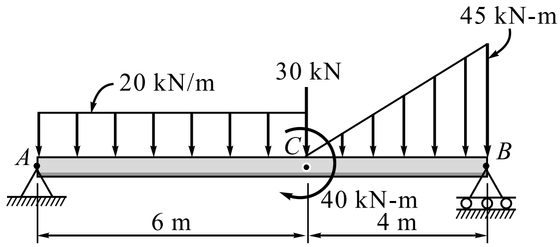


Fig. 3.24(a)

Solution

(i) Consider the FBD of Beam AB [Fig. 3.24(b)]

(ii) $\Sigma M_A = 0$

$$-120 \times 3 - 30 \times 6 - 40 - 90 \times 8.67 + R_B \times 10 = 0$$

$$R_B = 136.03 \text{ kN } (\uparrow)$$

(iii) $\Sigma F_x = 0$

$$H_A = 0$$

($\because$ there is no horizontal force acting)

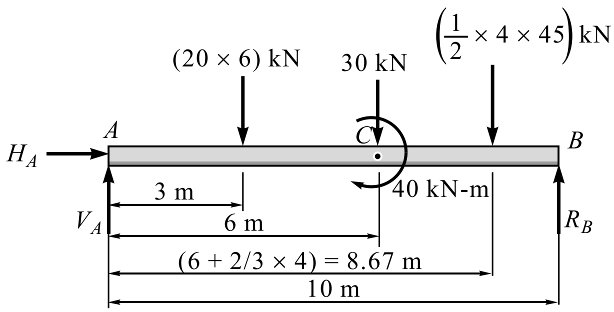


Fig. 3.24(b)

(iv) $\Sigma F_y = 0$

$$V_A - 120 - 30 - 90 + 136.03 = 0$$

$$V_A = 103.97 \text{ kN } (\uparrow) \quad \textbf{\textit{Ans.}}$$

Problem 25

Find the support reactions at A and B for the beam loaded as shown in Fig. 3.25(a).

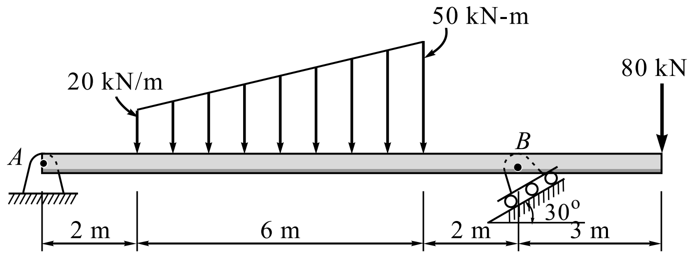


Fig. 3.25(a)

Solution

(i) Consider the FBD of Beam AB [Fig. 3.25(b)].

(ii) $\Sigma M_A = 0$

$$R_B \sin 60^\circ \times 10 - 120 \times 5 - 90 \times 6 - 80 \times 13 = 0$$

$$R_B = 251.73 \text{ kN } (60^\circ)$$

(iii) $\Sigma F_x = 0$

$$H_A - 251.73 \cos 60^\circ = 0$$

$$H_A = 125.87 \text{ kN } (\rightarrow)$$

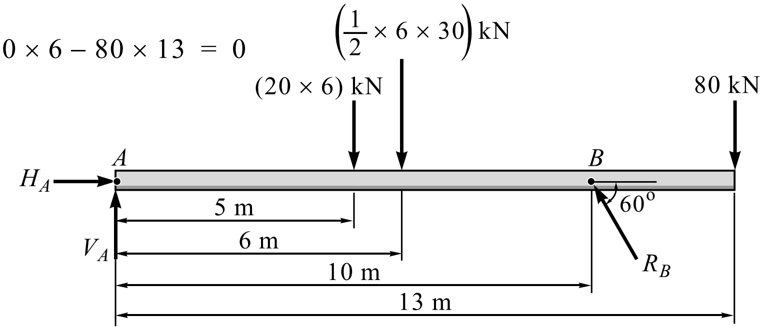


Fig. 3.25(b)

(iv) $\Sigma F_y = 0$

$$V_A - 120 - 90 + 251.73 \sin 60^\circ - 80 = 0$$

$$V_A = 72 \text{ kN } (\uparrow) \quad \textbf{\textit{Ans.}}$$

Problem 26

Find analytically the support reaction at B and the load P, for the beam shown in Fig. 3.26(a), if the reaction of support A is zero.

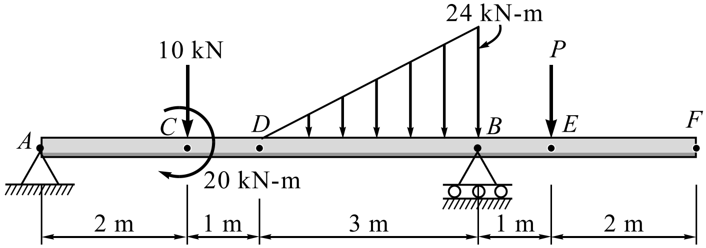


Fig. 3.26(a)

Solution

(i) Consider the FBD of Beam AF

(ii) $\Sigma F_y = 0$

$V_A + R_B - 10 - 36 - P = 0 \quad (V_A = 0 \text{ given})$

$R_B - P = 46 \qquad \ldots\text{(I)}$

(iii) $\Sigma M_A = 0$

$R_B \times 6 - 10 \times 2 - 20 - 36 \times 5 - P \times 7 = 0$

$6\,R_B - 7\,P = 220 \qquad \ldots\text{(II)}$

(iv) Solving Eqs. (I) and (II)

$R_B = 102 \text{ kN } (\uparrow)$ ***Ans.***

(v) From Eq. (I)

$P = 102 - 46$

$P = 56 \text{ kN } (\downarrow)$ ***Ans.***

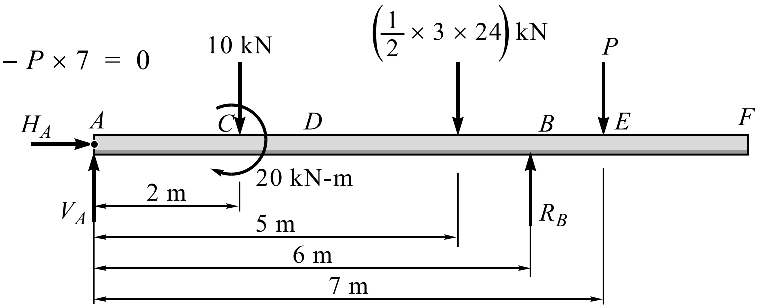


Fig. 3.26(b) : F.B.D of Beam *AF*

Problem 27

Find the support reactions at A and F for the given Fig. 3.27(a).

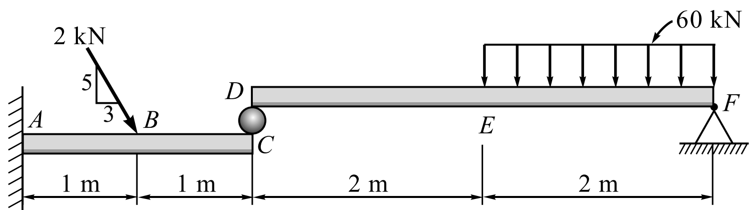


Fig. 3.27(a)

Solution

(i) **Consider the FBD of Beam *DF*** [Fig. 3.27(b)]

$\Sigma M_F = 0$

$120 \times 1 - R_D \times 4 = 0 \quad \therefore \; R_D = 30 \text{ kN}$

$\Sigma F_x = 0 \qquad \therefore \; H_F = 0$

$\Sigma F_y = 0$

$R_D + V_F - 120 = 0$

$V_F = 90 \text{ kN } (\uparrow)$ ***Ans.***

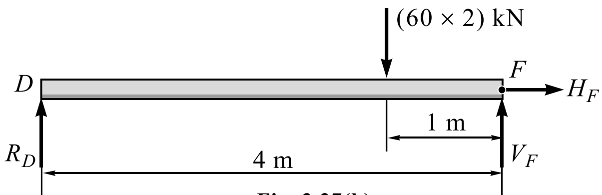


Fig. 3.27(b)

(ii) **Consider the FBD of Beam *AC*** [Fig. 3.27(c)]

$\Sigma M_A = 0$

$M_A - 2 \sin 59.04^\circ \times 1 - 30 \times 2 = 0$

$M_A = 61.72 \text{ kN-m } (\circlearrowleft)$

$\Sigma F_x = 0$

$2 \cos 59.04^\circ - H_A = 0$

$H_A = 1.03 \text{ kN } (\leftarrow)$

$\Sigma F_y = 0$

$V_A - 2 \sin 59.04^\circ - 30 = 0$

$V_A = 31.72 \text{ kN } (\uparrow)$ ***Ans.***

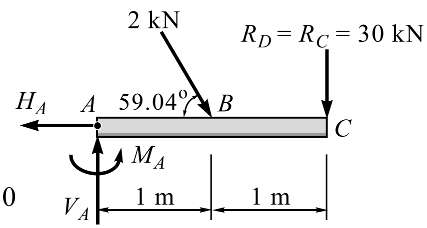


Fig. 3.27(c)

Problem 28

Two beams AB and CD are arranged as shown in Fig. 3.38(a). Find the support reactions at D.

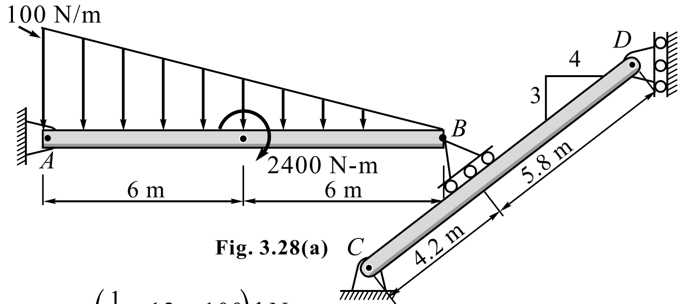


Fig. 3.28(a)

Solution

(i) Consider the FBD of Beam *AB*

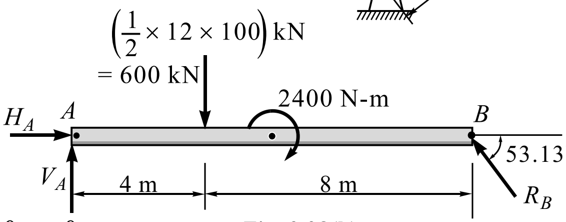


Fig. 3.28(b)

$$\Sigma M_A = 0$$

$$R_B \sin 53.13^\circ \times 12 - 600 \times 4 - 2400 = 0$$

$$R_B = 500 \text{ N} \qquad \textbf{\textit{Ans.}}$$

(ii) Consider the FBD of Beam *CD*

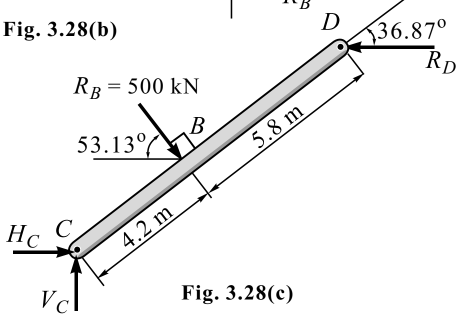


Fig. 3.28(c)

$$\Sigma M_C = 0$$

$$R_D \sin 36.87^\circ \times 10 - 500 \times 4.2 = 0$$

$$R_D = 350 \text{ N } (\leftarrow) \qquad \textbf{\textit{Ans.}}$$

Problem 29

Determine the intensity of distributed load W at the end C of the beam ABC as shown in Fig. 3.29(a), for which the reaction at C is zero. Also calculate the reaction at B.

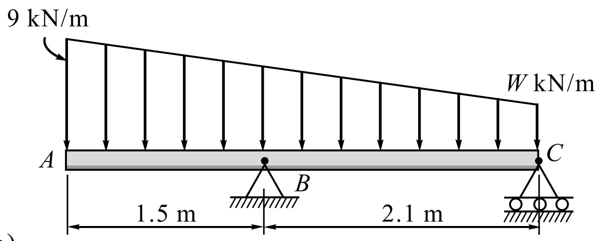


Fig. 3.29(a)

Solution

(i) Consider the FBD of beam ABC with equivalent point load shown in Fig. 3.29(b).

(ii) $\Sigma M_B = 0$

$$\frac{1}{2} \times 3.6 \times (9 - W) \times 0.3 + R_C \times 0 - W \times 3.6 \times 0.3 = 0$$

$$4.86 - 0.54W - 1.08W = 0$$

$$1.62\ W = 4.86 \qquad \therefore W = 3 \text{ kN}$$

(iii) $\Sigma F_x = 0 \quad \therefore H_B = 0$

(iv) $\Sigma F_y = 0$

$$V_B - \frac{1}{2} \times 3.6 \times (9 - W) - W \times 3.6 + 0 = 0$$

$$V_B = 10.8 + 10.8$$

$$\therefore V_B = 21.6 \text{ kN } (\uparrow) \qquad \textbf{\textit{Ans.}}$$

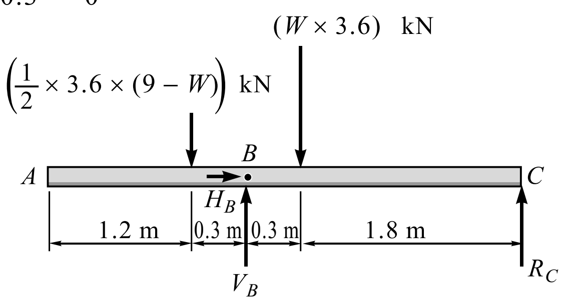


Fig. 3.29(b) : F.B.D of Beam *ABC*

Problem 30

Bars AB and CD are rigidly connected by welding at D as shown in Fig. 3.30(a). Bar AB weighs 5 kN/m whereas weight of bar CD is negligible.

Determine the support reactions.

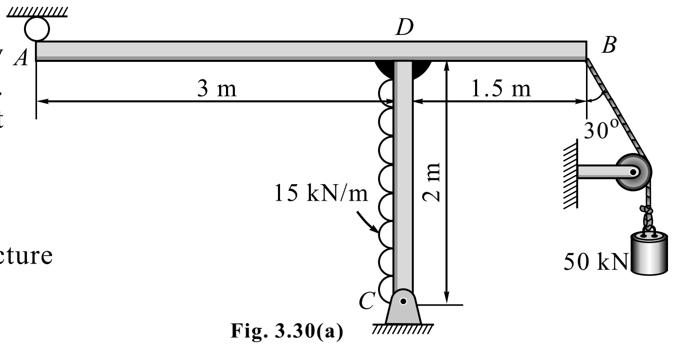


Fig. 3.30(a)

Solution

(i) Consider the FBD of the whole structure since it is a single rigid body.

(ii) $\Sigma M_C = 0$

$$R_A \times 3 + (4.5 \times 5) \times 0.75 - 50 \cos 30^\circ \times 1.5$$

$$- 50 \sin 30^\circ \times 2 - (15 \times 2) \times 1 = 0$$

$$R_A = 42.69 \text{ kN } (\downarrow) \textbf{ Ans.}$$

(iii) $\Sigma F_x = 0$

$$(15 \times 2) + 50 \sin 30^\circ - H_C = 0$$

$$H_C = 55 \text{ kN } (\leftarrow) \textbf{ Ans.}$$

(iv) $\Sigma F_y = 0$

$$V_C - R_A - (4.5 \times 5) - 50 \cos 30^\circ = 0$$

$$V_C = 108.49 \text{ kN } (\uparrow) \textbf{ Ans.}$$

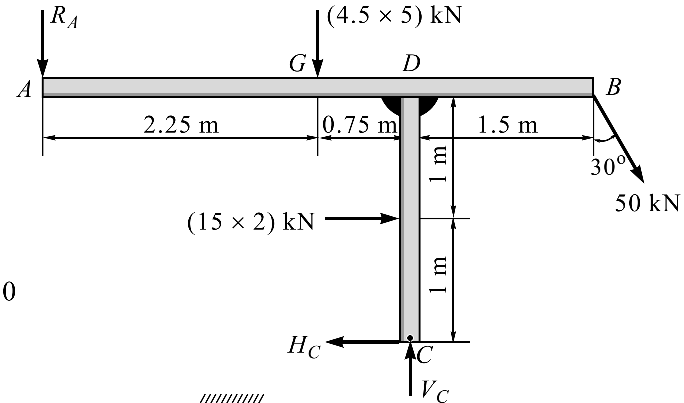


Fig. 3.30(b)

Problem 31

A single rigid bar ABC of 'L' shape is loaded and supported, as shown Fig. 3.31(a). Find the support reactions.

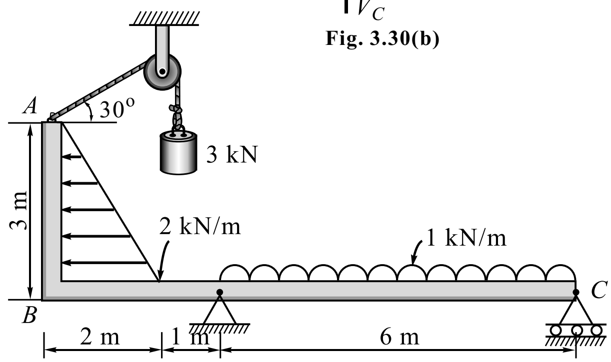


Fig. 3.31(a)

Solution

(i) Consider the FBD of Beam ABC.

(ii) $\Sigma M_D = 0$

$$R_C \times 6 - 3 \cos 30^\circ \times 3 - 3 \sin 30^\circ \times 3$$

$$- 6 \times 3 + \left(\frac{1}{2} \times 3 \times 2\right) \times 1 = 0$$

$$R_C = 4.55 \text{ kN } (\uparrow) \qquad \textbf{Ans.}$$

(iii) $\Sigma F_x = 0$

$$3 \cos 30^\circ - \left(\frac{1}{2} \times 3 \times 2\right) + H_D = 0$$

$$H_D = 0.4 \text{ kN } (\rightarrow) \qquad \textbf{Ans.}$$

(iv) $\Sigma F_y = 0$

$$V_D + 3 \sin 30^\circ + R_C - 6 = 0$$

$$V_D = -0.05 \text{ (Wrong assumed direction)}$$

$$V_D = 0.05 \text{ kN } (\downarrow) \textbf{ Ans.}$$

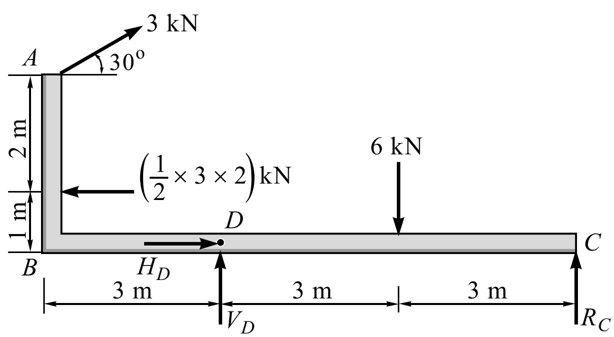


Fig. 3.31(b)

Problem 32

Find the support reactions of the beam shown in Fig. 3.32(a). E is internal hinge.

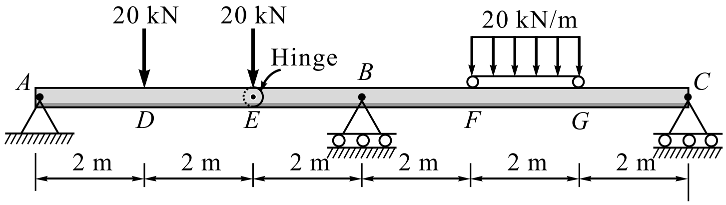


Fig. 3.32(a)

Solution

(i) Consider the FBD of Beam *AE*

$$\Sigma M_E = 0$$

$$20 \times 2 - V_A \times 4 = 0$$

$$V_A = 10 \text{ kN } (\uparrow) \quad \textbf{\textit{Ans.}}$$

$$\Sigma F_y = 0$$

$$V_A + V_E - 20 - 20 = 0$$

$$V_E = 30 \text{ kN } (\uparrow) \quad \textbf{\textit{Ans.}}$$

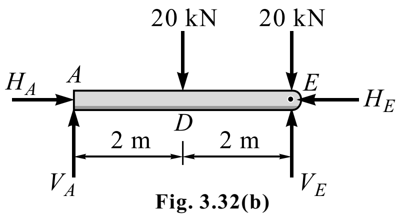


Fig. 3.32(b)

Since there is a horizontal or inclined external force acting, therefore, horizontal component of reaction will be zero.

$$H_A = H_E = 0$$

(ii) Method I

Consider the FBD of Beam *EC*

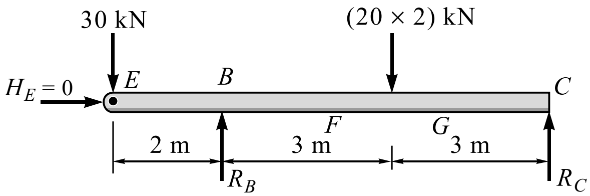


Fig. 3.32(c)

$$\Sigma M_B = 0$$

$$30 \times 2 - 20 \times 2 \times 3 + R_C \times 6 = 0$$

$$R_C = 10 \text{ kN } (\uparrow) \quad \textbf{\textit{Ans.}}$$

$$\Sigma F_y = 0$$

$$R_B + R_C - 30 - 20 - 20 = 0$$

$$R_B = 60 \text{ kN } (\uparrow) \quad \textbf{\textit{Ans.}}$$

(iii) Method II

Consider the FBD of Beam *AC*

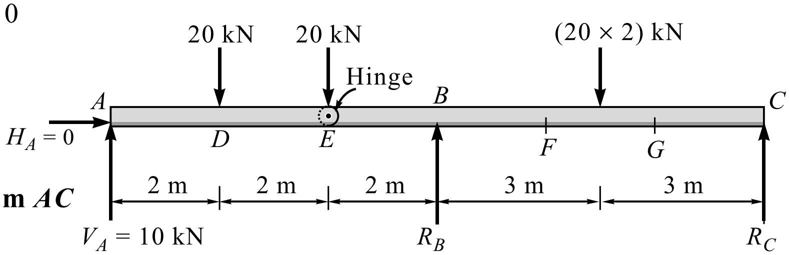


Fig. 3.32(d)

$$\Sigma M_B = 0$$

$$R_C \times 6 - 20 \times 2 \times 3 + 20 \times 2 + 20 \times 4 - 10 \times 6 = 0$$

$$R_C = 10 \text{ kN } (\uparrow) \quad \textbf{\textit{Ans.}}$$

$$\Sigma F_y = 0$$

$$10 + R_B + 10 - 20 - 20 - 20 \times 2 = 0$$

$$R_B = 60 \text{ kN } (\uparrow) \quad \textbf{\textit{Ans.}}$$

Exercises

[I] Problems

1. An electric light weighing 15 N hangs from a point C by the two strings AC and BC as shown in Fig. 3.E1. AC is inclined at 60° to the horizontal and BC at 45° to the vertical as shown. Using the Lami's theorem find the forces in the strings AC and BC.

[**Ans.** $T_{AC} = 10.98$ N and $T_{BC} = 7.76$ N.]

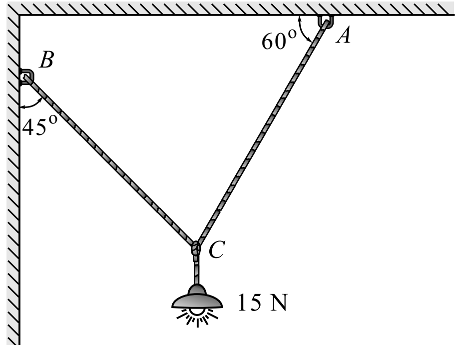


Fig. 3.E1

2. A force P is applied at O to the strings AOB as shown in Fig. 3.E2. If the tension in each string is 50 N, find the magnitude and direction of force P for equilibrium conditions.

[**Ans.** $\theta = 7.5^\circ$ and $P = 60.88$ N.]

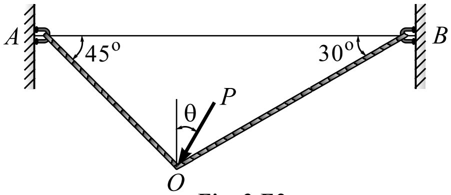


Fig. 3.E2

3. A smooth sphere of mass 2 kg is supported by a chain as shown in Fig. 3.E3. The length of chain AB is equal to the radius of the sphere. Draw free body diagram of each element and find the tension in the chain and reaction of the wall.

[**Ans.** $T_{AB} = 22.6$ N and $R_C = 11.33$ N.]

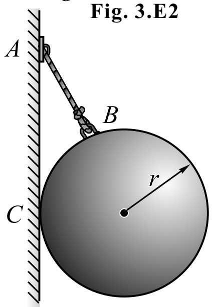


Fig. 3.E3

4. A smooth sphere of weight 500 N rests in a V shaped groove, whose sides are inclined at 25° and 65° to the horizontal, as shown in Fig. 3.E4. Find the reactions at A and B.

[**Ans.** $R_A = 453.15$ N (65°) and
$R_B = 211.31$ N (25°) .]

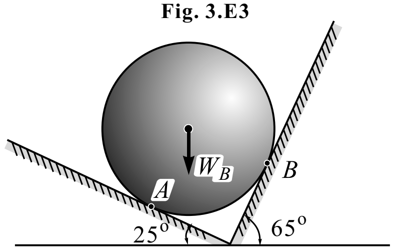


Fig. 3.E4

5. Determine the horizontal distance to which a 1 m long in extensible string holding weight of 500 N can be pulled before the string breaks. The string can withstand the maximum pull of 1000 N, as shown in Fig. 3.E5. Determine also the required force F.

[**Ans.** $F = 866$ N and $x = 0.86$ m.]

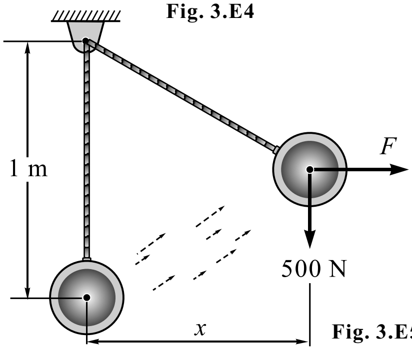


Fig. 3.E5

6. A bar AB of weight 1 kN is hinged to a vertical wall at A and supported by a cable BD as shown in Fig. 3.E6. Find the tension in the cable and the magnitude and direction of reaction at the hinge.

[***Ans.*** $T = 0.866$ kN and
$R_A = 0.5$ kN $\angle \theta = 30^\circ$.]

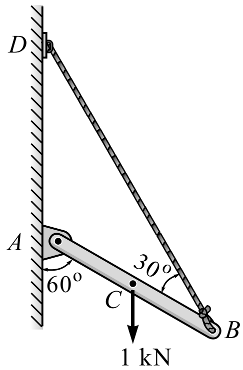


Fig. 3.E6

7. A string ACB of length l carries a small pulley C from which a load W is suspended as shown in Fig. 3.E7. Find the position of equilibrium as defined by the angle α.

Given : $d = \dfrac{l}{2}$, $h = \dfrac{l}{4}$.

[***Ans.*** $\alpha = 60^\circ$]

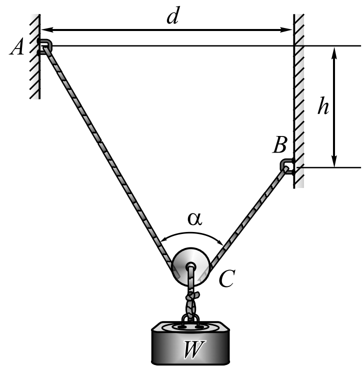


Fig. 3.E7

8. A system of connected flexible cables shown in Fig. 3.E8 is supporting two vertical forces 200 N and 250 N at points B and D. Determine the forces in various segments of the cable.

[***Ans.*** $T_{DE} = 224.14$ N, $T_{BD} = 183.01$ N
$T_{BC} = 336.6$ N and $T_{AB} = 326.79$ N.]

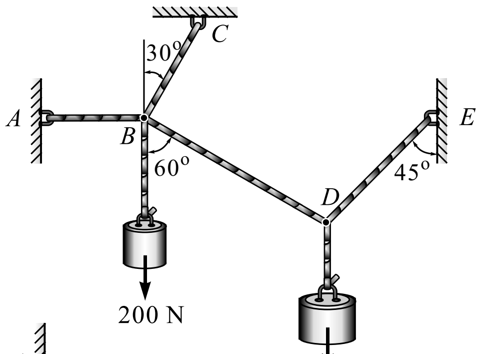


Fig. 3.E8

9. Two equal loads of 2500 N are supported by a flexible string $ABCD$ at points B and C as shown in Fig. 3.E9. Find the tension in the portion AB, BC, CD of the string.

[***Ans.*** $T_{AB} = 4330$ N, $T_{BC} = 2500$ N
and $T_{CD} = 2500$ N.]

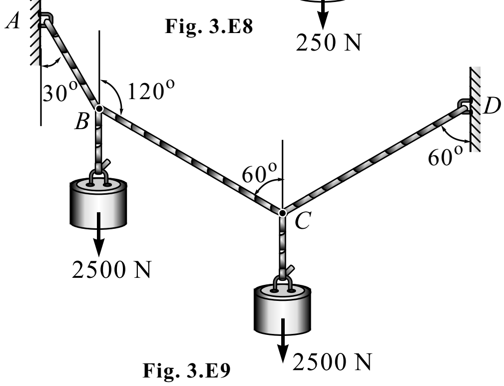


Fig. 3.E9

10. Two cables are tied together at C and loaded as shown in Fig. 3.E10. Determine the tensions in AC and BC.

[***Ans.*** $T_{AC} = 326$ N and $T_{BC} = 368$ N.]

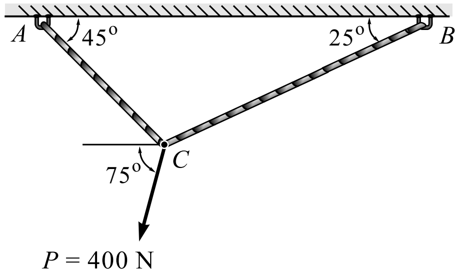


Fig. 3.E10

11. Two smooth spheres of weight 100 N and radius 250 mm each are in equilibrium in a horizontal channel of width 870 mm as shown in Fig. 3.E11. Find the reactions at the surfaces of contact A, B, C, D, assuming all the surfaces to be smooth.

[***Ans.*** $R_A = 133.3$ N (→), $R_B = 200$ N (↑), $R_C = 133.3$ N (←) and $R_D = 166.6$ N.]

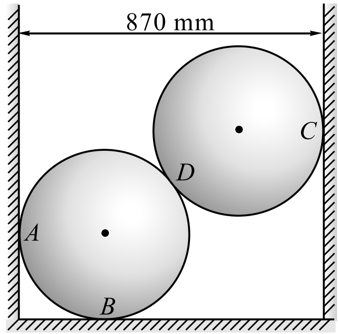


Fig. 3.E11

12. Two smooth cylinder with diameters 250 mm and 400 mm respectively are kept in a groove with slanting surfaces, making angles 60° and 30° respectively as shown in Fig. 3.E12. Determine the reactions at contact points A, B, and C.

[***Ans.*** $R_B = 297.37$ N, $R_B = 1125.85$ N and $R_A = 352.69$ N.]

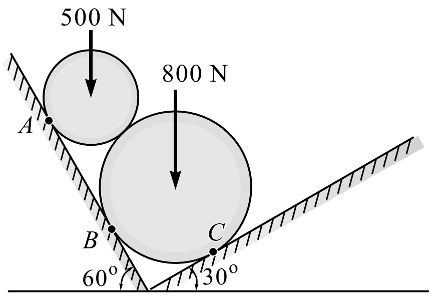


Fig. 3.E12

13. Two cylinders P and Q are in a channel as shown in Fig. 3.E13. The cylinder P has a diameter of 100 mm and weight 200 N and Q has 180 mm and 500 N. Determine the reaction at all the contact surfaces.

[***Ans.*** $R_1 = 134.2$ N (←), $R_2 = 240.8$ N, $R_3 = 154.9$ N ($\angle 30°$) and $R_4 = 622.5$ N (↑).]

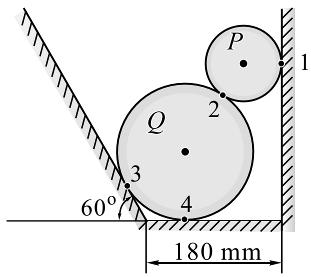


Fig. 3.E13

14. Two identical rollers each of weight 500 N are kept in a right-angle frame *ABC* having negligible weight as shown in Fig. 3.E14. Assuming smooth surfaces, find the reactions induced at the points *P*, *Q*, *R* and *S*. Also find the reactions at *B* and *C*.

Ans. $R_P = 500\text{ N}, R_Q = R_S = 433\text{ N},$
$R_R = 250\text{ N}, R_C = 246.41\text{ N}\ (60^\circ \nwarrow),$
$H_B = 123.21\text{ N}\ (\rightarrow)$ and $V_B = 786.6\text{ N}\ (\uparrow).$

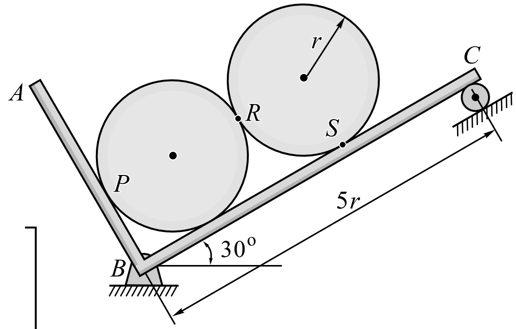


Fig. 3.E14

15. A light bar is suspended from a cable *BE* and supports a 200 N block at *C*. The extremities *A* and *D* of the bar are in contact with frictionless vertical walls, as shown in Fig. 3.E15. Determine the tension in cable *BE* and the reactions at *A* and *D*.

Ans. $T = 200\text{ N}, R_D = 75\text{ N}\ (\leftarrow)$
and $R_A = 75\text{ N}\ (\rightarrow).$

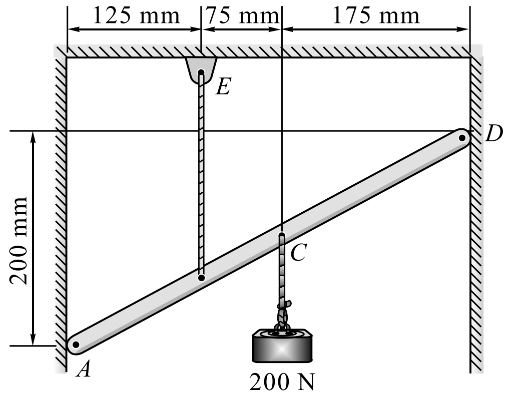


Fig. 3.E15

16. Neglecting friction, determine the tension in cable *ABD* and the reaction at support *C*, as shown in Fig. 3.E16.

Ans. $T = 80\text{ N},$
$H_C = 80\text{ N}\ (\rightarrow)$ and
$V_{BC} = 40\text{ N}\ (\uparrow).$

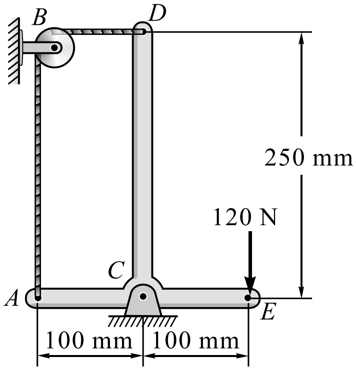


Fig. 3.E16

17. Determine the tension in cable *BC*, as shown in Fig. 3.E17. Neglect the self-weight of *AB*.

Ans. $V_A = 5850\text{ N},$
$H_A = 4070.3\text{ N},$
and $T = 4700\text{ N}.$

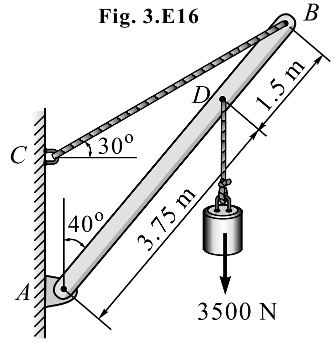


Fig. 3.E17

18. Determine the force P applied at 45^{o} to the horizontal just necessary to start a roller 100 cm diameter over an obstruction 25 cm high, if the roller weighs 1000 N, as shown in Fig. 3.E18. Also find the magnitude and direction of P when it is minimum.

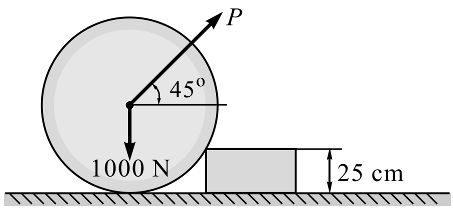


Fig. 3.E18

$$\left[\begin{array}{l}\textit{Ans.}\ \ \text{(a)}\ P = 866.48\ \text{N and} \\ \qquad\ \ \text{(b)}\ P_{\text{min}} = 866\ \text{N at } 60^{o} \text{ to horizontal.}\end{array}\right]$$

19. A roller of radius 400 mm, weighing 4 kN is to be pulled over a rectangular block of height 200 mm as shown in Fig. 3.E19, by a horizontal force applied at the end of a string wound round the circumference of the roller. Find the magnitude of the horizontal force P and the reaction at B, which will just turn the roller over the corner of the rectangular block. Also, determine the least force and its line of action at the roller centre, for turning the roller over the rectangular block.

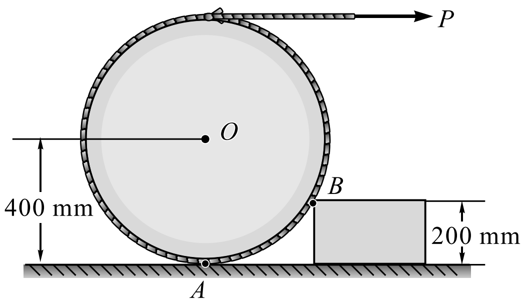


Fig. 3.E19

$$\left[\begin{array}{l}\textit{Ans.}\ \ \text{(a)}\ P = 2.31\ \text{kN and} \\ \qquad\ \ \text{(b)}\ P_{\text{min}} \text{ at roller centre} = 3.46\ \text{kN.}\end{array}\right]$$

20. Determine the magnitude and direction of the smallest force P required to start to wheel over the block as shown in Fig. 3.E20.

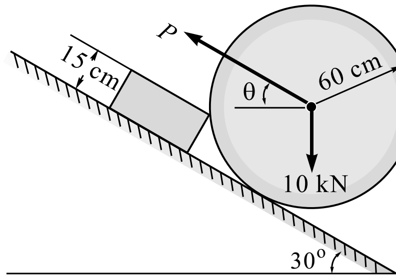


Fig. 3.E20

$$\left[\begin{array}{l}\textit{Ans.}\ \ P = 9.47\ \text{kN}\ (\theta) \text{ and} \\ \qquad\ \ \theta = 71.4^{o}.\end{array}\right]$$

21. A vertical pole is anchored in a cement foundation. Three wires are attached to the pole as shown in Fig. 3.E21. If the reaction at point A consists of the reactions as shown, find the tensions in the wires.

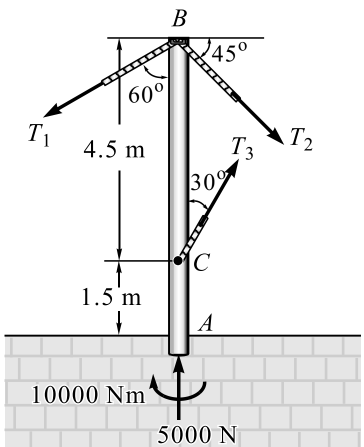


Fig. 3.E21

$$\left[\begin{array}{l}\textit{Ans.}\ \ T_1 = 8104.6\ \text{N}, \\ \qquad\ \ T_2 = 6784.3\ \text{N and} \\ \qquad\ \ T_3 = 4444.4\ \text{N.}\end{array}\right]$$

22. The spanner shown in Fig. 3.E22 is used to rotate a shaft. A pin fits in a hole at A, while a flat frictionless surface rests against the shaft at B. If the moment about C of the force exerted on the shaft at A is to be 87 N-m, find (a) the force P which should be exerted on the spanner at D and (b) the corresponding value of the force exerted on the spanner at B.

$\left[\textbf{Ans. } P = 240 \text{ N and } R_B = 1768 \text{ N } (\rightarrow).\right]$

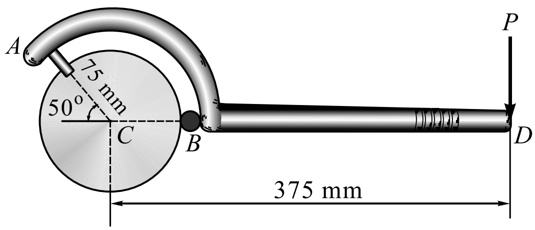


Fig. 3.E22

23. A smooth pipe rests against the wall at the points of contact A, B and, C, as shown in Fig. 3.E23. Determine the reactions at these points needed to support the vertical force of 200 N. Neglect the pipe's thickness in the calculation.

$$\left[\begin{array}{ll}\textbf{Ans. } R_C = 284 \text{ N,} \\ \quad R_B = 53.1 \text{ N and} \\ \quad R_A = 115.5 \text{ N.}\end{array}\right]$$

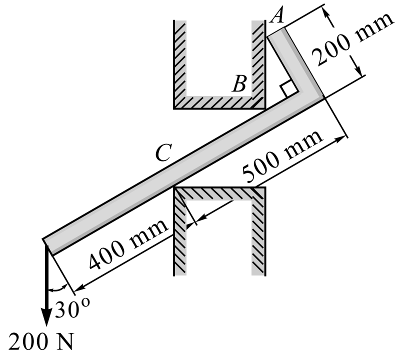


Fig. 3.E23

24. Two rollers of weights 50 N and 100 N are connected by a flexible string AB. The rollers rest on two mutually perpendicular surfaces DE and EF as shown in Fig. 3.E24. Find the tension in the string and the angle θ that it makes with the horizontal when the system is in equilibrium.

$\left[\textbf{Ans. } \theta = 10.9^\circ \text{ and } T = 66.15 \text{ N.}\right]$

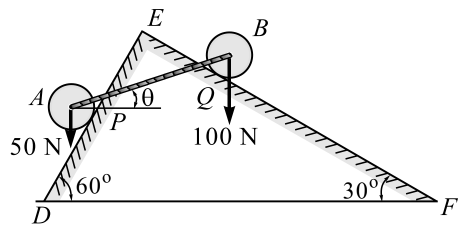


Fig. 3.E24

25. A vertical post PQ of a crane is pivoted at P and supported by a guide Q. Find the reactions at P and Q due to the loads acting, as shown in Fig. 3.E25.

$\left[\textbf{Ans. } R_Q = 4000 \text{ N and } R_P = 5656.8 \text{ N.}\right]$

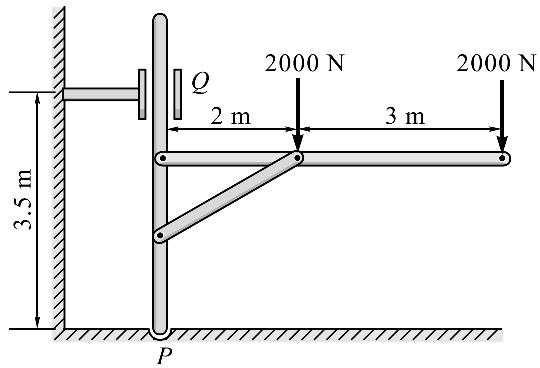


Fig. 3.E25

26. A and B are identical smooth cylinders having masses of 100 kg each as shown in Fig. 3.E26. Determine the maximum force P which can be applied without causing A to leave the floor.

[**Ans.** $P = 1699$ N.]

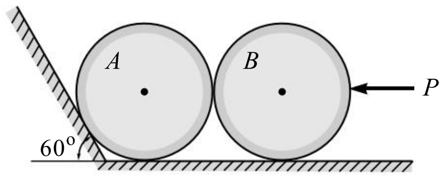


Fig. 3.E26

27. A man weighing 75 N stands on the middle rung of a 25 N ladder resting on a smooth floor and against a wall as shown in Fig. 3.E27. The ladder is prevented from slipping by a string OD. Find the tension in the string and reactions at A and B.

[**Ans.** $R_A = 120.26$ N, $R_B = 35.13$ N and $T = 40.56$ N.]

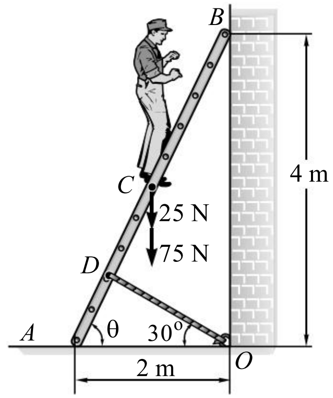


Fig. 3.E27

28. The frame BCD as shown in Fig. 3.E28 supports a 600 N cylinder. The frame is hinged at D. Determine the reactions at A, B, C and D.

[**Ans.** $R_A = 200$ N, $R_B = 600$ N, $R_C = 200$ N and $R_D = 632.46$ N.]

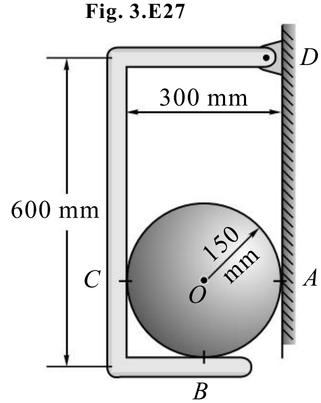


Fig. 3.E28

29. A square block of wood of mass M is hinged at A and rests on a roller at B as shown in Fig. 3.E29. It is pulled by means of a string attached at D and inclined at an angle 30° with the horizontal. Determine the force P required to be applied to string to just lift the block off the roller.

[**Ans.** $P = 0.366\ Mg$]

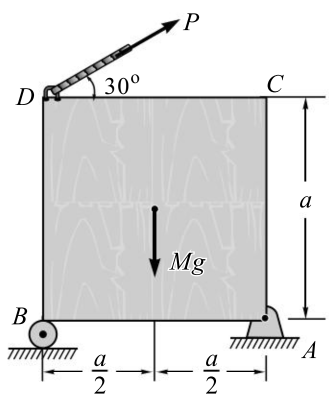


Fig. 3.E29

30. Figure 3.E30 shows several identical smooth rollers of weight W each stacked on an inclined plane. Determine (a) the maximum number of rollers which will lie in a single row as shown and (b) all forces acting on roller A under condition (a).

[***Ans.*** (a) 6 Nos
(b) $3.11\ W$ (θ) $\theta = 45^\circ$,
$0.062\ W$ (θ) $\theta = 60^\circ$,
$2.5\ W$ (θ) $\theta = 30^\circ$ and W ($\downarrow$).]

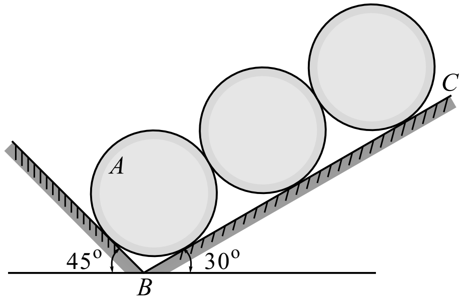


Fig. 3.E30

31. Two prismatic bar AB and CD are welded together in the form of a rigid T and are suspended in a vertical plane as shown in the Fig. 3.E31. Determine the angle θ that the bar will make with the vertical when a load of 100 N is applied at the end D. Two bars are identical and each weighing 50 N.

[***Ans.*** $\theta = 15.86^\circ$]

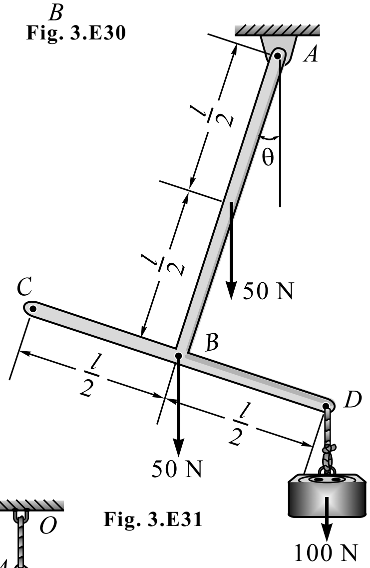


Fig. 3.E31

32. Two bars AB and BC of length 1 m and 2 m and weights 100 N and 200 N respectively, are rigidly joined at B and suspended by a string AO as shown in Fig. 3.E32. Find the inclination θ of the bar BC to the horizontal when the system is in equilibrium.

[***Ans.*** $\theta = 19.1^\circ$]

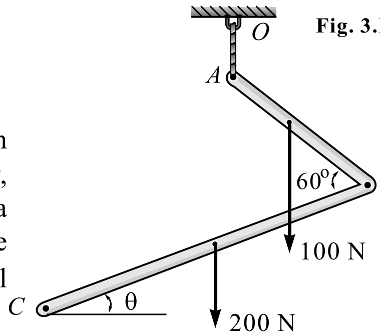


Fig. 3.E32

33. A pulley of 1 m radius, supporting a load of 500 N, is mounted at B on a horizontal beam as shown in Fig. 3.E33. If the beam weighs 200 N and pulley weighs 50 N, find the hinge force at C.

[***Ans.*** $R_C = 472$ N 32°]

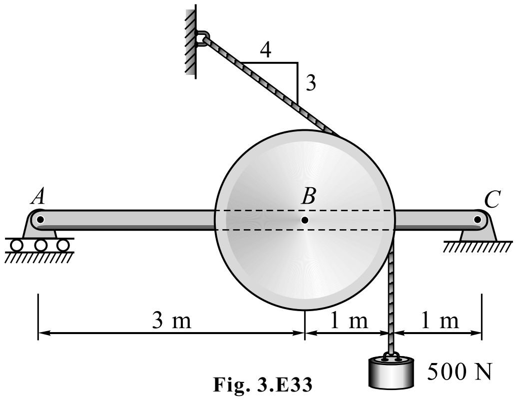


Fig. 3.E33

34. Determine the reactions at all the supports of the beam/ structure shown in Fig. 3.E34.

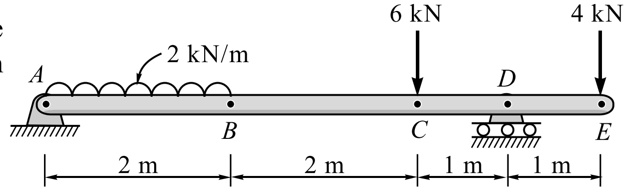


Fig. 3.E34

$$\left[\textbf{Ans.}\quad V_A = 3.6\text{ kN }(\uparrow),\quad V_D = 10.4\text{ kN }(\uparrow)\text{ and } H_A = 0.\right]$$

35. Determine the reactions at all the supports of the beam/ structure shown in Fig. 3.E35.

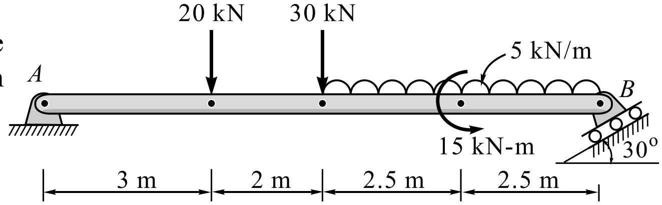


Fig. 3.E35

$$\left[\textbf{Ans.}\quad R_B = 44.17\text{ kN }(60^\circ),\quad V_A = 36.75\text{ kN }(\uparrow)\text{ and } H_A = 22.1\text{ kN }(\rightarrow).\right]$$

36. Determine the reactions at all the supports of the beam shown in Fig. 3.E36.

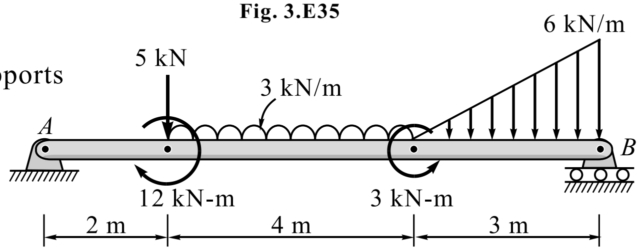


Fig. 3.E36

$$\left[\textbf{Ans.}\quad H_A = 0,\quad V_A = 10.56\text{ kN }(\uparrow)\text{ and } R_B = 15.44\text{ kN }(\uparrow).\right]$$

37. Determine the reactions at all the supports of the beam shown in Fig. 3.E37.

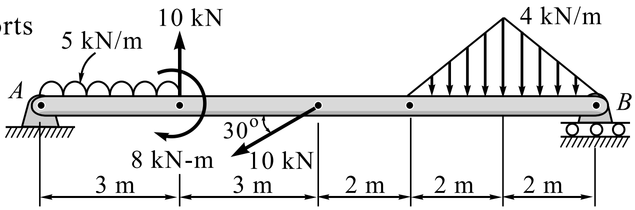


Fig. 3.E37

$$\left[\textbf{Ans.}\quad H_A = 8.66\text{ kN }(\rightarrow),\quad V_A = 8.79\text{ kN }(\uparrow)\text{ and } V_B = 9.21\text{ kN }(\uparrow).\right]$$

38. Determine the reactions at all the supports of the beam shown in Fig. 3.E38.

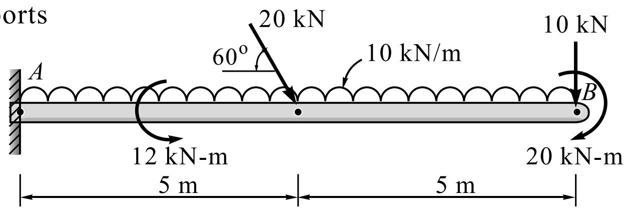


Fig. 3.E38

$$\left[\textbf{Ans.}\quad H_A = 10\text{ kN }(\leftarrow),\quad V_A = 127.32\text{ kN }(\uparrow)\text{ and } M_A = 694.6\text{ kNm }(\circlearrowleft).\right]$$

39. Calculate the reactions at A and B for the beam subjected to two linearly distributed loads as shown in Fig. 3.E39.

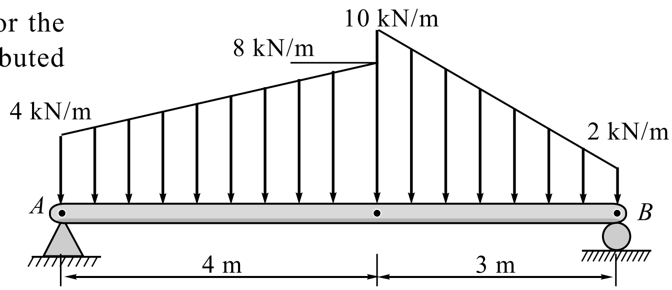


Fig. 3.E39

$$\left[\textbf{Ans.}\quad H_A = 0,\quad V_A = 21.1\text{ kN }(\uparrow)\text{ and } V_B = 20.9\text{ kN }(\uparrow).\right]$$

40. The beams AB and CF are arranged as shown in Fig. 3.E40. Determine the reactions at A, C and D due to the forces acting on the beam.

Ans. H_A = 28.28 kN (←),
V_A = 9.43 kN (↓),
R_D = 40.92 kN (↑),
H_C = 10 kN (→) and
V_C = 34.12 kN (↓).

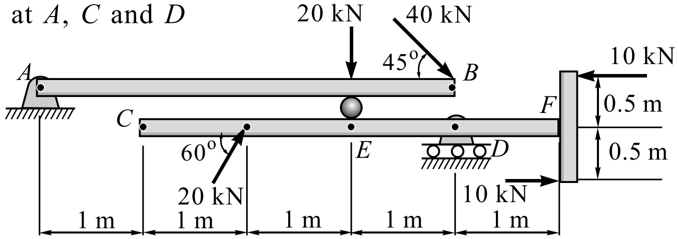


Fig. 3.E40

41. A beam weighs 4 kN/m is loaded and supported as shown in the Fig. 3.E41. Find the support reactions.

Ans. V_A = 70.81 kN (↑),
H_A = 5 kN (→) and
R_A = 72.15 kN (↓).

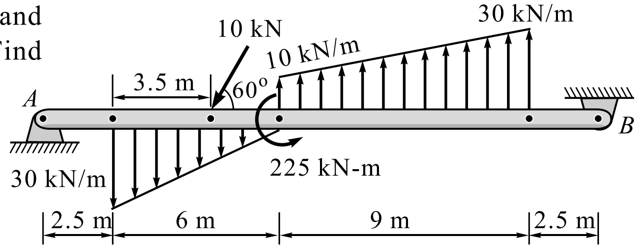


Fig. 3.E41

[II] Review Questions

1. What are the conditions of equilibrium for concurrent, parallel and general force system ?
2. Describe FBD and its importance in the analysis of problems.
3. Explain the types of supports and indicate the unknown reactions they offer.
4. What are the different types of loads ?
5. How do you identify the two-force member in a structure ?
6. Explain three-force member principle.
7. State and prove Lami's theorem.

[III] Fill in the Blanks

1. If a system is in equilibrium and acted by two forces, then these two forces must be ________ in magnitude, ________ in direction and collinear in action.
2. If three concurrent force system is in equilibrium, then the resultant of two forces should be ________ and _______ to the third force.
3. Full form of UDL is ___________________ and that of UVL is ___________________.
4. Fixed support is also named as ________ support.
5. Tensile force of a straight member in FBD is represented by drawing an arrow ________ from joint or body.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. When the resultant of force system acting on a body is zero, the body is said to be in ____.

(a) rotating condition **(b)** translation condition

(c) equilibrium condition **(d)** dynamic condition

2. Free body diagram represents _________.

(a) active forces **(b)** reactive forces **(c)** geometrical dimensions **(d)** all of these

3. Principle of transmissibility is used to represent _______ effect.

(a) internal **(b)** external **(c)** internal and external **(d)** none of these

4. Number of component of reaction at hinge support are ________.

(a) zero **(b)** one **(c)** two **(d)** three

5. In FBD a cable is always represented by ________ force.

(a) spring **(b)** compressive **(c)** tensile **(d)** normal

6. Freely sliding guide or collar is of ________ category support.

(a) fixed **(b)** hinge **(c)** roller **(d)** welded

7. Types of distributed load are ________.

(a) point load **(b)** uniformly distributed load

(c) uniformly varying load **(d)** both (b) and (c)

8. "If three concurrent coplanar forces acting on a body having same nature are in equilibrium, then each force is proportional to the sine of angle included between the other two forces". This is called ________.

(a) Newton's law **(b)** parallelogram law **(c)** Varignon's theorem **(d)** Lami's theorem

9. If a member is connected to other member at its extreme end by pin and no external force or couple is acting in between its end point then such member is identified as a ______ member.

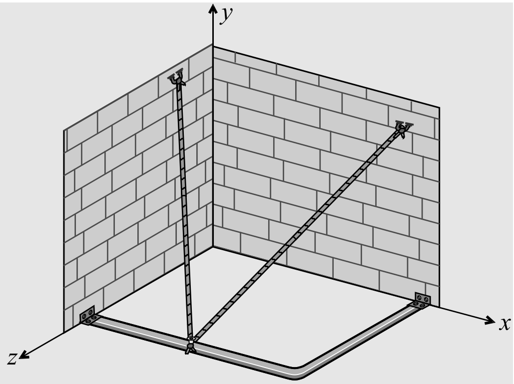
(a) one-force **(b)** two-force **(c)** three-force **(d)** four-force

10. The tension on both side of cord for frictionless and massless pulley are ________.

(a) same **(b)** different **(c)** infinite **(d)** zero

4

FORCES IN SPACE



4.1 Introduction

We know that there are two types of force system, viz. coplanar force system and non-coplanar force system. In the earlier chapters, our discussion was limited to coplanar force system where the line of action of force lies in same plane, which is a two-dimensional force system. In this chapter, we will deal with non-coplanar force system where the line of action of force lies in different planes which forms a three-dimensional force system. Such a system is also called a *space force system*. This chapter can be effectively exercised by vector approach.

4.2 Vectors

1. Basic Vector Operation

$$\bar{F}_1 = x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}$$

$$\bar{F}_2 = x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}$$

(a) *Dot Product*

$$\bar{F}_1 \cdot \bar{F}_2 = x_1x_2 + y_1y_2 + z_1z_2$$

(b) *Cross Product*

$$\bar{F}_1 \times \bar{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

2. Force Vector ($\overline{F}$)

If the line of action of force (F) in space passes through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then find the force vector $(\overline{F})$ by multiplying magnitude of force (F) and unit vector AB $(\overline{e}_{AB})$:

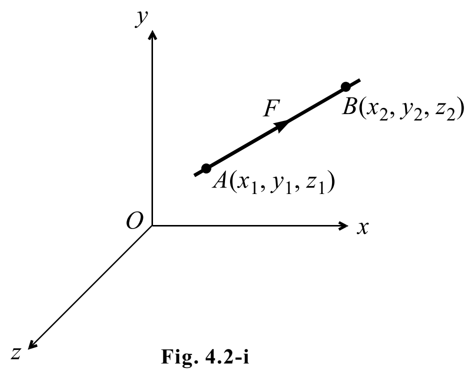


Fig. 4.2-i

$$\overline{F} = (F)(\overline{e}_{AB})$$

$$\overline{F} = (F)\left[\frac{(x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}\right]$$

$$\overline{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

Note : **i, j** and **k** printed in bold type represents unit vectors along x, y and z axis, respectively. Also all vector quantities in this entire chapter are printed in bold italic type with bar.

3. Moment Vector
(Moment of Force About a Given Point)

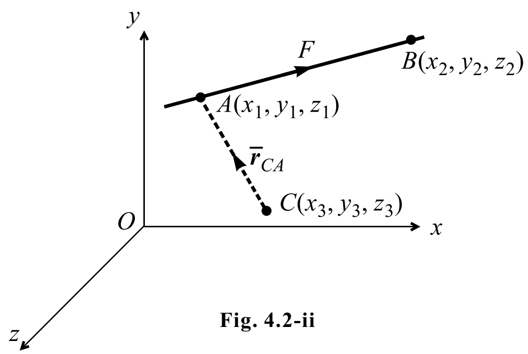


Fig. 4.2-ii

(a) Force Vector ($\overline{F}$)

Find the force vector as explained in previous article.

$$\overline{F} = (F)(\overline{e}_{AB})$$

$$\overline{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

(b) Position Vector ($\overline{r}_{CA}$)

Find the position vector $\overline{r}_{CA}$ extending from the moment centre C to any point on the line of action of force (A or B) :

$$\overline{r}_{CA} = (x_1 - x_3)\,\mathbf{i} + (y_1 - y_3)\,\mathbf{j} + (z_1 - z_3)\,\mathbf{k}$$

$$\overline{r}_{CA} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

(c) Moment Vector ($\overline{M}_C$)

Take the cross product of the position vector $\overline{r}_{CA}$ and the force vector $\overline{F}$ to obtain the moment vector $\overline{M}_C$:

$$\overline{M}_C = \overline{r}_{CA} \times \overline{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\overline{M}_C = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_z\,\mathbf{k}$$

where M_x, M_y and M_z are the component of $\overline{M}_C$ along x, y, and z axis, respectively.

Note : If the moment of force is required about the coordinate axis, then take the moment about origin, i.e., $\overline{M}_O = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_z\,\mathbf{k}$ where M_x, M_y and M_z are the moment of the given force about x, y and z axis, respectively.

4. Vector Component of Force Along a Given Line

(a) Force Vector ($\overline{F}$)

$$\overline{F} = (F)(\overline{e}_{AB})$$

$$\therefore \quad \overline{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

(b) Unit Vector ($\overline{e}_{CD}$)

Find the unit vector of the given line CD along which the vector component is required.

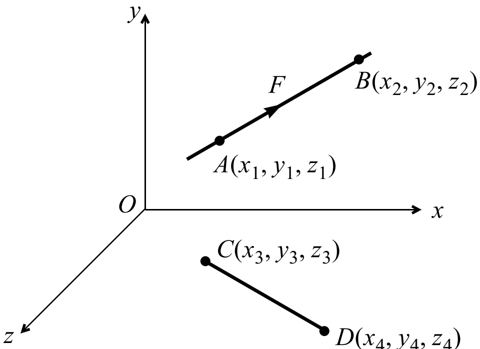


Fig. 4.2-iii

$$\overline{e}_{CD} = \frac{(x_4 - x_3)\,\mathbf{i} + (y_4 - y_3)\,\mathbf{j} + (z_4 - z_3)\,\mathbf{k}}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}}$$

$$\overline{e}_{CD} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

(c) Scalar Component (F_{CD})

Find the magnitude of force along the given line (scalar component) by taking the dot product of force vector $\overline{F}$ and unit vector of the given line $\overline{e}_{CD}$

$$F_{CD} = \overline{F} \cdot \overline{e}_{CD}$$

(d) Vector Component ($\overline{F}_{CD}$)

Find the vector component of force along the given line by multiplying scalar component F_{CD} and the unit vector of the given line $\overline{e}_{CD}$.

$$\overline{F}_{CD} = (F_{CD})(\overline{e}_{CD})$$

5. Moment of Force About a Given Line (or Axis)

(a) Force Vector ($\overline{F}$)

$$\overline{F} = (F)\,(\overline{e}_{AB})$$

$$\overline{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

(b) Moment Vector ($\overline{M}_C$)

Find the moment of force about any point (C or D) on the given line as explained in **3** above.

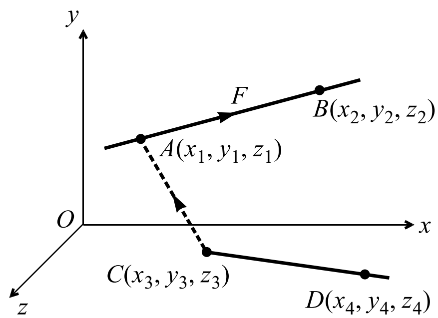


Fig. 4.2-iv

(i) $\overline{r}_{CA} = (x_1 - x_3)\,\mathbf{i} + (y_1 - y_3)\,\mathbf{j} + (z_1 - z_3)\,\mathbf{k}$

$$\overline{r}_{CA} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

(ii) $\overline{M}_C = \overline{r}_{CA} \times \overline{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$

$$\overline{M}_C = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_z\,\mathbf{k}$$

(c) Unit Vector ($\bar{e}_{CD}$)

Find the unit vector of the given line *CD* along which the vector component of moment is required.

$$\bar{e}_{CD} = \frac{(x_4 - x_3)\,\mathbf{i} + (y_4 - y_3)\,\mathbf{j} + (z_4 - z_3)\,\mathbf{k}}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}}$$

(d) Scalar Component (M_{CD})

Find the magnitude of moment along the given line (scalar component) by taking the dot product of moment vector $\bar{M}_C$ and unit vector of the given line $\bar{e}_{CD}$.

$$M_{CD} = \bar{M}_C \cdot \bar{e}_{CD}$$

(e) Vector Component ($\bar{M}_{CD}$)

Find the vector component of moment along the given line by multiplying scalar component M_{CD} and unit vector of the given line $\bar{e}_{CD}$.

$$\bar{M}_{CD} = (M_{CD})(\bar{e}_{CD})$$

6. Magnitude of Force and Direction Angles

If F is magnitude of force making angles θ_x, θ_y, θ_z with x, y and z respectively, then their components are given as follows :

$$F_x = F \cos \theta_x$$

$$F_y = F \cos \theta_y$$

$$F_z = F \cos \theta_z$$

Force in vector form is

$$\bar{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

Magnitude of force $F = \sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2}$

Aangle's θ_x, θ_y and θ_z are known as the *force directions* and their values lie between 0 to 180°. By direction cosine rule, we have

$$\mathbf{\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1}$$

Note : If the value of force components is negative, then corresponding angle is more than 90°.

7. Rectangular Component of Force

For the purpose of analysis, it is desirable to resolve a force $\bar{F}$ in space into three components F_x , F_y and F_z along three mutually perpendicular x, y and z axes respectively, F_x , F_y and F_z are scalars.

Vectors of unit magnitude along the positive x, positive y and positive z axes are known as unit vectors. These unit vectors are denoted by i, j and k respectively.

Hence the force vector $\bar{F}$ is represented as

$$\bar{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

F_x, F_y and F_z are the three rectangular scalar components of a force $\overline{\boldsymbol{F}}$.

Resolving the force vector $\boldsymbol{F}$ along x, y and z axes we have

$$F_x = F\cos\theta_x \; ; \; F_y = F\cos\theta_y \; ; \; F_z = F\cos\theta_z$$

θ_x is the angle between force vector $\overline{\boldsymbol{F}}$ and the positive x-axis

θ_y is the angle between force vector $\overline{\boldsymbol{F}}$ and the positive y-axis

θ_z is the angle between force vector $\overline{\boldsymbol{F}}$ and the positive z-axis.

The cosines of the angles θ_x, θ_y and θ_z are known as *direction cosines* of the force vector $\overline{\boldsymbol{F}}$.

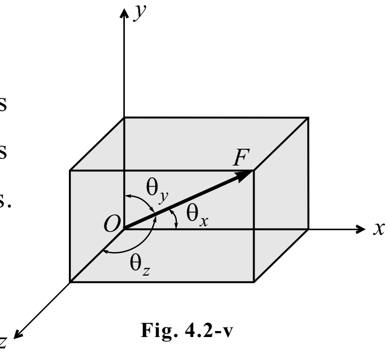


Fig. 4.2-v

We have the following relations :

(i) Magnitude of force $F = \sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2}$

(ii) Unit vector $e = \sqrt{(e_x)^2 + (e_y)^2 + (e_z)^2}$

(iii) Direction cosine rule : $\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$

4.3 Solved Problems

Problem 1

Resolve the given force (shown in Fig. 4.1) into components along x, y and z axes and also express in vectorial form.

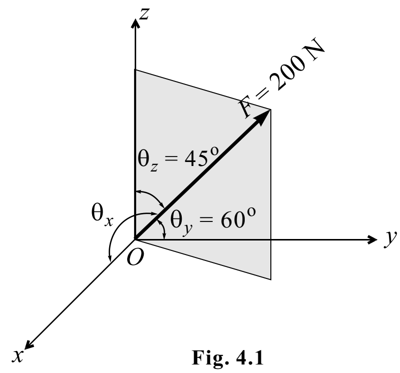


Fig. 4.1

Solution

Given magnitude of force $F = 200$ N

$$\theta_y = 60^\circ$$

$$\theta_z = 45^\circ$$

By direction cosine rule, we have

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$\cos^2\theta_x + \cos^2(60^\circ) + \cos^2(45^\circ) = 1$$

$$\therefore \cos^2\theta_x = 0.25$$

$$\therefore \cos\theta_x = \pm 0.5$$

$$\therefore \cos\theta_x = 0.5 \quad \text{or} \quad \cos\theta_x = -0.5$$

$$\therefore \theta_x = 60^\circ \quad \text{or} \quad \theta_x = 120^\circ$$

From Fig. 4.1, $\theta_x = 60^\circ$ because F_x is in +ve direction.

$F_x = F\cos\theta_x$	$F_y = F\cos\theta_y$	$F_z = F\cos\theta_z$
$F_x = 200\cos 60^\circ$	$F_y = 200\cos 60^\circ$	$F_z = 200\cos 45^\circ$
$F_x = 100$ N	$F_y = 100$ N	$F_z = 141.42$ N

$\overline{\boldsymbol{F}} = 100\,\mathbf{i} + 100\,\mathbf{j} + 141.42\,\mathbf{k}$ ***Ans.***

Problem 2

Force of 800 N acts along AB, $A(3, 2, -4)$ and $B(8, -5, 6)$. Write the force vector.

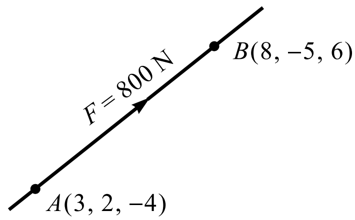


Fig. 4.2

Solution

Unit vector $\bar{e}_{AB}$

$$\bar{e}_{AB} = \frac{(8-3)\,\mathbf{i} + (-5-2)\,\mathbf{j} + (6-(-4))\,\mathbf{k}}{\sqrt{(5)^2 + (-7)^2 + (10)^2}}$$

$$\bar{e}_{AB} = \frac{5\,\mathbf{i} - 7\,\mathbf{j} + 10\,\mathbf{k}}{\sqrt{174}}$$

$$\bar{F}_{AB} = (F)(\bar{e}_{AB}) = (800)\left(\frac{5\,\mathbf{i} - 7\,\mathbf{j} + 10\,\mathbf{k}}{\sqrt{174}}\right)$$

$$\bar{F}_{AB} = 303.24\,\mathbf{i} - 424.54\,\mathbf{j} + 606.48\,\mathbf{k} \quad \textbf{\textit{Ans.}}$$

Problem 3

Force of magnitude 50 kN is acting at point $A(2, 3, 4)$ m towards point $B(6, -2, -3)$ m. Find **(i)** Vector component of this force along the line AC. Point C is (5, 1, 2) m and **(ii)** Moment of the given force about a point $D(-1, 1, 2)$ m.

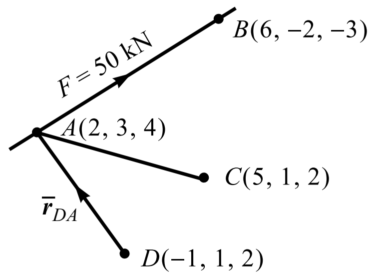


Fig. 4.3

Solution

(i) Vector component of force along the line AC ($\bar{F}_{AC}$)

(a) Force vector ($\bar{F}$)

$$\bar{F} = (F)(\bar{e}_{AB})$$

$$\bar{F} = (50)\left(\frac{4\,\mathbf{i} - 5\,\mathbf{j} - 7\,\mathbf{k}}{\sqrt{4^2 + (-5)^2 + (-7)^2}}\right)$$

$$\bar{F} = 21.08\,\mathbf{i} - 26.35\,\mathbf{j} - 36.89\,\mathbf{k}$$

(b) Unit vector ($\bar{e}_{AC}$)

$$\bar{e}_{AC} = \frac{\overline{AC}}{|\overline{AC}|} = \frac{3\,\mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k}}{\sqrt{3^2 + 2^2 + 2^2}}$$

$$\bar{e}_{AC} = 0.727\,\mathbf{i} - 0.485\,\mathbf{j} - 0.485\,\mathbf{k}$$

(c) Scalar component (F_{AC})
(Magnitude of force along given line AC)

$$F_{AC} = \bar{F} \cdot \bar{e}_{AC} \qquad \text{(Dot product)}$$

$$F_{AC} = (21.08\,\mathbf{i} - 26.35\,\mathbf{j} - 36.89\,\mathbf{k}) \cdot (0.727\,\mathbf{i} - 0.485\,\mathbf{j} - 0.485\,\mathbf{k})$$

$$F_{AC} = 44.95 \text{ kN}$$

(d) Vector component of force along the line AC ($\overline{F}_{AC}$)

$$\overline{F}_{AC} = (F_{AC})(\overline{e}_{AC})$$

$$\overline{F}_{AC} = (44.95)(0.727\,\mathbf{i} - 0.485\,\mathbf{j} - 0.485\,\mathbf{k})$$

$$\overline{F}_{AC} = 33.44\,\mathbf{i} - 22.31\,\mathbf{j} - 22.31\,\mathbf{k}\ \text{(kN)} \quad \textbf{\textit{Ans.}}$$

(ii) Moment of $\overline{F}$ about a point D ($\overline{M}_D$)

(a) Position vector ($\overline{r}_{DA}$)

$$\overline{r}_{DA} = \overline{DA}$$

$$\overline{r}_{DA} = 3\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$$

(b) Force vector ($\overline{F}$)

$$\overline{F} = 21.08\,\mathbf{i} - 26.35\,\mathbf{j} - 36.89\,\mathbf{k}$$

(c) Moment of $\overline{F}$ about a point D ($\overline{M}_D$)

$$\overline{M}_D = \overline{r}_{DA} \times \overline{F} \quad \text{(cross product)}$$

$$\overline{M}_D = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 21.08 & -26.35 & -36.89 \end{vmatrix}$$

$$\overline{M}_D = -21.08\,\mathbf{i} + 152.83\,\mathbf{j} - 121.21\,\mathbf{k}\ \text{(kN-m)} \quad \textbf{\textit{Ans.}}$$

Problem 4

A force $\overline{F}$ = (3 i – 4 j + 12 k) N acts at a point A, whose coordinates are (1, –2, 3) m. Find **(i)** Moment of the force about the origin. **(ii)** Moment of the force about the point B(2, 1, 2) m. **(iii)** The vector component of the force $\overline{F}$ along line AB and the moment of this force about the origin.

Solution

(i) Moment of $\overline{F}$ about the origin O ($\overline{M}_O$)

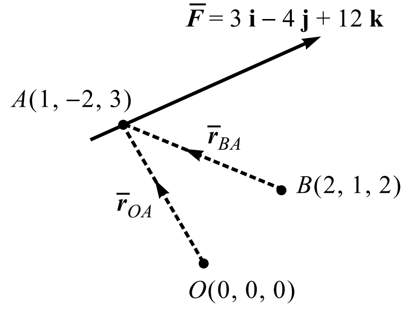


Fig. 4.4

(a) Position vector ($\overline{r}_{OA}$)

$$\overline{r}_{OA} = \overline{OA}$$

$$\overline{r}_{OA} = \mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$$

(b) Force vector ($\overline{F}$)

$$\overline{F} = 3\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}$$

(c) Moment vector ($\overline{M}_O$)

$$\overline{M}_O = \overline{r}_{OA} \times \overline{F} \quad \text{(cross product)}$$

$$\overline{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 3 & -4 & 12 \end{vmatrix}$$

$\overline{M}_O$ = $-12\,\mathbf{i} - 3\,\mathbf{j} + 2\,\mathbf{k}$ (N-m) ***Ans.***

(ii) Moment of $\overline{F}$ about the point B ($\overline{M}_B$)

(a) Position vector ($\overline{r}_{BA}$)

$$\overline{r}_{BA} = \overline{BA}$$

$$\overline{r}_{BA} = -\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$$

(b) Force vector ($\overline{F}$)

$$\overline{F} = 3\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}$$

(c) Moment vector ($\overline{M}_B$)

$\overline{M}_B = \overline{r}_{BA} \times \overline{F}$ (cross product)

$$\overline{M}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & 1 \\ 3 & -4 & 12 \end{vmatrix}$$

$\overline{M}_B = -32\,\mathbf{i} + 15\,\mathbf{j} + 13\,\mathbf{k}$ (N-m)

(iii) (I) Vector component of force $\overline{F}$ along the line AB ($\overline{F}_{AB}$)

(a) Unit vector ($\overline{e}_{AB}$)

$$\overline{e}_{AB} = \frac{\overline{AB}}{|\overline{AB}|} = \frac{\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}}{\sqrt{1^2 + 3^2 + 1^2}}$$

$$\overline{e}_{AB} = 0.3015\,\mathbf{i} + 0.9045\,\mathbf{j} - 0.3015\,\mathbf{k}$$

(b) Scalar component of $\overline{F}$ along the line AB (F_{AB})

$F_{AB} = \overline{F} \cdot \overline{e}_{AB}$ (dot product)

$$F_{AB} = (3\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}) \cdot (0.3015\,\mathbf{i} + 0.9045\,\mathbf{j} - 0.3015\,\mathbf{k})$$

$$F_{AB} = -6.3315 \text{ N}$$

(c) Vector component of $\overline{F}$ along the line AB ($\overline{F}_{AB}$)

$$\overline{F}_{AB} = (F_{AB})(\overline{e}_{AB})$$

$$\overline{F}_{AB} = (-6.3315)(0.3015\,\mathbf{i} + 0.9045\,\mathbf{j} - 0.3015\,\mathbf{k})$$

$$\overline{F}_{AB} = -1.909\,\mathbf{i} - 5.727\,\mathbf{j} + 1.909\,\mathbf{k}$$

(II) Moment of $\overline{F}_{AB}$ about the origin O $(\overline{M}_O)_{AB}$

$$(\overline{M}_O)_{AB} = \overline{r}_{OA} \times \overline{F}_{AB} \quad \text{(cross product)}$$

$$(\overline{M}_O)_{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ -1.909 & -5.727 & 1.909 \end{vmatrix}$$

$$(\overline{M}_O)_{AB} = 13.363\ \mathbf{i} - 7.636\ \mathbf{j} - 9.545\ \mathbf{k} \ \text{(N-m)}$$ ***Ans.***

Problem 5

A 500 N force passes through points whose position vectors are $\overline{r}_1 = 10\ \mathbf{i} - 3\ \mathbf{j} + 12\ \mathbf{k}$ and $\overline{r}_2 = 3\ \mathbf{i} - 2\ \mathbf{j} + 5\ \mathbf{k}$. What is the moment of this force about a line in the xy plane, passing through the origin and inclined at an angle of 30° with the x-axis ?

Solution

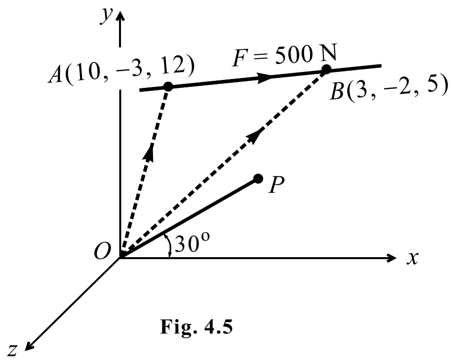


Fig. 4.5

(i) Position vector

$$\overline{r}_1 = \overline{OA} = 10\ \mathbf{i} - 3\ \mathbf{j} + 12\ \mathbf{k}$$

$$\overline{r}_2 = \overline{OB} = 3\ \mathbf{i} - 2\ \mathbf{j} + 5\ \mathbf{k}$$

$\therefore$ Coordinate of point $A(10, -3, 12)$ and $B(3, -2, 5)$.

(ii) Force vector $(\overline{F})$

$$\overline{F} = (500)(\overline{e}_{AB})$$

$$\overline{F} = (500)\left(\frac{-7\ \mathbf{i} + \mathbf{j} - 7\ \mathbf{k}}{\sqrt{7^2 + 1^2 + 7^2}}\right)$$

$$\overline{F} = -351.76\ \mathbf{i} + 50.25\ \mathbf{j} - 351.76\ \mathbf{k}$$

(iii) Moment of $(\overline{F})$ about the origin O $(\overline{M}_O)$

$$\overline{M}_O = \overline{r}_1 \times \overline{F} \quad \text{(cross product)}$$

$$\overline{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & -3 & 12 \\ -351.76 & 50.25 & -351.76 \end{vmatrix}$$

$$\overline{M}_O = 452.28\ \mathbf{i} - 703.52\ \mathbf{j} - 552.78\ \mathbf{k}$$

(iv) Unit vector $(\overline{e}_{OP})$

$$\overline{e}_{OP} = (\cos 30^\circ)\ \mathbf{i} + (\cos 60^\circ)\ \mathbf{j} + (\cos 90^\circ)\ \mathbf{k}$$

(v) Scalar component (M_{OP})

(Magnitude of moment of force about given line OP)

$$M_{OP} = \overline{M}_O \cdot \overline{e}_{OP} \quad \text{(dot product)}$$

$$M_{OP} = (452.28\,\mathbf{i} - 703.52\,\mathbf{j} - 552.78\,\mathbf{k}) \cdot (0.866\,\mathbf{i} + 0.5\,\mathbf{j} + 0\,\mathbf{k})$$

$$M_{OP} = 39.91 \text{ (N-m)}$$

(vi) Vector component ($\overline{\boldsymbol{M}}_{OP}$)

(Moment of force about given line *OP* in vector form)

$$\overline{\boldsymbol{M}}_{OP} = (M_{OP})\,(\bar{\boldsymbol{e}}_{OP})$$

$$\overline{\boldsymbol{M}}_{OP} = (39.91)(0.866\,\mathbf{i} + 0.5\,\mathbf{j})$$

$$\overline{\boldsymbol{M}}_{OP} = 34.56\,\mathbf{i} + 19.96\,\mathbf{j} \text{ (N-m)}$$ ***Ans.***

Problem 6

A rectangular platform *OCDE* is hinged to a vertical wall at *A* and *B* and supported by a cable which passes over a smooth hook at *F*. (Refer to Fig. 4.6). If the tension in the cable is 355 N, find the moment about each of the coordinate axes of the force exerted by the cable at *D*.

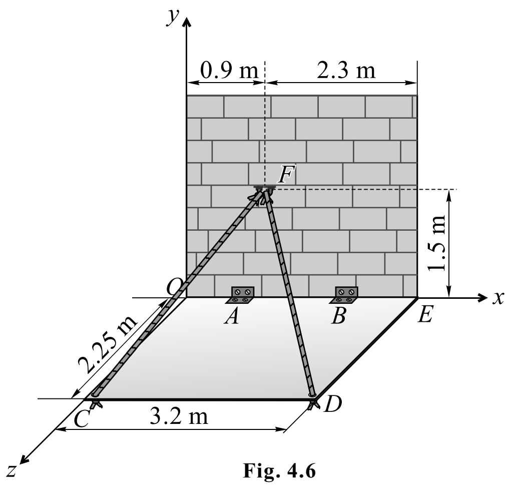


Fig. 4.6

Solution

(i) Coordinates

$O(0, 0, 0)$, $D(3.2, 0, 2.25)$, $F(0.9, 1.5, 0)$

(ii) Force vector

$$\overline{\boldsymbol{F}} = (355)(\bar{\boldsymbol{e}}_{DF})$$

$$\overline{\boldsymbol{F}} = (355)\left[\frac{-2.3\,\mathbf{i} + 1.5\,\mathbf{j} - 2.25\,\mathbf{k}}{\sqrt{2.3^2 + 1.5^2 + 2.25^2}}\right]$$

$$\overline{\boldsymbol{F}} = -230\,\mathbf{i} + 150\,\mathbf{j} - 225\,\mathbf{k}$$

(iii) Position vector

$$\bar{\boldsymbol{r}}_{OD} = 3.2\,\mathbf{i} + 2.25\,\mathbf{k}$$

(iv) Moment vector

$$\overline{\boldsymbol{M}}_O = \bar{\boldsymbol{r}}_{OD} \times \overline{\boldsymbol{F}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.2 & 0 & 2.25 \\ -230 & 150 & -225 \end{vmatrix}$$

$$\overline{\boldsymbol{M}}_O = -337.5\,\mathbf{i} + 202.5\,\mathbf{j} + 480\,\mathbf{k} \text{ (N-m)}$$

(v) Moment of force exerted by cable at point *D* about the coordinate axis are as follows :

$$M_x = -337.5 \text{ (N-m)}$$

$$M_y = 202.5 \text{ (N-m)}$$

$$M_z = 480 \text{ (N-m)}$$ ***Ans.***

Problem 7

Tension T of magnitude 15 kN is applied to a cable AB attached to the top A of a rigid mass and secured to the ground at B. Determine the moment M of tension T about z-axis passing through the base O. Refer Fig. 4.7.

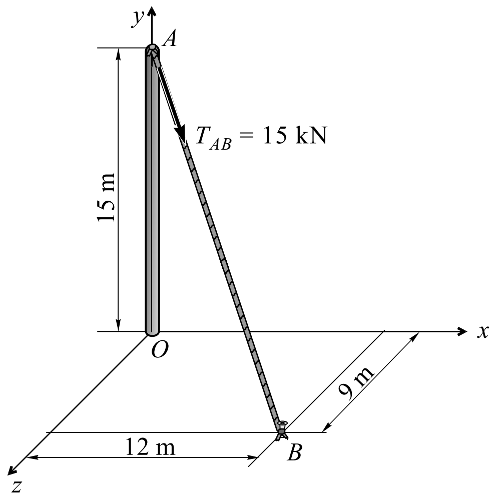


Fig. 4.7

Solution

(i) Coordinates

$O(0, 0, 0)$, $A(0, 15, 0)$, $B(12, 0, 9)$

(ii) Force vector ($\overline{\boldsymbol{T}}_{AB}$)

$$\overline{\boldsymbol{T}}_{AB} = (15)(\overline{\boldsymbol{e}}_{AB})$$

$$= (15)\left[\frac{(12-0)\,\mathbf{i} + (0-15)\,\mathbf{j} + (5-0)\,\mathbf{k}}{\sqrt{12^2 + (-15)^2 + 5^2}}\right]$$

$$\overline{\boldsymbol{T}}_{AB} = 9.07\,\mathbf{i} - 11.335\,\mathbf{j} + 6.801\,\mathbf{k}\ \text{(kN)}$$

(iii) Moment of the force $\overline{\boldsymbol{T}}_{AB}$ about z-axis ($\overline{\boldsymbol{M}}_O$)

$$\overline{\boldsymbol{M}}_O = \overline{\boldsymbol{r}}_{OA} \times \overline{\boldsymbol{T}}_{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 15 & 0 \\ 9.07 & -11.335 & 6.801 \end{vmatrix}$$

$$\overline{\boldsymbol{M}}_O = 102.02\,\mathbf{i} - 0\,\mathbf{j} - 136.05\,\mathbf{k}$$

(iv) Moment of the force about the z-axis

$$\overline{\boldsymbol{M}}_O \cdot \overline{\boldsymbol{e}}_{z\text{-axis}} = (102.02\,\mathbf{i} - 0\,\mathbf{j} - 136.05\,\mathbf{k}) \cdot (\mathbf{k})$$

$$= -136.05\ \text{kN-m}$$ ***Ans.***

Problem 8

A force acts at the origin in a direction defined by the angle $\theta_y = 65^\circ$ and $\theta_z = 40^\circ$. Knowing that the x-component of the force is -750 kN, determine **(i)** the value of θ_x, **(ii)** magnitude of the force and **(iii)** the other component.

Solution

(i) By direction cosine rule, we have

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$\cos^2\theta_x + \cos^2 65^\circ + \cos^2 40^\circ = 1$$

$$\cos^2\theta_x = 0.234$$

Consider the negative value because $\overline{\boldsymbol{F}}_x = -750$ kN, which is negative

$$\cos\theta_x = -0.484$$

$$\therefore\ \theta_x = 118.95^\circ$$ ***Ans.***

(ii) x-component of a force (F_x)

$$F_x = F \cos \theta_x$$

$$-750 = F \cos 118.95^\circ$$

$$F = 1549.44 \text{ kN} \quad \textbf{\textit{Ans.}}$$

(iii) $F_y = F \cos \theta_y$ $\qquad$ $F_z = F \cos \theta_z$

$$F_y = 1549.44 \cos 65^\circ \qquad F_z = 1549.44 \cos 40^\circ$$

$$F_y = 654.82 \text{ kN} \quad \textbf{\textit{Ans.}} \qquad F_z = 1186.94 \text{ kN} \quad \textbf{\textit{Ans.}}$$

Problem 9

A force of 5 kN is acting along AB, where position vector of point A is $\overline{\boldsymbol{r}}_A = (2\,\mathbf{i} + 3\,\mathbf{j} - 5\,\mathbf{k})$ m and $\overline{\boldsymbol{r}}_B = (5\,\mathbf{j} - 3\,\mathbf{k})$ m. Another force acting at point B 6 kN and it makes angles 30° and 75° with x and y axes respectively. Find **(i)** resultant of the two forces, **(ii)** component of resultant along line BD where coordinates of D are (3, 2, –1) m and **(iii)** moment of the resultant about line CD where coordinate of C are (–2, 5, 0) m.

Solution

(i) Resultant of the two forces

(a) Coordinates

$A(2, 3, -5)$ m, $B(0, 5, -3)$ m

(b) Force vector ($\overline{\boldsymbol{F}}_1$)

Magnitude $F_1 = 5$ kN

$$\overline{\boldsymbol{F}}_1 = (5)(\overline{\boldsymbol{e}}_{AB})$$

$$\overline{\boldsymbol{F}}_1 = (5)\left[\frac{-2\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}}{\sqrt{2^2 + 2^2 + 2^2}}\right]$$

$$\overline{\boldsymbol{F}}_1 = -2.89\,\mathbf{i} + 2.89\,\mathbf{j} + 2.89\,\mathbf{k}$$

(c) Force vector ($\overline{\boldsymbol{F}}_2$)

Magnitude $F_2 = 6$ kN ; $\theta_x = 30^\circ$, $\theta_y = 75^\circ$

By direction cosine rule, we have

$$\cos \theta_z = +\sqrt{1 - \cos^2 \theta_x - \cos^2 \theta_y}$$

$$\therefore \theta_z = 64.67^\circ$$

$$\overline{\boldsymbol{F}}_2 = (6)\left[(\cos 30^\circ)\,\mathbf{i} + (\cos 75^\circ)\,\mathbf{j} + (\cos 64.67^\circ)\right]\mathbf{k}$$

$$\overline{\boldsymbol{F}}_2 = 5.196\,\mathbf{i} + 1.553\,\mathbf{j} + 2.567\,\mathbf{k}$$

(d) Resultant force vector

$$\overline{\boldsymbol{R}} = \overline{\boldsymbol{F}}_1 + \overline{\boldsymbol{F}}_2$$

$$\overline{\boldsymbol{R}} = 2.306\,\mathbf{i} + 4.443\,\mathbf{j} + 5.457\,\mathbf{k} \text{ (kN)} \quad \textbf{\textit{Ans.}}$$

(ii) Component of resultant along line *BD* (R_{BD})

(a) Coordinates

$B(0, 5, -3)$ m , $D(3, 2, 1)$ m

(b) Unit vector ($\bar{e}_{BD}$)

$$\bar{e}_{BD} = \frac{\overline{BD}}{|\overline{BD}|} = \frac{3\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{3^2 + 3^2 + 4^2}}$$

$$\bar{e}_{BD} = 0.514\,\mathbf{i} - 0.514\,\mathbf{j} + 0.686\,\mathbf{k}$$

(c) Scalar component (R_{BD})

$$R_{BD} = \bar{R} \cdot \bar{e}_{BD} \quad \text{(dot product)}$$

$$R_{BD} = (2.306\,\mathbf{i} + 4.443\,\mathbf{j} + 5.457\,\mathbf{k}) \cdot (0.514\,\mathbf{i} - 0.514\,\mathbf{j} + 0.686\,\mathbf{k})$$

$R_{BD} = 2.645$ kN ***Ans.***

(iii) Moment of resultant about line *CD* (M_{CD})

(a) Coordinates

$B(0, 5, -3)$ m ,$C(-2, 5, 0)$ m , $D(3, 2, 1)$ m

(b) Moment vector ($\bar{M}_C$)

Resultant force vector

$$\bar{R} = 2.306\,\mathbf{i} + 4.443\,\mathbf{j} + 5.457\,\mathbf{k}$$

Position vector

$$\bar{r}_{CB} = \overline{CB} = 2\,\mathbf{i} + 0\,\mathbf{j} - 3\,\mathbf{k}$$

$$\bar{M}_C = \bar{r}_{CB} \times \bar{R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 2.306 & 4.44 & 5.457 \end{vmatrix}$$

$$\bar{M}_C = 13.32\,\mathbf{i} - 17.818\,\mathbf{j} + 8.88\,\mathbf{k}$$

(c) Unit vector ($\bar{e}_{CD}$)

$$\bar{e}_{CD} = \frac{\overline{CD}}{|\overline{CD}|} = \frac{5\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}}{\sqrt{5^2 + 3^2 + 1^2}}$$

$$\bar{e}_{BD} = 0.845\,\mathbf{i} - 0.507\,\mathbf{j} + 0.169\,\mathbf{k}$$

(d) Moment of resultant about line CD (M_{CD})

$$M_{CD} = \bar{M}_C \cdot \bar{e}_{CD} \quad \text{(dot product)}$$

$$M_{CD} = (13.32\,\mathbf{i} - 17.818\,\mathbf{j} + 8.88\,\mathbf{k}) \cdot (0.845\,\mathbf{i} - 0.507\,\mathbf{j} + 0.169\,\mathbf{k})$$

$M_{CD} = 21.79$ kN-m ***Ans.***

4.4 Resultant of Concurrent Force System in Space

Resultant of concurrent force system in space is a single force $\overline{\boldsymbol{R}}$ and it acts through point of concurrency.

Refer to Fig. 4.4-i, where $\overline{\boldsymbol{F}}_1$, $\overline{\boldsymbol{F}}_2$, $\overline{\boldsymbol{F}}_3$, $\overline{\boldsymbol{F}}_4$, $\overline{\boldsymbol{F}}_5$ are the force vectors passing through point A.

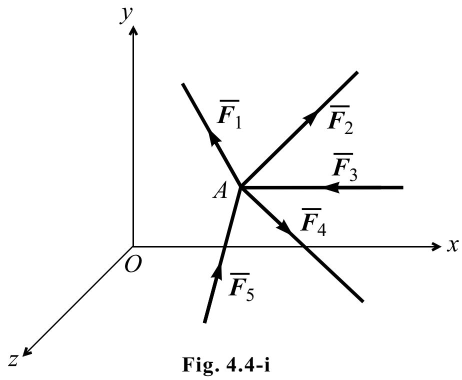


Fig. 4.4-i

Resultant force vector $\overline{\boldsymbol{R}}$ is the summation of all force vectors.

$$\overline{\boldsymbol{R}} = \overline{\boldsymbol{F}}_1 + \overline{\boldsymbol{F}}_2 + \overline{\boldsymbol{F}}_3 + \overline{\boldsymbol{F}}_4 + \overline{\boldsymbol{F}}_5$$

$$\overline{\boldsymbol{R}} = (\Sigma F_x)\,\mathbf{i} + (\Sigma F_y)\,\mathbf{j} + (\Sigma F_z)\,\mathbf{k}$$

$$\overline{\boldsymbol{R}} = R_x\,\mathbf{i} + R_y\,\mathbf{j} + R_z\,\mathbf{k}$$

Magnitude of resultant $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$

Directions

$$\theta_x = \cos^{-1}\left(\frac{R_x}{R}\right)$$

$$\theta_y = \cos^{-1}\left(\frac{R_y}{R}\right)$$

$$\theta_z = \cos^{-1}\left(\frac{R_z}{R}\right)$$

Note :
- If resultant is acting along x-axis then $R_x = \Sigma F_x = R$

  $R_y = \Sigma F_y = 0$ and $R_z = \Sigma F_z = 0$
- If resultant is acting along y-axis then $R_y = \Sigma F_y = R$

  $R_x = \Sigma F_x = 0$ and $R_z = \Sigma F_z = 0$
- If resultant is acting along z-axis then $R_z = \Sigma F_z = R$

  $R_x = \Sigma F_x = 0$ and $R_y = \Sigma F_y = 0$

Problem 10

The lines of actions of three forces concurrent at origin O pass respectively through point A, B, and C having coordinates

$$x_a = 1 \qquad y_a = +2 \qquad z_a = +4$$

$$x_b = +3 \qquad y_b = 0 \qquad z_b = -3$$

$$x_c = +2 \qquad y_c = -2 \qquad z_c = +4$$

The magnitude of the forces are $F_a = 40$ N, $F_b = 10$ N and $F_c = 30$ N. Find the magnitude and direction of their resultant.

Solution

(i) Force vectors

$$\overline{\boldsymbol{F}}_a = (F_a)(\overline{\boldsymbol{e}}_{OA})$$

$$= (40)\left[\frac{\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{1^2 + 2^2 + 4^2}}\right]$$

$$\overline{\boldsymbol{F}}_a = 8.73(\,\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k})$$

$$\overline{\boldsymbol{F}}_a = 8.73\,\mathbf{i} + 17.46\,\mathbf{j} + 34.92\,\mathbf{k}$$

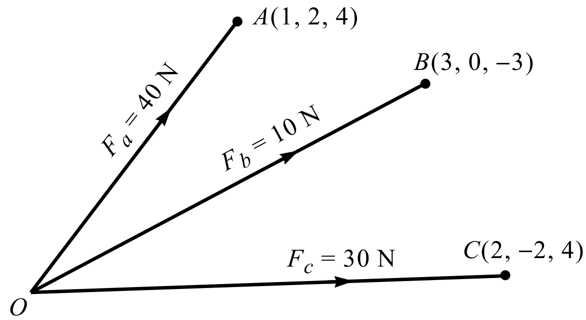


Fig. 4.10

$$\overline{\boldsymbol{F}}_b = (F_b)(\overline{\boldsymbol{e}}_{OB})$$

$$\overline{\boldsymbol{F}}_b = (10)\left[\frac{3\,\mathbf{i} + 0\,\mathbf{j} - 3\,\mathbf{k}}{\sqrt{3^2 + 0^2 + 3^2}}\right]$$

$$\overline{\boldsymbol{F}}_b = 2.36(3\,\mathbf{i} + 0\,\mathbf{j} - 3\,\mathbf{k})$$

$$\overline{\boldsymbol{F}}_b = 7.08\,\mathbf{i} + 0\,\mathbf{j} - 7.08\,\mathbf{k}$$

$$\overline{\boldsymbol{F}}_c = (F_c)(\overline{\boldsymbol{e}}_{OC})$$

$$\overline{\boldsymbol{F}}_c = (30)\left[\frac{2\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{2^2 + 2^2 + 4^2}}\right]$$

$$\overline{\boldsymbol{F}}_c = 6.12(2\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k})$$

$$\overline{\boldsymbol{F}}_c = 12.24\,\mathbf{i} - 12.24\,\mathbf{j} + 24.48\,\mathbf{k}$$

(ii) Resultant vector

$$\overline{\boldsymbol{R}} = \overline{\boldsymbol{F}}_a + \overline{\boldsymbol{F}}_b + \overline{\boldsymbol{F}}_c$$

$$\overline{\boldsymbol{R}} = (8.73 + 7.08 + 12.24)\,\mathbf{i} + (17.46 + 0 - 12.24)\,\mathbf{j} + (34.92 - 7.08 + 24.48)\,\mathbf{k}$$

$$\overline{\boldsymbol{R}} = 28.05\,\mathbf{i} + 5.22\,\mathbf{j} + 53.32\,\mathbf{k}$$

(iii) Magnitude of resultant

$$R = \sqrt{(28.05)^2 + (5.22)^2 + (53.32)^2}$$

$$R = 59.59\text{ N}$$

(iv) Direction of R_x, R_y and R_z

We know for $\overline{\boldsymbol{R}} = R_x\,\mathbf{i} + R_y\,\mathbf{j} + R_z\,\mathbf{k}$

$$\theta_x = \cos^{-1}\left(\frac{R_x}{R}\right) \qquad \theta_y = \cos^{-1}\left(\frac{R_y}{R}\right) \qquad \theta_z = \cos^{-1}\left(\frac{R_z}{R}\right)$$

$$\theta_x = \cos^{-1}\left(\frac{28.05}{59.59}\right) \qquad \theta_y = \cos^{-1}\left(\frac{5.22}{59.59}\right) \qquad \theta_z = \cos^{-1}\left(\frac{53.32}{59.59}\right)$$

$$\theta_x = 61.92^{\circ} \qquad \theta_y = 84.97^{\circ} \qquad \theta_z = 26.52^{\circ}$$

Problem 11

Knowing that the tension in AC is, T_{AC} = 20 kN, determine the required values of tension T_{AB} and T_{AD} so that the resultant of the three forces applied at A is vertical and calculate resultant.

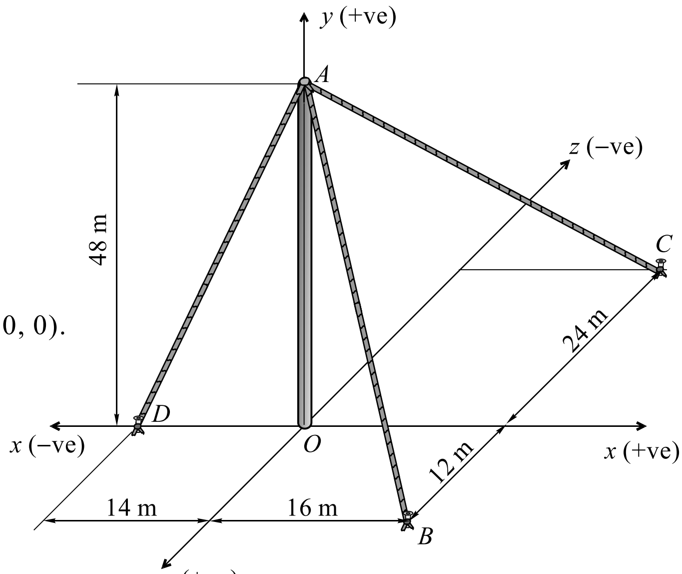


Fig. 4.11

Solution

(i) Coordinate $O(0, 0, 0)$, $A(0, 48, 0)$, $B(16, 0, 12)$, $C(16, 0, -24)$, $D(-14, 0, 0)$.

(ii) Force vector

$$\overline{\boldsymbol{T}}_{AC} = (T_{AC})(\overline{\boldsymbol{e}}_{AC})$$

$$= (20)\left[\frac{16\,\mathbf{i} - 48\,\mathbf{j} - 24\,\mathbf{k}}{\sqrt{16^2 + 48^2 + 24^2}}\right]$$

$$\overline{\boldsymbol{T}}_{AC} = \frac{20(16\,\mathbf{i} - 48\,\mathbf{j} - 24\,\mathbf{k})}{\sqrt{3136}}$$

$$\overline{\boldsymbol{T}}_{AC} = 5.712\,\mathbf{i} - 17.136\,\mathbf{j} - 8.568\,\mathbf{k} \qquad \ldots \text{(I)}$$

$$\overline{\boldsymbol{T}}_{AB} = (T_{AB})(\overline{\boldsymbol{e}}_{AB})$$

$$\overline{\boldsymbol{T}}_{AB} = (T_{AB})\left[\frac{16\,\mathbf{i} - 48\,\mathbf{j} + 12\,\mathbf{k}}{\sqrt{16^2 + 48^2 + 12^2}}\right] = \frac{T_{AB}}{52}(16\,\mathbf{i} - 48\,\mathbf{j} + 12\,\mathbf{k})$$

$$\overline{\boldsymbol{T}}_{AB} = T_{AB}(0.308\,\mathbf{i} - 0.923\,\mathbf{j} + 0.231\,\mathbf{k}) \qquad \ldots \text{(II)}$$

$$\overline{\boldsymbol{T}}_{AD} = (T_{AD})(\overline{\boldsymbol{e}}_{AD})$$

$$\overline{\boldsymbol{T}}_{AD} = (T_{AD})\left[\frac{-14\,\mathbf{i} - 48\,\mathbf{j} + 0\,\mathbf{k}}{\sqrt{14^2 + 48^2 + 0^2}}\right] = \frac{T_{AD}}{50}(-14\,\mathbf{i} - 48\,\mathbf{j} + 0\,\mathbf{k})$$

$$\overline{\boldsymbol{T}}_{AD} = T_{AD}(-0.28\,\mathbf{i} - 0.96\,\mathbf{j} + 0\,\mathbf{k}) \qquad \ldots \text{(III)}$$

(iii) Since the resultant is along y-axis

$\therefore\ \Sigma F_z = 0$ and $\Sigma F_x = 0$

(iv) For $\Sigma F_z = 0$; add **k** terms of Eqs. (I), (II) and (III)

$-8.568 + 0.231\,T_{AB} + 0 = 0 \qquad \therefore\ T_{AB} = 37.09$ kN

(v) For $\Sigma F_x = 0$; add **i** terms of Eqs. (I), (II) and (III)

$5.712 + 0.308\,T_{AB} + (-0.28)\,T_{AD} = 0 \qquad \therefore\ T_{AD} = 61.12$ kN

(vi) Resultant $R = \Sigma F_y$; add **j** terms of Eqs. (I), (II) and (III)

$$R = -17.136 + (-0.923)\,T_{AB} + (-0.96)\,T_{AD} = -17.136 - 0.923 \times 37.09 - 0.96 \times 61.12$$

$R = -110.05$ kN; $\qquad \therefore\ R = 110.05$ kN $(\downarrow)$ ***Ans.***

Problem 12

The resultant of the three concurrent space forces at A is $\overline{\boldsymbol{R}} = (-788\ \mathbf{j})$ N. Find the magnitude of F_1, F_2 and F_3 forces. Refer to Fig. 4.12.

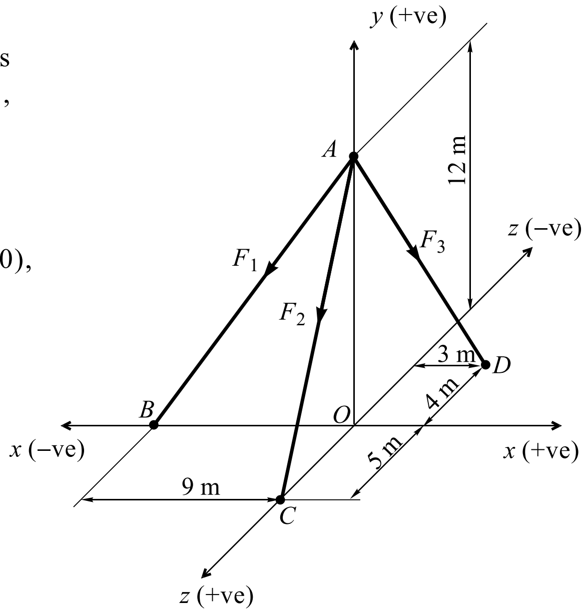


Fig. 4.12

Solution

(i) Coordinate $O(0, 0, 0)$, $A(0, 12, 0)$, $B(-9, 0, 0)$, $C(0, 0, 5)$, $D(3, 0, -4)$.

(ii) Force vector

$$\overline{\boldsymbol{F}}_1 = (F_1)(\overline{\boldsymbol{e}}_{AB})$$

$$\overline{\boldsymbol{F}}_1 = (F_1)\left[\frac{-9\ \mathbf{i} - 12\ \mathbf{j} + 0\ \mathbf{k}}{\sqrt{9^2 + 12^2 + 0^2}}\right]$$

$$\overline{\boldsymbol{F}}_1 = (F_1)(-0.6\ \mathbf{i} - 0.8\ \mathbf{j})$$

$$\overline{\boldsymbol{F}}_2 = (F_2)(\overline{\boldsymbol{e}}_{AC})$$

$$\overline{\boldsymbol{F}}_2 = (F_2)\left[\frac{0\ \mathbf{i} - 12\ \mathbf{j} + 5\ \mathbf{k}}{\sqrt{0^2 + 12^2 + 5^2}}\right]$$

$$\overline{\boldsymbol{F}}_2 = (F_2)(0\ \mathbf{i} - 0.923\ \mathbf{j} + 0.385\ \mathbf{k})$$

$$\overline{\boldsymbol{F}}_3 = (F_3)(\overline{\boldsymbol{e}}_{AD})$$

$$\overline{\boldsymbol{F}}_3 = (F_3)\left[\frac{3\ \mathbf{i} - 12\ \mathbf{j} - 4\ \mathbf{k}}{\sqrt{3^2 + 12^2 + 4^2}}\right]$$

$$\overline{\boldsymbol{F}}_3 = (F_3)(0.231\ \mathbf{i} - 0.923\ \mathbf{j} - 0.308\ \mathbf{k})$$

(iii) Resultant $\overline{\boldsymbol{R}} = -788\ \mathbf{j}$ (Given)

We know for concurrent force system resultant is given by

$$\overline{\boldsymbol{R}} = \overline{\boldsymbol{F}}_1 + \overline{\boldsymbol{F}}_2 + \overline{\boldsymbol{F}}_3$$

$$-788\ \mathbf{j} = (F_1)(-0.6\ \mathbf{i} - 0.8\ \mathbf{j}) + (F_2)(-0.923\ \mathbf{j} + 0.385\ \mathbf{k}) + (F_3)(0.231\ \mathbf{i} - 0.923\ \mathbf{j} - 0.308\ \mathbf{k})$$

(iv) Equating $\mathbf{i}$ and $\mathbf{k}$ to zero and $\mathbf{j} = -788$, we get

$$-0.6\ F_1 + 0.23\ F_3 = 0 \qquad \ldots \text{(I)}$$

$$-0.8\ F_1 - 0.923\ F_2 - 0.923\ F_3 = -788 \qquad \ldots \text{(II)}$$

$$0.385\ F_2 - 0.308\ F_3 = 0 \qquad \ldots \text{(III)}$$

Solving Eqs. (I), (II) and (III), we get

$F_1 = 154$ N, $F_2 = 320$ N, $F_3 = 400$ N ***Ans.***

4.5 Resultant of Parallel Force System in Space

Problem 13

Five vertical forces are acting on a horizontal plate as shown in Fig. 4.13. Find resultant of the forces and point of application w.r.t. origin.

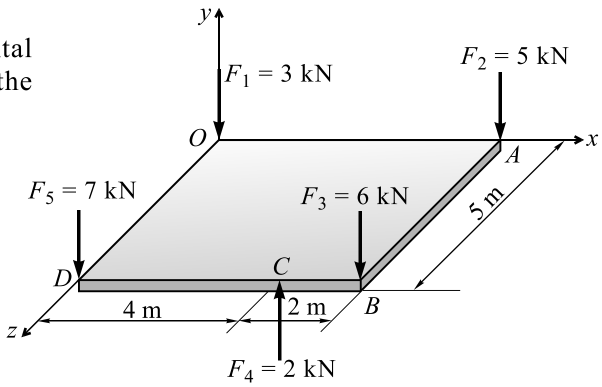


Fig. 4.13

Solution

(i) Resultant force (R)

$R = -3 - 5 - 6 + 2 - 7$

$R = -19$ kN (↓)

(ii) Taking moment about x-axis and applying Varignon's theorem

$\Sigma M_x = R \times z$

$(-3)(0) + (-5)(0) + (-6)(5) + (2)(5) + (-7)(5) = (-19)(z)$

$\therefore\ z = 2.89$ m

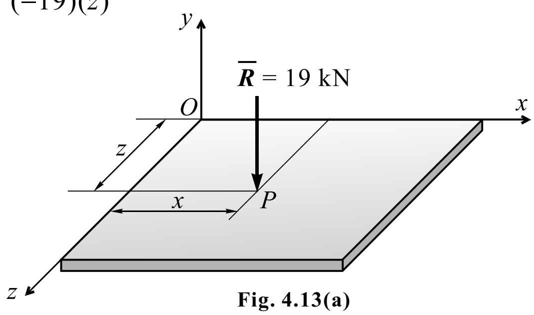


Fig. 4.13(a)

(iii) Taking moment about z-axis and applying Varignon's theorem

$\Sigma M_z = R \times x$

$(-3)(0) + (-5)(6) + (-6)(6) + (2)(4) + (-7)(0)$

$= (-19)(x)$

$\therefore\ x = 3.05$ m

(iv) Resultant and its point of application w.r.t. origin

$\bar{\boldsymbol{R}} = -19\,\mathbf{j}$ (kN) acts at point $P(3.05, 0, 2.89)$ m ***Ans.***

Problem 14

Replace the given force system (Fig. 4.14) by a force and a couple.

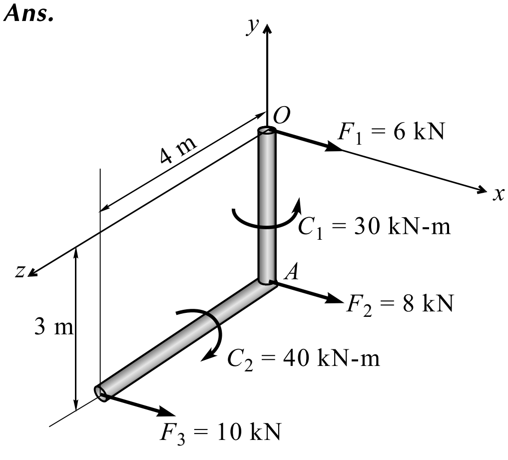


Fig. 4.14

Solution

(i) Resultant force (R)

$R = 6 + 8 + 10 \quad \therefore\ R = 24$ kN

(ii) Taking moment about y-axis and applying Varignon's theorem

$\Sigma M_x = R \times z$

$(6)(0) + (8)(0) + (10)(4) + 30 = 24(z)$

$\therefore\ z = 2.917$ m

(iii) Taking moment about z-axis and applying Varignon's theorem

$\Sigma M_z = R \times y$

$(6)(0) + (8)(0) + (10)(3) - 40 = 24(-y)$

$\therefore\ y = -0.583$ m

(iv) Resultant and its point of application w.r.t. origin

$\bar{\boldsymbol{R}} = 24\,\mathbf{j}$ (kN) acts at point $P(0, -0.583, 2.917)$ m ***Ans.***

Problem 15

Determine the loads to be applied at A and F, if the resultant of all six loads is to pass through the centre of the foundation of hexagonal shape of side (3 m) as shown in Fig. 4.15.

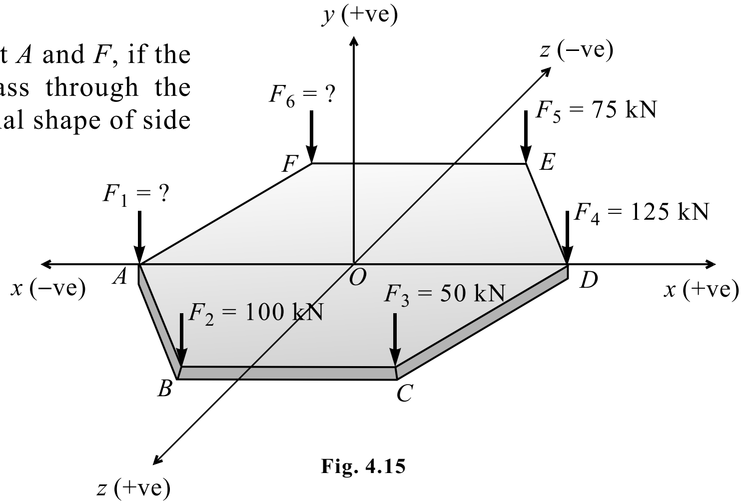


Fig. 4.15

Solution

(i) Draw the top view of hexagonal plate [Fig. 4.15(a)].

$\Delta\ OBC$ is an equilateral triangle

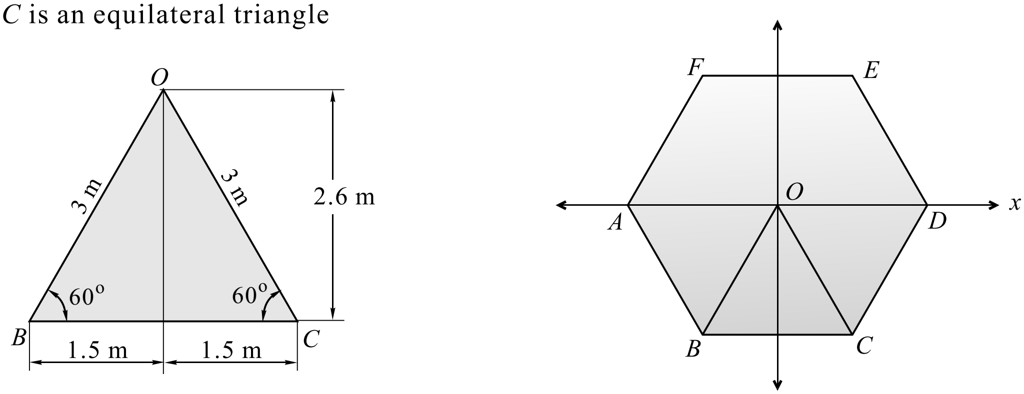


Fig. 4.15(a) : Top view

$\therefore$ Coordinates are as follows :

$O(0, 0, 0)$; $A(-3, 0, 0)$; $B(-1.5, 0, 2.6)$; $C(1.5, 0, 2.6)$;
$D(3, 0, 0)$; $E(1.5, 0, -2.6)$; $F(-1.5, 0, -2.6)$.

(ii) Taking the moment about x-axis and applying Varignon's theorem

$$\Sigma M_x = R \times z$$

$$(-F_1)(0) + (-100)(2.6) + (-50)(2.6) + (-125)(0) + (-75)(-2.6) + (-F_6)(-2.6) = R \times 0$$

$\{ \because R$ is passing through origin $O \;\therefore\; z = 0 \}$

$\therefore\ F_6 = 75$ kN $(\downarrow)$ ***Ans.***

(iii) Taking the moment about z-axis and applying Varignon's theorem

$$\Sigma M_z = R \times x$$

$$(-F_1)(-3) + (-100)(-1.5) + (-50)(1.5) + (-125)(3) + (-75)(1.5) + (-75)(-1.5) = R \times 0$$

$\{ \because R$ is passing through origin $O \;\therefore\; x = 0 \}$

$\therefore\ F_1 = 100$ kN $(\downarrow)$ ***Ans.***

4.6 Resultant of General Force System in Space

1. Write all the forces in vector form and add them.

   Resultant force vector $\overline{\boldsymbol{R}} = \Sigma\,\overline{\boldsymbol{F}} = \overline{\boldsymbol{F}}_1 + \overline{\boldsymbol{F}}_2 + \overline{\boldsymbol{F}}_3 + ...$

2. Take moments of all forces about the origin or given point and add them.

   Resultant moment (couple) vector $\Sigma\,\overline{\boldsymbol{M}}_O = \overline{\boldsymbol{M}}_1 + \overline{\boldsymbol{M}}_2 + \overline{\boldsymbol{M}}_3 + ...$

Note :
- General force system in space cannot be reduced to a single force. Therefore, the resultant is expressed in two components :
  (i) Resultant force component
  (ii) Resultant moment (couple) component.
- Varignon's theorem is not applicable to general force system.

Procedure

1. Coordinates
2. Force vectors $\overline{\boldsymbol{F}}_1, \overline{\boldsymbol{F}}_2, \overline{\boldsymbol{F}}_3, ...$
3. Position vectors (w.r.t. origin or given point) $\overline{r}_1, \overline{r}_2, \overline{r}_3, ...$
4. Moment vectors $\overline{\boldsymbol{M}}_1, \overline{\boldsymbol{M}}_2, \overline{\boldsymbol{M}}_3, ... + \overline{\boldsymbol{C}}_1 + \overline{\boldsymbol{C}}_2 + \overline{\boldsymbol{C}}_3 + ...$
5. Resultant force vector $\overline{\boldsymbol{R}} = \overline{\boldsymbol{F}}_1 + \overline{\boldsymbol{F}}_2 + \overline{\boldsymbol{F}}_3 + ...$
6. Resultant moment (couple) vector $\Sigma\,\overline{\boldsymbol{M}} = \overline{\boldsymbol{M}}_1 + \overline{\boldsymbol{M}}_2 + \overline{\boldsymbol{M}}_3 + ... + \overline{\boldsymbol{C}}_1 + \overline{\boldsymbol{C}}_2 + \overline{\boldsymbol{C}}_3 + ...$

Problem 16

Determine the resultant force and resultant couple moment at point $A(3, 1, 2)$ m of the following force systems :

$$\overline{\boldsymbol{F}}_1 = 5\,\mathbf{i} + 8\,\mathbf{k} \text{ (N) acting at point } B(8, 3, -1) \text{ m}$$

$$\overline{\boldsymbol{F}}_2 = 3\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k} \text{ (N) acting at point } O(0, 0, 0) \text{ m}$$

$$\overline{\boldsymbol{M}} = 12\,\mathbf{i} - 20\,\mathbf{j} + 9\,\mathbf{k} \text{ (N-m).}$$

Solution

(i) Coordinates : $A(3, 1, 2)$; $B(8, 3, -1)$.

(ii) Force vector

$$\overline{\boldsymbol{F}}_1 = 5\,\mathbf{i} + 8\,\mathbf{k} \text{ (N)}; \quad \overline{\boldsymbol{F}}_2 = 3\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k} \text{ (N).}$$

(iii) Position vectors w.r.t. point A

$$\overline{\boldsymbol{r}}_1 = \overline{AB} = 5\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}; \quad \overline{\boldsymbol{r}}_2 = \overline{AO} = -3\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$$

(iv) Moment vectors

$$\overline{M}_1 = \overline{r}_1 \times \overline{F}_1 \qquad \overline{M}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -3 \\ 5 & 0 & 8 \end{vmatrix}$$

$$\overline{M}_1 = 16\,\mathbf{i} - 55\,\mathbf{j} - 10\,\mathbf{k} \text{ (N-m)}$$

$$\overline{M}_2 = \overline{r}_2 \times \overline{F}_2 \qquad \overline{M}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

$$\overline{M}_2 = 8\,\mathbf{i} - 18\,\mathbf{j} - 3\,\mathbf{k} \text{ (N-m)}$$

$$\therefore \overline{M} = 12\,\mathbf{i} - 20\,\mathbf{j} + 9\,\mathbf{k} \text{ (N-m)}$$

(v) Resultant force vector

$$\overline{R} = \overline{F}_1 + \overline{F}_2 = 5\,\mathbf{i} + 8\,\mathbf{k} + 3\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}$$

$$\overline{R} = 8\,\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k} \text{ (N)}$$

(vi) Resultant moment vector

$$\Sigma\,\overline{M}_A = \overline{M}_1 + \overline{M}_2 + \overline{M}$$

$$\Sigma\,\overline{M}_A = 26\,\mathbf{i} - 93\,\mathbf{j} - 4\,\mathbf{k} \text{ (N-m)}$$

Problem 17

Forces $F_1 = 1$ kN, $F_2 = 3$ kN, $F_3 = 2$ kN, $F_4 = 5$ kN and $F_5 = 2$ kN act along the line joining the corners of the parallelopiped whose sides are 2.5 m, 2 m and 1.5 m respectively as shown in Fig. 4.17. Find the resultant force and the moment of the resultant couple at the origin O.

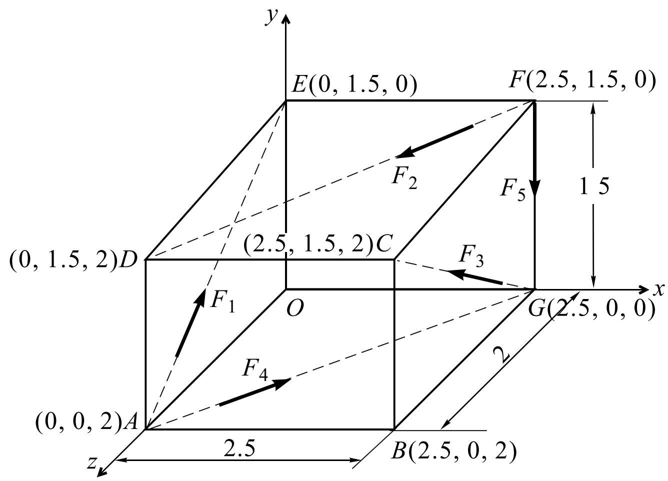


Fig. 4.17

Solution

Given : $F_1 = 1$ kN $(A \rightarrow E)$;
$F_2 = 3$ kN $(F \rightarrow D)$; $F_3 = 2$ kN $(G \rightarrow C)$;
$F_4 = 5$ kN $(A \rightarrow G)$; $F_5 = 2$ kN $(F \rightarrow G)$.

(i) Coordinates : $O(0, 0, 0)$; $A(0, 0, 2)$; $B(2.5, 0, 2)$; $C(2.5, 1.5, 2)$; $D(0, 1.5, 2)$; $E(0, 1.5, 0)$; $F(2.5, 1.5, 0)$; $G(2.5, 0, 0)$.

(ii) Force vector

$$\overline{F}_1 = (F_1)(\overline{e}_{AE}) = (1)\left[\frac{1.5\,\mathbf{j} - 2\,\mathbf{k}}{\sqrt{(1.5)^2 + 2^2}}\right] \qquad \therefore \overline{F}_1 = 0.6\,\mathbf{j} - 0.8\,\mathbf{k}$$

$$\overline{F}_2 = (F_2)(\overline{e}_{FD}) = (3)\left[\frac{-2.5\,\mathbf{i} + 2\,\mathbf{k}}{\sqrt{(2.5)^2 + 2^2}}\right] \qquad \therefore \overline{F}_2 = -2.34\,\mathbf{i} + 1.87\,\mathbf{k}$$

$$\overline{F}_3 = (F_3)(\overline{e}_{GC}) = (2)\left[\frac{1.5\,\mathbf{j} + 2\,\mathbf{k}}{\sqrt{(1.5)^2 + 2^2}}\right] \qquad \therefore \quad \overline{F}_3 = 1.2\,\mathbf{j} + 1.6\,\mathbf{k}$$

$$\overline{F}_4 = (F_4)(\overline{e}_{AG}) = (5)\left[\frac{2.5\,\mathbf{i} - 2\,\mathbf{k}}{\sqrt{(2.5)^2 + 2^2}}\right] \qquad \therefore \quad \overline{F}_4 = 3.90\,\mathbf{i} - 3.12\,\mathbf{k}$$

$$\overline{F}_5 = (F_5)(\overline{e}_{FG}) = (2)\left[\frac{-1.5\,\mathbf{j}}{\sqrt{(1.5)^2}}\right] \qquad \therefore \quad \overline{F}_5 = -2\,\mathbf{j}$$

(iii) Position vectors

$\overline{r}_1 = 2\,\mathbf{k}$; $\overline{r}_2 = 1.5\,\mathbf{j} + 2\,\mathbf{k}$; $\overline{r}_3 = 2.5\,\mathbf{i}$; $\overline{r}_4 = 2\,\mathbf{k}$; $\overline{r}_5 = 2.5\,\mathbf{i}$

(iv) Moment vectors

$$\overline{M}_1 = \overline{r}_1 \times \overline{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & 0.6 & -0.8 \end{vmatrix} \qquad \therefore \quad \overline{M}_1 = -1.2\,\mathbf{i}$$

$$\overline{M}_2 = \overline{r}_2 \times \overline{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1.5 & 2 \\ -2.34 & 0 & 1.87 \end{vmatrix} \qquad \therefore \quad \overline{M}_2 = 2.8\,\mathbf{i} - 4.68\,\mathbf{j} + 3.51\,\mathbf{k}$$

$$\overline{M}_3 = \overline{r}_3 \times \overline{F}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2.5 & 0 & 0 \\ 0 & 1.2 & 1.6 \end{vmatrix} \qquad \therefore \quad \overline{M}_3 = -4\,\mathbf{j} + 3\,\mathbf{k}$$

$$\overline{M}_4 = \overline{r}_4 \times \overline{F}_4 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 3.9 & 0 & -3.12 \end{vmatrix} \qquad \therefore \quad \overline{M}_4 = 7.8\,\mathbf{j}$$

$$\overline{M}_5 = \overline{r}_5 \times \overline{F}_5 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2.5 & 0 & 0 \\ 0 & -2 & 0 \end{vmatrix} \qquad \therefore \quad \overline{M}_5 = -5\,\mathbf{k}$$

(v) Resultant force vector

$$\overline{R} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4 + \overline{F}_5$$

$$\overline{R} = 1.56\,\mathbf{i} - 0.2\,\mathbf{j} - 0.45\,\mathbf{k} \quad \textbf{\textit{Ans.}}$$

(vi) Resultant moment vector

$$\Sigma\,\overline{M} = \overline{M}_1 + \overline{M}_2 + \overline{M}_3 + \overline{M}_4 + \overline{M}_5$$

$$\Sigma\,\overline{M} = 1.6\,\mathbf{i} - 0.88\,\mathbf{j} + 1.51\,\mathbf{k} \quad \textbf{\textit{Ans.}}$$

Problem 18

Determine the resultant force and the resultant couple of the force system shown in Fig. 4.18, where F_1 = 100 N, F_2 = 20 $\sqrt{2}$ N, F_3 = 40 N, F_4 = 40 N, C_1 = 250 N-m and C_2 = 100 N-m.

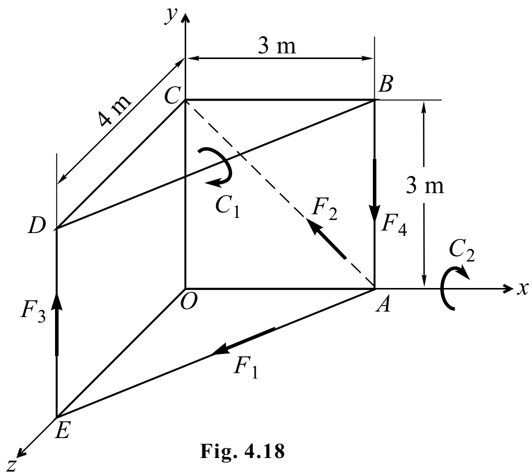


Fig. 4.18

Solution

Given : $F_1 = 100 \text{ N } (A \rightarrow E)$;

$F_2 = 20\sqrt{2} \text{ N } (A \rightarrow C)$; $F_3 = 40 \text{ N } (E \rightarrow D)$;

$F_4 = 40 \text{ N } (B \rightarrow A)$

$C_1 = 250 \text{ N-m } (D \rightarrow B)$; $C_2 = 100 \text{ N-m } (A \rightarrow O)$;
[By right-hand-thumb rule]

(i) Coordinates : $O(0, 0, 0)$; $A(3, 0, 0)$; $B(3, 3, 0)$; $C(0, 3, 0)$; $D(0, 3, 4)$; $E(0, 0, 4)$.

(ii) Force vector

$$\overline{F}_1 = (F_1)(\overline{e}_{AE}) = (100)\left[\frac{-3\ \mathbf{i} + 4\ \mathbf{k}}{\sqrt{9 + 16}}\right]$$

$$\therefore\ \overline{F}_1 = -60\ \mathbf{i} + 80\ \mathbf{k}$$

$$\overline{F}_2 = (F_2)(\overline{e}_{AC}) = (20\sqrt{2})\left[\frac{-3\ \mathbf{i} + 3\ \mathbf{j}}{\sqrt{9 + 9}}\right]$$

$$\therefore\ \overline{F}_2 = -20\ \mathbf{i} + 20\ \mathbf{j}$$

$$\overline{F}_3 = (F_3)(\overline{e}_{ED}) = 40\ \mathbf{j}$$

$$\overline{F}_4 = (F_4)(\overline{e}_{BA}) = -40\ \mathbf{j}$$

[By right-hand-thumb rule]

$$\overline{C}_1 = (C_1)(\overline{e}_{DB}) = (250)\left[\frac{3\ \mathbf{i} - 4\ \mathbf{k}}{\sqrt{9 + 16}}\right]$$

$$\therefore\ \overline{C}_1 = 150\ \mathbf{i} - 200\ \mathbf{k}$$

$$\overline{C}_2 = (C_2)(\overline{e}_{AO}) = (100)\left[\frac{-3\ \mathbf{i}}{\sqrt{9}}\right]$$

$$\therefore\ \overline{C}_2 = -100\ \mathbf{i}$$

(iii) Position vectors

$$\overline{r_1} = 3\ \mathbf{i}\ ;\ \overline{r_2} = 3\ \mathbf{i}\ ;\ \overline{r_3} = 4\ \mathbf{k}\ ;\ \overline{r_4} = 3\ \mathbf{i}$$

(iv) Moment vectors

$$\overline{M}_1 = \overline{r}_1 \times \overline{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 0 \\ -60 & 0 & 80 \end{vmatrix} = \mathbf{i}\ (0) - \mathbf{j}\ (240) + \mathbf{k}\ (0) \qquad \therefore\ \overline{M}_1 = -240\ \mathbf{j}$$

$$\overline{M}_2 = \overline{r}_2 \times \overline{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 0 \\ 20 & 20 & 0 \end{vmatrix} = \mathbf{i}\,(0) - \mathbf{j}\,(0) + \mathbf{k}\,(60) \qquad \therefore\ \overline{M}_2 = 60\,\mathbf{k}$$

$$\overline{M}_3 = \overline{r}_3 \times \overline{F}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ 0 & 40 & 0 \end{vmatrix} = \mathbf{i}\,(-160) - \mathbf{j}\,(0) + \mathbf{k}\,(0)$$

$$\therefore\ \overline{M}_3 = -160\,\mathbf{i}$$

$$\overline{M}_4 = \overline{r}_4 \times \overline{F}_4 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 0 \\ 0 & -40 & 0 \end{vmatrix} = \mathbf{i}\,(0) - \mathbf{j}\,(0) + \mathbf{k}\,(-120)$$

$$\therefore\ \overline{M}_4 = -120\,\mathbf{k}$$

From point (ii), we have

$\overline{C}_1 = 150\,\mathbf{i} - 200\,\mathbf{k}$ and $\overline{C}_2 = -100\,\mathbf{i}$

[Direction of couple is taken by right-hand-thumb rule]

(v) Resultant force vector

$$\overline{R} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4$$

$$\overline{R} = (-60\,\mathbf{i} + 80\,\mathbf{k}) + (-20\,\mathbf{i} + 20\,\mathbf{j}) + 40\,\mathbf{j} - 40\,\mathbf{j}$$

$$\overline{R} = -80\,\mathbf{i} + 20\,\mathbf{j} + 80\,\mathbf{k}$$ ***Ans.***

(vi) Resultant moment (couple) vector

$$\Sigma\,\overline{M} = \overline{M}_1 + \overline{M}_2 + \overline{M}_3 + \overline{M}_4 + \overline{C}_1 + \overline{C}_2$$

$$\Sigma\,\overline{M} = -240\,\mathbf{j} + 60\,\mathbf{k} - 160\,\mathbf{i} - 120\,\mathbf{k} + (150\,\mathbf{i} - 200\,\mathbf{k}) - 100\,\mathbf{i}$$

$$\Sigma\,\overline{M} = -110\,\mathbf{i} - 240\,\mathbf{j} - 260\,\mathbf{k}$$ ***Ans.***

4.7 Equilibrium of Forces in Space

In equilibrium, the resultant of force system is equal to zero. It means resultant force $\overline{F}$ and resultant moment $\overline{M}$ both are zero.

Equation of Equilibrium

$$\Sigma F_x = 0 \qquad \Sigma M_x = 0$$

$$\Sigma F_y = 0 \qquad \Sigma M_y = 0$$

$$\Sigma F_z = 0 \qquad \Sigma M_z = 0$$

Types of Support

1. Ball and Socket Support

In a socket a ball is set in such a way that it supports the end of the rod and allows rotation in all the three directions (Fig. 4.7-i).

Hinge is an example for coplanar but ball and socket is an example for three dimensional freedom.

A rod one end which is inside the socket will not be allowed to move in any linear direction. This restriction will result in a form of reaction component along x, y and z axis.

$$\therefore\ \Sigma F_x = 0\ ;\ \Sigma F_y = 0\ ;\ \Sigma F_z = 0$$

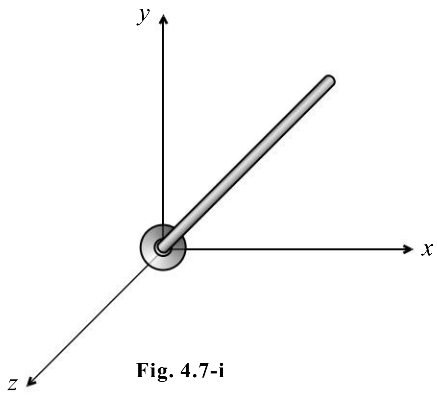


Fig. 4.7-i

2. Fixed End Support

In this case the end of the rod is inbuilt, i.e. fixed at one end. It does not allow any linear moment in x, y and z directions, also it does not allow any rotation moment along x, y, and z axes.

$$\therefore\ \Sigma F_x = 0\ ;\ \Sigma F_y = 0\ ;\ \Sigma F_z = 0$$

and $\Sigma M_x = 0\ ;\ \Sigma M_y = 0\ ;\ \Sigma M_z = 0$

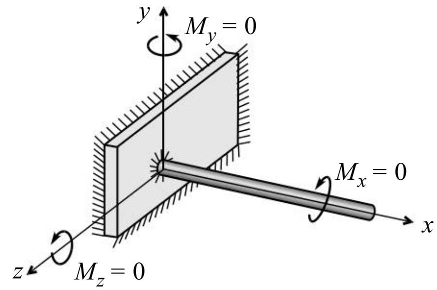


Fig. 4.7-ii

Equilibrium Condition of Concurrent Space Force System

In equilibrium, the resultant of force system is equal to zero. It means resultant force ***F*** and resultant moment ***M*** both are zero.

$$\therefore\ \Sigma F_x = 0\ ;\ \Sigma F_y = 0\ ;\ \Sigma F_z = 0$$

Problem 19

Three members are connected to each other at end D carrying a load 5 kN as shown in Fig. 4.19. Find forces in all members, consider all joints to be of ball and socket type.

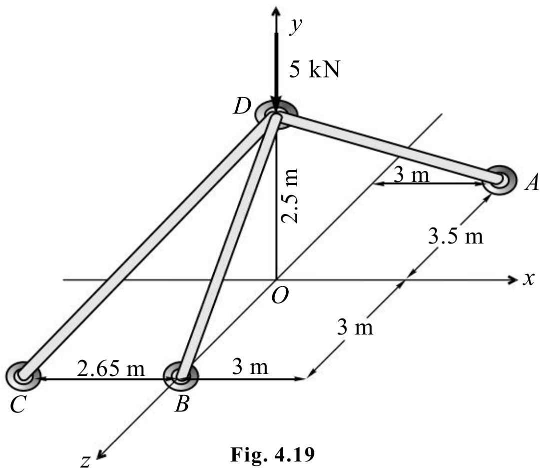


Fig. 4.19

Solution

Given coordinates : $A(3, 0, -2.5)$
$B(0, 0, 3)$
$C(-2.65, 0, 3)$
$D(0, 2.5, 0)$

(ii) Force vector

$$\overline{\boldsymbol{F}}_{AD} = (F_{AD})(\overline{\boldsymbol{e}}_{AD})$$

$$= F_{AD}\left[\frac{3\,\mathbf{i} - 2.5\,\mathbf{j} - 2.5\,\mathbf{k}}{\sqrt{3^2 + 2.5^2 + 2.5^2}}\right]$$

$$\overline{\boldsymbol{F}}_{AD} = F_{AD}(0.647\,\mathbf{i} - 0.539\,\mathbf{j} - 0.539\,\mathbf{k})$$

$$\overline{\boldsymbol{F}}_{BD} = (F_{BD})(\overline{\boldsymbol{e}}_{BD})$$

$$= F_{BD}\left[\frac{0\,\mathbf{i} - 2.5\,\mathbf{j} + 3\,\mathbf{k}}{\sqrt{0^2 + 2.5^2 + 3^2}}\right]$$

$$\overline{\boldsymbol{F}}_{BD} = F_{BD}(0\,\mathbf{i} - 0.649\,\mathbf{j} - 0.768\,\mathbf{k})$$

$$\overline{\boldsymbol{F}}_{CD} = (F_{CD})(\overline{\boldsymbol{e}}_{CD})$$

$$= F_{CD}\left[\frac{-2.65\,\mathbf{i} - 2.5\,\mathbf{j} + 3\,\mathbf{k}}{\sqrt{2.65^2 + 2.5^2 + 3^2}}\right]$$

$$\overline{\boldsymbol{F}}_{CD} = F_{CD}(-0.56\,\mathbf{i} - 0.53\,\mathbf{j} - 0.636\,\mathbf{k})$$

Load 5 kN is $0\,\mathbf{i} - 5\,\mathbf{j} + 0\,\mathbf{k}$

(iii) Considering equilibrium condition, we have

$$\Sigma F_x = 0$$

$$0.647\,F_{AD} - 0.56\,F_{CD} = 0 \qquad \text{... (I)}$$

$$\Sigma F_y = 0$$

$$-0.539\,F_{AD} - 0.64\,F_{BD} - 0.53\,F_{CD} - 5 = 0 \qquad \text{... (II)}$$

$$\Sigma F_z = 0$$

$$-0.539\,F_{AD} - 0.768\,F_{BD} - 0.636\,F_{CD} = 0 \qquad \text{... (III)}$$

(iv) Solving equations (I), (II) and (III) we get

$F_{AD} = -5.06$ kN (Compression)

$F_{BD} = 1.29$ kN (Tension)

$F_{CD} = -5.84$ kN (Compression) ***Ans.***

Exercises

[I] Problems

1. A force of magnitude 50 kN is acting at point $A(2, 3, 4)$ m towards point $B(6, -2, 3)$ m. Find the (a) vector component of this force along the line AC. Point C is $(5, -1, 2)$ m and (b) moment of the given force about a point $D(-1, 1, 2)$ m.

[***Ans.*** (a) $25.07\,\mathbf{i} - 33.45\,\mathbf{j} - 16.7\,\mathbf{k}$ (kN) and
(b) $\overline{\boldsymbol{M}}_D = -21.08\,\mathbf{i} + 152.83\,\mathbf{j} - 121.21\,\mathbf{k}$ (kN-m).]

2. A force $\overline{\boldsymbol{F}} = 4\,\mathbf{i} - 3\,\mathbf{j} + 8\,\mathbf{k}$ (N) acts at a point $A(2, -1, 3)$. Find the
(a) vector component of F along the line AB, the coordinates of point B are $(3, 2, 3)$ m and
(b) moment of the vector component of F about the origin.

[***Ans.*** (a) $-0.5\,\mathbf{i} - 1.5\,\mathbf{j}$ (N) and (b) $4.5\,\mathbf{i} - 1.5\,\mathbf{j} - 3.5\,\mathbf{k}$ (N-m)]

3. A force of 200 N acts from $A(4, -2, 2)$ m, towards $B(-1, 2, 3)$ m. Find the scalar and vector components of moment of this force about a line CD where the coordinates of C and D are $C(2, 4, -3)$ and $D(6, -1, 1)$ m. Also find the moment of the above force about the origin.

[***Ans.*** $M_{CD} = -235.2$ (N),
$\overline{\boldsymbol{M}}_{CD} = -124.63\,\mathbf{i} + 155.2\,\mathbf{j} + 124.63\,\mathbf{k}$ (N-m) and
$\overline{\boldsymbol{M}}_O = -308.6\,\mathbf{i} + 432.04\,\mathbf{j} + 185.16\,\mathbf{k}$ (N-m).]

4. A 700 N force passes through two points $A(-5, -1, 4)$ towards $B(1, 2, 6)$ m. Find the
(a) moment of force about a point $C(2, -2, 1)$ m and
(b) scalar moment of the force about line OC where O is the origin.

[***Ans.*** (a) $-700\,\mathbf{i} + 3200\,\mathbf{j} - 2700\,\mathbf{k}$ (N-m) and (b) 3500 (Nm).]

5. A force $\overline{\boldsymbol{F}} = 30\,\mathbf{i} + 40\,\mathbf{j} + 20\,\mathbf{k}$ N acts at a point $A(-2, 3, 2)$ m. Find its moment about a line OC lying in the x-y plane passing through origin and making an angle of 45° with positive axis.

[***Ans.*** 56.6 (Nm)]

6. A force 1000 N forms angles of 60°, 45° and 120° with x, y and z-axes, respectively. Write the equation of the force in the vector form.

[***Ans.*** $\overline{\boldsymbol{F}} = 500\,\mathbf{i} + 707.1\,\mathbf{j} - 500\,\mathbf{k}$ (N)]

7. Force vector F is given in the form $\overline{\boldsymbol{F}} = 20\,\mathbf{i} - 30\,\mathbf{j} + 60\,\mathbf{k}$ (N). Find θ_x, θ_y and θ_z.

[***Ans.*** $\theta_x = 73.4^\circ$, $\theta_y = 115.4^\circ$ and $\theta_z = 31^\circ$.]

8. Determine the magnitude and the direction of a force, $\overline{\boldsymbol{F}} = 345\,\mathbf{i} + 150\,\mathbf{j}\ 290\,\mathbf{k}$.

[***Ans.*** $F = 475$ units, $\theta_x = 43.4^\circ$, $\theta_y = 71.6^\circ$ and $\theta_z = 127.62^\circ$.]

9. The lines of actions of three forces concurrent at origin O pass respectively through A, B, C having coordinates : $x_A = -1$ $y_A = +2$ $z_A = +4$

$x_B = +3$ $y_B = 0$ $z_B = -3$

$x_C = +2$ $y_C = -2$ $z_C = +4$

The magnitude of the forces are $F_A = 40$ N, $F_B = 10$ N, $F_C = 30$ N Find the magnitude and direction of their resultant.

[**Ans.** $R = 5364$ N, $\theta_x = 78.62^\circ$, $\theta_y = 84.41^\circ$ and $\theta_z = 12.69^\circ$.]

10. The cable exerts forces $F_{AB} = 100$ N and $F_{AC} = 120$ N on the ring at A, as shown in the Fig. 4.E10. Determine the magnitude of the resultant force acting at A.

[**Ans.** $R = 217$ N]

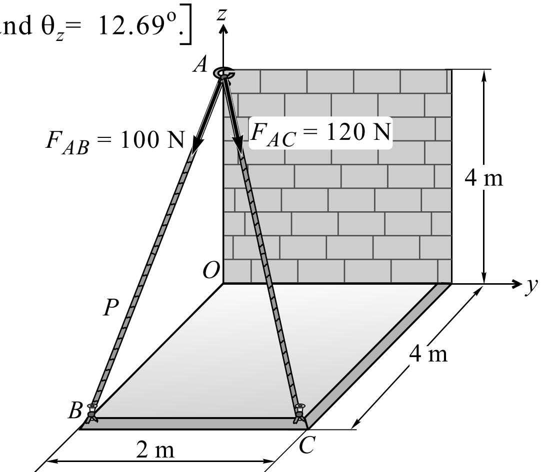


Fig. 4.E10

11. Knowing that the tension in cable AD is 2520 N, as shown in Fig. 4.E11. Determine the required value of tension in each of the cables AB and AC so that the resultant of the three forces applied by the cables at A is vertical.

[**Ans.** $T_{AC} = 3131$ N and $T_{AB} = 2277$ N.]

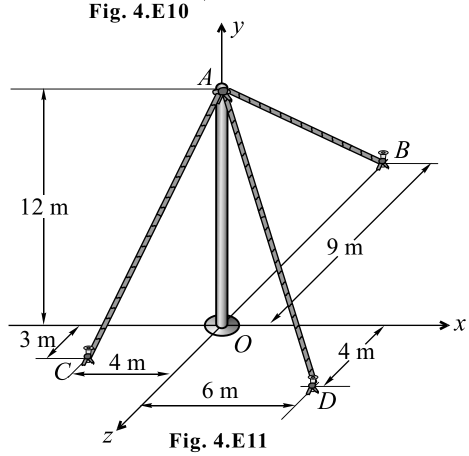


Fig. 4.E11

12. Figure 4.E12 shows a rectangular parallelopiped subjected to four forces in the direction shown. Reduce them to a resultant force at the origin and a moment. $OC = 5$ m ; $OA = 3$ m ; $OD = 4$ m.

[**Ans.** $\overline{\boldsymbol{R}} = 109.6\,\mathbf{i} + 1024.2\,\mathbf{j} + 1167\,\mathbf{k}$ (N) and
$\overline{\boldsymbol{M}}_O = 1103.1\,\mathbf{i} - 2121.2\,\mathbf{k}$ (Nm).]

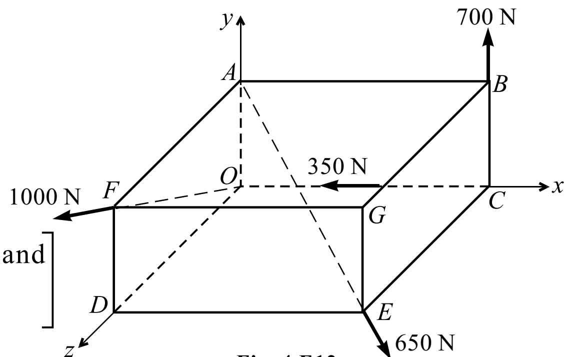


Fig. 4.E12

13. Replace the three forces shown on a bent in Fig. 4.E13 by a force-moment system at the origin.

[**Ans.** $\overline{\boldsymbol{R}} = 50\ \mathbf{k}$ (N) and

$\overline{\boldsymbol{M}}_O = -25\ \mathbf{j} - 12.5\ \mathbf{k}$ (N-m).]

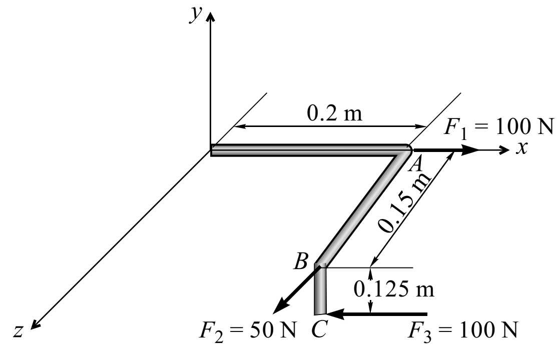


Fig. 4.E13

14. Find the resultant of the forces system as shown in Fig. 4.E14 at the origin.

[**Ans.** $\overline{\boldsymbol{R}} = 9\ \mathbf{i} + 28\ \mathbf{j} + \mathbf{k}$ (N) and

$\overline{\boldsymbol{M}}_O = 40.39\ \mathbf{i} + 47\ \mathbf{j} - 68.39\ \mathbf{k}$ (N-m).]

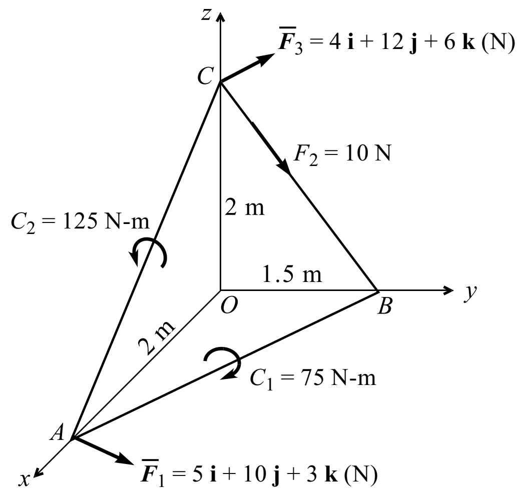


Fig. 4.E14

15. Determine the resultant of the force and couple system, which acts on the rectangular solid in Fig. 4.E15.

[**Ans.** $\overline{\boldsymbol{R}} = 0$ and

$\overline{\boldsymbol{M}}_O = 10\ \mathbf{i}$ (N-m).]

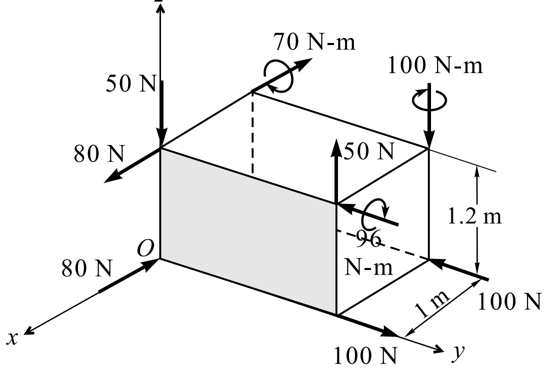


Fig. 4.E15

16. The concrete slab supports the six vertical loads shown in Fig. 4.E16. Determine the x and y coordinates of the point on the slab through which the resultant of the loading system passes. Also find the resultant.

[***Ans.*** R = 184 kN (↓) and
x = 2.92 m, y = 6.33 m.]

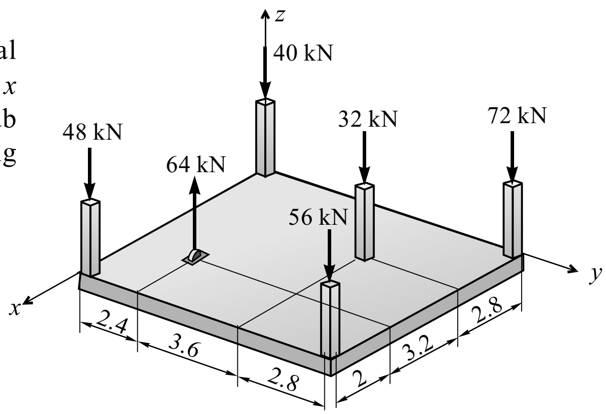


Fig. 4.E16

17. Determine the x and y coordinates of a point through which the resultant of the parallel forces passes in Fig. 4.E17. Also determine the resultant.

[***Ans.*** $\overline{\boldsymbol{R}} = -450\,\mathbf{k}$ (N) and
acts at (22.22, −53.33, 0) mm.]

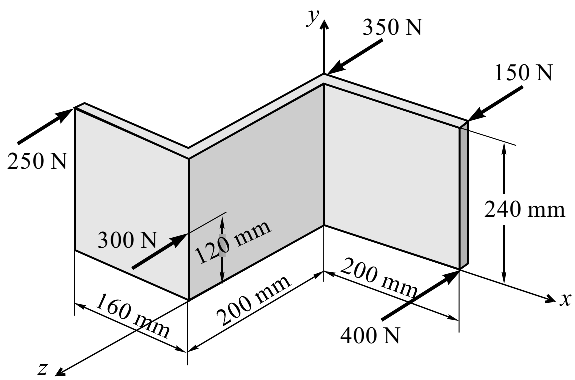


Fig. 4.E17

18. Three parallel bolting forces act on the rim of the circular cover plate as shown in Fig. 4.E18. Determine the magnitude and direction of a resultant force equivalent to the given force system and locate its point of application, P on the cover plate.

[***Ans.*** R = 650 N (↓) and
P(0.24, −0.12, 0) m.]

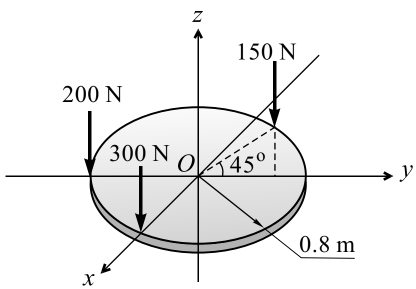


Fig. 4.E18

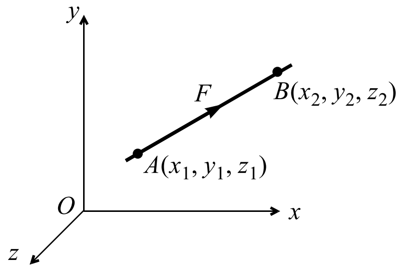
[II] Review Questions

1. Find the dot product and cross product of

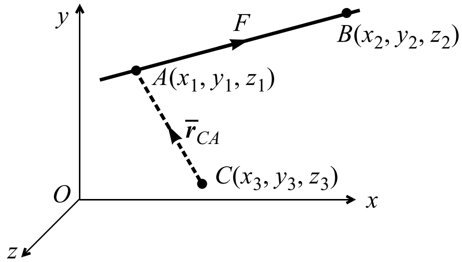
$\overline{\boldsymbol{F}}_1 = x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}$ and

$\overline{\boldsymbol{F}}_2 = x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}$.

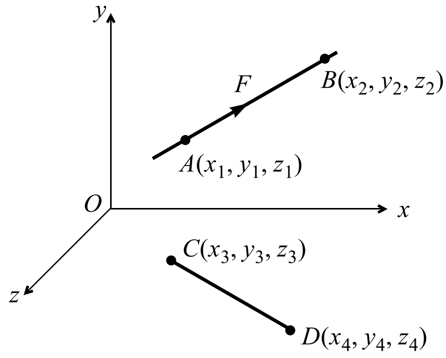
2. Find the force vector $\overline{\boldsymbol{F}}$ if the line of action of force in space passes through $A_1(x_1, y_1, z_1)$ and $B_2(x_2, y_2, z_2)$. Refer to the given figure.



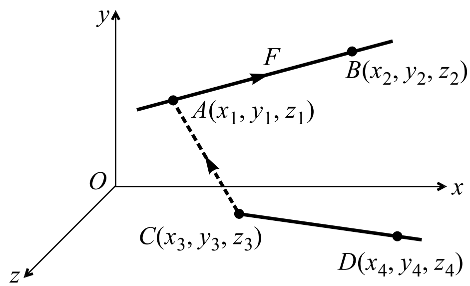
3. In the given figure, find the moment vector of force $\overline{\boldsymbol{F}}$ about a given point C.



4. In the given figure, find the vector component of force $\overline{\boldsymbol{F}}$ about a given line CD.



5. In the given figure, find the moment of force $\overline{\boldsymbol{F}}$ about a given line CD.



[III] Fill in the Blanks

1. By direction cosine rule, we have $\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z$ = _______ .
2. Scalar component of force vector $\overline{\boldsymbol{F}}$ along the line AB is the _______ product of $\overline{\boldsymbol{F}}$ and $\overline{\boldsymbol{e}}_{AB}$.
3. For concurrent force system if resultant is along x-axis, then $\Sigma F_y = \Sigma F_z$ = _______.
4. Position of resultant of parallel force system is found by ________ theorem.
5. Varignon's theorem is _______ applicable to general force system.

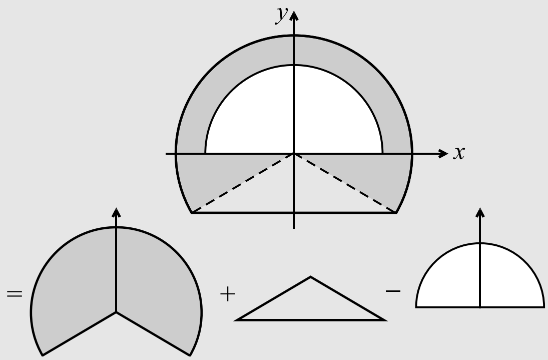
[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. If angle made by the force with y and z axes are $\theta_y = 60^o$ and $\theta_z = 45^o$ then θ_x = _______.
   (a) 130^o **(b)** 90^o **(c)** 105^o **(d)** 120^o
2. Which of the following actions of forces does not produce a moment ? _________
   (a) Opening a door **(b)** Opening a water tap
   (c) Paddling a bicycle **(d)** Compressing a spring
3. Which of the following conditions is a scalar quantity ? _________.
   (a) Moment of force about an axis **(b)** Moment of force about a point other than origin
   (c) Moment of force about the origin **(d)** Moment of couple
4. Sign convention for anticlockwise moment is considered +ve ________.
   (a) since they point along $-z$ direction **(b)** since they point along $+z$ direction
   (c) just for name sake **(d)** none of the above
5. The magnitude of moment is ________ when line of action of force is perpendicular to the lever.
   (a) 0 **(b)** minimum **(c)** maximum **(d)** negative
6. A force couple system can be converted into single force only when the resultant force and couple are ________ to each other.
   (a) inclined at 60^o **(b)** perpendicular **(c)** inclined at 30^o **(d)** parallel
7. Which of the following system of forces cannot be reduced to a single force ? _____.
   (a) Parallel forces in plane **(b)** Parallel forces in space
   (c) Non-concurrent forces in plane **(d)** Non-concurrent forces in space

5

CENTRE OF GRAVITY



5.1 Introduction

Take a body of any size, shape and mass m. Fix the nails at three different positions say A, B and C as shown in Fig. 5.1-i. Suspend the body through point A. The body will be in equilibrium under the action of the tension in the cord and the resultant W of the gravitational force acting on all particles of the body. This resultant is clearly collinear with the cord, and it will be assumed that we mark vertical dotted line of action through A. Repeat the experiment by suspending the same body from other points such as B and C and in each case, mark the vertical dotted lines of action of the resultant force. It can be observed that this line of action will be concurrent at a single point G, which is known as the ***centre of gravity*** of the body.

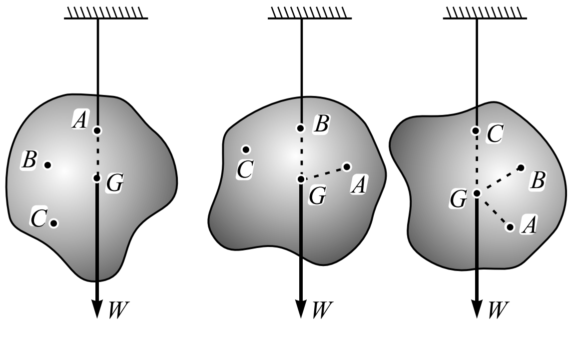


Fig. 5.1-i

Every body consists of particles. These particles are attracted towards the centre of earth. The force with which each particle of the body is attracted towards the centre of earth is called the *weight of that particle*. All these weights form a system of parallel forces acting towards the centre of the earth, which is far away from the body. Refer to Fig. 5.1-ii.

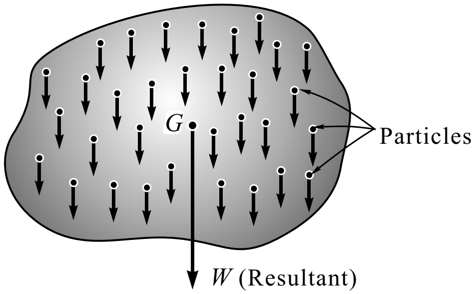


Fig. 5.1-ii

The attraction exerted by the Earth on a rigid body can be represented by a single force W. ***Centre of gravity of a body*** is *a point through which the resultant force of gravity acts irrespective of the orientation of the body*.

5.2 Centre of Gravity

Centre of gravity is a point where the whole weight of the body is assumed to act, i.e., *it is a point where entire distribution of gravitational force (weight) is suppose to be concentrated.*

The term ***centre of gravity*** is usually denoted by '*G*' for all three-dimensional rigid bodies, e.g., sphere, table, vehicle, dam, human, etc.

Centroid *is a point where the whole area of a plane lamina (figure) is assumed to act. It is a point where the entire length, area and volume is supposed to be concentrated.*

In other words, *centroid is the geometrical centre of a figure*. We use the term *centroid* for *two-dimensional figures*, i.e., areas, e.g., rectangle, triangle, circle, semicircle, sector, etc.

Centre of mass *is a point where the entire distribution of mass is supposed to be concentrated.*

Note : The method of finding the centroid or centre of mass or centre of gravity is very similar. In many books, these terms are treated equivalent. At surface of Earth, centre of mass and centre of gravity are considered to be same. It will slightly differ if body is too large as compared to earth which is hypothetical.

5.2.1 Centre of Gravity of a Flat Plate

Consider a flat plate having a uniform thickness (t) and lying in xy-plane as shown in Fig. 5.2.1-i. The plate can be divided into small elements having weights W_1, W_2, W_3, ..., W_n. The coordinates of elements are denoted by (x_1, y_1), (x_2, y_2), ..., (x_n, y_n).

The resultant of the elementary forces is the total weight W of the plate which acts through point G having coordinates $(\bar{x}, \bar{y})$ and can be given as

$$W = W_1 + W_2 + ... + W_n$$

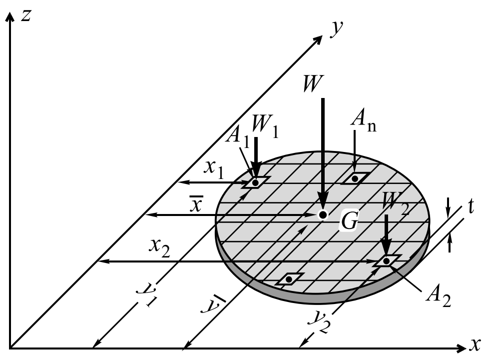


Fig. 5.2.1-i

Taking moment about y-axis and applying Varignon's theorem, we have

$$W(\bar{x}) = W_1 x_1 + W_2 x_2 + ... + W_n x_n$$

$$\bar{x} = \frac{W_1 x_1 + W_2 x_2 + ... + W_n x_n}{W_1 + W_2 + ... + W_n} \quad ...(5.1)$$

Centroid of an Area

Let t be the uniform thickness, ρ be the mass density and A be the total surface area of the plate.

We know, $W = m\,g = v\,\rho\,g = A\,t\,\rho\,g$. Similarly, considering the small elements of the plate, we get $W_1 = A_1\,t\,\rho\,g,\ W_2 = A_2\,t\,\rho\,g,\ ......,\ W_n = A_n\,t\,\rho\,g$

Putting the above relation in Eq. (5.1), we get

$$\bar{x} = \frac{A_1\,t\,\rho\,g\,x_1 + A_2\,t\,\rho\,g\,x_2 + ... + A_n\,t\,\rho\,g\,x_n}{A_1\,t\,\rho\,g + A_2\,t\,\rho\,g + ... + A_n\,t\,\rho\,g}$$

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2 + ... + A_n x_n}{A_1 + A_2 + ... + A_n} = \frac{\Sigma A_i x_i}{\Sigma A_i} \quad ...(5.2)$$

Similarly, taking moment about x-axis and applying Varignon's theorem, we get

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + ... + A_n y_n}{A_1 + A_2 + ... + A_n} = \frac{\Sigma A_i y_i}{\Sigma A_i} \quad ...(5.3)$$

Mathematically, equations (5.2) and (5.3) can also be represented as follows :

$$\bar{x} = \frac{\int x\, dA}{\int dA} \quad ; \quad \bar{y} = \frac{\int y\, dA}{\int dA} \quad ...(5.4)$$

Points to Remember

1. Centroid is the geometrical centre of a figure.
2. For a symmetric figure, centroid lies on the axis of symmetry.
3. If the figure is symmetric about x-axis then y coordinate of C.G., i.e., $\bar{y} = 0$.
4. If the figure is symmetric about y-axis then x coordinate of C.G., i.e., $\bar{x} = 0$.
5. If a figure has more than one axis of symmetry then the intersection of axis of symmetry is the centroid of the given figure.
6. Centroid may or may not lie on the given figure.
7. If the xis of symmetry is inclined at 45° to horizontal line then $\bar{x} = \bar{y}$.

Freely Suspended Plane Lamina or Bent Wire in Equilibrium

For a freely suspended plane lamina or a bent wire in equilibrium, the centroid or centre of gravity will lie vertically below the point of suspension.

On the other hand, if a plane lamina or a bent wire is freely suspended exactly through the centroid or centre of gravity point, it will remain in equilibrium in any position.

Centroid of Areas

1. Centroid of Triangular Area

Consider the triangular area with base b and height h. (Fig. 5.2.1-ii)

Consider an elemental strip of thickness dy at a distance of y from base.

Now ΔADE and ΔABC, by property of similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h}$$

$$l = \frac{(h-y)b}{h}$$

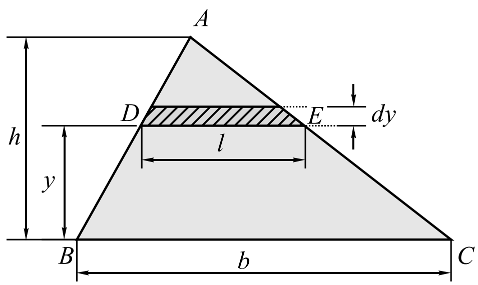


Fig. 5.2.1-ii

From basic principle of centroid, we know

$$\bar{y} = \frac{\int y\, dA}{\int dA} \quad \text{where } dA = \text{Area of elemental strip}$$

$$y = \text{Centroid distance of strip from base}$$

$$= \frac{\int_0^h y\, l\, dy}{\int_0^h l\, dy} = \frac{\int_0^h \frac{h-y}{h} b\, y\, dy}{\int_0^h \frac{(h-y)b}{h} dy} = \frac{\frac{b}{h}\int_0^h (hy - y^2)\, dy}{\frac{b}{h}\int_0^h (h-y)\, dy}$$

$$= \frac{\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h}{\left[hy - \frac{y^2}{2}\right]_0^h} = \frac{\left[\frac{h^3}{2} - \frac{h^3}{3}\right]}{\left[h^2 - \frac{h^2}{2}\right]} = \frac{\frac{h^3}{6}}{\frac{h^2}{2}}$$

$$\therefore \quad \bar{y} = \frac{h}{3}$$

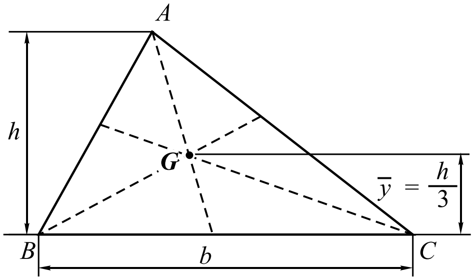


Fig. 5.2.1-iii

2. Centroid of Semicircular Area

Consider a semicircular lamina with radius r. (Fig. 5.2.1-iv)

Here semicircular area is symmetric about y-axis, therefore $\bar{x} = 0$.

Select an elementary sector as shown in Fig. 5.2-iv, which can be considered as a triangle whose base is $r\, d\theta$ and altitude is r.

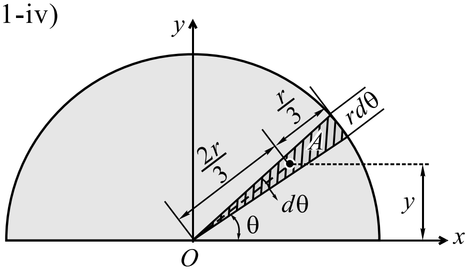


Fig. 5.2.1-iv

The location of the centroid of the elementary sector is A.

$\because$ Distance $OA = \frac{2r}{3}$

$\therefore$ y coordinate of the centroid of element is $y = \frac{2}{3} r \sin\theta$

Area of the elementary sector $dA = \frac{1}{2} \times r \times r\, d\theta = \frac{1}{2} r^2\, d\theta$

From basic principle of centroid, we have

$$\bar{y} = \frac{\int_0^\pi y\, dA}{\int_0^\pi dA} = \frac{\int_0^\pi \frac{2}{3} r \sin\theta \cdot \frac{1}{2} r^2\, d\theta}{\int_0^\pi \frac{1}{2} r^2\, d\theta} = \frac{\frac{r^3}{3}\int_0^\pi \sin\theta\, d\theta}{\frac{r^2}{2}\int_0^\pi d\theta}$$

$$= \frac{\frac{r^3}{3}\left[-\cos\theta\right]_0^\pi}{\frac{r^2}{2}\left[\theta\right]_0^\pi} = \frac{\frac{r^3}{3} \times 2}{\frac{r^2}{2}\pi}$$

$$\therefore \quad \bar{y} = \frac{4r}{3\pi}$$

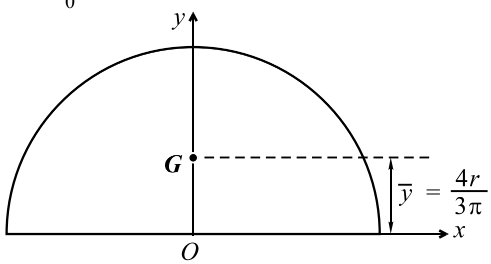


Fig. 5.2.1-v

3. Centroid of Quarter Circular Area

Consider a quarter circular area with radius r. Taking the similar consideration as that of semicircular area, we have

$$dA = \frac{1}{2} \times r \times r\,d\theta = \frac{1}{2} r^2\,d\theta$$

$$x = \frac{2}{3} r\cos\theta$$

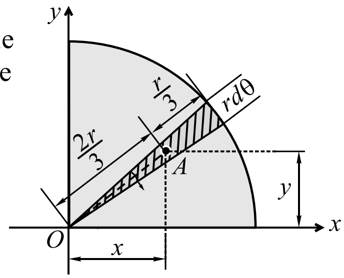


Fig. 5.2.1-vi

From the basic principle of centroid, we have

$$\bar{x} = \frac{\int_0^{\pi/2} x\,dA}{\int_0^{\pi/2} dA} = \frac{\int_0^{\pi/2} \frac{2}{3} r\cos\theta \cdot \frac{1}{2} r^2\,d\theta}{\int_0^{\pi/2} \frac{1}{2} r^2\,d\theta}$$

$$= \frac{\frac{r^3}{3}\int_0^{\pi/2} \cos\theta\,d\theta}{\frac{r^2}{2}\int_0^{\pi/2} d\theta} = \frac{\frac{r^3}{3}\left[\sin\theta\right]_0^{\pi/2}}{\frac{r^2}{2}\left[\theta\right]_0^{\pi/2}} = \frac{\frac{r^3}{3}}{\frac{r^2}{2}\,\frac{\pi}{2}}$$

$$\therefore \quad \bar{x} = \frac{4r}{3\pi}$$

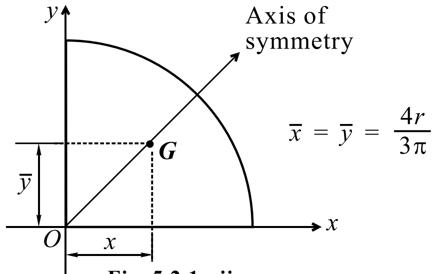


Fig. 5.2.1-vii

As axis of symmetry is inclined at 45° to horizontal line

$$\therefore \quad \bar{x} = \bar{y} = \frac{4r}{3\pi}$$

4. Centroid of Sector of a Circle

Consider the sector of a circle which subtends an angle 2α at the centre. Here the figure is symmetric about x-axis, therefore, $\bar{y} = 0$.

For finding $\bar{x}$, select an elementary sector which can be considered as a triangle whose base is $r\,d\theta$ and altitude is r.

The location of the centroid of the elementary sector is A as shown in Fig. 5.2.1-viii.

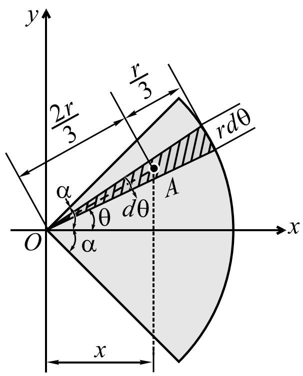


Fig. 5.2.1-viii

$$dA = \frac{1}{2} \times r \times r\,d\theta = \frac{1}{2} r^2\,d\theta \quad \text{and} \quad x = \frac{2}{3} r\cos\theta$$

From the basic principle of centroid, we have

$$\bar{x} = \frac{\int_{-\alpha}^{\alpha} x\,dA}{\int_{-\alpha}^{\alpha} dA} = \frac{\int_{-\alpha}^{\alpha} \frac{2}{3} r\cos\theta \cdot \frac{1}{2} r^2\,d\theta}{\int_{-\alpha}^{\alpha} \frac{1}{2} r^2\,d\theta} = \frac{\frac{r^3}{3}\int_{-\alpha}^{\alpha} \cos\theta\,d\theta}{\frac{r^2}{2}\int_{-\alpha}^{\alpha} d\theta}$$

$$= \frac{\frac{r^3}{3}\left[\sin\theta\right]_{-\alpha}^{\alpha}}{\frac{r^2}{2}\left[\theta\right]_{-\alpha}^{\alpha}} = \frac{\frac{2}{3} r[\sin\alpha - \sin(-\alpha)]}{\alpha - (-\alpha)} = \frac{\frac{2r}{3}\,2\sin\alpha}{2\alpha}$$

$$\therefore \quad \bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

Procedure to Locate Centroid of Composite Area

1. Place the given figure into the first quadrant touching the outer edge of figure to the axis selected, if the axes are not given.
2. Divide the given composite figure into standard geometrical shapes such as rectangle, triangle, circle, semicircle, quarter circle, sector of circle, etc.
3. Mark the centroids G_1, G_2, ..., etc., on the composite figure and find their coordinates w.r.t. the given axes, i.e. x_1, x_2, ..., etc., and y_1, y_2, ..., etc.
4. Find the coordinates of centroid using the following equations:

$$\bar{x} = \frac{\Sigma A_i x_i}{\Sigma A_i} = \frac{A_1 x_1 + A_2 x_2 + \dots + A_n x_n}{A_1 + A_2 + \dots + A_n}$$

$$\bar{y} = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{A_1 y_1 + A_2 y_2 + \dots + A_n y_n}{A_1 + A_2 + \dots + A_n}$$

5.3 Solved Problems

Problem 1

Find the centroid of the shaded area in Fig. 5.1(a).

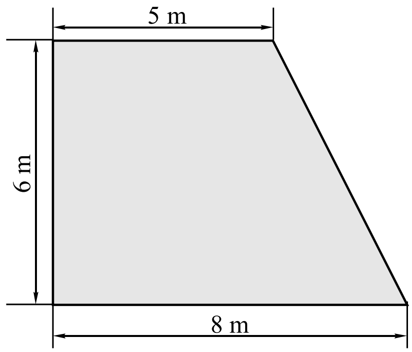
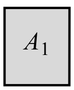


Fig. 5.1(a)

Solution

(i) The area can be viewed as a rectangle and a triangle combined together. The areas and centroidal coordinates for each of these shapes can be determined by referring to Fig. 5.1(b).

(ii) Consider rectangle *OABD*

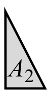


$$A_1 = 5 \times 6 = 30 \text{ m}^2$$

$$x_1 = \frac{5}{2} = 2.5 \text{ m}$$

$$y_1 = \frac{6}{2} = 3 \text{ m}$$

(iii) Consider triangle *DBC*



$$A_2 = \frac{1}{2} \times 6 \times 3 = 9 \text{ m}^2$$

$$x_2 = 5 + \frac{3}{3} = 6 \text{ m}$$

$$y_2 = \frac{6}{3} = 2 \text{ m}$$

(iv) Coordinates of the centroid of shaded area

$$\bar{x} = \frac{30 \times 2.5 + 9 \times 6}{30 + 9} = 3.308 \text{ m}$$

$$\bar{y} = \frac{30 \times 3 + 9 \times 2}{30 + 9} = 2.769 \text{ m}$$

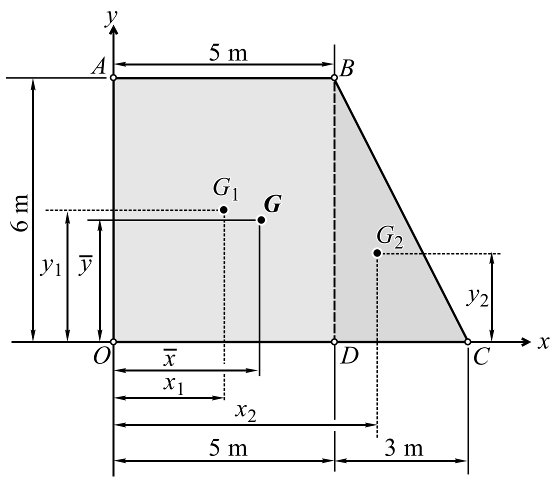


Fig. 5.1(b)

$\therefore$ Coordinates of centroid w.r.t. origin O are G (3.308, 2.769) m. ***Ans.***

Problem 2

Three plates ABC and $BCDE$ and DEF are welded together as shown in Fig. 5.2(a). Circle of diameter 1.5 m is cut from the composite plate. Determine the centroid of the remaining area.

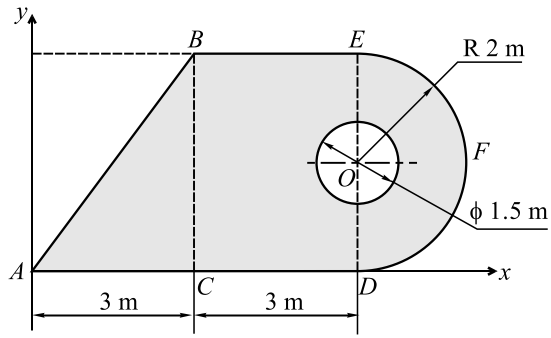


Fig. 5.2(a)

Solution

(i) The composite area can be viewed as a triangle ⊕ rectangle ⊕ semicircle ⊖ circle. The areas and centroidal coordinates for each of these shapes can be determined by referring to Fig. 5.2(b).

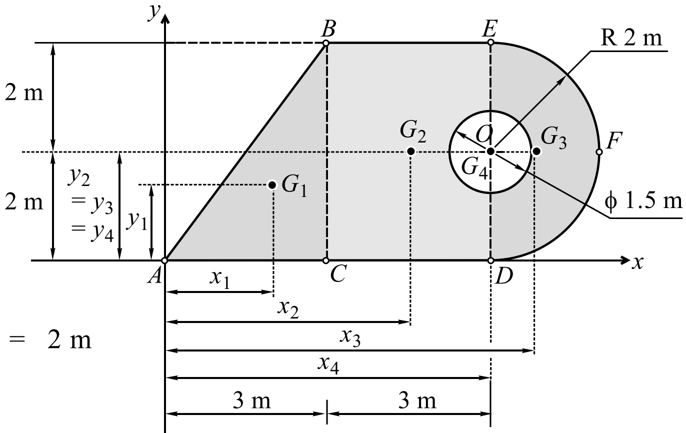
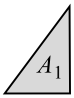


Fig. 5.2(b)

(ii) Consider triangle ABC

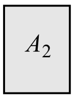


$$A_1 = \frac{1}{2} \times 3 \times 4 = 6 \text{ m}^2$$

$$x_1 = \frac{2}{3} \times 3 = 2 \text{ m}$$

$$y_1 = \frac{1}{3} \times 4 = 1.33 \text{ m}$$

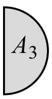
(iii) Consider rectangle $BCDE$



$$A_2 = 3 \times 4 = 12 \text{ m}^2$$

$$x_2 = 3 + 1.5 = 4.5 \text{ m and } y_2 = 2 \text{ m}$$

(iv) Consider semicircle EFD

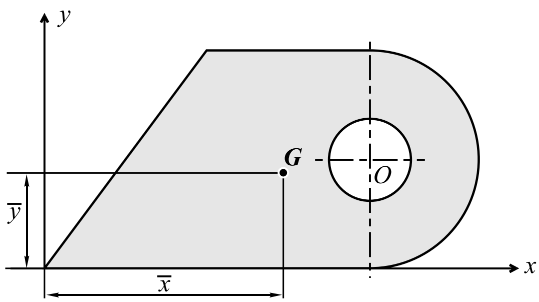


$$r = 2 \text{ m}$$

$$A_3 = \frac{\pi \times 2^2}{2} = 6.283 \text{ m}^2$$

$$x_3 = 3 + 3 + \frac{4 \times 2}{3\pi} = 6.848 \text{ m and}$$

$$y_3 = 2 \text{ m}$$



(v) Consider circle with centre O and radius r

$$r = 0.75 \text{ m}$$

$$-A_4 = -(\pi \times 0.75^2) = -1.767 \text{ m}^2$$

$$x_4 = 6 \text{ m and } y_4 = 2 \text{ m}$$

(vi) Centroid of the given shaded area is given as

$$\bar{x} = \frac{6 \times 2 + 12 \times 4.5 + 6.283 \times 6.848 - 1.767 \times 6}{6 + 12 + 6.283 - 1.767}$$

$$\bar{x} = 4.37 \text{ m}$$

$$\bar{y} = \frac{6 \times 1.33 + 12 \times 2 + 6.283 \times 2 - 1.767 \times 2}{6 + 12 + 6.283 - 1.767}$$

$$\bar{y} = 1.82 \text{ m}$$

∴ Coordinates of centroid w.r.t. origin A are $\boldsymbol{G}$ (4.37, 1.82) m. ***Ans.***

Problem 3

Find the coordinates of the centroid of the area shown in Fig. 5.3(a).

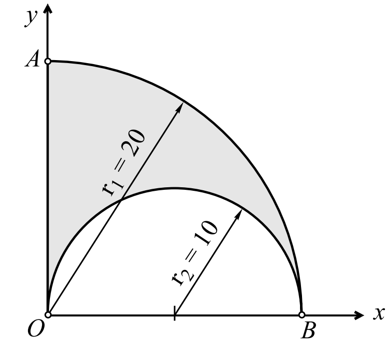


Fig. 5.3(a)

[All dimensions are in cm]

Solution

(i) Divide the given area into two subareas as shown.

(ii) Quarter circle *OAB* : part ①

$$A_1 = \frac{\pi r_1^2}{4} = \frac{\pi \times 20^2}{4} = 314.16 \text{ cm}^2$$

$$x_1 = y_1 = \frac{4r_1}{3\pi} = \frac{4 \times 20}{3\pi}$$

$$\therefore x_1 = y_1 = 8.49 \text{ cm}$$

(iii) Semicircle *OB* : part ②

$$A_2 = \frac{\pi r_2^2}{2} = \frac{\pi \times 10^2}{2} = 157.08 \text{ cm}^2$$

$$x_2 = 10 \text{ cm}$$

$$y_2 = \frac{4r_2}{3\pi} = 4.244 \text{ cm}$$

(iv) Centroid of the given shaded area is given as

$$\bar{x} = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2} = \frac{314.16 \times 8.49 - 157.08 \times 10}{314.16 - 157.08}$$

$$\bar{x} = 6.98 \text{ cm}$$

$$\bar{y} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{314.16 \times 8.49 - 157.08 \times 4.244}{314.16 - 157.08}$$

$$\bar{y} = 12.74 \text{ mm}$$

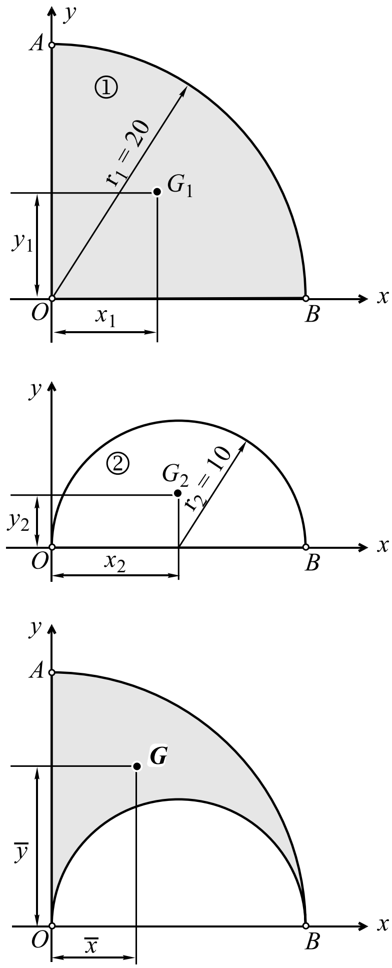


Fig. 5.3(b)

$\therefore$ Coordinates of centroid w.r.t. origin *O* are ***G*** (6.98, 12.74) cm. ***Ans.***

Problem 4

Find the centroid of the shaded area shown in Fig. 5.4(a).

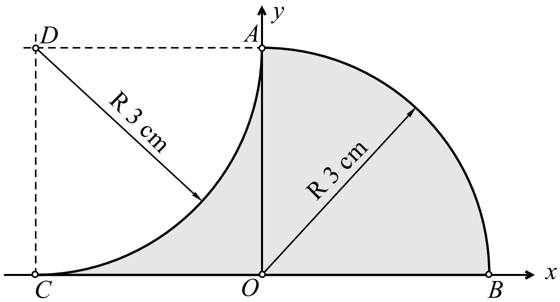


Fig. 5.4(a)

Solution

(i) Divide the shaded area into three parts as shown.

(ii) Quarter circle : part ①

$$A_1 = \frac{\pi \times 3^2}{4} = 7.07 \text{ cm}^2$$

$$x_1 = y_1 = \frac{4 \times 3}{3\pi} = 1.27 \text{ cm}$$

(iii) Square : part ②

$$A_2 = 3 \times 3 = 9 \text{ cm}^2$$

$$x_1 = -1.5 \text{ cm} \; ; \; y_2 = 1.5 \text{ cm}$$

(iv) Quarter circle : part ③

$$-A_3 = -\frac{\pi \times 3^2}{4} = -7.07 \text{ cm}^2$$

$$x_3 = -\left(3 - \frac{4 \times 3}{3\pi}\right) = -1.73 \text{ cm}$$

$$y_3 = 3 - \frac{4 \times 3}{3\pi} = 1.73 \text{ cm}$$

(v) Centroid of the given shaded area is given as

$$\bar{x} = \frac{7.07 \times 1.27 + 9 \times (-1.5) + (-7.07) \times (-1.73)}{7.07 + 9 - 7.07}$$

$$\bar{x} = 0.8566 \text{ cm}$$

$$\bar{y} = \frac{7.07 \times 1.27 + 9 \times 1.5 + (-7.07) \times 1.73}{7.07 + 9 - 7.07}$$

$$\bar{y} = 1.139 \text{ cm}$$

∴ Coordinates of centroid w.r.t. origin O are $\boldsymbol{G}$ (0.8566, 1.139) cm. ***Ans.***

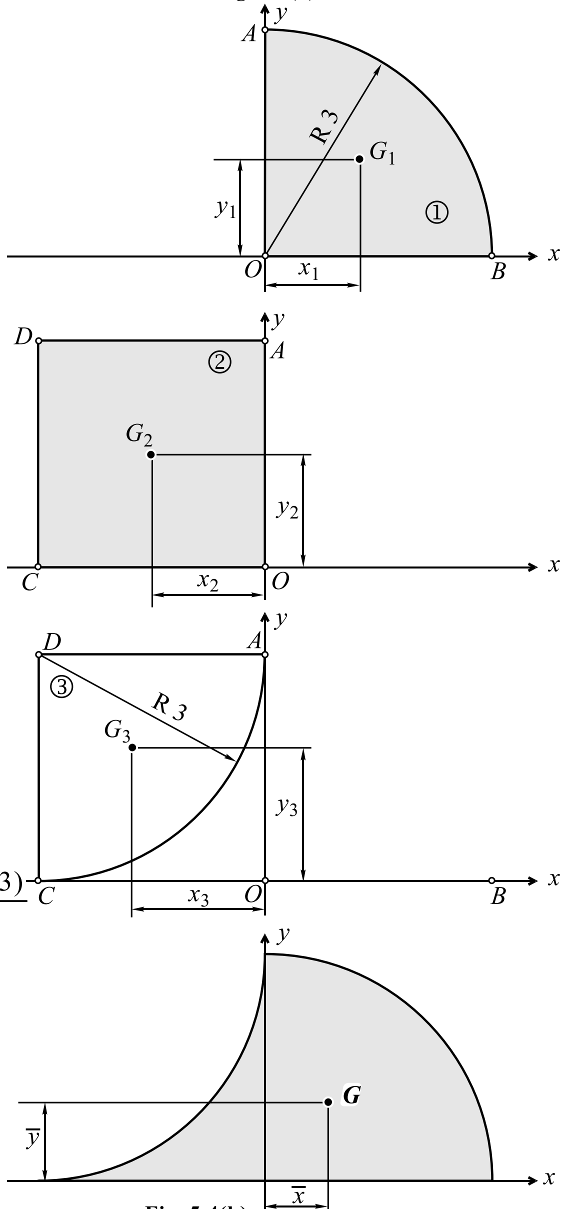


Fig. 5.4(b)

Problem 5

Calculate numerically the centroid of the shaded area shown in Fig. 5.5(a).

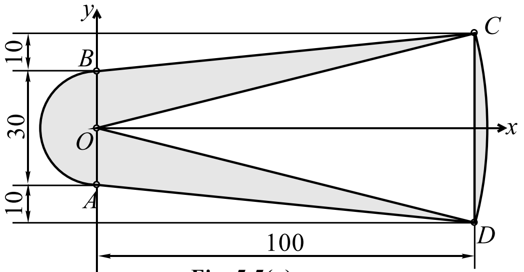


Fig. 5.5(a)

[All dimensions are in cm]

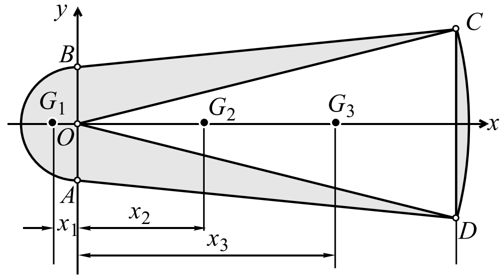


Fig. 5.5(b)

Solution

(i) The given figure is symmetric about x-axis

$$\therefore \quad \bar{y} = 0.$$

(ii) Consider semicircle : part ①

$$A_1 = \frac{\pi \times 15^2}{2} = 353.43 \text{ cm}^2$$

$$x_1 = \frac{-4 \times 15}{3\pi} = -6.37 \text{ cm}$$

(iii) Consider two equal triangles : part ②

$$2(A_2) = 2\left(\frac{1}{2} \times 15 \times 100\right) = 2(750) \text{ cm}^2$$

$$x_2 = \frac{100}{3} = 33.33 \text{ cm}$$

(iv) Consider a sector of circle : part ③

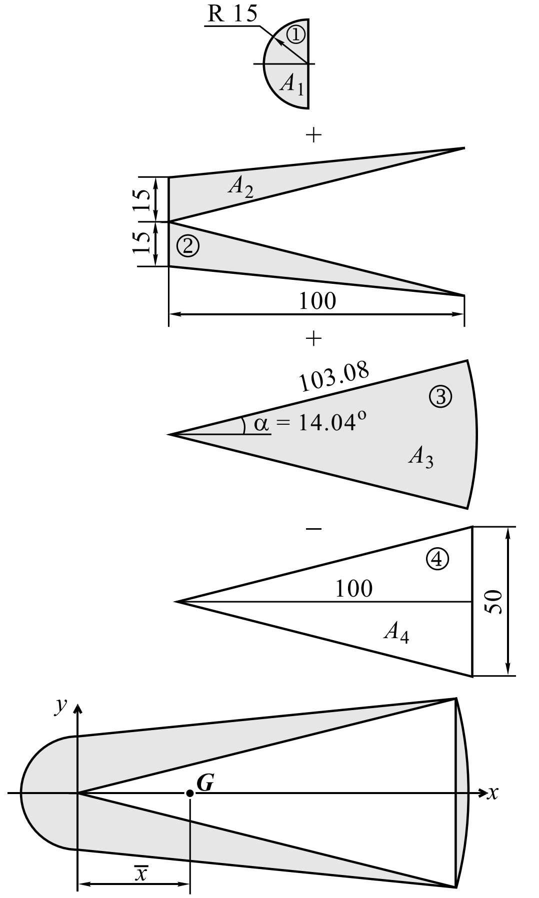
$$A_3 = 103.08^2 \times 14.04 \times \frac{\pi}{180} = 2603.71 \text{ cm}^2$$

$$x_3 = \frac{2 \times 103.08 \sin 14.04}{3 \times 14.04 \times \dfrac{\pi}{180}} = 68.03 \text{ cm}$$

(v) Consider triangle : part ④

$$-A_4 = -\left(\frac{1}{2} \times 50 \times 100\right) = -2500 \text{ cm}^2$$

$$x_4 = \frac{2}{3} \times 100 = 66.67 \text{ cm}$$



(vi) Coordinates of the centroid

$$\bar{x} = \frac{353.43 \times (-6.37) + 2(750 \times 33.33) + 2603.71 \times 68.03 + (-2500) \times (66.67)}{353.43 + 2(750) + 2603.71 - 2500} = 29.74 \text{ cm}$$

$\therefore$ Coordinates of centroid w.r.t. origin O are $\boldsymbol{G}$ (29.74, 0) cm. ***Ans.***

Problem 6

A thin homogenous semicircular plate of radius r is suspended from is corner A as shown in Fig. 5.6(a). Find the angle made by its straight edge AB with the vertical.

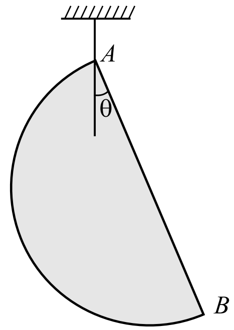


Fig. 5.6(a)

Solution

If the plate is suspended through point A then its centroid lies on the vertical line passing through A.

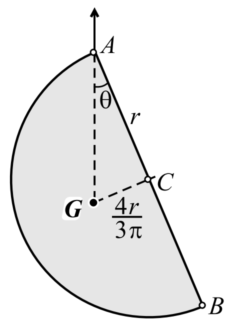


Fig. 5.6(b)

In ΔACG,

$$\tan\theta = \frac{\left(\frac{4r}{3\pi}\right)}{r}$$

$$\tan\theta = \frac{4}{3\pi}$$

$$\therefore\ \theta = 23^{\circ} \quad \textbf{\textit{Ans.}}$$

Problem 7

A thin homogenous composite plate formed by a semicircular and triangular shape as shown in Fig. 5.7(a), is freely suspended from point A. If the side BC remains horizontal in equilibrium condition then find side BC.

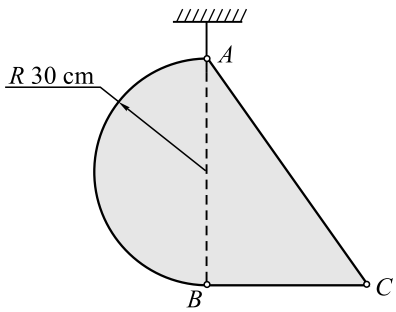


Fig. 5.7(a)

Solution

If the plate is suspended through A then its centroid lies on the vertical line passing through A.

Consider y-axis through A which passes through C.G. of plate.

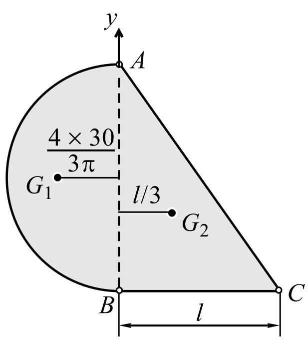


Fig. 5.7(b)

$$\therefore\ \bar{x} = 0$$

$$\bar{x} = 0 = \left(\frac{\pi \times 30^2}{2}\right)\left(\frac{-4 \times 30}{3\pi}\right) + \left(\frac{1}{2} \times l \times 60\right)\left(\frac{l}{3}\right)$$

$$\frac{-4 \times 30^3}{6} + 10\,l^2 = 0$$

$$10\,l^2 = \frac{4}{6} \times 30^3$$

$$l = 42.43 \text{ cm} \quad \textbf{\textit{Ans.}}$$

Problem 8

An isosceles triangle is to be cut from one edge of a square plate of side 1 m such that the remaining part of the plate remains in equilibrium in any position when suspended from the apex for the triangle. Find the area of the triangle to be removed.

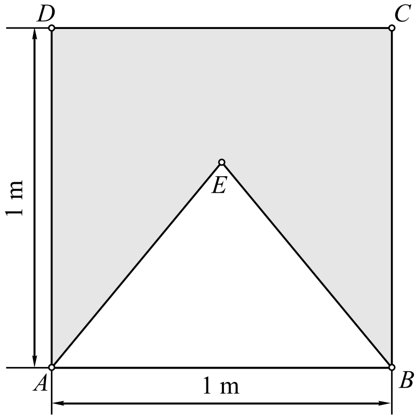


Fig. 5.8(a)

Solution

If the plane lamina is suspended exactly through the centroid, then it remains in equilibrium in any position.

(i) Therefore, as per the given condition in the problem the apex of triangle E must act as the centroid of remaining part.

Let h be the height of the triangle

$\therefore\ \bar{y} = h$

(ii) Rectangle $ABCD$: part ①

$$A_1 = 1 \times 1 = 1 \text{ m}^2$$

$$y_1 = 0.5 \text{ m}$$

(iii) Triangle ABE : part ②

$$-A_2 = -\frac{1}{2} \times 1 \times h = -0.5\,h \text{ m}^2$$

$$y_2 = \frac{h}{3} \text{ m}$$

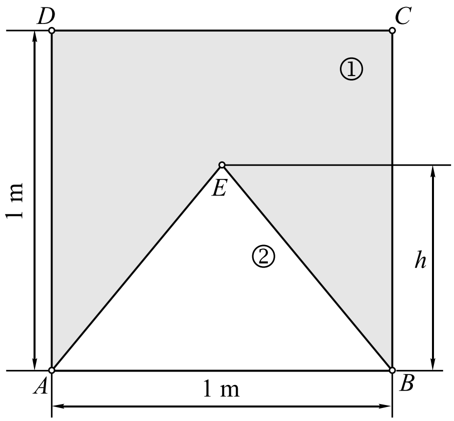


Fig. 5.8(b)

(iv) Coordinates of the centroid of given shaded area can be calculated as

$$\therefore\ \bar{y} = h = \frac{1 \times 0.5 - 0.5h \times \dfrac{h}{3}}{1 - 0.5h}$$

$$h(1 - 0.5\,h) = 0.5 - \frac{0.5h^2}{3}$$

$$h - 0.5h^2 = \frac{1.5 - 0.5h^2}{3}$$

$$3h - 1.5h^2 = 1.5 - 0.5h^2$$

$$h^2 - 3h + 1.5 = 0$$

$$h = 0.634 \text{ m or } h = 2.37 \text{ m}$$

$h = 0.634$ m (because h cannot be greater than 1 m)

$\therefore$ Area of triangle to be removed $= \dfrac{1}{2} \times 1 \times 0.634 = 0.317 \text{ m}^2$ ***Ans.***

5.4 Solved Problems Using Integration

Problem 9

Find the centroid of the shaded area OPQ, shown in Fig. 5.9(a). The curve OQ is parabolic.

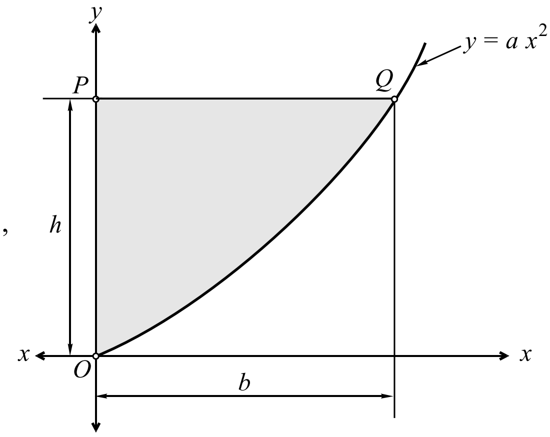


Fig. 5.9(a)

Solution

Step I : To find $\bar{x}$

(i) *Differential Element*

Consider an elementary strip, parallel to y-axis, of thickness dx as shown in Fig. 5.22(b).

(ii) *Area and Moment Arm*

The area of the elementary strip is

$dA = (h - y)\, dx$

Its moment arm is at a distance x from y-axis.

(iii) *Integration*

Taking moment of area about y-axis, we have

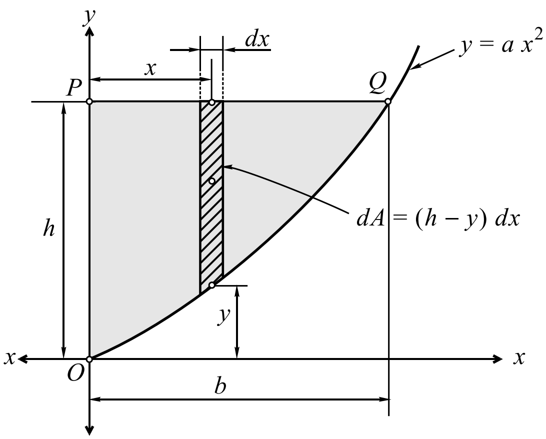


Fig. 5.9(b)

$$\bar{x} = \frac{\int x\, dA}{\int dA} = \frac{\int_0^b x\,(h-y)\,dx}{\int_0^b (h-y)\,dx}$$

$$\therefore\ \bar{x} = \frac{\int_0^b x\,(h - a\,x^2)\,dx}{\int_0^b (h - a\,x^2)\,dx} \qquad \left\{\begin{matrix} y = a\,x^2 \text{ equation} \\ \text{of the curve.}\end{matrix}\right\}$$

$$\therefore\ \bar{x} = \frac{\int_0^b x\,h\,dx - \int_0^b a\,x^3\,dx}{\int_0^b h\,dx - \int_0^b a\,x^2\,dx} = \frac{\left[\frac{h\,x^2}{2}\right]_0^b - \left[a\,\frac{x^4}{4}\right]_0^b}{\left[h\,x\right]_0^b - \left[a\,\frac{x^3}{3}\right]_0^b}$$

$$\therefore\ \bar{x} = \frac{\left(\frac{h\,b^2}{2} - \frac{a\,b^4}{4}\right)}{\left(h\,b - \frac{a\,b^3}{3}\right)} = \frac{\left(\frac{2h\,b^2 - a\,b^4}{4}\right)}{\left(\frac{3h\,b - a\,b^3}{3}\right)}$$

$$\therefore\ \bar{x} = \frac{b^2(2h - a\,b^2)}{4} \times \frac{3}{b(3h - a\,b^2)}$$

$$\therefore\ \bar{x} = \frac{3b(2h - a\,b^2)}{4(3h - a\,b^2)} \qquad \textit{\textbf{Ans.}}$$

Step II : To find $\bar{y}$

(i) *Differential Element*

Consider a small elementary strip parallel to x-axis of thickness dy as shown in Fig. 5.9(c).

Width of the element = x

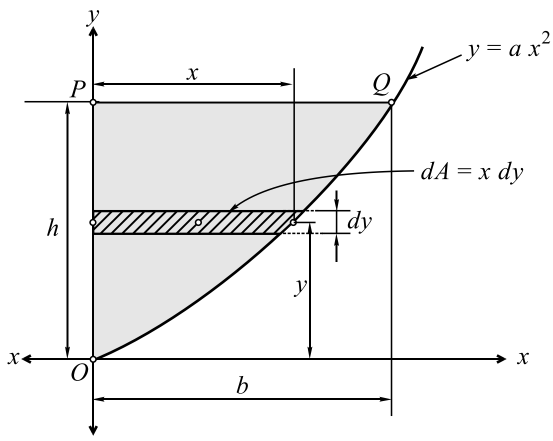


Fig. 5.9(c)

(ii) *Area and Moment Arm*

The area of the elementary strip is

$dA = x\,dy$

Its moment arm is at a distance y from x-axis.

(iii) *Integration*

Taking moment of area about x-axis, we have

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int_0^h y\,x\,dy}{\int_0^h x\,dy}$$

$$\bar{y} = \frac{\int_0^h y\,\frac{1}{\sqrt{a}}\,\sqrt{y}\,dy}{\int_0^h \frac{1}{\sqrt{a}}\,\sqrt{y}\,dy}$$

$$\therefore\ \bar{y} = \frac{\frac{1}{\sqrt{a}}\int_0^h y^{3/2}\,dy}{\frac{1}{\sqrt{a}}\int_0^h y^{1/2}\,dy}$$

$$\therefore\ \bar{y} = \frac{\left[\frac{2}{5}\,y^{5/2}\right]_0^h}{\left[\frac{2}{3}\,y^{3/2}\right]_0^h}$$

$$\therefore\ \bar{y} = \frac{\frac{2}{5}\times h^{5/2}}{\frac{2}{3}\times h^{3/2}}$$

$$\therefore\ \bar{y} = \frac{2}{5}\times\frac{3}{2}\times h$$

$$\therefore\ \bar{y} = \frac{3}{5}\,h \quad \textbf{\textit{Ans.}}$$

Problem 10

Locate the centroid of the area bounded by the curve as shown in Fig. 5.10(a).

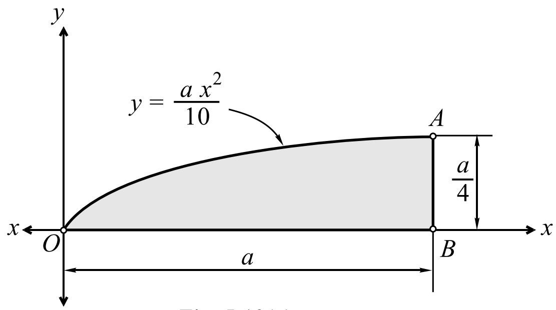


Fig. 5.10(a)

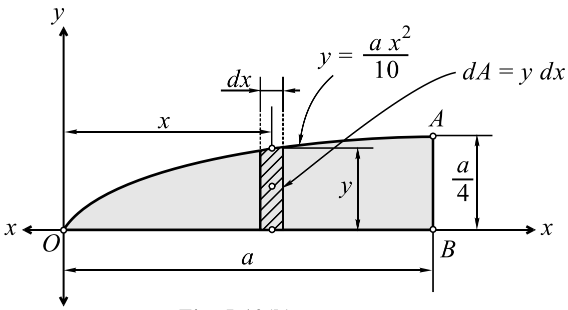


Fig. 5.10(b)

Solution

Step I : To locate $\bar{x}$

(i) *Differential Area Element*

Consider a small elemental strip of thickness dx at a distance x from y-axis and parallel to y-axis as shown in Fig. 5.10(b).

The area of the elementary strip is

$$dA = y\,dx$$

(ii) *Moment About the y-axis*

Taking moment of area of the elemental strip about y-axis, we have

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^a x\,y\,dx}{\int_0^a y\,dx}$$

$$\therefore\ \bar{x} = \frac{\int_0^a x\,\frac{ax^2}{10}\,dx}{\int_0^a \frac{ax^2}{10}\,dx}$$

$$\therefore\ \bar{x} = \frac{\int_0^a \frac{a}{10}x^3\,dx}{\int_0^a \frac{a}{10}x^2\,dx}$$

$$\therefore\ \bar{x} = \frac{\frac{a}{10}\left[\frac{x^4}{4}\right]_0^a}{\frac{a}{10}\left[\frac{x^3}{3}\right]_0^a}$$

$$\therefore\ \bar{x} = \frac{a^4}{4} \times \frac{3}{a^3}$$

$$\therefore\ \bar{x} = \frac{3}{4}a \quad \textbf{\textit{Ans.}}$$

Step II : To locate $\bar{y}$

(i) *Differential Area Element*

Consider a small elemental strip of thickness *dy,* which is at a distance y from x-axis and parallel to x-axis, as shown in Fig. 5.10(c).

Width of the element $= (a - x)$

The area of the elementary strip is

$dA = (a - x)\, dy$

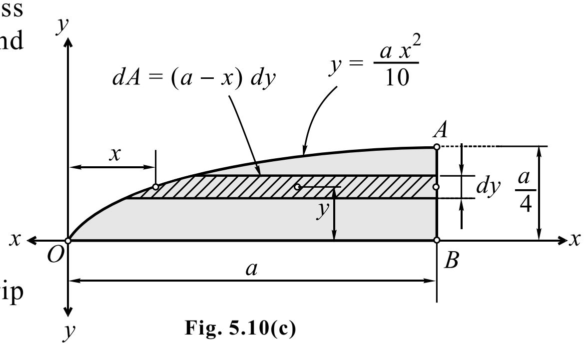


Fig. 5.10(c)

(ii) *Moment About the x-axis*

Taking moment of area of the elemental strip about the x-axis, we have

$$\bar{y} = \frac{\int y\, dA}{\int dA} = \frac{\int_0^{a/4} y\,(a - x)\, dy}{\int_0^{a/4} (a - x)\, dy}$$

$$\therefore \quad \bar{y} = \frac{\int_0^{a/4} y\left(a - \frac{\sqrt{10}\sqrt{y}}{\sqrt{a}}\right) dy}{\int_0^{a/4} \left(a - \frac{\sqrt{10}\sqrt{y}}{\sqrt{a}}\right) dy} = \frac{\int_0^{a/4} a\, y\, dy - \int_0^{a/4} \frac{\sqrt{10}}{\sqrt{a}}\, y^{3/2}\, dy}{\int_0^{a/4} a\, dy - \int_0^{a/4} \frac{\sqrt{10}}{\sqrt{a}}\, y^{1/2}\, dy}$$

$$\therefore \quad \bar{y} = \frac{\left[a\,\frac{y^2}{2}\right]_0^{a/4} - \left[\frac{\sqrt{10}}{\sqrt{a}}\, y^{5/2} \times \frac{2}{5}\right]_0^{a/4}}{\left[a\,y\right]_0^{a/4} - \left[\frac{\sqrt{10}}{\sqrt{a}}\, y^{3/2} \times \frac{2}{3}\right]_0^{a/4}} = \frac{\left[\frac{a}{2} \times \frac{a^2}{16}\right] - \left[\frac{2\sqrt{10}}{3\sqrt{a}} \times \frac{a^{5/2}}{30}\right]}{\left[a \times \frac{a}{4}\right] - \left[\frac{2\sqrt{10}}{3\sqrt{a}} \times \frac{a^{3/2}}{8}\right]}$$

$$\therefore \quad \bar{y} = \frac{\left[\frac{a^3}{32}\right] - \left[\frac{\sqrt{10}}{80} \times a^2\right]}{\left[\frac{a^2}{4}\right] - \left[\frac{\sqrt{10}}{12} \times a\right]}$$

$$\therefore \quad \bar{y} = \frac{\left(\frac{5a^3 - 2\sqrt{10}\, a^2}{160}\right)}{\left(\frac{3a^2 - \sqrt{10}\, a}{12}\right)} = \frac{\frac{a^2}{160}\,(5a - 2\sqrt{10}\,)}{\frac{a}{12}\,(3a - \sqrt{10}\,)}$$

$$\therefore \quad \bar{y} = \frac{a^2}{160}(5a - 2\sqrt{10}\,) \times \frac{12}{a} \times \frac{1}{(3a - \sqrt{10}\,)}$$

$$\therefore \quad \bar{y} = \frac{3a}{40}\;\frac{(5a - 2\sqrt{10}\,)}{(3a - \sqrt{10}\,)} \quad \textbf{\textit{Ans.}}$$

Problem 11

Show that the coordinates of the centroid G of the area between the parabola $y = \dfrac{x^2}{a}$ and the straight line $y = x$ are $\bar{x} = \dfrac{a}{2}$, $\bar{y} = \dfrac{2a}{5}$.

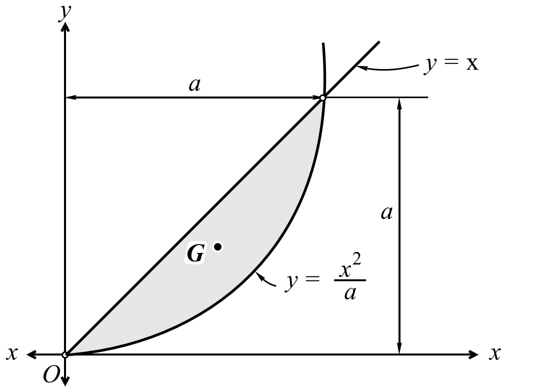


Fig. 5.11(a)

Solution

Step I : To show that $\bar{x} = \dfrac{a}{2}$

(i) *Differential Area Element*

Consider a small elemental strip of thickness dx at a distance x from y-axis as shown in Fig. 5.11(b).

The height of the element is $(y_2 - y_1)$.

$\therefore$ Area of the elementary strip is

$$dA = (y_2 - y_1)\, dx$$

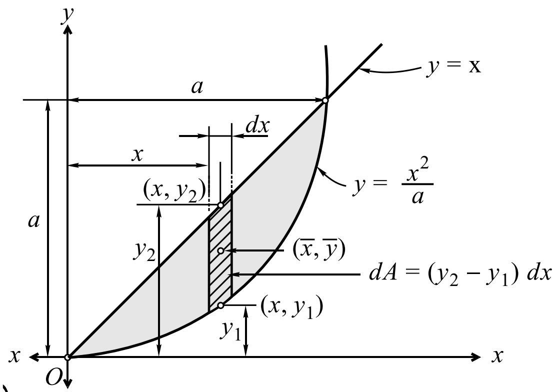


Fig. 5.11(b)

(ii) *Moment of Area About y-axis*

Taking moment of area of the elemental strip about y-axis, we have

$$\bar{x} = \frac{\int x\, dA}{\int dA} = \frac{\int_0^a x\,(y_2 - y_1)\, dx}{\int_0^a (y_2 - y_1)\, dx}$$

$$\therefore\ \bar{x} = \frac{\int_0^a x\left(x - \frac{x^2}{a}\right)dx}{\int_0^a \left(x - \frac{x^2}{a}\right)dx} \qquad \left\{\begin{array}{l} y = \frac{x^2}{a} \text{ for curve} \\ y_1 = \frac{x^2}{a} \\ y = x \text{ for straight line} \\ y_2 = x \end{array}\right\}$$

$$\therefore\ \bar{x} = \frac{\int_0^a x^2\, dx - \int_0^a \frac{x^3}{a}\, dx}{\int_0^a x\, dx - \int_0^a \frac{x^2}{a}\, dx} = \frac{\left[\frac{x^3}{3} - \frac{x^4}{4a}\right]_0^a}{\left[\frac{x^2}{2} - \frac{x^3}{3a}\right]_0^a}$$

$$\therefore\ \bar{x} = \frac{\frac{a^3}{3} - \frac{a^3}{4}}{\frac{a^2}{2} - \frac{a^2}{3}} = \frac{\left(\frac{4a^3 - 3a^3}{12}\right)}{\left(\frac{3a^2 - 2a^2}{6}\right)} = \frac{a^3}{12} \times \frac{6}{a^2}$$

$$\therefore\ \bar{x} = \frac{a}{2} \qquad \textbf{\textit{Proved.}}$$

Step II : To show that $\bar{y} = \dfrac{2a}{5}$

(i) *Differential Area Element*

Consider a small elemental strip parallel to x-axis at a distance y from x-axis, having thickness dy as shown in Fig. 5.11(c).

The width of the element is $(x_2 - x_1)$.

$\therefore$ Area of the elementary strip is

$dA = (x_2 - x_1)\, dy$

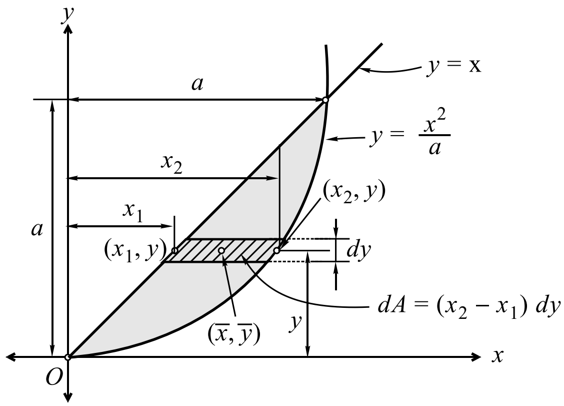


Fig. 5.11(c)

(ii) *Moment of Area About x-axis*

Taking moment of area of the elemental strip about x-axis, we have

$$\bar{y} = \frac{\int y\, dA}{\int dA} = \frac{\int_0^a y\,(x_2 - x_1)\, dy}{\int_0^a (x_2 - x_1)\, dy}$$

$$\therefore\ \bar{y} = \frac{\int_0^a y(\sqrt{ay} - y)\, dy}{\int_0^a (\sqrt{ay} - y)\, dy} \qquad \left\{ \begin{array}{l} y = x \text{ for straight line} \\ y = x_1 \\ y = \dfrac{x^2}{a} \text{ for curve} \\ y = \dfrac{{x_2}^2}{a} \\ \therefore\ x_2 = \sqrt{a\,y} \end{array} \right\}$$

$$\therefore\ \bar{y} = \frac{\int_0^a (\sqrt{a}\, y^{3/2} - y^2)\, dy}{\int_0^a (\sqrt{a}\, y^{1/2} - y)\, dy} = \frac{\int_0^a \sqrt{a}\, y^{3/2}\, dy - \int_0^a y^2\, dy}{\int_0^a \sqrt{a}\, y^{1/2} - \int_0^a y\, dy}$$

$$\therefore\ \bar{y} = \frac{\left[\sqrt{a}\, \dfrac{y^{5/2}}{5/2} - \dfrac{y^3}{3}\right]_0^a}{\left[\sqrt{a}\, \dfrac{y^{3/2}}{3/2} - \dfrac{y^2}{2}\right]_0^a} = \frac{\left(\dfrac{2}{5}\, a^{1/2} \times a^{5/2}\right) - \dfrac{a^3}{3}}{\left(\dfrac{2}{3}\, a^{1/2} \times a^{3/2}\right) - \dfrac{a^2}{2}}$$

$$\therefore\ \bar{y} = \frac{\dfrac{2}{5}\, a^3 - \dfrac{a^3}{3}}{\dfrac{2}{3}\, a^2 - \dfrac{a^2}{2}} = \frac{\dfrac{6a^3 - 5a^3}{15}}{\dfrac{4a^2 - 3a^2}{6}} = \frac{a^3}{15} \times \frac{6}{a^2}$$

$\therefore\ \bar{y} = \dfrac{2}{5}\, a$ ***Proved.***

5.5 Pappus-Guldinus Theorems

Greek geometer Pappus and Swiss mathematician Guldinus contributed to the concept of surfaces and volumes of revolution. Therefore, it is known as the theorem of Pappus-Guldinus.

First Theorem

The area of a surface of revolution is equal to the length of the generating curve times the distance travelled by the centroid of the generating curve while the surface is generated.

Proof

Consider a right angle triangle having hypotenuse as a straight line L generating curve. Let the vertical line along y-axis be an imaginary axis and horizontal line along x-axis as base radius of cone. If it revolves completely, i.e., 0 to 360° about y-axis then the hypotenuse L will generate the surface of revolution and a cone is obtained.

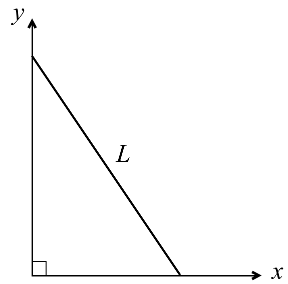


Fig. 5.5-i : Straight Line L Generating Curve

Consider an elemental length dl of the generator L generates a surface area dA when revolves about y-axis.

$$dA = 2\pi x \times dl$$

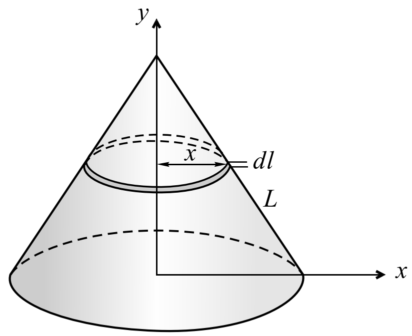


Fig. 5.5-ii : Surface of Cone Generated

Integrating for entire surface area generated by the given curve we have

$$A = \int dA = \int_0^L 2\pi x \, dl$$

$$A = 2\pi \, x_G \, L$$

By definition of centroid of length, we have

$$x_G \, L = \int x \, dl$$

Hence, the area of the surface generated is given by the product of $2\pi \; x_G$ and the length of the surface L as if the entire length of the generating curve was concentrated at the radius x_G.

Second Theorem

The volume of a body of revolution is equal to the generating area times the distance travelled by the centroid of the area while the body is generated.

Proof

Consider elemental area dA of the area A, which generates a volume dv when revolved about the y-axis.

$$dv = 2\pi x \times dA$$

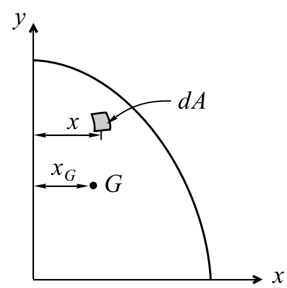


Fig. 5.5-iii

Integrating for entire volume generated by the given area, we have

$$v = \int dv = \int 2\pi x \, dA$$

By definition of centroid of an area, we have

$$x_G A = \int x \, dA$$

Hence, the volume of body generated is given by the product of $2\pi\ x_G$ and the area of the surface A as if the entire area of generating surface was concentrated at its centroid of radius x_G.

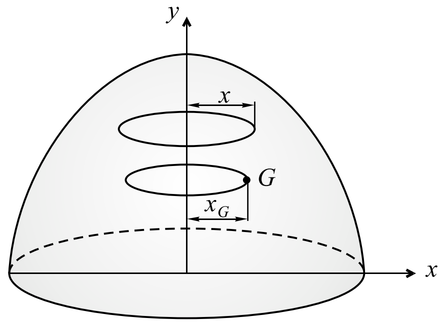


Fig. 5.5-iv

Problem 12

Derive the expression for volume of a sphere of radius R.

Solution

Consider the semicircular area with radius R and diameter along y-axis, we know G lies on symmetric of semicircle at a distance

$$x_G = \frac{4R}{3\pi}$$

Area of semicircle is $A = \dfrac{\pi R^2}{2}$

Applying Pappus-Guldinus theorem, by rotating semicircle about y-axis, we get

$$v = 2\pi\, x_G\, A$$

$$v = 2\pi \frac{4R}{3\pi} \times \frac{\pi R^2}{2}$$

$$\therefore \quad v = \frac{4\pi R^3}{3}$$

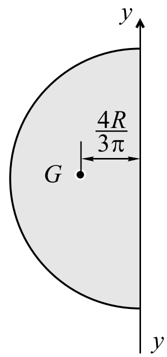


Fig. 5.12

Problem 13

Find the centroid of cone having base radius R and height H. The cone is having axis of symmetry, so centroid must lie on the axis.

Solution

Consider an elementary disc of radius r and width dy at a distance y from apex of cone.

Volume of elementary disc $dv = \pi r^2 dy$

For the entire volume of the cone, we have

$$y_G = \frac{\int y \, dv}{v}$$

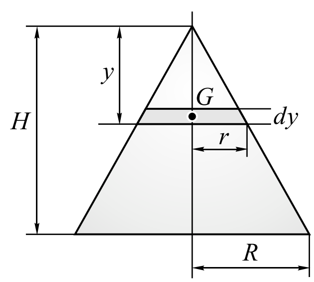


Fig. 5.13

$$\therefore \quad y_G = \frac{\int_0^H y \, \pi \, r^2 \, dy}{v}$$

By property of triangle $r = \dfrac{Ry}{H}$

Volume of cone $v = \dfrac{\pi R^2 H}{3}$

$$\therefore \quad y_G = \frac{\int_0^H y \dfrac{\pi R^2 y^2}{H^2} dy}{\dfrac{\pi R^2 H}{3}}$$

$$\therefore \quad y_G = \frac{\pi R^2 H^2}{H^2} \times \frac{3}{\pi R^2 H}$$

$$\therefore \quad y_G = \frac{3}{4} H$$

$\therefore$ Centroid of cone from base on axis is at one-fourth height.

Problem 14

A composite solid built by cone, cylinder and hemisphere is shown in Fig. 5.14. Locate the centroid w.r.t. origin.

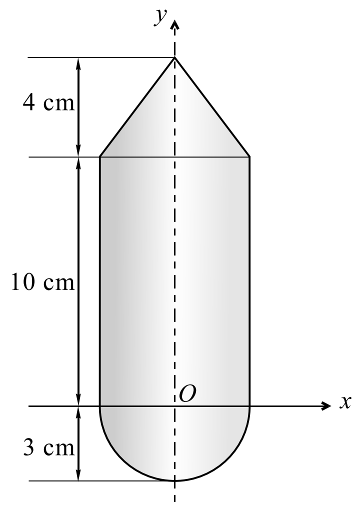


Fig. 5.14

Solution

The given solid is symmetric about y-axis.

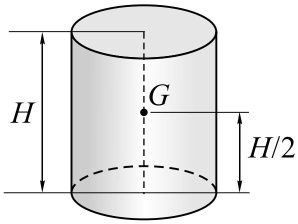
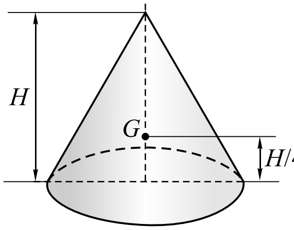
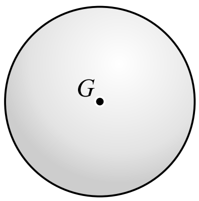
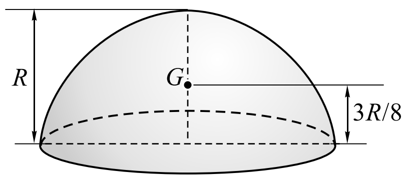
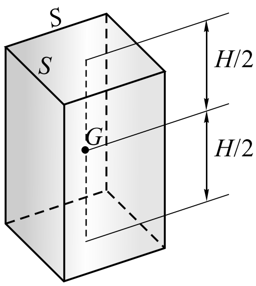
$$\therefore \quad \bar{x} = 0$$

$$\bar{y} = \frac{v_{\text{cone}} \, y_1 + v_{\text{cylinder}} \, y_2 + v_{\text{hemisphere}} \, y_3}{v_{\text{cone}} + v_{\text{cylinder}} + v_{\text{hemisphere}}}$$

$$\therefore \quad \bar{y} = \frac{\dfrac{\pi \times 3^2 \times 4}{3} \times 11 + \pi \times 3^2 \times 10 \times 5 + \dfrac{2\pi \times 3^3}{3}\left(\dfrac{-3 \times 3}{8}\right)}{\dfrac{\pi \times 3^2 \times 4}{3} + \pi \times 3^2 \times 10 + \dfrac{2\pi \times 3^3}{3}}$$

$$\therefore \quad \bar{y} = 4.68 \text{ cm}$$

Centroid of Volume of Solids

Solid	Geometrical Shape	Centroid	Volume
1. Cylinder		$\frac{H}{2}$	$\pi R^2 H$
2. Cone		$\frac{H}{4}$	$\frac{\pi R^2 H}{3}$
3. Sphere		0	$\frac{4\pi R^3}{3}$
4. Hemisphere		$\frac{3R}{8}$	$\frac{2\pi R^3}{3}$
5. Prism		$\frac{H}{2}$	$S^2 H$

Exercises

[I] Problems

Find the centroid of the following shaded plane areas shown in **1** to **15**. *[All dimensions are in mm]*

1.

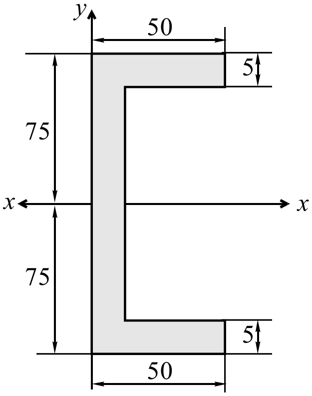


Fig. 5.E1

[***Ans.*** $\bar{x}$ = 11.875 mm and $\bar{y}$ = 0.]

2.

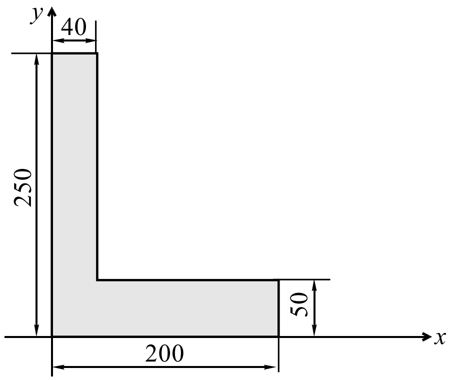


Fig. 5.E2

[***Ans.*** $\bar{x}$ = 64.4 mm and $\bar{y}$ = 80.5 mm.]

3.

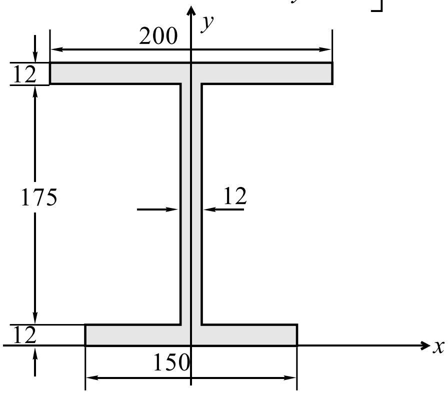


Fig. 5.E3

[***Ans.*** $\bar{x}$ = 0 and $\bar{y}$ = 108.4 mm.]

4.

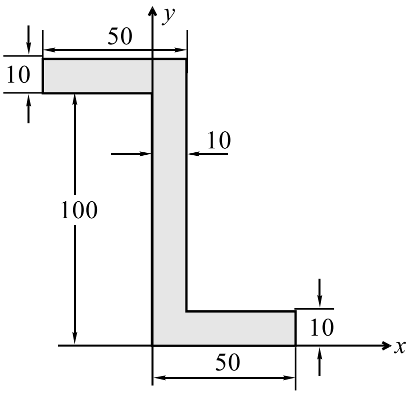


Fig. 5.E4

[***Ans.*** $\bar{x}$ = 5 mm and $\bar{y}$ = 55 mm.]

5.

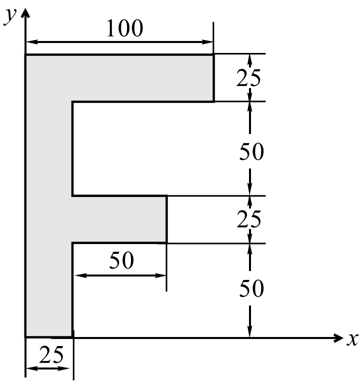


Fig. 5.E5

[***Ans.*** $\bar{x}$ = 32.95 mm and $\bar{y}$ = 89.77 mm.]

6.

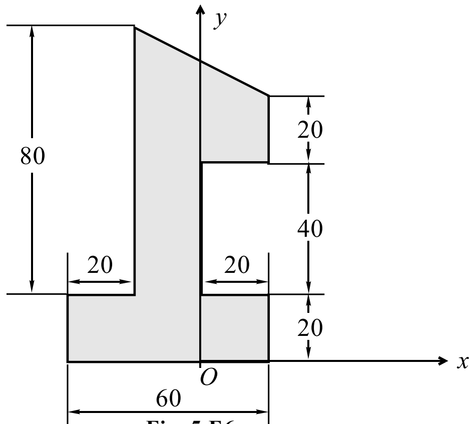


Fig. 5.E6

[***Ans.*** $\bar{x}$ = −7.1 mm and $\bar{y}$ = 42.1 mm.]

7.

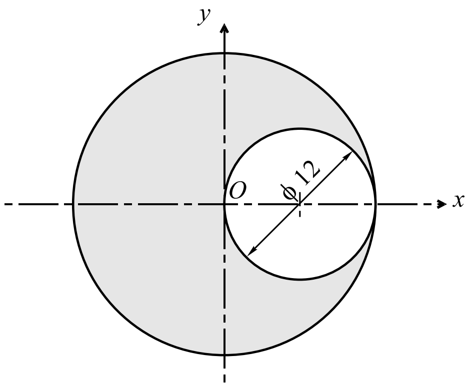


Fig. 5.E7

$\left[\textbf{\textit{Ans.}}\ \ \bar{x} = -2 \text{ mm and } \bar{y} = 0.\right]$

8.

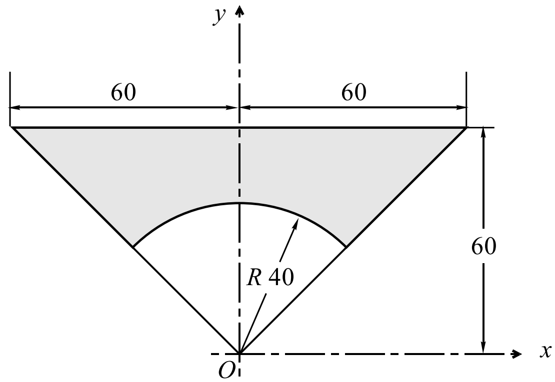


Fig. 5.E8

$\left[\textbf{\textit{Ans.}}\ \ \bar{x} = 0 \text{ and } \bar{y} = 48.6 \text{ mm.}\right]$

9.

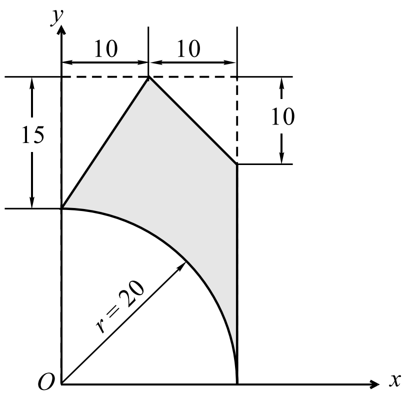


Fig. 5.E9

$\left[\textbf{\textit{Ans.}}\ \ \bar{x} = 12.46 \text{ mm and } \bar{y} = 22.04 \text{ mm.}\right]$

10.

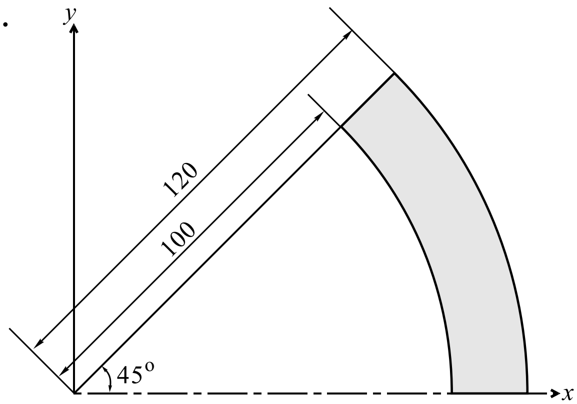


Fig. 5.E10

$\left[\textbf{\textit{Ans.}}\ \ \bar{x} = 99.3 \text{ mm and } \bar{y} = 41.13 \text{ mm.}\right]$

11.

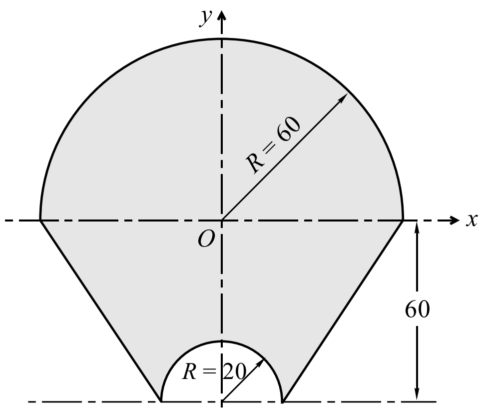


Fig. 5.E11

$\left[\textbf{\textit{Ans.}}\ \ \bar{x} = 0 \text{ mm and } \bar{y} = 5.8 \text{ mm.}\right]$

12.

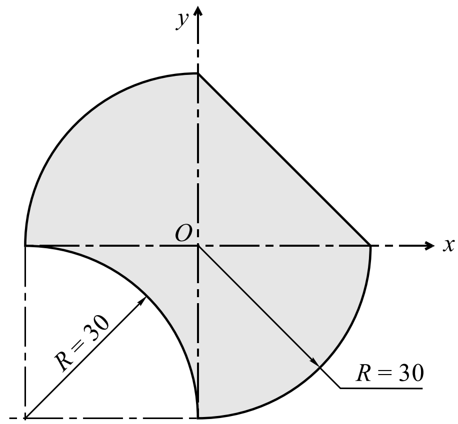


Fig. 5.E12

$\left[\textbf{\textit{Ans.}}\ \ \bar{x} = \bar{y} = 1.563 \text{ mm.}\right]$

13.

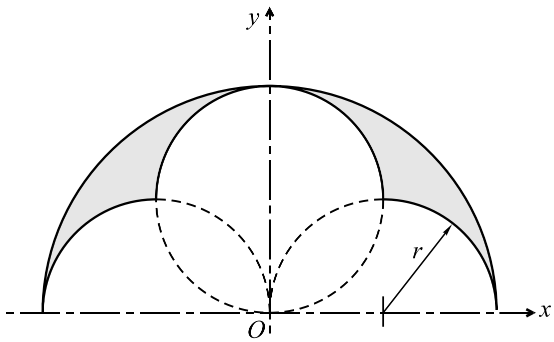


Fig. 5.E13

$\left[\textbf{Ans. } \bar{x} = 0 \text{ and } \bar{y} = 1.25\, r.\right]$

14.

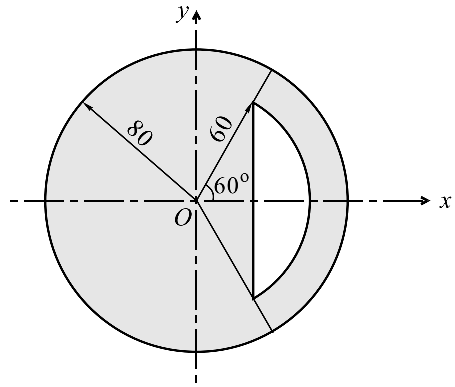


Fig. 5.E14

$\left[\textbf{Ans. } \bar{x} = -5.23 \text{ mm and } \bar{y} = 0.\right]$

15.

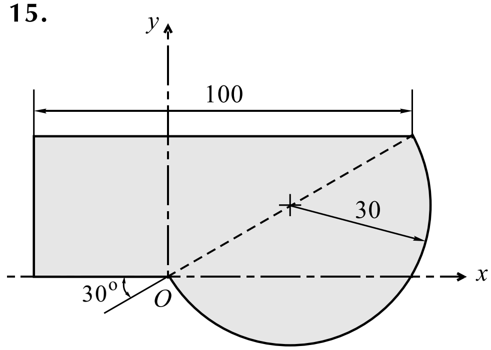


Fig. 5.E15

$\left[\textbf{Ans. } \bar{x} = 6.76 \text{ mm and } \bar{y} = 11.79 \text{ mm.}\right]$

16. If the dimensions of *a* and *b* of the plane figure shown in Fig. 5.E16 are fixed, what should be the dimension of *c* in order that the centroid of the shaded area will lie on the line *AB* ?

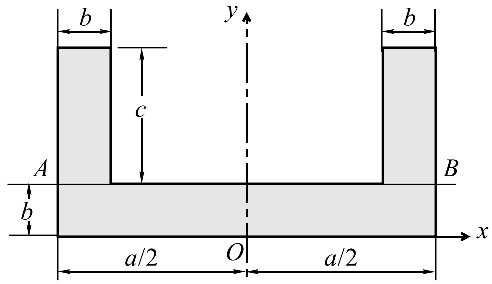


Fig. 5.E16

$\left[\textbf{Ans. } c = \sqrt{ab/2}\,.\right]$

17. Prove that the centroid of the shaded area, in Fig. 5.E17, with respect to the *x* and *y* axes is given by

$$\bar{x} = \frac{\frac{2}{3}\, r \sin^3 \alpha}{\left(\alpha - \frac{\sin 2\alpha}{2}\right)}$$

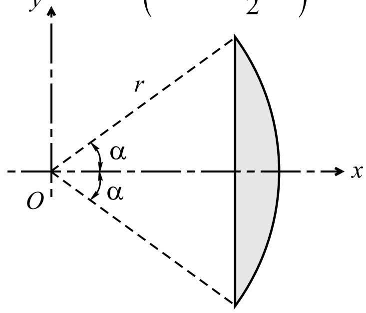


Fig. 5.E17

18. A plane lamina is hung freely from point *D* in Fig. 5.E18. Find the angle made by *BD* with the vertical.

$\left[\textbf{Ans. } \theta = 29.62°\right]$

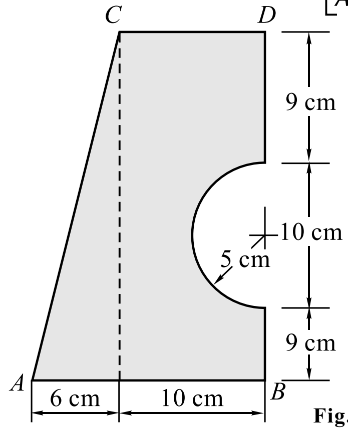


Fig. 5.E18

19. Determine the position of C.G. of the shaded area OBD shown in Fig. 5.E19. The curve OD is a parabola.

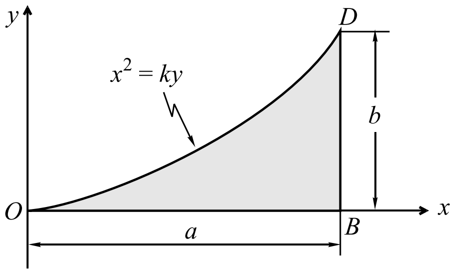


Fig. 5.E19

$\left[\textbf{Ans. } \bar{x} = 3a/4 \text{ and } \bar{y} = 3b/10.\right]$

20. Find by direct integration, the C.G. of the shaded area under the curve $y = kx^n$ shown in Fig. 5.E20.

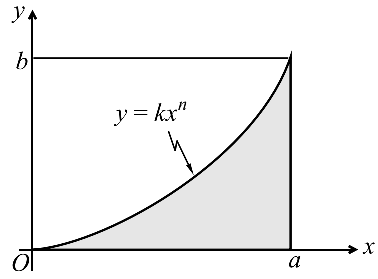


Fig. 5.E20

$\left[\textbf{Ans. } \bar{x} = \frac{(n+1)a}{(n+2)} \text{ and } \bar{y} = \left(\frac{n+1}{2n+1}\right)\frac{b}{2}.\right]$

21. Find the C.G. of shaded area shown in Fig. 5.E21.

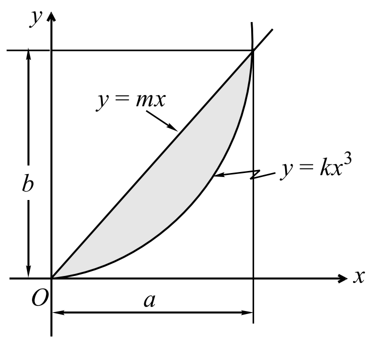


Fig. 5.E21

$\left[\textbf{Ans. } \bar{x} = 8a/15 \text{ and } \bar{y} = 8b/21.\right]$

22. Find C.G. of the shaded area shown in Fig. 5.E22.

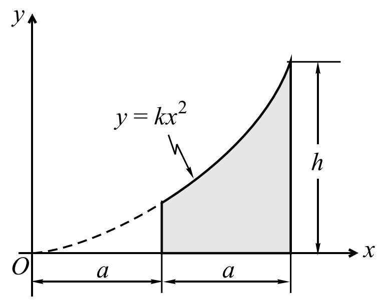


Fig. 5.E22

$\left[\textbf{Ans. } \bar{x} = 1.6\,a \text{ and } \bar{y} = 0.332\,h.\right]$

[II] Review Questions

1. Define centre of gravity.
2. Define centroid.
3. Derive an equation for centroid of the following areas :
   (a) Triangle (b) Semicircle (c) Quarter circle (d) Sector of a circle
4. Derive an equation for centre of gravity of bent wire of the following shapes :
   (a) Semicircular (b) Quarter circular (c) Arc of a circle
5. Describe the method of finding centroid of composite areas.
6. Describe the method of finding centre of gravity of composite bent wires.

[III] Fill in the Blanks

1. For a freely suspended plate in equilibrium, the __________ will lie vertically below the point of suspension.
2. Median of any triangle contains the ________ of that triangle.
3. The centroid of equilateral triangle with each side s from any of the three sides will be at _________ distance.
4. If the figure is symmetric about the y-axis then $\bar{x}$ = ______.
5. Centroid of semicircle on symmetry lies _______ distance from diameter.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. A point at which whole weight of the body is suppose to be concentrated is called _________.

   (a) moment of inertia **(b)** moment centre **(c)** centre of mass **(d)** centre of gravity

2. Centroid _________ lie on the given figure.

   (a) must **(b)** must not **(c)** may or may not **(d)** None of these

3. If axis of symmetry is inclined at 45^{o} to the horizontal then _______.

   (a) $\bar{x} = 0$ **(b)** $\bar{y} = 0$ **(c)** $\bar{x} = \bar{y} = 0$ **(d)** $\bar{x} = \bar{y}$

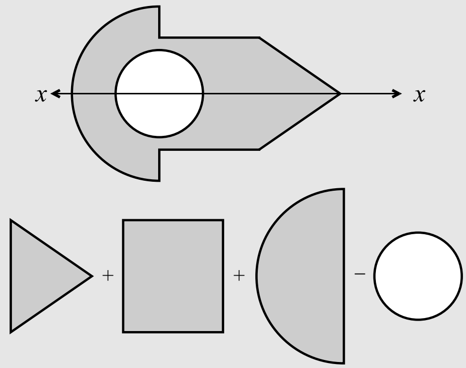
4. Centroid of quarter circular area is ________.

   (a) $\bar{x} = \dfrac{4r}{3\pi}$ **(b)** $\bar{y} = \dfrac{4r}{3\pi}$ **(c)** $\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$ **(d)** None of these

5. Centroid of sector of a circle is given by the relation ________.

   (a) $\dfrac{2r \sin \alpha}{3\alpha}$ **(b)** $\dfrac{r \sin \alpha}{3\alpha}$ **(c)** $\dfrac{2r \sin \alpha}{\alpha}$ **(d)** $\dfrac{3r \sin \alpha}{2\alpha}$

6



MOMENT OF INERTIA

6.1 Introduction

Moment of Inertia of a Body : The concept which gives a quantitative estimate of the relative distribution of area and mass of a body with respect to some reference axis is termed as the *moment of inertia of a body*.

Area Moment of Inertia : The moment of inertia of an area is called the *area moment of inertia* or the *second moment of area*.

Mass Moment of Inertia : The moment of inertia of a mass of the body is called the *mass moment of inertia*.

Application : The concept of moment of inertia (MI) is useful in the study of strength of materials. This concept is important in studying the load-carrying capacity of any structural member subjected to bending.

The moment of inertia determination of the cross-sectional areas of structure members is carried out in the study of subjects like structural design, strength of materials, machine design and many more.

6.2 Moment of Inertia of an Area

In the region bounded by two reference axes Ox and Oy, consider a plane lamina of area A as shown in Fig. 6.2-i.

It consists of a number of small elemental areas such that the sum of these elemental areas is equal to the area A.

Let dA be the area of any element situated at a distance of x and y from the axis as shown in the figure.

The moment of inertia of A with respect to x-axis and y-axis, which is also called the *second moment of area*, is defined by the integral

$$I_{xx} = \int y^2 \, dA \quad \text{and} \quad I_{yy} = \int x^2 \, dA \qquad \ldots(6.1)$$

where

I_{xx} is moment of inertia of area A about the x-axis and

I_{yy} is moment of inertia of area A about the y-axis.

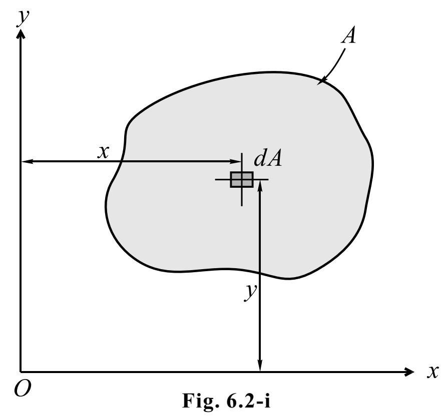


Fig. 6.2-i

The units of moment of inertia are mm^4, cm^4 or m^4 depending on the unit of linear measurement.

6.3 Perpendicular Axis Theorem

Polar Moment of Inertia

Let Oz be the axis perpendicular to the axis Ox and Oy passing through the origin O (pole).

Consider the elemental area dA at a distance of x and y from the axis Oy and Ox respectively, as shown in Fig. 6.3-i. Let r be the distance of CG of elemental area from Oz axis.

Moment of inertia of area of plane lamina about Oz axis is given by

$$I_{zz} = \int r^2\, dA \quad \text{but} \quad r^2 = x^2 + y^2$$

$$\therefore\ I_{zz} = \int x^2\, dA + \int y^2\, dA$$

$$\therefore\ I_{zz} = I_{xx} + I_{yy} \qquad \ldots(6.2)$$

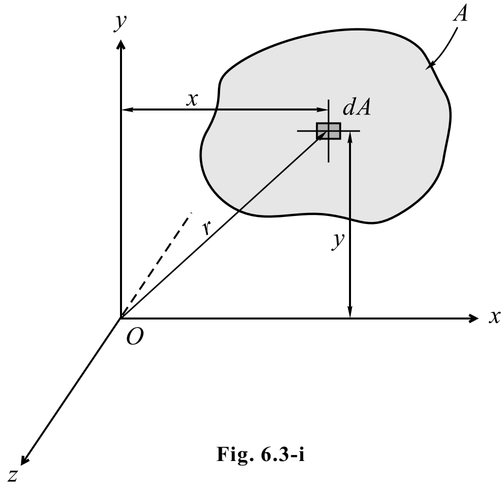


Fig. 6.3-i

Thus, perpendicular axis theorem states that *the moment of inertia of an area with respect to an axis perpendicular to the x-y plane and passing through origin will be equal to the sum of moment of inertia of the same area about x-x and y-y axis*.

It is also termed as the *polar moment of inertia*.

6.4 Parallel Axis Theorem (Transfer Formula)

Let A be the area of the plane lamina as shown in Fig. 6.4-i. Consider a small elemental area dA at a distance of y from the axis passing through the centroid of the area A. Let r be the distance between the parallel centroidal axis and the reference axis AB.

Moment of inertia of the elemental area dA about the reference axis AB is

$$I_{AB} = \int (y + r)^2\, dA$$

$$I_{AB} = \int (y^2 + 2r\,y + r^2)\, dA$$

$$I_{AB} = \int y^2\, dA + 2r\int y\, dA + r^2\int dA$$

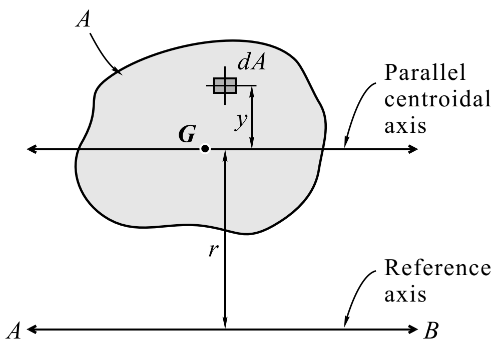


Fig. 6.4-i

Let us analyse the above three terms in the above equations.

1. $\int y^2\, dA$ = Moment of inertia of given area A about centroidal axis.

$$\therefore \int y^2\, dA = I_G$$

2. $2r\int y\, dA = 2rA\int y\, dA$

Now, $\int y\, dA = A\overline{y} = 0$ (as the distance of the centroid from the centroidal axis is zero, since the centroid axis passes through ***G***.)

3. $r^2 \int dA = A\, r^2$

The above simplification gives

$$I_{AB} = I_G + Ar^2 \qquad \text{...(6.3)}$$

Thus, parallel axis theorem states that *the moment of inertia of a plane area with respect to any reference axis in its plane is equal to the sum of moment of inertia w.r.t. a parallel centroidal axis and product of the total area and the square of the distance between the two axes.*

6.5 Radius of Gyration

Radius of gyration of a body is defined as *the distance from the reference axis at which the given area is assumed to be compressed and kept as a thin strip, such that there is no change in its moment of inertia*.

Figure 6.5-i shows a plane figure of area A. Let I_{AB} be its moment of inertia about reference axis AB. Assume the figure to be compressed into a thin strip of the same area A at a distance k from the reference axis AB such that it has same moment of inertia (i.e., I_{AB}) as shown in Fig. 6.5-ii.

Then from the definition

$$I_{AB} = Ak^2 \qquad \text{...(6.4)}$$

where k is known as the *radius of gyration*.

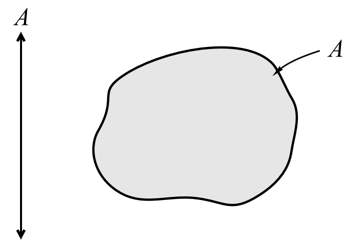


Fig. 6.5-i

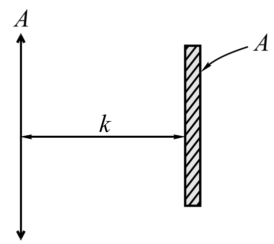


Fig. 6.5-ii

6.6 Moment of Inertia of Some Standard Geometrical Shapes

1. MI of Rectangle About its Base

Consider a rectangle of base b and height h.

AB is the reference axis through base.

Consider an elemental strip of width dy located at a distance y from the reference axis AB, as shown in Fig. 6.6-i.

Using basic principle of MI

$$I = \int r^2\, dA$$

Area of elemental strip $dA = b\, dy$

$$I_{AB} = \int_0^h y^2\, b\, dy = b \int_0^h y^2\, dy = b \left[\frac{y^3}{3}\right]_0^h$$

$$I_{AB} = \frac{bh^3}{3}$$

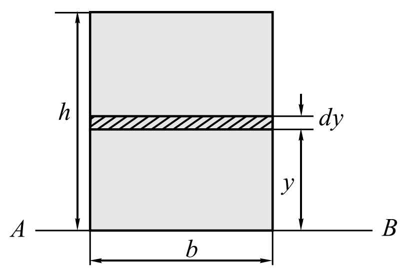


Fig. 6.6-i

MI of rectangle about parallel centroidal axis to the base

By parallel axis theorem

$$I_{AB} = I_G + Ad^2$$

where d = Distance between the reference axis and parallel centroidal axis.

$$I_G = I_{AB} - Ad^2$$

$$I_G = \frac{bh^3}{3} - b \times h \times \left(\frac{h}{2}\right)^2 = \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$I_G = \frac{bh^3}{12}$$

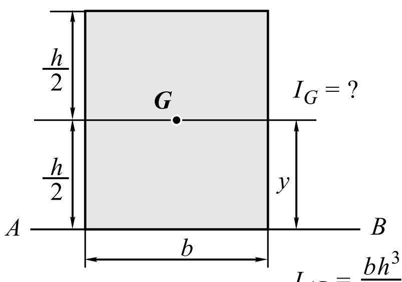


Fig. 6.6-ii

2. MI of Triangle About its Base

Consider a triangle of base b and height h.

AB is the reference axis through base, consider an elemental strip of width dy located at a distance of y from the reference axis, as shown in Fig. 6.6-iii.

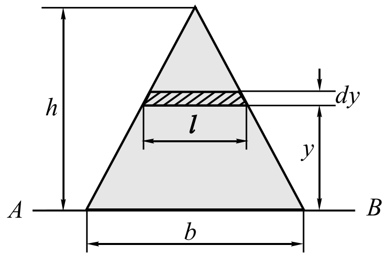


Fig. 6.6-iii

By property of similar triangle, we have

$$\frac{h}{b} = \frac{h-y}{l}$$

$$\therefore\ l = \frac{b}{h}(h-y)$$

Area of elemental strip $dA = \dfrac{b}{h}(h-y)\,dy$

Using the basic principle of MI

$$I = \int r^2\,dA$$

$$I_{AB} = \int_0^h y^2 \frac{b}{h}(h-y)\,dy = \frac{b}{h}\int_0^h (y^2h - y^3)\,dy$$

$$= \frac{b}{h}\left[\frac{y^3h}{3} - \frac{y^4}{4}\right]_0^h = \frac{b}{h}\left[\frac{h^4}{3} - \frac{h^4}{4}\right]$$

$$I_{AB} = \frac{bh^3}{12}$$

MI of triangle about parallel centroidal axis to the base.

By parallel axis theorem

$$I_{AB} = I_G + Ad^2$$

$$I_G = I_{AB} - Ar^2$$

$$I_G = \frac{bh^3}{12} - \frac{1}{2} \times b \times h \times \left(\frac{h}{3}\right)^2 = \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$I_G = \frac{bh^3}{36}$$

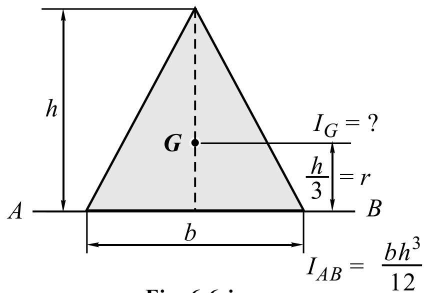


Fig. 6.6-iv

3. MI of Circle About Diametrical Axis

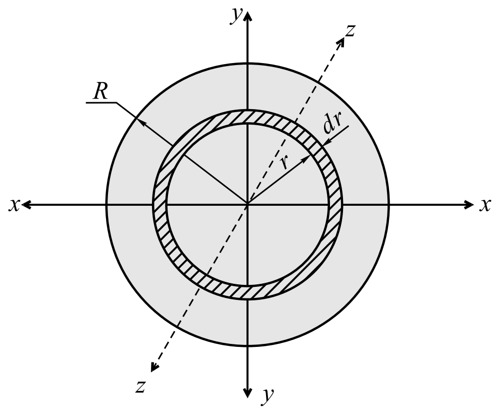


Fig. 6.6-v

Consider an elemental thin circular ring of width dr and radius r, as shown in Fig. 6.6-v.

So area of ring $dA = 2\pi r\, dr$.

By polar moment of inertia, we have

$$I_{zz} = \int r^2\, dA$$

$$= \int_0^R r^2\, 2\pi r\, dr = 2\pi \int_0^R r^3\, dr = 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$I_{zz} = \frac{\pi R^4}{2}$$

Since circle is symmetric about x-x and y-y axis, we have

$$I_{xx} = I_{yy}$$

By perpendicular axis theorem, we have

$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = 2I_{xx} = 2I_{yy} \quad \therefore \; I_{xx} = I_{yy} = \frac{I_{zz}}{2}$$

$$\therefore \; I_{xx} = \frac{\pi R^4}{4} \text{ and } I_{yy} = \frac{\pi R^4}{4}$$

4. Moment of Inertia of Semicircle About Diametrical Axis

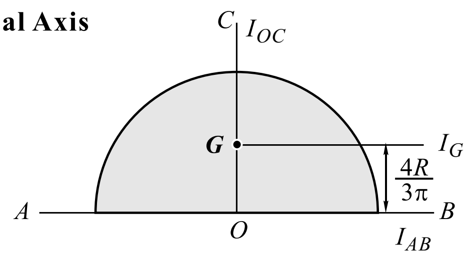


Fig. 6.6-vi

Considering the above similar procedure

$$\because \text{ Semicircle } = \frac{1}{2} \times \text{Circle}$$

$$\therefore \; I_{xx} = I_{yy} = \frac{\pi R^4}{8}$$

$$I_{zz} = \frac{\pi R^4}{4}$$

By parallel axis theorem

$$I_{AB} = I_G + Ad^2$$

$$I_G = 0.11\, R^4 \; \left(\begin{array}{l}\text{MI about parallel centroidal}\\ \text{axis to diametrical axis}\end{array}\right)$$

$$I_{OC} = \frac{\pi R^4}{8} \; \left(\begin{array}{l}\text{MI about symmetrical}\\ \text{centroidal, i.e., perpendicular}\\ \text{to diametrical axis.}\end{array}\right)$$

5. Moment of Inertia of Quarter Circle

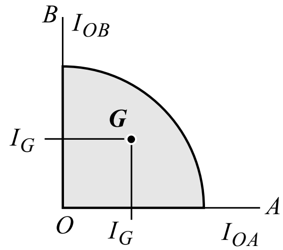


Fig. 6.6-vii

Considering the above similar procedure, we have

$$I_{OA} = I_{OB} = \frac{\pi R^4}{16}$$

$$I_G = 0.055\, R^4$$

6.7 Solved Problems

Problem 1

Find the moment of inertia of shaded area shown in Fig. 6.1(a) about *x-x* axis and *y-y* axis.

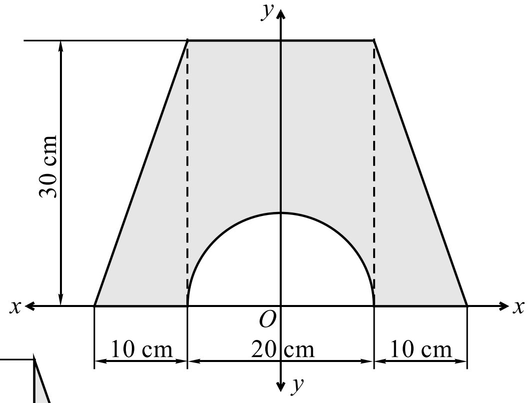


Fig. 6.1(a)

Solution

(i) Moment of inertia about *x-x* axis

Refer to Fig. 6.1(b).

The composite area can be viewed as

triangle ⊕ rectangle ⊕ triangle ⊖ semicircle.

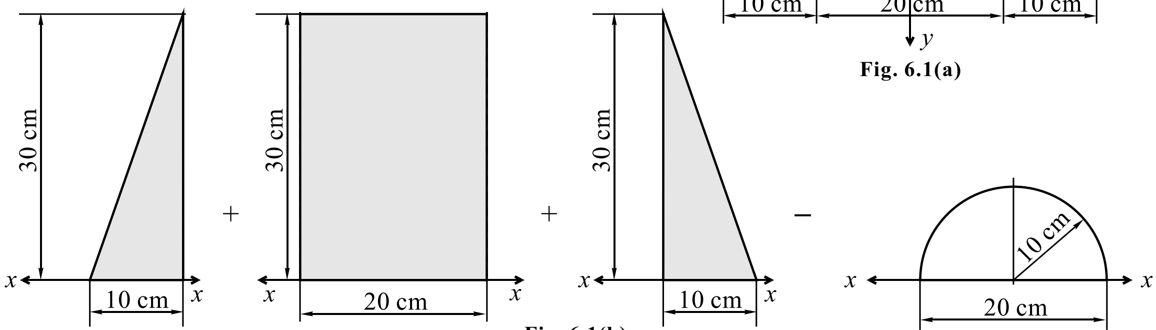


Fig. 6.1(b)

$$\therefore\ I_{xx} = \frac{10 \times 30^3}{12} + \frac{20 \times 30^3}{3} + \frac{10 \times 30^3}{12} - \frac{\pi \times 10^4}{8} = 22500 + 180000 + 22500 - 3926.99$$

$$\therefore\ I_{xx} = 221073.01 \text{ cm}^4 \quad \textbf{\textit{Ans.}}$$

(ii) Moment of inertia about *y-y* axis. Refer to Fig. 6.1(c).

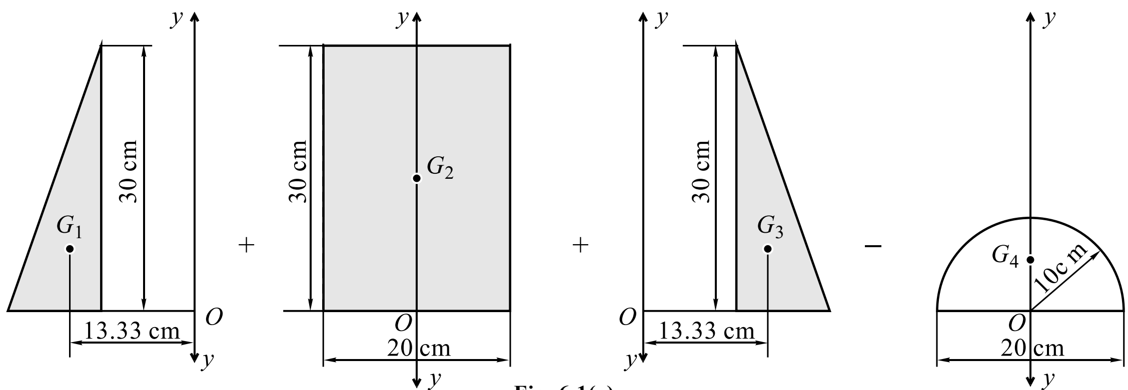


Fig. 6.1(c)

Applying parallel axis theorem for both the triangular areas.

$$\therefore\ I_{yy} = 2\left(\frac{30 \times 10^3}{36} + \frac{1}{2} \times 30 \times 10 \times 13.33^2\right) + \frac{30 \times 20^3}{12} - \frac{\pi \times 10^4}{8}$$

$$\therefore\ I_{yy} = 54973.34 + 20000 - 3926.99$$

$$\therefore\ I_{yy} = 71046.35 \text{ cm}^4 \quad \textbf{\textit{Ans.}}$$

Problem 2

Find the moment of inertia of shaded area shown in Fig. 6.2(a) about OX and OY axis. Also find the centroid of the shaded area.

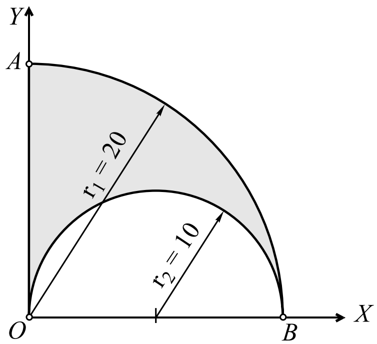


Fig. 6.2(a)

[All dimensions are in cm]

Solution

(i) Divide the given area into two subareas quarter circle and semicircle, as shown in Fig. 6.2(b).

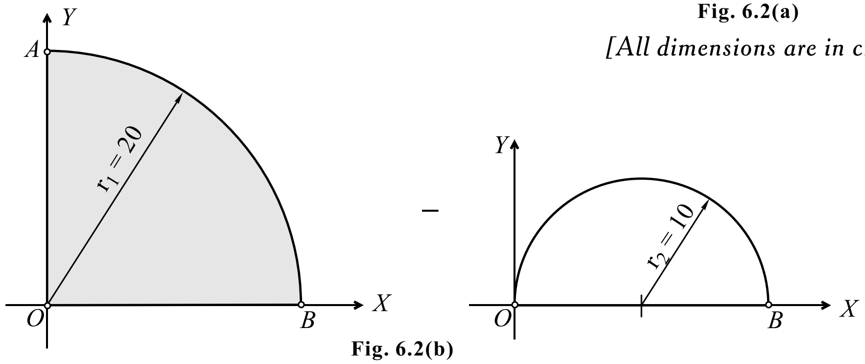


Fig. 6.2(b)

(ii) Moment of inertia about *OX* axis. Refer to Fig. 6.2(b).

$$I_{OX} = \frac{\pi r_1^4}{16} - \frac{\pi r_2^4}{8}$$

$$\therefore I_{OX} = \frac{\pi \times 20^4}{16} - \frac{\pi \times 10^4}{8} = 31415.93 - 3926.99$$

$$\therefore I_{OX} = 27488.94 \text{ cm}^4 \quad \textit{Ans.}$$

(iii) Moment of inertia about *OY* axis. Refer to Fig. 6.2(b).

$$I_{OY} = \frac{\pi r_1^4}{16} - \frac{\pi r_2^4}{8} = \frac{\pi \times 20^4}{16} - \frac{\pi \times 10^4}{8} = 31415.93 - 3926.99$$

$$\therefore I_{OY} = 27488.94 \text{ cm}^4 \quad \textit{Ans.}$$

(iv) $\overline{x}$ w.r.t. *O*

$$\overline{x} = \frac{\left(\frac{\pi \times 20^2}{4}\right)\left(\frac{4 \times 20}{3\pi}\right) - \left(\frac{\pi \times 10^2}{2}\right)(10)}{\left(\frac{\pi \times 20^2}{4}\right) - \left(\frac{\pi \times 10^2}{2}\right)} = \frac{2666.67 - 1570.8}{314.16 - 157.08} = 6.977 \text{ cm} \quad \textit{Ans.}$$

(v) $\overline{y}$ w.r.t. *O*

$$\overline{y} = \frac{\left(\frac{\pi \times 20^2}{4}\right)\left(\frac{4 \times 20}{3\pi}\right) - \left(\frac{\pi \times 10^2}{2}\right)\left(\frac{4 \times 10}{3\pi}\right)}{\left(\frac{\pi \times 20^2}{4}\right) - \left(\frac{\pi \times 10^2}{2}\right)} = \frac{2666.67 - 666.67}{314.16 - 157.08} = 12.732 \text{ cm} \quad \textit{Ans.}$$

Problem 3

Find the moment of inertia of shaded area shown in Fig. 6.3(a) about x-axis and y-axis.

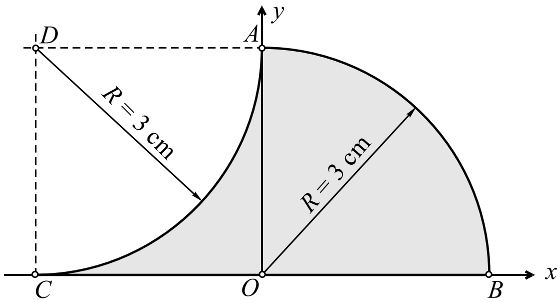


Fig. 6.3(a)

[All dimensions are in cm]

Solution

(i) Moment of inertia about *x*-axis

The composite area can be viewed as quarter circle ⊕ square ⊖ quarter circle. Refer to Fig. 6.3(b).

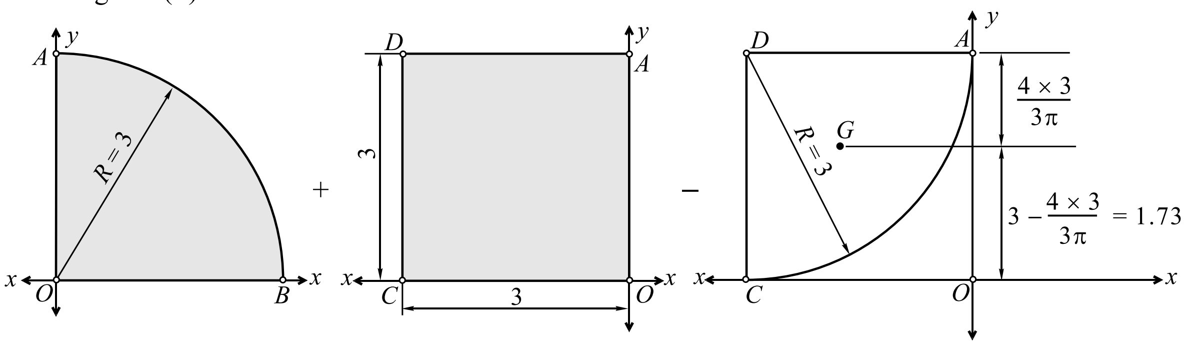


Fig. 6.3(b)

(Apply parallel axis theorem)

$$\therefore\ I_{xx} = \frac{\pi \times 3^4}{16} + \frac{3 \times 3^3}{3} - \left(0.055 \times 3^4 + \frac{\pi \times 3^2}{4} \times 1.73^2\right)$$

$$= 15.9 + 27 - 25.61$$

$$\therefore\ I_{xx} = 17.29\ \text{cm}^4 \quad \textbf{\textit{Ans.}}$$

(ii) Moment of inertia about *y*-axis. Refer to Fig. 6.3(c).

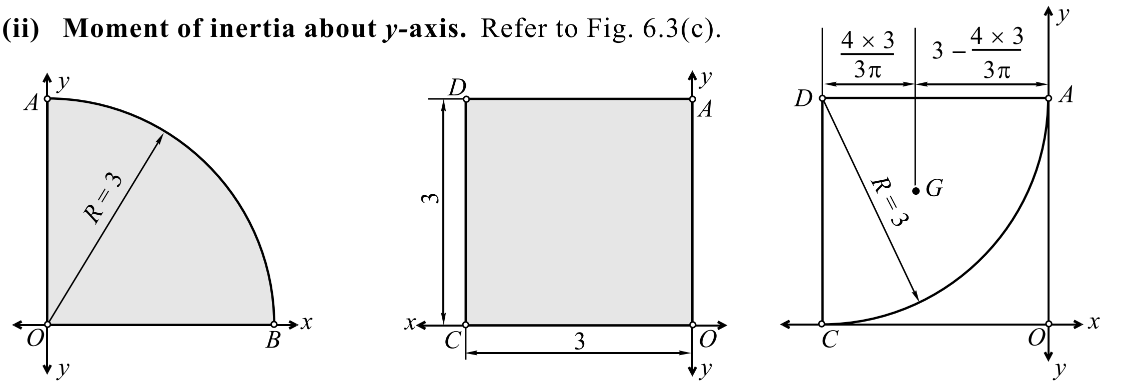


Fig. 6.3(c)

(Apply parallel axis theorem)

$$\therefore\ I_{yy} = \frac{\pi \times 3^4}{16} + \frac{3 \times 3^3}{3} - \left(0.055 \times 3^4 + \frac{\pi \times 3^2}{4} \times 1.73^2\right)$$

$$= 15.9 + 27 - 25.61$$

$$\therefore\ I_{yy} = 17.29\ \text{cm}^4 \quad \textbf{\textit{Ans.}}$$

Problem 4

Find centroid of plane area shown in Fig. 6.4(a) and also MI about centroidal axis.

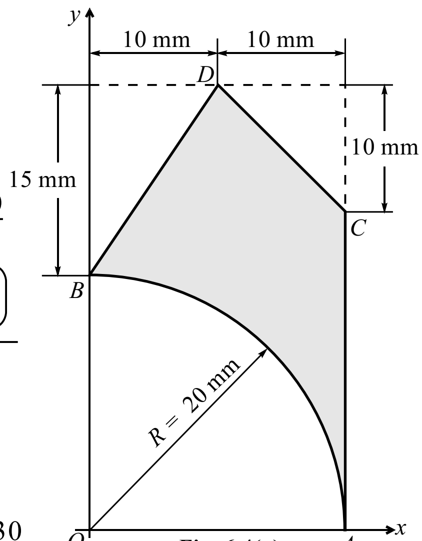


Fig. 6.4(a)

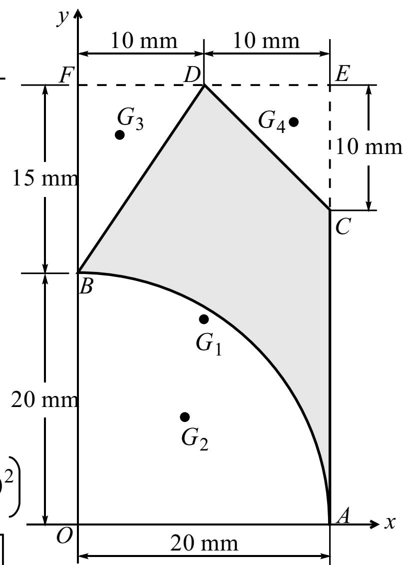


Fig. 6.4(b)

Solution

(i) To find $\bar{x}$ and $\bar{y}$

$$\bar{x} = \frac{20 \times 35 \times 10 - \dfrac{\pi \times 20^2}{4} \times \dfrac{4 \times 20}{3\pi} - \dfrac{1}{2} \times 10 \times 15 \times \dfrac{10}{3} - \dfrac{1}{2} \times 10 \times 10\left(20 - \dfrac{10}{3}\right)}{20 \times 35 - \dfrac{\pi \times 20^2}{4} - \dfrac{1}{2} \times 10 \times 15 - \dfrac{1}{2} \times 10 \times 10}$$

$$\bar{x} = \frac{3250}{260.84} \qquad \therefore\ \bar{x} = 12.46 \text{ mm} \quad \textbf{\textit{Ans.}}$$

$$\bar{y} = \frac{20 \times 35 \times 17.5 - \dfrac{\pi \times 20^2}{4} \times \dfrac{4 \times 20}{3\pi} - \dfrac{1}{2} \times 10 \times 15 \times 30 - \dfrac{1}{2} \times 10 \times 10\left(35 - \dfrac{10}{3}\right)}{20 \times 35 - \dfrac{\pi \times 20^2}{4} - \dfrac{1}{2} \times 10 \times 15 - \dfrac{1}{2} \times 10 \times 10}$$

$$\bar{y} = \frac{5750}{260.84} \qquad \therefore\ \bar{y} = 22.04 \text{ mm} \quad \textbf{\textit{Ans.}}$$

(ii) Area of shaded region

$$A = 20 \times 35 - \frac{\pi \times 20^2}{4} - \frac{1}{2} \times 10 \times 15 - \frac{1}{2} \times 10 \times 10$$

$$A = 260.84 \text{ mm}^2$$

(iii) To find I_{xOx} and I_{yOy}

$$I_{xOx} = \frac{20 \times 35^3}{3} - \frac{\pi \times 20^4}{16} - \left(\frac{10 \times 15^3}{36} + \frac{1}{2} \times 10 \times 15 \times 30^2\right) - \left[\frac{10 \times 10^3}{36} + \frac{1}{2} \times 10 \times 15 \times \left(35 - \frac{10}{3}\right)^2\right]$$

$$I_{xOx} = 135552.68 \text{ mm}^4$$

$$I_{yOy} = \frac{35 \times 20^3}{3} - \frac{\pi \times 10^4}{16} - \frac{15 \times 10^3}{12} - \left[\frac{10 \times 10^3}{36} + \frac{1}{2} \times 10 \times 10\left(20 - \frac{10}{3}\right)^2\right]$$

$$I_{yOy} = 46500.74 \text{ mm}^4$$

(iv) To find I_{xGx} and I_{yGy}

$$I_{xGx} = I_{xOx} - A(\bar{y})^2$$

$$I_{xGx} = 135552.6 - 260.84 \times (22.04)^2$$

$$I_{xGx} = 8846.62 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

$$I_{yGy} = I_{yOy} - A(\bar{x})^2$$

$$I_{yGy} = 46500.74 - 260.84 \times (12.46)^2$$

$$I_{yGy} = 6004.91 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

Problem 5

Determine the moment of inertia of the plane area shown in Fig. 6.5(a) about the centroidal x-axis and centroidal y-axis.

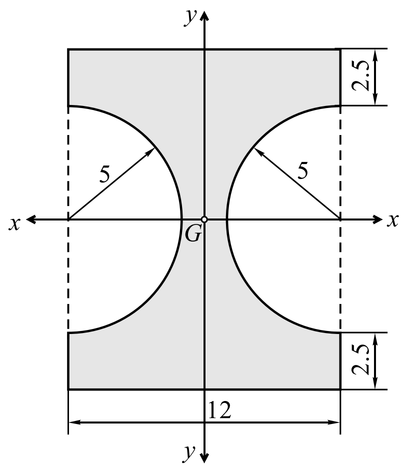


Fig. 6.5(a)

[All dimensions are in cm]

Solution

(i) The composite area can be viewed as rectangle ⊖ two equal semicircles. Refer to Fig. 6.5(b).

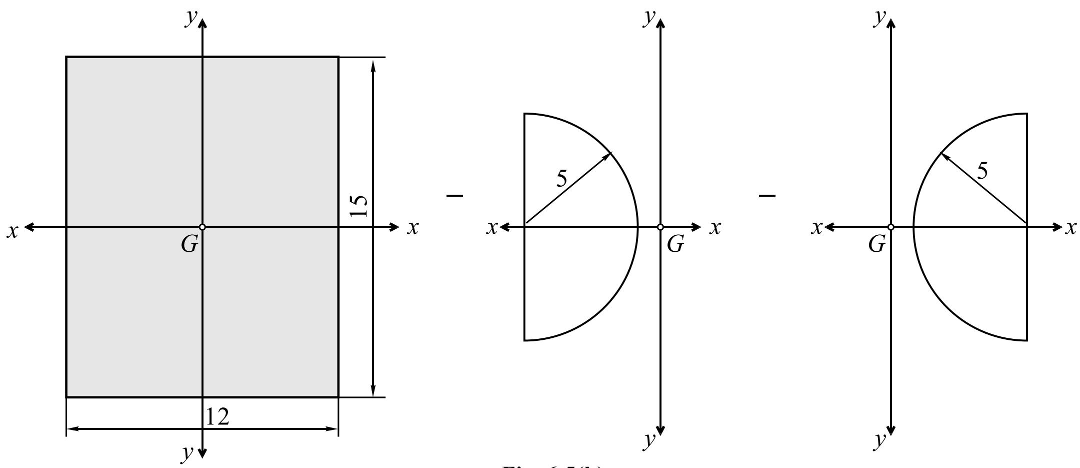


Fig. 6.5(b)

(ii) Moment of inertia about x-axis

$$\therefore I_{xx_G} = \frac{12 \times 15^3}{12} - 2\left(\frac{\pi \times 5^4}{8}\right)$$

$$= 3375 - 490.87$$

$$\therefore I_{xx_G} = 2884.13 \text{ cm}^4$$ ***Ans.***

(iii) Moment of inertia about y-axis

Applying parallel axis theorem for both the semicircular areas.

$$\therefore I_{yy_G} = \frac{15 \times 12^3}{12} - 2\left[0.11 \times 5^4 + \frac{\pi \times 5^2}{2} \times \left(6 - \frac{4 \times 5}{3\pi}\right)^2\right]$$

$$= 2160 - 1318.61$$

$$\therefore I_{yy_G} = 841.39 \text{ cm}^4$$ ***Ans.***

Problem 6

For the given shaded area shown in Fig. 6.6(a) find,

(i) MI about the reference axes (i.e., Ox and Oy axis).

(ii) MI about the centroidal axis.

(iii) Polar moment of inertia about the origin O.

(iv) Radius of gyration about reference axes.

(v) Radius of gyration about the centroidal axis.

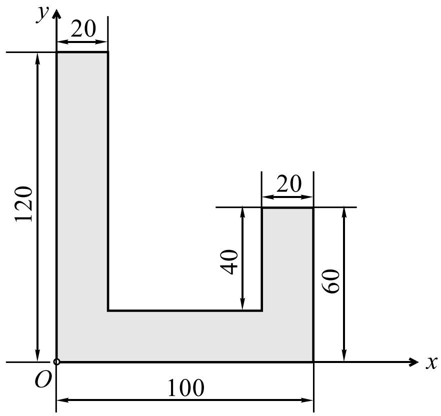


Fig. 6.6(a)

[All dimensions are in mm]

Solution

(i) MI about the reference axes

$$I_{Ox} = \frac{20 \times 120^3}{3} + \frac{60 \times 20^3}{3} + \frac{20 \times 60^3}{3}$$

$$\therefore\ I_{Ox} = 13120000 \quad \textbf{Ans.}$$

$$I_{Oy} = \frac{120 \times 20^3}{3} + \left(\frac{20 \times 60^3}{12} + 20 \times 60 \times 50^2\right) + \left(\frac{60 \times 20^3}{12} + 60 \times 20 \times 90^2\right)$$

$$I_{Oy} = 320000 + 3360000 + 3760000$$

$$\therefore\ I_{Oy} = 13440000 \quad \textbf{Ans.}$$

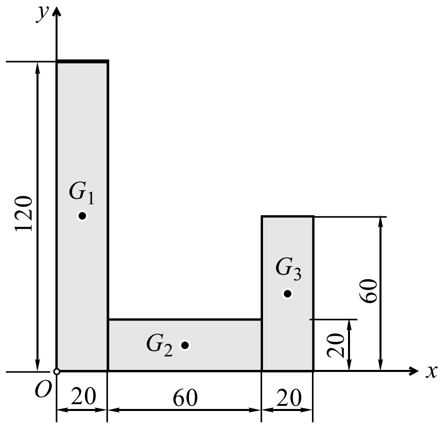


Fig. 6.6(b)

(ii) MI about the centroidal axes

For non-symmetrical section if MI about centroidal axis is required then find centroid w.r.t. reference axes (i.e., $\bar{x}$ and $\bar{y}$) and by applying parallel axis theorem between the reference axis and parallel centroidal axis, find MI about the centroidal axis.

$$\bar{x} = \frac{(120 \times 20 \times 10) + (60 \times 20 \times 50) + (20 \times 60 \times 90)}{(120 \times 20) + (60 \times 20) + (20 \times 60)}$$

$$\bar{x} = 40 \text{ mm}$$

$$\bar{y} = \frac{(120 \times 20 \times 60) + (60 \times 20 \times 10) + (20 \times 60 \times 30)}{(120 \times 20) + (60 \times 20) + (20 \times 60)}$$

$$\bar{y} = 40 \text{ mm}$$

Shaded area, $A = (120 \times 20) + (60 \times 20) + (20 \times 60)$

$$A = 4800 \text{ mm}^2$$

By parallel axis theorem, we have

$$I_{Ox} = I_{xx_G} + A\,\bar{y}^2$$

$$I_{xx_G} = I_{Ox} - A\,\bar{y}^2 = 13120000 - (4800 \times 40^2)$$

$$I_{xx_G} = 5440000 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

$$I_{Oy} = I_{yy_G} + A\,\bar{x}^2$$

$$I_{yy_G} = I_{Oy} - A\,\bar{x}^2 = 13440000 - (4800 \times 40^2)$$

$$I_{yy_G} = 5760000 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

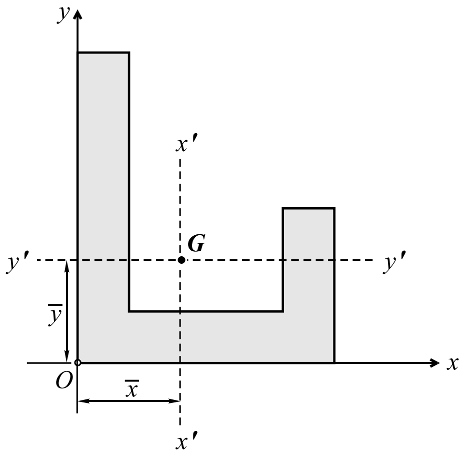


Fig. 6.6(c)

(iii) Polar moment of inertia about origin *O*

$$I_P = I_{Ox} + I_{Oy}$$

$$I_P = 13120000 + 13440000$$

$$I_P = 26560000 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

(iv) Radius of gyration about the reference axes (i.e., *Ox* and *Oy* axis) $I = Ak^2$

$$k_{Ox} = \sqrt{\frac{I_{Ox}}{A}} = \sqrt{\frac{13120000}{4800}} = 52.28 \text{ mm} \quad \textbf{\textit{Ans.}}$$

$$k_{Oy} = \sqrt{\frac{I_{Oy}}{A}} = \sqrt{\frac{13440000}{4800}} = 52.92 \text{ mm} \quad \textbf{\textit{Ans.}}$$

(v) Radius of gyration about the centroidal axis

$$k_{xx_G} = \sqrt{\frac{I_{xx_G}}{A}} = \sqrt{\frac{5440000}{4800}} = 33.67 \text{ mm} \quad \textbf{\textit{Ans.}}$$

$$k_{yy_G} = \sqrt{\frac{I_{yy_G}}{A}} = \sqrt{\frac{5760000}{4800}} = 34.64 \text{ mm} \quad \textbf{\textit{Ans.}}$$

Problem 7

Find the MI of a section shown in Fig. 6.7(a) about

(i) x-axis

(ii) y-axis

(iii) Parallel centroidal x-axis

(iv) Parallel centroidal y-axis.

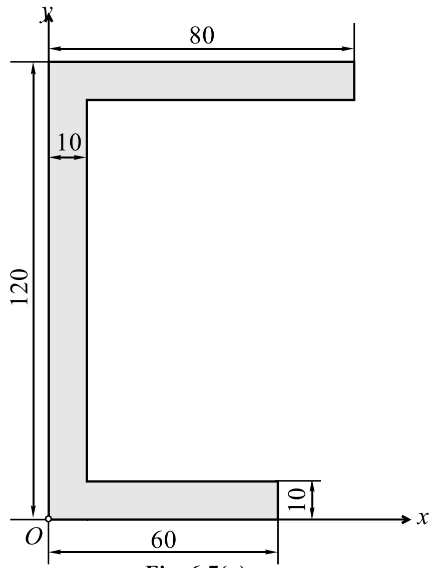


Fig. 6.7(a)

[All dimensions are in mm]

Solution

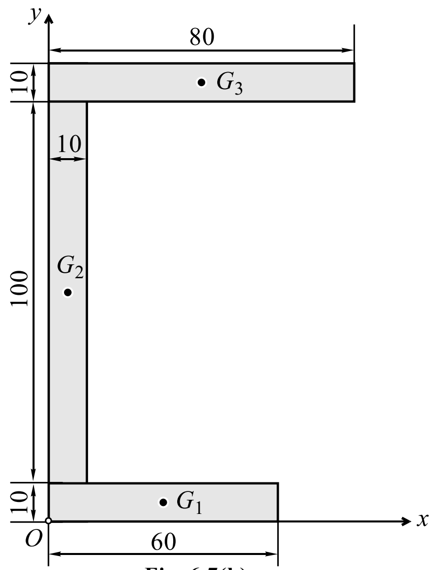


Fig. 6.7(b)

(i) MI about *x*-axis

$$I_{xx} = I_{x_1} + I_{x_2} + I_{x_3}$$

$$I_{xx} = \frac{60 \times 10^3}{3} + \left(\frac{10 \times 100^3}{12} + 10 \times 100 \times 60^2\right)$$

$$+ \left(\frac{80 \times 10^3}{12} + 80 \times 10 \times 115^2\right)$$

$$= 20000 + 443333.33 + 10586666.67$$

$$\therefore\ I_{xx} = 15040000 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

(ii) MI about *y*-axis

$$I_{yy} = I_{y_1} + I_{y_2} + I_{y_3}$$

$$I_{yy} = \frac{10 \times 60^3}{3} + \frac{100 \times 10^3}{3} + \frac{10 \times 80^3}{3}$$

$$\therefore\ I_{yy} = 2460000 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

(iii) To find $\bar{x}$ and $\bar{y}$

$$\bar{x} = \frac{(10 \times 60 \times 30) + (100 \times 10 \times 5) + (80 \times 10 \times 40)}{(10 \times 60) + (100 \times 10) + (80 \times 10)} = 22.92 \text{ mm}$$

$$\bar{y} = \frac{(10 \times 60 \times 5) + (100 \times 10 \times 60) + (80 \times 10 \times 115)}{(10 \times 60) + (100 \times 10) + (80 \times 10)} = 64.58 \text{ mm}$$

Shaded area, $A = (10 \times 60) + (100 \times 10) + (80 \times 10) = 2400 \text{ mm}^2$

(iv) To find I_{xx_G} and I_{yy_G}

By parallel axis theorem, we have

$$I_{xx} = I_{xx_G} + A\,\bar{y}^2$$

$$I_{xx_G} = I_{xx} - A\,\bar{y}^2 = 15040000 - [2400 \times (64.58)^2]$$

$$I_{xx_G} = 5030616.64 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

$$I_{yy} = I_{yy_G} + A\,\bar{x}^2$$

$$I_{yy_G} = I_{yy} - A\,\bar{x}^2 = 2460000 - [2400 \times (22.92)^2]$$

$$I_{yy_G} = 1199216.64 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

Problem 8

Figure 6.8(a) shows a plane area. Find the MI of the section about *x-x* and *y-y* axes passing through the CG of the section.

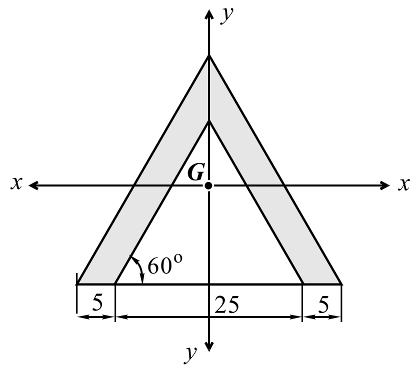


Fig. 6.8(a)

[All dimensions are in cm]

Solution

(i) Shaded area = Big Δ – Small Δ shown in Fig. 6.8(a).

Height of big triangle,

$$\tan 60^\circ = \frac{H}{17.5} \qquad \therefore\ H = 30.31 \text{ cm}$$

Height of small triangle,

$$\tan 60^\circ = \frac{h}{12.5} \qquad \therefore\ h = 21.65 \text{ cm}$$

We should first find distance of $\bar{y}$ from *x-x* axis.

$$\bar{y} = \frac{\left(\frac{1}{2}\times 35\times 30.31\times\frac{30.31}{3}\right)-\left(\frac{1}{2}\times 25\times 21.65\times\frac{21.65}{3}\right)}{\left(\frac{1}{2}\times 35\times 30.31\right)-\left(\frac{1}{2}\times 25\times 21.65\right)}$$

$$\bar{y} = \frac{3406.05}{259.8} \qquad \therefore\ \bar{y} = 13.11 \text{ cm}$$

$$\text{Shaded area } A = \left(\frac{1}{2}\times 35\times 30.31\right)-\left(\frac{1}{2}\times 25\times 21.65\right)$$

$$A = 259.8 \text{ cm}^2$$

(ii) MI about *y-y* axis

Since the given figure is symmetric about *y*-axis, finding I_{yy} is easy. Consider the MI valued twice

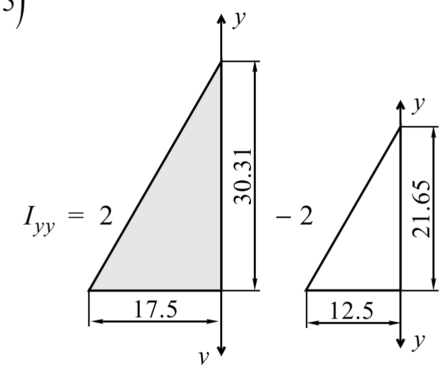


Fig. 6.8(b)

$$I_{yy} = 2\times\left(\frac{30.31\times 17.5^3}{12}\right) - 2\times\left(\frac{21.65\times 12.5^3}{12}\right)$$

$$I_{yy} = 27073.78 - 7047.53$$

$$I_{yy} = 20026.25 \text{ cm}^4 \quad \textit{\textbf{Ans.}}$$

(iii) MI about *AB* axis

First find MI about the baseline, say *AB*

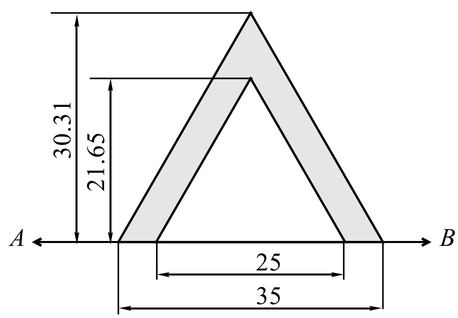


Fig. 6.8(c)

$$I_{AB} = \frac{35\times 30.31^3}{12} - \frac{25\times 21.65^3}{12}$$

$$I_{AB} = 60075.23 \text{ cm}^4 \quad \textit{\textbf{Ans.}}$$

By parallel axis theorem, we have

$$I_{AB} = I_{xx_G} + A\,\bar{y}^2 \qquad (A = \text{ Shaded area})$$

$$\therefore\ I_{xx_G} = I_{AB} - A\,\bar{y}^2 = 60075.23 - (259.8\times 13.11^2)$$

$$I_{xx_G} = 15422.86 \text{ cm}^4 \quad \textit{\textbf{Ans.}}$$

Problem 9

Calculate the polar MI of shaded area shown in Fig. 6.9(a) about the point O.

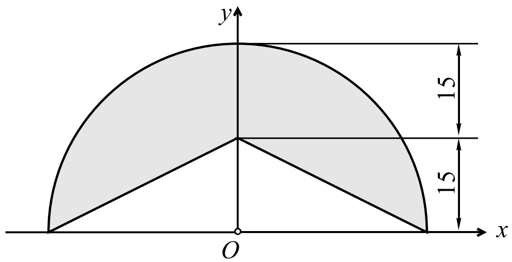


Fig. 6.9(a)

[All dimensions are in mm]

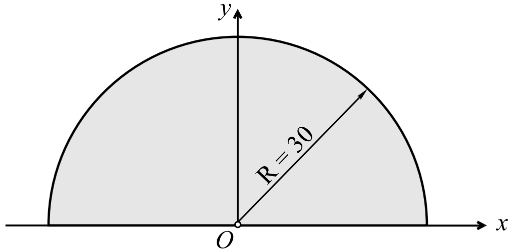


Fig. 6.9(b)

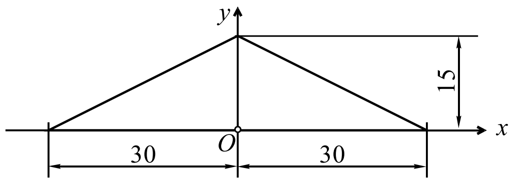


Fig. 6.9(c)

Solution

(i) For semicircle

$$I_{xx} = I_{yy} = \frac{\pi R^4}{8} = \frac{\pi \times 30^4}{8} = 318086.26 \text{ mm}^4$$

$$\therefore \text{ Polar MI, } P_O = I_{xx} + I_{yy} = 2 \times 318086.26$$

$$\therefore P_O = 636172.52 \text{ mm}^4$$

(ii) For triangle (Negative)

$$I_{xx} = \frac{60 \times 15^3}{12} = 16875 \text{ mm}^4$$

$$I_{yy} = 2 \times \frac{15 \times 30^3}{12} = 67500 \text{ mm}^4$$

$$\therefore \text{ Polar MI, } P_O = I_{xx} + I_{yy} = 16875 + 67500$$

$$\therefore P_O = 84375 \text{ mm}^4$$

(iii) Polar MI of shaded area

$$P_O = P_{O(semicircle)} - P_{O(triangle)}$$

$$= 636172.52 - 84375$$

$$\therefore P_O = 551797.52 \text{ mm}^4 \quad \textbf{\textit{Ans.}}$$

Problem 10

ABC is an isosceles triangle of base 100 mm and height 90 mm. A square of side a is removed from the triangular lamina as shown in Fig. 6.10, such that the moment of inertia of the triangle about the base reduces by 20% of the original value. Find the size of the square.

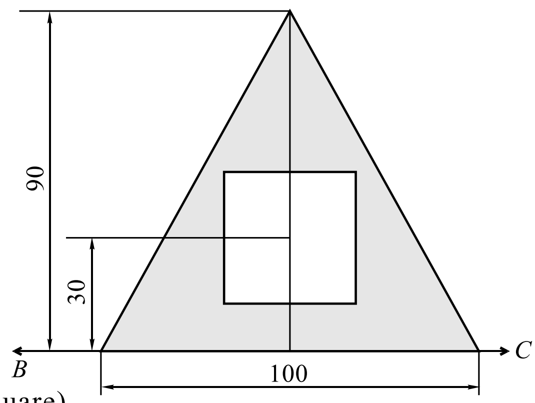


Fig. 6.10

[All dimensions are in mm]

Solution

MI of the triangle

$$I_{BC} = \frac{100 \times 90^3}{12} = 6075000 \text{ mm}^4$$

Given : *MI* of the triangle when square removed reduces by 20%.

$$\therefore I_{BC} = 4860000 \text{ mm}^4 \qquad I_{BC} = 6075000 \times 0.8$$

But $I_{BC} = I_{\Delta} - I_{\square}$

$$4860000 = \frac{100 \times 90^3}{12} - \left[\frac{a \times a^3}{12} + (a \times a \times 30^2)\right]$$

(*a* is the side of square)

$$\frac{a^4}{12} + a^2 \times 30^2 = 1215000$$

$$a^4 + a^2 \times 30^2 \times 12 = 1215000 \times 12$$

$$(a^2)^2 + a^2\,(10800) - 14580000 = 0$$

Solving quadratic equation, we get

$a = 34.84$ mm ***Ans.***

Problem 11

Two channels are kept as shown in Fig. 6.11(a), at a distance d between them to form the cross section of a column. Find the value of the distance d if the centroidal moment of inertia I_x and I_y of the area are equal.

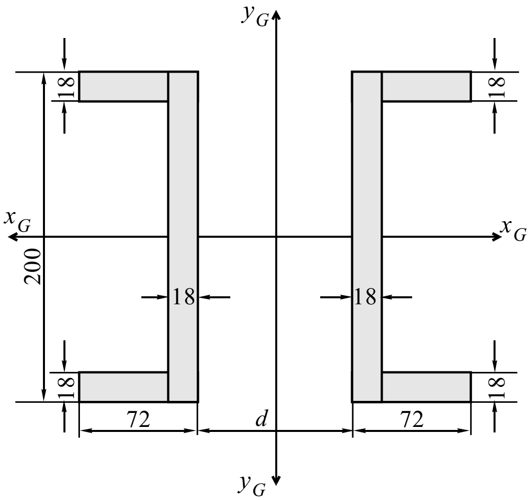


Fig. 6.11(a)
[All dimensions are in mm]

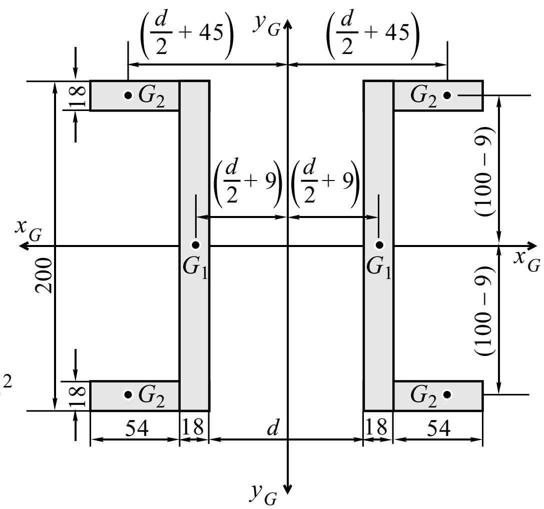


Fig. 6.11(b)

Solution

(i) MI about *x*-axis

$$I_{x_G} = 2\left[\frac{18\times 200^3}{12}\right] + 4\left[\frac{54\times 18^3}{12} + (54\times 18)(100-9)^2\right]$$

$$\therefore I_{x_G} = 56.3\times 10^6 \text{ mm}^4$$

(ii) MI about *y*-axis

$$I_{y_G} = 2\left[\frac{200\times 18^3}{12} + (200\times 18)\left(\frac{d}{2}+9\right)^2\right] + 4\left[\frac{18\times 54^3}{12} + (18\times 54)\left(\frac{d}{2}+45\right)^2\right]$$

Put $\frac{d}{2} = a$

$$\therefore I_{y_G} = 1.14\times 10^6 + 3888(a+45)^2 + 7200(a+9)^2$$

$$= 1.14\times 10^6 + 3888(a^2+90\,a+2025) + 7200(a^2+18\,a+81)$$

$$= 9.6\times 10^6 + 11088\,a^2 + (479.52\times 10^3)\,a$$

Given : $I_{x_G} = I_{y_G}$

$$56.3\times 10^6 = 9.6\times 10^6 + 11088\,a^2 + (479.52\times 10^3)\,a$$

$$\therefore 11088\,a^2 + (479.52\times 10^3)\,a - 46.7\times 10^6 = 0$$

$$\therefore a = 46.78 \text{ or } -90.16$$

$a = -90.16$ is meaningless

$$\therefore a = 46.78 \text{ mm}$$

$$a = \frac{d}{2} = 46.78 \text{ mm}$$

$\therefore d = 93.56$ mm ***Ans.***

6.8 Mass Moment of Inertia of Standard Shapes (Transfer Formula for Mass Moment of Inertia)

1. MI of a Uniform (Slender) Thin Rod About an Axis Perpendicular to the Length and Passing through the Centre of Gravity

Consider a uniform rod AB of length l with its mid point O as shown in Fig. 6.8-i(a).

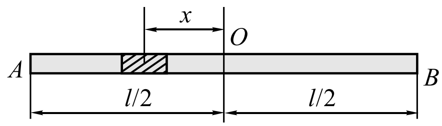


Fig. 6.8-i(a)

Let m = Mass of the rod per unit length

M = Total mass of the rod $= ml$

Consider an elementary strip of length dx at a distance x from the axis through centre O.

By the definition of MI, we have

MI of strip length

$$I_O = (\text{Mass of strip length}) \times (\text{Distance from } G)^2$$

$$I_O = m\,dx \times x^2 = mx^2\,dx$$

$\therefore$ MI of slender rod

$$I_O = \int_{-l/2}^{l/2} mx^2\,dx = m\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{ml^3}{12} = \frac{mll^2}{12}$$

$$\therefore\ I_O = \frac{Ml^2}{12} \qquad (\because\ ml = M)$$

MI of a Uniform Thin Rod About an Axis Perpendicular to the Length and Passing through the End *O*

By parallel axis theorem, we have

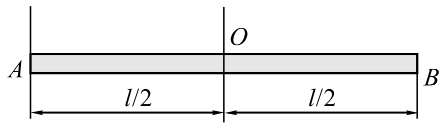


Fig. 6.8-i(b)

$$I_A = I_O + M\left(\frac{l^2}{2}\right) = \frac{Ml^2}{12} + \frac{Ml^2}{4}$$

$$\therefore\ I_A = \frac{Ml^2}{3}$$

2. Mass Moment of Inertia of Thin Circular Ring

Consider a thin circular ring of radius r and centre O as shown in Fig. 6.8-ii.

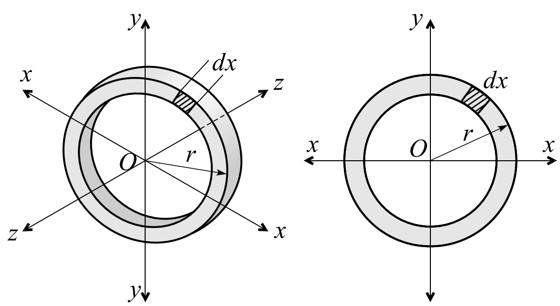


Fig. 6.8-ii

Let m = Mass of the ring per unit peripheral length

M = Total mass of the ring $= 2\pi rm$

Consider an elementary strip of length dx as shown.

MI of the elementary strip $I_{zOz} = (m\,dx) \times r^2$

$\therefore$ MI of the total ring $I_{zOz} = (2\pi rm) \times r^2$

$$\therefore\ I_{zOz} = M\,r^2$$

By perpendicular axis theorem

$$I_{zOz} = I_{xOx} + I_{yOy}$$

$$\therefore \; I_{xOx} = I_{yOy} = \frac{I_{zOz}}{2} = \frac{M\,r^2}{2}$$

3. Mass Moment of Inertia of Thin Disc (Circular Lamina)

Let m = Mass of the lamina per unit area of lamina

M = Total mass of the lamina $= \pi r^2 m$

Consider an elementary ring of centre O, radius x and thickness dx as shown in Fig.6.8-iii.

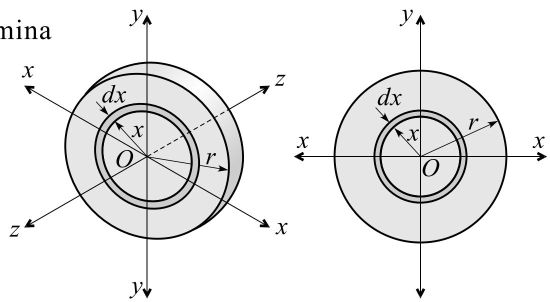


Fig. 6.8-iii

By the definition of MI

$$I_{zOz} = 2\pi x \cdot dx \cdot m \times x^2$$

MI of total circular lamina

$$I_{zOz} = \int_0^r 2\pi\, m\, x^3\, dx = \frac{\pi\, m\, r^4}{2} = \frac{\pi\, r^2\, m\, r^2}{2}$$

$$I_{zOz} = \frac{M\,r^2}{2} \qquad (\because \pi\, r^2\, m = M)$$

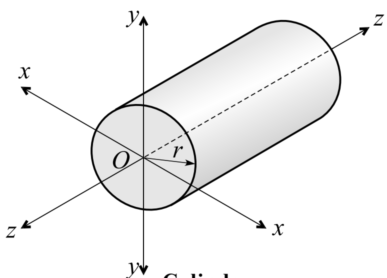
By perpendicular axis theorem, we have

$$I_{zz} = I_{xx} + I_{yy}$$

$$\therefore \; I_{xOx} = I_{yOy} = \frac{I_{zOz}}{2} = \frac{M\,r^2}{4}$$

Similarly, MI of the cylinder will be

$$I_{zOz} = \frac{M\,r^2}{2}$$



Cylinder

4. Mass Moment of Inertia of Sphere

Consider a solid sphere of centre G and radius r, as shown in Fig. 6.8-iv.

Let m = Mass of the sphere per unit volume

M = Total mass of the sphere

$$= \left(\frac{4\pi\, r^3}{3}\right) \times m \qquad \left(\because \; v = \frac{4\pi\, r^3}{3}\right)$$

Consider an elementary thin disc at a distance x from the centre of the sphere G and thickness dx as shown in Fig. 6.8-iv.

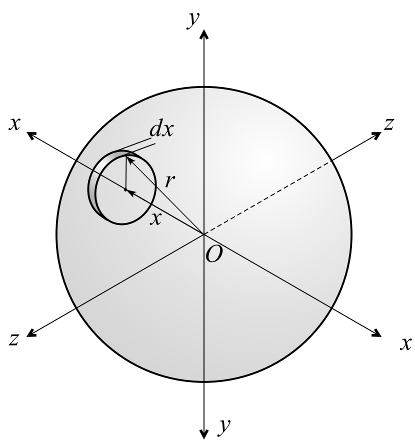


Fig. 6.8-iv

Radius of thin disc $= \sqrt{r^2 - x^2}$

Mass of thin disc $= m\,\pi \left(\sqrt{r^2 - x^2}\right)^2 dx$

$$= m\,\pi\,(r^2 - x^2)\,dx$$

MI of the thin disc about an axis perpendicular to the plane of disc and passing through centre of the disc O.

$$I_{xOx} = \frac{\text{Mass} \times (\text{Radius})^2}{2} = \frac{m\,\pi\,(r^2 - x^2)\,dx \times \left(\sqrt{r^2 - x^2}\right)^2}{2}$$

$$I_{xOx} = \frac{m\,\pi}{2}(r^4 + x^4 - 2r^2x^2)\,dx$$

$$\therefore\ I_{xOx} = \int_{-r}^{r} \frac{m\pi}{2}\,(r^4 + x^4 - 2r^2x^2)\,dx = \frac{8m\pi\,r^5}{15}$$

$$\therefore\ I_{xOx} = \frac{2}{5}\,M\,r^2 \qquad \left(\because\ M = \frac{4\pi\,r^3 m}{3}\right)$$

For sphere, due to all symmetry the mass moment of the sphere about any axis passing through the centre G is same.

$$\therefore\ I_{xOx} = I_{yOy} = I_{zOz} = \frac{2M\,r^2}{5}$$

5. Mass Moment of Inertia of a Rectangular Lamina

Consider a rectangular lamina of length l, width b and centre O as shown in Fig. 6.8-v.

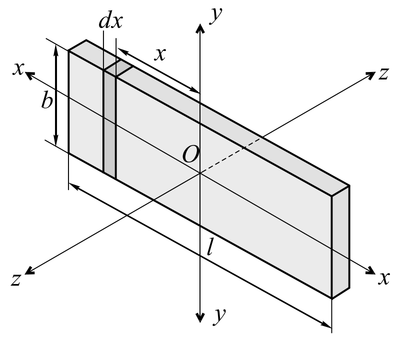


Fig. 6.8-v

Let m = Mass of the lamina per unit area

M = Total mass of the lamina = blm

Consider an elementary strip of width b and thickness dx, at a distance x from the centre as shown in the figure.

By the definition of MI, we have

$$I_{yOy} = (m)(b\,dx)\,x^2$$

MI of total rectangular lamina

$$I_{yOy} = mb\int_{-l/2}^{l/2} x^2\,dx = mb\,\frac{l^3}{12}$$

$$I_{yOy} = \frac{M\,l^2}{12} \qquad (\because\ mbl = M)$$

Similarly, MI about x-axis

$$I_{xOx} = \frac{M\,b^2}{12}$$

By perpendicular axis theorem, we have

$$I_{zOz} = I_{xOx} + I_{yOy}$$

$$\therefore\ I_{zOz} = \frac{M(b^2 + l^2)}{12}$$

6. Mass Moment of Inertia of a Triangular Plate About Centroidal Axis Parallel to Base

Consider a triangular plate of base b, height h and thickness t as shown in Fig. 6.8-vi.

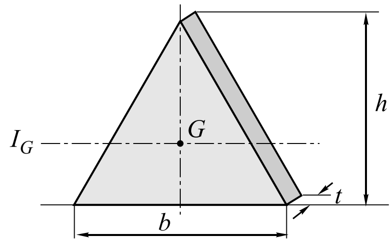


Fig. 6.8-vi

Let M be the mass of the plate.

By the definition of MI, we have

$$I_G = \frac{bh^3}{36} \rho\, t \qquad \text{(where } \rho \text{ is the density)}$$

$$I_G = \left(\frac{1}{2} bht\, \rho\right)\left(\frac{h^2}{18}\right)$$

$$\therefore\ I_G = \frac{Mh^2}{18}$$

7. Mass Moment of Inertia of a Right Circular Cone

Consider a cone with base radius R, height H and mass density ρ as shown in Fig. 6.8-vii.

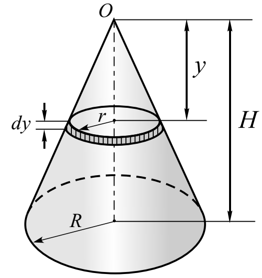


Fig. 6.8-vii

Let us consider a thin circular disc of radius r located at distance y from apex O with thickness dy.

Let the mass of the disc be dm.

$$dm = \pi r^2\, dy\, \rho \quad \text{By geometry } \frac{R}{H} = \frac{r}{y} \quad \therefore\ r = \frac{Ry}{H}$$

$$\therefore\ dm = \pi \frac{R^2 y^2}{H^2} dy\, \rho$$

$\therefore$ Mass moment of inertia about axis

$$I_{axis} = \frac{1}{2} dm\, r^2$$

$$I_{axis} = \frac{1}{2} \frac{\pi R^2 y^2\, dy\, \rho}{H^2} \frac{R^2 y^2}{H^2}$$

$$I_{axis} = \frac{1}{2} \frac{\rho\, \pi\, R^4 y^4\, dy}{H^4}$$

Integrating for entire cone

$$I_{axis} = \int_0^H \frac{1}{2} \frac{\rho\, \pi\, R^4 y^4}{H^4} dy = \frac{1}{2} \frac{\rho\, \pi\, R^4}{H^4} \left(\frac{y^5}{5}\right)_0^H = \frac{1}{2} \frac{\rho\, \pi\, R^4 H^5}{5H^4}$$

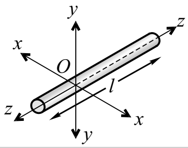
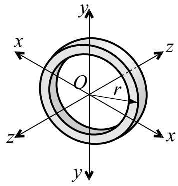
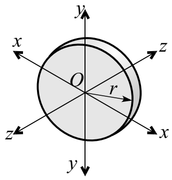
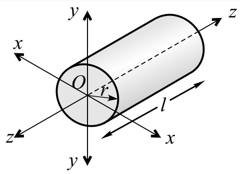
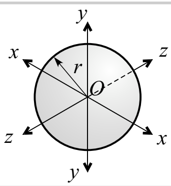
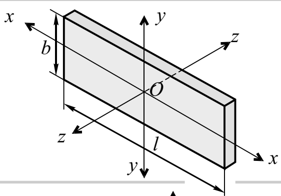
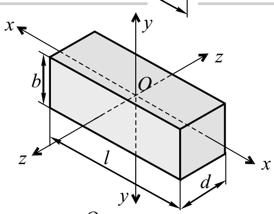
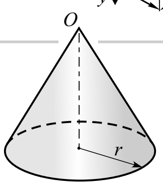
$$I_{axis} = \frac{\pi\, \rho\, R^4 H}{10}$$

But mass of cone $M = \dfrac{1}{3} \pi R^2 H \rho$

$$\therefore\ I_{axis} = \left(\frac{1}{3} \pi R^2 H \rho\right)\left(\frac{R^3\, 3}{10}\right)$$

$$I_{axis} = \frac{3M R^2}{10}$$

Tabulating the Results

Solid	Figure	Mass Moment of Inertia
1. Uniform Thin Rod		$I_{xOx} = I_{yOy} = \dfrac{Ml^2}{12}$
2. Thin Circular Ring		$I_{zOz} = M\,r^2$ $I_{xOx} = I_{yOy} = \dfrac{M\,r^2}{2}$
3. Circular Lamina		$I_{zOz} = \dfrac{M\,r^2}{2}$ $I_{xOx} = I_{yOy} = \dfrac{M\,r^2}{4}$
4. Cylinder		$I_{zOz} = \dfrac{M\,l^2}{2}$ $I_{xOx} = I_{yOy} = \dfrac{M\,r^2}{4}$
5. Sphere		$I_{xOx} = I_{yOy} = I_{zOz} = \dfrac{2M\,r^2}{5}$
6. Rectangular Plate		$I_{xOx} = \dfrac{M\,b^2}{12}$; $I_{yOy} = \dfrac{M\,l^2}{12}$ $I_{zOz} = \dfrac{M(b^2 + l^2)}{12}$
7. Rectangular Prism		$I_{xOx} = \dfrac{M(d^2 + b^2)}{12}$ $I_{yOy} = \dfrac{M(l^2 + d^2)}{12}$ $I_{zOz} = \dfrac{M(b^2 + l^2)}{12}$
8. Cone		$I_{axis} = \dfrac{3M\,r^2}{10}$

Problem 12

Determine the mass moment of inertia of a mild steel rectangular plate of size 15 cm × 30 cm and thickness 10 mm about the centroidal axis, which is parallel to shorter side. Mass density of mild steel is 8000 kg/m^3.

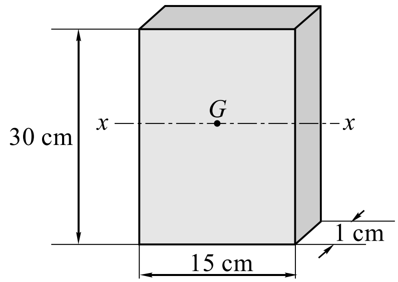


Fig. 6.12

Solution

Method I

$$I_{xGx} = \frac{bh^3}{12}\ \rho\ t$$

$$I_{xGx} = \frac{0.15 \times 0.3^3 \times 8000 \times 0.01}{12}$$

$$I_{xGx} = 0.027 \text{ kg.m}^2 \quad \textbf{\textit{Ans.}}$$

Method II

$$I_{xGx} = \frac{Mh^2}{12}$$

$$I_{xGx} = \frac{(0.15 \times 0.3 \times 0.01 \times 8000) \times 0.3^2}{12}$$

$$I_{xGx} = 0.027 \text{ kg.m}^2 \quad \textbf{\textit{Ans.}}$$

Problem 13

A concrete column of diameter 40 cm and height 3.3. m is standing vertically. For mass density of concrete 2500 kg/m^3, find the mass moment of inertia about its axis.

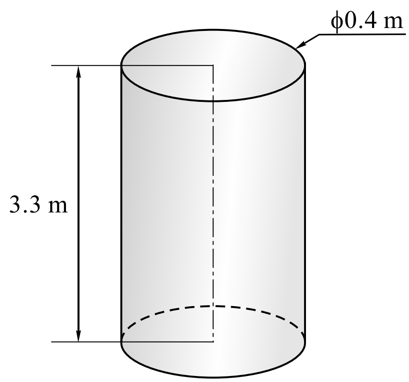


Fig. 6.13

Solution

Relevant formula to find mass moment of cylinder

$$I_{axis} = \frac{M\,R^2}{2}$$

$$I_{axis} = (\pi R^2\, h\, \rho)\ \frac{R^2}{2}$$

$$I_{axis} = (\pi \times 0.2^2 \times 3.3 \times 2500)\left(\frac{0.2^2}{2}\right)$$

$$I_{axis} = 20.74 \text{ kg.m}^2 \quad \textbf{\textit{Ans.}}$$

Problem 14

For a steel triangular plate of base 50 cm and height 100 cm and thickness 1 cm, find mass moment of inertia about its **(i)** base and **(ii)** parallel centroidal axis. Consider density of steel as 7900 kg/m^3.

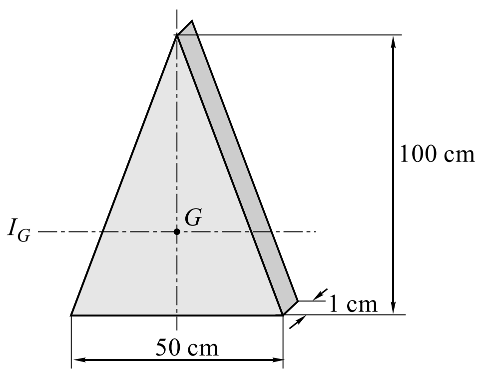


Fig. 6.14

Solution

(i) $$I_{base} = \frac{bh^3}{12}\ \rho\ t$$

$$I_{base} = \frac{0.5 \times 1^3 \times 7900 \times 0.01}{12}$$

$$I_{base} = 3.292 \text{ kg.m}^2 \quad \textbf{\textit{Ans.}}$$

(ii) $I_G = \dfrac{bh^3}{36}\,\rho\, t$

$$I_G = \frac{0.5 \times 1^3 \times 7900 \times 0.01}{36}$$

$$I_G = 1.097 \text{ kg.m}^2 \quad \textbf{\textit{Ans.}}$$

Problem 15

Two uniform rods, each of mass 40 kg and length 3 m are welded together to form a T shape assembly, which is pin suspended at O, as shown in Fig. 6.15. Determine the mass moment of inertia of the assembly about pin axis at O.

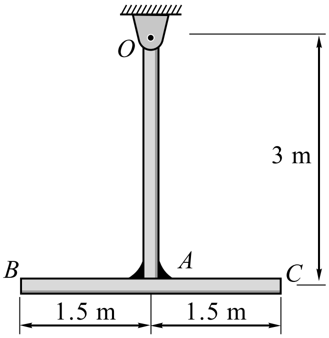


Fig. 6.15

Solution

$$I_O = I_{\text{Rod } OA} + I_{\text{Rod } BC}$$

$$I_O = \frac{M\,l^2}{3} + \left[\frac{M\,l^2}{12} + M(d)^2\right]$$

$$I_O = \frac{40 \times 3^2}{3} + \left[\frac{40 \times 3^2}{12} + 40 \times 3^2\right]$$

$$\therefore\ I_O = 510 \text{ kg.m}^2 \quad \textbf{\textit{Ans.}}$$

Problem 16

A uniform rod of mass 10 kg is pinned at O and its ends are welded by sphere of mass 20 kg at upper end and circular disc of mass 15 kg at lower end, as shown in Fig. 6.16. Find MI about pinned axis.

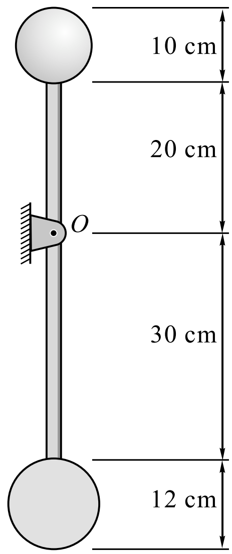


Fig. 6.16

Solution

$$I_O = I_{\text{Rod}} + I_{\text{Sphere}} + I_{\text{Disc}}$$

$$I_O = \left[\frac{M_1\,l^2}{12} + M_1 d_1^{\,2}\right] + \left[\frac{2M_2\,r_2^{\,2}}{12} + M_2 d_2^{\,2}\right] + \left[\frac{M_3\,r_3^{\,2}}{2} + M_3 d_3^{\,2}\right]$$

$$I_O = \left[\frac{10 \times 50^2}{12} + 10 \times 5^2\right] + \left[\frac{2 \times 20 \times 50^2}{5} + 20 \times 25^2\right] + \left[\frac{15 \times 6^2}{2} + 15 \times 36^2\right]$$

$$I_O = 2333.33 + 12700 + 19710$$

$$\therefore\ I_O = 34743.33 \text{ kg.cm}^2 \quad \textbf{\textit{Ans.}}$$

Exercises

[I] Problems

1. Find the MI about the centroidal axis in Fig. 6.E1.

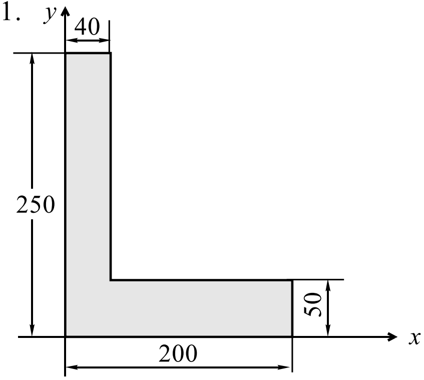


Fig. 6.E1 *[All dimensions are in mm]*

$$\left[\textbf{Ans. } \bar{x} = 64.4 \text{ mm}, \ \bar{y} = 80.5 \text{ mm}, \ I_{xx} = 98.18 \times 10^6 \text{ mm}^4 \text{ and } I_{yy} = 62.83 \times 10^6 \text{ mm}^4.\right]$$

2. Find the MI about the centroidal axis in Fig. 6.E2.

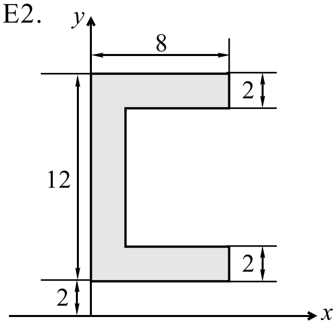


Fig. 6.E2 *[All dimensions are in cm]*

$$\left[\textbf{Ans. } I_{xx} = 14.89 \text{ cm}^4\right]$$

3. Find the moment of inertia w.r.t. the centroidal x and y axes in Fig. 6.E3.

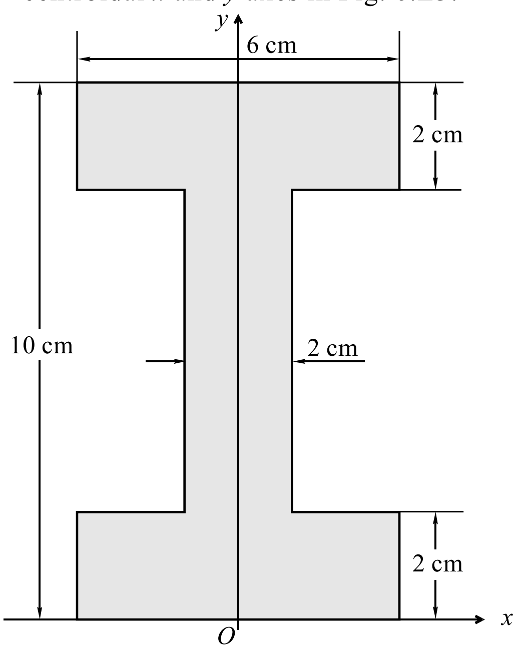


Fig. 6.E3

$$\left[\textbf{Ans. } I_{x_G} = 428 \text{ cm}^4 \text{ and } I_{y_G} = 76 \text{ cm}^4.\right]$$

4. Find the centroid of the unequal I- section shown in Fig. 6.E4 and calculate MI about the centroidal x and y axis. Also find MI about base.

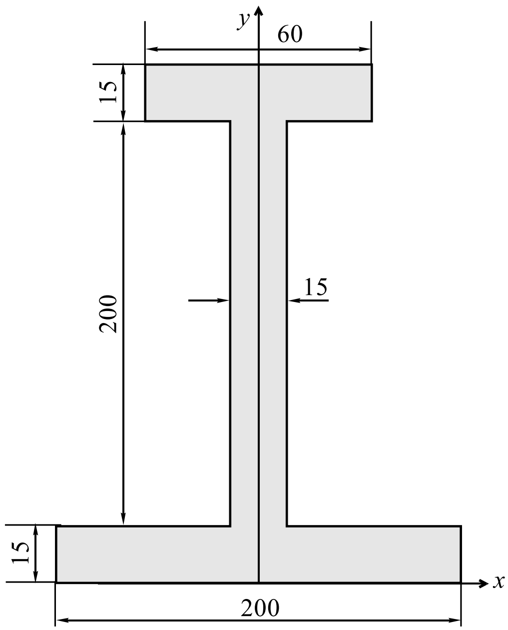


Fig. 6.E4 *[All dimensions are in mm]*

$$\left[\textbf{Ans. } \bar{x} = 0, \ \bar{y} = 82.28 \text{ mm}, \ I_{x_G} = 47.756 \times 10^6 \text{ mm}^4, \ I_{y_G} = 10.326 \times 10^6 \text{ mm}^4 \text{ and } I_{\text{Base}} = 94.47 \times 10^6 \text{ mm}^4.\right]$$

5. Find the MI about the centroidal axis in Fig. 6.E5.

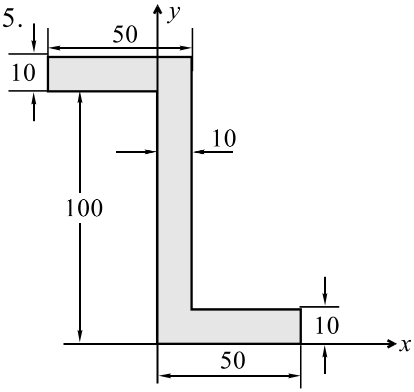


Fig. 6.E5 *[All dimensions are in mm]*

[**Ans.** $\bar{x} = 5$ mm, $\bar{y} = 55$ mm,
$I_{xx} = 3.12 \times 10^6$ mm^4 and
$I_{yy} = 0.526 \times 10^6$ mm^4.]

6. Find the MI about the centroidal axis in Fig. 6.E6.

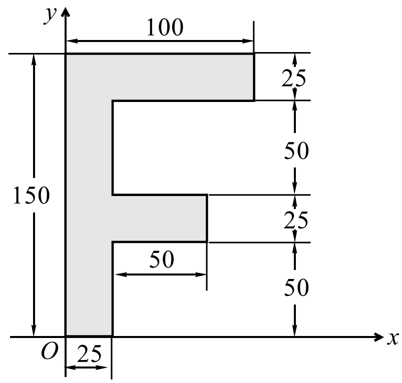


Fig. 6.E6 *[All dimensions are in mm]*

[**Ans.** $\bar{x} = 32.95$ mm, $\bar{y} = 89.77$ mm,
$I_{x_G} = 13.213 \times 10^5$ mm^4 and
$I_{y_G} = 49.035 \times 10^5$ mm^4.]

7. Find the MI about the centroidal axis in Fig. 6.E7.

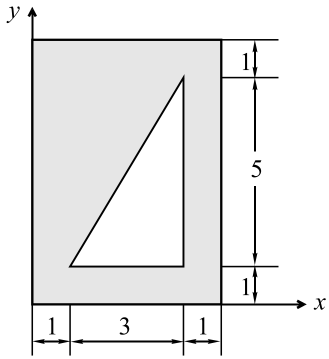


Fig. 6.E7 *[All dimensions are in cm]*

[**Ans.** $I_{xx} = 125.92$ cm^4 and $I_{yy} = 66.815$ cm^4.]

8. Find the MI about the centroidal axis in Fig. 6.E8.

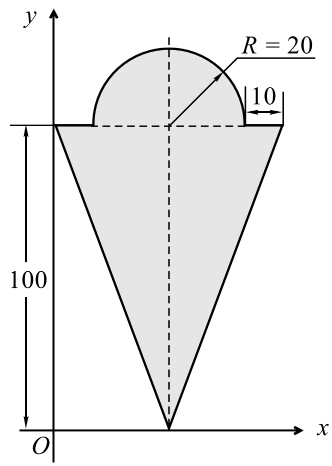


Fig. 6.E8 *[All dimensions are in cm]*

[**Ans.** $I_{xx} = 2.59 \times 10^6$ cm^4 and
$I_{yy} = 512831$ cm^4.]

9. Calculate the MI about the centroidal x and y for the X-section shown in Fig. 6.E9.

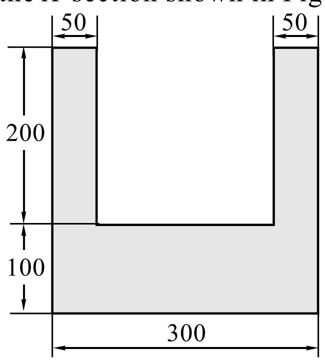


Fig. 6.E9 *[All dimensions are in cm]*

[**Ans.** $I_{x_G} = 3.617 \times 10^8$ mm^4 and
$I_{y_G} = 5.417 \times 10^8$ mm^4.]

10. Determine the polar MI of the shaded area in Fig. 6.E10 with respect to axis and through the origin.

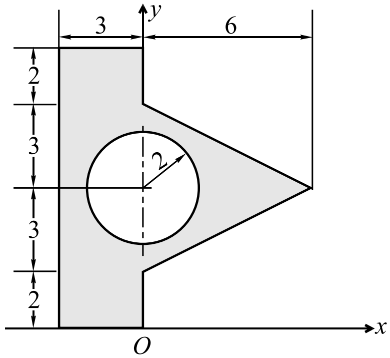


Fig. 6.E10 *[All dimensions are in cm]*

$\left[\textbf{Ans.}\ \ 1336\ \text{cm}^4\right]$

11. Find the MI about the centroidal axis in Fig. 6.E11.

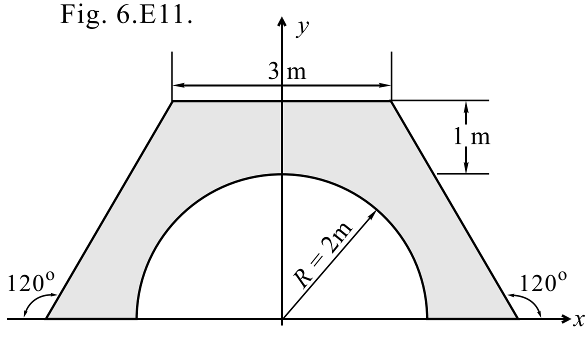


Fig. 6.E11

$\left[\textbf{Ans.}\ \ I_{xx} = 28.51\ \text{m}^4 \text{ and } I_{yy} = 23.757\ \text{m}^4.\right]$

12. Determine the polar radius of gyration about point A for the shaded area shown in Fig. 6.E12.

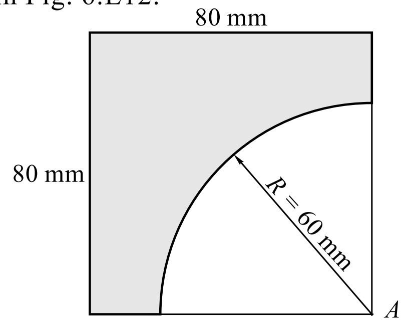


Fig. 6.E12

$\left[\textbf{Ans.}\ \ k = 78.9\ \text{mm}\right]$

13. Find the thickness t of the top plate so that the centroid of the built up section is 130 mm from the bottom. Determine moment of inertia of the section about centroidal x and y axes from Fig. 6.E13.

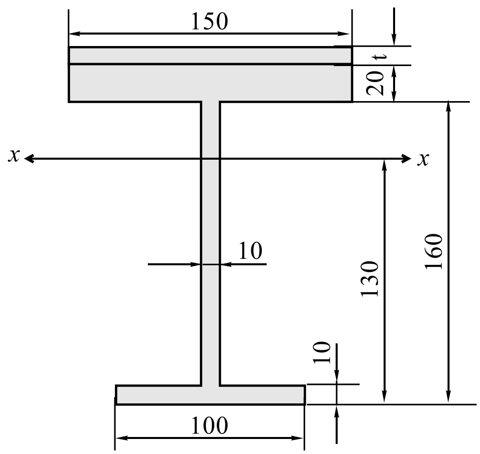


Fig. 6.E13 *[All dimensions are in cm]*

$$\left[\begin{array}{l}\textbf{Ans.}\ \ I_{x_G} = 30.334 \times 10^6\ \text{mm}^4 \text{ and} \\ \qquad\quad I_{y_G} = 8.966 \times 10^6\ \text{mm}^4.\end{array}\right]$$

14. Find I_{AB} in Fig. 6.E14.

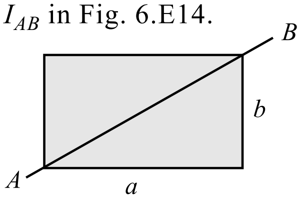


Fig. 6.E14

$$\left[\textbf{Ans.}\ \ I_{AB} = \frac{a^3\, b^3}{6(a^2 + b^2)}\right]$$

15. For the H-beam section shown in Fig. 6.E15, determine the flange width b that will make the moment of inertia about the central x and y axes equal.

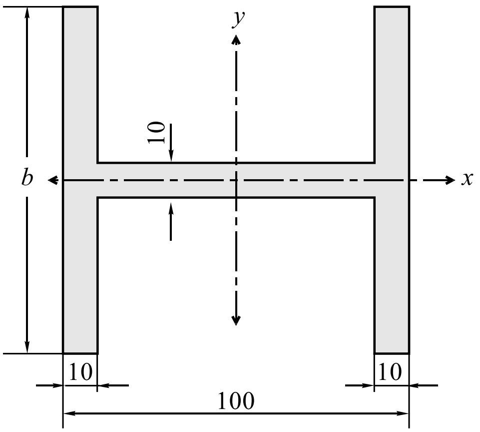


Fig. 6.E15 *[All dimensions are in mm]*

[**Ans.** b = 161.1 mm]

16. The shaded area shown in Fig. 6.E16 is 125 cm^2. If I_{xx} = 35000 cm^4, $I_{x'x'}$ = 70000 cm^4 and d_2 = 7.5 cm, determine the distance d_1 and MI of the area w.r.t. centroidal axis parallel to the axis x-x.

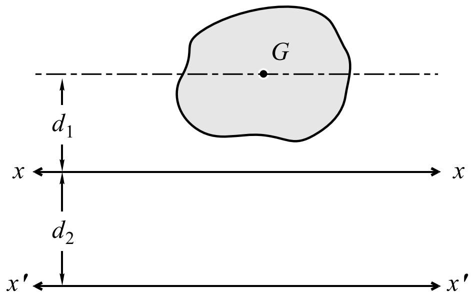


Fig. 6.E16

[**Ans.** d_1 = 14.92 cm and I_G = 7174.2 cm^4.]

[II] Review Questions

1. Define the term Moment of Inertia.
2. State and prove Perpendicular Axis Theorem.
3. State and prove Parallel Axis Theorem.
4. Define the term Radius of Gyration.
5. Describe the method of finding moment of inertia of composite areas.
6. Derive an equation for moment of inertia of the following areas :

   (a) About centroidal axis (d) Circle
   (b) Rectangle (e) Semicircle
   (c) Triangle (f) Quarter circle

[III] Fill in the Blanks

1. Perpendicular axis theorem is also called the _________.
2. Moment of inertia of semicircular area about diametrical line is given by a relation ________.
3. If unit of area is cm^2 and the distance of centroid of area is cm then the unit of moment of inertia will be _________.
4. If a figure does not have any axis of symmetry then one has to find _______ before finding moment of inertia about centroidal axis.
5. Area moment of inertia is also mathematically called the _______ moment of area.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. Concept of moment of inertia is based on distribution of _________.

(a) only mass **(b)** only area **(c)** both mass and area **(d)** None of these

2. Perpendicular axis theorem is used while deriving the moment of inertia of _________.

(a) rectangle **(b)** triangle

(c) circle **(d)** square

3. The relation used by parallel axis theorem is _______.

(a) $I = I_G + Ar^2$ **(b)** $I = I_{xx} + I_{yy}$ **(c)** $I_{zz} = I_{xx} + I_{yy}$ **(d)** $I = I_G + Ar$

4. The relation used by radius of gyration is ________.

(a) $I = Ak$ **(b)** $I = A^2k$ **(c)** $I = A^2k^2$ **(d)** $I = Ak^2$

5. Moment of inertia of triangular area about its base is given by a relation ________.

(a) $I = \frac{bh}{12}$ **(b)** $I = \frac{bh^2}{12}$ **(c)** $I = \frac{bh^3}{12}$ **(d)** $I = \frac{bh^3}{36}$

6. If the axis of symmetry for given shaded area is inclined at 45^o to horizontal then ______ .

(a) $I_{xx} = 0$ **(b)** $I_{yy} = 0$ **(c)** $I_{zz} = 0$ **(d)** $I_{xx} = I_{yy}$

7. If figure is symmetric about *x-x* axis and *y-y* axis then for finding moment of inertia about centroidal axis we require to find ______.

(a) $\overline{x}$ **(b)** $\overline{y}$ **(c)** $\overline{x}$ and $\overline{y}$ **(d)** None of these

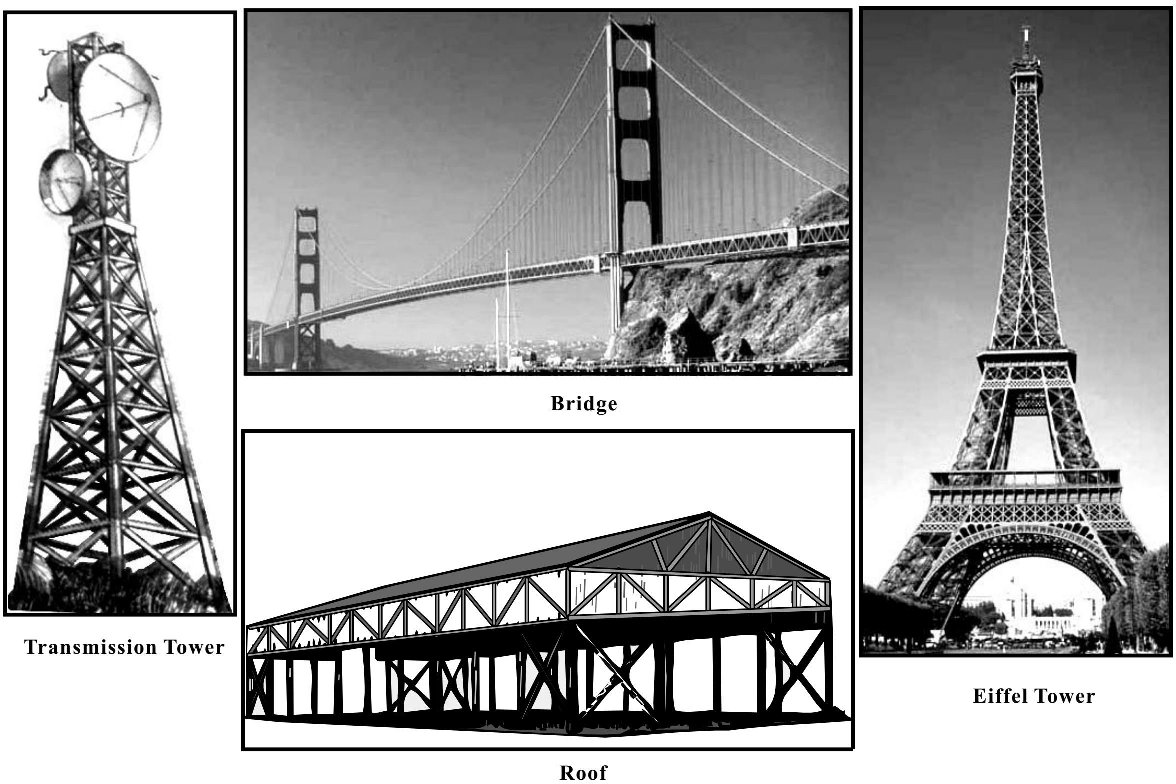
7

TRUSSES

7.1 Introduction

A ***truss*** *is a structure that is made of straight slender bars that are joined together at their ends by frictionless pins to form a pattern of triangle*. The loads act only at joints and not on the members. Thus, every member of truss is identified as two-force member.

Applications : Trusses are usually designed to transmit forces over relatively long spans, common examples being bridge trusses, roof trusses, transmission towers, etc.



Transmission Tower

Bridge

Eiffel Tower

Roof

Fig. 7.1-i : Applications of Truss

7.2 Types of Trusses

Some common types of trusses are shown in Fig. 7.2-i.

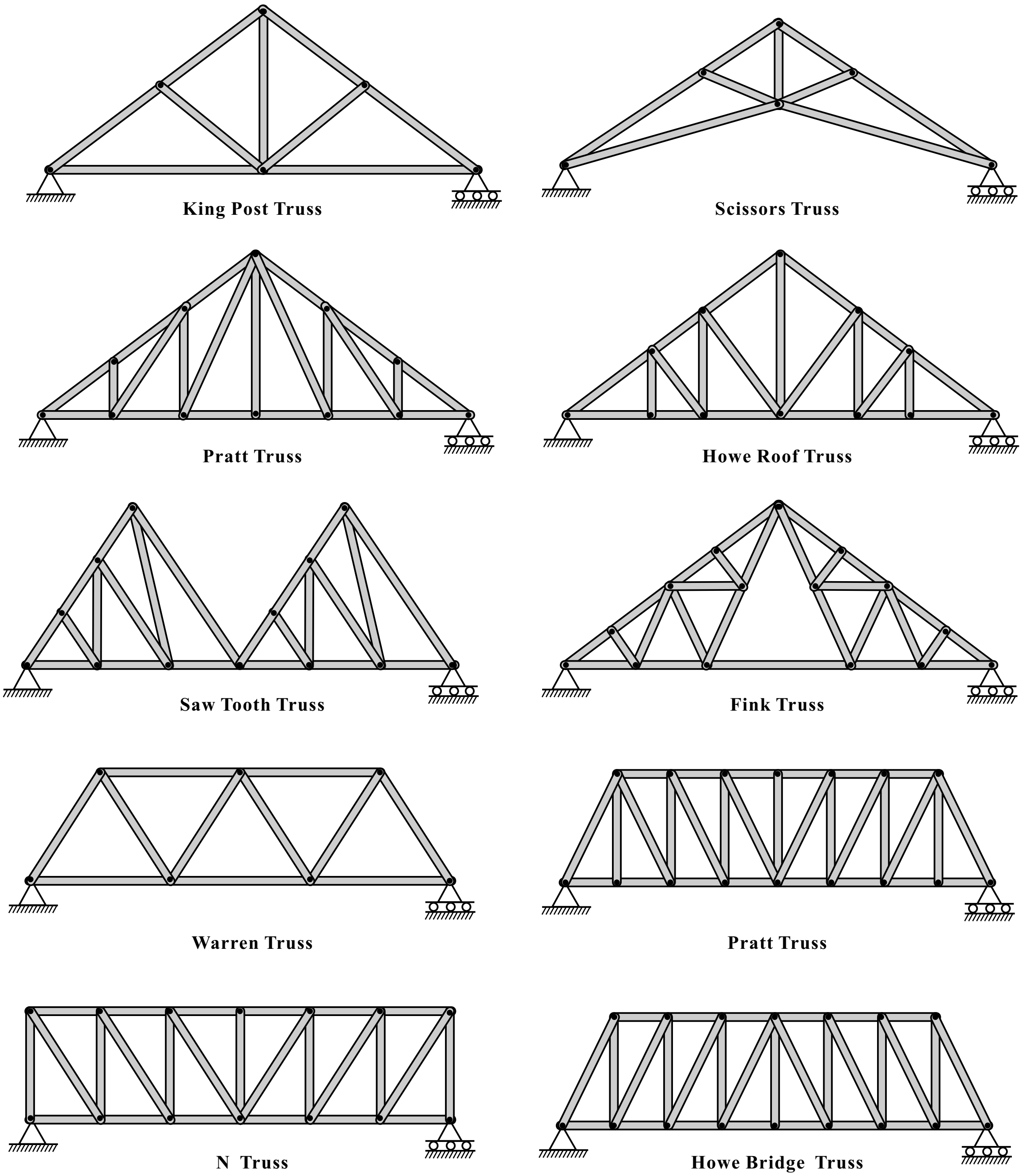


Fig. 7.2-i : Common Types of Trusses

Types of Trusses

- Perfect Truss
- Imperfect Truss
  - Deficient Truss
  - Redundant Truss

7.2.1 Perfect Truss

A pin jointed truss which has got just sufficient number of members to resist the load without undergoing any deformation in shape is called a ***perfect truss***.

Triangular frame is the simplest perfect truss and has three joints and three members.

It may be observed that to increase one joint in a perfect truss, two more members are required. Hence, the following expression may be written down as the relationship between the number of members *m,* number of joints *j* and number of support reaction components *r.*

$$m = 2j - r \quad \text{...(7.1)}$$

Example 1

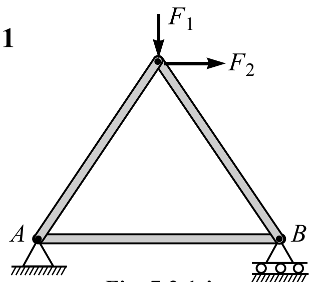


Fig. 7.2.1-i

Here, members $m = 3$, joints $j = 3$ and reactions $r = 3$ (i.e., H_A, V_A and R_B).

$$m = 2j - r$$

$$3 = 2 \times 3 - 3 \Rightarrow 3 = 3$$

Example 2

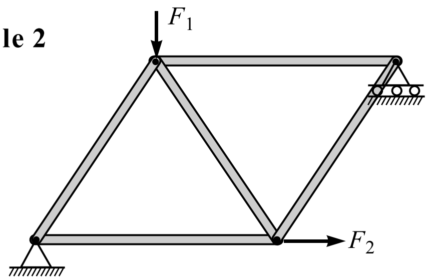


Fig. 7.2.1-ii

Here, members $m = 5$, joints $j = 4$ and reactions $r = 3$.

$$m = 2j - r$$

$$5 = 2 \times 4 - 3 \Rightarrow 5 = 5$$

Example 3

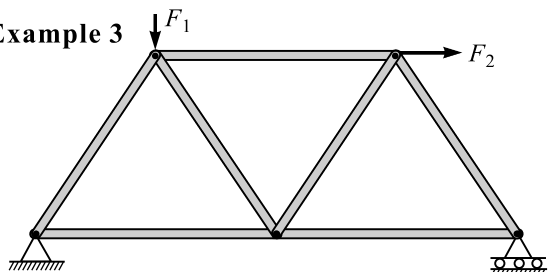


Fig. 7.2.1-iii

Here, members $m = 7$, joints $j = 5$ and reactions $r = 3$.

$$m = 2j - r$$

$$7 = 2 \times 5 - 3 \Rightarrow 7 = 7$$

Example 4

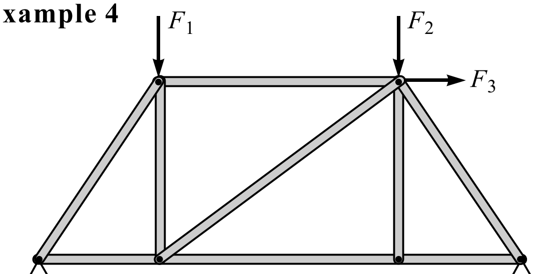


Fig. 7.2.1-iv

Here, members $m = 9$, joints $j = 6$ and reactions $r = 3$.

$$m = 2j - r$$

$$9 = 2 \times 6 - 3 \Rightarrow 9 = 9$$

A truss which satisfies the relation $m = 2j - r$ is called a *perfect truss*. Figures 7.2.1(i - iv) and iv are examples of perfect truss. Perfect truss can be completely analysed by static equilibrium condition. Therefore, it is also called a *determinate structure*.

7.2.2 Imperfect Truss

A truss which does not satisfies the relation $m = 2j - r$ is called an ***imperfect truss***.
Following are two subimperfect trusses.

1. **Imperfect Deficient Truss :** A truss which satisfies the relation $m < 2j - r$ is called a *deficient truss*. It is unstable and may collapse under external forces.

Example

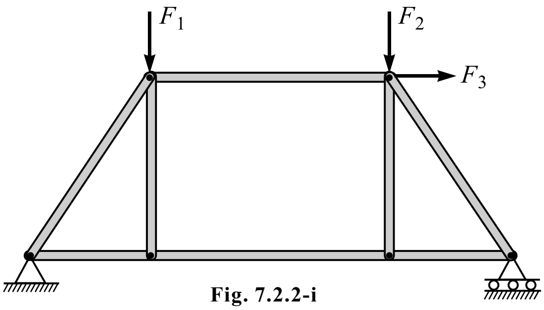


Fig. 7.2.2-i

Here, members $m = 8$, joints $j = 6$ and reactions $r = 3$.

$$m = 2j - r$$
$$8 = 2 \times 6 - 3$$
$$8 \neq 9$$
$$8 < 9$$

i.e., $m < 2j - r \Rightarrow$ deficient truss

2. **Imperfect Redundant Truss :** A truss which satisfies the relation $m > 2j - r$ is called a *redundant truss*. It is over rigid truss. It cannot be completely analysed by static equilibrium condition. Therefore, it is an indeterminate structure.

Example

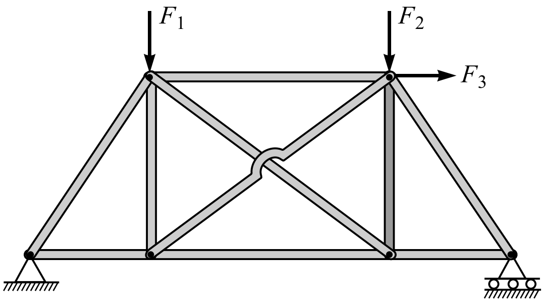


Fig. 7.2.2-ii

Here, members $m = 10$, joints $j = 6$ and reactions $r = 3$.

$$m = 2j - r$$
$$10 = 2 \times 6 - 3$$
$$10 \neq 9$$
$$10 > 9$$

i.e., $m > 2j - r \Rightarrow$ redundant truss

Assumptions for a Perfect Truss

1. All the members of truss are straight and connected to each other at their ends by frictionless pins.
2. All loading (external forces) on truss are acting only at pins.
3. All the members are assumed to be weightless.
4. All the members of truss and external forces acting at pins lies in same plane.
5. Static equilibrium condition is applicable for analysis of perfect truss (i.e., $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$).

7.3 Two-Force Member Concept

The previous assumption that all the members of a truss are straight, connected to each other at their ends by frictionless pins and no external force is acting in between their joint, identifies each truss member as a two-force member which may be in tension or compression.

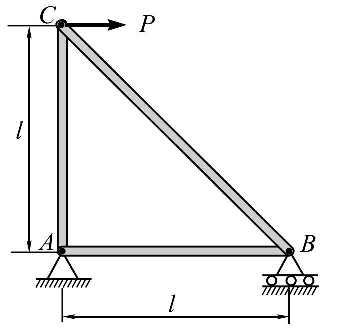


Fig. 7.3-i(a)

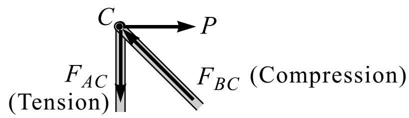


Fig. 7.3-i(b) : FBD of Pin *C*

Consider a simply supported truss as shown in Fig. 7.3-i(a) while Fig. 7.3-i(b) shows the FBD of pin *C*.

Tensile force is represented by an arrow drawn away from the pin.

Compressive force is represented by an arrow drawn towards the pin.

The two common techniques for computing the internal forces in a truss are the *method of joints* and the *method of sections*, each of which is discussed in the following sections.

7.4 Method of Joints

Procedure for Method of Joints

1. For simply supported truss, consider the FBD of the entire truss. Applying condition of equilibrium (i.e., $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$) find support reactions.

2. Consider the FBD of joint (pin) from the truss at which not more than two members with unknown force exists.

3. Assume the member to be in tension or compression by simple inspection and applying condition of equilibrium (i.e., $\Sigma F_x = 0$ and $\Sigma F_y = 0$) to find the answers.

4. The assumed sense can be verified from the obtained numerical results. A positive answer indicate that the sense is correct, whereas a negative answer indicates that the sense shown on the FBD must be changed.

5. Select the new FBD of joint with not more than two unknowns in a member and repeat points 3, 4 and 5 for complete analysis.

6. Tabulate the answer representing the member, magnitude of force and their nature.

Analysis in Method of Joints

While using the method of joints to calculate the forces in the member of truss, the equilibrium equations are applied to individual joints (or pins) of the truss. Because the members are two-force members, the FBD of a joint forms concurrent force system.

Consequently, two independent equilibrium equations (i.e., $\Sigma F_x = 0$ and $\Sigma F_y = 0$) are available for each joint.

To illustrate this method of analysis, consider the truss shown in Fig. 7.4-i(a).

Observe the FBD of pin C shown in Fig. 7.4-i(b) having three forces, one known and two unknown. We have two equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$, so force in member F_{BC} and force in member F_{AC} can be calculated.

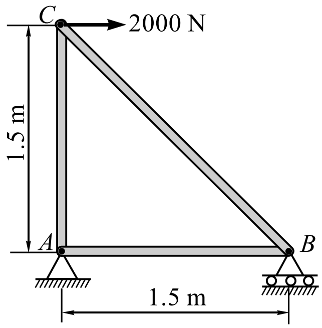


Fig. 7.4-i(a)

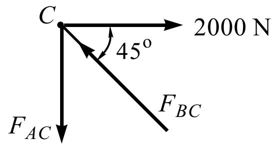


Fig. 7.4-i(b) : FBD of Pin *C*

$$\Sigma F_x = 0$$

$$2000 - F_{BC} \cos 45^{\circ} = 0$$

$$F_{BC} = 2828.43 \text{ N (Compression)}$$

$$\Sigma F_y = 0$$

$$F_{BC} \sin 45^{\circ} - F_{AC} = 0$$

$$F_{AC} = 2000 \text{ N (Tension)}$$

Note : Observe the FBD of joint C, 2000 N horizontal external force is acting towards right, by simple inspection we can assume F_{BC} in compression because its horizontal component will act towards left to balance 2000 N. Now try to inspect F_{AC}, it should be in tension because vertical component of F_{BC} is upward so to balance it F_{AC} should act in vertical downward direction.

Special Conditions

1. **Identification of zero force member simply by inspection (Without calculation) :** If any joint is identified without external force (load) acting on it such that joint is formed by three members and two of them are collinear, then the third non-collinear member should be identified as zero force member.

Example 1

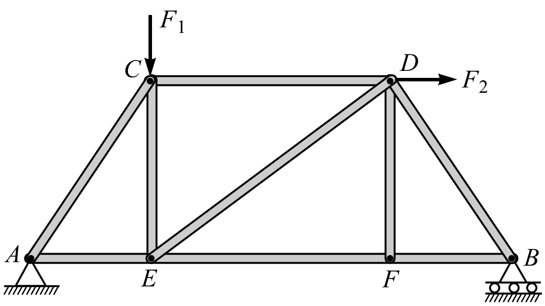


Fig. 7.4-ii(a)

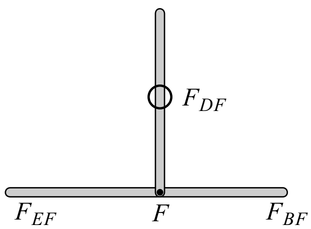


Fig. 7.4-ii(b) : Joint *F*

Observe the joint F shown in Fig. 7.4-ii(b). Force in member EF and BF are going to balance each other (irrespective of tension or compression) because they are collinear. The third non-collinear member DF is identified as zero-force member.

i.e., $F_{EF} = F_{BF}$ because they are collinear and

$$F_{DF} = 0$$

Example 2

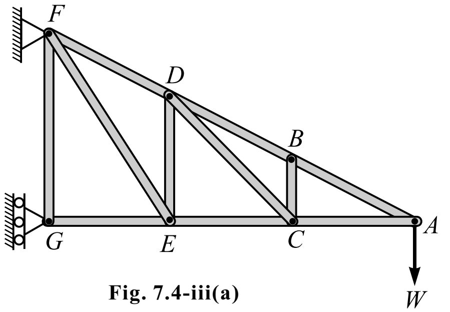


Fig. 7.4-iii(a)

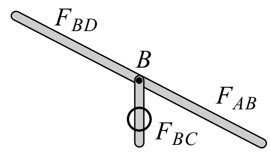


Fig. 7.4-iii(b) : Joint *B*

$F_{BD} = F_{AC}$ because they are collinear and

$F_{BC} = 0$

If any joint is formed by three members such that two are collinear and no external force is acting at that joint, then the third non-collinear member is identified as a zero-force member without any calculation. Now considering $F_{BC} = 0$ observe the joint *C*, member *CE* and *AC* are collinear therefore member *CD* should also be identified as a zero-force member $F_{CD} = 0$. On similar condition we have $F_{DE} = F_{EF} = 0$.

2. If any joint is formed by two non-collinear members without any external force acting on it then both the members are identified as zero-force members.

Example 1

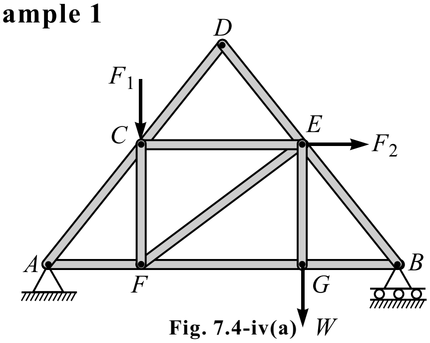
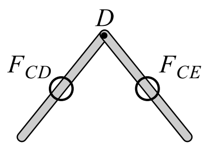


Fig. 7.4-iv(a)



$F_{CD} = F_{DE} = 0$

Fig. 7.4-iv(b) : Joint *D*

Consider joint *D* we have, $F_{CD} = F_{DE} = 0$

Example 2

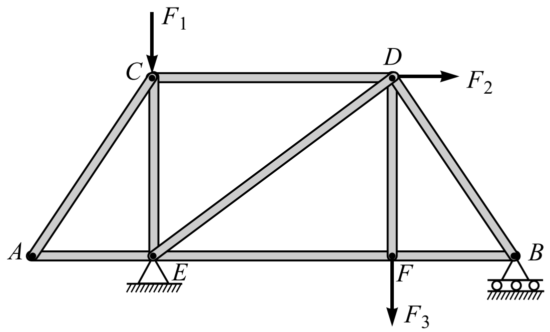


Fig. 7.4-v(a)

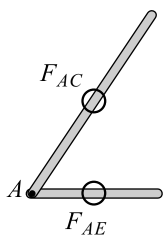


Fig. 7.4-v(b) : Joint *A*

Observe joint *A*, $F_{AC} = F_{AE} = 0$

3. If any joint is formed such that only four forces are acting and are collinear in pairs then each collinear forces are equal.

Example

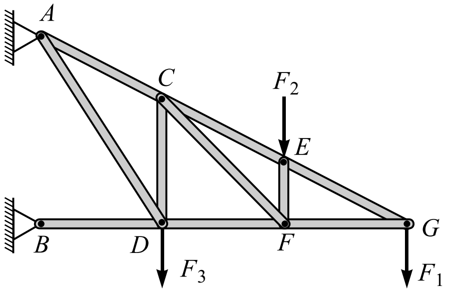


Fig. 7.4-vi(a)

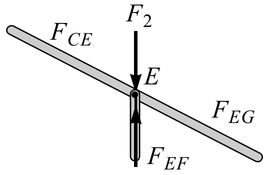


Fig. 7.4-vi(b) : Joint *E*

Observe joint E, $F_2 = F_{EF}$ and $F_{CE} = F_{EG}$

4. If a given truss is symmetrical in geometry as well as in loading and support then support reactions are symmetrical and forces in members on half side of symmetric are equal to the forces in members on the other half.

Example

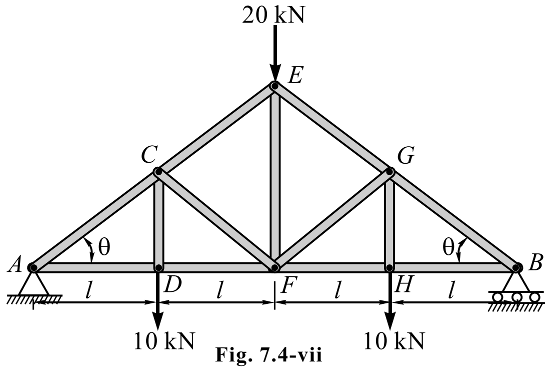


Fig. 7.4-vii

Observe the truss in Fig. 7.4-vii.

Vertical load is 10 + 20 + 10 = 40 kN.

$\therefore\ R_A = R_B = 20$ kN ($\uparrow$)

$F_{AC} = F_{BG}$, $F_{AD} = F_{BH}$, $F_{CD} = F_{GH}$,

$F_{CE} = F_{EG}$, $F_{DF} = F_{HF}$ and $F_{CF} = F_{GF}$.

5. **Cantilever Truss :** In cantilever truss, support reactions are not required for the analysis of truss. Analysis can be started from extreme end of the truss.

Example

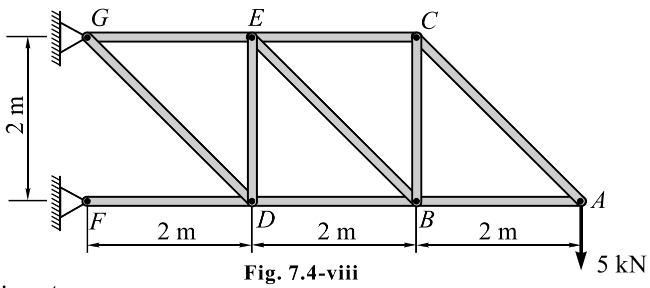
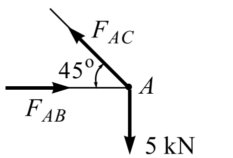
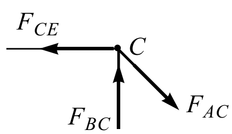


Fig. 7.4-viii

Consider the FBD of joint A



Consider the FBD of joint C



Consider the FBD of joint B. So on of all the joints.

7.4.1 Solved Problems on Method of Joints

Problem 1

Find the force and its nature in member AD and BC for given cantilever truss loaded by 40 kN as shown (AB = 4 m and AD = 5 m) in Fig. 7.1(a).

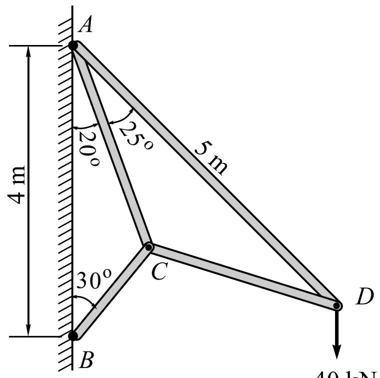


Fig. 7.1(a)

Solution

(i) For the given cantilever truss, we can start by considering the FBD of joint D. But before this we should workout the geometrical angles.

In $\Delta\ ABC$

By sine rule, we have

$$\frac{AB}{\sin 130^\circ} = \frac{AC}{\sin 30^\circ}$$

$$AC = 2.61 \text{ m}$$

In $\Delta\ ACD$

By cosine rule, we have

$$CD = \sqrt{(AC)^2 + (AD)^2 - 2(AC)(AD)\cos 25^\circ}$$

$$CD = \sqrt{(2.61)^2 + (5)^2 - 2(2.61)(5)\cos 25^\circ}$$

$$CD = 2.86 \text{ m}$$

In $\Delta\ ACD$, by sine rule, we have

$$\frac{CD}{\sin 25^\circ} = \frac{AC}{\sin \alpha}$$

$$\frac{2.86}{\sin 25^\circ} = \frac{2.61}{\sin \alpha} \qquad \therefore\ \alpha = 22.69^\circ$$

By geometry $\alpha + \beta = 45^\circ \qquad \therefore\ \beta = 22.31^\circ$

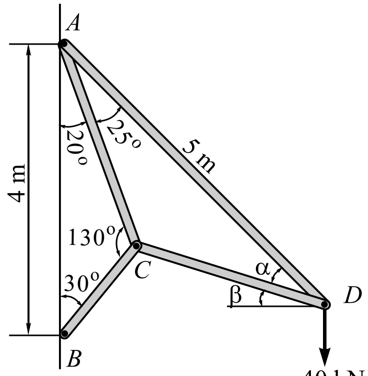


Fig. 7.1(b)

(ii) Consider the FBD of Joint D

By Lami's theorem

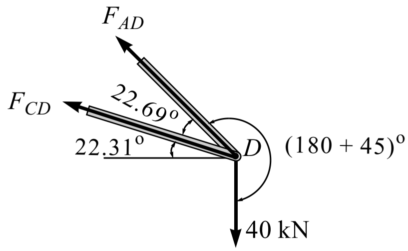
$$\frac{40}{\sin 22.69^\circ} = \frac{F_{AD}}{\sin (90 + 22.31)^\circ} = \frac{F_{CD}}{\sin (180 + 45)^\circ}$$

$\therefore\ F_{AD} = 95.93$ kN (T)

$F_{CD} = -73.32$ kN (Wrong assumed direction)

$\therefore\ F_{CD} = 73.32$ kN (C)

$\Sigma F_y = 0$



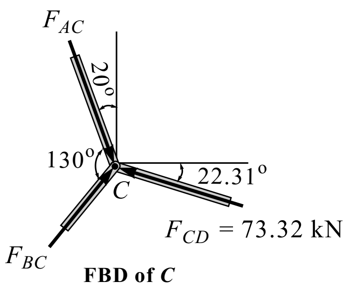
FBD of D

(iii) Consider the FBD of Joint C

By Lami's theorem

$$\frac{F_{CD}}{\sin 130^\circ} = \frac{F_{BC}}{\sin (20 + 90 + 22.31)^\circ}$$

$\therefore\ F_{BC} = 70.78$ kN (C)



FBD of C

Problem 2

Find the forces in the members *DF*, *DE*, *CE* and *EF* by method of joints only for the pin-jointed frame shown in Fig. 7.2.

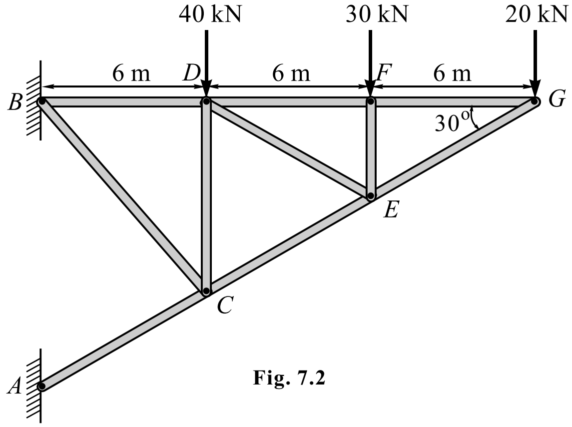


Fig. 7.2

Solution

(i) Consider the FBD of Joint *G*

$$\Sigma F_y = 0$$

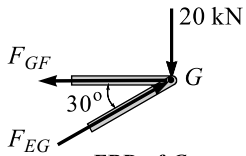
$$-20 + F_{EG} \sin 30^\circ = 0$$

$$\therefore F_{EG} = 40 \text{ kN (C)}$$

$$\Sigma F_x = 0$$

$$-F_{GF} + F_{EG} \cos 30^\circ = 0$$

$$\therefore F_{GF} = 34.64 \text{ kN (T)}$$



FBD of *G*

(ii) Consider the FBD of Joint *F*

$$\Sigma F_x = 0$$

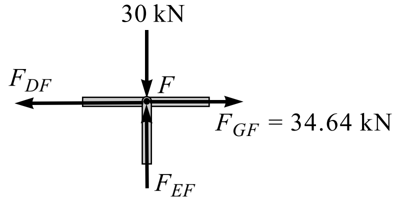
$$F_{GF} - F_{DF} = 0$$

$$\therefore F_{DF} = 34.64 \text{ kN (T)}$$

$$\Sigma F_y = 0$$

$$-30 + F_{EF} = 0$$

$$\therefore F_{EF} = 30 \text{ kN (C)}$$



FBD of *F*

(iii) Consider the FBD of Joint *E*

$$\Sigma F_y = 0$$

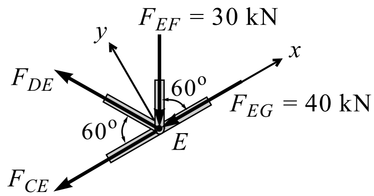
$$F_{DE} \sin 60^\circ - 30 \sin 60^\circ = 0$$

$$\therefore F_{DE} = 30 \text{ kN (T)}$$

$$\Sigma F_x = 0$$

$$F_{CE} + F_{DE} \cos 60^\circ + 30 \cos 60^\circ + 40 = 0$$

$$\therefore F_{CE} = 70 \text{ kN (C)}$$



FBD of *E*

Problem 3

Refer to Fig. 7.3(a) and determine the

(i) support reactions and

(ii) forces in members *AE*, *BC* and *EC*.

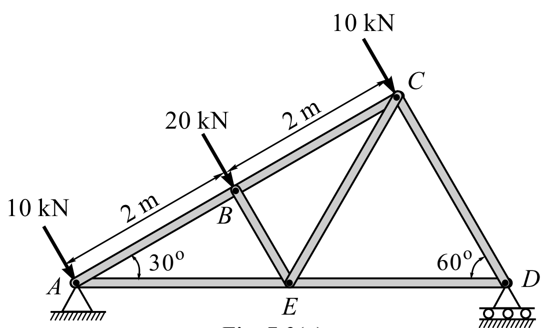


Fig. 7.3(a)

Solution

(i) In $\Delta\, ADC$

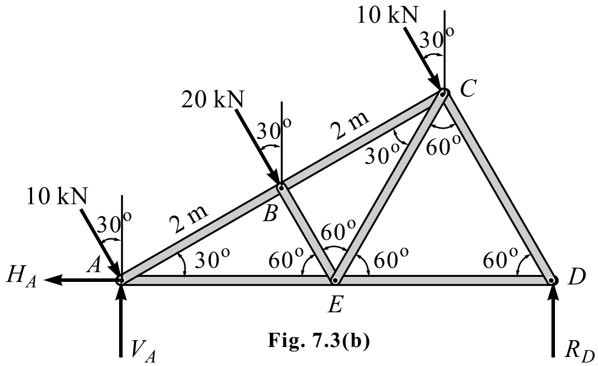


Fig. 7.3(b)

$$\sin 60^{\circ} = \frac{AC}{AD}$$

$$AD = \frac{4}{\sin 60^{\circ}}$$

$$AD = 4.62 \text{ m}$$

$$\Sigma M_A = 0$$

$$-20 \times 2 - 10 \times 4 + R_D \times 4.62 = 0$$

$$R_D = 17.32 \text{ kN } (\uparrow)$$

$$\Sigma F_y = 0$$

$$-10 \cos 30^{\circ} - 20 \cos 30^{\circ} - 10 \cos 30^{\circ} + V_A + R_D = 0$$

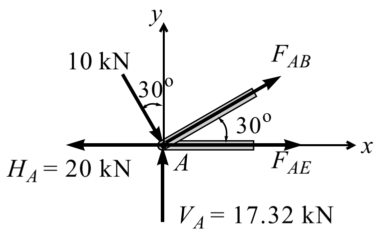
$$V_A = 17.32 \text{ kN } (\uparrow)$$

$$\Sigma F_x = 0$$

$$-H_A + 10 \sin 30^{\circ} + 20 \sin 30^{\circ} + 10 \sin 30^{\circ} = 0$$

$$\therefore H_A = 20 \text{ kN } (\leftarrow)$$

(ii) Consider the FBD of Joint *A*



FBD of *A*

$$\Sigma F_y = 0$$

$$-10 \cos 30^{\circ} + 17.32 + F_{AB} \sin 30^{\circ} = 0$$

$$F_{AB} = -17.32 \text{ kN (Wrong assumed direction)}$$

$$\therefore F_{AB} = 17.32 \text{ kN (C)}$$

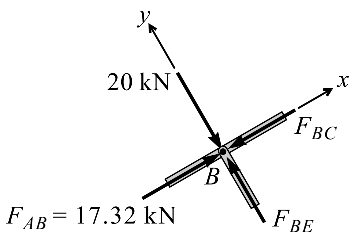
$$\Sigma F_x = 0$$

$$-20 + 10 \sin 30^{\circ} + F_{AB} \cos 30^{\circ} + F_{AE} = 0$$

$$F_{AE} = 30 \text{ kN (T)}$$

(iii) Consider the FBD of Joint *B*

For convenience, consider the *x* and *y* axis as shown in



FBD of *B*

$$\Sigma F_x = 0$$

$$F_{BC} - F_{AB} = 0$$

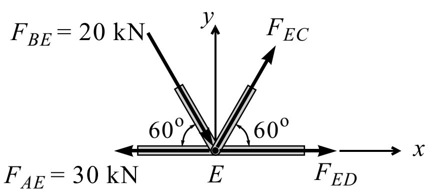
$$F_{BC} = 17.32 \text{ kN (C)}$$

$$\Sigma F_y = 0$$

$$-20 + F_{BE} = 0$$

$$F_{BE} = 20 \text{ kN (C)}$$

(iv) Consider the FBD of Joint *E*



FBD of *E*

$$\Sigma F_y = 0$$

$$-20 \sin 60^{\circ} + F_{EC} \sin 60^{\circ} = 0$$

$$F_{EC} = 20 \text{ kN (T)}$$

Problem 4

Find out the forces in the member of the truss, loaded and supported, as shown in Fig. 7.4(a). You are free to use any method for the analysis of this truss. State the nature of the force in each member.

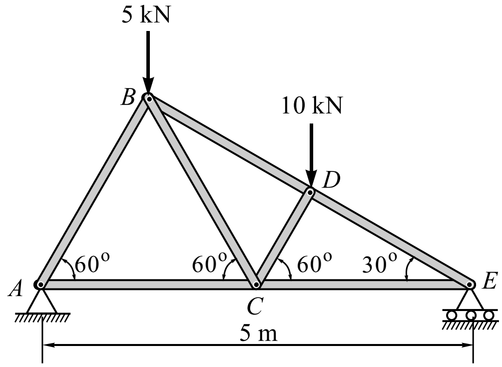


Fig. 7.4(a)

Solution

For finding forces in all members, it is preferable to use joint method. First let us find support reaction and some geometrical distance for simplicity.

(i) Consider the FBD of the entire truss

In $\Delta\, ABE$

$$\sin 30^\circ = \frac{AB}{AE}$$

$$AB = 2.5 \text{ m}$$

$$AB = AC = CE = 2.5 \text{ m}$$

Drop perpendicular line from B and D and mark P and Q respectively as shown in Fig. 7.4(b).

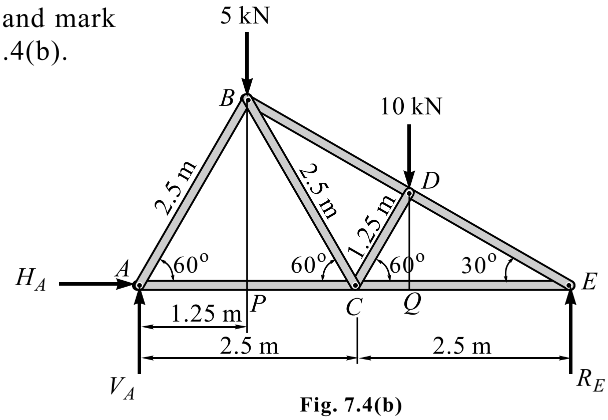


Fig. 7.4(b)

In $\Delta\, ABP$

$$\cos 60^\circ = \frac{AP}{AB}$$

$$\therefore\ AP = 1.25 \text{ m}$$

In $\Delta\, CDE$

$$\sin 30^\circ = \frac{CD}{CE} \qquad \therefore\ CD = 1.25 \text{ m}$$

In $\Delta\, CQD$

$$\cos 60^\circ = \frac{CQ}{CD} \qquad \therefore\ CQ = 0.625 \text{ m}$$

$$\Sigma M_A = 0$$

$$R_E \times 5 - 5 \times 1.25 - 10 \times (2.5 + 0.625) = 0$$

$$R_E = 7.5 \text{ kN } (\uparrow)$$

$$\Sigma F_x = 0$$

$$H_A = 0$$

$$\Sigma F_y = 0$$

$$V_A + R_E - 5 - 10 = 0$$

$$V_A = 7.5 \text{ kN } (\uparrow)$$

(ii) Consider the FBD of Joint *A*

$\Sigma F_y = 0$

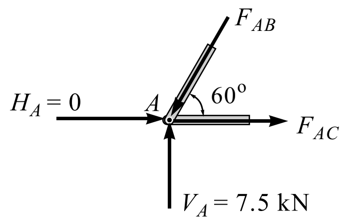
$V_A - F_{AB} \sin 60^o = 0$

$F_{AB} = 8.66$ kN (C)

$\Sigma F_x = 0$

$F_{AC} - F_{AB} \cos 60^o = 0$

$F_{AC} = 4.33$ kN (T)



FBD of *A*

(iii) Consider the FBD of Joint *E*

$\Sigma F_y = 0$

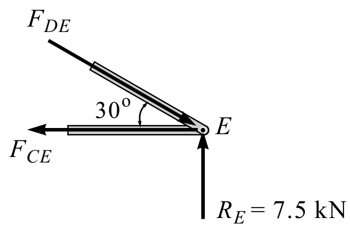
$R_E - F_{DE} \sin 30^o = 0$

$F_{DE} = 15$ kN (C)

$\Sigma F_x = 0$

$F_{DE} \cos 30^o - F_{CE} = 0$

$F_{CE} = 12.99$ kN (T)



FBD of *E*

(iv) Consider the FBD of Joint *D*

Note : Orientation of axes are changed for easy solution.

$\Sigma F_y = 0$

$F_{CD} - 10 \sin 60^o = 0$

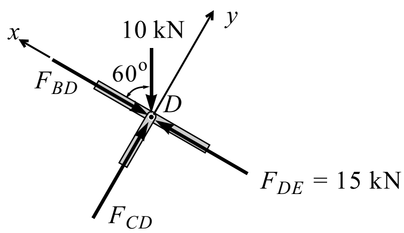
$F_{CD} = 8.66$ kN (C)

$\Sigma F_x = 0$

$F_{DE} - F_{BD} - 10 \cos 60^o = 0$

$F_{BD} = 15 - 10 \cos 60^o$

$F_{BD} = 10$ kN (C)



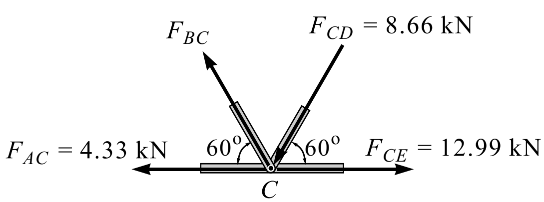
FBD of *D*

(v) Consider the FBD of Joint *C*

$\Sigma F_y = 0$

$F_{BC} \sin 60^o - F_{CD} \sin 60^o = 0$

$F_{BC} = 8.66$ kN (T)



FBD of *C*

The results are tabulated as follows :

Member	*AB*	*AC*	*CE*	*DE*	*BD*	*CD*	*BC*
Force (kN)	8.66	4.33	12.99	15	10	8.66	8.66
Type (T/C)	C	T	T	C	C	C	T

Ans.

7.5 Method of Sections (Method of Moments)

Procedure for Method of Joints

1. Consider the FBD of entire truss and find the support reactions applying equilibrium conditions.

2. Select the cutting section to cut the truss into two parts such that it should not cut more than three unknown members.

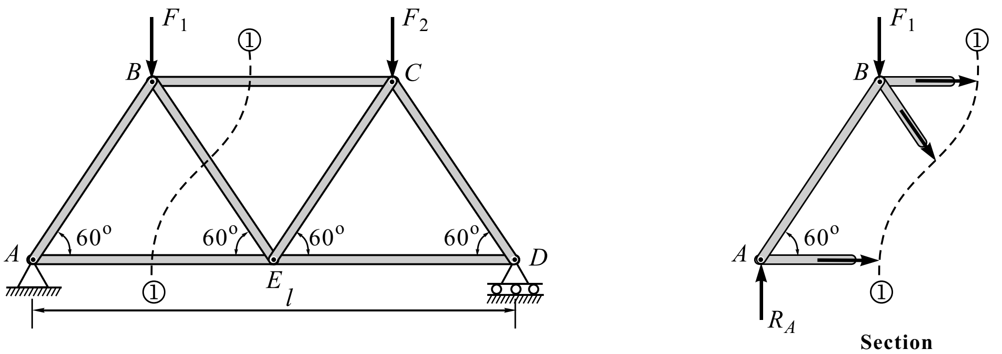
3. Select the FBD of any one of the two parts considering all active and reactive force acting on that part.

4. Assume tension or compression in the cut members and by applying equilibrium condition its numerical values can be obtained. If the obtained value is negative, do the required change in nature of force (T/C).

5. Though three equations of equilibrium are available (i.e., $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$) preferably use $\Sigma M = 0$ by selecting appropriate point for moment such that two known passes through that point. Moment of centre may or may not lie on the FBD of truss.

6. Do not consider the effect of uncut member in FBD.

Example



Special Case

In general, we should not cut more than three members because we have three equations of equilibrium to find three unknowns. But in exceptional cases we can do this, especially where many members are collinear or concurrent. We may overcome this condition of section and can cut more than three members. We should select an appropriate moment of centre through which line of action of all the cut members excluding one are passing and answer the required unknown.

Advantages of Section Method

In section method, we do not have to analyse the entire truss if any intermediate member force is desired to be obtained. It can directly be obtained by selecting proper position of section, so it is less time consuming as compared to joint method.

7.5.1 Solved Problems on Method of Sections

Problem 5

For the truss loaded as shown in Fig. 7.5(a), find the force in members *CE* and *CF* by method of sections only.

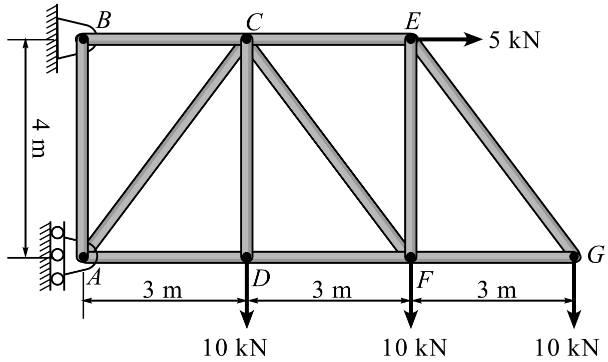


Fig. 7.5(a)

Solution

Consider the FBD of the right part of truss, section along S-S

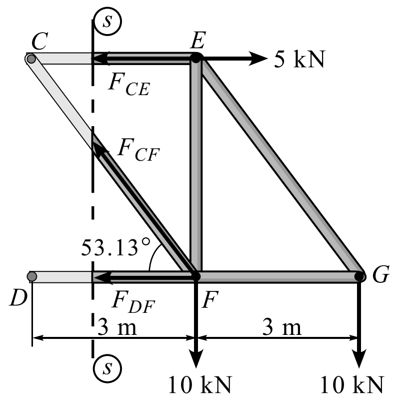


Fig. 7.5(b)

(i) $\Sigma F_y = 0$

$$F_{CF} \sin 53.13^o - 10 - 10 = 0$$

$\therefore$ $F_{CF} = 25$ kN (Tension)

(ii) $\Sigma M_F = 0$

$$F_{CE} \; 4 - 5 \times 4 - 10 \times 3 = 0$$

$\therefore$ $F_{CE} = 12.5$ kN (Tension)

Problem 6

For the pin jointed truss loaded as shown in Fig. 7.6(a). Find

(i) all the reactions at A and B and

(ii) forces in member EC, ED, DF by method of sections only.

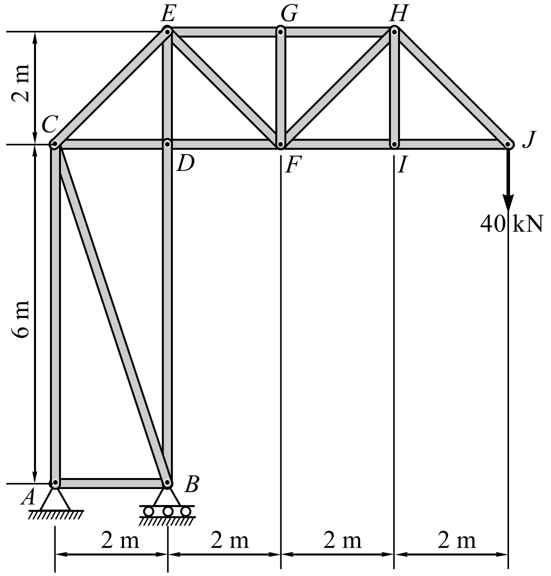


Fig. 7.6(a)

Solution

To find the support reactions.

(i) Consider the FBD of the entire truss

$$\Sigma M_A = 0$$

$$R_B \times 2 - 40 \times 8 = 0$$

$$R_B = 160 \text{ kN } (\uparrow)$$

$$\Sigma F_x = 0$$

$$H_A = 0$$

$$\Sigma F_y = 0$$

$$V_A + R_B - 40 = 0$$

$$V_A = 40 - 160$$

$$V_A = -120 \text{ kN (Wrong assumed direction)}$$

$$V_A = 120 \text{ kN } (\downarrow)$$

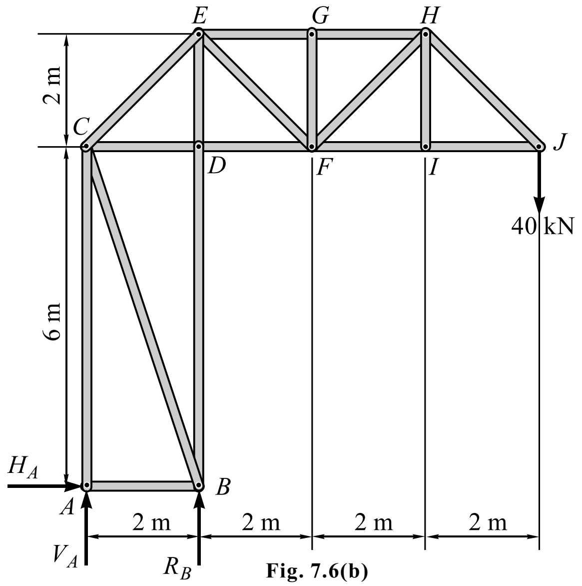


Fig. 7.6(b)

(ii) Consider the FBD of the right part of Section ① - ①

$$\Sigma M_C = 0$$

$$-F_{ED} \times 2 - 40 \times 8 = 0$$

$$F_{ED} = -160 \text{ kN (Wrong assumed direction)}$$

$$F_{ED} = 160 \text{ kN (C)}$$

$$\Sigma M_E = 0$$

$$-F_{DF} \times 2 - 40 \times 6 = 0$$

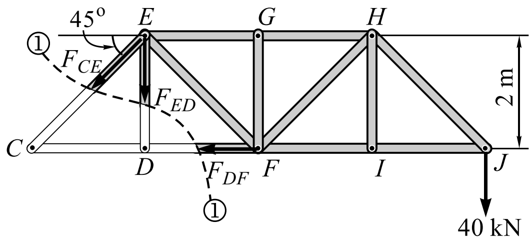
$$F_{DF} = -120 \text{ kN (Wrong assumed direction)}$$

$$F_{DF} = 120 \text{ kN (C)}$$

$$\Sigma M_D = 0$$

$$F_{CE} \cos 45^\circ \times 2 - 40 \times 6 = 0$$

$$F_{CE} = 169.71 \text{ kN (T)}$$



FBD of Section ① - ①

Exercises

[I] Problems

1. Determine the forces in the members of the truss as shown in Fig. 7.E1.

> ***Ans.***
> F_{AE} = 13.39 kN (T), F_{CE} = 12.928 kN (T),
> F_{DE} = 6.46 kN (T), F_{AB} = 14.78 kN (C),
> F_{BC} = 0.928 kN (C), F_{CD} = 12.9 kN (C) and
> F_{BE} = 12.928 kN (C).

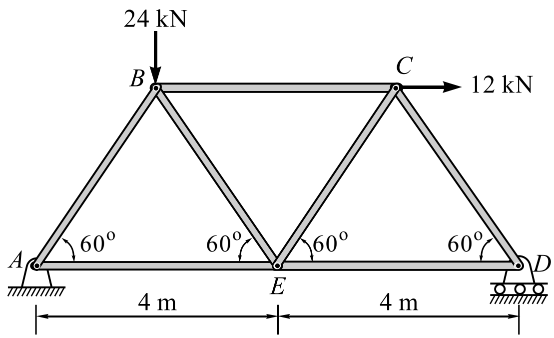


Fig. 7.E1

2. Determine the forces in the members of the truss as shown in Fig. 7.E2.

> ***Ans.***
> F_{AB} = 21.21 kN (C), F_{BC} = 0,
> F_{CD} = 18 kN (C), F_{BE} = 21.21 kN (C),
> F_{AF} = 25 kN (T), F_{DE} = 10 kN (T),
> F_{CE} = 15 kN (T), F_{EF} = 25 kN (T) and
> F_{CE} = 30 kN (T).

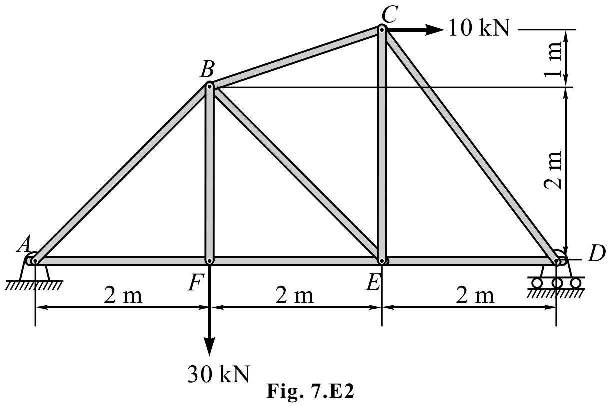


Fig. 7.E2

3. A simple plane truss is shown in Fig. 7.E3. Two 1000 N loads are shown acting on pins C and E. Determine the force in all the members using method of joints.

> ***Ans.***
> F_{AB} = 1414 N (C), F_{CE} = 1000 N (T),
> F_{AC} = 1000 N (T), F_{DE} = 1000 N (T),
> F_{BC} = 1000 N (T), F_{EF} = 1000 N (T),
> F_{BD} = 1000 N (C), F_{DF} = 1414 N (C) and
> F_{CD} = 0.

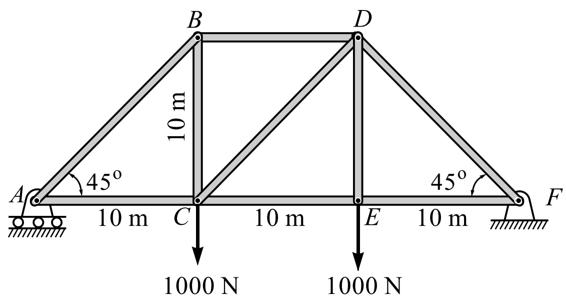


Fig. 7.E3

4. Find the forces in all the members of the truss shown in Fig. 7.E4.

Ans.
$V_D = V_F = 23$ kN (↑), $H_F = 0$,
$F_{AD} = F_{CF} = 7$ kN (C),
$F_{BD} = F_{BF} = 34$ kN (C),
$F_{DF} = F_{EF} = 30$ kN (T),
$F_{BE} = 8$ kN (T) and $F_{AB} = F_{BC} = 0$.

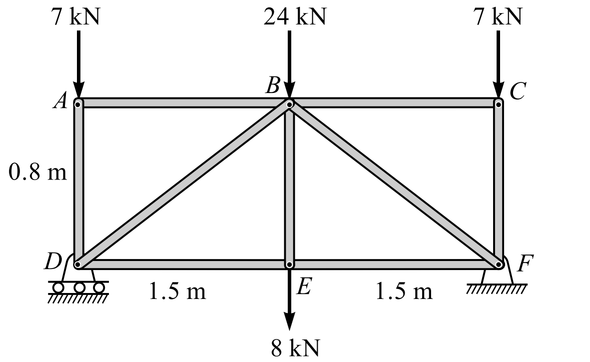


Fig. 7.E4

5. Find the forces in all the members of the truss shown in Fig. 7.E5.

Ans.
$F_{AD} = 15$ kN (T), $F_{CD} = 16$ kN (C),
$F_{BD} = 9$ kN (C), $F_{DE} = 4$ kN (C),
$F_{BE} = 5$ kN (T) and $F_{AB} = 4$ kN (T).

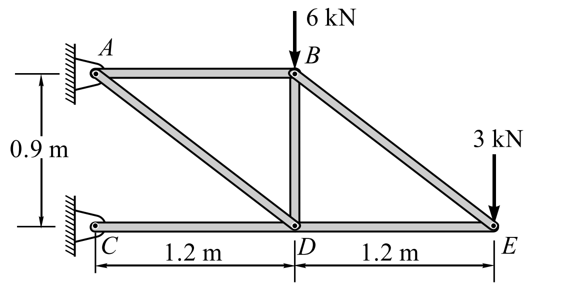


Fig. 7.E5

6. Find the forces in all the members of the truss shown in Fig. 7.E6.

Ans.
$F_{AD} = 44.72$ kN (T), $F_{DE} = 20$ kN (T),
$F_{CD} = 10$ kN (T), $F_{CE} = 22.36$ kN (C),
$F_{BC} = 20$ kN (C) and $F_{BD} = 22.36$ kN (C).

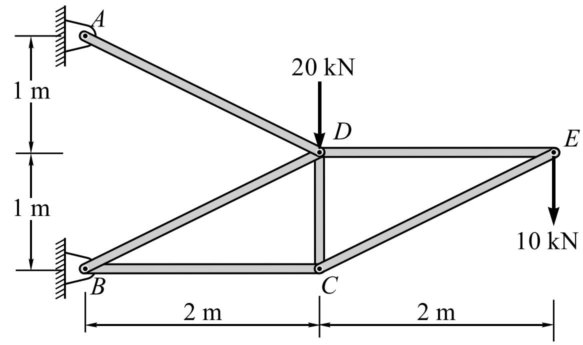


Fig. 7.E6

7. Find the forces in all the members of the truss shown in Fig. 7.E7.

Ans.
$F_{BD} = 10.51$ kN (T), $F_{DF} = 6$ kN (T),
$F_{FG} = 6$ kN (T), $F_{BC} = 4.21$ kN (T),
$F_{DE} = 4.75$ kN (C), $F_{GE} = 6.33$ kN (C),
$F_{CE} = 11.08$ kN (C), $F_{DC} = 3.5$ kN (C),
$F_{EF} = 3$ kN (C) and $F_{AC} = 14.77$ kN (C).

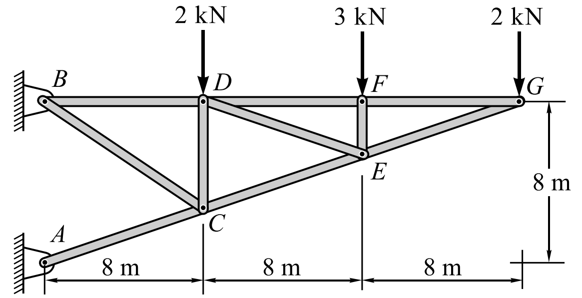


Fig. 7.E7

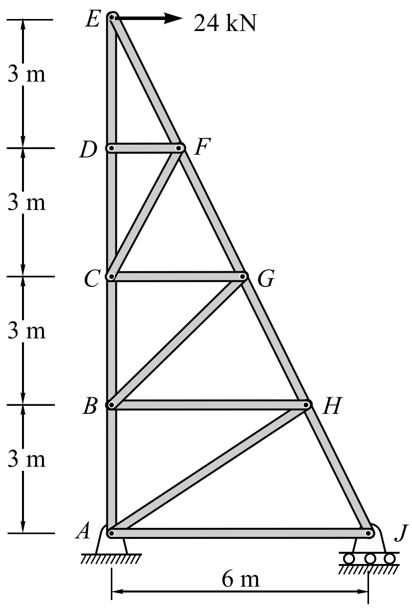


Fig. 7.E8

8. Find the forces in all the members of the truss shown in Fig. 7.E8.

Ans.
$F_{AB} = F_{BC} = F_{CD} = F_{DE} = 48$ kN (T),
$F_{EF} = F_{FG} = F_{GH} = F_{HJ} = 53.667$ kN (C),
$F_{AJ} = 24$ kN (T) and
$F_{DF} = F_{CF} = F_{CG} = F_{BG} = F_{BH} = F_{AH} = 0.$

[II] Review Questions

1. What are trusses ?

2. State the applications of trusses.

3. What are the assumptions of a perfect truss ?

4. Describe how trusses are analysed by the method of joints.

5. Describe how trusses are analysed by the method of sections.

6. How to identify a zero-force member ?

7. Distinguish between:

(a) Perfect truss and imperfect truss

(b) Deficient truss and redundant truss

(c) Simply supported truss and cantilever truss

[III] Fill in the Blanks

1. If we want to find forces in all member of truss then we would prefer the method of ________.

2. In the members of truss, the members are either in ________ or in ________.

3. A truss which satisfies the relation $m = 2j - r$ is called a _________ truss.

4. A perfect truss is considered as a _________ structure.

5. As per the assumption of perfect truss, all loadings (external forces) on truss act only at _______ .

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. All the members of truss are _________ force member.

   (a) zero **(b)** one **(c)** two **(d)** three

2. If truss satisfies $m > 2j - r$ ($m \Rightarrow$ number of members, $j \Rightarrow$ number of joints and $r \Rightarrow$ number of components of support reactions) then the truss is assumed to be _________ truss.

   (a) perfect **(b)** redundant **(c)** deficient **(d)** None of these

3. All the members are assumed to be _______.

   (a) weightless **(b)** straight **(c)** weightless and straight **(d)** None of these

4. Method of section is also called as ________.

   (a) Method of moment **(b)** Moment of inertia **(c)** Method of joint **(d)** None of these

5. Imperfect deficient truss holds the ________ relation.

   (a) $m = 2j - r$ **(b)** $m < 2j - r$ **(c)** $m > 2j - r$ **(d)** $m < j - 2r$

6. Over rigid truss is also called a ________ truss.

   (a) perfect **(b)** redundant **(c)** deficient **(d)** unstable

7. If a joint is formed by three members such that two are collinear and no external force is acting at joint then the third non collinear member is identified as a ______.

   (a) zero-force member **(b)** one-force member

   (c) two-force member **(d)** three-force member

8

FRICTION

8.1 Introduction

In most of the equilibrium problems that we have analysed up to this point, surfaces of contact have been assumed to be frictionless. The concept of a frictionless surface is, of course, an idealisation. All real surfaces have some roughness.

*When a body moves or tends to move over another body, a force opposing the motion develops at the contact surfaces. Whenever a tendency exists for one contacting surfaces to slide along another surface, tangential force is generated between contacting surface. This force which opposes the movement or tendency of movement is called a **frictional force** or simply **friction**.*

When two bodies in contact are in motion then the frictional force is opposite to the relative motions.

Cause of Friction : Friction is due to *the resistance offered to motion by minutely projecting portions at the contact surfaces*. These microscopic projections get interlocked. To a small extent, the material of two bodies in contact also produces resistance to motion due to intra-molecular force of attraction, i.e., adhesive properties.

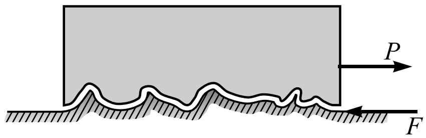


Fig. 8.1-i : Magnified Microscopic View of Rough Surface

8.2 Types of Friction

Dry Friction : Dry friction develops when *the unlubricated surface of two solids are in contact under a condition of sliding or a tendency to slide*. A frictional force tangent to the surfaces of contact is developed both during the interval leading up to impending slippage and while slippage takes place. The direction of the frictional force always opposes the relative motion or impending motion. This type of friction is also known as *Coulomb friction*.

Fluid Friction : Fluid friction is developed when *adjacent layers in a fluid (liquid or gas) are moving at different velocities.* This motion gives rise to frictional forces between fluid elements and these forces depend on the relative velocity between layers. Fluid frictions depends not only on the velocity gradients within the fluid but also on viscosity of fluid, which is a measure of its resistance to shearing action between fluid layers. Fluid friction is treated in the study of fluid mechanics and is beyond the scope of this text. So we are going to deal with dry friction only.

8.3 Mechanism of Friction

Consider a block of weight W resting on a horizontal surface as shown in Fig. 8.3-i. The contacting surface possesses a certain amount of roughness. Let P be the horizontal force applied which will vary continuously from zero to a value sufficient to just move the block and then to maintain the motion. The free body diagram of the block shows active forces (i.e., Applied force P and weight of block W) and reactive forces (i.e., normal reaction N and tangential frictional force F).

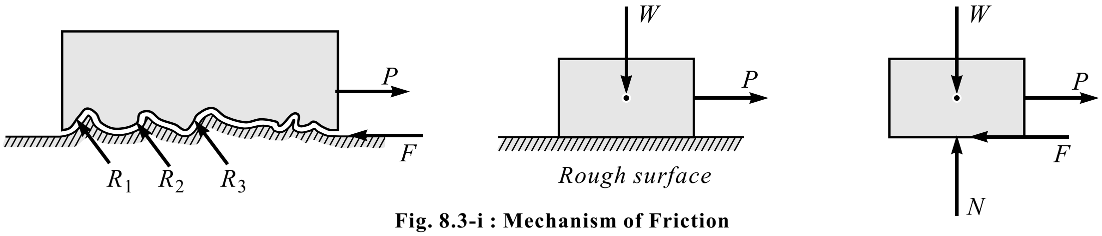


Fig. 8.3-i : Mechanism of Friction

Frictional force F has the remarkable property of adjusting itself in magnitude equal to the applied force P till the limiting equilibrium condition.

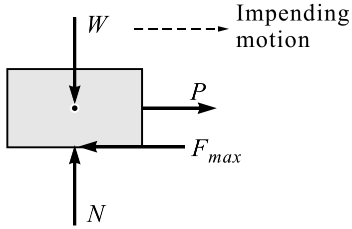
Limiting Equilibrium Condition : As applied force P increases, the frictional force F is equal in magnitude and opposite in direction. However, there is a limit beyond which the magnitude of the frictional force cannot be increased. If the applied force is more than this maximum frictional force, there will be movement of one body over the other. Once the body begins to move, there is decrease in frictional force F from maximum value observed under static condition. The frictional force between the two surface when the body is in motion is called *kinetic* or *dynamic friction* F_K.

Limiting Frictional Force (F_{max}) : It is the maximum frictional force developed at the surface when the block is at the verge of motion (impending motion).

Coefficient of Friction : By experimental evidence it is proved that limiting frictional force is directly proportional to normal reaction.

$$F_{max} \propto N$$

$$F_{max} = \mu_S N \quad \Rightarrow \quad \mu_S = \frac{F_{max}}{N} \qquad \text{...(8.1)}$$

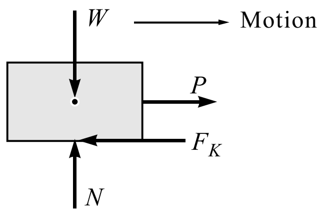


Coefficient of Static Friction : The ratio of limiting frictional force (F_{max}) and normal reaction (N) is a constant. This constant is called the *coefficient of static friction* (μ_S).

If a body is in motion, we have

$$F_K \propto N$$

$$F_K = \mu_K N \quad \Rightarrow \quad \mu_K = \frac{F_K}{N} \qquad \text{...(8.2)}$$



Coefficient of Kinetic Friction : The ratio of kinetic frictional force (F_K) and normal reaction (N) is a constant. This constant is known as *coefficient of kinetic friction* (μ_K).

Kinetic friction is always less than limiting friction.

8.4 Laws of Friction

1. The frictional force is always tangential to the contact surface and acts in a direction opposite to that in which the body tends to move.
2. The magnitude of frictional force is self-adjusting to the applied force till the limiting frictional force is reached and at the limiting frictional force the body will have impending motion.
3. Limiting frictional force F_{max} is directly proportional to normal reactions (i.e., $F_{max} = \mu_S N$).
4. For a body in motion, kinetic frictional force F_K developed is less than that of limiting frictional force F_{max} and the relation $F_K = \mu_K N$ is applicable.
5. Frictional force depends upon the roughness of the surface and the material in contact.
6. Frictional force is independent of the area of contact between the two surfaces.
7. Frictional force is independent of the speed of the body.
8. Coefficient of static friction μ_S is always greater than the coefficient of kinetic friction μ_K.

Angle of Friction : It is the angle made by the resultant of the limiting frictional force F_{max} and the normal reaction N with the normal reactions.

Consider the block with weight W and applied force P.

When the block is at the verge of motion, limiting frictional force F_{max} will act in opposite direction of applied force and normal reaction N will act perpendicular to surface, as shown in Fig. 8.4-i. We can replace the F_{max} and N by resultant reaction R which acts at an angle ϕ to the normal reaction.

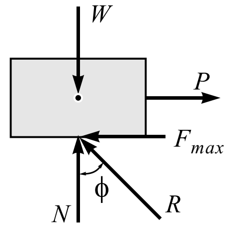


Fig. 8.4-i

This angle ϕ is called as the *angle of friction*.

From Fig. 8.4-ii, we have

$$F_{max} = R \sin \phi$$

$$\mu_S N = R \sin \phi \quad \text{... (I)} \quad (\because F_{max} = \mu_S N)$$

$$N = R \cos \phi \quad \text{... (II)}$$

Dividing Eq. (I) by Eq. (II), we get

$$\tan \phi = \mu_S \text{ or } \phi = \tan^{-1} \mu_S \quad \text{...(8.3)}$$

Angle of Repose : It is the minimum angle of inclination of a plane with the horizontal at which the body so kept will just begin to slide down on it without the application of any external force (due to self-weight).

Consider the block with weight W resting on an inclined plane, which makes an angle θ with horizontal as shown in Fig. 8.4-ii. When θ is small the block will rest on the plane. If θ is increased gradually a slope is reached at which the block is about to start sliding. This angle θ is called the *angle of repose*.

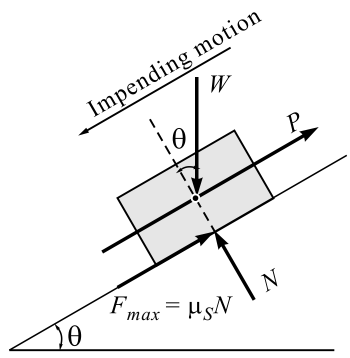


Fig. 8.4-ii

For limiting equilibrium condition, we have

$$\Sigma F_x = 0$$

$$\mu_S N - W \sin\theta = 0$$

$$W \sin\theta = \mu N \qquad \text{... (I)}$$

$$\Sigma F_y = 0$$

$$N - W \cos\theta = 0$$

$$W \cos\theta = N \qquad \text{... (II)}$$

Dividing Eq. (I) by Eq. (II), we get

$$\tan\theta = \mu_S \qquad \text{...(8.4)}$$

In previous discussion, we had $\tan\phi = \mu_S$, which shows

Angle of friction ϕ = Angle of repose θ

The above relation also shows that the angle of repose is independent of the weight of the body and it depends on μ.

Cone of Friction : When the applied force P is just sufficient to produce the impending motion of given body, angle of friction ϕ is obtained which is the angle made by resultant of limiting friction force and normal reaction with normal reaction as shown in Fig. 8.4-iii. If the direction of applied force P is gradually changed through 360^o, the resultant R generates a right circular cone with semi vertex angle equal to ϕ. This is called the *cone of friction*.

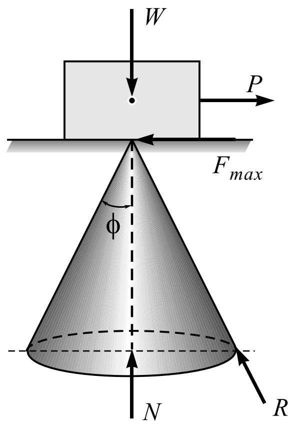


Fig. 8.4-iii

8.5 Types of Friction Problems

The above discussion can be represented by a graph with applied force P v/s frictional force F as shown in Fig. 8.5-i.

Referring to the graph we recognize three distinct types of friction problems. Here, we have static friction, limiting friction and kinetic friction.

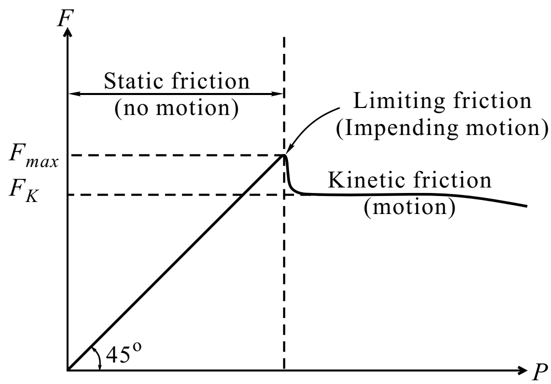


Fig. 8.5-i

1. **Static Friction :** If there is neither the condition of impending motion nor that of motion then to determine the actual force, we first assume static equilibrium and take F as a frictional force required to maintain the equilibrium condition.

Here, we have three possibilities

(i) $F < F_{max}$ $\Rightarrow$ Body is in static equilibrium condition which means body is purely at rest.

(ii) $F = F_{max}$ $\Rightarrow$ Body is in limiting equilibrium condition which means impending motion and hence $F = F_{max} = \mu_S N$ is valid equation.

(iii) $F > F_{max}$ $\Rightarrow$ Body is in motion which means $F = F_K = \mu_K N$ is valid equation (this condition is impossible, since the surfaces cannot support more force than the maximum frictional force. Therefore, the assumption of equilibrium is invalid, the motion occurs).

2. **Limiting Friction :** The condition of impending motion is known to exist. Here a body which is in equilibrium is on the verge of slipping which means the body is in limiting equilibrium condition.

   $F_{max} = \mu_S N$ is valid equation.

3. **Kinetic Friction :** The condition of relative motion is known to exist between the contacting surfaces. So, the body is in motion.

   Kinetic friction takes place $F_K = \mu_K N$ is valid equation.

8.6 Applications of Friction

1. The running vehicle is controlled by applying brake to its tyre because of friction. The vehicle moves with better grip and does not slip due to appropriate friction between the tyre and the road.
2. One can walk comfortably on the floor because of proper gripping between floor and the sole of the shoes. It is difficult to walk on oily or soapy floor.
3. Belt and pulley arrangement permits loading and unloading of load effectively because of suitable friction.
4. Lift moves smoothly without slipping due to proper rope and pulley friction combination.
5. A simple lifting machine like screw jack functions effectively based on principle of wedge friction.

8.7 Solved Problems

8.7.1 Body Placed on Horizontal Plane

Problem 1

Determine the frictional force developed on the block shown in Fig. 8.1 when

(i) $P = 40$ N,

(ii) $P = 80$ N. Coefficient of static friction between the block and floor is $\mu_S = 0.3$ and $\mu_K = 0.25$ and

(iii) Also find the value of P when the block is about to move.

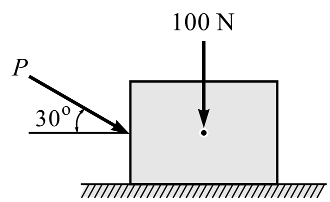


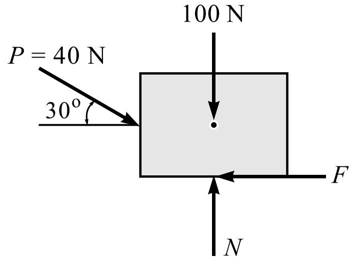
Fig. 8.1

Solution

(i) When P = 40 N

Consider the FBD of the block

Let F be the frictional force required to maintain the static equilibrium condition.



FBD of Block

$$\Sigma F_y = 0$$

$$N - 100 - 40 \sin 30^o = 0$$

$$N = 120 \text{ N}$$

$$\Sigma F_x = 0$$

$$40 \cos 30^o - F = 0$$

$$F = 34.64 \text{ N}$$

For limiting equilibrium condition, we have

$$F_{max} = \mu_S \times N = 0.3 \times 120$$

$$F_{max} = 36 \text{ N}$$

Since $F < F_{max}$ therefore block is in static equilibrium condition.

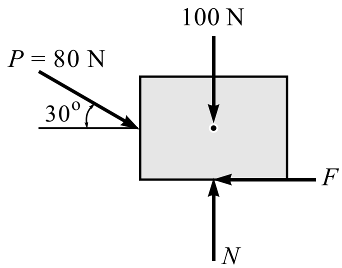
Therefore, actual frictional force is $F = 34.64$ N ***Ans.***

Here block is not moving.

(ii) When P = 80 N

Consider the FBD of the block

Let F be the frictional force required to maintain the static equilibrium condition.



FBD of Block

$$\Sigma F_y = 0$$

$$N - 100 - 80 \sin 30^o = 0$$

$$N = 140 \text{ N}$$

$$\Sigma F_x = 0$$

$$80 \cos 30^o - F = 0$$

$$F = 69.28 \text{ N}$$

For limiting equilibrium condition, we have

$$F_{max} = \mu_S \times N = 0.3 \times 140$$

$$F_{max} = 42 \text{ N}$$

$\because$ $F > F_{max}$

$\therefore$ The block is in motion and kinetic friction is considered.

$\therefore$ $F_K = \mu_K N$ is applicable and $F_{max} = \mu_S N$ is not applicable.

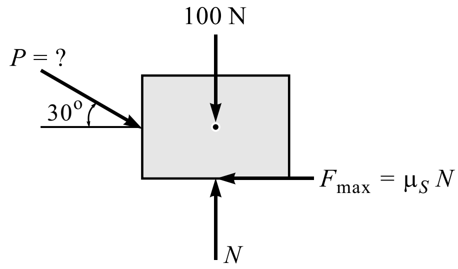
$$F_K = 0.25 \times 140$$

$$F_K = 35 \text{ N}$$

$\therefore$ Actual frictional force acting at surface is F_K = 35 N and block is in motion. ***Ans.***

(iii) Find P = ?

For limiting equilibrium condition, consider the FBD of the block.



FBD of Block

$$\Sigma F_y = 0$$

$$N - 100 - P \sin 30^\circ = 0$$

$$N = 100 + P \sin 30^\circ \qquad \ldots \text{(I)}$$

$$\Sigma F_x = 0$$

$$P \cos 30^\circ - \mu_S N = 0$$

From Eq. (I)

$$P \cos 30^\circ - 0.3\,(100 + P \sin 30^\circ) = 0$$

$$P \cos 30^\circ - 0.3\, P \sin 30^\circ = 0.3 \times 100$$

$$P = 41.9 \text{ N}$$

When $P = 41.9$ N the block is about to move (Impending motion). ***Ans.***

Problem 2

A wooden block of mass 40 kg is on rough inclined plane as shown in Fig. 8.2. Find the frictional force at surface in contact if $\mu_S = 0.4$ and $\mu_K = 0.35$.

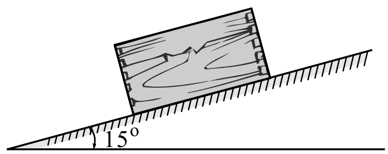


Fig. 8.2

Solution

$$\tan \phi = \mu_S$$

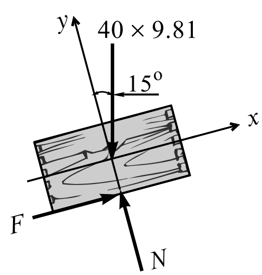
$$\therefore \ \phi = 21.8^\circ$$

We know angle of friction is equal to angle of repose for limiting equilibrium condition where self-weight of block is just sufficient to slide down without any external force acting on it.

In the above case, inclination of surface at an angle 15° is less than angle of friction $\phi = 21.8^\circ$. Therefore, the block will be in static equilibrium condition (i.e., stationary).

(i) Consider the FBD of the block

Let F be the frictional force required to maintain the static equilibrium condition.



FBD of Block

$$\Sigma F_x = 0$$

$$F - 40 \times 9.81 \sin 15^\circ = 0$$

$$F = 101.56 \text{ N} \quad \textit{Ans.}$$

Problem 3

For Problem 2, what is the external force required to be applied parallel to the inclined plane in downward direction for impending motion ?

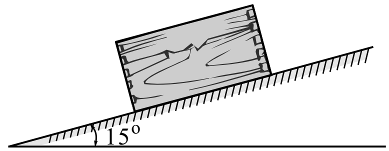


Fig. 8.3

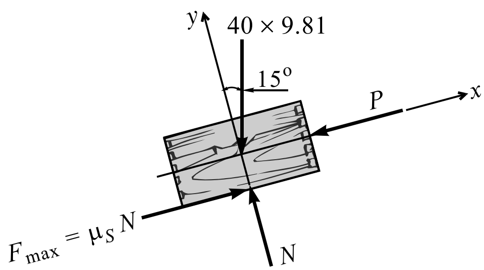
Solution

Impending motion means limiting equilibrium condition is applicable, i.e.,

$$F_{max} = \mu N$$

(i) Consider the FBD of the block

Let P be the force required to be applied to develop impending motion.



FBD of Block

$$\Sigma F_y = 0$$

$$N - (40 \times 9.81 \cos 15^\circ) = 0$$

$$N = 379.03 \text{ N}$$

$$\Sigma F_x = 0$$

$$\mu_S N - P - (40 \times 9.81 \sin 15^\circ) = 0$$

$$P = 0.4 \times 379.03 - 40 \times 9.81 \sin 15^\circ$$

$$P = 50.05 \text{ N } (15^\circ)$$ ***Ans.***

Problem 4

A support block is acted upon by two forces as shown in Fig. 8.4, knowing that the coefficient of friction between the block and incline are $\mu_S = 0.35$, $\mu_K = 0.25$, determine the force P required

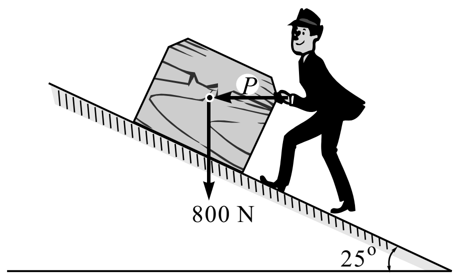


Fig. 8.4

(i) to start the block moving up the plane,

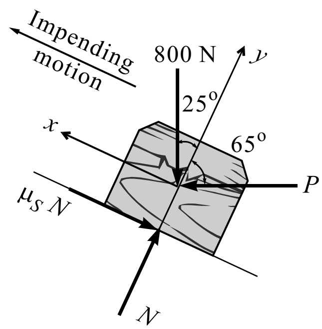
(ii) to keep it moving up, and

(iii) to prevent it from sliding down.

Solution

(i) Force *P* required to start the block moving up the plane

Consider the FBD of the block



FBD of Block

$$\Sigma F_y = 0$$

$$N - 800 \cos 25^\circ - P \cos 65^\circ = 0$$

$$N = 800 \cos 25^\circ + P \cos 65^\circ$$

$\Sigma F_x = 0$

$P \sin 65^\circ - \mu_S N - 800 \sin 25^\circ = 0$

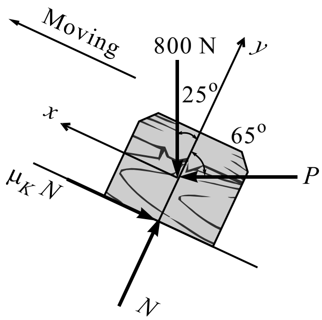
$P \sin 65^\circ - 0.35(800 \cos 25^\circ + P \cos 65^\circ) - 800 \sin 25^\circ = 0$

$P \sin 65^\circ - 0.35 \times P \cos 65^\circ - 0.35 \times 800 \cos 25^\circ - 800 \sin 25^\circ = 0$

$P (\sin 65^\circ - 0.35 \cos 65^\circ) = 0.35 \times 800 \cos 25^\circ + 800 \sin 25^\circ$

$P = 780.42$ N ***Ans.***

(ii) Force *P* required to keep it moving up



FBD of Block

Consider the FBD of the block

$\Sigma F_y = 0$

$N - 800 \cos 25^\circ - P \cos 65^\circ = 0$

$N = 800 \cos 25^\circ + P \cos 65^\circ$

$\Sigma F_x = 0$

$P \sin 65^\circ - \mu_K N - 800 \sin 25^\circ = 0$

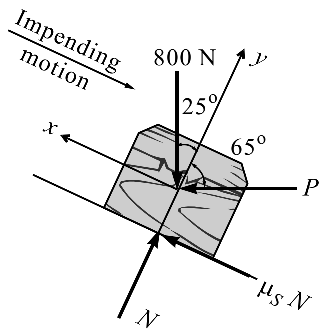
$P \sin 65^\circ - 0.25(800 \cos 25^\circ + P \cos 65^\circ) - 800 \sin 25^\circ = 0$

$P \sin 65^\circ - 0.25 \times P \cos 65^\circ - 0.25 \times 800 \cos 25^\circ - 800 \sin 25^\circ = 0$

$P (\sin 65^\circ - 0.25 \cos 65^\circ) = 0.25 \times 800 \cos 25^\circ + 800 \sin 25^\circ$

$P = 648.67$ N ***Ans.***

(iii) Force *P* required to prevent it from sliding down



FBD of Block

Consider the FBD of the block

$\Sigma F_y = 0$

$N - 800 \cos 25^\circ - P \cos 65^\circ = 0$

$N = 800 \cos 25^\circ + P \cos 65^\circ$

$\Sigma F_x = 0$

$\mu_S N + P \sin 65^\circ - 800 \sin 25^\circ = 0$

$0.35(800 \cos 25^\circ + P \cos 65^\circ) + P \sin 65^\circ - 800 \sin 25^\circ = 0$

$0.35 \times 800 \cos 25^\circ - 0.35 \times P \cos 65^\circ + P \sin 65^\circ - 800 \sin 25^\circ = 0$

$P (0.35 \cos 65^\circ + \sin 65^\circ) = 800 \cos 25^\circ - 0.35 \times 800 \sin 25^\circ$

$P = 80$ N ***Ans.***

Problem 5

Two blocks W_1 and W_2, resting on two inclined planes, are connected by a horizontal bar AB, as shown in Fig. 8.5. If W_1 equals 1000 N, determine the maximum value of W_2 for which the equilibrium can exist. The angle of limiting friction is 20° at all rubbing faces.

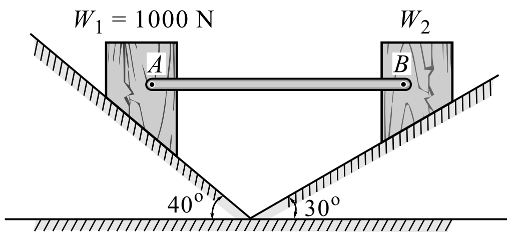


Fig. 8.5

Solution

For maximum weight of the block B in limiting equilibrium condition, tendency of block B will be to impend upwards.

∴ Impending motion of block A will be downward.

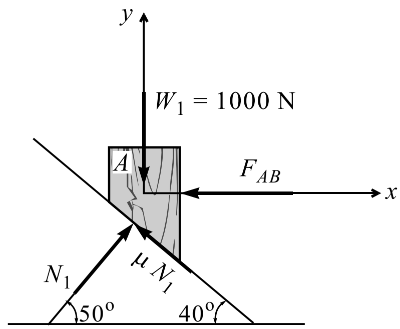
(i) Consider the FBD of Block A

$$\Sigma F_y = 0$$

$$N_1 \sin 50^\circ + \mu N_1 \sin 40^\circ - 1000 = 0$$

$$N_1 = \frac{1000}{\sin 50^\circ + \tan 20^\circ \sin 40^\circ}$$

$$N_1 = 1000 \text{ N}$$



FBD of Block A

$$\Sigma F_x = 0$$

$$N_1 \cos 50^\circ - \mu N_1 \cos 40^\circ - F_{AB} = 0$$

$$F_{AB} = 1000 \cos 50^\circ - \tan 20^\circ \times 1000 \cos 40^\circ$$

$$F_{AB} = 363.97 \text{ N}$$

(ii) Consider the FBD of Block B

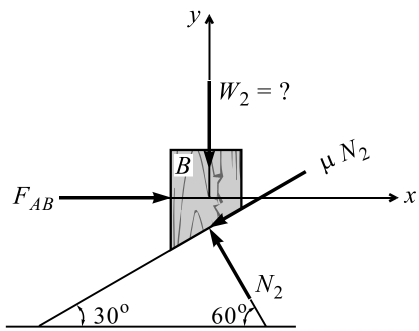
$$\Sigma F_x = 0$$

$$F_{AB} - \mu N_2 \cos 30^\circ - N_2 \cos 60^\circ = 0$$

$$363.97 - \tan 20^\circ N_2 \cos 30^\circ - N_2 \cos 60^\circ = 0$$

$$N_2 = \frac{363.97}{(\cos 60^\circ + \tan 20^\circ \cos 30^\circ)}$$

$$N_2 = 446.48 \text{ N}$$



FBD of Block B

$$\Sigma F_y = 0$$

$$N_2 \sin 60^\circ - \mu N_2 \sin 30^\circ - W_2 = 0$$

$$W_2 = 446.48 \sin 60^\circ - \tan 20^\circ \times 446.48 \sin 30^\circ$$

$$W_{2(\max)} = 305.41 \text{ N} \quad \textbf{\textit{Ans.}}$$

Problem 6

Two blocks W_1 and W_2 which are connected by a horizontal bar AB are supported on rough planes as shown in Fig. 8.6. The coefficient of friction for the block A = 0.4. The angle of friction for the block B is 20^o. Find the smallest weight W_1 of the block A for which the equilibrium can exist, if W_2 = 2250 N.

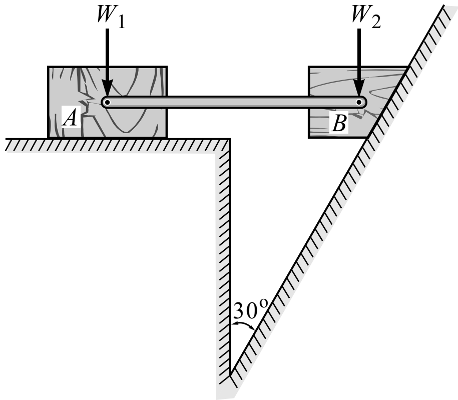


Fig. 8.6

Solution

(i) Consider the FBD of Block *B*

$$\Sigma F_y = 0$$

$$N_1 \sin 30^o + \mu_1 N_1 \sin 60^o - 2250 = 0$$

$$N_1 (\sin 30^o + \tan 20^o \sin 60^o) = 2250$$

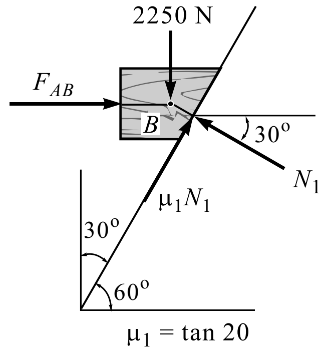
$$N_1 = 2760.03 \text{ N}$$

$$\Sigma F_x = 0$$

$$F_{AB} + \mu_1 N_1 \cos 60^o - N_1 \cos 30^o = 0$$

$$F_{AB} = 2760.03 \cos 30^o - \tan 20^o \times 2760.03 \cos 60^o$$

$$F_{AB} = 1887.97 \text{ N}$$



FBD of Block *B*

(ii) Consider the FBD of Block *A*

$$\Sigma F_x = 0$$

$$\mu_2 N_2 - F_{AB} = 0$$

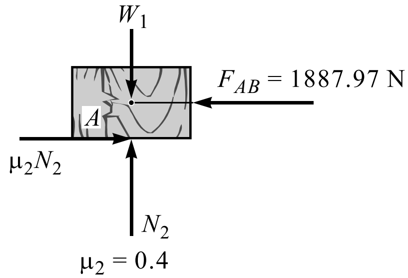
$$0.4\, N_2 = 1887.97$$

$$N_2 = 4719.93 \text{ N}$$

$$\Sigma F_y = 0$$

$$N_2 - W_1 = 0$$

$\therefore$ $W_1 = 4719.93$ N ***Ans.***



FBD of Block *A*

Problem 7

Two blocks A = 100 N and B = W are connected by a rod at their ends by frictionless hinges as shown in Fig. 8.7. Find the weight of block B (W) required for limiting equilibrium of the system if coefficient of friction at all sliding surface is 0.3.

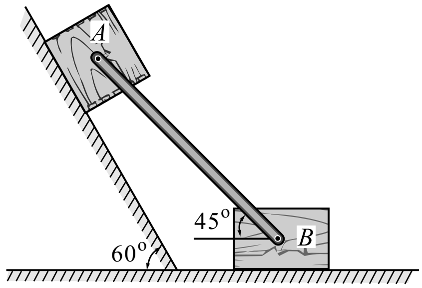


Fig. 8.7

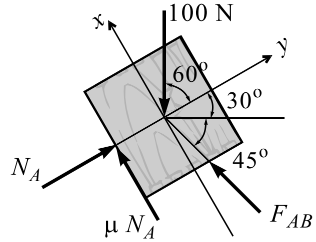
Solution

(i) Consider the FBD of Block A

$$\Sigma F_y = 0$$

$$N_A - 100 \cos 60^\circ - F_{AB} \cos 75^\circ = 0$$

$$N_A = 100 \cos 60^\circ + F_{AB} \cos 75^\circ$$



FBD of Block A

$$\Sigma F_x = 0$$

$$\mu N_A + F_{AB} \sin 75^\circ - 100 \sin 60^\circ = 0$$

$$0.3(100 \cos 60^\circ + F_{AB} \cos 75^\circ) + F_{AB} \sin 75^\circ - 100 \sin 60^\circ = 0$$

$$(0.3 \times F_{AB} \cos 75^\circ) + F_{AB} \sin 75^\circ + (0.3 \times 100 \cos 60^\circ) - 100 \sin 60^\circ = 0$$

$$F_{AB} (0.3 \cos 75^\circ + \sin 75^\circ) = 100 \sin 60^\circ - (0.3 \times 100 \cos 60^\circ)$$

$$F_{AB} = \frac{100 \sin 60^\circ - (0.3 \times 100 \cos 60^\circ)}{(0.3 \cos 75^\circ + \sin 75^\circ)}$$

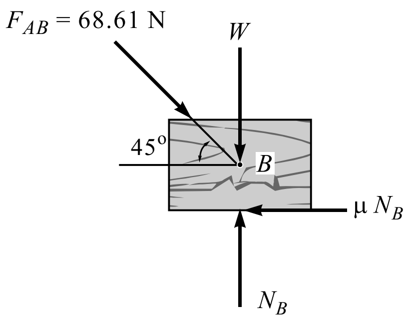
$$F_{AB} = 68.61 \text{ N (rod is under compression)}$$

(ii) Consider the FBD of Joint B

$$\Sigma F_y = 0$$

$$N_B - W - F_{AB} \sin 45^\circ = 0$$

$$N_B = W + F_{AB} \sin 45^\circ$$



FBD of Block B

$$\Sigma F_x = 0$$

$$F_{AB} \cos 45^\circ - \mu N_B = 0$$

$$68.61 \cos 45^\circ - 0.3(W + 68.61 \sin 45^\circ) = 0$$

$$0.3\, W = 33.96$$

W = 113.2 N ***Ans.***

Problem 8

Two identical blocks A and B are connected by a rod and rest against vertical and horizontal planes, respectively, as shown in Fig. 8.8. If sliding impends when $\theta = 45^o$, determine the coefficient of friction μ, assuming it to be the same at both floor and wall.

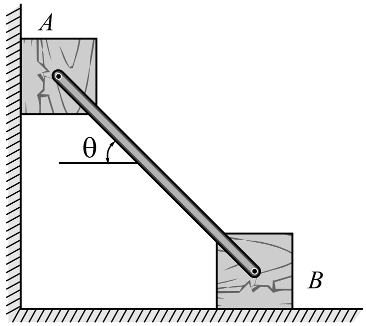
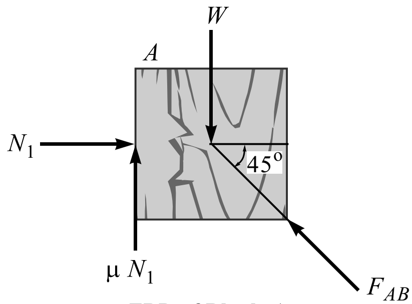


Fig. 8.8

Solution

(i) Consider the FBD of Block A



FBD of Block A

$$\Sigma F_x = 0$$

$$N_1 - F_{AB} \cos 45^o = 0$$

$$N_1 = F_{AB} \cos 45^o$$

$$\Sigma F_y = 0$$

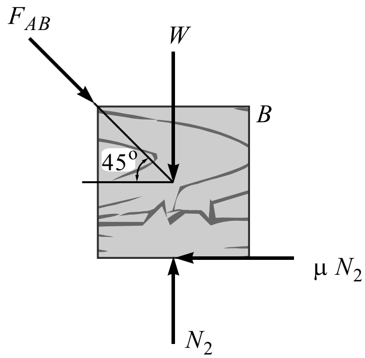
$$\mu N_1 + F_{AB} \sin 45^o - W = 0$$

$$W = \mu N_1 + F_{AB} \sin 45^o$$

$$W = \mu F_{AB} \cos 45^o + F_{AB} \sin 45^o$$

$$W = F_{AB} (\mu \cos 45^o + \sin 45^o) \qquad \text{... (I)}$$

(ii) Consider the FBD of Block B



FBD of Block B

$$\Sigma F_y = 0$$

$$N_2 - W - F_{AB} \sin 45^o = 0$$

$$N_2 = W + F_{AB} \sin 45^o$$

$$\Sigma F_x = 0$$

$$F_{AB} \cos 45^o - \mu N_2 = 0$$

$$F_{AB} \cos 45^o - \mu (W + F_{AB} \sin 45^o) = 0$$

$$F_{AB} \cos 45^o - \mu W - \mu F_{AB} \sin 45^o = 0$$

$$W = \frac{F_{AB} \cos 45^o - \mu F_{AB} \sin 45^o}{\mu}$$

$$W = \frac{F_{AB} (\cos 45^o - \mu \sin 45^o)}{\mu} \qquad \text{... (II)}$$

Equating Eqs. (I) and (II),

$$F_{AB} (\mu \cos 45^o + \sin 45^o) = \frac{F_{AB} (\cos 45^o - \mu \sin 45^o)}{\mu}$$

$$\mu^2 \cos 45^o + \mu \sin 45^o = \cos 45^o - \mu \sin 45^o$$

$$\mu^2 \cos 45^o + 2\mu \sin 45^o - \cos 45^o = 0$$

$$\mu^2 + 2\mu - 1 = 0$$

Solving the quadratic equation, we get $\mu = 0.414$ ***Ans.***

Problem 9

Determine the force P to cause motion to impend. Take masses of blocks A and B as 9 kg and 4 kg respectively and the coefficient of sliding friction as 0.25. The force P and the rope are parallel to the inclined plane as shown in Fig. 8.9. Assume pulley to be frictionless.

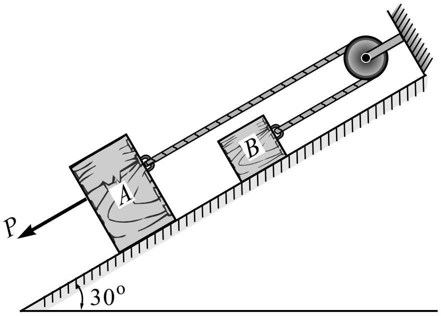


Fig. 8.9

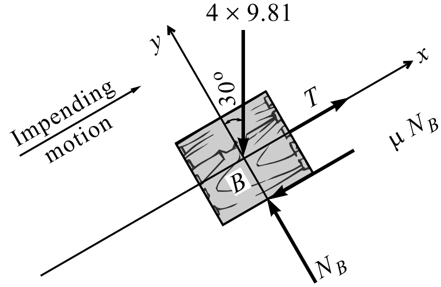
Solution

(i) Consider the FBD of Block *B*

$$\Sigma F_y = 0$$

$$N_B - (4 \times 9.81 \cos 30^\circ) = 0$$

$$N_B = 33.98 \text{ N}$$



FBD of Block *B*

$$\Sigma F_x = 0$$

$$T - \mu N_B - (4 \times 9.81 \sin 30^\circ) = 0$$

$$T = (0.25 \times 33.98) + (4 \times 9.81 \sin 30^\circ)$$

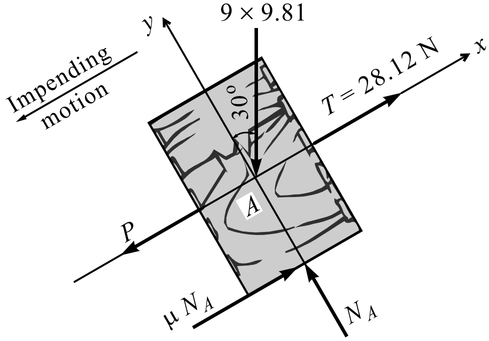
$$T = 28.12 \text{ N}$$

(ii) Consider the FBD of Block *A*

$$\Sigma F_y = 0$$

$$N_A - (9 \times 9.81 \cos 30^\circ) = 0$$

$$N_A = 76.46 \text{ N}$$



FBD of Block *A*

$$\Sigma F_x = 0$$

$$T + \mu N_A - P - (9 \times 9.81 \sin 30^\circ) = 0$$

$$P = 28.12 + (0.25 \times 76.46) - (9 \times 9.81 \sin 30^\circ)$$

$$P = 3.09 \text{ N} \quad \textit{Ans.}$$

Problem 10

Two blocks A and B of weight 500 N and 750 N, respectively are connected by a cord that passes over a frictionless pulley, as shown in Fig. 8.10. The coefficient of friction between the block A and the inclined plane is 0.4 and that between the block B and the inclined plane is 0.3. Determine the force P to be applied to block B to produce the impending motion of block B down the plane.

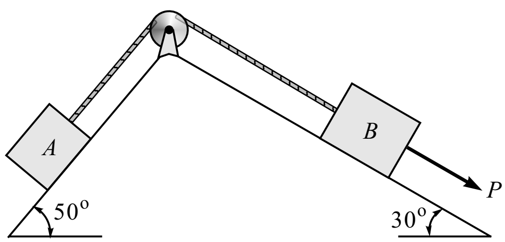


Fig. 8.10

(i) Consider the FBD of Block *A*

$$\Sigma F_y = 0$$

$$N_A - 500 \cos 50^\circ = 0$$

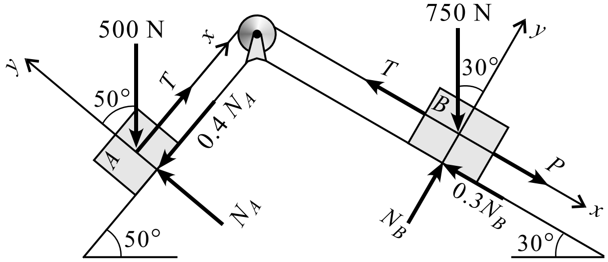
$$N_A = 500 \cos 50^\circ$$

$$\Sigma F_x = 0$$

$$T - 0.4\, N_A - 500 \sin 50^\circ = 0$$

$$T = 0.4 \times 500 \cos 50^\circ + 500 \sin 50^\circ$$

$$T = 511.58 \text{ N}$$



FBD of Block *A* and Block *B*

(ii) Consider the FBD of Block *B*

$$\Sigma F_y = 0$$

$$N_B - 750 \cos 30^\circ = 0$$

$$N_B = 750 \cos 30^\circ$$

$$\Sigma F_x = 0$$

$$P - 0.3\, N_B - T + 750 \sin 30^\circ = 0$$

$$P = 0.3 \times 750 \cos 30^\circ + 511.58 - 750 \sin 30^\circ$$

$$P = 331.44 \text{ N} \left(\searrow 30^\circ\right)$$ ***Ans.***

Problem 11

Find the value of θ if the blocks A and B shown in Fig. 8.11 have impending motion. Given block $A = 20$ kg, block $B = 20$ kg, $\mu_A = \mu_B = 0.25$.

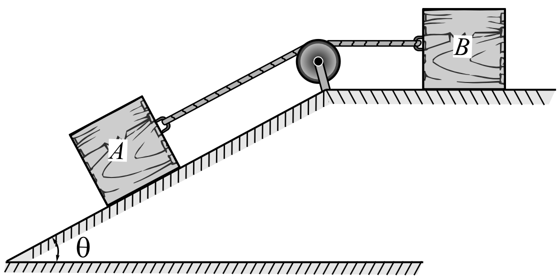


Fig. 8.11

Solution

(i) Consider the FBD of Block *B*

$$\Sigma F_y = 0$$

$$N_B - (20 \times 9.81) = 0$$

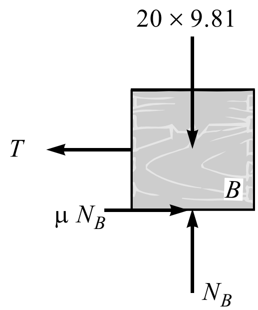
$$N_B = 20 \times 9.81$$

$$\Sigma F_x = 0$$

$$\mu\, N_B - T = 0$$

$$(0.25 \times 20 \times 9.81) - T = 0$$

$$T = 0.25 \times 20 \times 9.81$$



FBD of Block *B*

(ii) Consider the FBD of Block *A*

$\Sigma F_y = 0$

$N_A - (20 \times 9.81) \cos \theta = 0$

$N_A = (20 \times 9.81) \cos \theta$

$\Sigma F_x = 0$

$\mu N_A + T - (20 \times 9.81) \sin \theta = 0$

$(0.25 \times 20 \times 9.81) \cos \theta + (0.25 \times 20 \times 9.81)$

$- (20 \times 9.81) \sin \theta = 0$

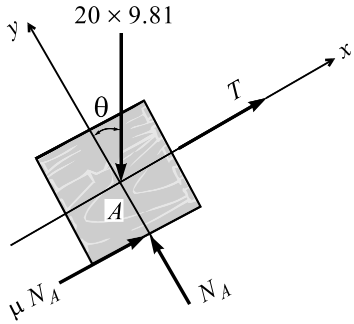
$0.25 \cos \theta + 0.25 - \sin \theta = 0$ (Multiply by 4)

$\cos \theta + 1 = 4 \sin \theta$

$2 \cos^2 (\theta/2) = 4 [2 \sin (\theta/2) . \cos (\theta/2)]$

$\frac{1}{4} = \tan \frac{\theta}{2} \Rightarrow \frac{\theta}{2} = \tan^{-1} 0.25$

$\therefore \theta = 28.07^o$ ***Ans.***



FBD of Block *A*

Problem 12

Find force P required to pull block B (shown in Fig. 8.12). Coefficient of friction between A and B is 0.3 and between B and floor is 0.25. Weights of A = 20 kg and B = 30 kg.

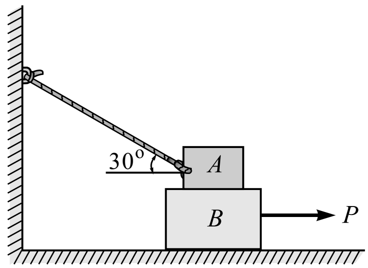


Fig. 8.12

Solution

(i) Consider the FBD of Block *A*

$\Sigma F_y = 0$

$N_A + T \sin 30^o - 20 \times 9.81 = 0$

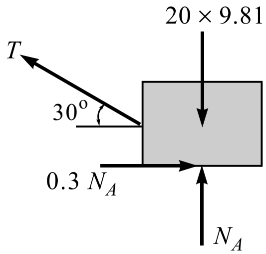
$N_A = 20 \times 9.81 - T \sin 30^o$

$\Sigma F_x = 0$

$T \cos 30^o - 0.3 N_A = 0$

$T \cos 30^o - 0.3(20 \times 9.81 - T \sin 30^o) = 0$

$T = 57.93$ N and $N_A = 167.235$ N



FBD of Block *A*

(ii) Consider the FBD of Block *B*

$\Sigma F_y = 0$

$N_B - 30 \times 9.81 - 167.06 = 0$

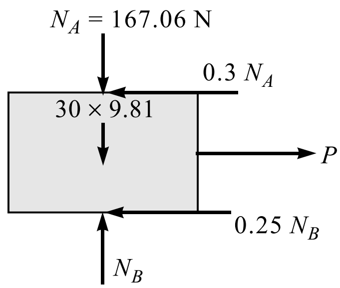
$N_B = 461.535$ N ***Ans.***

$\Sigma F_x = 0$

$P - 0.3 N_A - 0.25 N_B = 0$

$P = (0.3 \times 167.06) + (0.25 \times 461.36)$

$P = 165.55$ N ***Ans.***



FBD of Block *B*

Problem 13

Two blocks A = 100 N and B = 150 N are resting on ground as shown in Fig. 8.13. Coefficient of friction between ground and block B is 0.10 and that between block B and A is 0.30. Find the minimum value of weight P in the pan so that motion starts. Find whether B is stationary w.r.t. ground and A moves or B is stationary w.r.t. A.

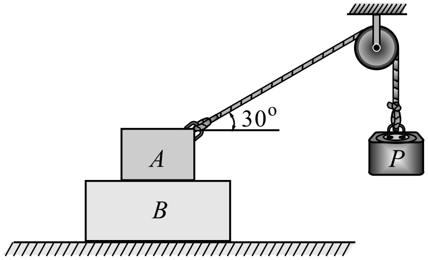


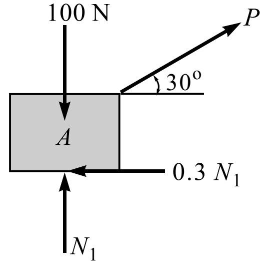
Fig. 8.13

Solution

Case I : B is stationary w.r.t. ground and A moves.

Consider given A is in limiting equilibrium which means block A moves over the surface of B.

Consider the FBD of Block *A*



FBD of Block *A*

$$\Sigma F_y = 0$$

$$N_1 + P \sin 30^{\circ} - 100 = 0$$

$$N_1 = 100 - P \sin 30^{\circ}$$

$$\Sigma F_x = 0$$

$$P \cos 30^{\circ} - 0.3\ N_1 = 0$$

$$P \cos 30^{\circ} - 0.3\ (100 - P \sin 30^{\circ}) = 0$$

$$P \cos 30^{\circ} - (0.3 \times 100) + 0.3\ P \sin 30^{\circ} = 0$$

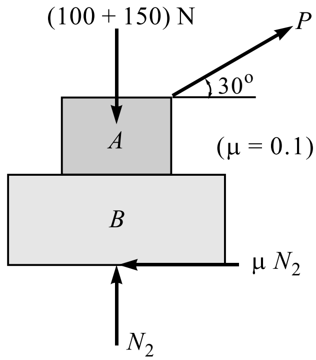
$$P\ (\cos 30^{\circ} + 0.3 \sin 30^{\circ}) = 0.3 \times 100$$

$$P = \frac{0.3 \times 100}{\cos 30^{\circ} + 0.3 \sin 30^{\circ}} \quad \therefore\ P = 29.53 \text{ N} \quad \textbf{\textit{Ans.}}$$

Case II : B is stationary w.r.t. A

Consider both blocks A and B moving together.

Consider the FBD of *A* and *B* together



FBD of Block *A* and *B* Together

$$\Sigma F_y = 0$$

$$N_2 - 250 + P \sin 30^{\circ} = 0$$

$$N_2 = 250 - P \sin 30^{\circ}$$

$$\Sigma F_x = 0$$

$$P \cos 30^{\circ} - \mu\ N_2 = 0$$

$$P \cos 30^{\circ} - 0.1\ (250 - P \sin 30^{\circ}) = 0$$

$$P \cos 30^{\circ} - (0.1 \times 250) + 0.1\ P \sin 30^{\circ} = 0$$

$$P\ (\cos 30^{\circ} + 0.1 \sin 30^{\circ}) = 0.1 \times 250$$

$$P = \frac{0.1 \times 250}{\cos 30^{\circ} + 0.1 \sin 30^{\circ}} \quad \therefore\ P = 27.29 \text{ N} \quad \textbf{\textit{Ans.}}$$

Referring to the answers of both the cases, we can declare 'Case II' is initiated first and therefore, minimum value of $P_{\min} = 27.29$ N.

Problem 14

Block A of mass 30 kg rests on block B of mass 40 kg, as shown in Fig. 8.14. Block A is restrained form moving by a horizontal rope tied at point C. What force P applied parallel to the plane inclined at 30° with horizontal is necessary to start block B sliding down the plane. Take coefficient of friction for all surfaces as 0.35.

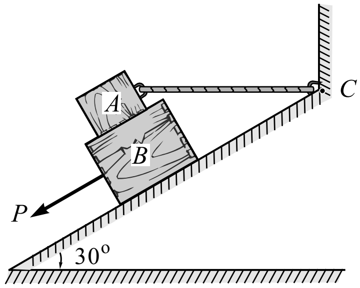


Fig. 8.14

Solution

(i) Consider the FBD of Block A

$$\Sigma F_y = 0$$

$$N_1 - 30 \times 9.81 \cos 30^\circ - T \sin 30^\circ = 0$$

$$N_1 = 30 \times 9.81 \cos 30^\circ + T \sin 30^\circ \quad \ldots \text{(I)}$$

$$\Sigma F_x = 0$$

$$T \cos 30^\circ - \mu N_1 - 30 \times 9.81 \sin 30^\circ = 0$$

From Eq. (I)

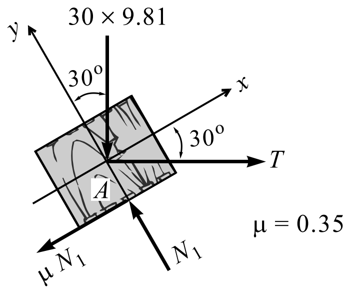
$$T \cos 30^\circ - 0.35\,(30 \times 9.81 \cos 30^\circ + T \sin 30^\circ) - 30 \times 9.81 \sin 30^\circ = 0$$

$$T = 342.04 \text{ N}$$

Substituting T in Eq. (I)

$$N_1 = 30 \times 9.81 \cos 30^\circ + 342.04 \sin 30^\circ$$

$$N_1 = 425.89 \text{ N}$$



FBD of Block A

(ii) Consider the FBD of Block B

$$\Sigma F_y = 0$$

$$N_2 - N_1 - 40 \times 9.81 \cos 30^\circ = 0$$

$$N_2 = 425.89 + 40 \times 9.81 \cos 30^\circ$$

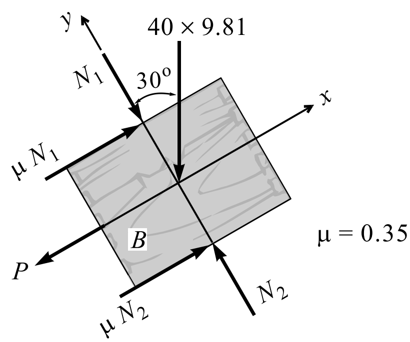
$$N_2 = 765.72 \text{ N}$$

$$\Sigma F_x = 0$$

$$\mu N_1 + \mu N_2 - 40 \times 9.81 \sin 30^\circ - P = 0$$

$$P = 0.35\,(425.89 + 765.72) - 40 \times 9.81 \sin 30^\circ$$

$P = 220.86$ N ***Ans.***



FBD of Block B

Problem 15

Three blocks are placed on the surface one above the other as shown in Fig. 8.15. The static coefficient of friction between the blocks and block C and surface is also shown. Determine the maximum value of P that can be applied before any slipping takes place.

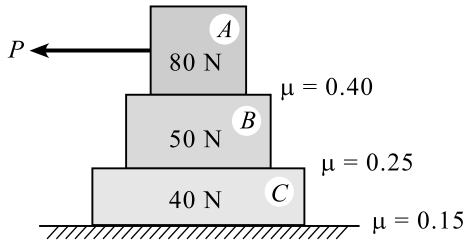


Fig. 8.15

Solution

For P there are three possibilities.

(i) Block A has impending motion and blocks B and C remain intact with each other and surface.

Consider the FBD of Block *A*

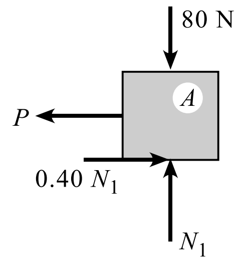
$\Sigma F_y = 0$

$N_1 - 80 = 0 \qquad \therefore\ N_1 = 80$ N

$\Sigma F_x = 0$

$0.4N_1 - P = 0$

$P = 0.4 \times 80 \qquad \therefore\ P = 32$ N $(\leftarrow)$



FBD of Block *A*

(ii) Blocks A and B together have impending motion and block C remains intact with surface.

FBD of Blocks *A* and *B* together

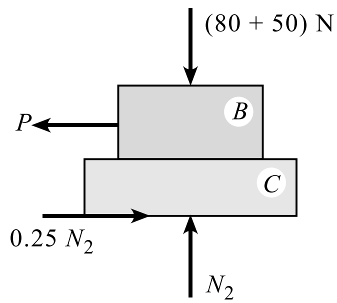
$\Sigma F_y = 0$

$N_2 - (80 + 50) = 0 \qquad \therefore\ N_2 = 130$ N

$\Sigma F_x = 0$

$0.25N_2 - P = 0$

$P = 0.25 \times 130 \qquad \therefore\ P = 32.5$ N $(\leftarrow)$



FBD of Block *A* and Block *B* Together

(iii) All the three blocks A, B and C together have impending motion.

FBD of Blocks *A*, *B* and *C* together

$\Sigma F_y = 0$

$N_3 - (50 + 80 + 40) = 0$

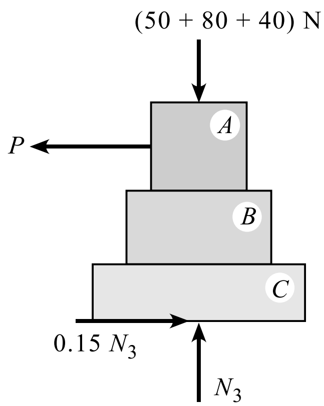
$N_3 = 170$ N

$\Sigma F_x = 0$

$0.15\ N_3 - P = 0$

$P = 0.15 \times 170$

$P = 25.5$ N



FBD of Block *A* and Block *B* and Block *C* Together

(iv) Comparing all three cases, we conclude that $P_{max} = 25.5$ N before any slipping takes place. ***Ans.***

Problem 16

Find the maximum height at which P should be applied so that the body would just slide without tipping. Also state magnitude of P. Refer to Fig. 8.16.

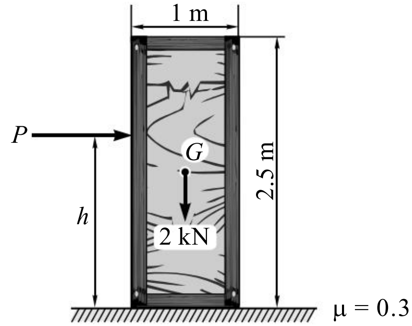
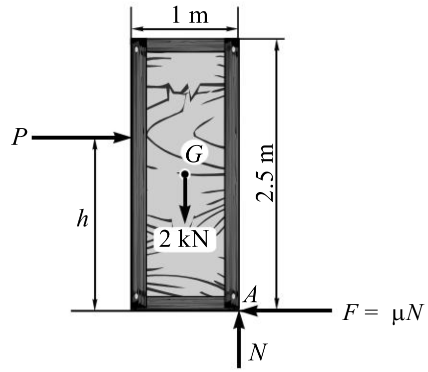


Fig. 8.16

Solution

Consider the FBD of the block

$$\Sigma F_y = 0$$
$$N - 2 = 0$$
$$N = 2 \text{ kN}$$
$$\Sigma F_x = 0$$
$$P - \mu N = 0$$
$$P = 0.3 \times 2$$
$$P = 0.6 \text{ kN} \quad \textbf{\textit{Ans.}}$$
$$\Sigma M_A = 0$$
$$2 \times 0.5 - P \times h = 0 \quad \Rightarrow \quad h = \frac{2 \times 0.5}{0.6}$$
$$h = 1.67 \text{ m} \quad \textbf{\textit{Ans.}}$$



FBD of Block

Problem 17

A homogeneous block A of weight W rests upon an inclined plane as shown in Fig. 8.17. $\mu = 0.3$. Determine the greatest height at which a force P parallel to the inclined plane may be applied so that the block will slide up the plane without tipping over.

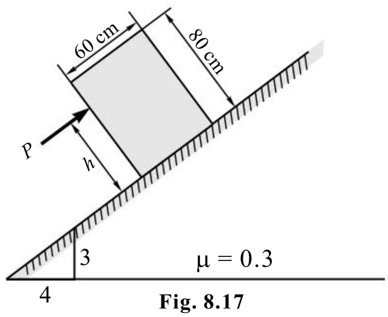
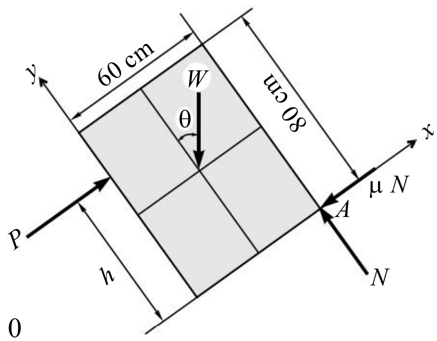


Fig. 8.17

Solution

Consider the FBD of Block *A*

$$\Sigma F_y = 0$$
$$N - W \cos \theta = 0$$
$$N = W \cos 36.87^\circ$$
$$\Sigma F_x = 0$$
$$P - \mu N - W \sin \theta = 0$$
$$P = 0.3 \times W \cos 36.87^\circ + W \sin 36.87^\circ$$
$$P = 0.84\, W \text{ N}$$
$$\Sigma M_A = 0$$
$$W \cos \theta \times 30 + W \sin \theta \times 40 - P \times h = 0$$
$$W \cos 36.87^\circ \times 30 + W \sin 36.87^\circ \times 40 - 0.84\, W \times h = 0$$
$$h = 57.14 \text{ cm} \quad \textbf{\textit{Ans.}}$$



FBD of Block *A*

8.8 Solved Problems Based on Wedge, Ladder and Screw Jack

Wedge

A tapper shaped block (with very less angle of inclination) which are used for lifting or shifting or holding the heavy block by very less effort is called a *wedge*. The lifting or shifting of the distance is very small. While installing heavy machinery horizontal levelling is required with zero error. It is possible to adjust the small height by inserting wedge as a packing. Sometimes, combination of wedge is also used to push or shift heavy bodies by little distance. Simple lifting machine such as screw jack is based on principle of wedge which is used to raise or lower the heavy load by small effort.

Problem 18

A block of mass 150 kg is raised by a 10^o wedge weighing 50 kg under it and by applying a horizontal force at it as shown in Fig. 8.18. Taking coefficient of friction between all surfaces of contact as 0.3, find minimum force that should be applied to raise the block.

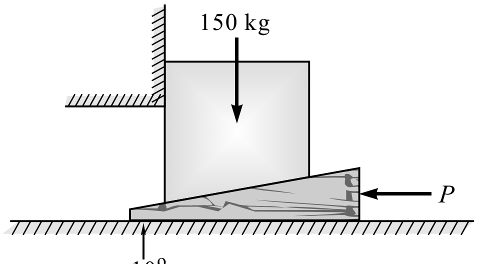
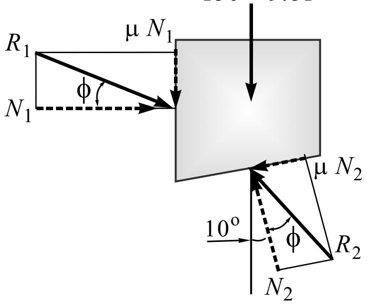


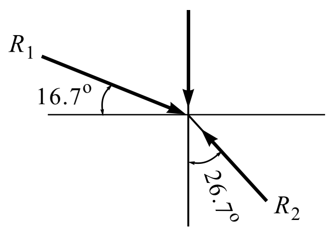
Fig. 8.18

Solution

(i) Consider the FBD of 150 kg block



FBD of Block



$$\tan \phi = \mu = 0.3$$

$$\therefore \phi = 16.7^o$$

By Lami's theorem, we have

$$\frac{R_2}{\sin (90 - 16.7)^o} = \frac{150 \times 9.81}{\sin (90 + 16.7 + 26.7)^o}$$

$$\therefore R_2 = 1939.84 \text{ N}$$

(ii) Consider the FBD of the wedge

$$\Sigma F_y = 0$$

$$N_2 - (50 \times 9.81) - 1929.84 \cos 26.7^o = 0$$

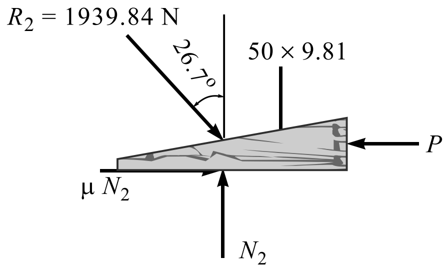
$$N_2 = 2223.5 \text{ N}$$

$$\Sigma F_x = 0$$

$$\mu N_2 + 1939.84 \sin 26.7^o - P = 0$$

$$P = (0.3 \times 2223.5) + 1939.84 \sin 26.7^o$$

$P = 1538.66$ N ***Ans.***



FBD of Wedge

Problem 19

The block, as shown in Fig. 8.19, supports a load W = 5000 N and is to be raised by forcing the wedge B under it. The angle of friction for all surface for contact is ϕ = 15°. Determine the force P which is necessary to start the wedge under the block. The block and wedge have negligible weight.

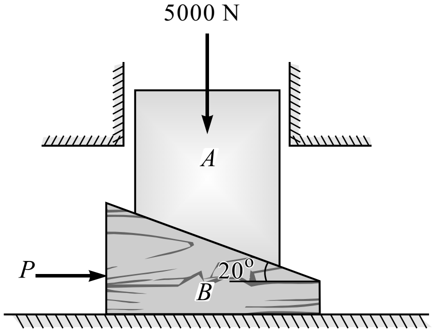
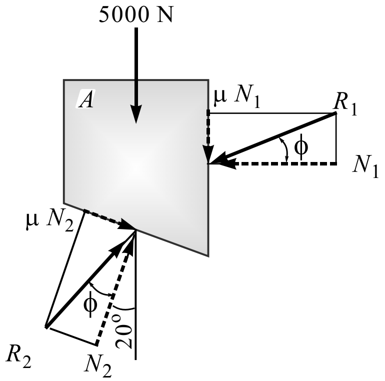


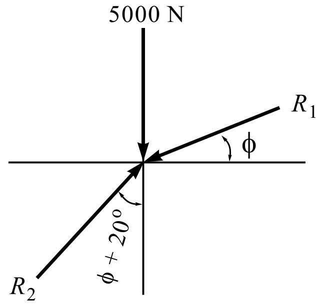
Fig. 8.19

Solution

(i) Consider the FBD of Block *A*



FBD of Block *A*



By Lami's theorem, we have

$$\frac{5000}{\sin(\phi + 20 + 90 + \phi)^{\circ}} = \frac{R_2}{\sin(90 - \phi)^{\circ}}$$

$$\therefore\ R_2 = \frac{5000 \sin 75^{\circ}}{\sin 140^{\circ}}$$

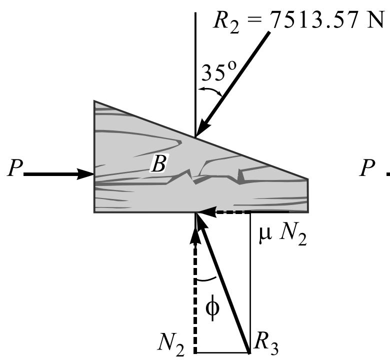
$$\therefore\ R_2 = 7513.57 \text{ N}$$

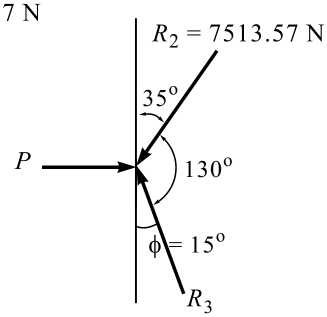
(ii) Consider the FBD of Wedge *B*

By Lami's theorem, we have

$$\frac{P}{\sin 130^{\circ}} = \frac{7513.57}{\sin 105^{\circ}}$$

$\therefore\ P$ = 5958.77 N ***Ans.***





FBD of Wedge

Problem 20

Two 6° wedges are used to push a block horizontally, as shown in Fig. 8.20. Calculate the minimum force required to push the block of weight 10 kN. Take $\mu = 0.25$ for all contact surfaces.

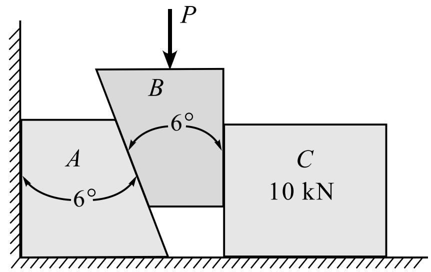
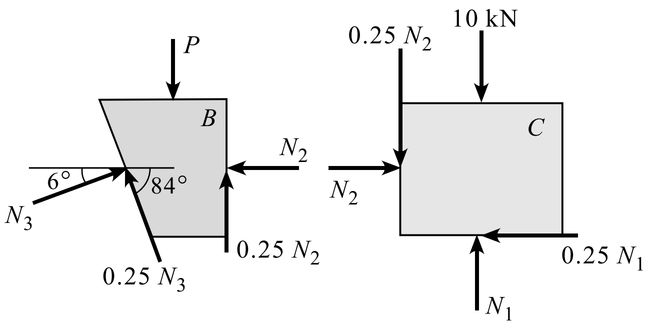


Fig. 8.20

Solution



FBD of Wedge *B* and Block *C*

(i) Consider the FBD of Block *C*

$\Sigma F_y = 0$

$N_1 - 10 - 0.25N_2 = 0$

$\therefore N_1 = 10 + 0.25N_2$

$\Sigma F_x = 0,$

$N_2 - 0.25N_1 = 0$

$N_2 - 0.25(10 + 0.25N_2) = 0$

$\therefore N_2 = 2666.7$

(ii) Consider the FBD of Wedge *B*

$\Sigma F_x = 0$

$N_3 \cos 6^\circ - N_2 - 0.25N_3 \cos 84^\circ = 0$

$0.9684N_3 = N_2$

$N_3 = 2.754$

$\Sigma F_y = 0$

$N_3 \sin 6^\circ + 0.25N_3 \sin 84^\circ + 0.25\ N_2 - P = 0$

$P = 1.639$ kN ***Ans.***

Problem 21

Determine the force P required to move the block A of weight 5000 N up the inclined plane. Coefficient of friction between all contact surfaces is 0.25. Neglect the weight of the wedge and the wedge angle is 15 degrees.

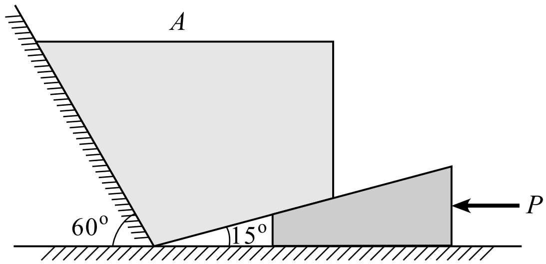


Fig. 8.21

Solution

(i) Consider the FBD of Block A

$$\Sigma F_x = 0$$

$$N_1 \cos 30^\circ + 0.25 N_1 \cos 60^\circ - 0.25 N_2 \cos 15^\circ - N_2 \cos 75^\circ = 0$$

$$0.860 N_1 + 0.125 N_1 - 0.241 N_2 - 0.2588 N_2 = 0$$

$$0.991 N_1 = 0.499 N_2$$

$$\therefore N_1 = 0.5165 N_2$$

$$\Sigma F_y = 0$$

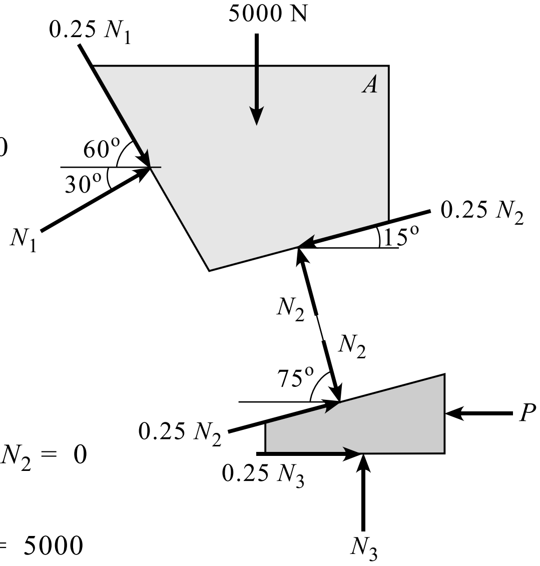
$$N_2 \sin 75^\circ + N_1 \sin 30^\circ - 5000 - 0.25 N_1 \sin 60^\circ - 0.25 N_2 \sin 15^\circ = 0$$

$$0.966 N_2 + 0.5 N_1 - 5000 - 0.2165 N_1 - 0.0647 N_2 = 0$$

Substituting the value of N_1

$$0.966 N_2 + 0.2583 N_2 - 0.1118 N_2 - 0.0647 N_2 = 5000$$

$$\therefore N_2 = 4772.3585 \text{ N}$$



FBD of Block A and Wedge

(ii) Consider the FBD of Wedge B

$$\Sigma F_y = 0$$

$$N_3 + 0.25 N_2 \sin 15^\circ - N_2 \sin 75^\circ = 0$$

$$N_3 + 308.79 - 4609.744 = 0$$

$$N_3 = 4300.95 \text{ N}$$

$$\Sigma F_y = 0$$

$$0.25 N_3 + 0.25 N_2 \cos 15^\circ + N_2 \cos 75^\circ - P = 0$$

$$1075.23 + 1152.436 + 123.17 = P$$

$P = 3462.84$ N ***Ans.***

Ladder

Many a times, we come across the uses of ladder for reaching the higher height. Ladders are used by painters and carpenters who want to peg a nail in the wall for mounting a photo frame. We observe that care is taken to place the ladder at an appropriate angle with respect to ground and wall. We try to adjust the friction offered by the ground and wall in contact with ladder. Also, sometimes we prefer to hold the ladder by a person for safety purposes. The forces acting on the ladder are normal reactions; frictional forces between the ground, the wall and the ladder, weight of the ladder and the weight of the man climbing the ladder. Considering the free body diagram of ladder, we get general force system. The system may be simplified by considering that equilibrium condition can be worked out by following equations :

$$\Sigma F_x = 0 ; \quad \Sigma F_y = 0 \text{ and } \Sigma M = 0$$

Problem 22

A uniform ladder weighing 100 N and 5 meters long has lower end B resting on the ground and upper end A resting against a vertical wall as shown in Fig. 8.22. The inclination of the ladder with horizontal is 60°. If the coefficient of friction at all surfaces of contact is 0.25, determine how much distance (up along the ladder) a man weighing 600 N can ascent without causing it to slip.

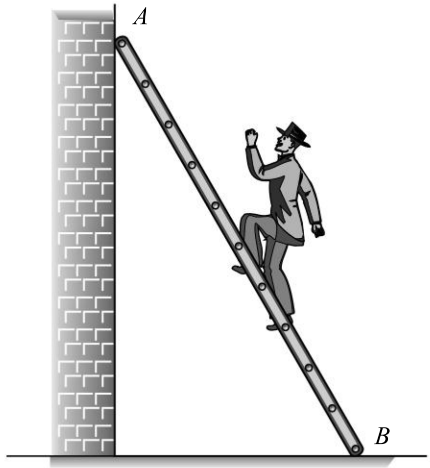


Fig. 8.22

Solution

Consider the FBD of the ladder

$$\Sigma F_x = 0$$

$$N_A - \mu N_B = 0$$

$$N_B = 4 N_A$$

$$\Sigma F_y = 0$$

$$\mu N_A + N_B - 100 - 600 = 0$$

$$0.25 N_A + 4 N_A = 700$$

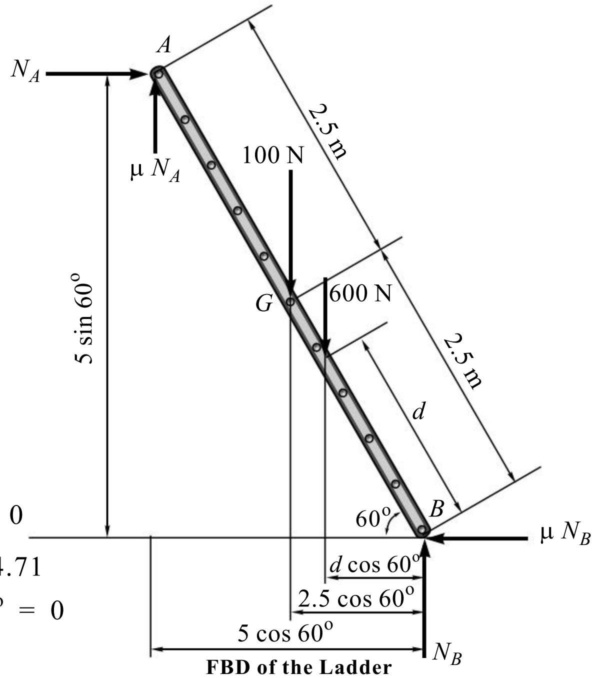
$$N_A = 164.71$$

$$\Sigma M_B = 0$$

$$100 \times 2.5 \cos 60^\circ + 600 \times d \cos 60^\circ - N_A \times 5 \sin 60^\circ - \mu N_A \times 5 \cos 60^\circ = 0$$

$$100 \times 2.5 \cos 60^\circ + 600 \times d \cos 60^\circ - 164.71 \times 5 \sin 60^\circ - 0.25 \times 164.71 \times 5 \cos 60^\circ = 0$$

$d = 2.304$ m ***Ans.***



FBD of the Ladder

Problem 23

A 100 N uniform rod AB is held in the position as shown in Fig. 8.23. If coefficient of friction is 0.15 at A and B. Calculate range of values of P for which equilibrium is maintained.

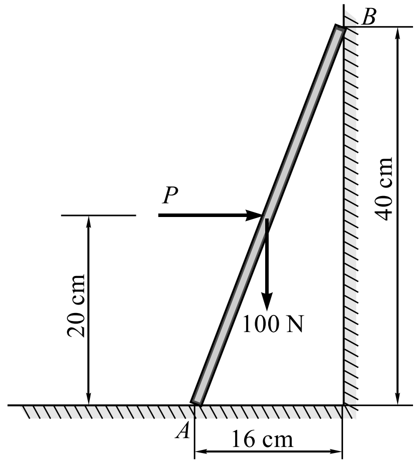


Fig. 8.23

Solution

Case I : For $P_{minimum}$

Consider FBD of rod AB when P is minimum and in limiting equilibrium condition the tendency of rod will be to slip in downward direction.

$$\Sigma F_x = 0$$

$$P_{min} + \mu N_A - N_B = 0$$

$$P_{min} = N_B - 0.15 N_A \qquad \text{... (I)}$$

$$\Sigma F_y = 0$$

$$N_A + \mu N_B - 100 = 0$$

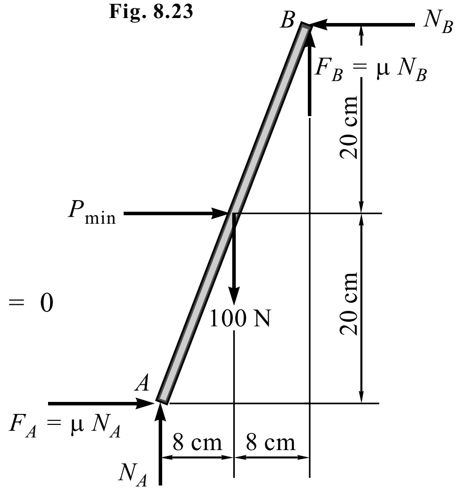
$$N_A + 0.15 N_B = 100 \qquad \text{... (II)}$$

$$\Sigma M_A = 0$$

$$\mu N_B \times 16 + N_B \times 40 - 100 \times 8 - P_{min} \times 20 = 0$$

$$0.15 N_B \times 16 + N_B \times 40 - 800 - (N_A - 0.15 N_A) \times 20 = 0$$

$$3 N_A + 22.4 N_B = 800 \qquad \text{... (III)}$$



FBD of Rod AB for P_{min}

Solving Eqs. (II) and (III),

$$N_A = 96.58 \text{ N}$$

$$N_B = 22.78 \text{ N}$$

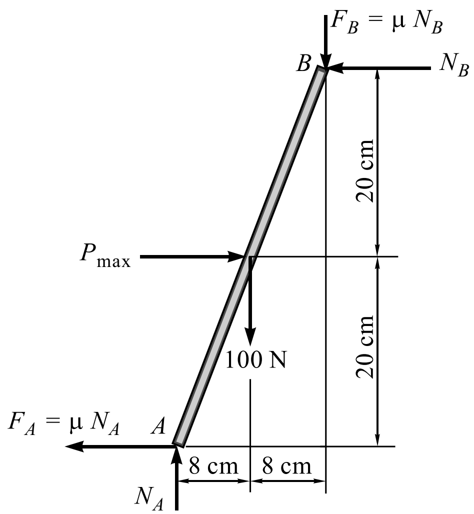
From Eq. (I),

$$P_{min} = 22.78 - 0.15 \times 96.58$$

$$P_{min} = 8.29 \text{ N} \quad \textit{Ans.}$$

Case II : For $P_{maximum}$

Consider FBD of rod AB when P is maximum and in limiting equilibrium condition the tendency of rod will be to slip in upward direction.



FBD of Rod AB for P_{max}

$$\Sigma F_x = 0$$

$$P_{max} - \mu N_A - N_B = 0$$

$$P_{max} = 0.15 N_A + N_B \qquad \text{... (IV)}$$

$$\Sigma F_y = 0$$

$$N_A - \mu N_B - 100 = 0$$

$$N_A - 0.15 N_B = 100 \qquad \text{... (V)}$$

$$\Sigma M_A = 0$$

$$N_B \times 40 - \mu N_B \times 16 - P_{max} \times 20 - 100 \times 8 = 0$$

$$N_B \times 40 - 0.15 N_B \times 16 - (0.15 N_A + N_B) \times 20 - 800 = 0$$

$$17.6 N_B - 3 N_A = 800 \quad \text{... (VI)}$$

Solving Eqs. (V) and (VI),

$$N_A = 109.62 \text{ N and } N_B = 64.14 \text{ N}$$

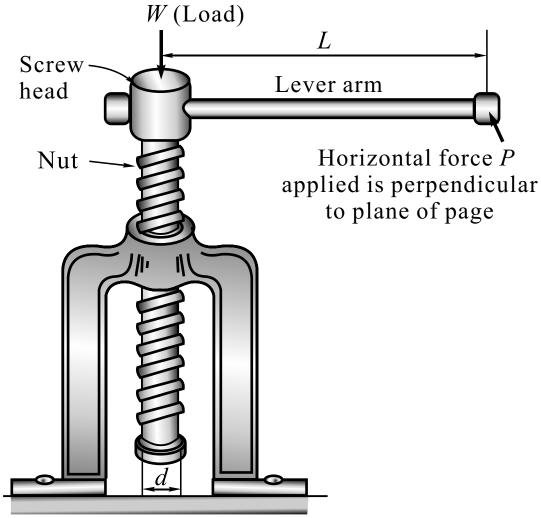
From Eq. (IV),

$P_{max} = 80.58$ N ***Ans.***

Screw Jack

It is a simple lifting machine, which is used for lifting heavy loads and its principle is same as that of inclined plane.

The machine consist of a screw and nut. The screw head carries the load W. The nut is an integral part of the screw jack. The screw is rotated by means of a lever at the end of which effort P is applied [Fig. 8.8(b)].

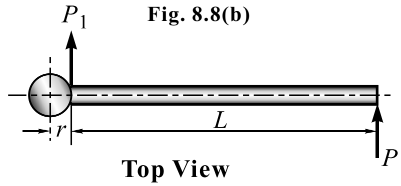


Front View
Fig. 8.8(b)

Lead : The axial advancement of screw when it completes one revolution is called *lead*.

Pitch : The distance between consecutive threads is called *pitch*. If the screw is single threaded, then lead of the screw is equal to the pitch. If the screw is double threaded then lead of the screw is twice the pitch, and so on.

Let L be the length of lever and r be the mean radius of the screw [see Fig. 8.8(c)].



Top View
Fig. 8.8(c)

If an effort P is applied at the end of lever, it is equivalent to an effort P_1 applied at the surface of the screw and is given by

$$P_1 \times r = P \times L$$

$$P = \frac{r}{L} P_1$$

Now, consider one complete revolution of the lever. The load W is lifted up by a distance p (pitch) equal to the lead of the screw (considering single threaded screw).

This can be compared to the case of an inclined plane on which a load W is moved up by a horizontal force P_1.

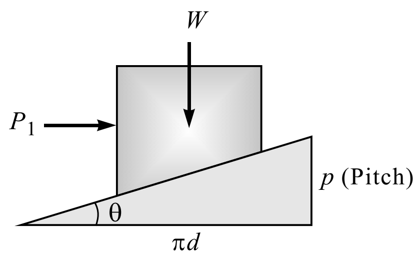
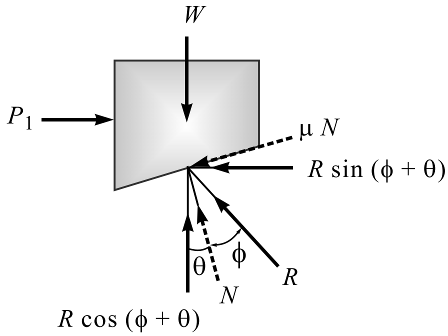


Fig. 8.8(d)

The inclination of this inclined plane with the horizontal is given by θ [Fig. 8.8(d)].

Consider FBD of Block with Load W



FBD of Block

$$\Sigma F_x = 0$$

$$P_1 = R \sin(\phi + \theta) \quad \dots \text{(I)}$$

$$\Sigma F_y = 0$$

$$W = R \cos(\phi + \theta) \quad \dots \text{(II)}$$

Dividing equation (I) by (II) gives

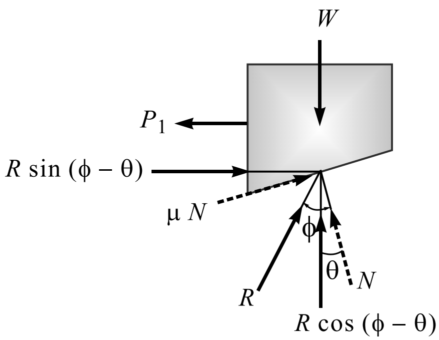
$$\frac{P_1}{W} = \tan(\phi + \theta)$$

or $P_1 = W \tan(\phi + \theta)$

From equation (I)

$$P_{(\text{Lift})} = \frac{r}{L} \times W \tan(\phi + \theta)$$

Consider FBD of Block with Load W



FBD of Block

$$\Sigma F_x = 0$$

$$P_1 = R \sin(\phi - \theta) \quad \dots \text{(III)}$$

$$\Sigma F_y = 0$$

$$W = R \cos(\phi - \theta) \quad \dots \text{(IV)}$$

Dividing equation (III) by (IV) gives

$$P_1 = W \tan(\phi - \theta)$$

But

$$P = \frac{r}{L} P_1$$

$$P_{(\text{Lower})} = \frac{r}{L} \times W \tan(\phi - \theta)$$

where P is the effort required to lower the load.

Note : If the load W lifted by the screw jack moves in the reverse direction on removal of effort P, it is called as *reversible* otherwise *self locking*. Exceptionally, if the screw jack is reversible, some effort is necessary to apply to the jack to prevent the load from moving down.

Check for reversibility of screw jack

If $\phi > \theta$, screw jack is self locking

$\phi < \theta$, screw jack is reversible

$\phi = \theta$, screw jack is at the point of reversing or self locking

For maximum size of pitch if screw jack is self-locking, then $\phi = \theta$.

Differential Screw Jack

Compared to simple screw jack, differential screw jack has advantages and more efficiency.

The arrangement is with two screws, screw inside the main screw. Main screw passes through the nut which is fixed to the body. Another screw runs inside the main screw, which means main screw is having external as well as internal threading. External threading matches with fixed nut of body and smaller screws external thread matches with internal threading on main screw.

A special lever mechanism is used with attached movable nut. When movable nut is rotated with the help of lever, main screw rotates. Rotation of inner smaller screw is prevented by special mechanism.

Consider D and d, the mean diameter of the main screw and smaller screw respectively. Let P and p be the pitch of main screw and smaller screw respectively.

For one complete rotation of lever, the axial advancement of main screw will be P, whereas simultaneously smaller screw will move axially in reverse direction by p. Since P is greater than p, therefore the net axial advancement of lifting of load will be $P - p$.

If effort is applied at the end of lever with radial length L from the axis of screw the velocity ratio is given by the relation

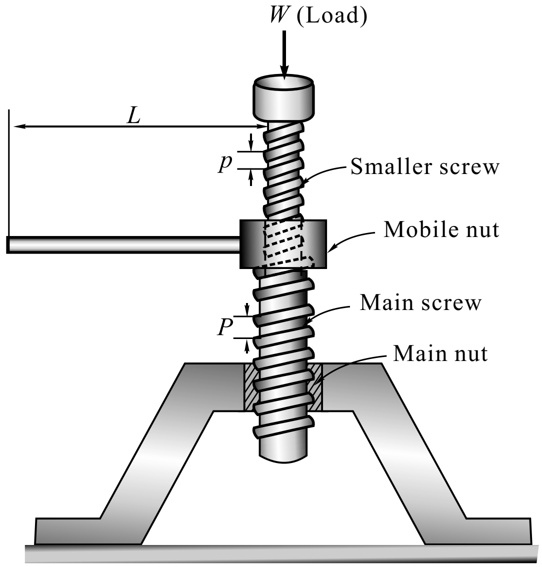


Fig. 8.8(iii)

$$VR = \frac{2\pi L}{P - p}$$

$$\text{Velocity Ratio } VR = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}}$$

$$\text{Mechanical Advantage } MA = \frac{\text{Load}}{\text{Effort}}$$

$$\text{Efficiency} = \frac{MA}{VR}$$

Problem 24

The screw press, shown in Fig. 8.24, is used in book binding. The screw bas a mean radius of 10 mm and its pitch is 5 mm. The static coefficient of friction between the threads is 0.18. If a clamping force of 1000 N is applied to the book, determine **(1)** The torque that was applied to the handle of the press. **(2)** The torque required to loosen the press.

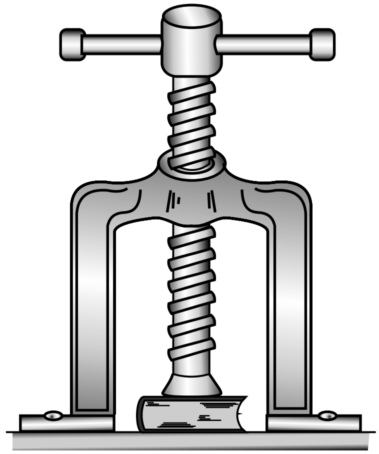


Fig. 8.24

Solution

The lead angle of the screw is

$$\theta = \tan^{-1}\frac{p}{2\pi r} = \tan^{-1}\frac{5}{2\pi \times 10} = 4.550^\circ$$

The friction angle is

$$\phi_S = \tan^{-1}\mu_S = \tan^{-1} 0.18 = 10.204^\circ$$

(1) Torque required to apply the force $W = 1000$ N

$$\begin{aligned} C_0 &= W\,r \tan(\phi_S + \theta) \\ &= 1000\,(0.01) \tan(10.204^\circ + 4.550^\circ) \\ &= 2.63 \text{ N-m} \quad \textbf{\textit{Ans.}} \end{aligned}$$

(2) Torque needed to loosen the press

$$\begin{aligned} C_0 &= W\,r \tan(\phi_S - \theta) \\ &= 1000\,(0.01) \tan(10.204^\circ - 4.550^\circ) \\ &= 0.990 \text{ N-m} \quad \textbf{\textit{Ans.}} \end{aligned}$$

Problem 25

A screw jack has square threads with a mean radius 38 mm and pitch of 15 mm. Consider $\mu = 0.06$. If the lever is of length 0.4 m find the force that has to be applied at the end of the lever to lift up a load of 8 kN. Is the screw jack self locking ? If not, find the force required at the end of the lever to prevent the load from descending.

Solution

Given : $W = 8000$ N, $\mu = 0.06$, $l = 0.4$ m,

pitch $p = 15$ mm, $r = 38$ mm.

$\therefore$ Helix angle, $\theta = \tan^{-1}\dfrac{p}{2\pi r} = \tan^{-1}\dfrac{15}{2\pi \times 38} = 3.59^\circ$

Angle of friction,

$$\phi = \tan^{-1}\mu = \tan^{-1} 0.06 = 3.43^\circ$$

Since $\theta > \phi$, the screw jack is 'not self locking'.

$$P_{\text{lift}} = \frac{W\,r}{l} \tan(\phi + \theta)$$

$$= \frac{8000 \times 0.038}{0.4} \tan(3.43^\circ + 3.59^\circ) = 93.58 \text{ N} \quad \textbf{\textit{Ans.}}$$

To prevent the load from descending,

$$P = \frac{W\,r}{l} \tan(\theta - \phi)$$

$$= \frac{8000 \times 0.038}{0.4} \tan(3.59^\circ - 3.43^\circ) = 2.12 \text{ N} \quad \textbf{\textit{Ans.}}$$

Problem 26

A screw jack with single start square threads has outside and inside diameter of the thread as 68 mm and 52 mm respectively. The coefficient of friction is 0.1 for all pairs of surfaces in contact. If the length of the lever is 0.5 m, find the force required to lift a load of 2 kN.

Solution

Given : $W = 2000$ N, $\mu = 0.1$, $l = 0.5$ m

For the screw jack, $d_0 = 68$ mm and $d_i = 52$ mm

For square threads,

$$\text{Mean diameter, } d = \frac{d_0 + d_i}{2} = \frac{68 + 52}{2} = 60 \text{ mm}$$

$$\therefore \text{ Mean radius, } r = 30 \text{ mm}$$

$$\text{Pitch} = d_0 - d_i = 68 - 52 = 16 \text{ mm}$$

$$\therefore \text{ Helix angle, } \theta = \tan^{-1} \frac{p}{2\pi\,r} = \tan^{-1} \frac{16}{2\pi \times 30} = 4.859^\circ$$

Angle of friction,

$$\phi = \tan^{-1} \mu = \tan^{-1} 0.1 = 5.713^\circ$$

Since $\phi > \theta$, the screw jack is 'self locking'.

$$P_{\text{lift}} = \frac{W\,r}{l} \tan(\phi + \theta)$$

$$= \frac{2000 \times 0.03}{0.5} \tan(5.71^\circ + 4.85^\circ) = 22.37 \text{ N} \quad \textbf{\textit{Ans.}}$$

Problem 27

A screw jack has threads with mean diameter 58 cm. Considering the coefficient of friction between the screw and the nut to be 0.12, find the maximum pitch of the thread so that the screw jack is self locking.

Solution

Given : $r = 29$ cm, $\mu = 0.12$

For a self locking screw jack $\phi > \theta$

$$\therefore \tan^{-1} \mu > \tan^{-1} \frac{p}{2\pi\,r}$$

For the range of values of μ commonly encountered,

$y = \tan^{-1} x$ is an increasing function.

$$\therefore \quad \mu > \frac{p}{2\pi r}$$

$$\therefore \; 0.12 > \frac{p}{2\pi \times 30}$$

$$\therefore \; 0.12 > p \text{ (Pitch)}$$

Hence for being self-locking, the maximum permissible pitch is 21.87 cm.

Exercises

[I] Problems

1. A wooden block rests on a horizontal plane, as shown in Fig. 8.E1. Determine the force P required to just impend motion. Assume the weight of block as 100 N and the coefficient of friction $\mu = 0.4$.

[***Ans.*** $P = 37.4$ N]

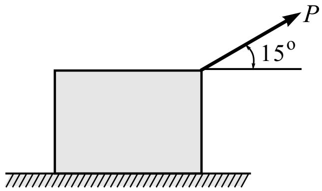


Fig. 8.E1

2. A 100 N force acts, as shown in Fig. 8.E2, on a 30.6 kg block on a inclined plane. The coefficient of friction (static and kinetic) between the block and the plane are $\mu_s = 0.25$ and $\mu_k = 0.20$, respectively. Determine whether the block is in equilibrium and find the value of the friction force. Take $g = 9.81$ m/sec^2.

[***Ans.*** $F = 48$ N , block will slide down.]

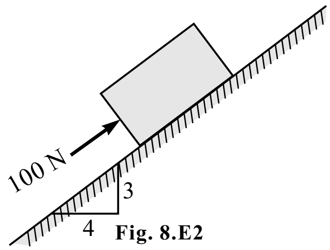


Fig. 8.E2

3. Block of weight 1000 N is kept on an inclined plane as shown in Fig. 8.E3. A force P is applied parallel to plane to keep the body in equilibrium. Determine range of values of P for which the block will be in equilibrium.

[***Ans.*** $370.1 \leq P \leq 629.9$ N]

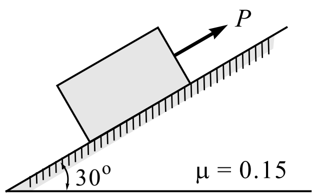


Fig. 8.E3

4. Block A of weight 2000 N is kept on an inclined plane at 35° as shown in Fig. 8.E4. It is connected to weight B by an inextensible string passing over smooth pulley. Determine the weight of B so that B just moves down. $\mu = 0.2$.

[***Ans.*** 1463.1 N]

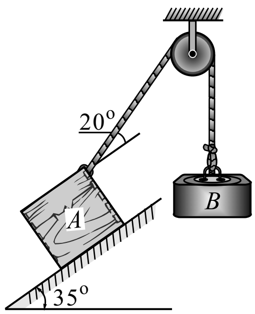


Fig. 8.E4

5. In Fig. 8.E5, the two blocks (W_1 = 30 N and W_2 = 50 N) are placed on rough horizontal plane. Coefficient of friction between the block A and plane is 0.3 and that between bock B and plane is 0.2. Find the minimum value of the force P to just move the system. Also find the tension in the string.

[**Ans.** P = 19.67 N and T = 9 N.]

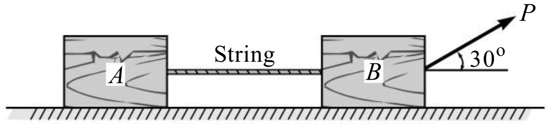


Fig. 8.E5

6. In Fig. 8.E6, weights of two blocks A and B are 100 N and 150 N respectively. Find the smallest value of the horizontal force F to just move the lower block B if

(a) the block is restrained by a string and

(b) when string is removed.

[**Ans.** (a) 82.5 N and (b) 62.5 N.]

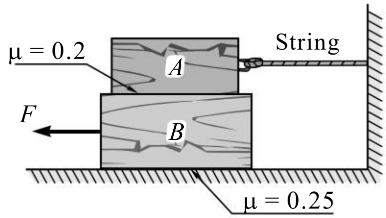


Fig. 8.E6

7. Determine the necessary force P acting parallel to the plane to cause motion to impend, as shown in Fig. 8.E7. Assume coefficient of friction as 0.25 and the pulley to be smooth.

[**Ans.** P = 98.85 N]

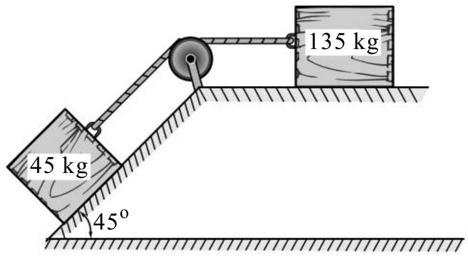


Fig. 8.E7

8. A weight 500 N just starts moving down a rough inclined plane supported by a force of 200 N acting parallel to the plane and it is at the point of moving up the plane when pulled by a force of 300 N parallel to the plane. Find the inclination of the plane and the coefficient of friction between the inclined plane and the weight.

[**Ans.** 30° and 0.1155.]

9. A cord connects two bodies A and B of weights 450 N and 900 N. The two bodies are laced on an inclined plane and the cord is parallel to inclined plane. The coefficient of friction for body - A is 0.16 and that for B is 0.42. Determine the inclination of the plane to the horizontal and tension in the cord when motion is about to take place down the plane.

[**Ans.** θ = 18.434° and T = 73.98 N.]

10. Two rectangular blocks of weights W_1 and W_2 are connected by a flexible cord and rest upon a horizontal and inclined plane respectively with the cord passing as shown in Fig. 8.E10. Taking a particular case, where W_1 = W_2 and coefficient of friction μ is same for all contact surfaces, find the angle of inclination of the inclined plane at which motion of the system will impend.

[**Ans.** $\alpha = 2 \tan^{-1} \mu$]

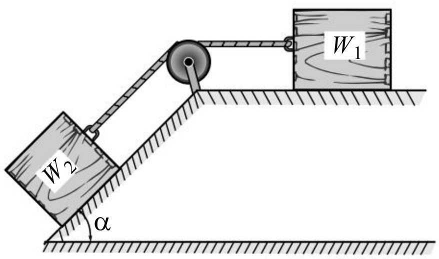


Fig. 8.E10

11. Two inclined planes AC and BC inclined at 60^o and 30^o to the horizontal meet at a ridge C, as shown in Fig. 8.E11. A mass of 100 kg rests on the inclined plane BC and is tied to a rope, which passes over a smooth pulley at the ridge, the other end of the rope, being connected to a block of W kg mass resting on the plane AC. Determine the least and greatest value of W for the equilibrium of the whole system.

[**Ans.** W_{min} = 243.88 N and W_{max} = 973.05 N.]

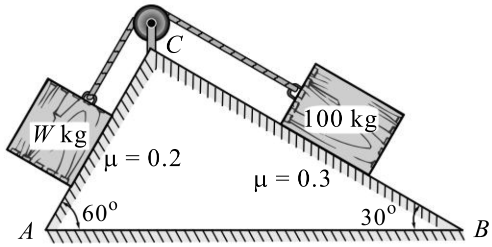


Fig. 8.E11

12. Find the tensions in the cords of the inclined plane system shown in Fig. 8.E12.

[**Ans.** T_1 = 165.36 kN and T_2 = 80.51 kN.]

Frictionless pulley

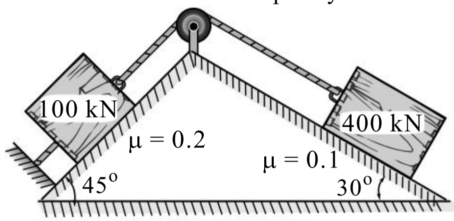


Fig. 8.E12

13. A 2.04 kg block A and a 3.06 kg block B are supported by an inclined plane, which is held in position shown in Fig. 8.E13. Knowing that the coefficient of friction is 0.15 between the two blocks and zero between block B and the incline, determine the value of θ for which motion is impending.

[**Ans.** θ = 31^o]

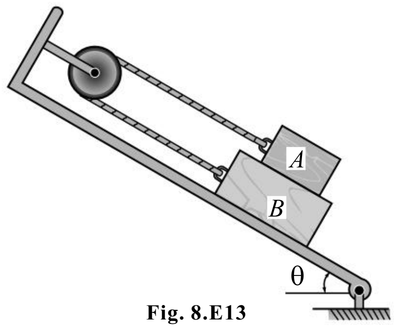


Fig. 8.E13

14. Find the least value of P that will just start the system of blocks shown in Fig. 8.E14. moving to the right. The coefficient of friction under each block is 0.30.

[**Ans.** P_{min} = 247.12 N at α = 16.7^o]

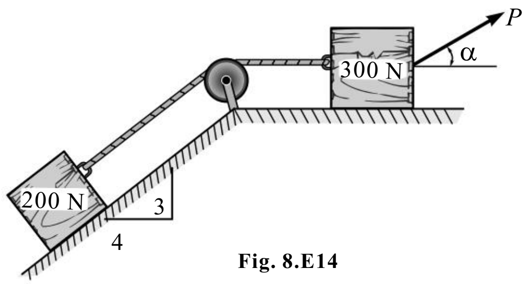


Fig. 8.E14

15. Find the weight W_B if the weight W_A = 20 kN is to be kept in equilibrium with pin-connected rod AB in horizontal position. Find also maximum value of W_B for the same purpose. Find, therefore, the range of values of axial force in the rod AB. Refer to Fig. 8.E15.

[**Ans.** $W_{B(min)}$ = 4.511 kN,
$W_{B(max)}$ = 26.365 kN and
6.766 kN (C) to 17.577 kN (C).]

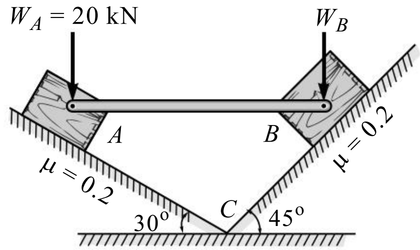


Fig. 8.E15

16. Two blocks *A* and *B* weighing 800 N and 1000 N, respectively rest on two inclined planes, each inclined at 30° to the horizontal. They are connected by a rope passing through a smooth pulley at the valley.

Ropes carrying loads W_1 and 5000 N (W_2) and passing over pulleys at the tops of the planes are also connected to the two blocks, as shown in Fig. 8.E16. Coefficient of friction μ may be taken as 0.1 and 0.2 for blocks *A* and *B*, respectively. Determine the least and greatest value of W_1 for the equilibrium of the whole system.

[***Ans.*** $W_{1(min)}$ = 4657.52 N and $W_{1(max)}$ = 5142.48 N.]

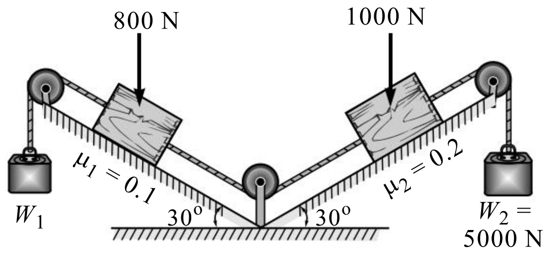


Fig. 8.E16

17. Two slender rods of negligible weight are pin-connected at *A* and attached to the 200 N block *B* and the 600 N block *C*, as shown. The coefficient of static friction is 0.6 between all surfaces of contact. Determine the range of values of *P* for which equilibrium is maintained. Refer to Fig. 8.E17.

[***Ans.*** 1913.9 N to 3150 N]

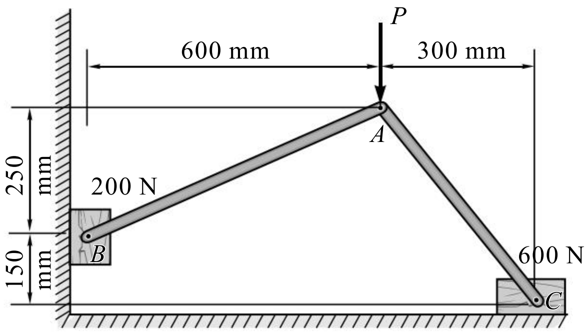


Fig. 8.E17

18. Determine the range of values of *P* for which equilibrium of the block, shown in Fig. 8.E18, is maintained (μ_S = 0.25, μ_K = 0.2).

[***Ans.*** 143.03 ≤ *P* ≤ 483.46 N]

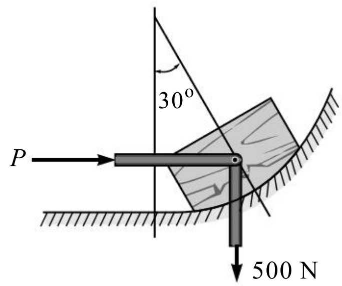


Fig. 8.E18

19. Block *A* has a mass of 20 kg and block *B* has a mass of 10 kg in Fig. 8.E19. Knowing that μ_s = 0.15 between all surfaces of contact, determine the value of θ for which motion will impend. Take $g = 10$ m/s^2.

[***Ans.*** θ = 46.4°]

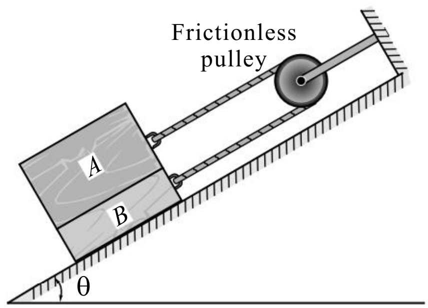


Fig. 8.E19

20. Find the least force P to start motion of any part of the system of three blocks resting upon one another, as shown in Fig. 8.E20. The weights of the blocks are W_A = 300 N, W_B = 100 N and W_C = 200 N. The coefficient of friction between A and B is 0.3, between B and C is 0.2 and between C and the ground is 0.1.

[*Ans.* 57.143 N]

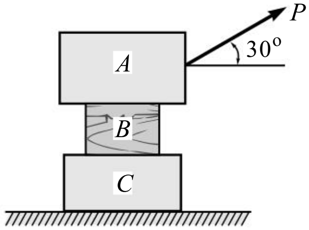


Fig. 8.E20

21. The three flat blocks are positioned on the 30° incline, as shown in Fig. 8.E21, and a force P parallel to the incline is applied to the middle block. The upper block is prevented from moving by a wire which attaches it to the fixed support. The coefficient of static friction for each of the three pairs of surfaces is shown. Determine the maximum value which P may have before any slipping takes place. Take $g = 10$ m/s^2.

[*Ans.* P = 95.58 N]

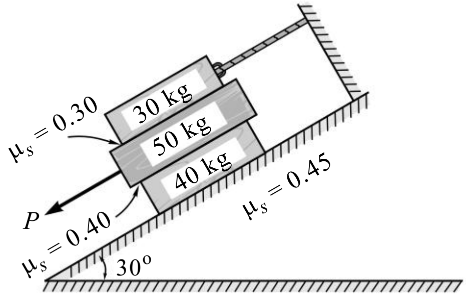


Fig. 8.E21

22. Two rectangular blocks of weight, W_1 = 150 N and W_2 = 100 N, are connected by a string and rest on an inclined plane and on a horizontal surface as shown in Fig. 8.E22. The coefficient of friction for all contiguous surfaces is μ = 0.2. Find the magnitude and direction of the least force P at which the motion of the blocks will impend.

[*Ans.* P = 161.7 N and θ = 11.31°.]

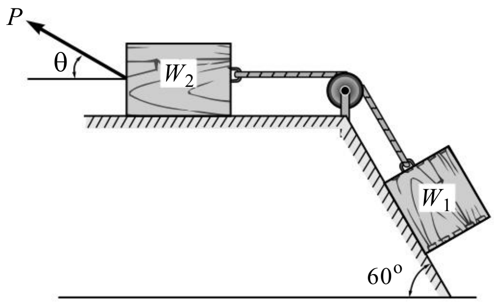


Fig. 8.E22

23. A 1500 N cupboard is to be shifted to the right by a horizontal force P, as shown in Fig. 8.E23. Find the force P required to just cause the motion and the maximum height up to which it can be applied. Take μ = 0.25.

[*Ans.* P = 375 N and h = 1.75 m.]

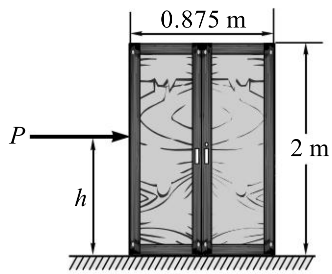


Fig. 8.E23

24. Two block A and B each weighing 1500 N are connected by a uniform horizontal bar which weighs 1000 N, as shown in Fig. 8.E24. If the angle of limiting friction under each block is 15°, find the force P directed parallel to the 60° inclined plane that will cause motion impending to the right.

[*Ans.* P = 1856.4 N]

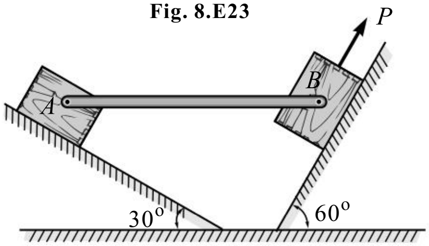


Fig. 8.E24

25. Refer to Fig. 8.E25, where a 8.15 kg block is attached to link *AB* and rests on a moving belt. Knowing that $\mu_s = 0.25$ and $\mu_k = 0.20$, determine the magnitude of the horizontal force *P* which should be applied to the belt to maintain its motion (a) to the right, (b) and to the left.

[***Ans.*** (a) $P = 18.08$ N and (b) $P = 14.34$ N.]

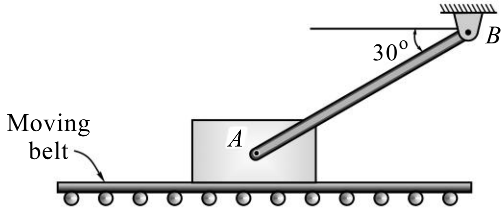


Fig. 8.E25

26. Block *A* has a mass of 20 kg and block *B* has a mass of 10 kg, as shown in Fig. 8.E26. Coefficient of static friction between the blocks is 0.15 and between the block *B* with the slope is zero. Find the existing frictional force between the blocks. What is the force in the string ?

[***Ans.*** $T = 75$ N and $F = 25$ N.]

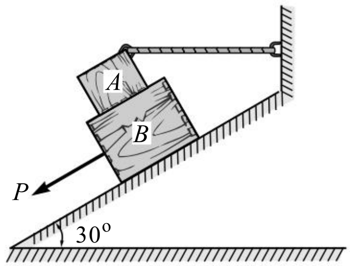


Fig. 8.E26

27. Refer to Fig. 8.E27, where a 45 kg disk rests on the surface for which the coefficient of static friction is $\mu = 0.2$, determine the largest couple moment *M* that can be applied to the bar without causing motion.

[***Ans.*** $M = 77.3$ Nm]

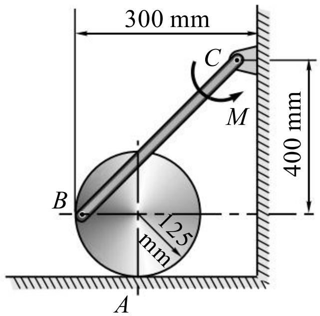


Fig. 8.E27

28. A uniform rod *AB* of length 10 m and weight 280 N is hinged at *B* and end *A* rests on a block weighing 400 N, as shown in Fig. 8.E28. If $\mu = 0.4$ for all contact surfaces, find horizontal force *P* required to start moving 400 N block.

[***Ans.*** $P = 320$ N]

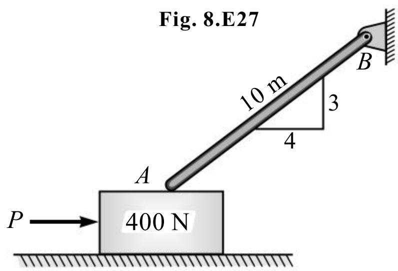


Fig. 8.E28

29. The beam *AB* has a negligible mass and thickness and is subjected to a triangular distributed loading. It is supported at one end by a pin and at the other end by a post having a mass of 50 kg and negligible thickness, as shown in Fig. 8.E29. Determine the minimum force *P* needed to move the post. The coefficients of static friction at *B* and *C* are $\mu_B = 0.4$ and $\mu_C = 0.2$, respectively.

[***Ans.*** $P = 355$ N]

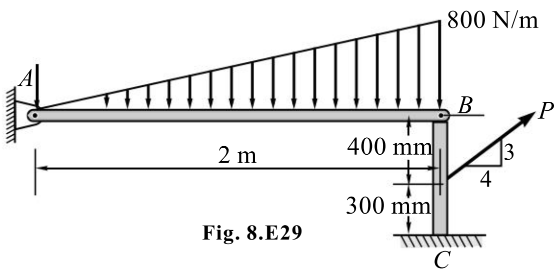


Fig. 8.E29

30. A 60 kg cupboard is to be shifted to the right, μ_s between cupboard and floor is 0.35, as shown in Fig. 8.E30. Determine

(a) the force P required to move the cupboard and

(b) The largest allowable value of h if the cupboard is not to tip over.

[***Ans.*** P = 206 N and h = 714 mm.]

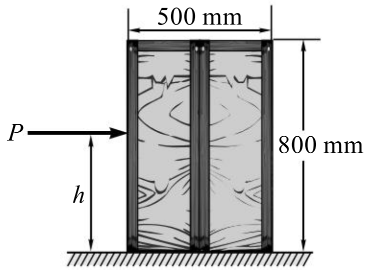


Fig. 8.E30

31. A cupboard of 750 N weight is placed over an inclined plane with μ = 0.20, as shown in Fig. 8.E31. Find the range of values of h where force P may be applied parallel to inclined plane to hold it in equilibrium.

[***Ans.*** 0.213 m and 2.014 m.]

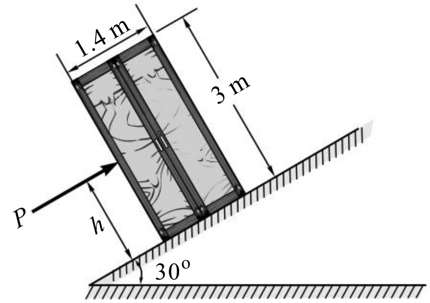


Fig. 8.E31

32. Referring to Fig. 8.E32, the coefficients of friction are as follows : 0.25 at the floor, 0.3 at the wall and 0.2 between the blocks. Find the minimum values of a horizontal force P, applied to the lower block that will hold the system in equilibrium.

[***Ans.*** P = 81.02 N]

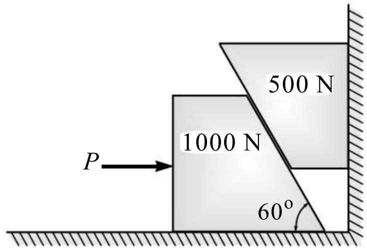


Fig. 8.E32

33. Block A weighs 25 kN and block B 18 kN in Fig. 8.E33. μ for all surfaces is 0.11. For what range of values of P will the system be in equilibrium.

[***Ans.*** P_{min} = 45.8 kN and P_{max} = 124 kN.]

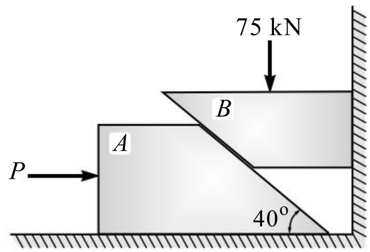


Fig. 8.E33

34. Refer to Fig. 8.E34 and draw the FBD for different bodies and find the minimum value of force F to move the block A up the plane.

[***Ans.*** 134.6 N]

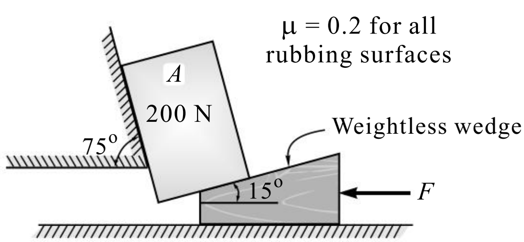


Fig. 8.E34

35. Determine the force P required to start the motion of wedge shown in Fig. 8.E35.
Take $\mu = 0.26$ for all surfaces.

[*Ans.* $P = 1464.33$ N]

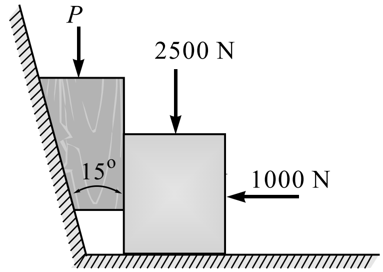


Fig. 8.E35

36. Calculate the force P required to initiate the motion of the 24 kg block up the 10° incline, as shown in Fig. 8.E36. The coefficient of static friction for each pair of surfaces is 0.3. Assume $g = 10$ m/s^2.

[*Ans.* $P = 224.36$ N]

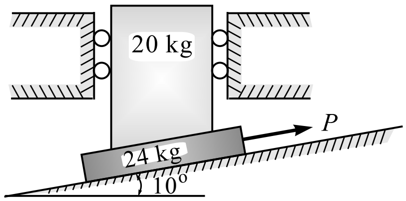


Fig. 8.E36

37. What horizontal force P on the wedge B and C is necessary to raise 200 kN resting on A, as shown in Fig. 8.E37? Assume that coefficient of friction μ between the wedges and the ground is 0.25 and between wedges and A is 0.2. Also assume symmetry.

[*Ans.* $P = 55.665$ N]

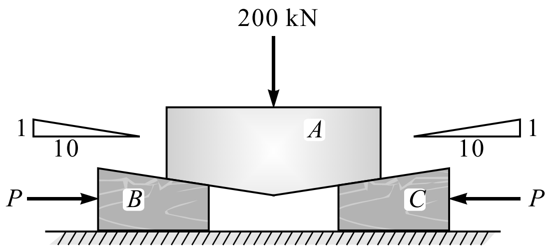


Fig. 8.E37

38. A horizontal force of 5 kN is acting on the wedge, as shown in Fig. 8.E38. The coefficient of friction at all rubbing surfaces is 0.25. Find the load W which can be held in position. The weight of block B may be neglected.

[*Ans.* $W = 22.89$ kN]

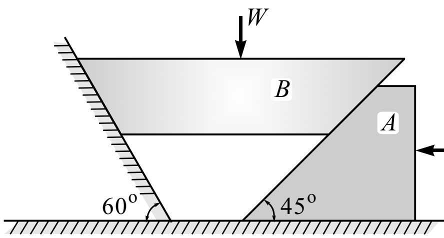


Fig. 8.E38

39. A 15° wedge of negligible weight is driven to tighten a body B which is supporting a vertical load of 1000 N, as shown in Fig. 8.E39. If the coefficient of friction for all contacting surfaces be 0.25, find the minimum force P required to drive the wedge.

[*Ans.* $P = 232$ N]

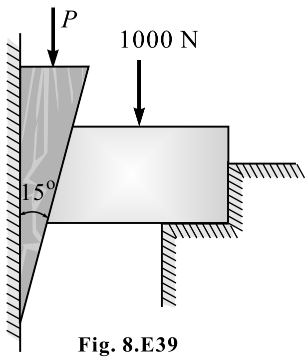


Fig. 8.E39

40. A ladder of length 4 m weighing 200 N is placed against a vertical wall, as shown in Fig. 8.E40. The coefficient of friction between the wall and the ladder is 0.2 and that between the ladder and the floor is 0.3. The ladder in addition to its own weight has to support a man weighing 600 N at a distance of 3 m from A. Calculate the minimum horizontal force to be applied at A to prevent slipping.

[***Ans.*** $P = 61.76$ N]

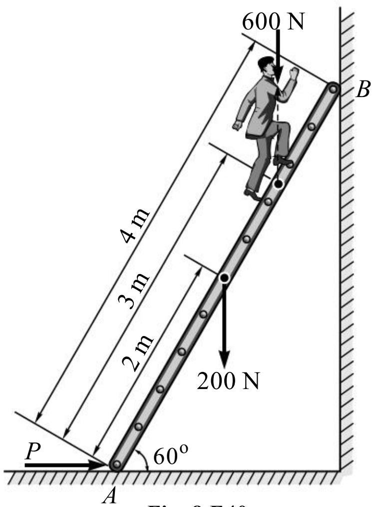


Fig. 8.E40

41. The ladder shown in Fig. 8.E41 is 6 m long and is supported by a horizontal floor and vertical wall. The coefficient of friction between the floor and the ladder is 0.4 and between wall and ladder is 0.25. The weight of ladder is 200 N and may be considered a concentrated at G. The ladder also supports a vertical load of 900 N at C, which is at a distance of 1 m from B. Determine the least value of α at which the ladder may be placed without slipping. Determine the reaction at that

[***Ans.*** $N_A = 1000$ N, $F_A = 400$ N,
$N_B = 400$ N, $F_B = 100$ N and $\alpha = 61.927^\circ$.]

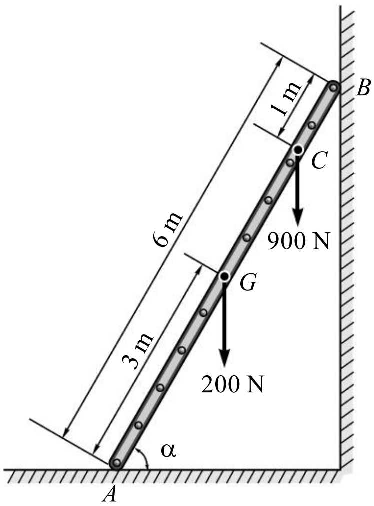


Fig. 8.E41

42. A horizontal bar 1 m long and of negligible weight rests on rough inclined plane, as shown in Fig. 8.E42. If the angle of friction is 15°, determine the minimum value of x at which the load Q = 200 N may be applied before slipping occurs.

[***Ans.*** $x = 0.35$ m]

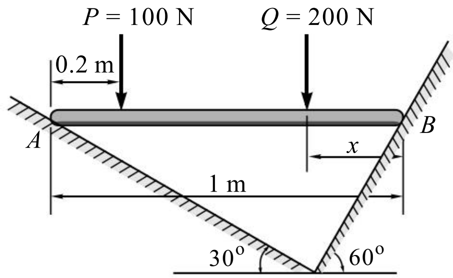


Fig. 8.E42

43. The pitch of the screw of a jack is 10 mm and mean diameter of the thread is 60 mm and the length of the lever is 500 mm. Find the effort required **(i)** to lift a load of 10 kN **(ii)** to lower the same load. Take $\mu = 0.08$.

[*Ans.* **(i)** 80.17 N **(ii)** 16.1 N]

44. A screw jack carries a load of 4000 N. The mean diameter of the screw rod is 50 mm and the pitch of the square threads is 20 mm. If the coefficient of friction is 0.22, find the torque required, **(i)** to raise the load **(ii)** to lower the load.

[*Ans.* **(i)** 35.73 Nm **(ii)** 9.015 Nm]

45. A square threaded screw press is used in book binding for compressing the books. The pitch of the screw is 5 mm and the mean diameter is 50 mm. The screw is double threaded. If a force of 100 N is applied horizontally, at the end of lever, 150 mm long, find the compressive force with which the books are pressed. Take $\mu_s = 0.08$.

[*Ans.* 4155 N]

46. A screw jack has square threads with 75 mm mean diameter and 15 mm pitch. The load on the jack revolves with the screw. The coefficient of friction at the screw threads is 0.05.

Find the tangential force applied to the jack at 360 mm radius so as to lift a load of 6000 N.

State whether the jack is self locking. If it is, find the torque required to lower the load. If not, find the torque which must be applied to keep the load from descending.

[*Ans.* 71.265 N, machine is not self locking, 30 64 Nmm]

[II] Review Questions

1. What is meant by frictional force ?

2. State the laws of friction.

3. Describe the following terms :

(a) Limiting friction (b) Static friction (c) Dynamic friction

(d) Coefficient of friction (e) Angle of friction (f) Angle of repose

(g) Cone of friction

4. Why is the coefficient of static friction greater than the coefficient of kinetic friction ?

5. Discuss the merits and demerits of friction.

6. What is the use of wedge ?

[III] Fill in the Blanks

1. A wedge is generally used to lift a heavy load by a ________ distance.

2. The inverted cone with semicentral angle equal to the limiting frictional angle ϕ is called the ________ .

3. The minimum angle of inclination of a plane with horizontal at which the body is about to slide on its own is called the ________.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. The cause of friction between two surfaces is _________.

(a) material **(b)** roughness **(c)** material and roughness **(d)** None of these

2. Limiting frictional force is directly proportional to _________.

(a) weight **(b)** mass **(c)** area **(d)** normal reaction

3. Frictional force is independent of the _______.

(a) only area **(b)** only speed **(c)** area and speed **(d)** None of these

4. Coefficient of static friction is always ________ than the coefficient of kinetic friction.

(a) greater **(b)** less **(c)** equal **(d)** zero

5. Angle of repose is ________ to angle of friction.

(a) less **(b)** equal **(c)** greater **(d)** zero

6. Angle made by the resultant of the limiting frictional force and the normal reaction with normal reaction is called the ________.

(a) angle of friction **(b)** angle of repose

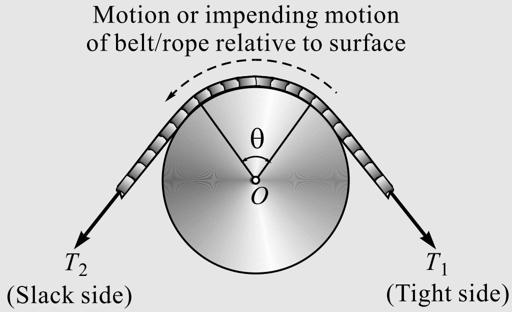
(c) angle of inclination **(d)** angle of limiting friction

7. If the inclination of the plane with horizontal is less than angle of friction then the block kept on the incline will ________ .

(a) move downward **(b)** move upward **(c)** be in equilibrium **(d)** be in motion

9

Belt and Rope Friction



9.1 Concept of Belt and Rope Friction

9.1.1 Transmission of Power Through Belt Drives

Belt are extensively used to transmit power from one shaft to another shaft. The shaft of pulley *A* rotates the pulley which then drives the pulley *B* due to the belt which runs around the two pulleys. Their working is dependent on the friction acting between the pulley surface and belt surface.

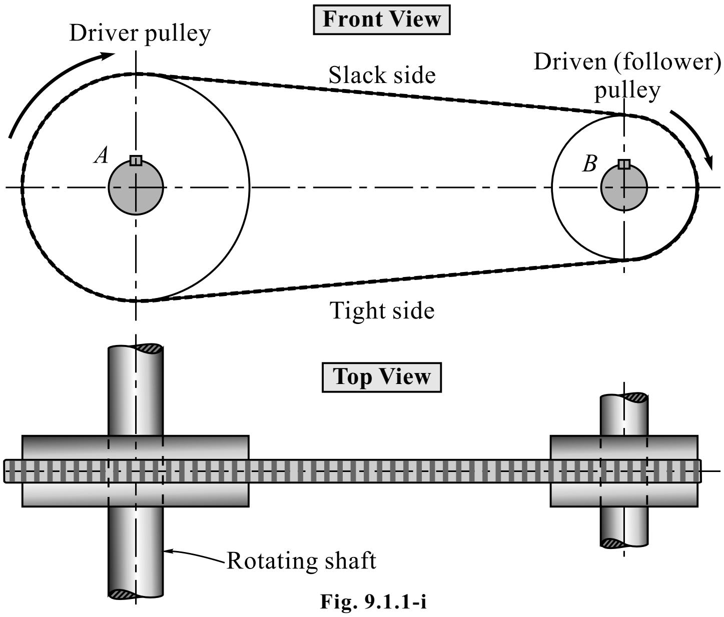


Fig. 9.1.1-i

In the above case, pulley *A* is called the *driver* and pulley *B* is called a *driven* (*follower*).

The belts may be flat belt, V-belt or circular (rope) belt.

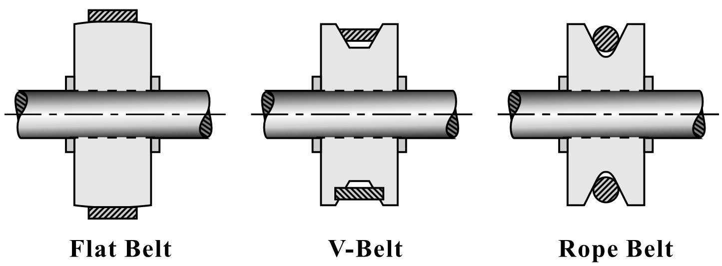


Fig. 9.1.1-ii

Types of Flat Belts

(1) Open-Belt Drive

(2) Cross-Belt Drive

(1) Open-Belt Drive

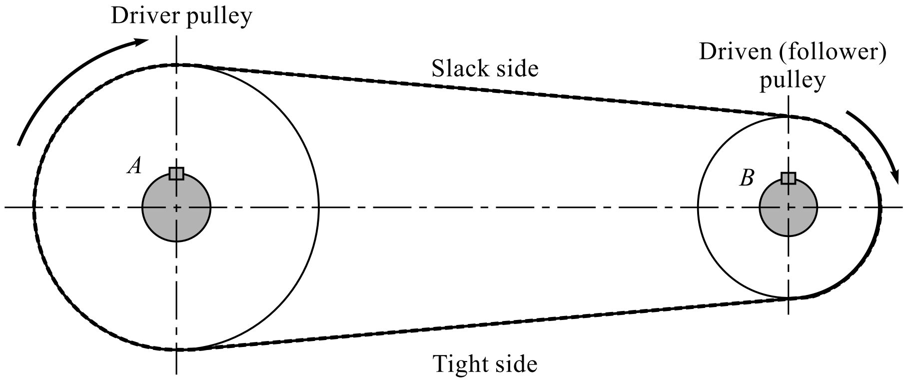


Fig. 9.1.1-iii

It consist of two pulleys *A* and *B*. One is known as driver pulley and is keyed to the rotating shaft. The other is known as driven (follower) pulley and is keyed to the shaft which is to be rotated.

Shafts are placed parallel to each other and both the pulleys rotate in the same direction. The belt is pulled by the driver from lower side and delivered to the upper side. This results in higher tension on the lower side than upper side.

Velocity Ratio

It is defined as the ratio of the velocity of the driven (follower) to the velocity of the driver.

Let N_1 = Speed of the driver in rpm

d_1 = Diameter of the driver

N_2 = Speed of the driven in rpm

d_2 = Diameter of the driven

Length of the belt passing over the two pulleys per minute will be the same

$$\pi d_1 N_1 = \pi d_2 N_2$$

$$\frac{N_1}{N_2} = \frac{d_2}{d_1}$$

$\therefore$ Velocity ratio is

$$\frac{N_2}{N_1} = \frac{d_1}{d_2}$$

The above relation shows that speed of the pulley is inversely proportional to the diameter of the pulley.

Length of the Open-Belt Drive

Total length of the belt is the sum of three parts,

1. the length of belt in contact with the driver (larger) pulley (*APB*),
2. the length of belt in contact with the driven (smaller) pulley (*CQD*), and
3. the length of belt NOT in contact with the driver and driven pulley ($BC + AD$).

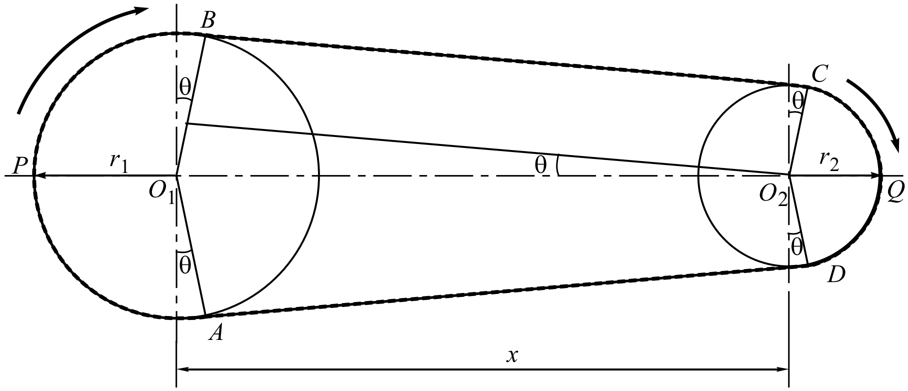


Fig. 9.1.1-iv : Length of the Open-Belt

Let x = Distance between the centres of the two pulleys (O_1 and O_2)

r_1 = Radius of the driver (larger) pulley

r_2 = Radius of the driven (smaller) pulley

L = Total length of the belt ($APB + BC + CQD + DA$)

From O_2 draw a line parallel to BC which will be perpendicular to O_1B at M

$$\therefore\ O_1M = r_1 - r_2$$

Consider the right angled ΔO_1MO_2, we have

$$BC = MO_2 = \sqrt{x^2 - (r_1 - r_2)^2} = x\sqrt{1 - \left(\frac{r_1 - r_2}{x}\right)^2}$$

$$BC = x\left[1 - \left(\frac{r_1 - r_2}{x}\right)^2\right]^{1/2}$$

By binomial theorem, we have

$$BC = x\left[1 - \frac{1}{2}\left(\frac{r_1 - r_2}{x}\right)^2 + \ldots\ldots\right] \quad \text{Further terms are negligible.}$$

$$BC = x - \frac{(r_1 - r_2)^2}{2x}$$

Total length, L = Arc APB + Arc CQD + BC + AD (But $BC = AD$)

$$L = 2(\text{Arc } PB + \text{Arc } CQ + BC)$$

$$L = 2\left[r_1\left(\frac{\pi}{2}+\theta\right)+r_2\left(\frac{\pi}{2}-\theta\right)+x-\frac{(r_1-r_2)^2}{2x}\right]$$

$$= 2\left[r_1\frac{\pi}{2}+r_2\frac{\pi}{2}+r_1\theta-r_2\theta+x-\frac{(r_1-r_2)^2}{2x}\right]$$

$$= \pi(r_1+r_2)+2\theta(r_1-r_2)+2x-\frac{(r_1-r_2)^2}{x}$$

But $\sin\theta = \frac{r_1-r_2}{x}$; since θ is very small we can assume $\sin\theta = \theta$

$$\therefore\ \theta = \frac{r_1-r_2}{x}$$

$$\therefore\ L = \pi(r_1+r_2)+2\,\frac{(r_1-r_2)^2}{x}+2x-\frac{(r_1-r_2)^2}{x}$$

$$\therefore\ L = \pi(r_1+r_2)+\frac{(r_1-r_2)^2}{x}+2x$$

Length of Cross-Belt Drive

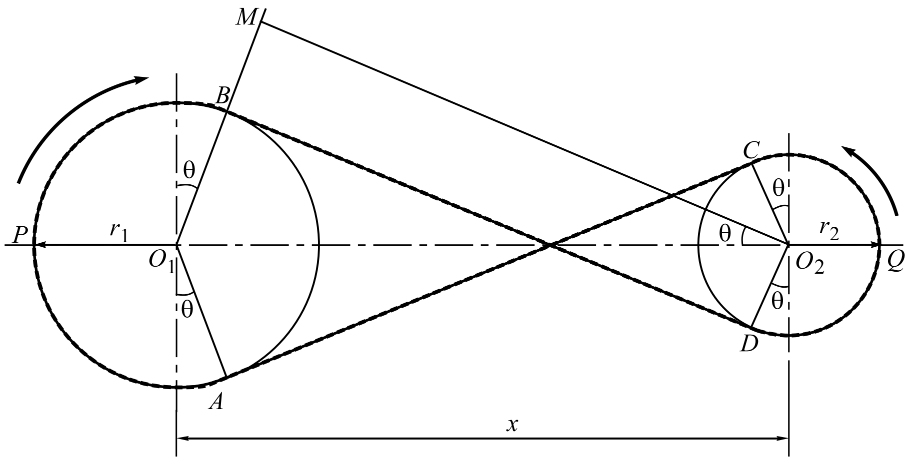


Fig. 9.1.1-v : Length of Cross-Belt Drive

Total length of the belt is sum of three parts :

1. the length of belt in contact with the driver (larger) pulley (*APB*),
2. the length of belt in contact with the driven (smaller) pulley (*CQD*) and
3. the length of belt NOT in contact with the driver and driven pulley (*BD* + *AC*).

Let x = Distance between the centres of the two pulleys (O_1 and O_2)

r_1 = Radius of the driver (larger) pulley

r_2 = Radius of the driven (smaller) pulley

L = Total length of the belt ($APB + BD + CQD + CA$)

From O_2 draw a line parallel to BC which will be perpendicular to O_1B at M

$$\therefore\ O_1M = (r_1 + r_2)$$

Consider the right angled ΔO_1MO_2, we have

$$BD = MO_2 = \sqrt{x^2 - (r_1 + r_2)^2} = x\sqrt{1 - \left(\frac{r_1 + r_2}{x}\right)^2}$$

$$BD = x\left[1 - \left(\frac{r_1 - r_2}{x}\right)^2\right]^{1/2}$$

By binomial theorem, we have

$$BD = x\left[1 - \frac{1}{2}\left(\frac{r_1 + r_2}{x}\right)^2 + ...\right] \quad \text{Further terms are negligible.}$$

$$BD = x - \frac{(r_1 + r_2)^2}{2x}$$

Total length, $L = \text{Arc } APB + \text{Arc } CQD + BD + AC$ (But $BC = AD$)

$$L = 2(\text{Arc } PB + \text{Arc } CQ + BD)$$

$$L = 2\left[r_1\left(\frac{\pi}{2} + \theta\right) + r_2\left(\frac{\pi}{2} + \theta\right) + x - \frac{(r_1 + r_2)^2}{2x}\right]$$

$$= 2\left[r_1\frac{\pi}{2} + r_2\frac{\pi}{2} + r_1\theta + r_2\theta + x - \frac{(r_1 + r_2)^2}{2x}\right]$$

$$= \pi(r_1 + r_2) + 2\theta(r_1 + r_2) + 2x - \frac{(r_1 + r_2)^2}{x}$$

But $\sin\theta = \dfrac{r_1 + r_2}{x}$; since θ is very small, we can assume $\sin\theta = \theta$

$$\therefore\ \theta = \frac{r_1 + r_2}{x}$$

$$\therefore\ L = \pi(r_1 + r_2) + 2\,\frac{(r_1 + r_2)^2}{x} + 2x - \frac{(r_1 + r_2)^2}{x}$$

$$\therefore\ L = \pi(r_1 + r_2) + \frac{(r_1 + r_2)^2}{x} + 2x$$

Belt Friction for Flat Belt

When a belt/rope passes over a fixed pulley with friction, tension in the belt/rope on two sides will be different due to friction.

Let T_1 = Tension in belt/rope on tight side

T_2 = Tension in belt/rope on slack side

μ = Coefficient of friction between belt/rope and pulley

θ = Angle of contact of belt with pulley

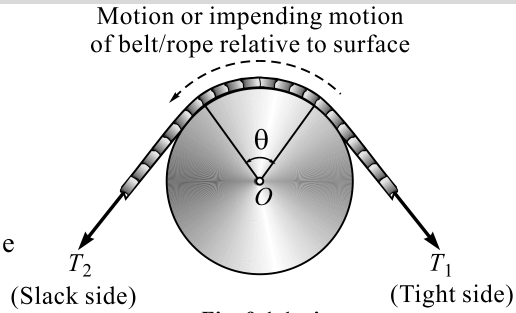


Fig. 9.1.1-vi

Relation between T_1 and T_2 can be given as

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

Derivation

Consider a belt/rope wrapped around a fixed pulley with friction. The slack side tension T_2 increases along the surface of contact to T_1 tight side tension because of friction.

Consider the FBD of an element of belt subtending an angle $d\theta$ at the centre. Let T be the tension on one face of element which increases to $(T + dT)$ on the other face.

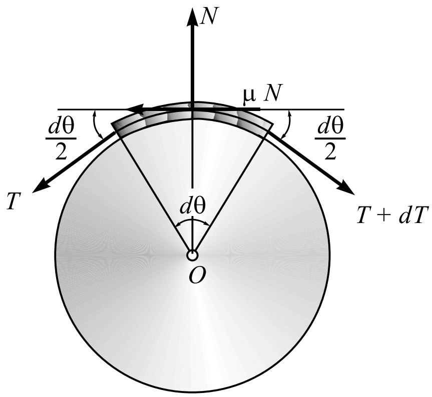


Fig. 9.1.1-vii

$$\Sigma F_x = 0$$

$$(T + dT)\cos\left(\frac{d\theta}{2}\right) - \mu N - T\cos\left(\frac{d\theta}{2}\right) = 0$$

$$dT\cos\left(\frac{d\theta}{2}\right) = \mu N$$

Now,

$$\cos\left(\frac{d\theta}{2}\right) = 1 \qquad \{\because\ d\theta \text{ is very small}\}$$

$$\therefore\ dT = \mu N \qquad \text{... (I)}$$

$$\Sigma F_y = 0$$

$$N - T\sin\left(\frac{d\theta}{2}\right) - (T + dT)\sin\left(\frac{d\theta}{2}\right) = 0$$

Now,

$$\sin\left(\frac{d\theta}{2}\right) = \frac{d\theta}{2} \qquad \{\because\ d\theta \text{ is very small}\}$$

$$T\left(\frac{d\theta}{2}\right) + (T + dT)\left(\frac{d\theta}{2}\right) = N$$

$$\therefore\ T\,d\theta = N \quad \{\text{neglecting product of } dT \text{ and } d\theta\} \qquad \text{... (II)}$$

From Eqs. (I) and (II), we get

$$dT = \mu\, T\, d\theta$$

$$\frac{dT}{T} = \mu\, d\theta$$

Integrating both sides

$$\int_{T_2}^{T_1} \frac{dT}{T} = \mu \int_0^{\theta} d\theta$$

$$\left[\log T\right]_{T_2}^{T_1} = \mu\left[\theta\right]_0^{\theta}$$

$$\log_e\left(\frac{T_1}{T_2}\right) = \mu\,\theta$$

$$\frac{T_1}{T_2} = e^{\mu\theta} \qquad \text{...(9.1)}$$

Belt Friction for V-Belt

Consider the V-belt placed in pulley with V-groove having angle of groove as 2α.

The normal reactions acted by V-groove pulley is shown in Fig. 9.1.1-viii.

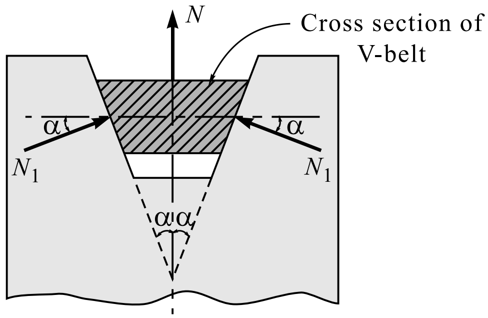


Fig. 9.1.1-viii

Total vertical component of the normal reaction

$$N = 2N_1 \sin \alpha$$

$$\therefore \; N_1 = \frac{N}{2 \sin \alpha}$$

For limiting equilibrium condition frictional force (F) is acting on both sides of V-groove. So net frictional force is twice.

We know, $F = \mu N_1$

$$\therefore \; F = \text{Twice } (\mu N_1)$$

$$F = 2\left(\mu \times \frac{N}{2 \sin \alpha}\right)$$

$$\therefore \; F = (\mu \operatorname{cosec} \alpha) N$$

Compare this equation with derivation used in above article (i.e., belt friction for flat belt)

we get μ replaced by ($\mu \operatorname{cosec} \alpha$)

$\therefore$ We conclude the new relation for V-belt as

$$\log_e\left(\frac{T_1}{T_2}\right) = (\mu \operatorname{cosec} \alpha)\, \theta$$

$$\therefore \; \frac{T_1}{T_2} = e^{(\mu \operatorname{cosec} \alpha)\theta} \qquad \ldots(9.2)$$

where T_1 = Tension in belt on tight side

T_2 = Tension in belt on slack side

θ = Angle of contact of belt with pulley

α = Semi V-groove angle

μ = Coefficient of friction between belt and pulley

Centrifugal Tension (T_c)

We know if a particle of mass M is rotated in a circular path of radius r at a uniform velocity v, a centrifugal force acts radially outwards and its magnitude is equal to

$$\frac{Mv^2}{r} = F_c$$

In pulley belt arrangement also as the belt material is moving with certain velocity on a pulley, it develops force which acts outwards from the centre of the pulley. This force is called the *centrifugal force* (F_c).

Let us consider an elemental length of belt which subtends an elemental angle $d\theta$ at the centre of the pulley as shown in Fig. 9.1.1-ix.

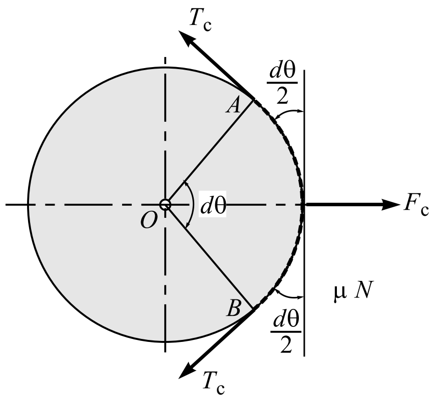


Fig. 9.1.1-ix

Let v = Velocity of the belt in m/s

r = Radius of pulley over which the belt runs

m = Mass of the belt per metre length

T_c = Centrifugal tension acting on belt

F_c = Centrifugal force acting radially outward

Elemental length of the belt $AB = r\,d\theta$

Mass of the belt $AB = m \times r\,d\theta$

Centrifugal force $F_c = \frac{Mv^2}{r} = mr\,d\theta\,\frac{v^2}{r}$

Resolving the forces horizontally and applying limiting equilibrium condition, we have

$$F_c = 2T_c \sin\frac{d\theta}{2}$$

$$\frac{mr\,d\theta\,v^2}{r} = 2T_c \sin\frac{d\theta}{2} \qquad \left[\begin{array}{l}\text{Since } \frac{d\theta}{2} \text{ is very small} \\ \therefore\ \sin\frac{d\theta}{2} = \frac{d\theta}{2}\end{array}\right]$$

$$m\,d\theta\,v^2 = T_c\,d\theta$$

$$T_c = mv^2$$

Note : When we consider the centrifugal tension, we have

Tension on tight side $= T_1 + T_c$

Tension on slack side $= T_2 + T_c$

$\therefore$ The maximum tension in the belt $T_{max} = T_1 + T_c$

For safety purpose T_{max} should not exceed the maximum permissible limit of tension otherwise belt will break.

Initial Tension in the Belt

Initially the belt is mounted such that a firm grip develops between the belt and pulley. Therefore, in stationary condition some amount of tension is developed in the belt. This tension is known as *initial tension*. This initial tension helps to prevent slipping of the belt on the pulley, which would result in loss of power and excessive wear. The initial tension is equal in both parts of the belt.

Under working condition, tension in the two side of the belt will change. The tight side of the belt stretches until the pull is increased from T_0 to T_1 and slack side shortens until the pull is decreased from T_0 to T_2.

$$\text{Increase of tension on tight side} = T_1 - T_0$$

$$\text{Decrease of tension on slack side} = T_0 - T_2$$

$$\text{Increase in length of tight side} = \text{Decrease in length of slack side}$$

$$\varepsilon(T_1 - T_0) = \varepsilon(T_0 - T_2)$$

$$T_1 - T_0 = T_0 - T_2$$

$$T_0 = \frac{T_1 + T_2}{2}$$

When centrifugal tension is also considered, we have

$$\text{Initial Tension} = \frac{T_1 + T_2 + 2T_c}{2}$$

Power Transmitted and Condition for Maximum Power Transmitted

$$\text{Let } T_1 = \text{Tension on tight side}$$

$$T_2 = \text{Tension on slack side}$$

$$v = \text{Linear velocity of belt}$$

$$\text{Power} = \text{Force} \times \text{Velocity}$$

$\therefore$ Power transmitted by the belt is given by

$$P = (T_1 - T_2)v$$

But we know that

$$\frac{T_1}{T_2} = e^{\mu\theta} \qquad \therefore\ T_2 = \frac{T_1}{e^{\mu\theta}}$$

$$P = \left(T_1 - \frac{T_1}{e^{\mu\theta}}\right)$$

$$P = T_1\left(1 - \frac{1}{e^{\mu\theta}}\right) v$$

$$P = T_1 k v \qquad \left(\text{where } k = 1 - \frac{1}{e^{\mu\theta}} = \text{constant}\right)$$

We know, maximum tension (T_{max}) on tight side will have a relation

$$T_{max} = T_1 + T_c$$

$$P = (T_{max} - T_c)\, vk$$

$$P = (T_{max} - mv^2)\, vk \qquad (\because\ T_c = mv^2)$$

$$P = (T_{max}\, v - mv^3)\, k$$

For power to be maximum, apply the maxima condition

$$\frac{dP}{dv} = 0$$

$$\frac{dP}{dv} = (T_{max} - 3mv^2) = 0$$

$$T_{max} - 3mv^2 = 0$$

$$T_{max} = 3mv^2$$

$$T_{max} = 3T_c \qquad (\because\ T_c = mv^2)$$

9.2 Solved Problems

Problem 1

Block *A* weighing 200 N is connected to another block *B* of weight *W*, by a cord passing over a rough fixed pulley. The weight *W* is slowly increased. Find its value for which motion just impends. Take coefficient of friction for all contacting surface = 0.2.

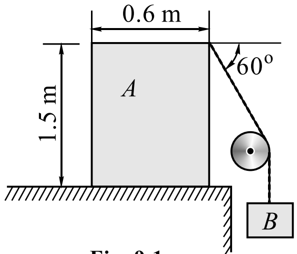


Fig. 9.1

Solution

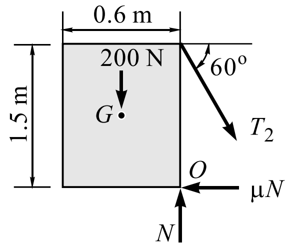
(i) Consider the FBD of Block *A*

Let T_2 be the tension acting on a block (slack side). As T_2 is acting at top of block tipping case will be initiated first about

$\therefore\ \Sigma M_O = 0$

$$200 \times 0.3 - T_2 \cos 60^\circ \times 1.5 = 0$$

$$T_2 = 80 \text{ N}$$

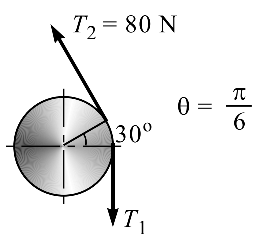


FBD of Block *A*

(ii) Consider the FBD of rough fixed pulley

$T_1 = ?\,(W)$, $T_2 = 80$ N,

$\theta = \frac{\pi}{6}$ and $\mu = 0.2$



FBD of fixed pulley

We have the relation $\dfrac{T_1}{T_2} = e^{\mu\theta}$

$$T_1 = T_2 \times e^{\mu\theta}$$

$$T_1 = 80 \times e^{0.2 \times \frac{\pi}{6}}$$

$T_1 = 88.83$ N, i.e., $W = 88.83$ N ***Ans.***

Problem 2

For given arrangement, find the angle α for which load begins to slip. Take $\mu = 0.3$ between fixed drum and rope.

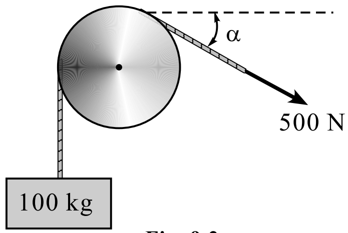


Fig. 9.2

Solution

(i) Using relation

$$\frac{T_1}{T_2} = e^{\mu\theta} \Rightarrow \frac{100 \times 9.81}{500} = e^{0.3\theta}$$

$$e^{0.3\theta} = 1.962$$

$$\theta = \frac{\log_e 1.962}{0.3} = 2.793 \text{ rad}$$

$$\theta = 2.247 \text{ rad}$$

(ii) In degree

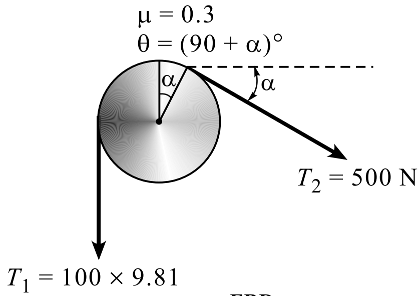
$$\theta = 2.247 \times \frac{180}{\pi}$$

$$\theta = 128.74$$

But $\theta = 90 + \alpha$

$$\therefore \alpha = 128.74 - 90$$

$$\therefore \alpha = 38.74^{\circ} \quad \textbf{\textit{Ans.}}$$



FBD

Problem 3

Block *A* shown in Fig. 9.3 weighs 2000 N. The chord attached to *A* passes over a fixed drum and supports a weight equal to 800 N. The value of coefficient of friction between *A* and horizontal plane is 0.25 and between the rope and the fixed drum is 0.1. Determine value of *P* if motion is impending towards left.

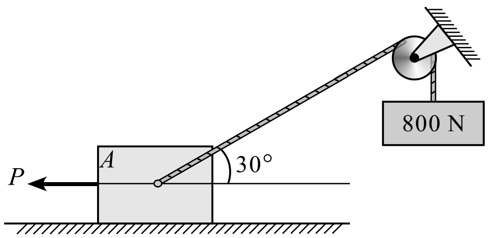


Fig. 9.3

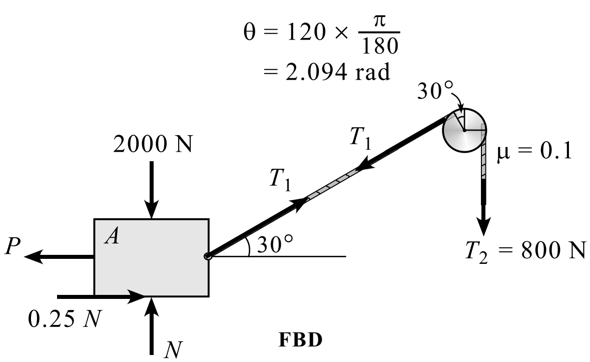
Solution :

(i) By relation

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

$$T_1 = 800 \times e^{0.1 \times 2.094}$$

$$T_1 = 986.35 \text{ N}$$



FBD

(ii) FBD of Block *A*

$$\Sigma F_y = 0$$

$$N - 2000 - T_1 \sin 30^{\circ} = 0$$

$$N = 2000 - 986.35 \sin 30^{\circ}$$

$$N = 1506.83 \text{ N}$$

$$\Sigma F_x = 0$$

$$T_1 \cos 30 + 0.25\,N - P = 0$$

$$P = 986.35 \cos 30^{\circ} + 0.25 \times 1506.83$$

$P = 1230.91$ N ***Ans.***

Problem 4

A lever *CD* is connected to cylindrical drum *A* through a belt, as shown in Fig. 9.4 such that the drum does not rotate. The coefficient of friction between the belt and the drum is 0.3. A boy exerts a 100 N upward push on the lever at *C*. Determine

(i) the maximum weight *W* that the boy can lift, and

(ii) the maximum weight *W* that the boy can hold.

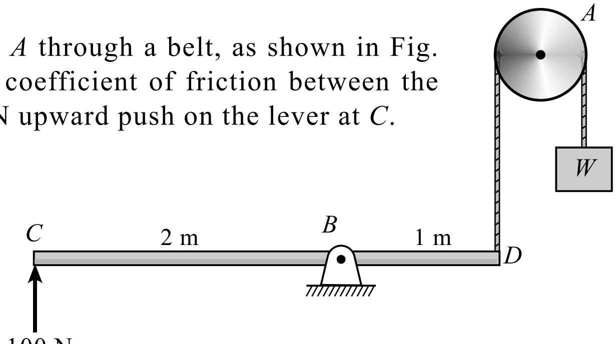


Fig. 9.4

Solution

Case (i)

$$\Sigma M_o = 0$$

$$T_1 \times 1 - 100 \times 2 = 0$$

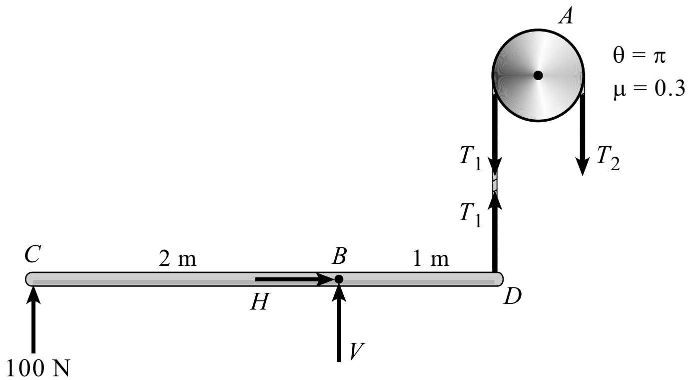
$$T_1 = 200 \text{ N}$$

The ratio of tensions is given by

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

$$\frac{200}{T_2} = e^{0.3 \times \pi}$$

$$T_2 = 77.93 \text{ N}$$



FBD

∴ The maximum weight that boy can lift is 77.93 N ***Ans.***

Case (ii)

$$\Sigma M_o = 0$$

$$T_2 \times 1 - 100 \times 2 = 0$$

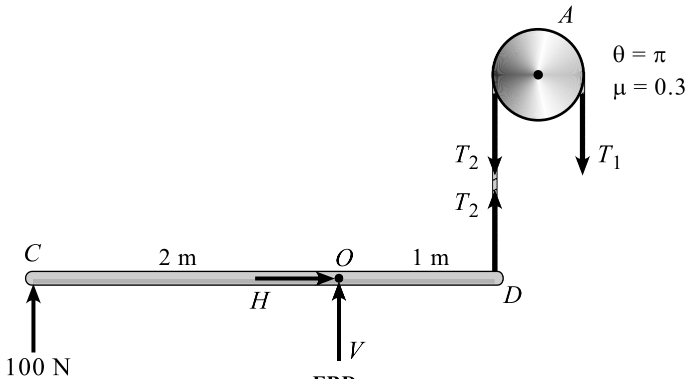
$$T_2 = 200 \text{ N}$$

The ratio of tensions is given by

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

$$\frac{T_1}{200} = e^{0.3 \times \pi}$$

$$T_1 = 513.27 \text{ N}$$



FBD

∴ The maximum weight that boy can hold is 513.27 N ***Ans.***

Problem 5

A V-belt drives a grooved pulley at a speed of 1440 m/min. The coefficient of friction between belt and pulley is 0.3, angle of groove is 45^o and angle of lap is 160^o. If the mass of belt is 0.45 kg/m and if permissible pull in the belt is 800 N. Calculate the power, in kilowatts, transmitted. Consider the centrifugal tension in the belt.

Solution

$$Given: v = \frac{1440}{60} = 24 \text{ m/s}, \ \mu = 0.3, \ \alpha = \frac{45}{2} = 22.5^o$$

$$\theta = 160^o = 160 \times \frac{\pi}{180} = 2.793 \text{ rad},$$

$$m = 0.45 \text{ kg/m and } T_{max} = 800 \text{ N}$$

$$T_c = mv^2 = 0.45 \times 25^2$$

$$T_c = 259.2 \text{ N}$$

$$T_{max} = T_1 + T_c$$

$$\therefore T_1 = T_{max} - T_c = 800 - 259.2$$

$$\therefore T_1 = 540.8 \text{ N}$$

We have the relation

$$\frac{T_1}{T_2} = e^{(\mu \text{ cosec } \alpha)\theta}$$

$$T_2 = \frac{T_1}{e^{(\mu \text{ cosec } \alpha)\theta}} = \frac{540.8}{e^{(0.3 \text{ cosec } 22.5^o) \times 2.793}}$$

$$\therefore T_2 = 60.55 \text{ N}$$

$$\text{Power} = (T_1 - T_2)v$$

$$P = (540.8 - 60.55) \times 24$$

$$P = 11526 \text{ watt} \quad \textbf{\textit{Ans.}}$$

Problem 6

A belt having cross section of dimension 50 mm × 6 mm has an angle of lap of 190^o on a pulley with diameter 800 mm. The pulley rotates at 240 rpm. If the material of the belt has a density of 1120 kg/m^3 and withstand a maximum tensile stress of 1500 kN/m^2. Find **(i)** centrifugal tension and **(ii)** power transmitted. Take $\mu = 0.3$ between the belt and pulley.

Solution

$$Given: \text{Area} = 50 \times 6 = 300 \text{ mm}^2 = 3 \times 10^{-4} \text{ m}^2$$

$$\theta = 190^o = 190 \times \frac{\pi}{180} = 3.316 \text{ rad}$$

$$r = 400 \text{ mm} = 0.4 \text{ m}$$

$$v = \frac{2\pi \times 0.4 \times 240}{180} = 10.05 \text{ m/s}$$

$$\sigma = 1500 \text{ kN/m}^2 = 1500 \times 10^3 \text{ N/m}^2$$

$$\rho = 1120 \text{ kg/m}^2$$

$$\mu = 0.3$$

$$\text{Stress, } \sigma = \frac{T_{max}}{\text{Area}}$$

$$T_{max} = \sigma \times A = 1500 \times 10^3 \times 3 \times 10^{-4}$$

$$T_{max} = 450 \text{ N}$$

Mass per unit length,

$$m = A \times l \times \rho \quad \text{(Assume } l \text{ as 1 metre length of belt)}$$

$$m = (3 \times 10^{-4}) \times 1 \times 1120$$

$$m = 0.336 \text{ kg/m}$$

Centrifugal tension,

$$T_c = mv^2 = 0.336 \times 10.05^2$$

$$\therefore T_c = 33.94 \text{ N}$$

$$T_{max} = T_1 + T_c$$

$$T_1 = T_{max} - T_c = 450 - 33.94$$

$$\therefore T_1 = 416.06 \text{ N}$$

We have the relation

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

$$T_2 = \frac{T_1}{e^{\mu\theta}} = \frac{416.06}{e^{0.3 \times 3.316}}$$

$$\therefore T_2 = 153.86 \text{ N}$$

$$\text{Power, } P = (T_1 - T_2)v$$

$$P = (416.06 - 153.86) \times 10.05$$

$P = 2635$ watt ***Ans.***

Problem 7

A leather belt transmits 30 kW from a pulley 75 cm diameter which runs at 500 rpm. The belt is in contact with the pulley over an arc of 150^o and coefficient of friction is 0.3. If the mass of belt is 1 kg per metre and if the safe stress is not to exceed 245 N/cm^2. Find the minimum area of the belt.

Solution

Given : Power, $P = 30{,}000$ watt

$$r = 37.5 \text{ cm} = 0.375 \text{ m}$$

$$v = \frac{2\pi \times 0.375 \times 500}{60} = 19.64 \text{ m/s}$$

$$\theta = 150 \times \frac{\pi}{180} = 2.618 \text{ rad}$$

$$\mu = 0.3$$

$$m = 1 \text{ kg/m}$$

$$\sigma = 245 \text{ N/cm}^2 = 245 \times 10^4 \text{ N/m}^2$$

Permissible stress,

$$\sigma = \frac{T_{\max}}{\text{Area}}$$

$$\therefore T_{\max} = 245 \times 10^4 \times A$$

$$\text{Power, } P = (T_1 - T_2)v \qquad \left[\because \frac{T_1}{T_2} = e^{\mu\theta} \;\therefore\; T_2 = \frac{T_1}{e^{\mu\theta}}\right]$$

$$P = \left(T_1 - \frac{T_1}{e^{\mu\theta}}\right) v$$

Centrifugal tension,

$$T_c = mv^2 = 1 \times 19.64^2$$

$$\therefore T_c = 385.73 \text{ N}$$

$$T_{\max} = T_1 + T_c$$

$$T_1 = T_{\max} - T_c$$

$$T_1 = 245 \times 10^4 \times A - 385.73$$

Now,

$$P = \left(T_1 - \frac{T_1}{e^{\mu\theta}}\right) v$$

$$30000 = \left(245 \times 10^4 \times A - 385.73 - \frac{245 \times 10^4 \times A - 385.73}{e^{0.3 \times 2.618}}\right) \times 19.64$$

$$\therefore \text{Area, } A = 13.03 \text{ cm}^2$$ ***Ans.***

Problem 8

A cross belt drive has the following details :

Thickness of belt = 8 mm

Maximum permissible stress in the belt = 2 MPascal

Diameter of driver pulley = 400 mm

Diameter of driven pulley = 800 mm

Density of belt material = 1500 kg/m^3

Coefficient of friction between belt and pulley = 0.25

If the angular velocity of driver pulley is 1200 rpm, find the power transmitted in watt per unit mm width of the belt. If it is desired to transmit power at maximum level, which is 16 kW, find the width of the belt and angular velocity in rpm of the driver pulley.

Solution

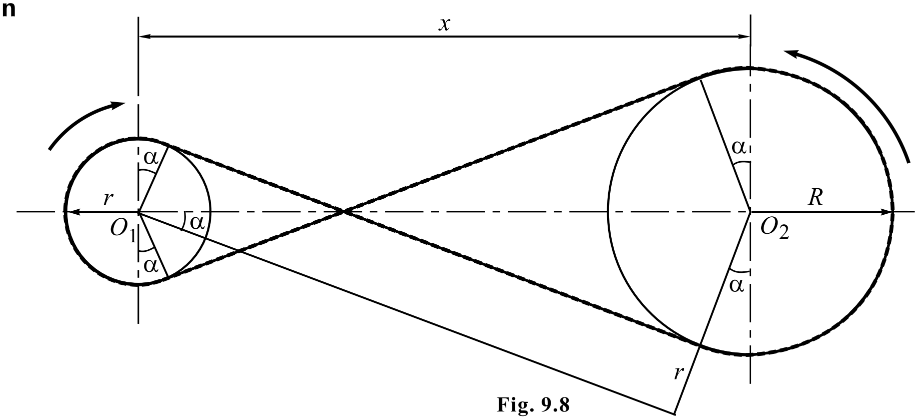


Fig. 9.8

Given : $\sigma = 2 \times 10^6 \text{ N/m}^2$

$r = 0.2 \text{ m}$

$R = 0.4 \text{ m}$

$x = 3.6 \text{ m}$

$$\alpha = \sin^{-1}\left(\frac{R+r}{x}\right) = \sin^{-1}\left(\frac{0.6}{3.6}\right)$$

$\alpha = 9.6^\circ$

$\rho = 1500 \text{ kg/m}^3$

$$\omega \text{ (driver)} = \frac{1200 \times 2\pi}{60} = 125.6 \text{ r/s}$$

$\mu = 0.25$

$v = r\,\omega = 0.2 \times 125.6$

$v = 25.13 \text{ m/s}$

θ for driver $= 180 + 2\alpha = 199.2^\circ$

$$\frac{T_1}{T_2} = e^{\mu\theta} = e^{\left(0.25 \times 199.2 \times \frac{\pi}{180}\right)}$$

$$\frac{T_1}{T_2} = 2.385$$

On per unit width basis,

let $w = 1 \times 10^{-3}$ m

$$T_{\max} = \sigma \times A = (2 \times 10^6)(1 \times 10^{-3} \times 8 \times 10^{-3})$$

$$T_{\max} = 16 \text{ N}$$

$$m = \rho \times (\text{Volume of 1 m length of belt})$$

$$m = 1500(1 \times 10^{-3} \times 8 \times 10^{-3} \times 1)$$

$$m = 0.012 \text{ kg}$$

Centrifugal tension, $T_c = mv^2 = 0.012 \times 25.13^2$

$$\therefore\ T_c = 7.578 \text{ N}$$

$$T_{max} = T_1 + T_c$$

$$T_1 = T_{max} - T_c = 16 - 7.578$$

$$\therefore\ T_1 = 8.42 \text{ N}$$

We have the relation, $\dfrac{T_1}{T_2} = 2.385 = \dfrac{8.42}{T_2}$

$$\therefore\ T_2 = 3.53 \text{ N}$$

Power, $P = (T_1 - T_2)v$

$$P = (8.42 - 3.53) \times 25.13$$

$$P = 122.87 \text{ watt (on per mm width basis)}$$

To transmit 16000 watt, width required is

$$\text{Width} = \frac{16000}{122.87} = 130.2 \text{ mm} \quad \textbf{\textit{Ans.}}$$

Problem 9

In an open-belt drive, the angle of lap on the smaller pulley is 172°. The smaller pulley has a diameter of 800 mm and rotates at 600 rpm. Consider μ between the pulley and belt as 0.3. Neglecting centrifugal tension, find the power that can be transmitted if the initial tension in the belt is 1200 N.

Solution

Given : $\theta = 172^\circ = 172 \times \dfrac{\pi}{180} = 3 \text{ rad}, r = 400 \text{ mm} = 0.4 \text{ m},$

$$v = \frac{2\pi rN}{60} = \frac{2\pi \times 0.4 \times 600}{60} = 25.13 \text{ m/s},$$

$$T_0 = 1200$$

$$\mu = 0.3$$

Initial tension, $T_0 = \dfrac{T_1 + T_2}{2}$

$$1200 = \frac{T_1 + T_2}{2}$$

$$T_1 + T_2 = 2400 \text{ N} \quad \text{.... (I)}$$

$$\frac{T_1}{T_2} = e^{\mu\theta} = e^{0.3 \times 3}$$

$$\frac{T_1}{T_2} = 2.46 \quad \text{.... (II)}$$

Solving Eqs. (I) and (II), we get

$$T_1 = 1706.36 \text{ N}$$

and $T_2 = 693.64$ N

$$\text{Power, } P = (T_1 - T_2)v$$

$$P = (1706.36 - 693.64) \times 25.13$$

$$P = 25449 \text{ watt} \quad \textit{\textbf{Ans.}}$$

Problem 10

Two parallel shaft, which are 6 m apart, are to be connected by a belt running over pulleys of diameter 46 cm and 30 cm respectively. Determine the length of the belt considering both open and crossed belt drives.

Solution

Given Distance between the centres of shaft, $x = 6$ m

Radius of larger pulley, $r_1 = \dfrac{46}{2} = 23 \text{ cm} = 0.23 \text{ m}$

Radius of smaller pulley, $r_2 = \dfrac{30}{2} = 15 \text{ cm} = 0.15 \text{ m}$

(i) For open belt, we have the relation

$$L = \pi(r_1 + r_2) + \frac{(r_1 - r_2)^2}{x} + 2x$$

$$L = \pi(0.23 + 0.15) + \frac{(0.23 - 0.15)^2}{6} + 2 \times 6$$

$$\therefore L = 13.195 \text{ m} \quad \textit{\textbf{Ans.}}$$

(ii) For cross belt, we have the relation

$$L = \pi(r_1 + r_2) + \frac{(r_1 + r_2)^2}{x} + 2x$$

$$L = \pi(0.23 + 0.15) + \frac{(0.23 + 0.15)^2}{6} + 2 \times 6$$

$$\therefore L = 13.22 \text{ m} \quad \textit{\textbf{Ans.}}$$

Problem 11

A belt 100 mm wide and 8 mm thick is transmitting power at a belt speed of 1600 m/minute. The angle of lap of the smaller pulley is 165° and coefficient of friction is 0.3. The maximum permissible stress in the belt is 2 N/mm^2 and the mass of the belt is 0.9 kg/m. Find the power transmitted and the initial tension in the belt. Also find the maximum power that can be transmitted and the corresponding belt speed.

Solution

(i) Width $w = 100$ mm $= 0.1$ m , Thickness $t = 8$ mm $= 0.008$ m.

Speed $v = 1600$ m/min $= 26.67$ m/s, Angle of lap $\theta = 165 \times \dfrac{\pi}{180} = 2.88$ rad

Coefficient of friction $\mu = 0.3$

(ii) Maximum permissible stress $\sigma_{max} = \dfrac{T_{max}}{\text{Area}}$

$T_{max} = 2 \times 10^6 \times 0.1 \times 0.008$

$T_{max} = 1600$ N

(iii) Mass of belt per unit length $m = 0.9$ kg/m

$T_{max} = T_1 + T_c$

$\therefore\ T_1 = T_{max} - T_c = 1600 - 640$

$\therefore\ T_1 = 960$ N

(iv) Power $P = T_1 \left(1 - \dfrac{1}{e^{\mu\theta}}\right) v$

$P = 960 \left(1 - \dfrac{1}{e^{0.3 \times 2.88}}\right) 26.67 \qquad \therefore\ P = 14812.15$ watt ***Ans.***

(v) $\dfrac{T_1}{T_2} = e^{\mu\theta}$

$T_2 = \dfrac{T_1}{e^{\mu\theta}} = \dfrac{960}{e^{0.3 \times 2.88}} \qquad \therefore\ T_2 = 404.6$ N

(vi) Initial tension, $T_0 = \dfrac{T_1 + T_2}{2}$

$T_0 = \dfrac{960 + 404.6}{2} \qquad \therefore\ T_0 = 682.31$ N ***Ans.***

(vi) For maximum power, we have the relation

$T_{max} = 3T_c$

$T_c = \dfrac{1600}{3} = 533.33$ N

$T_c = mv^2$

$v = \sqrt{\dfrac{533.33}{0.9}} \qquad \therefore\ v = 24.34$ m/s ***Ans.***

$T_1 = T_{max} - T_c$

$T_1 = 1600 - 533.33 = 1066.67$ N

$P_{max} = T_1 \left(1 - \dfrac{1}{e^{\mu\theta}}\right) v$

$P_{max} = 1066.67 \left(1 - \dfrac{1}{e^{0.3 \times 2.88}}\right) 24.34 \quad \therefore\ P_{max} = 15020.16$ watt ***Ans.***

Exercises

[I] Problems

1. A mass of 500 kg is to be maintained in position by pulling a rope taken over a half barrel and wrapped twice around a capstan as shown in Fig. 8.E1. If the coefficient of static friction is 0.2 for all contact surfaces, calculate the minimum force F required to maintain the load.

[***Ans.*** $F = 290$ N]

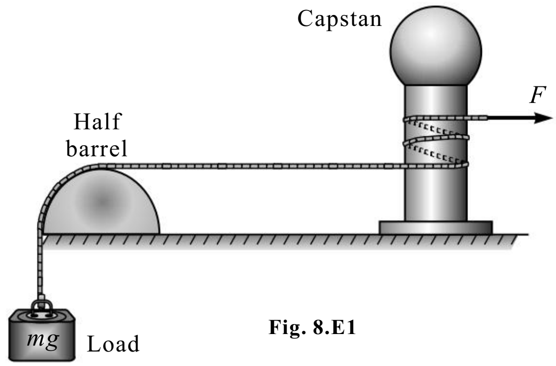


Fig. 8.E1

2. Determine minimum force P required to just lift the 30 kN load shown in Fig. 8.E2. Assume $\mu = 0.3$.

[***Ans.*** 123.3 kN]

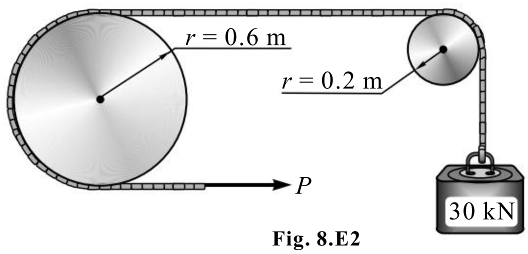


Fig. 8.E2

3. What force is necessary to hold a mass of 900 kg suspended on a rope wrapped twice around a post ? Assume $\mu = 0.20$. Determine the force P required to hold the weight of 500 N.

[***Ans.*** $P = 712.43$ N]

4. For the system shown in Fig. 8.E4, determine force P required to hold the weight of 500 N.

[***Ans.*** $P = 87$ N]

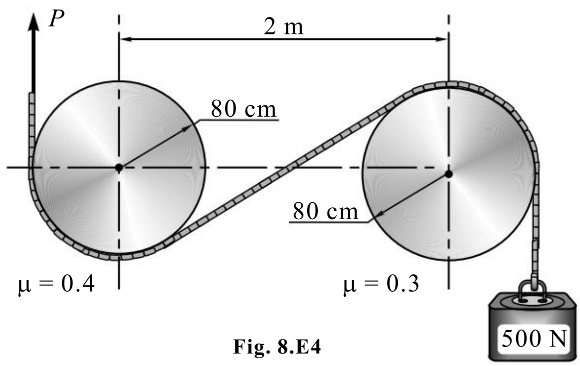


Fig. 8.E4

5. The maximum tension that can be developed in the belt, as shown in Fig. 8.E5, is 500 N. If the pulley at A is free to rotate and the coefficient of static friction at the fixed drums B and C is 0.25. Determine the largest mass of the cylinder that can be lifted by the belt.

[***Ans.*** $m = 15.7$ kg]

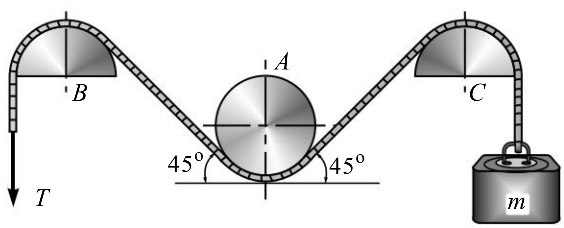


Fig. 8.E5

6. Two blocks A and B are to be held in position by means of an inextensible rope passing over a fixed drum as shown in Fig. 8.E6. The coefficient of friction between the blocks, as well as between the block and the inclined surface, between the rope and the drum is 0.2. The mass of B is 500 kg. Determine the minimum weight of A so that B is prevented from moving downwards.

[***Ans.*** $m_A = 126$ kg]

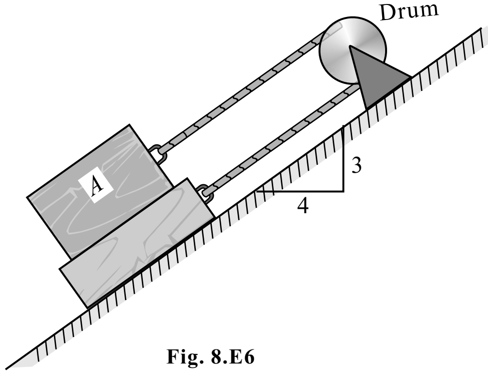


Fig. 8.E6

7. A mass of 55 kg is prevented by a rope wrapped ¼ turn around drum B and 1¼ turns around drum A, as shown in Fig. 8.E7. Assume that drum B is smooth and the coefficient of friction between the rope and A is 0.25. What is the value of the holding force F ?

[***Ans.*** $F = 75.7$ N]

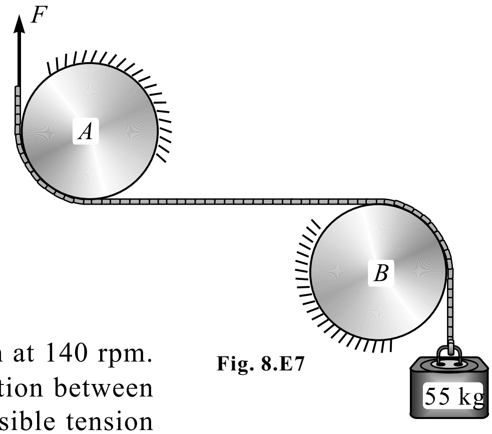


Fig. 8.E7

8. A belt is running over a pulley of diameter 800 mm at 140 rpm. The angle of contact is 165° and coefficient of friction between the belt and pulley is 0.25. If the maximum permissible tension in the belt is 2.5 kN, find the power transmitted by the belt.

[***Ans.*** 7524 watt]

9. The initial tension in an open flat belt is 2000 N. The angle of contact of smaller pulley is 135° and μ is 0.25. The smaller pulley's diameter is 36 cm and rotates at 300 rpm. Find the power transmitted by the belt.

[***Ans.*** $P = 6.46$ kW]

10. A flat belt transmits 15000 W from a shaft running at 300 rpm to another parallel shaft running at the same rpm. The maximum permissible tension in the belt is 3 kN. Find the radius of the pulley consider $\mu = 0.25$.

[***Ans.*** $r = 29.3$ cm]

11. Two pulleys, one having diameter 450 mm and other having diameter 200 mm are mounted on parallel shaft and are 2000 mm apart. For open-belt arrangement, find power transmitted if larger pulley rotates at 250 rpm and $\mu = 0.3$ and maximum tension in the belt is 1.5 kN.

[***Ans.*** $P = 5.71$ kW]

12. A belt of density 1.5 gm/cm^3 has a maximum permissible stress of 300 N/cm^2. Determine the maximum power that can be transmitted by a belt of 15 cm × 1.5 cm if the ratio of the tension is 2.5.

[***Ans.*** $P_{max} = 69.714$ kW]

13. For an open-belt drive transmitting 4 kW, the distance between the centre of pulley '*A*' and the centre of pulley '*B*' is 100 cm. Diameters of pulleys *A* and *B* are 30 cm and 20 cm respectively. The coefficient of friction between belt and pulley is 0.2. The pulley *A* is rotating at 200 rpm. If the permissible tensile force per cm width of belt is 180 N, determine the width of the belt necessary.

[***Ans.*** 15.38 cm]

14. A leather belt of 1 cm thick and 12.5 cm width drives a pulley of 120 cm diameter at 180 rpm. The angle of lap is 190^o. The density of belt material is 1.15 gm/cm^3. If the stress in the belt is limited to 200 N/cm^2, determine (a) power transmitted neglecting centrifugal tension and (b) considering centrifugal tension. Take $\mu = 0.3$.

[***Ans.*** (a) $P = 17.8$ kW and (b) 16.45 kW.]

15. What is the maximum power transmitted if the cross section of the belt is 10 cm^2 and maximum stress is limited to 2400 N/cm^2. Density of the belt material = 5 cm^3. The ratio of effective tension = 2.

[***Ans.*** $P_{\text{max}} = 320$ kW]

16. Determine the belt speed at which the maximum power is transmitted by the belt when maximum belt tension is limited to 2 kN. Also find out the power transmitted by the belt at this speed. Consider angle of lap 170^o, $\mu = 0.2$, groove angle 45^o, mass of belt 0.72 kg/m.

[***Ans.*** $v = 30.4$ m/s and $P = 31.95$ kW.]

17. The power transmitted by a belt drive is 18 kW. The diameter of one of the pulley is 180 cm which runs at 300 rpm. Permissible stress in the belt material is 300 N/cm^2. Thickness of the belt = 8 cm, $\mu = 0.3$, density $\rho = 0.95$ gm/cm^3. Determine the width of the belt required.

[***Ans.*** $w = 4.4$ cm]

[II] Review Questions

1. Derive the relation $T_1/T_2 = e^{\mu\theta}$.

2. What is a centrifugal tension ?

3. Derive $T_c = mv^2$.

[III] Fill in the Blanks

1. In relation $T_1 = T_2\, e^{\mu\theta}$, T_1 is the ______ side of rope.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

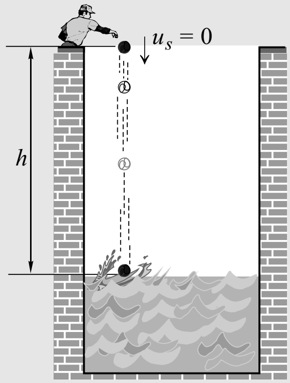
1. Centrifugal tension for belt friction is given by the relation ________.

(a) $T_c = 2mv^2$ **(b)** $T_c = 2m^2v$ **(c)** $T_c = m^2v$ **(d)** $T_c = mv^2$

2. Condition for maximum power transmission is ________ .

(a) $T_{max} = T_1 + T_c$ **(b)** $T_{max} = T_1 + T_0$ **(c)** $T_{max} = 3T_c$ **(d)** $T_{max} = 2T_c$

10



INTRODUCTION TO DYNAMICS

10.1 Revision of Mechanics

Before we discuss dynamics, let us revise some terms from Chapter 1 once again.

- Mechanics
- Statics
- Dynamics
- Kinematics
- Kinetics
- Newton's Law of Motion
- Particle
- Rigid body
- Space
- Time
- Mass
- Weight
- Concept of Force

Dynamics

- Kinematics
  - Kinematics of Particle
  - Kinematics of Rigid Body
- Kinetics
  - Kinetics of Particle
  - Kinetics of Rigid Body

10.2 Motion

A body is said to be in motion if it is changing its position with respect to a reference point.

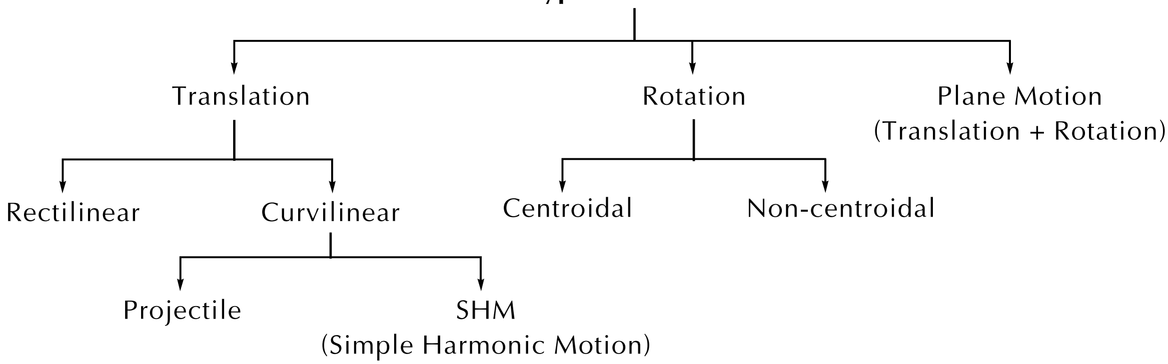
A person sitting in a running train is in motion when referred to the platform but two passengers in train are at rest when referred to the train itself. So in dynamics problem any fixed point on the Earth is an implied reference point. Other examples of implied reference points are centre of the Earth for the study of satellite motion, centre of the Sun for the study of motion of the solar system.

The above said reference point w.r.t. Earth is also called as *Newtonian frame of reference* or *Inertial frame of reference* or *datum*. Newton's laws are valid for such a reference frame.

In this book only engineering problems have been taken up and hence any fixed point on the Earth is an implied reference point while considering motion of the concerned body.

One can use three mutually perpendicular axes *x-y-z* placed at reference point called *reference axis*.

Types of Motion



10.2.1 Translation

If a straight line drawn on the moving body remains parallel to its original position then such motion is called *translation*.

In translation we have two subtypes :

1. Rectilinear motion
2. Curvilinear motion

1. **Rectilinear Motion :** During translation if the path followed by a point is a **straight line** then such motion is called *rectilinear motion*. It is also called a *linear motion*. Refer to Fig. 10.2.1-i.

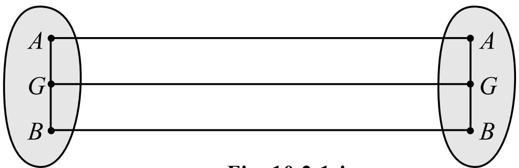


Fig. 10.2.1-i

2. **Curvilinear Motion :** During translation if the path followed by a point is a **curve** then such motion is called *curvilinear motion*. Refer to Fig. 10.2.1-ii.

Fig. 10.2.1-ii

10.2.2 Rotation

If all particles of a rigid body moves in a **concentric circle** then such motion is called *rotation*. In rotation we have two subtypes :

1. Centroidal rotation
2. Non-centroidal rotation

1. **Centroidal Rotation :** If the body is rotating about centroidal axis then it is called a *centroidal rotation*. For example, fan, motor, turbine, flywheel, etc.

2. **Non-centroidal Rotation :** If the body is rotating about non-centroidal axis then it is called a *non-centroidal rotation*. For example, pendulum bob, etc.

10.2.4 General Plane Motion

The general plane motion is a *combination of both translation and rotation*. For example, points on a wheel of moving car, ladder sliding down from its position against wall, etc.

10.2.5 Analysis of Dynamics

Problems of dynamics will have some parameters as given below.

- Displacement, velocity and acceleration
- The time interval
- The path followed
- The position occupied
- The active and reactive forces
- The relation between the force and the motion
- Diagrams of particles and/or rigid bodies in relation.

While analysing the problem in dynamics one should ask the following questions :

(i) Whether force and mass are considered ?

Ans. If **Yes** then *kinetics.* If **No** then *kinematics.*

(ii) Whether dimensions are considered ?

Ans. If **Yes** then *rigid body.* If **No** then *particle.*

(iii) Which type of motion is performed ? Translation or rotation or plane motion ?

Ans. **Particle** can perform only *translation motion*.

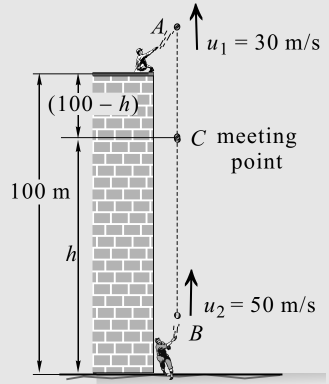
Rigid body may perform *translation* or *rotation* or *general plane motion*.

Important Steps Considered in a Solution :

- Given data.
- What is to be determined ?
- Necessary diagrams.
- Calculations.
- Answers and conclusions.

11

Kinematics of Particles - I Rectilinear Motion



Kinematics of Particles *is the study of geometry of translation motion without reference to the cause of motion. Force and mass are not considered.*
In this chapter, we shall study translation motion of a particle considering its position, displacement, velocity, acceleration and time. Particle cannot have rotational or general plane motion. The two major point of chapter are rectilinear motion and curvilinear motion.

11.1 Rectilinear Motion

If the particle is moving along straight path then it is called a *rectilinear motion*. For example, a train moving on a straight track, a stone released from the top of tower, etc.

Position

Position means the location of a particle with respect to a fixed reference point say origin O. The sketch (given below) shows position of particle at A as $S_A = 4$ m and at B as $S_B = -3$ m.

Example

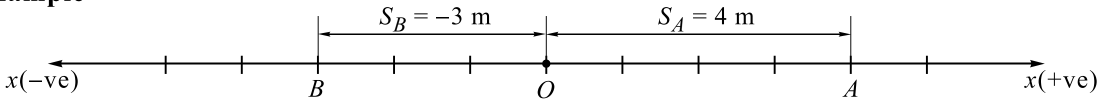


Fig. 11.1-i

Displacement

Displacement is a change in position of the particle. It is the difference between final position and initial position. It is a vector quantity connecting the initial position to the final position.

Example

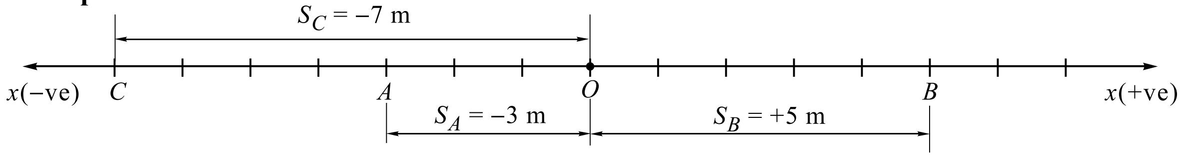


Fig. 11.1-ii

Initial position of particle is A ($S_A = -3$ m). It moves to position B ($S_B = +5$ m) and finally to position C ($S_C = -7$ m).

$$\therefore \text{ Displacement of a particle } = \text{ Final position } - \text{ Initial position}$$

$$S = S_C - S_A = -7 - (-3) = -4 \text{ m} = 4 \text{ m } (\leftarrow)$$

Thus, displacement depends only on initial and final position of the particle and its value may be positive or negative.

Distance

Distance is the total path travelled by a particle from initial position to final position. It is a scalar quantity. For example, refer to Fig. 11.1-ii,

$$d = AO + OB + BO + OC = 3 + 5 + 5 + 7$$

$$\therefore \ d = 20 \text{ m}$$

If the particle is moving along the straight line in the same direction then the distance covered is equal to displacement.

Velocity

The rate of change of displacement with respect to time is called velocity. It is a vector quantity.

If s is the displacement in time t, then the average velocity

$$v = \frac{s}{t}$$

Velocity of a particle at a given instant is called *instantaneous velocity* and is given by the limiting value of the ratio s/t at time t when both s and t are very small. Let δs be the small displacement in a small interval time δt.

The instantaneous velocity $v = \lim\limits_{\delta t \to 0} \dfrac{\delta s}{\delta t}$

$$v = \frac{ds}{dt}$$

The S.I. unit of velocity is m/s.

Conversion of kilometer per hour (kmph)

$$1 \text{ kmph} = \frac{1 \times 1000}{60 \times 60} = \frac{5}{18} \text{ m/sec}$$

For example, $v = 54 \text{ kmph} = 54 \times \dfrac{5}{18} \text{ m/sec}$

$$v = 15 \text{ m/sec}$$

Speed

The rate of change of distance with respect to time is called speed. It is a scalar quantity. *The magnitude of velocity is also known as speed.*

Example

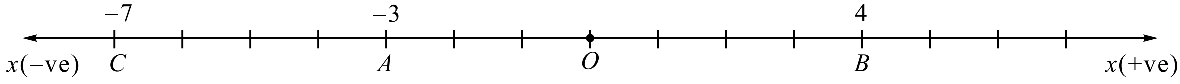


Fig. 11.1-iii

Consider a particle moving from A to B in time $t_1 = 4$ sec then from B to C in time $t_2 = 6$ sec. Find the average velocity and the average speed.

$$\text{Average velocity} = \frac{\text{Change in displacement}}{\text{Time interval}} = \frac{-7-(-3)}{4+6} = \frac{-4}{10}$$

$$\text{Average velocity} = 0.4 \text{ m/sec} \ (\leftarrow)$$

$$\text{Average speed} = \frac{\text{Distance travelled}}{\text{Time interval}} = \frac{3+4+4+7}{4+6} = \frac{20}{10}$$

$$\text{Average speed} = 2 \text{ m/sec}$$

Acceleration

The rate of change of velocity with respect to time is called acceleration.

The instantaneous acceleration $a = \lim\limits_{\delta t \to 0} \dfrac{\delta v}{\delta t}$

$$a = \frac{dv}{dt}$$

S.I. unit of acceleration 'a' is m/s^2

The acceleration may be positive or negative. *Positive* acceleration is simply called *acceleration* and *negative* acceleration is called *retardation* or *deceleration*.

Positive acceleration means magnitude of velocity increases w.r.t. time and particle moves faster in positive direction. Negative acceleration means particle moves slowly in the positive direction or moves faster in the negative direction.

In other words, if the acceleration is in the direction of velocity then velocity increases and if the direction of acceleration is in opposite to the direction of velocity then velocity decreases.

Acceleration can also be expressed as

1. $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2s}{dt^2}$

2. $a = \dfrac{dv}{dt} = \dfrac{vdv}{ds}$

11.2 Equations of Motion

1. Motion with uniform (constant) velocity :

$$\text{Displacement} = \text{Velocity} \times \text{Time}$$

$$s = v \times t$$

2. Motion with uniform (constant) acceleration : Consider a particle moving with constant acceleration.

Let u be the initial velocity, v be the final velocity and t be the time interval.

Acceleration is the rate of change of velocity with respect to time.

$$a = \frac{v-u}{t}$$

$$v = u + at \qquad \text{..... (11.1)}$$

$$\text{Displacement } s = \text{Average velocity} \times \text{Time}$$

$$s = \left(\frac{u+v}{2}\right)(t) \qquad \text{..... (11.2)}$$

Substituting the value of v from Eq. (11.1) in Eq. (11.2)

$$s = \left(\frac{u+u+at}{2}\right) \times t$$

$$s = ut + \frac{1}{2}at^2 \qquad \text{..... (11.3)}$$

From Eq. (11.1)

$$t = \frac{v-u}{a}$$

Substituting in Eq. (11.2)

$$s = \left(\frac{u+v}{2}\right)\left(\frac{v-u}{a}\right)$$

$$s = \frac{v^2-u^2}{2a}$$

$$v^2 = u^2 + 2as \qquad \text{..... (11.4)}$$

Thus, equations of motion of a particle moving with a constant acceleration are

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

3. Motion with variable acceleration

Motion with variable acceleration : If the rate of change of velocity is not uniform then it is called *variable acceleration motion*.

When the variation of acceleration or velocity or displacement with respect to time is known, we can solve such problems by differentiation or by integration with boundary conditions

$$a = \frac{dv}{dt} = v\,\frac{dv}{ds} \;;\quad v = \frac{ds}{dt}$$

4. Vertical motion under gravity : This is the special case of motion with uniform acceleration.

Equations of motion of a particle moving under gravity are

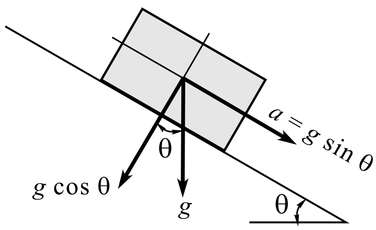
$$v = u + gt$$

$$h = ut + \frac{1}{2}gt^2$$

$$v^2 = u^2 + 2gh$$

where $g = 9.81 \text{ m/s}^2 (\downarrow)$ and $g = -9.81 \text{ m/s}^2 (\uparrow)$.

5. Motion along an inclined plane under gravity : If a block is sliding by its self-weight on a frictionless inclined plane then its constant acceleration is given as



$$a = g \sin\theta$$

11.3 Sign Convention

- The kinematics quantities such as displacement, velocity and acceleration are vector quantities.
- Therefore, we should use the proper sign convention while solving the problems.
- We shall consider the initial direction of motion as positive sign for displacement, velocity and acceleration.
- But retardation will be negative in initial direction of motion.

Example 1
A ball is thrown vertically up.
Sign Convention

- Initial direction of motion is upwards ($\uparrow$).
- Therefore, upward direction ($\uparrow$) will be positive.
- Velocity in upward direction ($\uparrow$) will be positive.
- Displacement in upward direction ($\uparrow$) will be positive.
- Acceleration due to gravity in upward direction ($\uparrow$) (retardation) will be negative (i.e., $g = -9.81$ m/s^2).

Example 2
A ball is thrown vertically down.
Sign Convention

- Initial direction of motion is downwards ($\downarrow$).
- Therefore, downward direction ($\downarrow$) will be positive.
- Velocity in downward direction ($\downarrow$) will be positive.
- Displacement in downward direction ($\downarrow$) will be positive.
- Acceleration due to gravity in downward direction ($\downarrow$) will be positive (i.e., $g = 9.81$ m/s^2).

Example 3
A car starting from rest moves towards right.
Sign Convention

- Initial direction of motion is towards right ($\rightarrow$).
- Therefore, all kinematic quantities, direction towards right ($\rightarrow$) will be positive but retardation will be negative.

Example 4
A car starting from rest moves towards left.
Sign Convention

- Initial direction of motion is towards left ($\leftarrow$).
- Therefore, all kinematic quantities, direction toward left ($\leftarrow$) will be positive but retardation will be negative.

11.4 Motion Curves

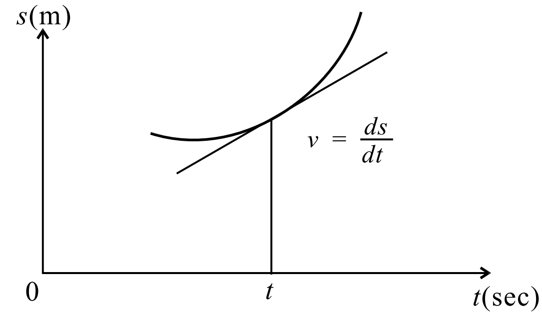
Motion curves are the graphical representation of displacement, velocity and acceleration with time.

1. **Displacement-Time Curve (*s-t* curve)**

In displacement-time curve, time is plotted along x-axis (abscissa) and displacement is plotted along y-axis (ordinate).

The velocity of particle at any instant of time t is the slope of s-t curve at that instant.

$$v = \frac{ds}{dt} \text{ (slope)}$$

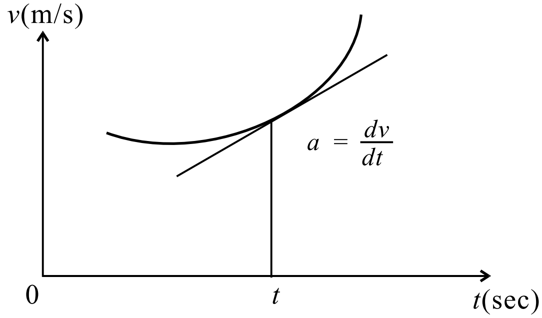


2. **Velocity-Time Curve (*v-t* curve)**

In velocity-time curve, time is plotted along x-axis (abscissa) and velocity is plotted along y-axis (ordinate).

(a) The acceleration of a particle at any instant of time t is the slope of v-t diagram at that instant.

$$a = \frac{dv}{dt} \text{ (slope)}$$



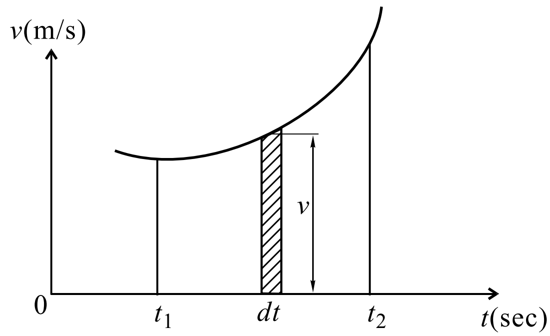
(b) Let the particle's position be s_1 at time t_1 and s_2 at time t_2. From v-t curve, we have

area of the elemental strip $= vdt$

$\therefore \int_{t_1}^{t_2} vdt$ represents the entire area under v-t curve between time t_1 and t_2.

$$\therefore \int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} vdt$$

$$\therefore s_2 - s_1 = \text{Area under } v\text{-}t \text{ curve}$$



$\therefore$ Change in displacement = Area under v-t curve

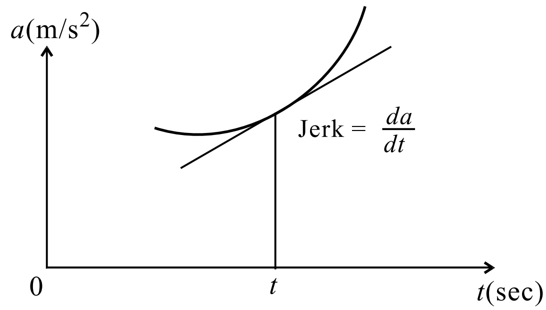
Thus, change in displacement of a particle in given interval of time is equal to area under v-t curve during the same interval of time.

3. **Acceleration-Time Curve (*a-t* curve)**

(a) In acceleration-time curve, time is plotted along x-axis (abscissa) and acceleration is plotted along y-axis (ordinate).

The slope of a-t curve is the jerk.

$$\text{Jerk} = \frac{da}{dt} \text{ (slope)}$$



(ii) Let the particles velocity be v_1 at time t_1 and v_2 at time t_2. From a-t curve, we have

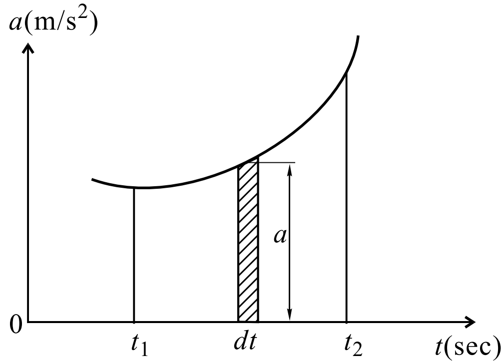
Area of the elemental strip $= adt$

$\therefore \int_{t_1}^{t_2} adt$ represents the entire area under a-t curve between time t_1 and t_2.

$$\therefore \int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} adt$$

$\therefore$ $v_2 - v_1$ = Area under a-t curve

$\therefore$ Change in velocity = Area under a-t curve

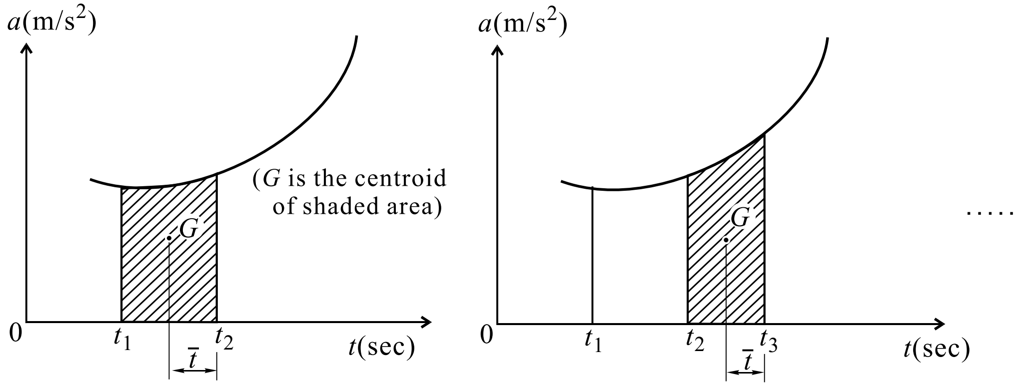


Thus, change in velocity of a particle in a given interval of time is equal to area under a-t curve during the same interval of time.

(iii) For finding displacement use moment area method.

- At time t_1 particles position and velocity is s_1 and v_1, respectively.
- At time t_2 particles position and velocity is s_2 and v_2, respectively.
- At time t_3 particles position and velocity is s_3 and v_3, respectively.

So on and so forth.



$s_2 - s_1 = v_1 (t_2 - t_1)$ + (Area under a-t curve between t_1 and t_2) ($\bar{t}$)

$s_3 - s_2 = v_2 (t_3 - t_2)$ + (Area under a-t curve between t_2 and t_3) ($\bar{t}$)

so on and so forth.

4. Velocity-Displacement Curve (v-s curve)

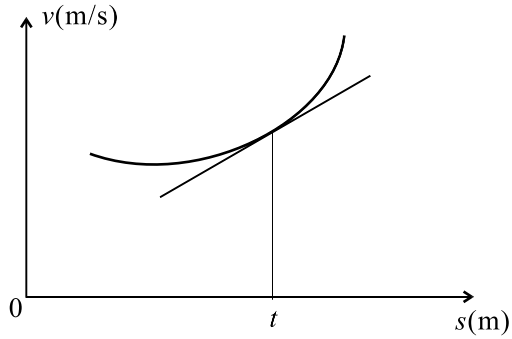
In velocity-displacement curve, displacement is plotted along x-axis (abscissa) and velocity is plotted along y-axis (ordinate).

From v-s curve, we have

$$\text{slope} = \frac{dv}{ds}$$

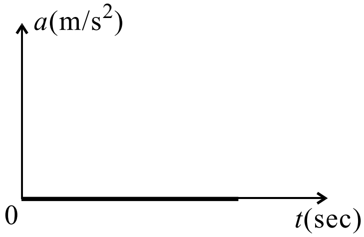
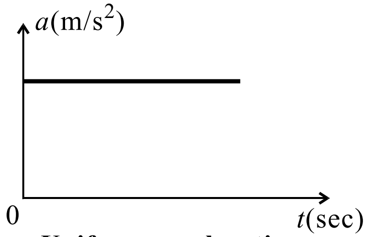
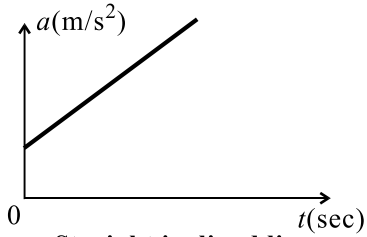
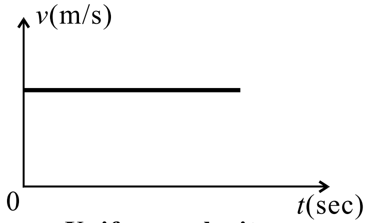
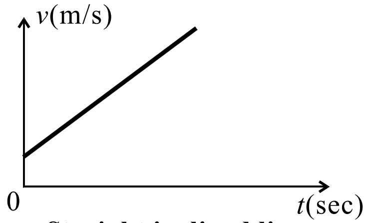
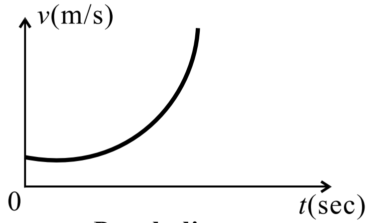
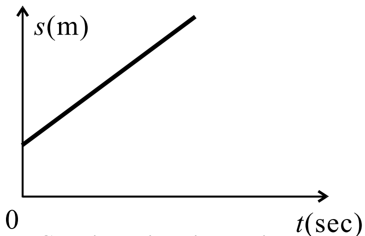
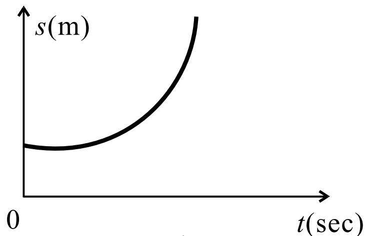
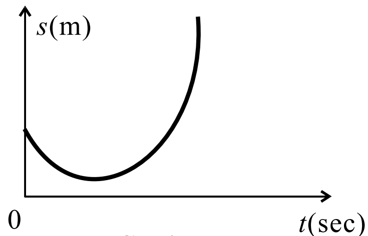
We know, $$a = v\frac{dv}{ds}$$

$$\therefore a = (v)\,(\text{slope of } v\text{-}s \text{ curve})$$

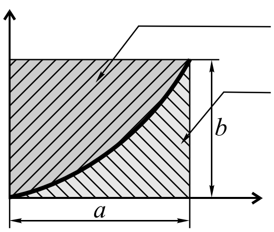


11.4.1 Important Relations of Motion Curve

1. If the acceleration is a polynomial of degree n, the degree of velocity is $(n + 1)$ and the degree of displacement is $(n + 2)$.
2. If the equation of displacement is given, then to find the velocity, displacement is differentiated w.r.t. time and to find acceleration, velocity is differentiated w.r.t. time.
3. If the equation of acceleration is given, then to find the velocity, integrate acceleration w.r.t. time and to find displacement, integrate velocity w.r.t. time. The sufficient conditions required to find the constants of integration will be give in the problem. If not mentioned then we can assume that the particle starts from rest from origin.

Uniform Velocity Motion Curve	Uniform Acceleration Motion Curve	Variable Acceleration Motion Curve
 **Zero acceleration**	 **Uniform acceleration degree = 0**	 **Straight inclined line degree = 1**
 **Uniform velocity**	 **Straight inclined line degree = 1**	 **Parabolic curve degree = 2**
 **Straight inclined line degree = 1**	 **Parabolic curve degree = 2**	 **Cubic curve degree = 3**

Area bounded by curve



$A_1 = \dfrac{nab}{n+1}$

$A_2 = \dfrac{ab}{n+1}$

$A = A_1 + A_2 = ab$

n = Curve of degree of polynomial

11.5 Solved Problems Based on Rectilinear Motion with Uniform (Constant) Acceleration and Uniform (Constant) Velocity

Problem 1

A particle starts moving along a straight line with initial velocity of 25 m/s, from O under a uniform acceleration of -2.5 m/s^2. Determine

(i) velocity, displacement and the distance travelled at $t = 5$ sec,

(ii) how long the particle moves in the same direction ? What are its velocity, displacement and distance covered then ?

(iii) the instantaneous velocity, displacement and the distance covered at $t = 15$ sec,

(iv) the time required to come back to O, velocity displacement and distance covered then and

(v) instantaneous velocity, displacement and distance covered at $t = 25$ sec.

Solution

Given : $u = 25$ m/s, $a = -2.5$ m/s^2, O is the origin.

(i) $t = 5$ sec. Refer to Fig. 11.1(a)

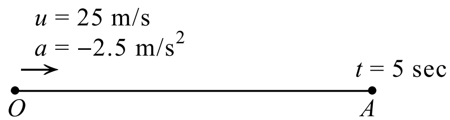


Fig. 11.1(a)

$v = u + at$

$v = 25 + (-2.5) \times 5$

$v = 12.5$ m/s $(\rightarrow)$

Now, $s = ut + \frac{1}{2}at^2$

$s = 25 \times 5 + \frac{1}{2} \times (-2.5) \times 5^2$

$s = 93.75$ m $(\rightarrow)$

Since velocity is positive, the particle is moving in the same direction and therefore displacement is equal to the distance travelled.

$\therefore$ $d = s = 93.75$ m. ***Ans.***

(ii) The particle moves in the same direction till it comes to rest because of negative acceleration and then its direction will reverse.

At the above-said instant velocity of particle will be zero

$v = 0$ (Point of reversal)

Let t be the time taken by the particle to move in the same direction [Refer to Fig. 11.1(b)], we have

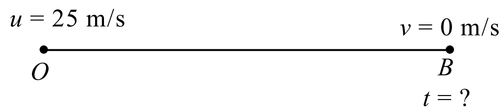


Fig. 11.1(b)

$v = u + at$

$0 = 25 + (-2.5)t$

$t = 10$ sec ***Ans.***

For displacement, we have

$s = ut + \frac{1}{2}at^2$

$s = 25 \times 10 + \frac{1}{2} \times (-2.5) \times 5^2$

$s = 218.75$ m $(\rightarrow)$

As particle has not reversed its direction, we have

Displacement = Distance travelled

$s = d = 218.75$ m. ***Ans.***

(iii) $t = 15$ sec. Refer to Fig. 11.1(c).

$v = u + at$

$v = 25 + (-2.5) \times 15 = -12.5$ m/s

$v = 12.5$ m/s $(\leftarrow)$

Now, $s = ut + \frac{1}{2}at^2$

$s = 25 \times 15 + \frac{1}{2} \times (-2.5) \times 15^2 = 93.75$ m $(\rightarrow)$

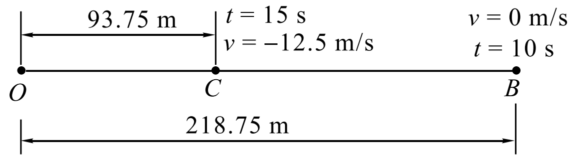


Fig. 11.1(c)

Particle is moving from O to B and then from B to C in $t = 15$ sec.

$\therefore$ Distance travelled $d = OB + BC \quad (BC = OB - OC)$

$d = 218.75 + (218.75 - 93.75)$

$d = 343.75$ m ***Ans.***

(iv) Let t be the time taken by particle to reach origin. Refer to Fig. 11.1(d).

$\therefore$ Displacement = 0

$s = ut + \frac{1}{2}at^2$

$0 = 25 \times t + \frac{1}{2} \times (-2.5) \times t^2$

$t = 20$ sec

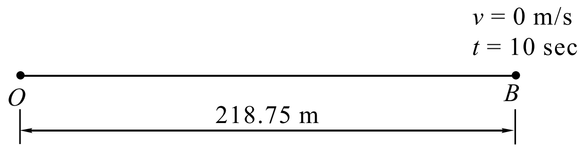


Fig. 11.1(d)

$v = u + at$

$v = 25 + (-2.5) \times 20 = -25$ m/s

$v = 25$ m/s $(\leftarrow)$

Distance covered $d = OB + BO = 218.75 + 218.75$

$d = 437.5$ m ***Ans.***

(v) At $t = 25$ sec. Refer to Fig. 11.1(e).

$v = u + at$

$v = 25 + (-2.5) \times 25$

$v = -37.5$ m/s

$v = 37.5$ m/s $(\leftarrow)$ ***Ans.***

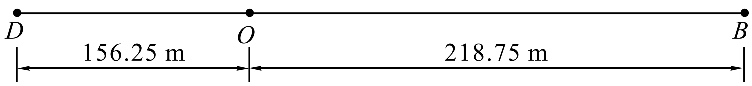


Fig. 11.1(e)

Now, $s = ut + \frac{1}{2}at^2$

$s = 25 \times 25 + \frac{1}{2} \times (-2.5) \times 25^2 = -156.25$ m

$s = 156.25$ m $(\leftarrow)$ ***Ans.***

Distance covered $d = OB + BO + OD = 218.75 + 218.75 + 156.25$

$d = 593.75$ m ***Ans.***

Problem 2

A particle travels along a straight line path such that in 4 sec it moves from an initial position $S_A = -8$ m to position $S_B = +3$ m. Then in another 5 sec it moves from S_B to $S_C = -6$ m. Determine the particle's average velocity and average speed during 9 sec interval.

Solution

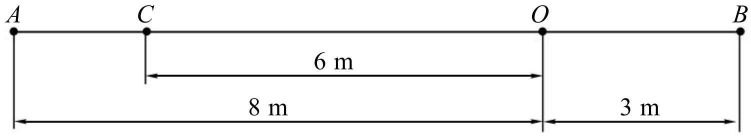


Fig. 11.2

(i) Average velocity $= \dfrac{\text{Final displacement} - \text{Initial displacement}}{\text{Time interval}}$

Average velocity $= \dfrac{-6-(-8)}{9} = \dfrac{2}{9}$

$v_{ave} = 0.2222$ m/s ***Ans.***

(ii) Average speed $= \dfrac{\text{Total distance travelled}}{\text{Time interval}}$

Average velocity $= \dfrac{AO + OB + BO + OC}{\text{Time interval}} = \dfrac{8+3+3+6}{9} = \dfrac{20}{9}$

Average speed $= 2.222$ m/s ***Ans.***

Problem 3

A motorist is travelling at 90 kmph, when he observes a traffic light 250 m ahead of him turns red. The traffic light is timed to stay red for 12 sec. If the motorist wishes to pass the light without stopping, just as it turns green. Determine **(i)** the required uniform deceleration of the motor and **(ii)** the speed of the motor as it passes the traffic light.

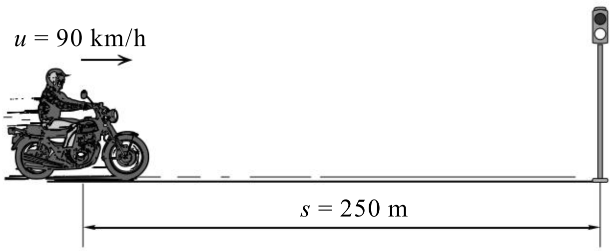


Fig. 11.3

Solution

Given : Initial velocity $u = 90$ km/hr $\therefore u = \dfrac{90 \times 5}{18} = 25$ m/s, Time $t = 12$ s and Displacement $s = 250$ m.

(i) $s = ut + \dfrac{1}{2}at^2$

$250 = 25 \times 12 + \dfrac{1}{2} \times a \times 12^2$

$a = -0.6944 \text{ m/s}^2$ ***Ans.*** (Negative sign indicates deceleration)

(ii) $v = u + at$

$v = 25 + (-0.6944) \times 12 = 16.67$ m/s $(\rightarrow)$ $\therefore v = 16.67 \times \dfrac{18}{5}$

$v = 60$ kmph ***Ans.*** (The speed of the motor cycle as it passes the traffic light)

Problem 4

Determine the time required for a car to travel 1 km along a road if the car starts from rest, reaches a maximum speed at some intermediate point, and then stops at the end of the road. The car can accelerate or decelerate at 1.5 m/s^2.

Solution

Method I : By using equation of motion

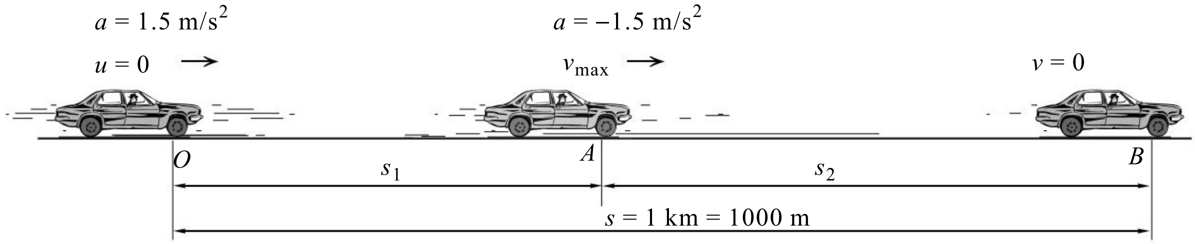


Fig. 11.4(a)

(i) Consider the motion of car from O to A [Refer to Fig. 11.4(a)].

Initial velocity $u = 0$; Final velocity $v = v_{max}$;

Acceleration $a = 1.5$ m/s^2 ; Time $t = t_1$;

Displacement $s = s_1$;

$$v = u + at$$

$$v_{max} = 0 + 1.5 \times t_1 \qquad \therefore \; t_1 = \frac{v_{max}}{1.5}$$

$$s = ut + \frac{1}{2}at^2$$

$$s_1 = (0)(t_1) + \frac{1}{2} \times 1.5 \times t_1^2$$

$$s_1 = 0.75\left(\frac{v_{max}}{1.5}\right)^2$$

(ii) Consider the motion of car from A to B.

Initial velocity $u = v_{max}$; Final velocity $v = 0$;

Acceleration $a = -1.5$ m/s^2 ; Time $t = t_2$;

Displacement $s = s_2$;

$$v = u + at$$

$$0 = v_{max} + (-1.5) \times t_2 \quad \therefore \; t_2 = \frac{v_{max}}{1.5}$$

$$s = ut + \frac{1}{2}at^2$$

$$s_2 = v_{max}\left(\frac{v_{max}}{1.5}\right) + \frac{1}{2} \times (-1.5)\left(\frac{v_{max}}{1.5}\right)^2$$

$$s_2 = v_{max}\left(\frac{v_{max}}{1.5}\right) - 0.75\left(\frac{v_{max}}{1.5}\right)^2$$

(iii) Total displacement $s = s_1 + s_2$

$$1000 = 0.75\left(\frac{v_{max}}{1.5}\right)^2 + \left[\frac{v_{max}^2}{1.5} - 0.75\left(\frac{v_{max}}{1.5}\right)^2\right]$$

$$1000 = \frac{v_{max}^2}{1.5}$$

$v_{max} = 38.73$ m/s ***Ans.***

(iv) Total time $t = t_1 + t_2$

$$t = \frac{v_{max}}{1.5} + \frac{v_{max}}{1.5}$$

$$t = \frac{38.73}{1.5} + \frac{38.73}{1.5}$$

$t = 51.64$ sec ***Ans.***

Method II : By using *v-t* diagram

(i) Consider the v-t diagram shown in Fig. 11.4(b).

Since the magnitude of acceleration is same, therefore, the slope is same.

$\therefore\ t_1 = t_2 \qquad \therefore\ t = 2t_1$

Slope = Acceleration = $1.5 = \dfrac{v_{max}}{t_1}$

$\therefore\ v_{max} = 1.5t_1$ .....(I)

Area under v-t diagram is displacement

$$\therefore\ 1000 = \frac{1}{2} \times 2t_1 \times v_{max}$$

$$v_{max} = \frac{1000}{t_1} \qquad .....(II)$$

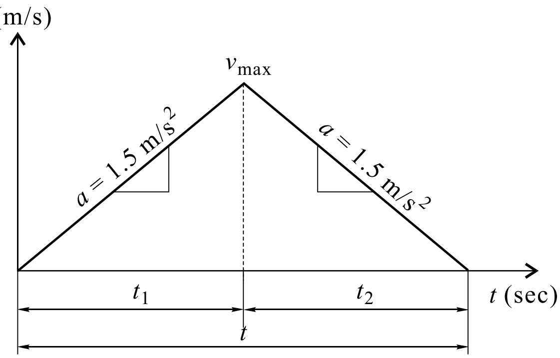


Fig. 11.4(b)

(ii) Equating Eqs. (I) and (II)

$$1.5t_1 = \frac{1000}{t_1}$$

$t_1 = 25.82$ sec

$\therefore\ t = 2 \times t_1 = 51.64$ sec

$v_{max} = 1.5 \times 25.82$

$v_{max} = 38.73$ m/s ***Ans.***

Problem 5

In Asian games, for 100 m event an athlete accelerates uniformly from the start to his maximum velocity in a distance of 4 m and runs the remaining distance with that velocity. If the athlete finishes the race in 10.4 seconds, determine **(i)** his initial acceleration and **(ii)** his maximum velocity.

Solution

Method I : By using equation of motion

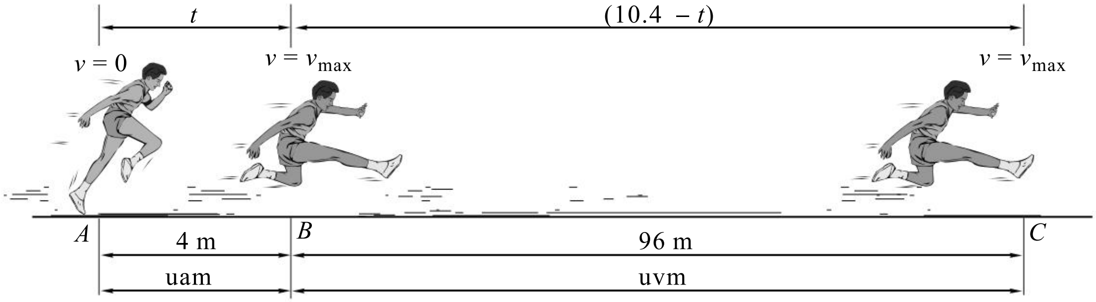


Fig. 11.5(a)

(i) Consider the uniform acceleration motion (uam) from A to B [Refer to Fig. 11.5(a)].

Initial velocity $u = 0$; Final velocity $v = v_{max}$; Displacement $s = 4$ m ;

Acceleration $a = ?$; Time $t = ?$

$v = u + at$

$v_{max} = 0 + at = at$ ..... (I)

$s = ut + \frac{1}{2}at^2 \quad \Rightarrow \quad 4 = 0 + \frac{1}{2}at^2 \quad \Rightarrow \quad 8 = (at)t$

$8 = (v_{max})t \quad \therefore \quad v_{max} = \frac{8}{t}$ ..... (II)

(ii) Consider the uniform velocity motion (uvm) from B to C.

Velocity $= v_{max}$; Time $= (10.4 - t)$

Displacement = Velocity × Time

$96 = v_{max}(10.4 - t)$

Substituting $v_{max} = \frac{8}{t}$ from Eq. (II), we get $96 = \frac{8}{t}(10.4 - t)$

$\therefore \; 96t = 8 \times 10.4 - 8t \quad \Rightarrow \quad 104t = 8 \times 10.4 \quad \therefore \; t = 0.8$ sec

(iii) From Eq. (II)

$v_{max} = \frac{8}{t} = \frac{8}{0.8} \qquad \therefore \; v_{max} = 10$ m/s ***Ans.***

(iv) From equation (I), $v_{max} = at$

$a = \frac{v_{max}}{t} = \frac{10}{0.8} \qquad \therefore \; a = 12$ m/s^2 ***Ans.***

Method II : By using *v-t* diagram

Consider the v-t diagram shown in Fig. 11.5(b).

(i) Let t be the time taken by athlete to attain maximum velocity (v_{max}) in a distance of 4 m.

$\therefore$ Time taken for remaining distance 96 m will be $(10.4 - t)$.

Area under *v-t* diagram is displacement, so we have

$$4 = \frac{1}{2} \times t \times v_{max} \quad \text{and} \quad 96 = v_{max}(10.4 - t)$$

$$\therefore \; v_{max} = \frac{8}{t} \quad \text{and} \quad v_{max} = \frac{96}{10.4 - t}$$

Equating v_{max}, we have

$$\therefore \; \frac{8}{t} = \frac{96}{10.4 - t} \qquad \therefore \; t = 0.8 \text{ sec}$$

(ii) $v_{max} = \frac{8}{t} = \frac{8}{0.8} = 10$ m/s ***Ans.***

(iii) Acceleration = Slope of *v-t* diagram

$$a = \frac{v_{max}}{t} = \frac{10}{0.8} \qquad \therefore \; a = 12.5 \text{ m/s}^2$$ ***Ans.***

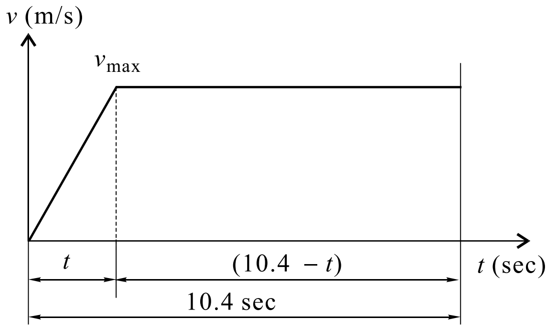


Fig. 11.5(b)

Problem 6

A radar-equipped police car observes a truck travelling at 110 kmph. The police car starts pursuit 30 sec after the observation, accelerates to 160 kmph in 20 sec. Assuming the speeds are maintained constant on a straight road, how far from the observation point, will the chase end ?

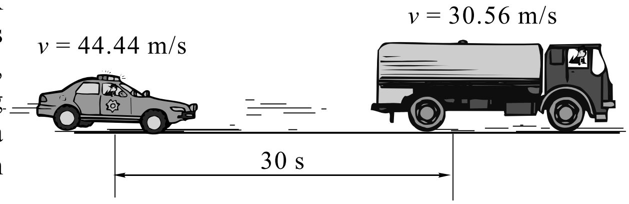


Fig. 11.6(a)

Solution

Observe the *v-t* diagram of the truck and police car, shown in Fig. 11.6(b) and (c).

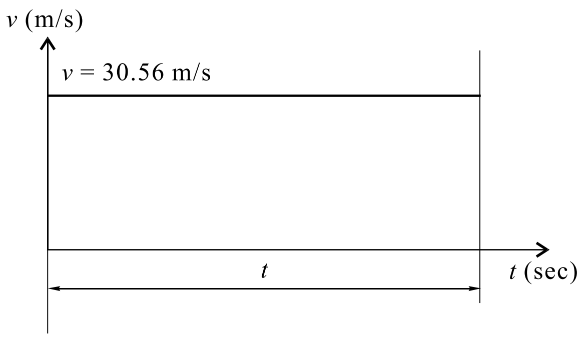


Fig. 11.6(b) : Truck

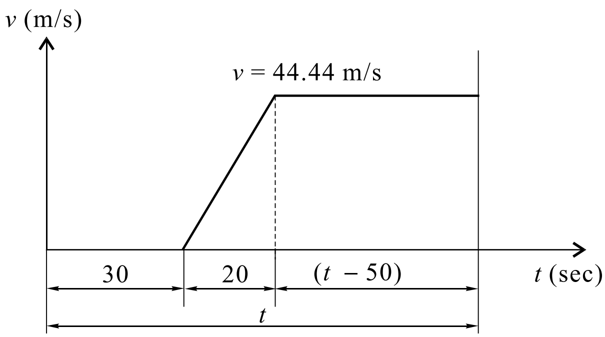


Fig. 11.6(c) : Police car

Let t be the time interval from the observation point to the point where the chase ends.

Distance covered by truck and police car will be same. We know, area under *v-t* diagram is distance covered.

Area under *v-t* diagram of truck = Area under *v-t* diagram of police car

$$(30.56)(t) = \frac{1}{2} \times 20 \times 44.44 + 44.44\,(t - 50)$$

$\therefore \; t = 128.07$ sec ***Ans.***

$\therefore$ Distance covered = 30.56×128.07

$s = 3913.82$ m ***Ans.***

Problem 7

A train travelling with a speed of 90 kmph slow down on account of work in progress, at a retardation of 1.8 kmph per second to 36 kmph. With this, it travels 600 m. Thereafter, it gains further speed with 0.9 kmph per second till getting the original speed. Find the delay caused.

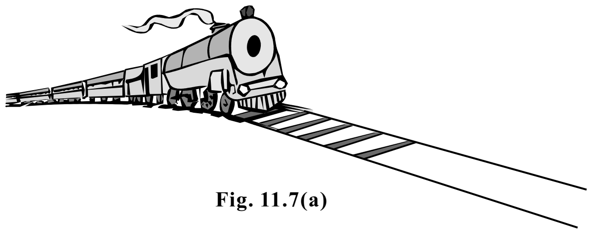


Fig. 11.7(a)

Solution

Observe the *v*-*t* diagram of train shown in Fig. 11.7(b).

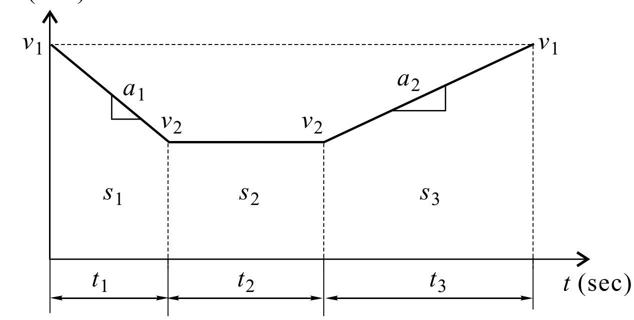


Fig. 11.7(b)

(i) *Given* : $v_1 = 90 \times \frac{5}{18} = 25$ m/s

$$a_1 = 1.8 \times \frac{5}{18} = 0.5 \text{ m/s}^2$$

$$v_2 = 36 \times \frac{5}{18} = 10 \text{ m/s}$$

$$a_2 = 0.9 \times \frac{5}{18} = 0.25 \text{ m/s}^2$$

$$s_2 = 600 \text{ m}$$

(ii) First phase of motion (constant acceleration)

$v = u + at$

$v_2 = v_1 + a_1 t_1$

$10 = 25 + (-0.5) \times t_1 \qquad \therefore\ t_1 = 30$ sec

Distance covered = Area under *v*-*t* diagram

$s_1 = \frac{1}{2}(v_1 + v_2)\, t_1 = \frac{1}{2}(25 + 10)\, 30 \qquad \therefore\ s_1 = 525$ m

(iii) Second phase of motion (constant velocity)

$s_2 = v_2 \times t_2$

$600 = 10 \times t_2 \qquad \therefore\ t_2 = 60$ seconds

(iv) Third phase of motion (constant acceleration)

$v_1 = v_2 + a_2 t_3$

$25 = 10 + (0.25) \times t_3 \qquad \therefore\ t_3 = 60$ seconds

Distance covered = Area under *v*-*t* diagram

$s_3 = \frac{1}{2}(v_2 + v_1)\, t_3 = \frac{1}{2}(10 + 25)\, 60$

$\therefore\ s_3 = 1050$ m

(v) Total distance covered = $s_1 + s_2 + s_3$

$d = 525 + 600 + 1050 \qquad \therefore\ d = 2175$ m

(vi) Total time taken = $t_1 + t_2 + t_3$

$t = 30 + 60 + 60 \qquad \therefore\ t = 150$ seconds

(vii) If there would have been no '*work in progress*', the speed would have been constant.

$v_1 = 25$ m/s

$\therefore$ Time required to travel would have been

$$t' = \frac{\text{Distance}}{\text{Speed}} = \frac{2175}{25}$$

$t' = 87$ sec

(viii) The delay caused = $t - t' = 150 - 87 = 63$ sec ***Ans.***

Problem 8

An elevator goes down a mine shaft 600 m deep in 60 seconds. For the first quarter of distance, only the speed is being uniformly accelerated and during the last quarter uniformly retarded, the acceleration and retardation being equal. Find the uniform speed of the elevator while travelling central portion of shaft.

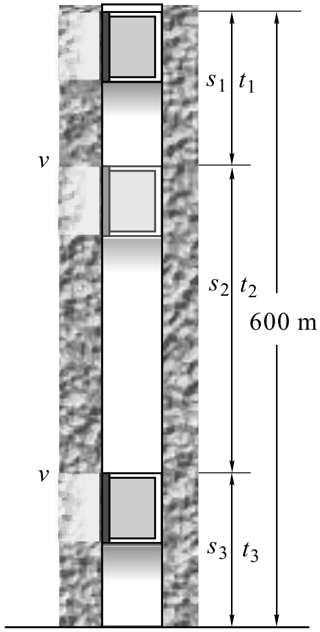


Fig. 11.8(a)

Solution

(i) Observe the v-t diagram of the elevator shown in Fig. 11.8(b).

As per given condition in problem, we have

Distance travelled in first quarter = Distance travelled in last quarter

$$s_1 = s_3 = \frac{600}{4} = 150 \text{ m}$$

$\therefore$ $s_2 = 300$ m

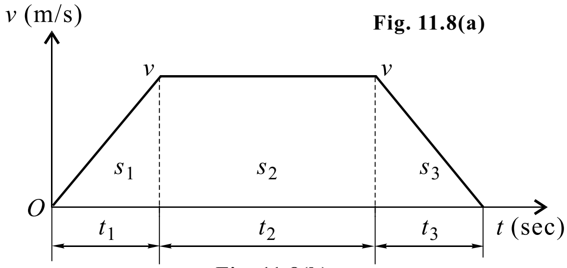


Fig. 11.8(b)

(ii) From v-t diagram, we have

Distance = Area under v-t diagram

$$s_1 = \frac{1}{2} \times t_1 \times v \quad ; \quad s_2 = v \times t_2 \quad ; \quad s_3 = \frac{1}{2} \times t_3 \times v$$

$$150 = \frac{vt_1}{2} \quad ; \quad s_2 = vt_2 \quad ; \quad 150 = \frac{vt_3}{2}$$

$$150 = \frac{vt_1}{2} = \frac{vt_3}{2} \quad \therefore \; t_1 = t_3$$

$$s_2 = 2s_1$$

$$vt_2 = 2 \times \frac{1}{2} \times vt_1 \quad \therefore \; t_1 = t_2$$

$$\therefore \; t_1 = t_2 = t_3 = \frac{60}{3} = 20 \text{ sec}$$

(iii) Uniform velocity motion during central portion of shaft

$$v = \frac{\text{Displacement}}{\text{Time}} = \frac{300}{20}$$

$v = 15$ m/s ***Ans.***

Problem 9

A burglar's car had a start with an acceleration 2 m/s^2. A police vigilant came in a van to the spot at a velocity of 20 m/s after 3.75 seconds and continued to chase the burglar's car with uniform velocity. Find the time in which the police van will overtake the burglar's car.

Solution

Given : Burglar's car $\Rightarrow$ $t = 0$; $a = 2$ m/s^2

Police van $\Rightarrow$ $t = 3.75$ sec ; $v = 20$ m/s (constant)

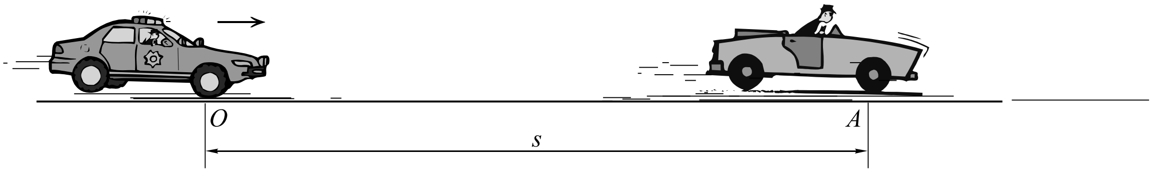


Fig. 11.9

(i) Let t be the time duration for motion of burglar's car. Motion of police van starts after 3.75 sec from the given spot. Therefore, time interval will be $(t - 3.75)$ seconds.

Motion of Burglar's car
(constant acceleration)

$$s = ut + \frac{1}{2}at^2$$

$$s = 0 + \frac{1}{2} \times 2 \times t^2$$

$$s = t^2 \qquad \text{..... (I)}$$

Motion of Police Van
(constant velocity)

$$s = v \times t$$

$$s = 20(t - 3.75) \qquad \text{..... (II)}$$

(ii) Equating Eqs. (I) and (II), we have

$$t^2 = 20(t - 3.75)$$

$$t^2 = 20t - 75$$

$$t^2 - 20t + 75 = 0$$

Solving the quadratic equation, we get

$$t = 5 \text{ sec and } t = 15 \text{ sec}$$

(iii) In the beginning, the velocity of burglar's car is less than police van. Therefore, at $t = 5$ sec the police van overtakes burglar's car.

Since burglar's car is moving with constant acceleration, thus as time progresses velocity of the car also increases, but velocity of the police van remains the same. $\therefore$ At $t = 15$ seconds burglar's car will overtake police van.

11.6 Solved Problems Based on Rectilinear Motion Under Gravity

Problem 10

A stone is dropped from the top of a tower. When it has fallen a distance of 10 m, another stone is dropped from a point 38 m below the top of the tower. If both the stones reach the ground at the same time, calculate

(i) the height of the tower and

(ii) the velocity of the stones when they reach the ground.

Solution

(i) Motion of Stone (1)

$$u_1 = \sqrt{2 \times 9.81 \times 10}$$

$$u_1 = 14 \text{ m/s}$$

$$h - 10 = u_1 t + \frac{1}{2} \times 9.81 \times t^2 \quad \text{..... (I)}$$

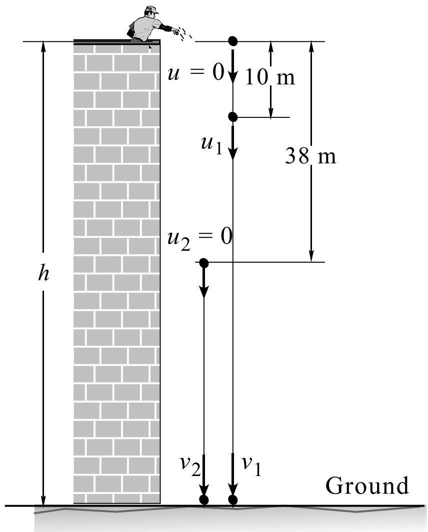


Fig. 11.10

(ii) Motion of Stone (2)

$$h - 38 = u_1 t + \frac{1}{2} \times 9.81 \times t^2 \quad \text{..... (II)}$$

(iii) Solving Eqs. (I) and (II)

$$h = 57.62 \text{ m}$$

$$t = 2 \text{ sec}$$

$$v_1 = \sqrt{2 \times 9.81 \times h}$$

$$v_1 = \sqrt{2 \times 9.81 \times 57.62}$$

$$v_1 = 33.62 \text{ m/s } (\downarrow) \quad \textit{Ans.}$$

$$v_2 = \sqrt{2 \times 9.81 \times (h - 38)}$$

$$v_2 = \sqrt{2 \times 9.81 \times 19.62}$$

$$v_2 = 19.62 \text{ m/s } (\downarrow) \quad \textit{Ans.}$$

Problem 11

A stone is dropped from a balloon at an altitude of 600 m. How much time is required for the stone to reach the ground if the balloon is

(i) Ascending with a velocity of 10 m/s,

(ii) Descending with a velocity of 10 m/s,

(iii) Stationary and

(iv) Ascending with a velocity of 10 m/s and an acceleration of 1 m/s^2 (Neglect the air resistance).

Solution : (i) Balloon is ascending with a velocity of 10 m/s

Method I

(a) Motion from *A* to *B* (↑)

Initial velocity of balloon (stone) = $u_b = u_s = 10$ m/s (↑) ;
Final velocity of stone = $v = 0$; $g = -9.81$ m/s^2
Time $t = t_1$; Displacement $h = h_1$.

$$v = u + gt$$
$$0 = 10 + (-9.81)(t_1)$$
$$t_1 = 1.02 \text{ sec}$$
$$h_1 = ut + \frac{1}{2}gt^2$$
$$h_1 = 10 \times 1.02 + \frac{1}{2}(-9.81) \times (1.02)^2$$
$$h_1 = 5.1 \text{ m}$$

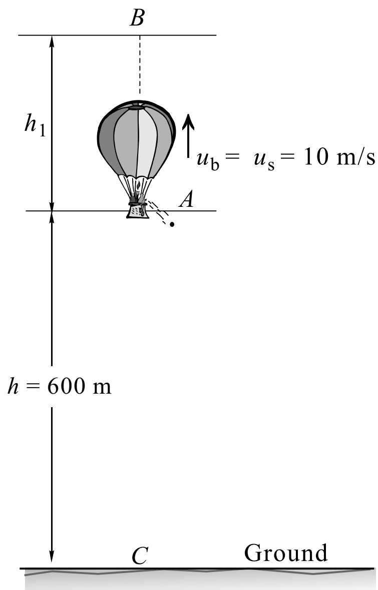


Fig. 11.11(a)

(b) Motion from *B* to *C* (↓)

Initial velocity of stone $u_s = 0$;
Displacement $h = 600 + 5.1 = 605.1$ m ;
Time $t = t_2$.

$$h = ut + \frac{1}{2}gt^2$$
$$605.1 = 0 \times t_2 + \frac{1}{2} \times 9.81 \times (t_2)^2$$
$$t_2 = 11.11 \text{ sec}$$

Total time $t = t_1 + t_2 = 1.02 + 11.11$

$t = 12.13$ sec ***Ans.***

Method II : Consider initial position *A* and final position *C*

Initial velocity of balloon (stone) = $u_b = u_s = 10$ m/s (↑) ; $g = -9.81$ m/s^2
Displacement $h = -600$ m ; Time $t = t$.

$$h = ut + \frac{1}{2}gt^2$$
$$-600 = 10 \times t + \frac{1}{2}(-9.81) \times t^2$$

$$4.905t^2 - 10t - 600 = 0$$

$$t = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(4.905)(-600)}}{2 \times 4.905}$$

$t = 12.13$ sec ***Ans.***

(ii) Balloon is descending with a velocity of 10 m/s (↓)

Initial velocity of balloon (stone)

$u_b = u_s = 10$ m/s (↓) ;

Displacement $h = 600$ m ; $g = 9.81$ m/s^2

Time $t = t$.

$$h = ut + \frac{1}{2}gt^2$$

$$600 = 10 \times t + \frac{1}{2} \times 9.81 \times t^2$$

$$4.905\,t^2 + 10\,t - 600 = 0$$

Solving quadratic equation, we get

$t = 10.09$ sec ***Ans.***

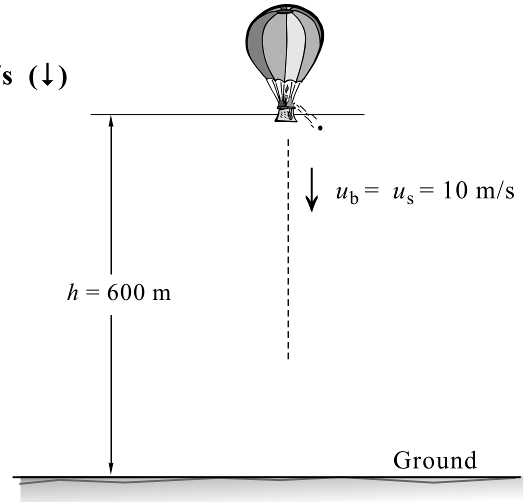


Fig. 11.11(b)

(iii) Balloon is stationary

Initial velocity of balloon (stone)

$u_b = u_s = 0$

Displacement $h = 600$ m ; $g = 9.81$ m/s^2

Time $t = t$.

$$h = ut + \frac{1}{2}gt^2$$

$$600 = 0 + \frac{1}{2} \times 9.81 \times t^2$$

$t = 11.06$ sec ***Ans.***

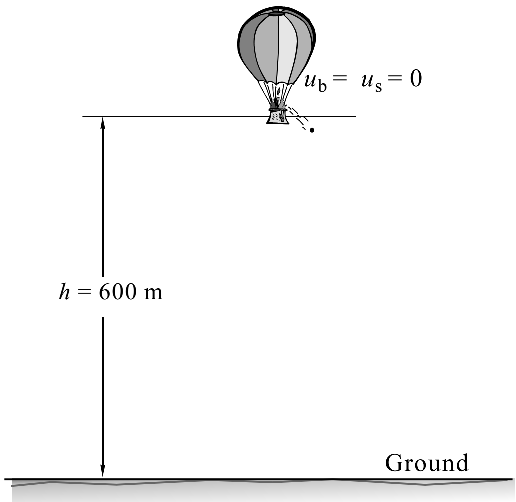


Fig. 11.11(c)

(iv) Balloon ascending with a velocity of 10 m/s and an acceleration of 1 m/s^2.

As long as the stone is attached to balloon, acceleration is influencing increase in velocity of balloon and stone simultaneously. At the instant when stone is dropped from balloon, gravity takes over and its motion is unaffected by acceleration of balloon. So, stone simply carries the instantaneous velocity of balloon.

Therefore, cases (i) and (iv) are similar.

$$h = ut + \frac{1}{2}gt^2$$

$$-600 = 10 \times t + \frac{1}{2}(-9.81) \times t^2$$

$$4.905\,t^2 - 10\,t - 600 = 0$$

$t = 12.13$ sec ***Ans.***

Problem 12

A body A is projected vertically upwards from the top of a tower with a velocity of 40 m/s, the tower being 180 m high. After t seconds, another body B is allowed to fall from the same point. Both the bodies reach the ground simultaneously. Calculate t and the velocities of A and B on reaching the ground.

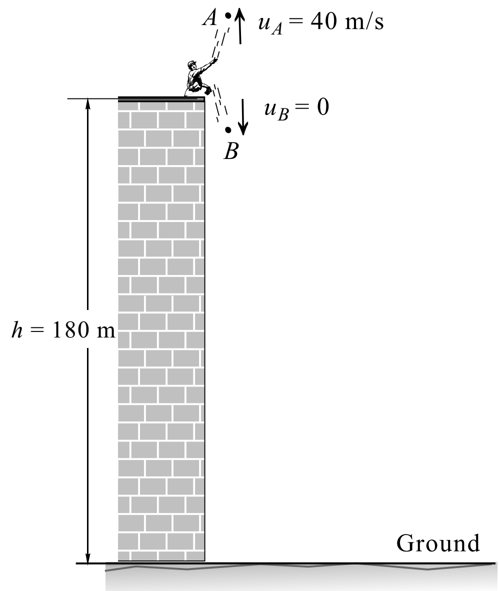


Fig. 11.12

Solution

(i) Motion of Body A

$$u = u_A = 40 \text{ m/s } (\uparrow);$$
$$h = -180 \text{ m}; \; g = -9.81 \text{ m/s}^2$$
$$t = t_A$$
$$h = ut + \frac{1}{2}gt^2$$
$$-180 = 40t_A + \frac{1}{2}(-9.81) \times (t_A)^2$$
$$4.905t_A^2 - 40t_A - 180 = 0$$
$$t_A = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(4.905)(-180)}}{2 \times 4.905}$$
$$t_A = 11.38 \text{ sec}$$

Motion of Body B

$$u = u_B = 0 \; (\downarrow)$$
$$h = 180 \text{ m}; \; g = 9.81 \text{ m/s}^2$$
$$t = t_B$$
$$h = ut + \frac{1}{2}gt^2$$
$$180 = 0 + \frac{1}{2}(9.81) \times (t_B)^2$$
$$t_B = 6.06 \text{ sec}$$

(ii)

$$v = u + gt$$
$$v_A = u_A + gt_A$$
$$v_A = 40 + (-9.81) \times (11.38)$$
$$v_A = -71.64 \text{ m/s}$$
$$v_A = 71.64 \text{ m/s } (\downarrow)$$

$$v = u + gt$$
$$v_B = u_B + gt_B$$
$$v_B = 0 + (9.81) \times (6.06)$$
$$v_B = 59.45 \text{ m/s } (\downarrow)$$

(iii) $t = t_A - t_B = 11.38 - 6.06$

$t = 5.32$ sec ***Ans.***

Problem 13

A ball is thrown vertically upwards at 30 m/s from the top of a tower 100 m high. Five seconds later another ball is thrown upwards from the base of the tower along the same vertical line at 50 m/s. Find when and where both balls will meet and their instantaneous velocity then.

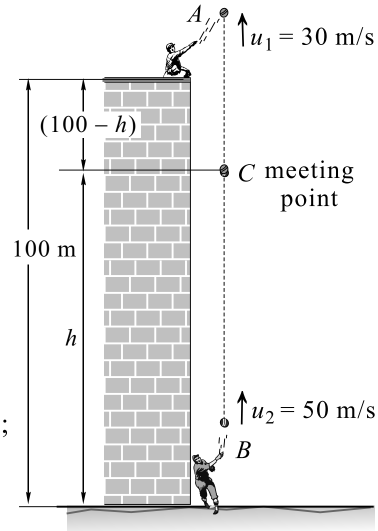


Fig. 11.13

Solution

(i) Motion of first ball from A to C

$u = u_1 = 30$ m/s (↑) ;

$h = -(100 - h)$;

$g = -9.81 \text{ m/s}^2$

$t = t$

$h = ut + \frac{1}{2}gt^2$

$-(100 - h) = 30t + \frac{1}{2}(-9.81) \times t^2$

$h = 30t - 4.905t^2 + 100$ ..... (I)

Motion of second ball from B to C

$u = u_2 = 50$ m/s (↑) ;

$h = h$;

$g = -9.81 \text{ m/s}^2$

$t = (t - 5)$

$h = ut + \frac{1}{2}gt^2$

$h = 50(t - 5) + \frac{1}{2}(-9.81) \times (t - 5)^2$

$h = 50(t - 5) - 4.905(t - 5)^2$ ..... (II)

(ii) Equating Eqs. (I) and (II)

$30t - 4.905t^2 + 100 = 50(t - 5) - 4.905(t - 5)^2$

$30t - 4.905t^2 + 100 = 50t - 250 - 4.905(t^2 - 10t + 25)$

$30t - 4.905t^2 + 100 = 50t - 250 - 4.905t^2 + 49.05t - 122.625$

$69.05t = 472.625$

$t = 6.85$ seconds ***Ans.***

(iii) From Eq. (I), we get

$h = 30 \times 6.85 - 4.905 \times (6.85)^2 + 100$

$h = 75.35$ m (Meeting point of balls from the ground)

(iv) Velocity of first ball

$v = u + gt$

$v = 30 + (-9.81) \times (6.85)$

$v = -37.52$ m/s

$v = 37.52$ m/s (↓) ***Ans.***

Velocity of second ball

$v = u + gt$

$v = 50 + (-9.81) \times (6.85 - 5)$

$v = 31.85$ m/s (↑) ***Ans.***

Problem 14

From the edge of a cliff, two stones are thrown at the same time, one vertically upwards and the other vertically downwards with the same speed of 20 m/s. The second stone reaches the ground in 5 seconds. How long will the first be in the air ? Also find the height of the cliff.

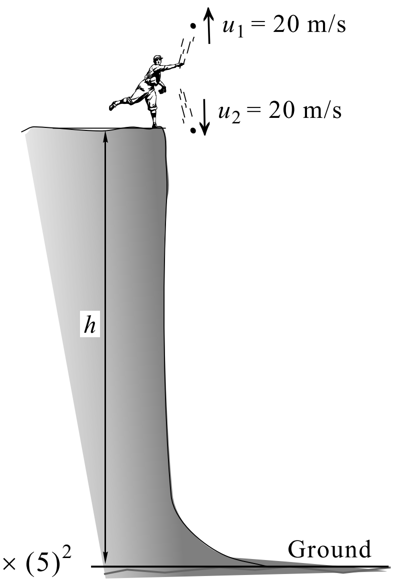


Fig. 11.14

Solution

(i) Motion of first stone

$$u = 20 \text{ m/s } (\uparrow) ;$$
$$h = -h ;$$
$$g = -9.81 \text{ m/s}^2$$
$$t = t$$
$$h = ut + \frac{1}{2} gt^2$$
$$-h = 20 \times t + \frac{1}{2}(-9.81) \times t^2$$
$$h = -20t + 4.905t^2 \quad \text{..... (I)}$$

Motion of second stone

$$u = 20 \text{ m/s } (\downarrow) ;$$
$$h = h ;$$
$$g = 9.81 \text{ m/s}^2$$
$$t = 5 \text{ sec}$$
$$h = ut + \frac{1}{2} gt^2$$
$$h = 20 \times 5 + \frac{1}{2}(9.81) \times (5)^2$$
$$h = 222.625 \text{ m} \quad \text{..... (II)}$$

(ii) Equating Eqs. (I) and (II)

$$-20t + 4.905t^2 = 222.625$$
$$4.905t^2 - 20t - 222.625 = 0$$
$$t = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4.905)(-222.625)}}{2 \times 4.905} \qquad \therefore \quad t = 9.08 \text{ seconds} \quad \textit{\textbf{Ans.}}$$

Ans. Height of the cliff, $h = 222.625$ m and the first stone will be in air for $t = 9.08$ seconds.

Problem 15

Two stones are projected vertically upwards at the same instant. One of them ascends 80 meters higher than the other and returns to the earth 4 seconds later. Find **(i)** the velocities of projection and **(ii)** the maximum heights reached by the stones.

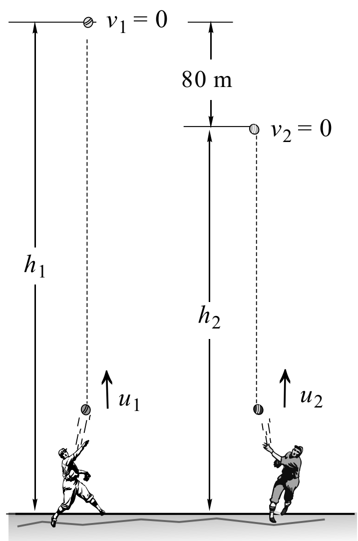


Fig. 11.15

Solution

(i) From the given condition of problem, we have

$$h_1 - h_2 = 80 \text{ m} \quad \text{..... (I)}$$
$$2t_1 - 2t_2 = 4$$
$$t_1 - t_2 = 2 \quad \text{..... (II)}$$

(ii) Motion of first stone

$$u = u_1 ;$$
$$v = v_1 = 0 ;$$
$$t = t_1 ;$$
$$h = h_1 ; g = -9.81 \text{ m/s}^2$$

Motion of second stone

$$u = u_2 ;$$
$$v = v_2 = 0 ;$$
$$t = t_2 ;$$
$$h = h_2 ; g = -9.81 \text{ m/s}^2$$

$$v = u + gt$$
$$0 = u_1 + (-9.81)t_1$$
$$u_1 = 9.81t_1 \quad \text{..... (III)}$$
$$h = ut + \frac{1}{2}gt^2$$
$$h_1 = u_1t_1 + \frac{1}{2}(-9.81)\,t_1^2$$
$$h_1 = 9.81t_1^2 - 4.905\,t_1^2$$
$$h_1 = 4.905\,t_1^2 \quad \text{..... (V)}$$

$$v = u + gt$$
$$0 = u_2 + (-9.81)t_2$$
$$u_2 = 9.81t_2 \quad \text{..... (IV)}$$
$$h = ut + \frac{1}{2}gt^2$$
$$h_2 = u_2t_2 + \frac{1}{2}(-9.81)\,t_2^2$$
$$h_2 = 9.81t_2^2 - 4.905\,t_2^2$$
$$h_2 = 4.905t_2^2 \quad \text{..... (VI)}$$

(iii) From Eq. (I), we have

$$4.905\,t_1^2 - 4.905t_2^2 = 80$$
$$t_1^2 - t_2^2 = 16.31$$
$$(t_1 - t_2)(t_1 + t_2) = 16.31$$

But from Eq. (II), we have $t_1 - t_2 = 2$

$$\therefore\ t_1 + t_2 = 8.155 \quad \text{..... (VII)}$$

(iv) Solving Eqs. (II) and (VII)

$$t_1 + t_2 = 8.155$$
$$t_1 - t_2 = 2$$

we get $t_1 = 5.0775$ seconds

$t_2 = 3.0775$ seconds

(v) From Eqs. (III) and (IV), we get

$u_1 = 9.81\,t_1$

$u_1 = 9.81 \times 5.0775$

$u_1 = 49.81$ m/s ***Ans.***

$u_2 = 9.81\,t_2$

$u_2 = 9.81 \times 3.0775$

$u_2 = 30.19$ m/s ***Ans.***

(vi) From Eqs. (V) and (VI), we get

$h_1 = 4.905\,t_1^2$

$h_1 = 4.905 \times (5.0775)^2$

$h_1 = 126.46$ m ***Ans.***

$h_2 = 4.905\,t_2^2$

$h_2 = 4.905 \times (3.0775)^2$

$h_2 = 46.46$ m ***Ans.***

Problem 16

In a flood relief area, a helicopter going vertically up with a constant velocity drops first batch of food packets which takes 4 seconds to reach the ground. No sooner this batch reaches the ground, second batch of food packets are released and this batch takes 5 seconds to reach the ground. From what height was the first batch released ? Also determine the velocity with which the helicopter is moving up ?

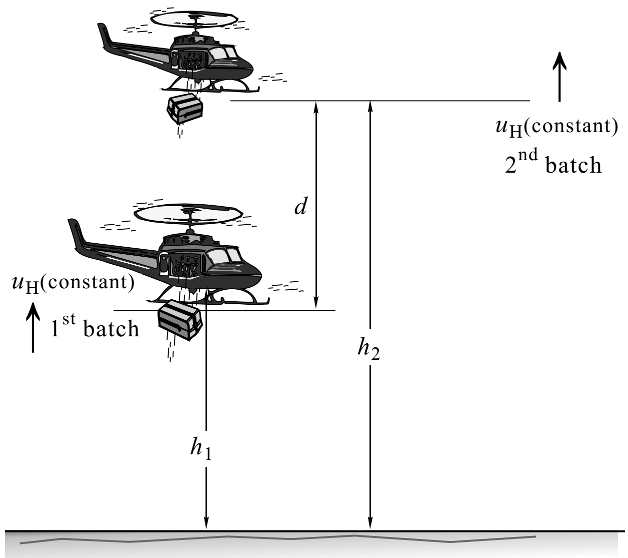


Fig. 11.16

Solution

(i)

Motion of first batch under gravity	**Motion of second batch under gravity**	**Motion of helicopter with constant velocity**
$u = u_H = u_1$ $(\uparrow)$;	$u = u_H = u_2$ $(\uparrow)$;	$u = u_H$ $(\uparrow)$
$h = -h_1$;	$h = -h_2$;	$t = 4$ sec
$t = 4$ sec ; $g = -9.81$ m/s^2	$t = 5$ sec ; $g = -9.81$ m/s^2	$d =$ Speed $\times$ Time
$h = ut + \frac{1}{2}gt^2$	$h = ut + \frac{1}{2}gt^2$	$d = u_H \times 4$
$-h_1 = u_H \times 4 + \frac{1}{2}(-9.81) \times (4)^2$	$-h_2 = u_H \times 5 + \frac{1}{2}(-9.81) \times (5)^2$	
$h_1 = -4u_H + 78.48$	$h_2 = -5u_H + 122.625$	

(ii) From Fig. 11.20, we have

$$d = h_2 - h_1 \qquad (u_1 = u_2 = u_H)$$

$$4u_H = (-5u_H + 122.625) - (-4u_H + 78.48)$$

$u_H = 8.829$ m/s is the velocity of helicopter ***Ans.***

(iii) Height from which first batch is released

$$h_1 = -4u_H + 78.48$$

$$h_1 = -4 \times 8.829 + 78.48$$

$h_1 = 43.164$ m ***Ans.***

Problem 17

The platform of an elevator moves down a mine shaft at an acceleration of 0.4 m/s^2 starting from rest from the top of the shaft. After the elevator has moved down by 20 m a ball is dropped from the top of the shaft. Find **(i)** the time at which the ball will hit the elevator and **(ii)** the distance moved by the elevator when the ball hits it.

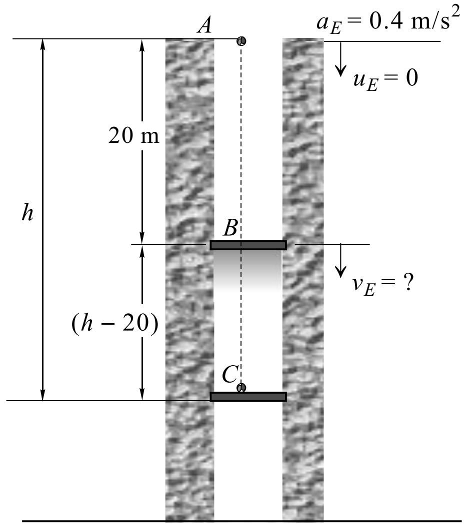


Fig. 11.17

Solution

(i) Motion of elevator from A to B

$$u = u_E = 0\ ;$$
$$s = 20\text{ m}\ ;$$
$$a = a_E = 0.4\text{ m/s}^2\ ;$$
$$v^2 = u^2 + 2as$$
$$v^2 = 0 + 2 \times 0.4 \times 20$$
$$v = 4\text{ m/s}$$

Motion of elevator from B to C

$$s = (h - 20)\ ;\ u = 4\text{ m/s}$$
$$t = t$$
$$s = ut + \frac{1}{2}at^2$$
$$h - 20 = 4 \times t + \frac{1}{2} \times 0.4 \times t^2$$
$$h = 4t + 0.2t^2 + 20 \quad \text{..... (I)}$$

Motion of ball from A to C

$$u = 0\ ;$$
$$h = h\ ;$$
$$t = t\ ;\ g = 9.81\text{ m/s}^2$$
$$h = ut + \frac{1}{2}gt^2$$
$$h = 0 + \frac{1}{2}(9.81) \times (t)^2$$
$$h = 4.905t^2 \quad \text{..... (II)}$$

(ii) Equating Eqs. (I) and (II), we have

$$4t + 0.2t^2 + 20 = 4.905t^2$$
$$4.705t^2 - 4t - 20 = 0$$
$$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4.705)(-20)}}{2 \times 4.705}$$

$t = 2.53$ seconds (Time taken by the ball to hit the elevator) ***Ans.***

$$h = 4.905t^2$$

$h = 31.4$ m (Distance moved by the elevator when the ball hit with respect to top position) ***Ans.***

Problem 18

A stone is dropped gently from the top of a tower. During its last one second of motion it falls through 64 % of the height. Find the height of the tower.

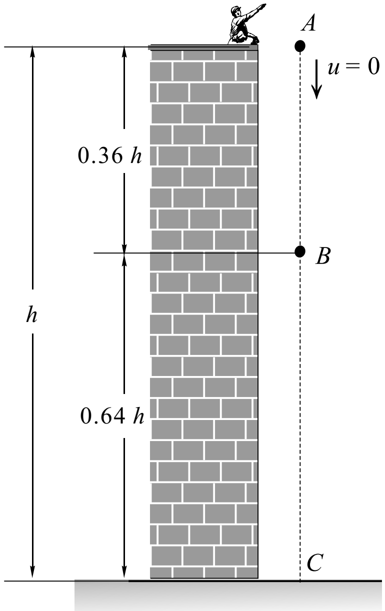


Fig. 11.18

Solution

(i) Let t be the time taken by stone to reach from A to C

$$h = ut + \frac{1}{2}gt^2 = 0 + 4.905t^2$$

$$h = 4.905t^2 \qquad \text{..... (I)}$$

(ii) Consider motion from A to B

Time taken by stone to travel 36 % will be $(t-1)$

$$h = ut + \frac{1}{2}gt^2$$

$$0.36\,h = 0 + \frac{1}{2} \times (9.81) \times (t-1)^2$$

$$h = \frac{4.905}{0.36} \times (t-1)^2 \qquad \text{..... (II)}$$

(iii) Equating Eqs. (I) and (II), we have

$$4.905t^2 = \frac{4.905}{0.36} \times (t-1)^2$$

$$0.36t^2 = t^2 - 2t + 1$$

$$0.64t^2 - 2t + 1 = 0$$

$$t = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(0.64)(1)}}{2 \times 0.64} \qquad \therefore\ t = 2.5 \text{ seconds} \quad \textbf{\textit{Ans.}}$$

(iv) From Eq. (I), we get

Height of tower $h = 4.905 \times t^2 = 4.905 \times 2.5^2$

$\therefore\ h = 30.66$ m ***Ans.***

Problem 19

A body is allowed to fall vertically under the action of gravity. It travels two points in it path, placed 45 m apart, in 1 second. Find from what height above the higher point was the body allowed to

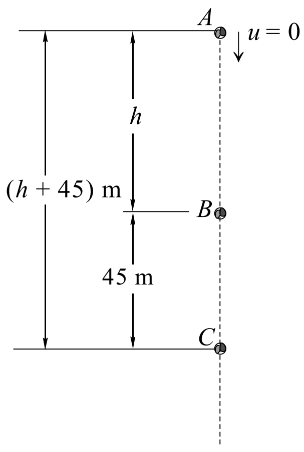


Fig. 11.19

Solution

(i) Motion from *A* to *B*

$$h = ut + \frac{1}{2}gt^2 = 0 + \frac{1}{2}(9.81) \times (t)^2$$

$$h = 4.905t^2 \qquad \text{..... (I)}$$

(ii) Motion from *A* to *C*

Height $= (h + 45)$ m, Time $= (t + 1)$ seconds

$$h = ut + \frac{1}{2}gt^2$$

$$(h + 45) = 0 + \frac{1}{2}(9.81) \times (t+1)^2$$

(iii) Substituting from Eq. (I), we get

$$4.905t^2 + 45 = 4.905(t^2 + 2t + 1)$$

$$4.905t^2 + 45 = 4.905t^2 + 9.81t + 4.905$$

$$45 - 4.905 = 9.81t$$

$$\therefore t = 4.09 \text{ seconds}$$

$$h = 4.905t^2 = 4.905 \times 4.09^2$$

$$\therefore h = 82.05 \text{ m}$$ ***Ans.***

Problem 20

A stone is dropped into a well with no initial velocity and 4.5 seconds later a splash is heard. If the velocity of sound is constant 330 m/s, find the depth of the well up to the water level.

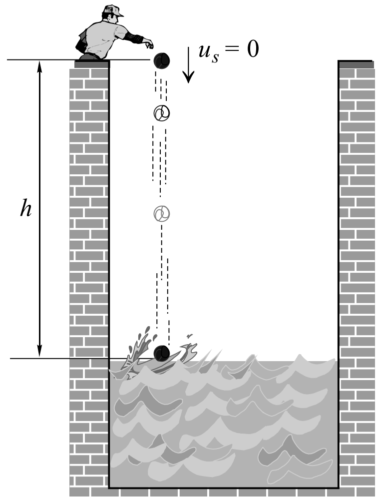


Fig. 11.20

Solution

Let t be the time taken by stone to reach the water surface.

(i) Motion of stone (Under gravity)

$u = u_S = 0$;

Height $= h$;

Time $= t$

$$h = ut + \frac{1}{2}gt^2$$

$$h = 0 + \frac{1}{2}(9.81) \times t^2$$

$$h = 4.905\, t^2 \qquad \text{..... (I)}$$

Motion of sound (With constant velocity)

Time $= (4.5 - t)$

Displacement = Velocity × Time

$$h = 330(4.5 - t) \qquad \text{..... (II)}$$

(ii) Equating Eqs. (I) and (II)

$$4.905t^2 = 330(4.5 - t)$$

$$4.905t^2 = 330 \times 4.5 - 330t$$

$$4.905t^2 + 330t - 1485 = 0$$

$$t = \frac{-330 \pm \sqrt{(330)^2 - 4\times(4.905)\times(-1485)}}{2 \times 4.905}$$

$t = 4.234$ sec or $t = -71.51$ seconds ($\because$ Time cannot be negative)

$\therefore t = 4.234$ sec ***Ans.***

(iii) Depth of the well up to water level $= h$

$$h = 4.905 \times t^2 = 4.905 \times 4.234^2$$

$$h = 87.93 \text{ m}$$ ***Ans.***

Problem 21

Drops of water fall from the roof of a building 16 m high, at regular interval of time. When first drop strikes the ground, at the same instant fifth drop starts its fall. Find the distance between individual drops in the air, the instant first drop reaches the ground.

Solution

(i) Motion of first drop

$$u = 0 \; ; h_1 = 16 \text{ m} \; ;$$

$$h = ut + \frac{1}{2}gt^2$$

$$\therefore \; h_1 = ut_1 + \frac{1}{2} \times (9.81) \times t_1^2 \quad \therefore \; 16 = 0 + \frac{1}{2} \times (9.81) \times t_1^2 \quad \therefore \; t_1 = 1.8 \text{ seconds}$$

(ii) Let Δt be the time interval to start the motion of each drop. In a time interval of $t_1 = 1.8$ seconds, four drops have started their motion at regular interval of time.

$$\Delta t = \frac{t_1}{4} = \frac{1.8}{4} = 0.45 \text{ seconds}$$

$$t_2 = t_1 - \Delta t = 1.8 - 0.45 = 1.35 \text{ second}$$

$$t_3 = t_2 - \Delta t = 1.35 - 0.45 = 0.9 \text{ second}$$

$$t_4 = t_3 - \Delta t = 0.9 - 0.45 = 0.45 \text{ second}$$

$$t_5 = t_4 - \Delta t = 0.45 - 0.45 = 0 \text{ second}$$

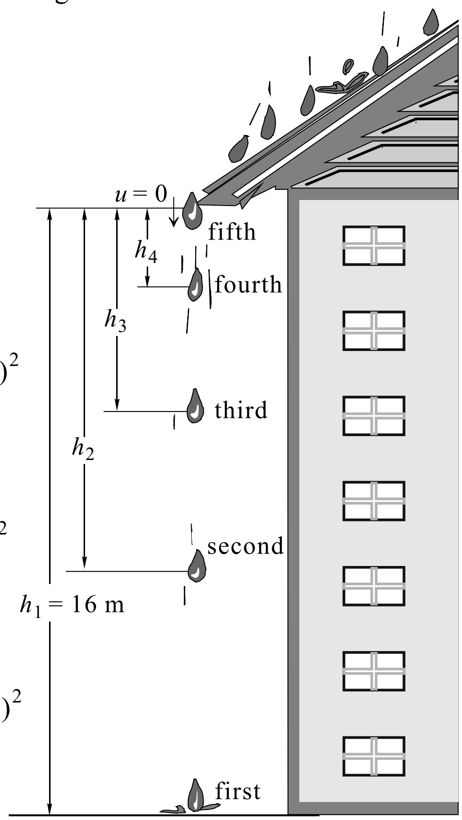


Fig. 11.21

(iii) Motion of second drop

$$h_2 = ut_2 + \frac{1}{2}gt_2^2 \qquad \therefore \; h_2 = 0 + \frac{1}{2} \times (9.81) \times (1.35)^2$$

$$\therefore \; h_2 = 8.94 \text{ m}$$

(iv) Motion of third drop

$$h_3 = ut_3 + \frac{1}{2}gt_3^2 \qquad \therefore \; h_3 = 0 + \frac{1}{2} \times (9.81) \times (0.9)^2$$

$$\therefore \; h_3 = 3.97 \text{ m}$$

(v) Motion of forth drop

$$h_4 = ut_4 + \frac{1}{2}gt_4^2 \qquad \therefore \; h_4 = 0 + \frac{1}{2} \times (9.81) \times (0.45)^2$$

$$\therefore \; h_4 = 0.99 \text{ m}$$

(vi) Distance between individual drops

Distance between first and second drop $= h_1 - h_2 = 7.06$ m

Distance between second and third drop $= h_2 - h_3 = 4.97$ m

Distance between third and fourth drop $= h_3 - h_4 = 2.98$ m

Distance between forth and fifth drop $= h_4 - h_5 = 0.99$ m ***Ans.***

Problem 22

Water drips from a tap at the rate of five drops per second. Determine the vertical separation between two consecutive drops after the lower drop has attained a velocity of 5 m/s.

Solution

(i) Since water drips from a tap at the rate of five drops per second.

$\therefore$ Time interval between each drop $= \frac{1}{5} = 0.2$ seconds

(ii) Motion of Drop *A*

$u = 0\ ;\ v_1 = 5$ m/s

$v = u + gt \quad \Rightarrow\ 5 = 0 + 9.81 \times t_1 \quad \therefore\ t_1 = 0.5097$ seconds

$$h_1 = ut_1 + \frac{1}{2}gt_1^2 = 0 + \frac{1}{2} \times 9.81 \times (0.5097)^2$$

$h_1 = 1.274$ m

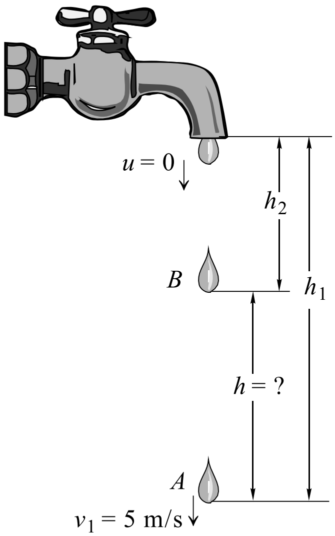


Fig. 11.22

(iii) Motion of Drop *B*

Time $t_2 = t_1 - 0.2 = 0.5097 - 0.2$

$t_2 = 0.3097$ seconds

$$h_2 = ut_2 + \frac{1}{2}gt_2^2 = 0 + \frac{1}{2} \times 9.81 \times (0.3097)^2$$

$h_2 = 0.4705$ m

(iv) The vertical separation between two consecutive drops

$h = h_1 - h_2 = 1.274 - 0.4705$

$h = 0.8035$ m ***Ans.***

Problem 23

A stone after falling 5 sec from rest breaks a glass pan and in breaking it losses 30 % of its velocity. How far will it fall in the next second ?

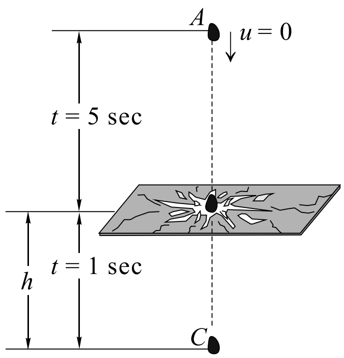


Fig. 11.23

Solution

(i) Velocity of stone before glass pan is broken $= v_1$

Velocity of stone after glass pan is broken $= v_2$

(ii) Motion of stone from rest to glass pan before breaking

$t = 5$ sec ; $u = 0$; $v = v_1$

$v = u + gt \quad \Rightarrow\ v_1 = 0 + 9.81 \times 5 \quad \therefore\ v_1 = 49.05$ m/s

(iii) Loss of 30 % of velocity after breaking the glass pan

$v_2 = 34.335$ m/s $\quad \therefore\ v_2 = (49.05) \times (0.7)$

(iv) Motion of stone after breaking of glass pan for next 1 second.

$u = v_2\ ;\ t = 1$ second ; displacement $= h$;

$$h = ut + \frac{1}{2}gt^2 = 34.335 \times 1 + \frac{1}{2} \times 9.81 \times 1^2$$

$h = 39.24$ m ***Ans.***

11.7 Solved Problems Based on Rectilinear Motion with Variable Acceleration

Problem 24

The motion of the particle along a straight line is governed by the relation $a = t^3 - 2t^2 + 7$ where a is the acceleration in m/s^2 and t is the time in seconds. At time $t = 1$ second. the velocity of the particle is 3.58 m/s and the displacement is 9.39 m. Calculate the displacement, velocity and acceleration at time $t = 2$ seconds.

Solution

(i) $a = t^3 - 2t^2 + 7$ .....(I)

$$\therefore \frac{dv}{dt} = t^3 - 2t^2 + 7$$

$$\therefore dv = (t^3 - 2t^2 + 7)\, dt$$

Integrating both sides, we get

$$\int dv = \int (t^3 - 2t^2 + 7)\, dt$$

$$\therefore v = \frac{t^4}{4} - \frac{2t^3}{3} + 7t + c_1$$

At $t = 1$ second; $v = 3.58$ m/s

$$\therefore 3.58 = \frac{(1)^4}{4} - \frac{2(1)^3}{3} + 7(1) + c_1$$

$$\therefore c_1 = -3$$

(ii) $v = \dfrac{t^4}{4} - \dfrac{2t^3}{3} + 7t - 3$ .....(II)

$$\therefore \frac{ds}{dt} = \frac{t^4}{4} - \frac{2t^3}{3} + 7t - 3$$

$$\therefore ds = \left[\frac{t^4}{4} - \frac{2t^3}{3} + 7t - 3\right] dt$$

Integrating both sides, we get

$$\therefore \int ds = \int \left[\frac{t^4}{4} - \frac{2t^3}{3} + 7t - 3\right] dt$$

(iii) $s = \dfrac{t^5}{20} - \dfrac{t^4}{6} + \dfrac{7t^2}{2} - 3t + c_2$

When $s = 9.39$ m, at $t = 1$ second.

$$\therefore 9.39 = \frac{(1)^5}{20} - \frac{(1)^4}{6} + \frac{7(1)^2}{2} - 3(1) + c_2$$

$$\therefore c_2 = 9$$

$$\therefore s = \frac{t^5}{20} - \frac{t^4}{6} + \frac{7t^2}{2} - 3t + 9 \quad .....(III)$$

(iv) At $t = 2$ seconds from Eqs. (I), (II) and (III), we have

$$\therefore\ s = \frac{(2)^5}{20} - \frac{(2)^4}{6} + \frac{7(2)^2}{2} - 3(2) + 9 \qquad \therefore\ s = 15.93 \text{ m}$$

$$\therefore\ v = \frac{(2)^4}{4} - \frac{2(2)^3}{3} + 7(2) - 3 \qquad \therefore\ v = 9.667 \text{ m/s}$$

$$\therefore\ a = (2)^3 - 2(2)^2 + 7 \qquad \therefore\ a = 7 \text{ m/s}^2$$

Problem 25

The acceleration of a particle is defined by the relation $a = 25 - 3x^2$ mm/s^2. The particle starts with no initial velocity at the position $x = 0$, determine **(i)** the velocity when $x = 2$ mm, **(ii)** the position when velocity is again zero and **(iii)** the position where the velocity is maximum and the corresponding maximum velocity.

Solution

$$a = 25 - 3x^2 \qquad \text{.....(I)}$$

$$v\frac{dv}{dx} = 25 - 3x^2$$

$$v\,dv = (25 - 3x^2)\,dx$$

Integrating both sides, we get

$$\frac{v^2}{2} = 25x - 3x^3 + c$$

At $x = 0$, $v = 0$ $\therefore$ $c = 0$

$$v^2 = 50x - 2x^3 \qquad \text{.....(II)}$$

(i) $v = ?$ when $x = 2$ mm

From Eq. (II)

$$v^2 = 50 \times 2 - 2 \times 2^3$$

$v = 9.17$ mm/s ***Ans.***

(ii) $x = ?$ when $v = 0$ (again)

From Eq. (II)

$$0 = 50x - 2x^3$$

$x = \pm 5$ mm ***Ans.***

(iii) $x = ?$ when v_{max} and $v_{max} = ?$

For velocity to be maximum, $\dfrac{ds}{dt} = 0 = a$

From Eq. (I)

$$0 = 25 - 3x^2$$

$x = 2.887$ mm

From Eq. (II)

$$v_{max}{}^2 = 50 \times 2.887 - 2 \times 2.887^3$$

$v_{max} = 9.809$ mm/s ***Ans.***

Problem 26

The displacement (x) of a particle moving in one direction, under the action of a constant force is related to the time t by the equation $t = \sqrt{x} + 3$ where x is in metres and t is in seconds. Find the displacement of the particle when its velocity is zero.

Solution

Given : $t = \sqrt{x} + 3 \quad \therefore \quad \sqrt{x} = t - 3$

Squaring both sides,

$$\therefore \quad (\sqrt{x})^2 = (t-3)^2$$

$$x = t^2 - 6t + 9$$

Differentiating both sides, we get

$$\frac{dx}{dt} = 2t - 6$$

$$\therefore \quad v = 2t - 6$$

When $v = 0$

$$\therefore \quad 0 = 2t - 6$$

$$t = 3 \text{ sec}$$

At $t = 3$ sec

$$x = t^2 - 6t + 9$$

$$x = (3)^2 - 6(3) + 9$$

$$x = 0 \quad \textbf{\textit{Ans.}}$$

Problem 27

A particle starting from rest at the position (5, 6, 2) m accelerates at $\bar{a} = 6t\,\bar{i} - 24t^2\,\bar{j} + 10\,\bar{k}$ m/s^2. Determine the acceleration, velocity and displacement of the particle at the end of 2 seconds.

Solution

Given : $\bar{a} = 6t\,\bar{i} - 24t^2\,\bar{j} + 10\,\bar{k}$ ..... (I)

(i) Integrating, we get

$$\bar{v} = (3t^2 + c_1)\,\bar{i} - (8t^3 + c_2)\,\bar{j} + (10t + c_3)\,\bar{k}$$

At $t = 0, v = 0 \quad \therefore \quad c_1 = c_2 = c_3 = 0$

$$\bar{v} = 3t^2\,\bar{i} - 8t^3\,\bar{j} + 10t\,\bar{k} \qquad \text{..... (II)}$$

Integrating, we get

$$\bar{r} = (t^3 + c_4)\,\bar{i} - (2t^4 + c_5)\,\bar{j} + (5t^2 + c_6)\,\bar{k}$$

At $t = 0, s\,(5, 6, 2) \quad \therefore \quad c_4 = 5, c_5 = 6, c_6 = 2$

$$\bar{r} = (t^3 + 5)\,\bar{i} - (2t^4 + 6)\,\bar{j} + (5t^2 + 2)\,\bar{k} \qquad \text{..... (III)}$$

(ii) Put $t = 2$ sec in Eqs. (I), (II) and (III), we get

$$\bar{a} = 12\,\bar{i} - 96\,\bar{j} + 10\,\bar{k} \qquad \text{magnitude } a = \sqrt{12^2 + 96^2 + 10^2} = 97.26 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

$$\bar{v} = 12\,\bar{i} - 64\,\bar{j} + 20\,\bar{k} \qquad \text{magnitude } v = \sqrt{12^2 + 64^2 + 20^2} = 68.12 \text{ m/s} \quad \textbf{\textit{Ans.}}$$

$$\bar{r} = 13\,\bar{i} - 38\,\bar{j} + 22\,\bar{k} \qquad \text{magnitude } r = \sqrt{13^2 + 38^2 + 22^2} = 45.79 \text{ m} \quad \textbf{\textit{Ans.}}$$

Problem 28

Motion of the particle along a straight line is defined by $v^3 = 64s^2$ where v is in m/s and s is in m. Determine the

(i) velocity when distance covered is 8 m,

(ii) acceleration when distance covered is 27 m and

(iii) acceleration when the velocity is 9 m/s.

Solution

(i) $v^3 = 64s^2$ At $s = 8$ m; $v = ?$

$\therefore\ v^3 = 64(8)^2 \quad \therefore\ v = 16$ m/s ***Ans.***

(ii) At $s = 27$ m; $a = ?$

$$v^3 = 64s^2$$

$$3v^2 \cdot \frac{dv}{dt} = 64(2s)\left(\frac{ds}{dt}\right)$$

$$3v^2 \cdot (a) = 128 \cdot s(v)$$

$$\therefore\ a = \frac{128 \cdot s}{3v}$$

At $s = 27$ m

$$v^3 = 64(27)^2$$

$$v = 36 \text{ m/s}$$

$$\therefore\ a = \frac{128 \times 27}{3 \times 36} \quad \therefore\ a = 32 \text{ m/s}^2$$ ***Ans.***

OR

$$v^3 = 64s^2$$

$$3v^2 \cdot \frac{dv}{ds} = 64(2s)$$

$$3\left[v \cdot \frac{dv}{ds}\right] v = 128\ s$$

$$3(a) \cdot v = 128\ s$$

$$a = \frac{128s}{3v}$$

$$\therefore\ a = \frac{128 \times 27}{3 \times 36} \quad \therefore\ a = 32 \text{ m/s}^2$$ ***Ans.***

(iii) At $v = 9$ m/s; $a = ?$

$$v^3 = 64s^2$$

$$(9)^3 = 64s^2 \quad \therefore\ s = 3.375 \text{ m}$$

$$\therefore\ a = \frac{128 \cdot s}{3v} = \frac{128 \times 3.375}{3 \times 9}$$

$$\therefore\ a = 16 \text{ m/s}^2$$ ***Ans.***

Problem 29

The velocity of a particle travelling in a straight line is given by $v = 6t - 3t^2$ m/s where t is in seconds. If $s = 0$ when $t = 0$, determine the particle's deceleration and position when $t = 3$ seconds. How far has the particle travelled during the 3 second time interval and what is its average speed?

Solution

(i) $v = 6t - 3t^2$ ..... (I)

Differentiating Eq. (I) w.r.t. time, we get

$v = 6t - 3t^2$ ..... (II)

Integrating Eq. (I), we get

$$s = \frac{6t^2}{2} - \frac{3t^3}{3} + c \qquad (\text{At } t = 0;\ s = 0\ \therefore c = 0)$$

$s = 3t^2 - t^3$ ..... (III)

(ii) Putting $t = 3$ seconds in Eqs. (II) and (III), we get

$a = 6 - 6 \times 3$

$a = -12$ m/s^2 ***Ans.***

$s = 3 \times 3^2 - 3^3$

$s = 0$ ***Ans.***

(iii) For distance travelled let us find time for point of reversal where $v = 0$

From Eq. (I)

$0 = 6t - 3t^2$

$t = 2$ seconds

Particle is travelling 2 seconds in same direction and reversing the direction.

Distance covered in $t = 2$ seconds from Eq. (III)

$s = 3 \times 2^2 - 2^3$

$s = 4$ m

In 3 seconds, displacement $s = 0$

$\therefore$ distance travelled $= 4 + 4$

$d = 8$ m ***Ans.***

(iv) $$\text{Average speed} = \frac{\text{Distance travelled}}{\text{Time}}$$

$$\text{Average speed} = \frac{8}{3}$$

$v = 2.667$ m/s ***Ans.***

11.8 Solved Problems Based on Motion Diagram (Graphical Solution)

Problem 30

The acceleration-time diagram for the linear motion is shown in Fig. 11.30(a). Construct velocity time and displacement time diagrams for the motion assuming that the motion starts with initial velocity of 5 m/s from the starting point.

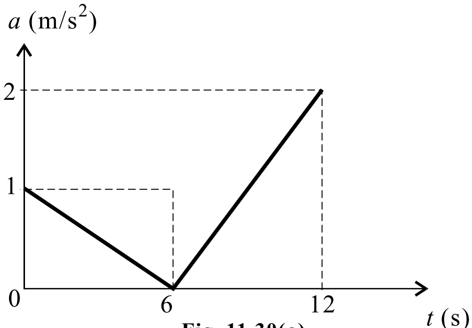


Fig. 11.30(a)

Solution

(i) Velocity-Time diagram

Change in velocity = Area under a-t diagram

(a) At $t = 6$ seconds

$$v_6 - v_0 = \frac{1}{2} \times 6 \times 1 \qquad (\because v_0 = 5 \text{ m/s})$$

$$v_6 = 5 + 3 = 8 \text{ m/s}$$

(b) At $t = 12$ seconds

$$v_{12} - v_6 = \frac{1}{2} \times 6 \times 2$$

$$v_{12} = 8 + \frac{1}{2} \times 6 \times 2 = 8 + 6$$

$$v_{12} = 14 \text{ m/s}$$

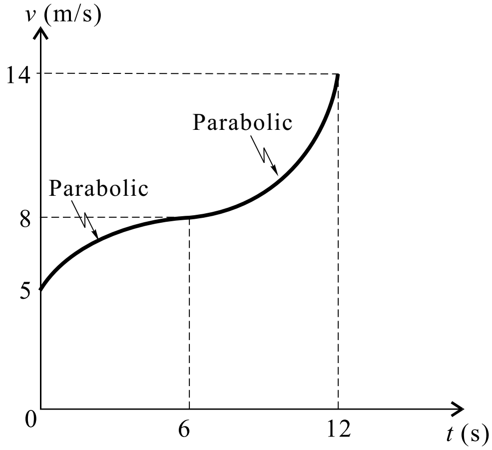


Fig. 11.30(b)

(ii) Displacement-Time diagram

Method I : Finding displacement

Change in displacement = Area under v-t diagram

(a) At $t = 6$ seconds

$$s_6 - s_0 = 6 \times 5 + \frac{2}{3} \times 6 \times 3 \qquad (\because s_0 = 0)$$

$$s_6 = 30 + 12 = 42 \text{ m}$$

(b) At $t = 12$ seconds

$$s_{12} - s_6 = 6 \times 8 + \frac{1}{3} \times 6 \times 6$$

$$s_{12} = 42 + 48 + 12 = 102 \text{ m}$$

Method II : Finding displacement - area moment

Change in displacement = $v_0 \times t$ + Moment of area under a-t diagram

(a) At $t = 6$ seconds

$$s_6 - s_0 = v_0 \times t + \text{Moment of area under } a\text{-}t \text{ diagram between 0 to 6}$$

$$s_6 = 5 \times 6 + \frac{1}{2} \times 6 \times 1 \times \frac{2}{3} \times 6 \qquad (\because s_0 = 0,\ v_0 = 5 \text{ m/s})$$

$$s_6 = 30 + 12 = 42 \text{ m}$$

(b) At $t = 12$ seconds

$$s_{12} - s_6 = v_6 \times t + \text{Moment of area under } a\text{-}t \text{ diagram between 6 to 12}$$

$$s_{12} = 42 + 8 \times 6 + \frac{1}{2} \times 6 \times 2 \times \frac{1}{3} \times 6$$

$$s_{12} = 42 + 48 + 12 = 102 \text{ m} \quad \textit{Ans.}$$

Problem 31

The acceleration-time diagram for the linear motion is shown in Fig. 11.31(a). Construct velocity time and displacement time diagrams for the motion assuming that the motion starts from rest.

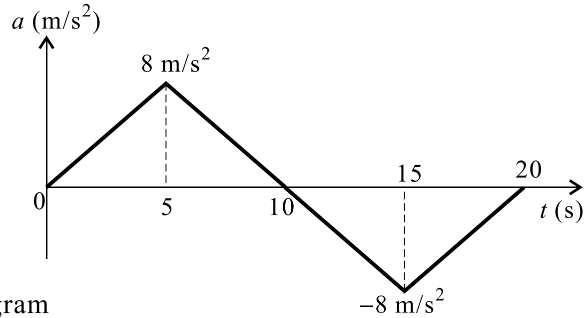


Fig. 11.31(a)

Solution

(i) Velocity-Time diagram

Change in velocity = Area under a-t diagram

(a) At t = 5 seconds

$$v_5 - v_0 = \frac{1}{2} \times 5 \times 8 \quad (\because v_0 = 0)$$

$$v_5 = 20 \text{ m/s}$$

(b) At t = 10 seconds

$$v_{10} - v_5 = \frac{1}{2} \times 5 \times 8$$

$$v_{10} = 20 + 20 = 40 \text{ m/s}$$

(c) At t = 15 seconds

$$v_{15} - v_{10} = \frac{1}{2} \times 5 \times (-8)$$

$$v_{15} = 40 - 20 = 20 \text{ m/s}$$

(d) At t = 20 seconds

$$v_{20} - v_{15} = \frac{1}{2} \times 5 \times (-8)$$

$$v_{15} = 20 - 20 = 0 \text{ m/s}$$

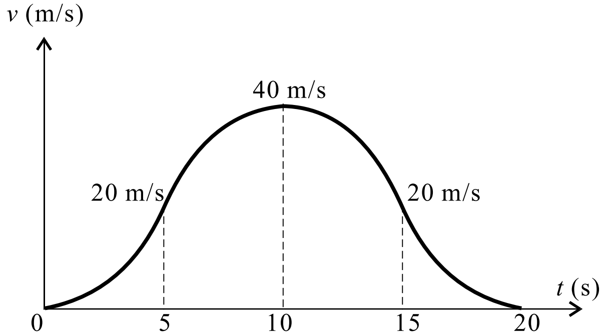


Fig. 11.31(b)

(ii) Displacement-Time diagram

Change in displacement = Area under v-t diagram

(a) At t = 5 seconds

$$s_5 - s_0 = \frac{1}{3} \times 5 \times 20$$

$$s_5 = 33.33 \text{ m}$$

(b) At t = 10 seconds

$$s_{10} - s_5 = 5 \times 20 + \frac{2}{3} \times 5 \times 20$$

$$s_{10} = 33.33 + 100 + 66.67 = 200 \text{ m}$$

(c) At t = 15 seconds

$$s_{15} - s_{10} = 5 \times 20 + \frac{2}{3} \times 5 \times 20$$

$$s_{15} = 200 + 100 + 66.67 = 366.67 \text{ m}$$

(d) At t = 20 seconds

$$s_{20} - s_{15} = \frac{1}{2} \times 5 \times 20$$

$$s_{20} = 366.67 + 33.33 = 400 \text{ m}$$

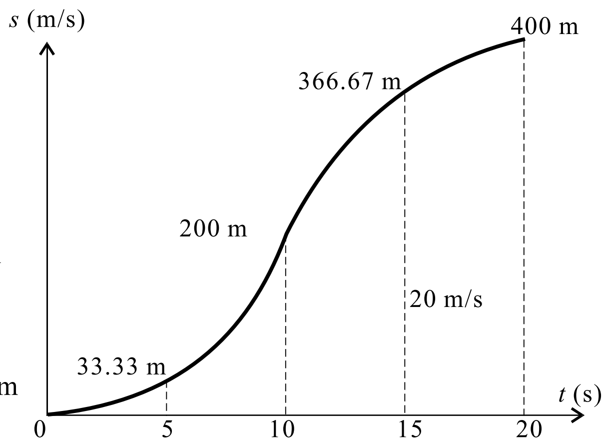


Fig. 11.31(c)

Problem 32

A particle moving with a velocity of 7.5 m/s is subjected to a retarding force which gives it a negative acceleration varying with time as shown in Fig. 11.32(a). For the first 3 seconds, after 3 seconds the acceleration remaining constant. Plot the *v-t* diagram for 6 seconds of the travel of particle. Determine the distance travelled by the particle from its position $t = 0$ to $t = 6$ seconds.

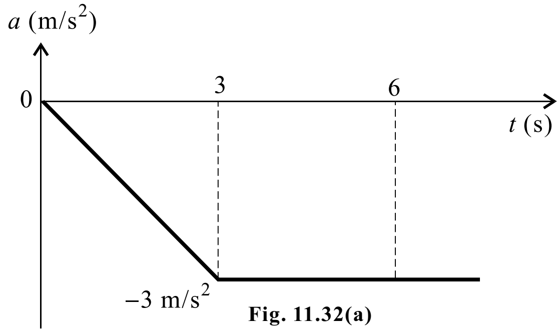


Fig. 11.32(a)

Solution

(i) Velocity-Time diagram

Change in velocity = Area under *a-t* diagram

(a) At $t = 3$ seconds

$$v_3 - v_0 = \frac{1}{2} \times 3 \times (-3)$$

$$v_3 = 7.5 - 4.5 \quad (\because v_0 = 7.5 \text{ m/s})$$

$$v_3 = 3 \text{ m/s}$$

(b) At $t = 6$ seconds

$$v_6 - v_3 = 3 \times (-3)$$

$$v_6 = 3 - 9 = -6 \text{ m/s}$$

By property of similar Δ, we have

$$\frac{3}{6} = \frac{d}{3-d}$$

$$\therefore \quad 3(3-d) = 6d$$

$$9 - 3d = 6d$$

$$9 = 9d \qquad \therefore \quad d = 1 \quad \textbf{Ans.}$$

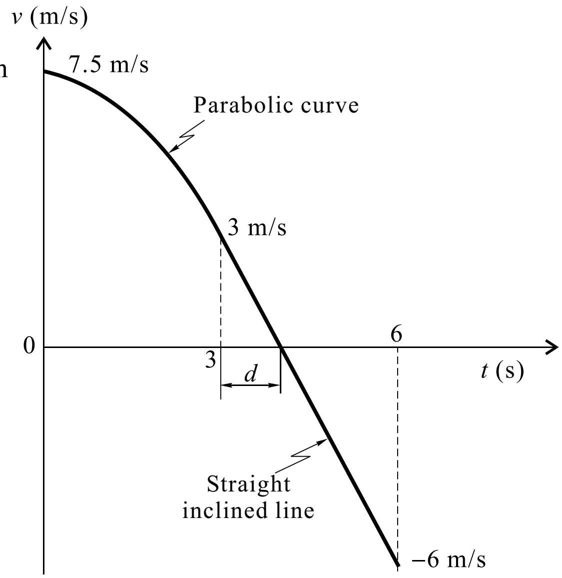


Fig. 11.32(b)

(ii) Displacement-Time diagram

Change in displacement = Area under *v-t* diagram

(a) At $t = 3$ seconds

$$s_3 - s_0 = 3 \times 3 + \frac{2}{3} \times 3 \times 4.5$$

$$s_3 = 9 + 9 = 18 \text{ m}$$

(b) At $t = 4$ seconds (Straight inclined line intersecting t axis $\because d = 1$)

$$s_4 - s_3 = \frac{1}{2} \times 1 \times 3$$

$$s_4 = 18 + 1.5 = 19.5 \text{ m}$$

(c) At $t = 6$ seconds

$$s_6 - s_4 = \frac{1}{2} \times 2 \times (-6)$$

$$s_6 = 19.5 - 6$$

$$s_6 = 13.5 \text{ m}$$

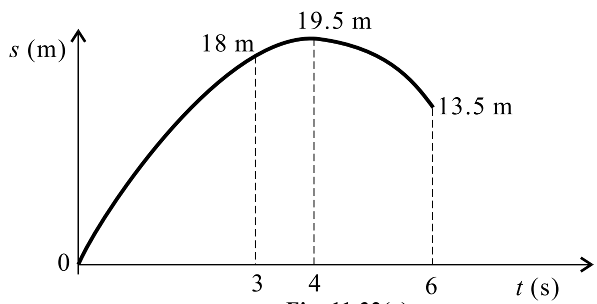


Fig. 11.32(c)

Distance travelled by particle form
$t = 0$ to $t = 6$ seconds

$$D = 19.5 + \frac{1}{2} \times 2 \times 6$$
$$D = 19.5 + 6$$
$$D = 25.5 \text{ m}$$

Problem 33

A particle moves in straight line with a velocity-time diagram shown in Fig. 11.33(a). Knowing that $s = -25$ m and $t = 0$, draw s-t and a-t diagram for $0 < t < 24$.

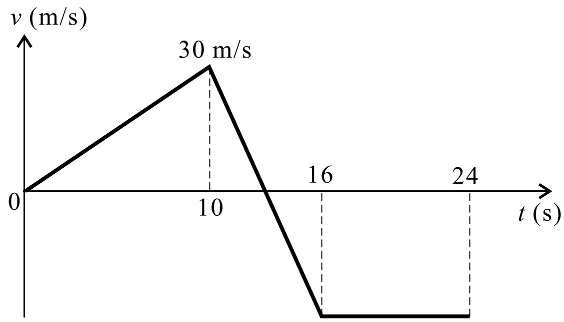


Fig. 11.33(a)

Solution

(i) Acceleration-Time diagram

Acceleration = Slope of v-t diagram

(a) $0 < t < 10$ seconds ;

$$\text{Slope} = a = \frac{30}{10} = 3 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

(b) 10 sec $< t < 16$ seconds ;

$$\text{Slope} = a = -\frac{30}{3}$$
$$= -10 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

(By geometry, slope is intersecting as a mid-point between $t = 10$ seconds to $t = 16$ seconds, i.e., $t = 13$ seconds)

(c) From $t = 16$ seconds onwards ;

$$\text{Slope} = a = 0 \quad \textbf{\textit{Ans.}}$$

(ii) Method for finding displacement

Change in displacement = Area under v-t diagram

(a) At $t = 10$ seconds

$$s_{10} - s_0 = \frac{1}{2} \times 10 \times 30 \qquad (\because s_0 = -25 \text{ m})$$
$$s_{10} = -25 + \frac{1}{2} \times 10 \times 30$$
$$s_{10} = 125 \text{ m} \quad \textbf{\textit{Ans.}}$$

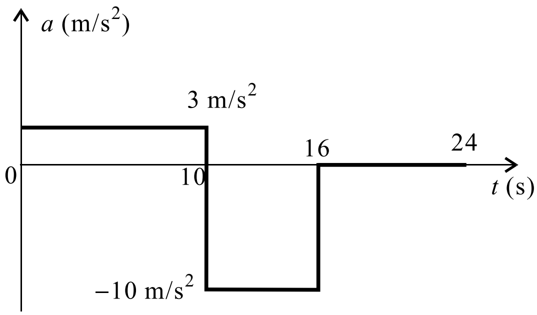


Fig. 11.33(b)

(b) At $t = 13$ seconds (t-axis intercept)

$$s_{13} - s_{10} = \frac{1}{2} \times 3 \times 30$$
$$s_{13} = 125 + 45$$
$$s_{13} = 170 \text{ m} \quad \textbf{\textit{Ans.}}$$

(c) At $t = 16$ seconds

$$s_{16} - s_{13} = \frac{1}{2} \times 3 \times (-30)$$

$$s_{16} = 170 - 45$$

$$s_{16} = 125 \text{ m} \quad \textit{Ans.}$$

(d) Due to negative velocity, the particle has reversed its direction and will reach again to its origin. Let t be the time taken by particle to reach the origin from $t = 16$ seconds.

$$s_0 - s_{16} = (-30) \times t \quad (s_0 = 0)$$

$$0 - 125 = -30\,t$$

$$t = 4.17 \text{ seconds} \quad \textit{Ans.}$$

$\therefore$ At $t = 16 + 4.17 = 20.17$ seconds, the particle will again pass through origin.

(e) At $t = 24$ seconds

$$s_{24} - s_{16} = 8 \times (-30)$$

$$s_{24} = 125 - 240$$

$$s_{24} = -115 \text{ m} \quad \textit{Ans.}$$

(f) Initially particle is having negative displacement, $s = -25$ m.

Let t be the time taken to cross origin, we know

$$25 = \frac{1}{3} \times t \times 30$$

$$\therefore \; t = 1.67 \text{ seconds} \quad \textit{Ans.}$$

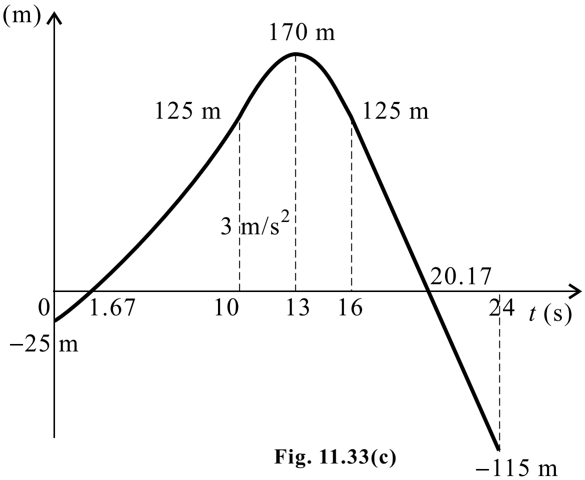


Fig. 11.33(c)

Exercises

[I] Problems

Based on Rectilinear Motion with Constant Acceleration and Constant Velocity

1. A motorist enters a freeway at 10 m/s and accelerates uniformly to 25 m/s. From the odometer in the car, the motorist knows that he travelled 200 m while accelerating. Determine **(a)** the acceleration of the car and **(b)** the time required to reach 25 m/s.

[***Ans.*** $a = 1.313$ m/s^2 and $t = 11.43$ seconds.]

2. A trust travels 164 m in 8 seconds while being decelerated at a constant rate of 0.5 m/s^2. Determine **(a)** its initial velocity, **(b)** its final velocity and **(c)** the distance travelled during the first 0.6 seconds.

[***Ans.*** $u = 22.5$ m/s, $v = 18.5$ m/s and $s = 13.41$ m.]

3. In travelling a distance of 3 km between points *A* and *D*, a car is driven at 100 km/hr from *A* to *B* for *t* seconds. If the brakes are applied for 4 sec between *B* and *C* to give a car uniform deceleration from 100 kmph to 60 kmph and it takes '*t*' seconds to move from *C* to *D* with a uniform speed of 60 kmph, determine the value of '*t*'.

[***Ans.*** t = 65.5 seconds.]

4. During a time interval of 90 minutes a car

(a) Runs at 50 km/hr for the first 20 min,

(b) Accelerates uniformly to 90 km/hr for the next 10 min,

(c) Runs at the speed of 90 km/hr in the next 40 min, and

(d) Decelerates uniformly to a stop in the remaining 20 min.

Calculate the **(a)** acceleration **(b)** deceleration **(c)** average speed during entire period.

[***Ans.*** **(a)** $a = 0.0185$ m/s^2, **(b)** $a = 0.021$ m/s^2 and **(c)** $v = 68.88$ km/hr.]

5. A train starting from rest accelerates uniformly for 3 minutes, runs at a constant speed for the next 5 minutes and then comes to rest in 2 minutes. If it covers a total distance of 9 km, find the retardation in m/s^2.

[***Ans.*** $a = 0.167$ m/s^2]

6. Track repairs are going on a 2 km length of a railway track. The maximum speed of a train is 90 km/hr. The speed over the repairs track is 36 km/hr. If the train on approaching the repair track decelerates uniformly from the full speed of 90 km/h to 36 km/hr in a distance of 200 m and after covering the repair track accelerates uniformly to full speed from 36 km/hr in a distance of 1600 m, find the time lost due to reduction of the speed in the repair track.

[***Ans.*** t = 150.86 sec]

7. The distance between two stations is 2.50 km. A locomotive starting from one station gives the train an acceleration reaching a speed of 36 km/hr in 30 seconds until the speed reaches 54 km/hr. This speed is maintained until the brakes are applied and the train is brought to rest at the second station under a retardation of 1 m/s^2. Find the time taken to perform the journey and the distances covered during the accelerated, uniform and retarded motion.

[***Ans.*** t = 196.67 seconds, s = 337.5 m, s = 2050 m and s = 112.5 m.]

8. Automobile *A* shown in Fig. 11.E8 starts from *O* and accelerates at the constant rate of 0.75 m/s^2. A short time later it is passed by bus *B* which is travelling in the opposite direction at constant speed of 6 m/s. Knowing that bus *B* passes point *O*, 20 seconds after automobile *A* started from there. Determine when and where the vehicles passed each other.

[***Ans.*** t = 11.6 seconds and s = 50.4 m.]

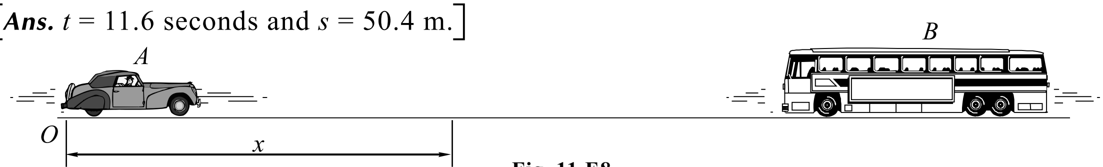


Fig. 11.E8

9. Two trains P and Q leave the same station on parallel lines. Train P starts at rest with uniform acceleration of 0.2 m/s^2 attains a speed of 10 m/s. Further the speed is kept constant. Train Q leaves 30 seconds later with uniform acceleration of 0.5 m/s^2 from rest and attains a maximum speed of 20 m/s. when will train Q overtake train P ?

[***Ans.*** After 75 seconds from the start of train P.]

10. Two trains P and Q start from rest simultaneously from stations A and B facing each other, with acceleration 0.5 m/s^2 and 2/3 m/s^2 reaching their maximum speeds of 90 km/hr and 72 km/hr, respectively. If they cross each other midway between the stations, find the distance between the stations and the time taken by each train.

[***Ans.*** s = 2000 m and t = 65 seconds.]

Based on Rectilinear Motion Under Gravity

11. From the top of a tower 49.05 m high, a stone is projected upwards with velocity of 19.62 m/s. Find **(a)** the time required for stone to reach the ground, **(b)** the time required for reach maximum elevation, **(c)** the maximum elevation, **(d)** the velocity with which stone strikes the grounds and **(e)** the time required for the velocity to attain a magnitude of 9.81 m/s.

[***Ans.*** **(a)** t = 5.742 seconds, **(b)** t = 2 seconds, **(c)** h = 19.62 from top of tower,
(d) v = 36.71 m/s and **(e)** t = 1 sec and 3 sec.]

12. A stone is dropped from the top of the tower 50 m high. At the same instant, another stone is thrown up from the foot of the same tower with a velocity of 25 m/s. At the distance from top and after how much time the two stones cross each other.

[***Ans.*** t = 2 seconds and h = 19.6 m from top.]

13. Two objects A and B, 130 m above the ground are projected vertically. A is projected vertically upwards with a velocity of 30 m/s while B is projected vertically downwards with the same velocity. Find the time taken by each object to reach the ground.

At what height the object A must be just released in order the two objects may hit the ground simultaneously ?

[***Ans.*** t_A = 9.05 seconds, t_B = 2.93 seconds and h = 42.04 m.]

14. A stone falls freely from rest and total distance covered by it in last second of its motion equals the distance covered by it is first three seconds of its motion. Determine the time in which the stone remains in air.

[***Ans.*** t = 5 seconds.]

15. A particle falls from rest and in the last second of its motion it passes 70 m. Find the height from which it fell and the time of its fall.

[***Ans.*** h = 286 m and t = 7.635 seconds.]

16. A particle moving with an acceleration of 10 m/s^2 travels a distance of 50 m during the 5th second of its travel, find its initial velocity.

[***Ans.*** v = 4.5 m/s]

17. A particle falling under gravity falls 30 m in a certain second. Find the time required to cover the next 30 m.

[***Ans.*** t = 0.775 seconds.]

18. A particle falling freely under the action of gravity passes 10 m apart vertically in 0.2 seconds. From what height above the higher point did it start to fall?

[***Ans.*** h = 122.5 m]

19. A stone is dropped into a well as shown in Fig. 11.E19. If a splash is heard 2.50 seconds later, determine depth of water surface assuming the velocity of sound as 330 m/s.

[***Ans.*** h = 28.49 m]

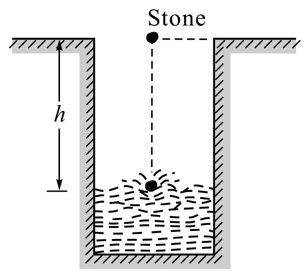


Fig. 11.E19

20. Water drips from a faucet at the rate of 5 drops per second as shown in Fig. 11.E20. Determine the vertical separation between two consecutive drops after the lower drop has attended a velocity of 3 m/s.

[***Ans.*** h = 404 mm]

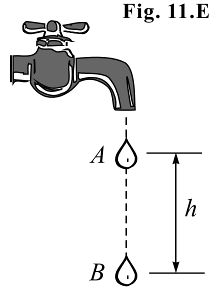


Fig. 11.E20

21. Water drips from a faucet at the rate of 6 drops per second. The faucet is 200 mm above the sink as shown in Fig. 11.E21. When one drop strikes the sink, how far is the next drop above the sink.

[***Ans.*** h = 194.55 mm]

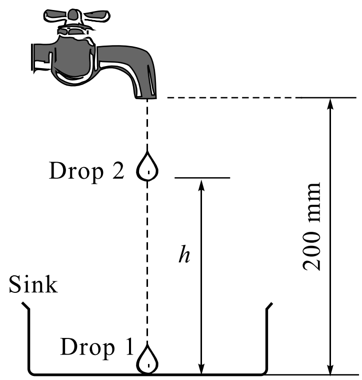


Fig. 11.E21

22. A balloon is rising vertically upward with a uniform velocity 'u' as shown in Fig. 11.E22. At a certain instant, a ball No. 1 is dropped from it, which takes a 4 seconds to reach the ground. The moment this ball strikes the ground, another ball is dropped from the balloon and this 2nd ball takes 5 seconds to reach the ground. Find the velocity of the balloon 'u' and height from which both the balls must have been dropped.

[***Ans.*** u = 8.82 m/s, h_1 = 43.12 m and h_2 = 78.4 m.]

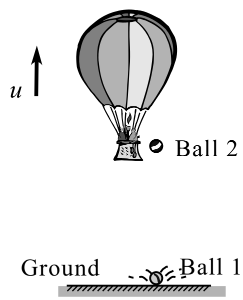


Fig. 11.E22

23. A helicopter is descending vertically downward with a uniform velocity. At a certain instant, a food packet is dropped from it, which takes 5 seconds to reach the ground. As this packet strikes the ground, another food packet is dropped from it, which takes 4 seconds to reach the ground. Find the velocity with which the helicopter is descending and its height, when first packet is dropped. Also, find the distance travelled by the helicopter during the interval of dropping the packets.

[***Ans.*** 11.04 m/s downwards at 177.8 m height and 55.2 m downwards.]

24. During a test, an elevator is travelling upward at 15 m/s and the hoisting cable is cut when it is 40 m from the ground. Determine the maximum height S_B reached by the elevator and its speed just before it hits the ground. During the entire time the elevator is in motion, it is subjected to a constant downward acceleration of 9.81 m/s^2 due to gravity. Neglect the effect of air resistance.

[***Ans.*** s_B = 51.5 m and v = 31.8 m/s.]

25. From a certain height, a helicopter starts going up with acceleration. In 10 seconds, it goes up by 1/6th of its original height. If a ball is dropped at this instant, it takes 5 seconds to reach the ground. What is the original height from which the helicopter has started and with what acceleration ?

[***Ans.*** h = 91.875 m and a = 0.30625 m/s^2.]

26. An elevator ascends with an upward acceleration of 1.2 m/s^2. At the instant when the upward speed is 2.4 m/s, a loose bolt drops from the ceiling of the elevator located 2.75 m from its floor. Calculate **(a)** the time of flight of the bolt from ceiling to floor of the elevator and **(b)** the displacement and the distance covered by the bolt during the free fall relative to the elevator shaft.

[***Ans.*** **(a)** t = 0.707 seconds and **(b)** 0.75 m , 1.34 m.]

27. A balloon starts moving upwards from the ground with constant acceleration of 1.6 m/s^2. Four seconds later, a stone is thrown from the same point. **(a)** What velocity should be imparted to the stone so that it just touches the ascending balloon ? and **(b)** At what height will the stone touch the balloon?

[***Ans.*** **(a)** v = 26.56 m/s and **(b)** h = 36 m.]

Based on Rectilinear Motion with Variable Acceleration

28. During a test, the car moves in a straight line such that for a short time its velocity is defined by $v = 0.3(9t^2 + 2t)$ m/s, where t is in seconds. Determine its position and acceleration when t = 3 seconds. Given at t = 0, s = 0.

[***Ans.*** s = 27 m and a = 16.8 m/s^2.]

29. The motion of a particle is defined by a relation $v = 4t - 3t - 1$ where v is in m/s and t is in sec. If the displacement $x = -4$ m at $t = 0$, determine the displacement and acceleration when t = 3 seconds. Find also the time when the velocity becomes zero and the distance traveled by the particle during that time.

[***Ans.*** x = 15.5 m, a = 21 m/s^2 and Δx = 1.17 m at t = 1 seconds.]

30. The acceleration of a particle is given by $a = k/x$, when $x = 250$ mm, v was 4 m/s. and when $x = 500$ mm, v was 3 m/s. Determine the velocity of the particle when $x = 750$ mm. Find the position of the particle when it comes to rest.

[***Ans.*** $v = 2.21$ m/s and $s = 1219$ mm.]

31. The acceleration is defined by $a = -kx^{-2}$. The particle starts with no initial velocity at $x = 0.8$ m and its velocity becomes 6 m/s when $x = 0.5$ m. Determine the value of k. Also determine the velocity of the particle when $x = 0.25$ m.

[***Ans.*** $k = 24$ and $v = 11.59$ m/s.]

32. A particle starting from rest, moves in a straight line, whose acceleration is given by $a = 10 - 0.006s^2$, where a is in m/s^2 and s is in metres. Determine **(a)** the velocity of the particle when it has travelled 50 m and **(b)** the distance travelled by the particle, when it comes to rest.

[***Ans.*** **(a)** $v = 22.36$ m/s and **(b)** $s = 70.71$ m.]

33. The acceleration of a particle moving along a straight line is given by the law, $a = 25 - 3s^2$ where a is m/s^2 and s is in metres. The particle starts from rest. Find **(a)** velocity when the displacement is 2 m, **(b)** The displacement when the velocity is again zero and **(c)** The displacement at maximum velocity.

[***Ans.*** **(a)** $v = 9.165$ m/s, **(b)** $s = 5$ m and **(c)** $s = 2.887$ m.]

34. A sphere is fired downward into a medium with an initial speed of 27 m/s. If it experiences a deceleration $a = -6t$ m/s^2 where t is in seconds, determine the distance travelled before it comes to rest.

[***Ans.*** $s = 54$ m]

35. A small projectile is fired vertically downward in to a fluid medium with an initial velocity of 60 m/s. If the projectile experiences a deceleration, $a = -0.4v^3$ m/s^2, where v is measured in m/s, determine the projectile velocity and position 4 seconds after it is fired.

[***Ans.*** $v = 0.559$ m/s and $s = 4.43$ m.]

36. A particle moving in a straight line has an acceleration, $a = \sqrt{v}$. Its displacement and velocity at time $t = 2$ seconds, are 128/3 m and 16 m/s, respectively. Find the displacement, velocity and acceleration at $t = 3$ seconds.

[***Ans.*** $s = 60.75$ m, $v = 20.25$ m/s and $a = 4.5$ m/s^2.]

37. The acceleration of the train, starting from rest, at any instant is given by the expression $a = \left[\dfrac{8}{v^2 + 1}\right]$ m/s^2, where v is the velocity of the train in m/sec. Find the velocity of the train when its displacement is 20 m and its displacement when velocity is 64.8 kmph.

[***Ans.*** $v = 4.931$ m/s and $s = 3300.75$ m.]

38. The acceleration of a particle is given by $a = -0.02v^{1.75}$ m/s^2 performing rectilinear motion. Knowing at $x = 0$, $v = 15$ m/s. Determine **(a)** the position where velocity is 14 m/s and **(b)** the acceleration when $x = 100$ m.

[***Ans.*** **(a)** $s = 6.73$ m and **(b)** $a = -0.29$ m/s^2.]

39. A projectile enters a resisting medium at $x = 0$ with an initial velocity $v_o = 360$ m/s and travels 100 mm before coming to rest. Assuming that the velocity of the projectile is defined by the relation $v = v_o - kx$, where v is expressed in metres/second and x in metres determine **(a)** initial acceleration and **(b)** the time required for the projectile to penetrate 94 mm into the resisting medium.

[***Ans.*** **(a)** $a = -1.291 \times 106$ m/s^2 and **(b)** $t = 7.815 \times 10^{-4}$ seconds.]

Based on Motion Diagram (Graphical Solution)

40. In travelling a distance of 3 km between points A and D, a car is driven at 100 km/hr from A to B for t seconds. If brakes are applied for 4 seconds between B and C to give a car uniform deceleration from 100 kmph to 60 kmph and it takes t seconds to move from C to D with a uniform speed of 60 kmph. Determine the value of t.

[***Ans.*** 65.5 sec]

41. Two trains P and Q leave the same station on parallel lines. Train P starts at rest with uniform acceleration of 0.2 m/s^2 attains a speed of 10 m/s. Further the speed is kept constant. Train Q leaves 30 seconds later with uniform acceleration of 0.5 m/s^2 from rest and attains a maximum speed of 20 m/s. When will train Q overtake train P?

[***Ans.*** After 75 seconds from the start of train P.]

42. A bus starts from rest at point A and accelerates at constant rate of 0.75 m/s^2 until it reaches a speed of 9 m/s. It then proceeds at 9 m/s until brakes are applied 27 m ahead of B, where it comes to rest. Assuming uniform deceleration and knowing that distance between A and B is 180 m. Determine the time required for bus to travel from A to B, and uniform deceleration. Sketch v-t and a-t diagrams and also find the average speed during entire period.

[***Ans.*** 29 seconds and 6.21 m/s.]

43. A point moves along a straight line such that its displacement is $s = 8t^2 + 2t$ where s is in metres, t in seconds. Plot the displacement, velocity and acceleration against time.

44. Figure 11.E44 shows a plot of acceleration versus time for a particle moving along straight line. What is the speed and the distance covered by the particle after 50 seconds? Also find the maximum speed and the time at which the speed is attained by the particle.

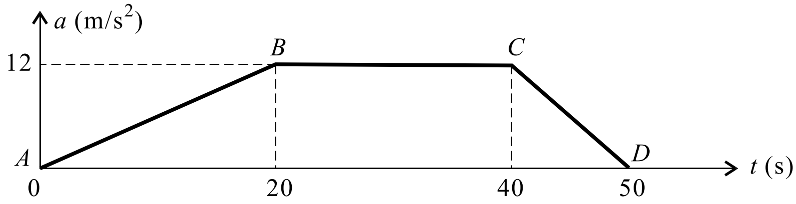


Fig. 11.E44

[***Ans.*** 420 m/s and 9600 m and v_{max} = 420 m/s at t = 50 seconds.]

45. A two-stage rocket is fired vertically from rest with acceleration, as shown in Fig. 11.E45. After 15 seconds the first stage A burns out and the second stage B ignites. Calculate the following:

(a) Velocity of the rocket at t = 15 seconds,

(b) Distance travelled by the rocket at t = 15 seconds,

(c) Velocity of the rocket at t = 40 seconds, and

(d) Distance travelled by the rocket from rest until t = 40 seconds.

[***Ans.*** **(a)** 112.5 m/s, **(b)** 2562.5 m, **(c)** 612.5 m/s and **(d)** 9625 m.]

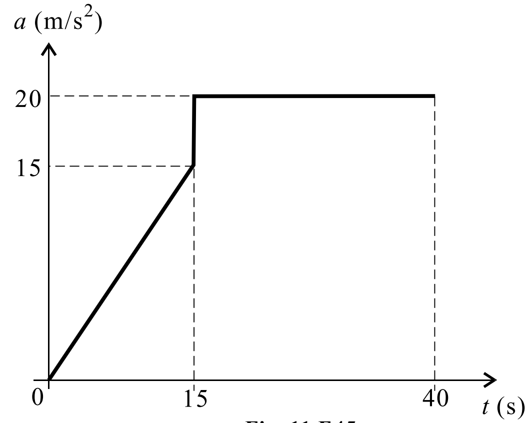


Fig. 11.E45

46. A rocket sled starts from rest and travels along a straight track such that it accelerates at a constant rate for 10 seconds and then decelerates at a constant rate, as shown in Fig. 11.E46. Draw the v-t curve and s-t curve and determine the time t' needed to stop the sled. How far the sled travelled.

[***Ans.*** t' = 60 sec and s = 3000 m.]

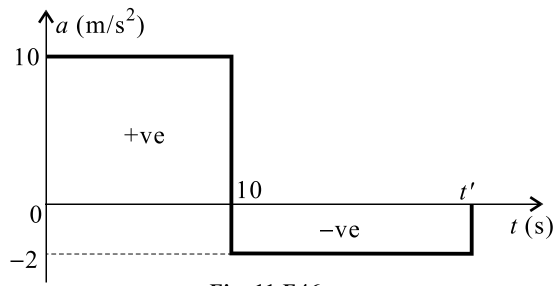


Fig. 11.E46

47. For the acceleration-time diagram of a particle, as is shown in Fig. 11.E47, calculate the velocity at the end of 3 seconds and the distance travelled in 4 seconds.

[***Ans.*** 0.5 m/s and 2 m.]

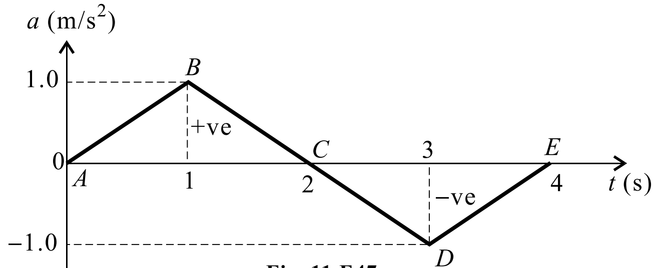


Fig. 11.E47

48. The v-x graph of a rectilinear moving particle is shown in Fig. 11.E48. Find acceleration of the particle at 20 m, 80 m and 200 m.

[***Ans.*** $a_{20} = 0.2$ m/s^2, $a_{80} = 0$ and $a_{200} = -0.4$ m/s^2.]

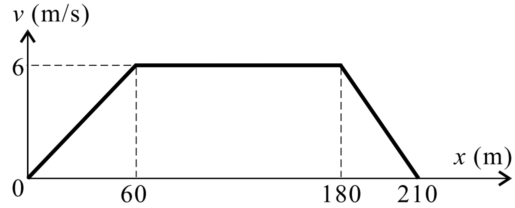


Fig. 11.E48

49. A particle starts from rest at $x = -2$ m and moves along x-axis with the velocity graph shown in Fig. 11.E49. Plot the acceleration and displacement graph for the first 2 seconds. Find the time t when the particle crosses the origin.

[***Ans.*** 0.92 seconds.]

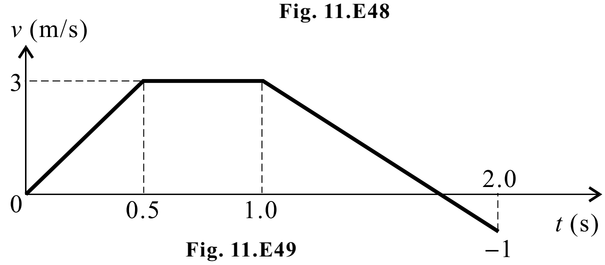


Fig. 11.E49

50. A dragster starting from rest travels along a straight road and for 10 seconds has acceleration, as shown in Fig. 11.E50. Construct the v-t graph that describes the motion and find the distance travelled in 10 seconds.

[***Ans.*** 114 m]

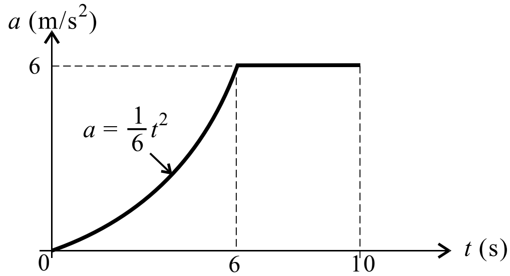


Fig. 11.E50

[II] Review Questions

1. Define the terms :

(i) Dynamics
(ii) Kinematics
(iii) Kinetics
(iv) Particle
(v) Rigid body

2. Distinguish between :

(i) Rectilinear motion and curvilinear motion.
(ii) Centroidal rotation and non-centroidal rotation

3. Explain the following :

(i) Translation motion
(ii) Rotational motion
(iii) General plane motion

4. What is meant by :

(i) Position
(ii) Displacement
(iii) Distance
(iv) Velocity
(v) Speed
(vi) Acceleration

5. What are the equations of various motions ?

[III] Fill in the Blanks

1. A body is said to be in motion if it is changing its ________ w.r.t. reference plane.
2. If a straight line drawn on the moving body remains parallel to its original position then such a motion is called __________ motion.
3. If all the particles of a rigid body move in a concentric circle then such a motion is called __________ motion.
4. __________ means the location of a particle w.r.t. origin.
5. The rate of change of distance w.r.t. time is called __________.
6. The rate of change of __________ w.r.t. time is called velocity.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. A particle can perform __________ motion.

   (a) only translation **(b)** only rotational **(c)** general plane **(d)** All of these

2. Study of kinematics of particle does not deals with _________.

   (a) displacement **(b)** velocity **(c)** acceleration **(d)** force

3. The value of a freely falling body from top of a tower is _______.

   (a) 9.81 cm/s^2 **(b)** –9.81 cm/s^2 **(c)** 9.81 m/s^2 **(d)** –9.81 m/s^2

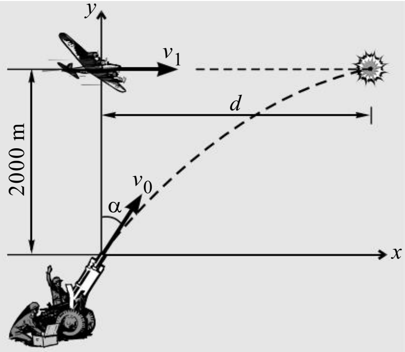
4. Slope of velocity-time curve is _______.

   (a) displacement **(b)** velocity **(c)** acceleration **(d)** distance

5. If acceleration-time diagram is represented by a horizontal straight line then displacement is a _______.

   (a) incline straight line **(b)** parabolic curve **(c)** cubic curve **(d)** zero

12



KINEMATICS OF PARTICLES - II

PROJECTILE MOTION

If a particle is freely thrown in air along any direction, other than vertical it will follow a curved path which is parabolic in nature. This motion is called *projectile motion* and the path traced by projectile is called its *trajectory* (neglecting the effect of air resistance).

Projectile motion is the combination of horizontal and vertical motion happening simultaneously. One particle is projected, we can say that the only force acting on the particle is the gravitational force, which is acting vertically downward. This gravitational force produces a change in the vertical component of velocity of the particle. However, the horizontal component of its velocity remains constant. Thus, we may conclude that for a projectile

$$a_x = 0 \text{ and } a_y = g = 9.81 \text{ m/s}^2 (\downarrow).$$

Thus, projectile motion is a combination of horizontal motion with constant velocity and vertical motion under gravity.

12.1 General Equation of Projectile Motion

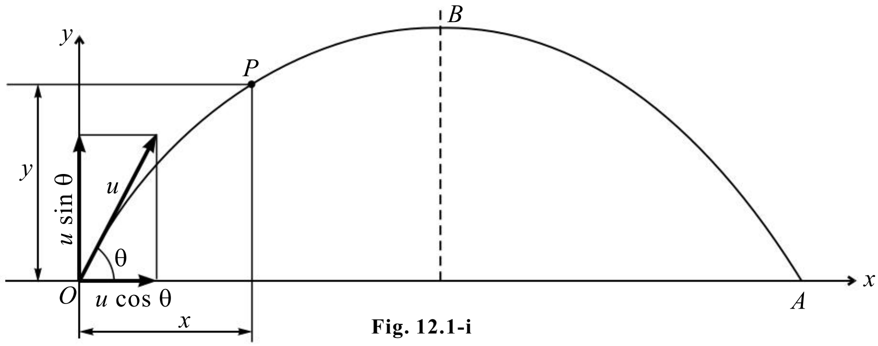


Fig. 12.1-i

Consider a particle to be freely thrown from point O (point of projection) at angle θ (angle of projection) with velocity u (velocity of projection). Let $P(x, y)$ be the projection of the particle after a time t.

Considering horizontal motion with constant velocity, we have

$$\text{Displacement} = \text{Velocity} \times \text{Time}$$

$$x = (u \cos \theta) \times (t)$$

$$\therefore \; t = \frac{x}{u \cos \theta}$$

Considering vertical motion under gravity, we have

$$h = ut - \frac{1}{2}gt^2$$

$$y = u \sin\theta \times \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

The above equation is of the form $y = Ax + Bx^2$ and represents a parabola. Thus, the path of projectile is a parabola and this equation is called the *general equation of projectile motion*.

Sign Convention for General Equation of Projectile Motion

y with respect to point of projection upward (+ve)

y with respect to point of projection downward (–ve)

θ angle of projection elevation θ (+ve)

θ angle of projection depression θ (–ve)

Some relations can be directly used when point of projection and point of target lies at same horizontal level.

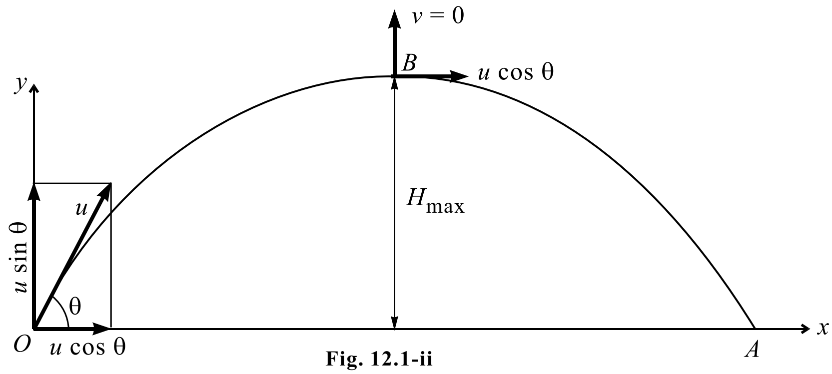


Fig. 12.1-ii

O = Point of projection

A = Point of target

OA = Range

B = Maximum height of projection

θ = Angle of projection

u = Velocity of projection

- **Time of Flight (T) :** The time taken by projectile to move from point of projection to point of target is called as *time of flight*.

  Consider vertical motion from O to A under gravity.

$$h = ut + \frac{1}{2}gt^2$$

$$0 = u\sin\theta \times T - \frac{1}{2}gT^2$$

$$T = \frac{2u\sin\theta}{g}$$

- **Range (R) :** The distance from point of projection to point of target is called a *range*.

  Consider horizontal motion from O to A with constant velocity.

$$\text{Displacement} = \text{Velocity} \times \text{Time}$$

$$R = u \sin\theta \times T = u \sin\theta \times \frac{2u \sin\theta}{g}$$

$$R = \frac{2u^2 \sin 2\theta}{g}$$

For maximum range

$$R_{max} = \frac{u^2}{g} \qquad \left[\sin 2\theta = 1 \Rightarrow 2\theta = 90^o \quad \therefore \quad \theta = 45^o \right]$$

- **Maximum Height (H_{max}) :** When projectile reaches to its maximum height vertical component of velocity at point B becomes zero. Consider vertical motion from O to B under gravity.

$$v^2 = u^2 + 2gh$$

$$0 = u^2 \sin^2\theta - 2gH_{max}$$

$$H_{max} = \frac{u^2 \sin^2\theta}{2g}$$

Note : If two projectiles having same velocity of projection but complementary angle of projection then range of both projectile will be same.

$$R_1 = \frac{u^2 \sin 2\theta}{g} \qquad R_2 = \frac{u^2 \sin 2(90 - \theta)}{g} = \frac{u^2 \sin (180 - 2\theta)}{g}$$

$$R_2 = \frac{u^2 \sin 2\theta}{g}$$

Projectile Motion is a Special Case of Curvilinear Motion

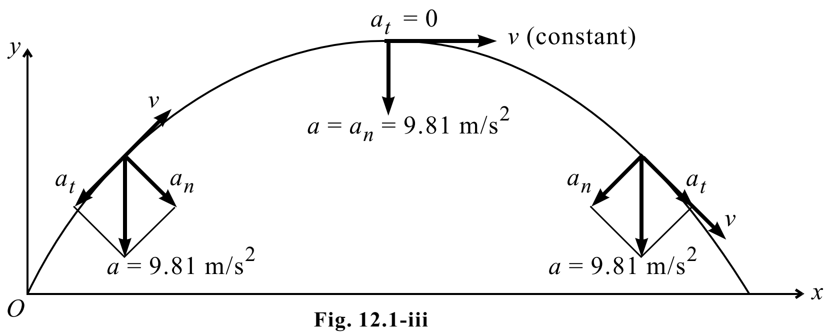


Fig. 12.1-iii

In projectile motion net acceleration is due to gravity, i.e., $g = 9.81$ m/s^2 ($\downarrow$).

Hence, a_t and a_n are the components of net acceleration $a = g = 9.81$ m/s^2 ($\downarrow$).

Velocity is always tangential.

$$\text{Radius of curvature } \rho = \frac{v^2}{a_n}$$

Radius of curvature will be minimum at topmost point of projectile since $a_n = g = 9.81$ m/s^2 is the maximum value

$$\rho_{min} = \frac{v^2}{9.81}$$

12.2 Solved Problems Based on Projectile Motion

Problem 1

A ball is thrown from horizontal level, such that it clears a 6 m high wall, situated at a horizontal distance of 35 m, as shown in Fig. 12.1. If the angle of projection is 60° with respect to the horizontal, what should be the minimum velocity of projection ?

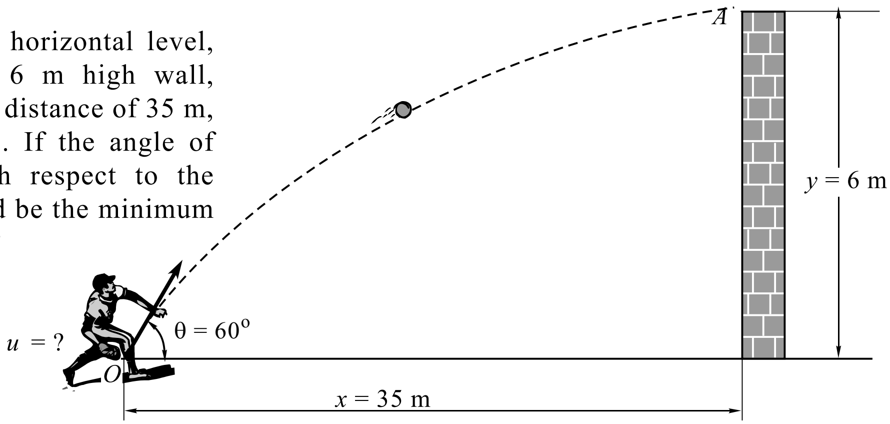


Fig.12.1

Solution

Using general equation of projectile motion

$$y = x \cdot \tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

$$6 = 35 \tan 60^\circ - \frac{9.81 \times 35^2}{2u^2}[1 + \tan^2(60^\circ)]$$

$u = 20.98$ m/s ***Ans.***

Problem 2

A ball thrown from top of building with a speed 12 m/s at angle of depression 30° with horizontal, strikes the ground 11.3 m horizontally from the foot of the building, as shown in Fig. 12.2. Determine the height of building.

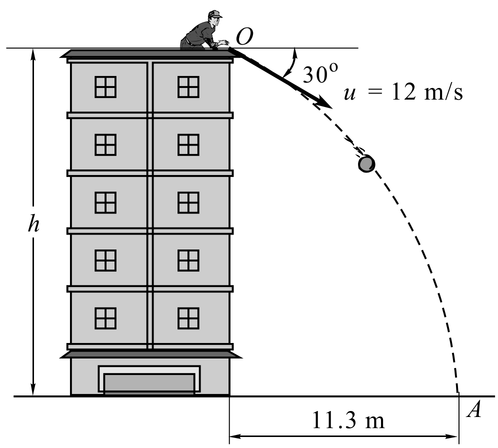


Fig. 12.2

Solution

Method I

Given : $y = -h$, $x = 11.3$m, $\theta = -30^\circ$, $u = 12$ m/s

By using general equation of projectile motion

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

$$-h = 11.3\tan(-30)^\circ - \frac{9.81 \times 11.3^2}{2 \times 12^2}[1 + \tan^2(-30)^\circ]$$

$\therefore h = 12.32$ m ***Ans.***

Method II

Let t be the time of flight. Consider horizontal motion with constant velocity

$s = v \times t$

$x = u\cos\theta \times t$

$11.3 = 12\cos 30^\circ \times t$

$t = 1.087$ seconds

Consider vertical motion under gravity

$$h = ut + \frac{1}{2}gt^2$$

$$h = 12 \sin 30^o \times 1.087 + \frac{1}{2} \times 9.81 \times (1.08)^2$$

$$h = 12.32 \text{ m} \quad \textbf{\textit{Ans.}}$$

Problem 3

An aeroplane is flying in horizontal direction with a velocity of 540 km/hr and at a height of 2200 m, as shown in Fig.12.3(a). When it is vertically above the point A on the ground, a body is dropped from it. The body strikes the ground at point B. Calculate the distance AB (ignore air resistance). Also find velocity at B and time taken to reach B.

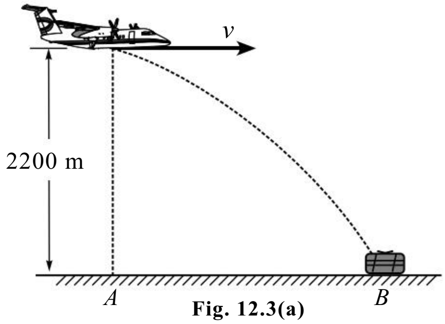


Fig. 12.3(a)

Solution

By general equation of projectile motion

$$y = x \tan \theta - \frac{gx^2}{2u^2} (1 + \tan^2 \theta)$$

$$-2200 = x \tan 0^o - \frac{9.81 \times x^2}{2 \times 150^2} (1 + \tan^2 0^o)$$

$$\therefore x = 3176.75 \text{ m (Distance } AB) \quad \textbf{\textit{Ans.}}$$

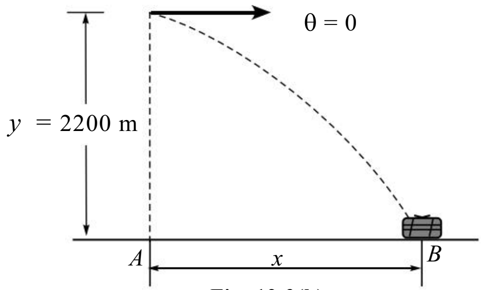


Fig. 12.3(b)

Consider horizontal motion with constant velocity

Displacement = Velocity × Time

$$3176.75 = 150 \times t$$

$$t = 21.18 \text{ seconds (Time taken to reach } B) \quad \textbf{\textit{Ans.}}$$

Considering vertical motion under gravity

$$v_{y_B}{}^2 = u^2 + 2gh$$

$$v_{y_B}{}^2 = 0 + 2 \times 9.81 \times 2200$$

$$v_{y_B} = 207.76 \text{ m/s} \quad (\downarrow)$$

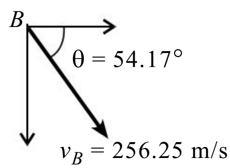
Considering horizontal constant velocity

$$v_{x_B} = 150 \text{ m/s} \quad (\rightarrow)$$

$$\tan \theta = \frac{v_{y_B}}{v_{x_B}} = \frac{207.76}{150}$$

$$\theta = 54.17^o$$

$$\therefore v_B = \sqrt{(v_{x_B})^2 + (v_{y_B})^2} = \sqrt{(150)^2 + (207.76)^2}$$



$$v_B = 256.25 \text{ m/s} \quad (\searrow\theta) \quad \textbf{\textit{Ans.}}$$

Problem 4

A ball thrown by a boy in the street, is caught by another boy on a balcony 4 m above the ground and 18 m away after 2 seconds as shown in Fig. 12.4. Calculate the initial velocity and the angle of projection.

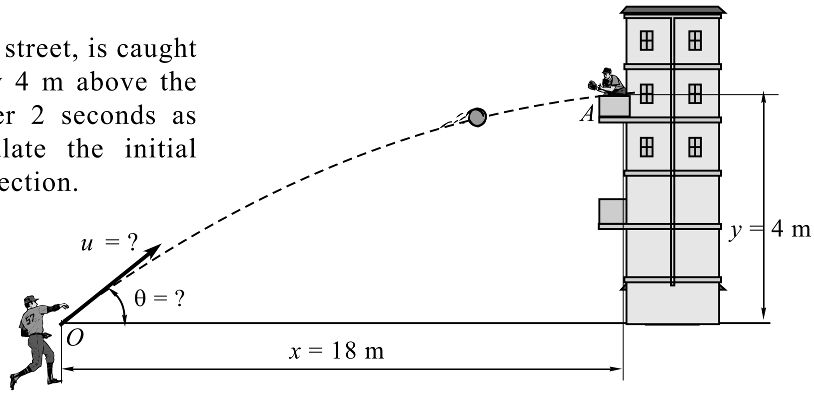


Fig. 12.4

Solution

Consider the vertical motion under gravity

$$h = ut + \frac{1}{2}gt^2$$

$$4 = 4 \sin\theta \times 2 - \frac{1}{2} \times 9.81 \times (2)^2$$

$$u \sin\theta = 11.81 \qquad \text{......... (I)}$$

Consider horizontal motion with constant velocity

$$s = v \times t$$

$$18 = u \cos\theta \times 2$$

$$u \cos\theta = 9 \qquad \text{......... (II)}$$

Dividing Eq. (I) by Eq. (II)

$$\tan\theta = \frac{11.81}{9} = 1.312 \qquad \therefore\ \theta = 52.69^\circ \quad \textbf{\textit{Ans.}}$$

From Eq. (1)

$$u \sin 52.69^\circ = 11.81$$

$$u = 14.85 \text{ m/s} \quad \textbf{\textit{Ans.}}$$

Problem 5

The water sprinkler positioned at the base of a hill releases a stream of water with a velocity of 6 m/s, as shown in Fig. 12.5(a). Determine the point $B(x, y)$ where the water particles strike the ground on the hill. Assume that the hill is defined by the equation $y = 0.2\,x^2$ m, and neglect the size of the sprinkler.

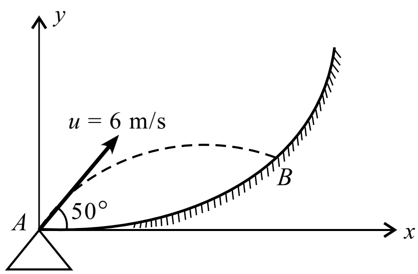


Fig. 12.5(a)

Solution

Given : $y = 0.2\,x^2$ ..... (I)

By general equation of projectile motion, we have

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

$$y = x \tan 50 - \frac{9.81 \times x^2}{2 \times 6^2}(1 + \tan^2 50)$$

$$y = 1.192\,x - 0.33\,x^2 \qquad \text{..... (II)}$$

From Eq. (I)

$$0.2\,x^2 = 1.192\,x - 0.33\,x^2$$

$$0.53\,x^2 = 1.192\,x$$

$$x = 2.25 \text{ m}$$

From Eq. (II)

$$y = 1.192 \times 2.25 - 0.33 \times 2.25^2 = 1.0114 \text{ m}$$

$\therefore$ $B(2.25, 1.0114)$ m ***Ans.***

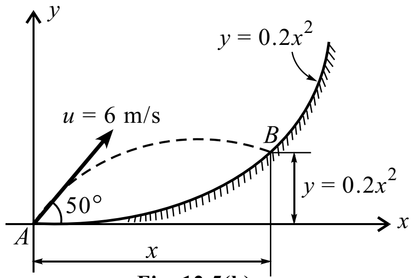


Fig. 12.5(b)

Problem 6

A ball is projected from the top of a tower of 110 m height with a velocity of 100 m/s and at an angle of elevation 25° to the horizontal, as shown in Fig. 12.6. Neglecting the air resistance, find **(i)** the maximum height the ball will rise from the ground, **(ii)** the horizontal distance it will travel just before it strikes the ground and **(iii)** the velocity with which it will strike the ground.

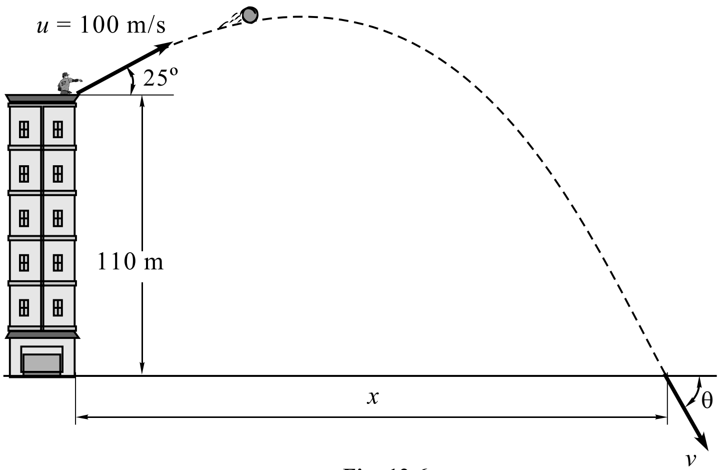


Fig. 12.6

Solution

$$H_{max} = \frac{u^2 \sin^2 \theta}{2g} = \frac{15^2 \sin^2 25^o}{2 \times 9.81}$$

$$H_{max} = 91.03 \text{ m}$$

$\therefore$ Maximum height the ball will rise from the ground

$$h = H_{max} + 110$$

$$h = 201.03 \text{ m} \quad \textbf{\textit{Ans.}}$$

Consider vertical motion under gravity

$$h = ut + \frac{1}{2}gt^2$$

$$-110 = 100 \sin 25^o \times t - \frac{1}{2} \times 9.81 \times t^2$$

$$t = 10.71 \text{ seconds}$$

Now, projectile horizontal motion happens with constant velocity

$\therefore$ Displacement = Velocity × Time

$$x = 100 \cos 25^o \times 10.71$$

$$x = 970.66 \text{ m} \quad \textbf{\textit{Ans.}}$$

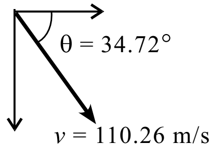
To find the velocity with which the ball strikes the ground

$v_x = 10 \cos 25^{\circ} = 90.63$ m/s $(\rightarrow)$

$-v_y = 100 \sin 25^{\circ} - 9.81 \times 10.71$

$v_y = 62.8$ m/s $(\downarrow)$

$\tan\theta = \dfrac{v_y}{v_x} = \dfrac{62.8}{90.63} \quad \therefore \; \theta = 34.72^{\circ}$



$\therefore \; v = \sqrt{v_x^{\,2} + v_y^{\,2}} = \sqrt{90.63^2 + 62.8^2}$

$v = 110.26$ m/s $\left(\theta\right)$ ***Ans.***

Problem 7

An object is projected so that it must clear two obstacles, each 7.5 m high, which are situated 50 m from each other, as shown in Fig. 12.7. If the time of passing between the obstacles is 2.5 seconds, calculate the complete range of projection and the initial velocity of projection.

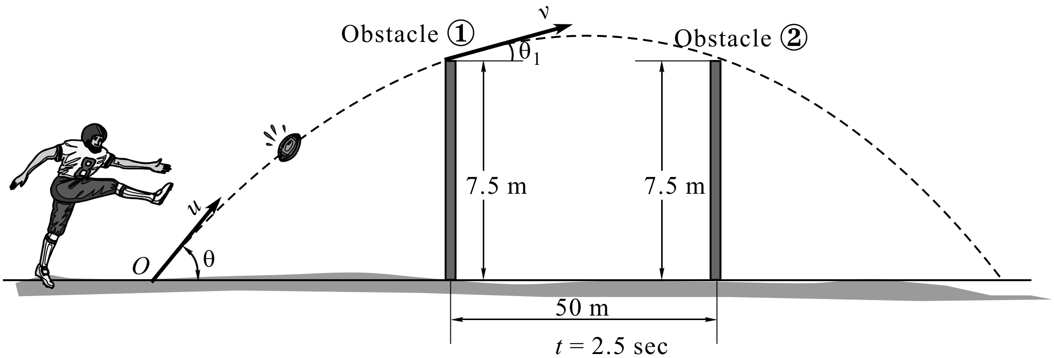


Fig. 12.7

Solution

Consider horizontal motion from obstacle ① to ②

$s = v \times t$

$50 = v\cos\theta_1 \times 2.5$

$v\cos\theta_1 = 20$ m/s

$v\cos\theta_1 = u\cos\theta = 20$ m/s .....(I) ...{Horizontal component of velocity is always constant}

Consider vertical motion from obstacle ① to ②

$h = ut + \dfrac{1}{2}gt^2$

$0 = v\sin\theta_1 \times 2.5 - \dfrac{1}{2} \times 9.81 \times 2.5^2$

$v\sin\theta_1 = 12.26$ m/s

Consider vertical motion from point O to obstacle ①

$v^2 = u^2 + 2gh$

$(12.26)^2 = (u\sin\theta)^2 - 2 \times 9.81 \times 7.5$

$u\sin\theta = 17.25$ ..... (II)

Dividing Eq. (II) by Eq. (I), we get

$$\tan\theta = \frac{17.25}{20} = 0.8625 \qquad \therefore\ \theta = 40.78^\circ$$

From Eq. (I), we have $u\cos\theta = 20$

$$u = \frac{20}{\cos 40.78^\circ} \qquad \therefore\ u = 26.41 \text{ m/s} \quad \textbf{\textit{Ans.}}$$

$$\text{Range } R = \frac{u^2 \sin 2\theta}{g} = \frac{(26.41)^2 \sin (2\times 40.78)^\circ}{9.81}$$

$\therefore\ R = 70.33$ m ***Ans.***

Problem 8

A boy throws a ball so that it may just clear a wall 3.6 m high. The boy is at a distance of 4.8 m from the wall. The ball was found to hit the ground at a distance of 3.6 m on the other side of the wall. Find the least velocity with which the ball can be thrown.

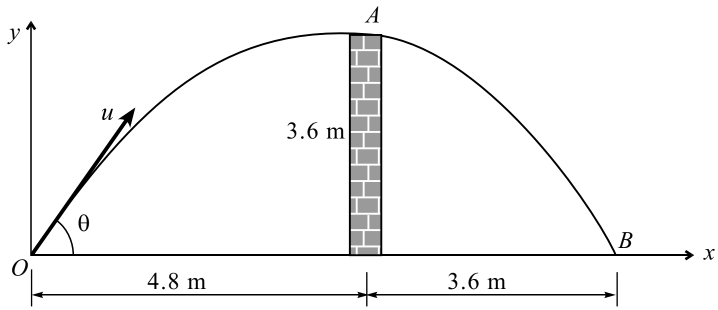


Fig. 12.8

Solution

Refer to Fig. 12.8.

Consider projectile motion

From O to A

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$3.6 = 4.8\tan\theta - \frac{9.81\times 4.8^2}{2u^2\cos^2\theta} \qquad \ldots \text{(I)}$$

From O to B

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$0 = 8.4\tan\theta - \frac{9.81\times 8.4^2}{2u^2\cos^2\theta}$$

$$2u^2\cos^2\theta = \frac{9.81\times 8.4^2}{8.4\tan\theta} \qquad \ldots \text{(II)}$$

From Eqs. (I) and (II), we get

$$3.6 = 4.8\tan\theta - \frac{9.81\times 4.8^2}{9.81\times 8.4^2}\times 8.4\tan\theta$$

$$3.6 = 4.8\tan\theta - 2.743\tan\theta$$

$$3.6 = 2.057\tan\theta \qquad \therefore\ \theta = 60.26^\circ$$

From Eq. (II), we have

$$u^2 = \frac{9.81\times 8.4^2}{8.4\tan\theta\times\cos^2\theta} = 95.66$$

$u = 9.78$ m/s $(\measuredangle 60.26^\circ)$ ***Ans.***

Problem 9

A particle projected from a point A, with the angle of projection equal to 15º falls short of a mark B on the horizontal plane through A by 22.5 and when the angle of projection is 45º it falls beyond B by the same distance, as shown in Fig. 12.9. Show that for the particle to fall exactly at B, the angle of projection must be $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)$, the velocity of projection being the same in all the three cases. Also determine the velocity of projection.

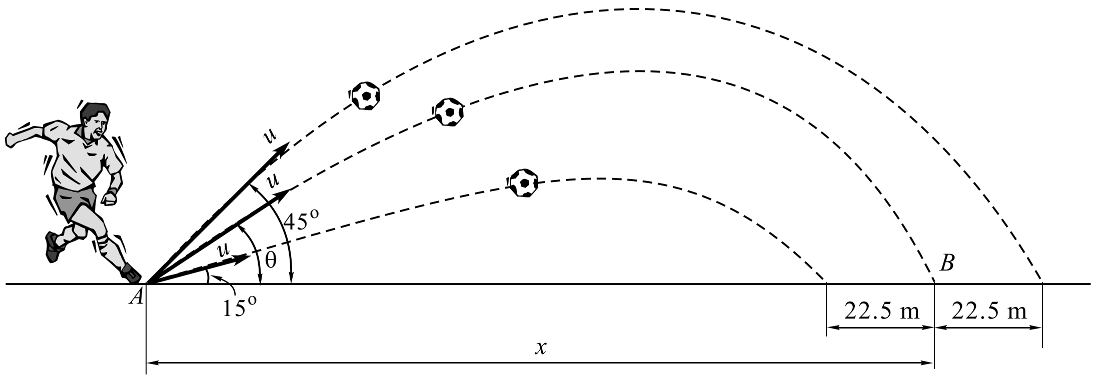


Fig. 12.9

Solution

When θ = 15º Range = (x – 22.5); when θ = 45º Range = (x + 22.5)

$$\text{Range} = \frac{u^2 \sin 2\theta}{g} \qquad \therefore\ x - 22.5 = \frac{u^2 \sin (2\times 15)^\circ}{9.81} \qquad \text{..... (I)}$$

$$\therefore\ x + 22.5 = \frac{u^2 \sin (2\times 45)^\circ}{9.81} \qquad \text{..... (II)}$$

Now, Dividing Eq. (I) by Eq. (II), we get

$$\frac{x - 22.5}{x + 22.5} = \frac{\sin 30^\circ}{\sin 90^\circ}$$

$$x - 22.5 = (x + 22.5)\, 0.5$$

$$x - 0.5x = 22.5 + 11.25$$

$$0.5x = 33.75 \qquad \therefore\ x = 67.5 \text{ m}$$

From Eq. (I)

$$67.5 - 22.5 = \frac{u^2 \sin 30^\circ}{9.81} \qquad \therefore\ u = 29.71 \text{ m/s}$$

$$\text{Now, Range } x = \frac{u^2 \sin 2\theta}{g}$$

$$67.5 = \frac{(29.71)^2 \sin 2\theta}{9.81}$$

$$\sin 2\theta = 0.75 = \frac{3}{4}$$

$$2\theta = \sin^{-1}\left(\frac{3}{4}\right) \qquad \theta = \frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)$$ ***Proved.***

Problem 10

A block of ice starts sliding down from the top of the inclined roof of a house (angle of inclination of roof is 30° with the horizontal) along a line of maximum slope as shown in Fig. 12.10. The highest and lowest points of the roof are at height of 10.9 m and 8.4 m, respectively, from the ground. At what horizontal distance from the starting point will the block hit the ground ? (Neglect friction)

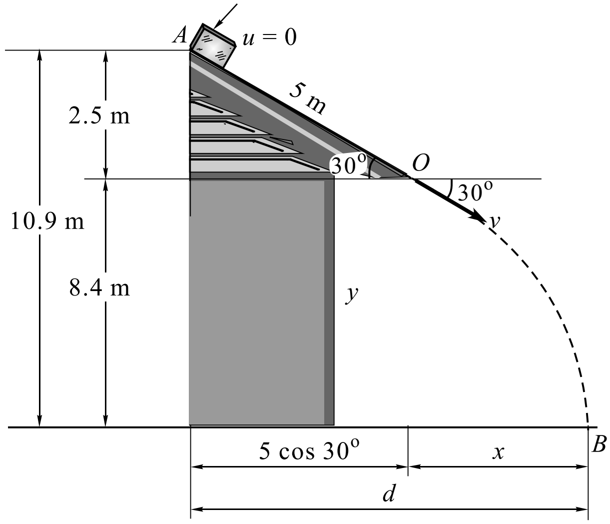


Fig. 12.10

Solution

Motion of ice block from A to O

$$v^2 = u^2 + 2as$$

$$v^2 = 0 + 2 \times 9.81 \sin 30^\circ \times 5$$

$$v = 7 \text{ m/s}$$

Motion from O to B is projectile motion

By general equation of projectile motion, we have

$$y = x \tan \theta - \frac{gx^2}{2u^2}(1 + \tan^2 \theta)$$

$$-8.4 = x \tan (-30)^\circ - \frac{9.81 \times (x)^2}{2 \times 7^2}[1 + \tan^2 (-30)^\circ]$$

$$x = 6.06 \text{ m}$$

Distance from starting point

$$d = 5 \cos 30^\circ + x$$

$$d = 5 \cos 30^\circ + 6.06$$

$d = 10.39$ m **_Ans._**

Problem 11

A shell bursts on contact with ground and fragments fly in all directions with speed up to 30 m/s. If a man is 40 m away from the spot, as shown in Fig. 12.11, find for how long he is in danger.

Solution

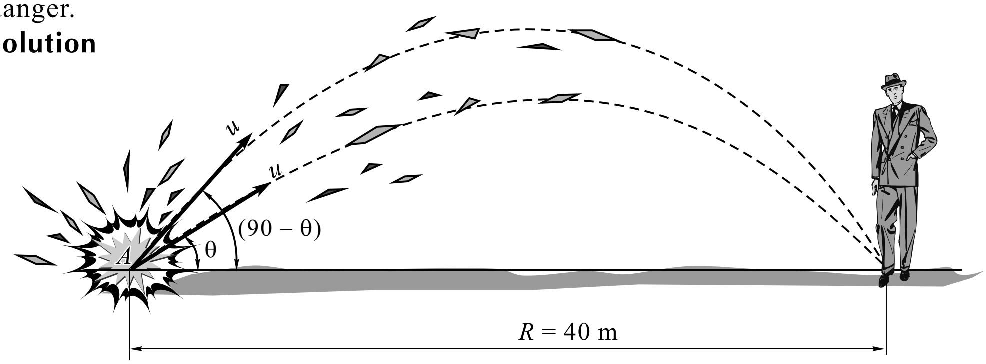


Fig. 12.11

Note : For same range and same initial velocity, θ and (90 – θ) are the two values of angle of projection but time of flight differs.

$$\text{Range } R = \frac{u^2 \sin 2\theta}{g}$$

$$40 = \frac{(30)^2 \sin 2\theta}{9.81}$$

$\therefore$ $\theta = 12.92^\circ$ $\qquad$ $\therefore$ $(90 - \theta) = 77.08^\circ$

Let t_1 be the time taken for earlier fragment to reach the man

$$t_1 = \frac{2u \sin \theta}{g}$$

$$t_1 = \frac{2 \times 30 \times \sin 12.92^\circ}{9.81}$$

$t_1 = 1.37$ seconds

Let t_2 be the time taken for last fragment to reach the man

$$t_2 = \frac{2u \sin (90 - \theta)}{g}$$

$$t_2 = \frac{2 \times 30 \times \sin 77.08^\circ}{9.81}$$

$t_2 = 5.96$ m

$\therefore$ Time for which man is in danger $T = t_2 - t_1$

$T = 5.96 - 1.37$

$T = 4.59$ seconds ***Ans.***

Problem 12

A ball is thrown upward from a high cliff with a velocity of 100 m/s at an angle of elevation of 30º with the horizontal, as shown in Fig. 12.12. The ball strikes the inclined ground at right angle. If the inclination of ground is 30º as shown, determine **(i)** the velocity with which it strikes the ground, **(ii)** the time after which the ball strikes the ground and **(iii)** coordinates (x, y) of a point of the strike w.r.t. point of projection.

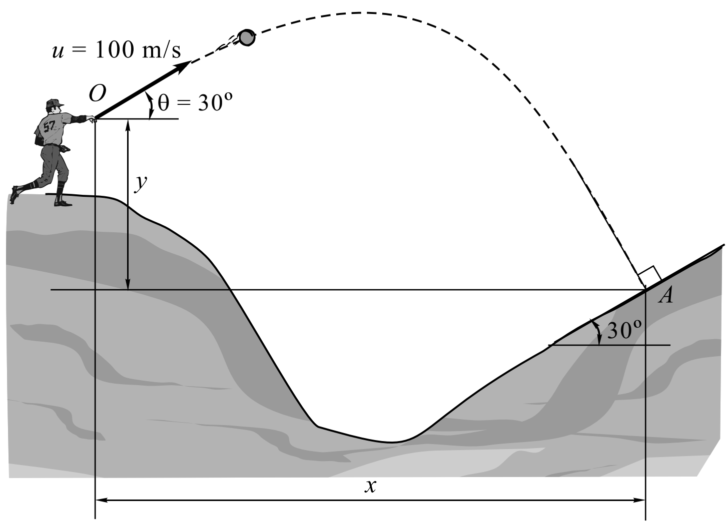


Fig. 12.12

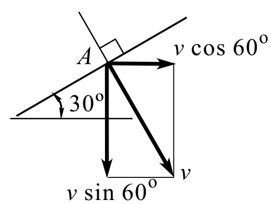
Solution

(i) We know in projectile motion, horizontal component of velocity remain constant

$\therefore$ $v \cos 60^\circ = u \cos \theta$

$v \cos 60^\circ = 100 \cos 30^\circ$

$v = 173.21$ m/s ***Ans.***



(ii) Time of flight (t)

Consider vertical motion under gravity

$v = u + gt$

$$-v \sin 60^\circ = 100 \sin 30^\circ - 9.81 \times t$$

$$\frac{-173.21 \sin 60^\circ - 100 \sin 30^\circ}{9.81} = -t$$

$$\therefore\ t = 20.39 \text{ sec} \quad \textbf{\textit{Ans.}}$$

(iii) Coordinates (x, y) consider horizontal motion with constant velocity

$$s = v \times t$$

$$x = 100 \cos 30^\circ \times 20.39$$

$$x = 1765.83 \text{ m}$$

Consider vertical motion under gravity

$$h = ut + \frac{1}{2} gt^2$$

$$-y = 100 \sin 30^\circ \times 20.39 - \frac{1}{2} \times 9.81 \times (20.39)^2$$

$$y = 1019.76 \text{ m} \quad \textbf{\textit{Ans.}}$$

Problem 13

Two guns are pointed at each other one upwards at an angle of elevation of 30º and the other at the same angle of depression and being 30 metres apart, as shown in Fig. 12.13. If the bullets leave the guns with velocity of 350 m/s and 300 m/s, respectively, find when and where they will meet.

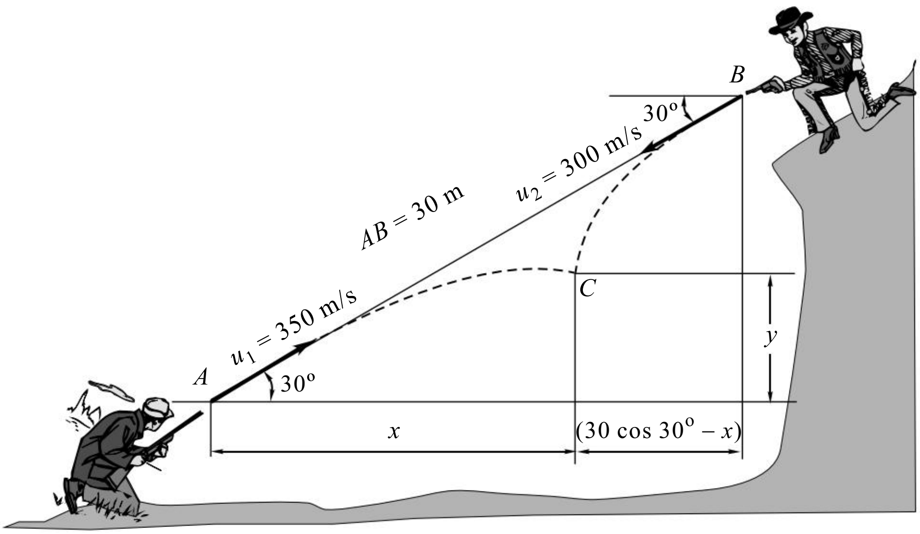


Fig. 12.13

Solution

Let 't' be the time taken to meet at C.

Consider horizontal motion with constant velocity

$$s = v \times t$$

$$x = 350 \cos 30^\circ \times t \quad ..... \text{(I)}$$

$$s = v \times t$$

$$30 \cos 30^\circ - x = 300 \cos 30^\circ \times t$$

$$x = -300 \cos 30^\circ \times t + 30 \cos 30^\circ \quad ..... \text{(II)}$$

Equating Eqs. (I) and (II), we get

$$350 \cos \times t = -300 \cos 30^{o} \times t + 30 \cos 30^{o}$$

$$t\,(350 \cos 30^{o} + 300 \cos 30^{o}) = 30 \cos 30^{o}$$

$$t = 0.04615 \text{ sec}$$

From Eq. (I)

$$x = 350 \cos 30^{o} \times 0.04615$$

$$x = 13.99 \text{ m}$$

Consider vertical motion under gravity of bullet A

$$h = ut + \frac{1}{2} gt^2$$

$$y = 350 \sin 30^{o} \times 0.04615 - \frac{1}{2} \times 9.81 \times (0.04615)^2$$

$y = 8.07$ m ***Ans.***

Problem 14

A ball is projected from point A with a velocity $u = 10$ m/s which is perpendicular to the incline, as shown in Fig. 12.14(a). Determine the range R when $\theta = 30^{o}$ solve from fundamentals.

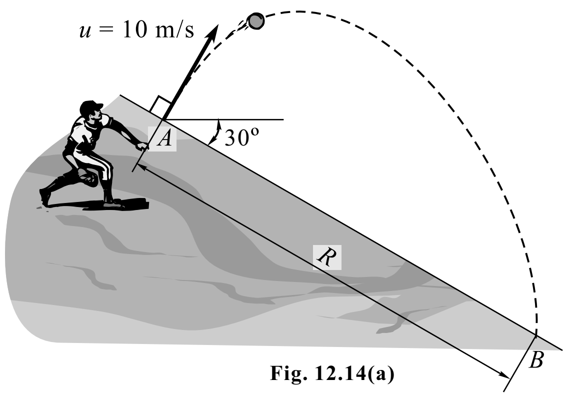


Fig. 12.14(a)

Solution

Let 't' be the time of flight from A to B

Horizontal motion with constant velocity

$$s = v \times t$$

$$R \cos 30^{o} = 10 \cos 60^{o} \times t$$

$$\therefore\ t = \frac{R \cos 30^{o}}{10 \cos 60^{o}} = R(0.1732)$$

Vertical motion under gravity

$$h = ut + \frac{1}{2} gt^2$$

$$-R \sin 30^{o} = 10 \sin 60^{o} \times R\,(0.173^2) - \frac{1}{2} \times 9.81\, R^2\,(0.1732)^2$$

$$-R\,0.5 = 1.5R - 0.147R^2$$

$$-2R = -\,0.147R^2$$

$R = \dfrac{2}{0.1471} = 13.6$ m ***Ans.***

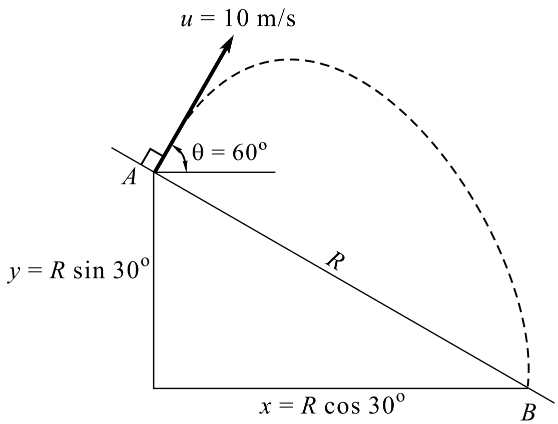


Fig. 12.14(b)

Problem 15

An object is projected with u = 10 m/s and $\theta = 30^{o}$ from point A, as shown in Fig. 12.15. Find the velocity with which it lands at B. Assume the ground has the shape of parabola as shown.

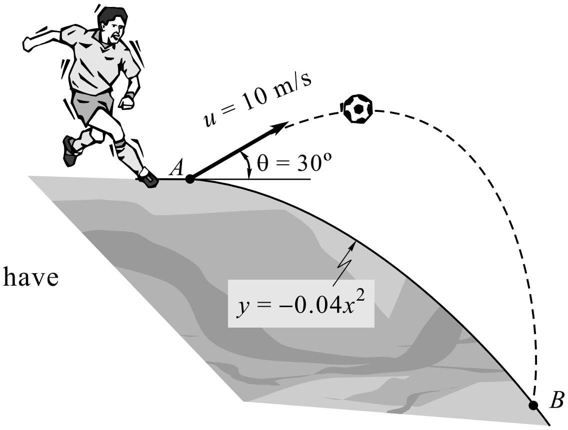


Fig. 12.15

Solution

By general equation of projectile motion, we have

$$y = x \tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

$$y = x \tan 30^{o} - \frac{9.81\, x^2}{2 \times 10^2}\,(1 + \tan^2 30^{o})$$

$$y = 0.577\, x - 0.065\, x^2 \qquad \text{..... (I)}$$

Equation of parabolic ground surface is given

$$y = -0.04\, x^2 \qquad \text{..... (II)}$$

Eq. (I) = Eq. (II)

$$0.577\, x - 0.065 x^2 = -0.04\, x^2$$

$$0.577\, x = 0.025\, x^2$$

$$\therefore\; x = \frac{0.577}{0.025} = 23.08 \text{ m}$$

From Eq. (II),

$$y = -0.04\,(23.08)^2 = -21.31 \text{ m}$$

$\therefore$ Coordinate of $B(x, y)$ = (23.08, –21.31) m.

Horizontal component of velocity is projectile motion is always constant

$$\therefore\; v_{Bx} = u \cos\theta$$

$$v_{Bx} = 10 \cos 30^{o} = 8.66 \text{ m/s } (\rightarrow)$$

Consider vertical motion under gravity

$$v^2 = u^2 + 2gh$$

$$v^2_{By} = (u \sin\theta)^2 + 2\,(-9.81) \times (-y)$$

$$v^2_{By} = (10 \sin 30^{o})^2 + 2 \times (-9.81) \times (-21.31)$$

$$v_{By} = -21.05 = 21.05 \text{ m/s } (\downarrow)$$

$$v_B = \sqrt{(v_{Bx})^2 + (v_{By})^2} = \sqrt{(8.66)^2 + (21.05)^2}$$

v_B = 22.76 m/s ***Ans.***

$\theta = \tan^{-1}\left(\frac{21.05}{8.66}\right) = 67.64^{o}$ ***Ans.***

Problem 16

A ball rebounds at A and strikes the inclined plane at point B at a distance 76 m, as shown in Fig. 12.16(a). If the ball rises to a maximum height h = 19 m above the point of projection, compute the initial velocity and the angle of projection α.

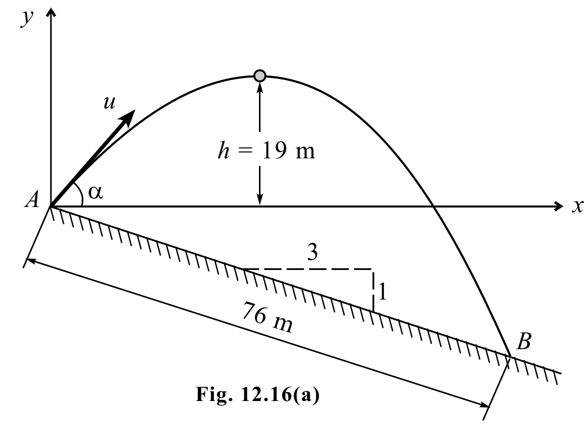


Fig. 12.16(a)

Solution

We know,

$$H_{\max} = \frac{u^2 \sin^2 \alpha}{2g}$$

$$19 = \frac{u^2 \sin^2 \alpha}{2 \times 9.81}$$

$$u \sin \alpha = 19.31 \qquad \ldots\ldots \text{(I)}$$

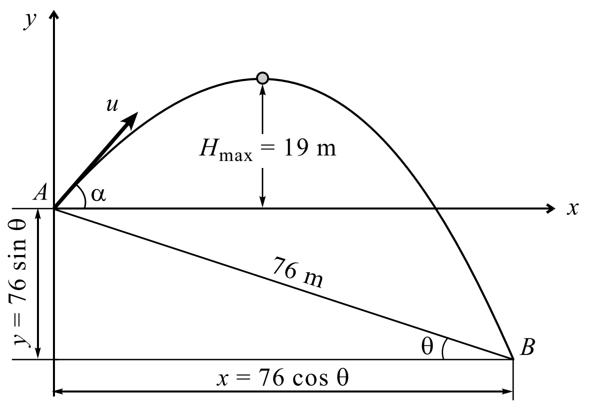


Fig. 12.16(b)

Consider vertical motion from A to B (under gravity)

$$h = ut + \frac{1}{2} gt^2$$

and from Eq. (I) we have

$$-76 \sin 18.44^{\circ} = u \sin \alpha t - \frac{1}{2} \times 9.81 \times t^2$$

$$-24.04 = 19.31 \times t - 4.905\, t^2$$

$$4.905\, t^2 - 19.31 \times t - 24.04 = 0$$

Solving quadratic equation, we get

$$t = 4.93 \text{ seconds}$$

Consider horizontal motion with constant velocity, we have

Displacement = Velocity × Time

$$x = u \cos \alpha \times t$$

$$76 \cos 18.44^{\circ} = u \cos \alpha \times 4.93$$

$$u \cos \alpha t = 14.62 \qquad \ldots\ldots \text{(II)}$$

Dividing Eq. (I) by Eq. (II), we get

$$\tan \alpha = \frac{19.31}{14.62}$$

$$\therefore\ \alpha = 52.87^{\circ}$$

$$u = 24.22\ (\measuredangle \alpha)$$ ***Ans.***

Problem 17

A boy throws a ball with an initial velocity 24 m/s, as shown in Fig. 12.17. Knowing that the boy throws the ball from a distance of 30 m from the building, determine **(i)** the maximum height *h* that can be reached by the ball and **(ii)** the corresponding angle α.

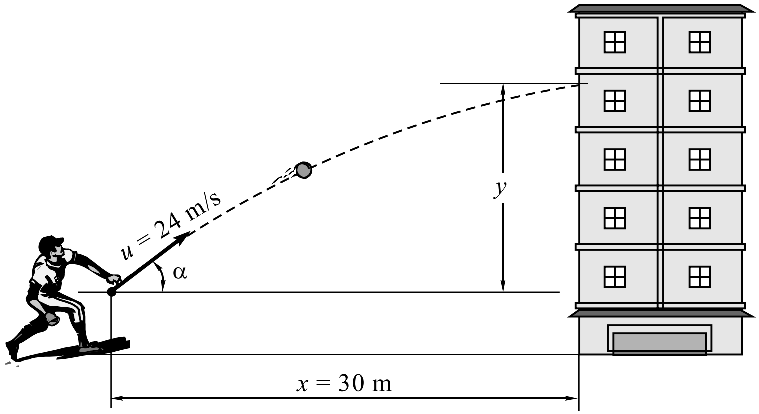


Fig. 12.17

Solution

By general equation of projectile motion, we have

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$$

$$y = 30\tan\alpha - \frac{9.81\times 30^2}{2\times 24^2}(1+\tan^2\alpha)$$

$$y = 30\tan\alpha - 7.66\,(1+\tan^2\alpha) \qquad \text{..... (I)}$$

For height to be maximum (y_{max}), apply maxima condition

i.e., $\dfrac{dy}{dx} = 0$

$$30\sec^2\alpha - 7.66\,(0 + 2\tan\alpha\sec^2\alpha) = 0$$

$$\sec^2\alpha\,(30 - 15.32\tan\alpha) = 0$$

$$\sec^2\alpha = 0$$

$$\tan\alpha = \frac{30}{15.32} \qquad \therefore\ \alpha = 62.95^{\circ}$$

From Eq. (I)

$$y_{max} = 30\tan 62.95^{\circ} - 7.66\,[1+\tan^2(61.95)^{\circ}]$$

$$\therefore\ y_{max} = h = 21.71 \text{ m}$$

The maximum height *h* that can be reached by the ball is 21.71 m and angle $\alpha = 62.95^{\circ}$.

Ans.

Problem 18

A projectile P is fired at a muzzle velocity of 200 m/s at an angle of elevation of 60º, as shown in Fig. 12.18. After some time a missile M is fired at 2000 m/s muzzle velocity and at an angle of elevation of 45º from the same point to destroy the projectile P. Find the **(i)** height, **(ii)** horizontal distance and **(iii)** time with respect to P at which the destruction takes place.

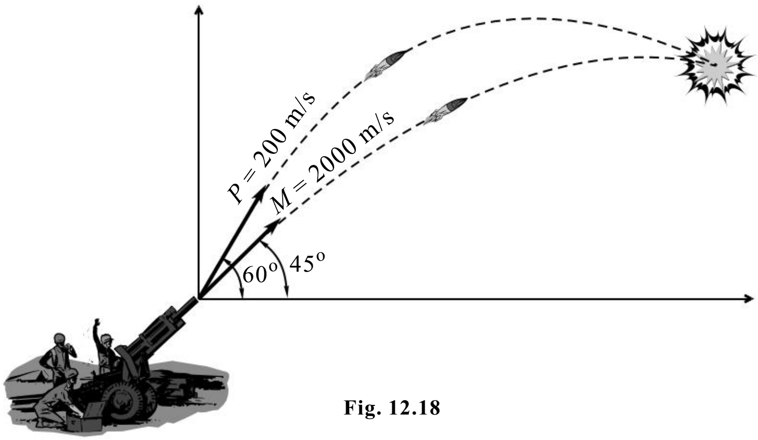


Fig. 12.18

Solution

At the time of destruction coordinates x and y of both the projectile will be same.

By general equation of projectile motion, we have

$$y = x \tan \theta - \frac{gx^2}{2u^2} (1 + \tan^2 \theta)$$

Projectile P

$$y = x \tan 60^o - \frac{9.81 \times x^2}{2 \times 200^2} [1 + \tan^2 (60)^o]$$

Projectile M

$$y = x \tan 45^o - \frac{9.81x^2}{2 \times 2000^2} [1 + \tan^2 (45)^o]$$

Equating both, we get

$$x (\tan 60^o - \tan 45^o) = \frac{9.81x^2}{2} \left[\frac{1 + \tan^2 (60)^o}{200^2} - \frac{1 + \tan^2 (45)^o}{2000^2} \right]$$

$$0.7321x = 4.905x^2 (1 \times 10^{-4} - 5 \times 10^{-7})$$

$$\therefore \; x = \frac{0.7321}{4.905 \times 9.95 \times 10^{-5}}$$

x = 1500 m (horizontal distance) ***Ans.***

$$y = x \tan 60^o - \frac{9.81x^2}{2 \times 200^2} [1 + \tan^2 (60)^o] \quad \text{Put } x = 1500 \text{ m}$$

$\therefore \; y$ = 1495 m (vertical height) ***Ans.***

For time, we have

$$x = u \cos \theta \times t$$

$\therefore \; t = \dfrac{1500}{200 \cos 60^o}$ = 15 sec with respect to projectile P. ***Ans.***

Curvilinear Motion

If a particle is moving along curved path then it is said to perform ***curvilinear motion****.*

12.3 Position, Velocity and Acceleration for Curvilinear Motion

Position

Consider the motion of a particle along a curved path, as shown in Fig. 12.3-i. It is represented by a position vector $\overline{r}$ which is drawn from the origin 'O' of the fixed reference axis to particle 'P'. The line OP is called *position vector*. As the particle will move along the curved path, the value of $\overline{r}$ will go on changing.

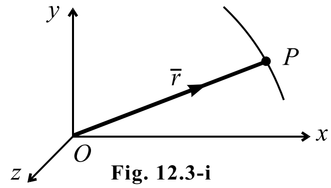


Fig. 12.3-i

Velocity

Consider after a short interval of time, Δt particle has occupied new position P' simultaneously the position vector $\overline{r}$ will change to $\overline{r}'$.

The vector joining P and P' is the change in position vector Δr during the time interval Δt.

$\therefore$ The average velocity $v = \dfrac{\Delta \overline{r}}{\Delta t}$

For very small interval of time $\Delta t \to 0$

Instantaneous velocity at P is $\lim\limits_{\Delta t \to 0} \dfrac{\Delta \overline{r}}{\Delta t}$

$$\therefore \; v = \frac{dr}{dt}$$

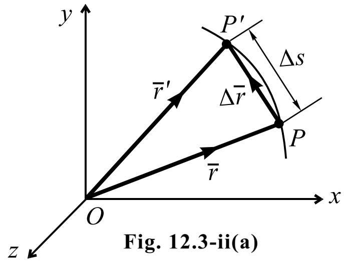


Fig. 12.3-ii(a)

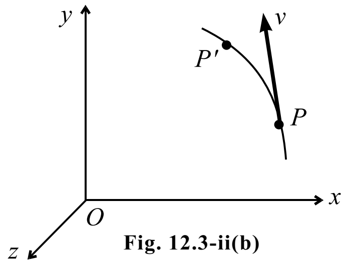


Fig. 12.3-ii(b)

Here during a small interval of time Δt, the particle moves a distance Δs along the curve.

$\therefore$ The magnitude of velocity called speed is given by relation

$$\text{Speed} = |\overline{v}| = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Note : In curvilinear motion, velocity of particle is always tangent to the curved path at every instant.

Acceleration

As the direction of velocity is continuously changing instant to instant in curvilinear motion it is responsible to develop acceleration also at every instant.

Consider the velocity of the particle at P to be v and at position P' be v'.

$\therefore$ Average acceleration $a = \dfrac{\Delta \overline{v}}{\Delta t}$

For very small interval of time $\Delta t \to 0$

Instantaneous velocity at P is $\lim\limits_{\Delta t \to 0} \dfrac{\Delta \overline{v}}{\Delta t}$

$$\therefore \; a = \frac{dv}{dt}$$

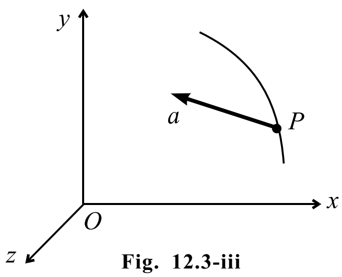


Fig. 12.3-iii

Note : In rectilinear motion displacement, velocity and acceleration are always directed along the path of particle. Whereas in curvilinear motion it change its direction instant to instant. Therefore, the analysis of curvilinear motion is done by considering different components system. There are two methods for analysis in terms of different component system.

(1) Curvilinear motion by Rectangular Component System.

(2) Curvilinear motion by Tangential and Normal Component System.

12.4 Curvilinear Motion by Rectangular Component System

If a particle is moving along a curved path, its motion can be splitted into x, y and z direction as independently performing rectilinear motions.

Thus for curvilinear motion we can have a relation as follows.

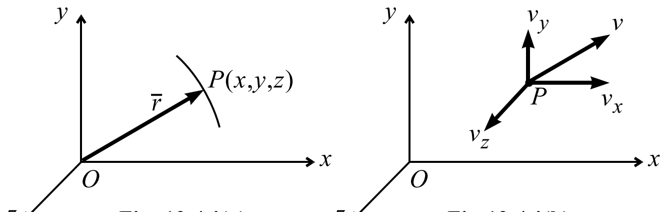


Fig. 12.4-i(a) Fig.12.4-i(b)

Vector Form	$\overline{r} = x\,\overline{i} + y\,\overline{j} + z\,\overline{k}$	$\overline{v} = \dfrac{d\overline{r}}{dt} = v_x\,\overline{i} + v_y\,\overline{j} + v_z\,\overline{k}$	$a = \dfrac{dv}{dt} = a_x\,\overline{i} + a_y\,\overline{j} + a_z\,\overline{k}$
Magnitude	$r = \sqrt{x^2 + y^2 + z^2}$	$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$	$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Direction is given by the relation

$$\cos\alpha = \frac{x}{r} = \frac{v_x}{v} = \frac{a_x}{a}\ ;\ \cos\beta = \frac{y}{r} = \frac{v_y}{v} = \frac{a_y}{a}\ ;\ \cos\gamma = \frac{z}{r} = \frac{v_z}{v} = \frac{a_z}{a}\ ;$$

While dealing with coplanar motion we can consider that the particle is moving in xy plane. Its rectangular component system will be as follows.

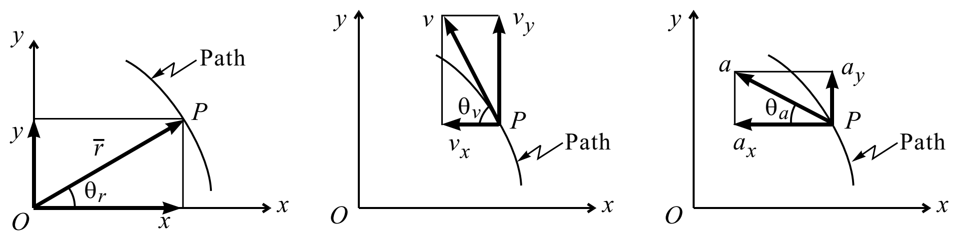


Fig. 12.4-ii(a) Fig. 12.4-ii(b) Fig. 12.4-ii(c)

Vector Form	$\overline{r} = x\,\overline{i} + y\,\overline{j}$	$\overline{v} = v_x\, i + v_y\, j$	$a = a_x\,\overline{i} + a_y\,\overline{j}$
Magnitude	$r = \sqrt{x^2 + y^2}$	$v = \sqrt{v_x^2 + v_y^2}$	$a = \sqrt{a_x^2 + a_y^2}$
	$\tan\theta_r = \dfrac{y}{x}$	$\tan\theta_v = \dfrac{v_y}{v_x}$	$\tan\theta_a = \dfrac{a_y}{a_x}$

The radius of curvature is calculated by following relation:

$$\rho = \left| \frac{(v_x^2 + v_y^2)^{3/2}}{v_x a_y - v_y a_x} \right|$$

12.5 Curvilinear Motion by Tangential and Normal Component System

In curvilinear motion the acceleration is splitted into two components, one along tangential direction (a_t) and another along normal direction (a_n).

$$\therefore\ \bar{a} = a_t\,\bar{e}_t + a_n\,\bar{e}_n$$

$$\text{Magnitude}\ \ a = \sqrt{a_t^2 + a_n^2}$$

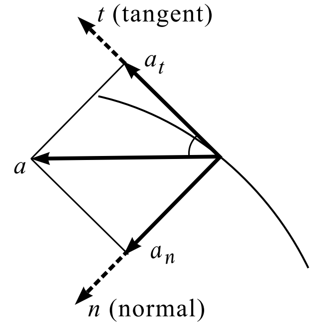


Fig. 12.5-i

Tangential component of acceleration a_t represents the rate of change of speed of a particle. The direction of a_t is along the velocity if speed is increasing, and is opposite to the direction of velocity if speed is decreasing. The direction of velocity is always tangential.

For constant speed in curvilinear motion

$$a_t = \frac{dv}{dt} = 0$$

and following equation of motion is applicable

$$v = u + a_t t$$

$$s = ut + \frac{1}{2}\,a_t t^2$$

$$v^2 = u^2 + 2a_t s$$

where s is the distance covered along curved path.

u is initial speed and v is final speed and

a_t is the component of acceleration along tangential direction.

Normal component of acceleration a_n represents the change in direction of motion and is always directed towards the centre of curvature of the path. It is also called as *centripetal acceleration*.

$$\text{Magnitude of } a_n = \frac{v^2}{\rho}$$

where v is the speed at the instant and ρ is the radius of curvature.

If curve is following a path defined by $y = f(x)$ then

$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\frac{d^2y}{dt^2}} \right|$$

Direction of velocity is always tangential

$$\therefore\ \text{Slope} = \frac{dy}{dx} = \tan\theta$$

$$\therefore\ v_x = v\cos\theta \ \text{ and } \ v_y = v\sin\theta$$

12.6 Solved Problems Based on Curvilinear Motion

Problem 19

A point moves along the path $y = \frac{1}{3}x^2$ with a constant speed of 8 m/s. What are the x and y components of the velocity when $x = 3$? What is the acceleration of the point when $x = 3$?

Solution

Given : $v = 8$ m/s is constant;

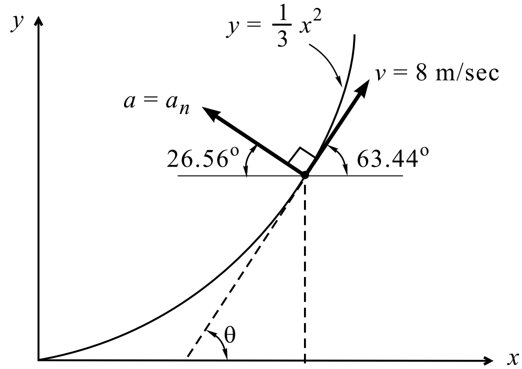


Fig. 12.19

$$a_t = 0 \qquad a_t = \frac{dv}{dt} = 0$$

$$\therefore \quad a_n = a \qquad [\because a_t = 0]$$

$$\therefore \quad a_n = \frac{v^2}{\rho}$$

$$y = \frac{1}{3}x^2$$

$$\frac{dy}{dx} = \frac{2}{3}x$$

$$\left(\frac{dy}{dx}\right)_{x=3} = \frac{2}{3} \times 3 = 2$$

$$\frac{d^2y}{dx^2} = \frac{2}{3}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=3} = \frac{2}{3} \qquad \textbf{\textit{Ans.}}$$

$$\therefore \quad \rho = \left|\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right| = \left|\frac{\left[1 + (2)^2\right]^{3/2}}{\frac{2}{3}}\right|$$

$$\rho = 16.77 \text{ m}$$

$$a_n = \frac{v^2}{\rho} = \frac{(8)^2}{16.77} = 3.82 \text{ m/s}^2$$

$$\tan\theta = \left(\frac{dy}{dx}\right)_{x=3} = 2$$

$$\theta = 63.44^\circ$$

$$v_x = v\cos\theta = 8\cos 63.44 = 3.58 \text{ m/s} \qquad \textbf{\textit{Ans.}}$$

$$v_y = v\sin\theta = 8\sin 63.44 = 7.15 \text{ m/s} \qquad \textbf{\textit{Ans.}}$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{0 + (3.82)^2}$$

$$a = 3.82 \text{ m/s}^2 \quad (26.56^\circ) \qquad \textbf{\textit{Ans.}}$$

Problem 20

A particle moves in the x-y plane with velocity components $v_x = 8t - 2$ and $v_y = 2$. If it passes through the point $(x, y) = (14, 4)$ at $t = 2$ seconds, determine the equation of the path traced by the particle. Find also the resultant acceleration at $t = 2$ seconds.

Solution

Given : $v_x = 8t - 2$; $v_y = 2$

$$\frac{dx}{dt} = 8t - 2 \; ; \qquad \frac{dy}{dt} = 2$$

Integrating we get

$\therefore\ x = 4t^2 - 2t + c_1$	$y = 2t + c_2$
At $t = 2$ seconds, $x = 14$ m	At $t = 2$ seconds, $y = 4$ m
$14 = 4(2)^2 - 2(2) + c_1$	$4 = 2(2) + c_2$
$c_1 = 2$	$c_2 = 0$
$\therefore\ x = 4t^2 - 2t + 2$	$y = 2t$

$$x = (2t)^2 - 2t + 2 = y^2 - y + 2 \quad (\because y = 2t)$$

$x = y^2 - y + 2$ is the equation of path. ***Ans.***

(Any equation of path does not have time)

$$\therefore\ \bar{v} = (8t - 2)\,\bar{i} + 2\,\bar{j}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = 8\,\bar{i} \text{ m/s}^2$$

$\therefore\ \bar{a} = 8\,\bar{i}$ m/s^2 ***Ans.***

Problem 21

A rocket follows a path such that its acceleration is given by $\bar{a} = (4\ i + t\ j)$ m/s^2 at $\bar{r} = 0$, it starts from rest. At $t = 10$ seconds. Determine **(i)** speed of the rocket, **(ii)** radius of curvature of its path and **(iii)** magnitude of normal and tangential components of acceleration.

Solution

(i) $\bar{a} = 4\,i + t\,j$ at $\bar{r} = 0, \bar{v} = 0, t = 0$

Integrating, we get

$$\bar{v} = 4\,t\,i + \frac{t^2}{2}\,j + c_1$$

At $t = 0, v = 0 \ \therefore\ c_1 = 0$

$$\bar{v} = 4\,t\,i + \frac{t^2}{2}\,j$$

At $t = 10$ seconds

$\bar{a} = 4\,i + 10\,j$	$\bar{v} = 40\,i + 50\,j$
$a = \sqrt{4^2 + 10^2}$	$v = \sqrt{40^2 + 50^2}$
$a = 10.77$ m/s^2 ***Ans.***	$v = 64.03$ m/s^2 ***Ans.***

$\therefore \quad a_x = 4 \quad \text{and} \quad a_y = 10$

(ii) Radius of curvature ρ

$$\rho = \frac{(v_x^2 + v_y^2)^{3/2}}{v_x a_y - v_y a_x} = \frac{(40^2 + 50^2)^{3/2}}{40 \times 10 - 50 \times 4}$$

$\rho = 1312.64 \text{ m}$ ***Ans.***

(iii) Component of normal acceleration a_n

$$a_n = \frac{v^2}{\rho} = \frac{64.03^2}{1312.64}$$

$a_n = 3.123 \text{ m/s}^2$ ***Ans.***

Component of tangential acceleration a_t

$$a_t = \sqrt{a^2 - a_n^2} = \sqrt{10.77^2 - 3.123^2}$$

$a_t = 10.31 \text{ m/s}^2$ ***Ans.***

Problem 22

A car travels along a vertical curve on a road, the equation of the curve being $x^2 = 200y$ (x-horizontal and y-vertical distances in m). The speed of the car is constant and equal to 72 km/hr. **(i)** Find its acceleration when the car is at the deepest point on the curve and **(ii)** What is the radius of curvature of the curve at this point ?

Solution

$x^2 = 200y$

$\therefore \quad y = \frac{x^2}{200}$

$v = 72 \text{ km/hr} = 72 \times \frac{5}{18} \text{ m/sec}$

$v = 20 \text{ m/s (constant)}$

Now, $a_t = 0$

$\therefore \quad y = \frac{x^2}{200}$

$$\frac{dy}{dx} = \frac{1}{200}(2x)$$

$$\left(\frac{dy}{dx}\right)_{\text{at } x=0} = 0 \; ; \quad \left(\frac{d^2y}{dx^2}\right)_{\text{at } x=0} = \frac{1}{100}$$

$$\therefore \quad \rho = \left| \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \right| = \left| \frac{\left[1 + (0)^2\right]^{3/2}}{\frac{1}{100}} \right|$$

$\rho = 100 \text{ m}$ ***Ans.***

$$a_n = \frac{v^2}{\rho} = \frac{(20)^2}{100}$$

$\therefore \quad a_n = 4 \text{ m/s}^2$ ***Ans.***

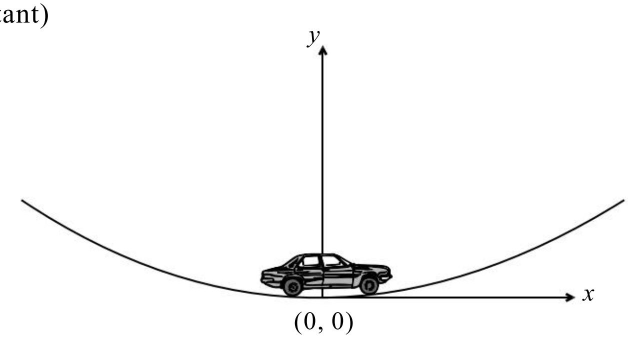


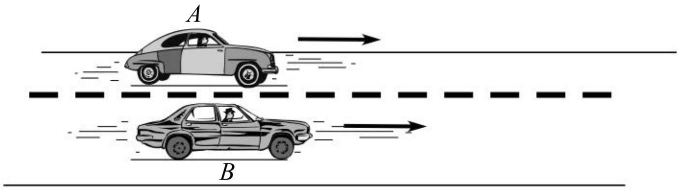
Fig. 12.22

Relative Motion

Generally, a moving body is observed by a person who is at rest. Considering the observers position at rest we are developing fixed axis reference. Such a set of fixed axes is defined as *absolute* or *Newtonian* or *inertial frame of reference*. For most of the moving bodies, the Earth is regarded as fixed although Earth itself is moving in space. Motion referred with such fixed axis is called an *absolute motion*, which we have already dealt in previous discussion.

However, if the axes reference is attached to a moving object then such motion is termed as *relative motion*. It means person in moving object is observing another object which is also in motion.

Example 1



Cars A and B are moving in the same direction on road parallel to each other. Car A is moving with a speed of 60 km/hr and car B is moving with 80 km/hr (These are the absolute speeds of the cars). Car A in relation to car B is moving backward with speed 20 km/hr whereas car B in relation to car A is moving forward with speed of 20 km/hr. Observation of drivers of car A and car B w.r.t. each is developing relative motion between them.

Example 2

If a pilot of fighter plane wants to target another moving plane, then relative motion analysis is must.

12.7 Relative Motion Between Two Particles

Consider two particles A and B moving on different path as shown in Fig. 12.7-i. Here xOy is the fixed frame of reference. Therefore, absolute position of A is given by $r_A = OA$ and of B is $r_B = OB$. Therefore, relative position of B w.r.t. A is written as $r_{B/A}$.

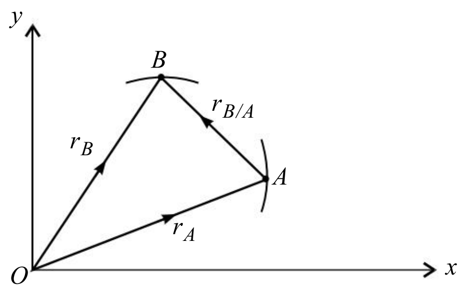


Fig. 12.7-i

By triangle law of vector addition, we have

$$r_A + r_{B/A} = r_B$$

$\therefore$ Relative position of B w.r.t. A $\quad r_{B/A} = r_B - r_A$ ..... (I)

Differentiating Eq. (I) w.r.t. t, we have

Relative velocity of B w.r.t. A $\quad v_{B/A} = v_B - v_A$ ..... (II)

Further differentiating Eq. (II) w.r.t. t, we have

Relative acceleration of B w.r.t. A $\quad a_{B/A} = a_B - a_A$ ..... (III)

12.8 Solved Problems Based on Relative Motion

Problem 23

Two cars A and B start from rest from point O at the same instant and travel towards right along a straight road as shown in Fig. 12.23. Car A moves with an acceleration of 4 m/s^2 and car B moves with an acceleration of 6 m/s^2. Find relative position, velocity and acceleration of car B w.r.t. car A 5 seconds from the start.

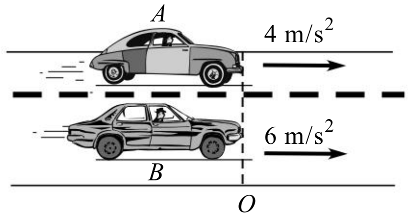


Fig. 12.23

Solution

Car *A*

$$u_A = 0$$
$$a_A = 4 \text{ m/s}^2$$
$$t = 5 \text{ seconds}$$
$$v_A = u_A + a_A t$$
$$v_A = 0 + 4 \times 5$$
$$v_A = 20 \text{ m/s}$$
$$s_A = u_A t + \frac{1}{2} a_A t^2$$
$$s_A = 0 + \frac{1}{2} \times 4 \times 5^2$$
$$s_A = 50 \text{ m}$$

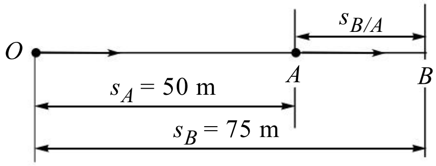
Car *B*

$$u_B = 0$$
$$a_B = 6 \text{ m/s}^2$$
$$t = 5 \text{ seconds}$$
$$v_B = u_B + a_B t$$
$$v_B = 0 + 6 \times 5$$
$$v_B = 30 \text{ m/s}$$
$$s_B = u_B t + \frac{1}{2} a_B t^2$$
$$s_B = 0 + \frac{1}{2} \times 6 \times 5^2$$
$$s_B = 75 \text{ m}$$

Relative position of car *B* w.r.t. car *A*

$$\boldsymbol{s_{B/A}} = \boldsymbol{s_B} - \boldsymbol{s_A}$$
$$\boldsymbol{s_{B/A}} = 75\,\boldsymbol{i} - 50\,\boldsymbol{i}$$
$$\boldsymbol{s_{B/A}} = 25\,\boldsymbol{i} \text{ m}$$

$$s_{B/A} = s_B - s_A$$
$$s_{B/A} = 75 - 50$$
$$s_{B/A} = 25 \text{ m } (\rightarrow)$$ ***Ans.***

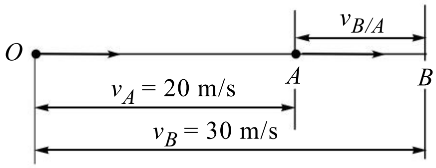


Relative Position of Car *B* w.r.t. Car *A*

Relative velocity of car *B* w.r.t. car *A*

$$\boldsymbol{v_{B/A}} = \boldsymbol{v_B} - \boldsymbol{v_A}$$
$$\boldsymbol{v_{B/A}} = 30\,\boldsymbol{i} - 20\,\boldsymbol{i}$$
$$\boldsymbol{v_{B/A}} = 10\,\boldsymbol{i} \text{ m/s}$$

$$v_{B/A} = v_B - v_A$$
$$v_{B/A} = 30 - 20$$
$$v_{B/A} = 10 \text{ m } (\rightarrow)$$ ***Ans.***

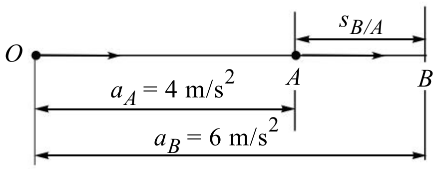


Relative Velocity of Car *B* w.r.t. Car *A*

Relative acceleration of car *B* w.r.t. car *A*

$$\boldsymbol{a_{B/A}} = \boldsymbol{a_B} - \boldsymbol{a_A}$$
$$\boldsymbol{a_{B/A}} = 6\,\boldsymbol{i} - 4\,\boldsymbol{i}$$
$$\boldsymbol{a_{B/A}} = 2\boldsymbol{i} \text{ m/s}^2$$

$$a_{B/A} = a_B - a_A$$
$$a_{B/A} = 6 - 4$$
$$a_{B/A} = 2 \text{ m/s}^2 (\rightarrow)$$ ***Ans.***



Relative Acceleration of Car *B* w.r.t. Car *A*

Problem 24

From point *O* in Fig. 12.24(a), a ship *A* travels in the North making an angle of 45° to the West with a velocity of 18 km/hr and ship *B* travels in the East with a velocity of 9 km/hr. Find the relative velocity of ship *B* w.r.t. ship *A*.

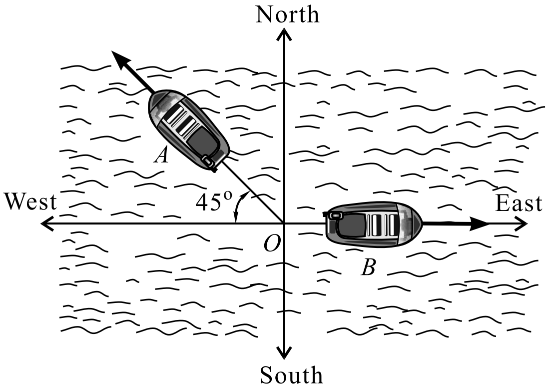


Fig. 12.24(a)

Solution

Refer to Fig. 12.24(b).

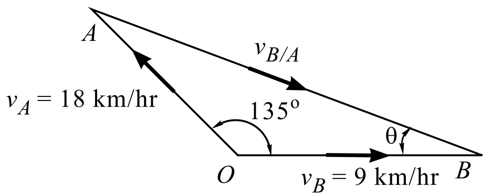


Fig. 12.24(b)

Consider the triangle law.

By cosine rule, we have

$$v_{B/A} = \sqrt{v_A^2 + v_B^2 - 2(v_A)(v_B)\cos 135^\circ}$$

$$v_{B/A} = \sqrt{18^2 + 9^2 - 2(18)(9)\cos 135^\circ}$$

$$v_{B/A} = 25.18 \text{ km/hr}$$

By sine rule, we have

$$\frac{v_{B/A}}{\sin 135^\circ} = \frac{v_A}{\sin\theta}$$

$$\frac{25.18}{\sin 135^\circ} = \frac{18}{\sin\theta}$$

$$\therefore\ \theta = 30.36^\circ$$

Relative velocity of ship *B* w.r.t. ship *A* is $v_{B/A} = 25.18 \text{ km/hr} \left(\searrow 30.36^\circ\right)$ ***Ans.***

Problem 25

Figure 12.25(a) shows cars *A* and *B* at a distance of 35 m. Car *A* moves with a constant speed of 36 kmph and car *B* starts from rest with an acceleration of 1.5 m/s^2. Determine the relative **(i)** position, **(ii)** velocity and **(iii)** acceleration of car *B* w.r.t. car *A* 5 seconds after car A crosses the intersection.

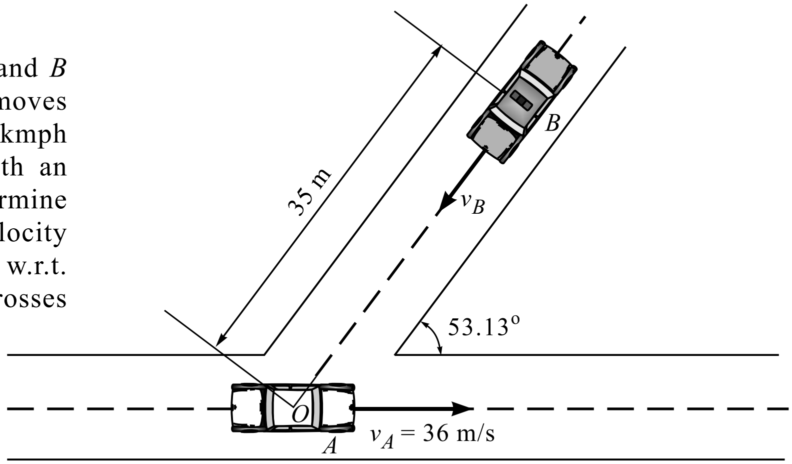


Fig. 12.25(a)

Solution

Car *A* (Uniform velocity)

$u_A = 10$ m/s, $t = 5$ seconds

$a_A = 0$

$s_A = u_A t$

$s_A = 10 \times 5$

$s_A = 50$ m

Car *B* (Uniform acceleration)

$u_B = 0$, $a_B = 1.5$ m/s^2, $t = 5$ seconds

Displacement of car *B*

$$s = u_B t + \frac{1}{2} a_B t^2$$

$$s = 0 + \frac{1}{2} \times 1.5 \times 5^2 = 18.75 \text{ m}$$

Initial distance from *O* is 35 m

Position of car *B* w.r.t. *O* after 5 seconds

$s_B = 35 - 18.75$

$s_B = 16.25$ m

$v_B = u_B + a_B t = 0 + 1.5 \times 5$

$v_B = 7.5$ m/s

(i) Relative position of car *B* w.r.t. car *A*

Consider the triangle law.

By cosine rule, we have

$$s_{B/A} = \sqrt{s_A^2 + s_B^2 - 2(s_A)(s_B)\cos 53.13^o}$$

$$s_{B/A} = \sqrt{50^2 + 16.25^2 - 2(50)(16.25)\cos 53.13^o}$$

$$s_{B/A} = 42.3 \text{ m}$$

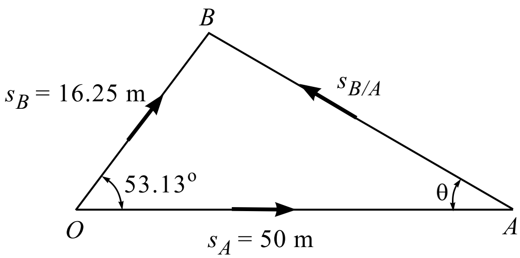


Fig. 12.25(b)

By sine rule, we have

$$\frac{s_A}{\sin\theta} = \frac{s_{B/A}}{\sin 53.13^o}$$

$$\frac{16.25}{\sin\theta} = \frac{42.3}{\sin 53.13^o} \quad \therefore \ \theta = 17.9^o$$

$\therefore$ Relative position of car *B* w.r.t. car *A* will be $s_{B/A} = 42.3$ m $\left(\underline{17.9^o}\right)$ ***Ans.***

(ii) Relative velocity of car *B* w.r.t. car *A*

Consider the triangle law.
By cosine rule, we have

$$v_{B/A} = \sqrt{v_A^2 + v_B^2 - 2(v_A)(v_B)\cos 126.87^\circ}$$

$$v_{B/A} = \sqrt{10^2 + 7.5^2 - 2(10)(7.5)\cos 126.87^\circ}$$

$$v_{B/A} = 15.69 \text{ m/s}$$

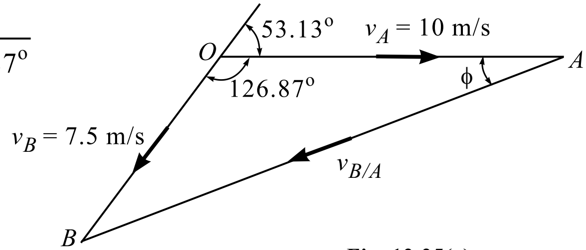


Fig. 12.25(c)

By sine rule, we have

$$\frac{v_{B/A}}{\sin 126.87^\circ} = \frac{v_B}{\sin\phi}$$

$$\frac{15.69}{\sin 126.87^\circ} = \frac{7.5}{\sin\phi} \qquad \therefore \phi = 22.48^\circ$$

$\therefore$ Relative velocity of car *B* w.r.t. car *A* will be $v_{B/A} = 15.69$ m/s (22.48°) ***Ans.***

(iii) Relative acceleration of car *B* w.r.t. car *A*

$$a_{B/A} = a_B - a_A = 1.5 - 0$$

$$a_{B/A} = 1.5 \text{ m/s}^2 \ (53.13^\circ)$$ ***Ans.***

Alternate Method

(i) Relative position of car *B* w.r.t. car *A*

$$\boldsymbol{s_A} = 50\,\boldsymbol{i} \text{ and } \boldsymbol{s_B} = 16.25\cos 53.13^\circ\,\boldsymbol{i} + 16.25\sin 53.13^\circ\,\boldsymbol{j}$$

$$\boldsymbol{s_{B/A}} = \boldsymbol{s_B} - \boldsymbol{s_A}$$

$$\boldsymbol{s_{B/A}} = (16.25\cos 53.13^\circ\,\boldsymbol{i} + 16.25\sin 53.13^\circ\,\boldsymbol{j}) - 50\,\boldsymbol{i} = -40.25\,\boldsymbol{i} + 13\,\boldsymbol{j}$$

Magnitude

$$s_{B/A} = \sqrt{(40.25)^2 + (13)^2}$$

$$s_{B/A} = 42.3 \text{ m}$$

Direction

$$\tan\theta = \frac{13}{40.25} \qquad \therefore \theta = 17.9^\circ$$

$\therefore s_{B/A} = 42.3$ m (17.9°) ***Ans.***

(ii) Relative velocity of car *B* w.r.t. car *A*

$$\boldsymbol{v_A} = 10\,\boldsymbol{i} \text{ and } \boldsymbol{v_B} = -7.5\cos 53.13^\circ\,\boldsymbol{i} - 7.5\sin 53.13^\circ\,\boldsymbol{j}$$

$$\boldsymbol{v_B} = -4.5\,\boldsymbol{i} - 6\,\boldsymbol{j}$$

$$\boldsymbol{v_{B/A}} = \boldsymbol{v_B} - \boldsymbol{v_A}$$

$$\boldsymbol{v_{B/A}} = (-4.5\,\boldsymbol{i} - 6\,\boldsymbol{j}) - (10\,\boldsymbol{i}) = -14.5\,\boldsymbol{i} - 6\,\boldsymbol{j}$$

Magnitude

$$v_{B/A} = \sqrt{(14.5)^2 + (6)^2}$$

$$v_{B/A} = 15.69 \text{ m/s}$$

Direction

$$\tan\phi = \frac{6}{14.5} \qquad \therefore \phi = 22.48^\circ$$

$\therefore v_{B/A} = 15.69$ m/s (22.48°) ***Ans.***

Problem 26

A car A is travelling along a straight highway, while a truck B is moving along a circular curve of 150 m radius. The speed of car A is increased at the rate of 1.5 m/s^2 and the speed of truck B is being decreased at the rate of 0.9 m/s^2. For the position shown in Fig. 12.26(a), determine the velocity of A relative to B and the acceleration of A relative to B. At this instant the speed of A is 75 km/hr and that of B is 40 km/hr.

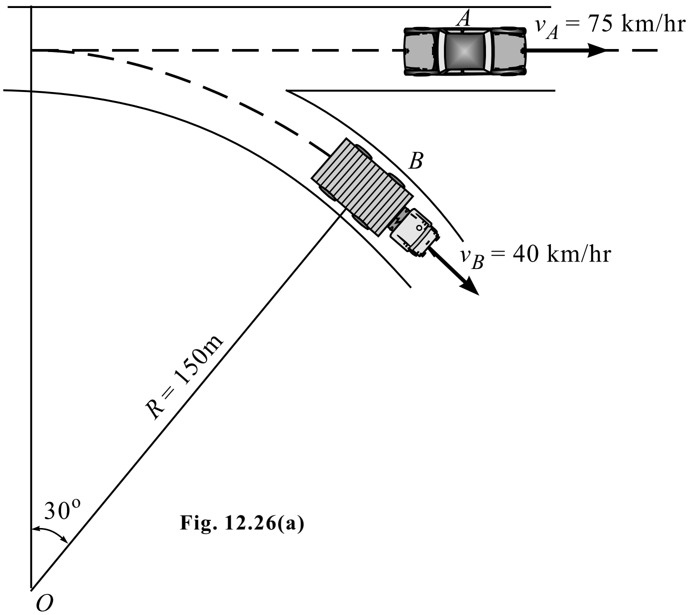


Fig. 12.26(a)

Solution

(i) Motion of car *A*

$$v_A = 75 \text{ km/hr} = 20.83 \text{ m/s}$$

$$\boldsymbol{v_A} = 20.83\, \boldsymbol{i}$$

$$\boldsymbol{a_A} = 1.5\, \boldsymbol{i}$$

Motion of truck *B*

$$v_B = 40 \text{ km/hr} = 11.11 \text{ m/s} \left(30^\circ\right)$$

$$\boldsymbol{v_B} = 11.11 \cos 30^\circ\, \boldsymbol{i} - 11.11 \sin 30^\circ\, \boldsymbol{j}$$

$$\boldsymbol{v_B} = 9.622\, \boldsymbol{i} - 5.55\, \boldsymbol{j}$$

Tangential component of acceleration,

$$a_t = 0.9 \text{ m/s}^2 \left(30^\circ\right)$$

Normal component of acceleration,

$$a_n = \frac{v^2}{\rho} = \frac{11.11^2}{150}$$

$$a_n = 0.823 \text{ m/s}^2 \left(60^\circ\right)$$

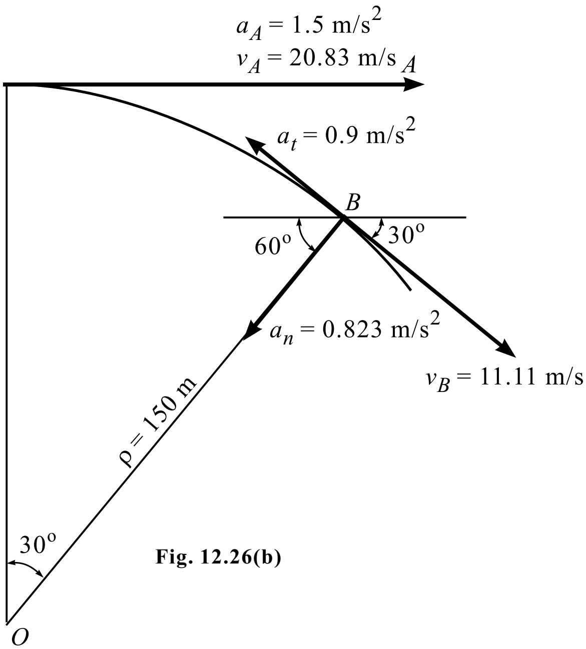


Fig. 12.26(b)

$$\boldsymbol{a_B} = (-0.9 \cos 30^\circ - 0.823 \cos 60^\circ)\, \boldsymbol{i} + (0.9 \sin 30^\circ - 0.823 \sin 60^\circ)\, \boldsymbol{j}$$

$$\boldsymbol{a_B} = -1.190\, \boldsymbol{i} - 0.267\, \boldsymbol{j}$$

(ii) Relative velocity of *A* w.r.t. *B*

$$\boldsymbol{v_{A/B}} = \boldsymbol{v_A} - \boldsymbol{v_B}$$

$$\boldsymbol{v_{A/B}} = 20.83\, \boldsymbol{i} - (9.622\, \boldsymbol{i} - 5.55\, \boldsymbol{j}) = 11.21\, \boldsymbol{i} + 5.55\, \boldsymbol{j}$$

Magnitude

$$v_{A/B} = \sqrt{(11.21)^2 + (5.55)^2}$$

$$v_{A/B} = 12.51 \text{ m/s}$$

Direction

$$\tan\theta = \frac{5.55}{11.21} \quad \therefore\ \theta = 26.36^\circ$$

$$\therefore\ v_{A/B} = 12.51 \text{ m/s } (\measuredangle 26.36^\circ) \quad \textit{Ans.}$$

(iii) Relative acceleration of *A* w.r.t. *B*

$$\boldsymbol{a}_{A/B} = \boldsymbol{a}_A - \boldsymbol{a}_B$$

$$\boldsymbol{a}_{A/B} = 1.5\,\boldsymbol{i} - (-1.190\,\boldsymbol{i} - 0.267\,\boldsymbol{j}) = 2.69\,\boldsymbol{i} + 0.2627\,\boldsymbol{j}$$

Magnitude

$$a_{A/B} = \sqrt{(2.69)^2 + (0.2627)^2}$$

$$a_{A/B} = 2.7 \text{ m/s}^2$$

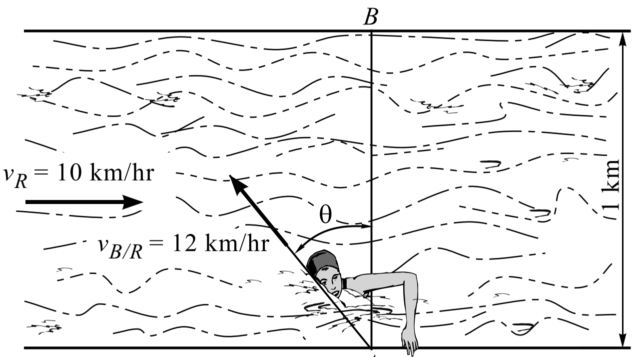
Direction

$$\tan\phi = \frac{0.2627}{2.69} \quad \therefore\ \phi = 5.58^\circ$$

$$\therefore\ a_{A/B} = 2.7 \text{ m/s}^2 \ (\measuredangle 5.58^\circ) \quad \textit{Ans.}$$

Problem 27

A boy wants to swim across a river of 1 km width which is flowing at 10 km/hr. The boy wants to reach the other side of bank *B* and so swims at 12 km/hr at an angle θ, as shown in Fig. 12.27(a). Determine **(i)** the angle θ at which the boy should swim to reach *B*, **(b)** the time taken to reach *B* and **(c)** if the boy is swimming straight at θ = 0 where would he have landed on the opposite bank and how much time is required.



v_B = Velocity of boy

v_R = Velocity of river

$v_{B/R}$ = Relative velocity of boy w.r.t. river

Fig. 12.27(a)

Solution

Refer to Fig. 12.27(b).

(i) Angle θ at which the boy should swim to reach *B*

$$\sin\theta = \frac{10}{12} \quad \therefore\ \theta = 56.44^\circ \quad \textit{Ans.}$$

(ii) The time taken to reach *B*

By Pythagoras theorem, we have

$$v_B = \sqrt{12^2 - 10^2}$$

$$\therefore\ v_B = 6.633 \text{ km/hr } (\uparrow)$$

Displacement = Velocity × Time

$$\therefore\ \text{Time} = \frac{1}{6.633} \times 3600$$

$$t = 542.74 \text{ seconds}$$

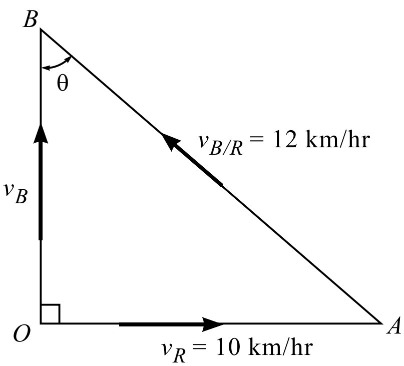


Fig. 12.27(b)

(iii) If the boy is swimming straight at $\theta = 0$. Refer to Fig. 12.27(c).

$$\tan\phi = \frac{12}{10} \quad \therefore \ \phi = 50.19^\circ$$

$$v_B = \sqrt{(10)^2 + (12)^2}$$

$$v_B = 15.62 \text{ km/hr}$$

$$\therefore \ \text{Time} = \frac{1}{12} \times 3600$$

$$t = 300 \text{ sec} \quad \textit{\textbf{Ans.}}$$

For distance

$$s_B = v_B \times t = 15.62 \times \frac{1000}{3600} \times 300$$

$$s_B = 1301.67 \text{ m} \quad \textit{\textbf{Ans.}}$$

$$d = \sqrt{(1301.67)^2 - (1000)^2}$$

$$d = 833.23 \text{ m} \quad \textit{\textbf{Ans.}}$$

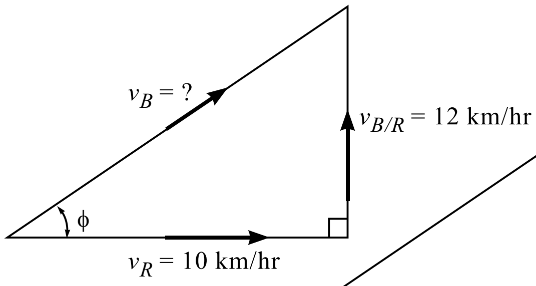


Fig. 12.108(c)

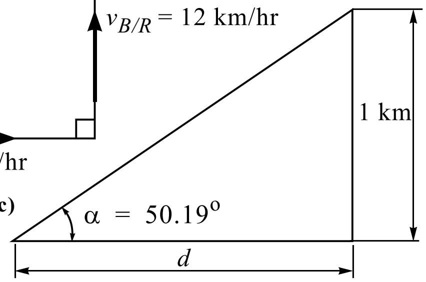


Fig. 12.27(d)

$$\sin\alpha = \frac{1000}{1301.67} \ \therefore \ \alpha = 50.19^\circ$$

Problem 28

A helicopter is moving horizontally at a height of 360 m above the ground. When the helicopter is at point O its speed is 100 m/s and it has an acceleration of 4 m/s^2. At the same instant, a packet is released from the helicopter. Find the position, velocity and acceleration of the particle w.r.t. the helicopter after 3 seconds.

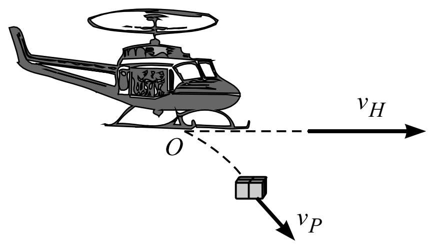


Fig. 12.28

Solution

Refer to Fig. 12.28.

Motion of helicopter

$u = 100$ m/s, $a = 4$ m/s^2, $t = 3$ sec

Displacement

$$s = ut + \frac{1}{2}at^2$$

$$s = 100 \times 3 + \frac{1}{2} \times 4 \times 3^2$$

$$\therefore \ s = 318 \text{ m } (\rightarrow)$$

Velocity

$$v = u + at$$

$$v = 100 + 4 \times 3$$

$$\therefore \ v = 112 \text{ m/s } (\rightarrow)$$

$$\therefore \ \boldsymbol{r_H} = 318\,\boldsymbol{i},$$

$$\boldsymbol{v_H} = 112\,\boldsymbol{i} \text{ and}$$

$$\boldsymbol{a_H} = 4\,\boldsymbol{i}$$

Motion of packet

Freely falling packet will perform projectile motion. Consider horizontal motion with constant velocity.

$$s_x = v_x \times t$$

$$s_x = 100 \times 3 = 300 \text{ m}$$

$$v_x = 100 \text{ m/s}$$

$$a_x = 0$$

Vertical motion under gravity.

$$s_y = u_y t + \frac{1}{2}gt^2$$

$$s_y = 0 + \frac{1}{2} \times 9.81 \times 3^2 = 44.145 \text{ m } (\downarrow)$$

Velocity

$$v_y = u_y + gt$$

$$v_y = 0 + 9.81 \times 3 = 29.43 \text{ m/s } (\downarrow)$$

Acceleration due to gravity in vertical direction

$$a_y = 9.81 \text{ m/s}^2 \ (\downarrow)$$

$\therefore$ **Motion of packet**

$$\boldsymbol{r_P} = 300\,\boldsymbol{i} + (-44.145)\,\boldsymbol{j}$$
$$\boldsymbol{v_P} = 100\,\boldsymbol{i} + (-29.43)\,\boldsymbol{j}$$
$$\boldsymbol{a_P} = 0\,\boldsymbol{i} + (-9.81)\,\boldsymbol{j}$$

Relative position after 3 seconds

$$\boldsymbol{r_{P/H}} = \boldsymbol{r_P} - \boldsymbol{r_H}$$
$$\boldsymbol{r_{P/H}} = (300\,\boldsymbol{i} - 44.145\,\boldsymbol{j}) - (318\,\boldsymbol{i})$$
$$\therefore\ \boldsymbol{r_{P/H}} = -18\,\boldsymbol{i} - 44.145\,\boldsymbol{j} \quad \textbf{\textit{Ans.}}$$

Relative velocity and acceleration after 3 seconds

$$\boldsymbol{v_{P/H}} = \boldsymbol{v_P} - \boldsymbol{v_H} \qquad \text{and} \qquad \boldsymbol{a_{P/H}} = \boldsymbol{a_P} - \boldsymbol{a_H}$$
$$\boldsymbol{v_{P/H}} = (100\,\boldsymbol{i} - 29.43\,\boldsymbol{j}) - (112\,\boldsymbol{i}) \qquad \boldsymbol{a_{P/H}} = -9.81\,\boldsymbol{j} - 4\,\boldsymbol{i}$$
$$\therefore\ \boldsymbol{v_{P/H}} = -12\,\boldsymbol{i} - 29.43\,\boldsymbol{j} \quad \textbf{\textit{Ans.}} \qquad \therefore\ \boldsymbol{a_{P/H}} = -4\,\boldsymbol{i} - 9.81\,\boldsymbol{j} \quad \textbf{\textit{Ans.}}$$

Problem 29

When a cyclist is riding at 12 km/hr, he finds the rain meeting him at an angle of 45° with the vertical. When he rides at 8 km/hr, he finds the rain meeting him at an angle of 30° with the vertical. What is the actual magnitude and direction of the rain ?

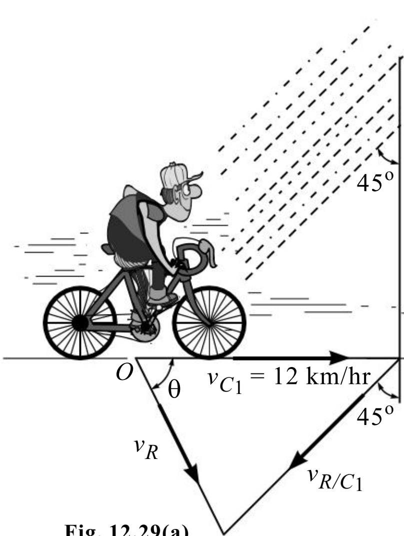


Fig. 12.29(a)

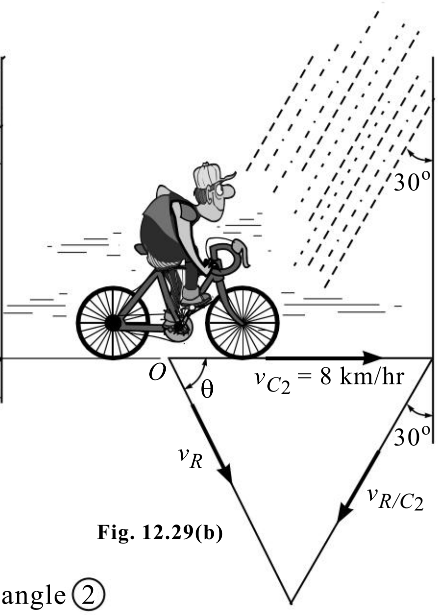


Fig. 12.29(b)

Solution

Refer to Figs. 12.29(a) and (b).

Superimposing both the triangles, we have triangle 1 and triangle 2

Consider triangle 1

By sine rule, we have

$$\frac{v_{R/C2}}{\sin 45^\circ} = \frac{4}{\sin 15^\circ} \qquad \therefore\ v_{R/C2} = 10.93 \text{ km/hr}$$

Consider triangle 2

By cosine rule, we have

$$v_R = \sqrt{8^2 + 10.93^2 - 2 \times 8 \times 10.93 \cos 60^\circ}$$
$$v_R = 9.799 \text{ m/s}$$

By sine rule, we have

$$\frac{\sin\theta}{10.93} = \frac{\sin 60^\circ}{9.799} \qquad \therefore\ \theta = 75^\circ$$

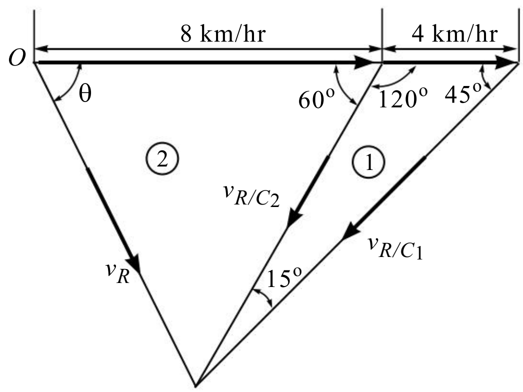


Fig. 12.29(c)

$\therefore$ Absolute velocity of rain, v_R = 9.799 km/hr $\left(75^\circ\right)$ ***Ans.***

Exercises

[I] Problems

Based on Projectile Motion

1. The basketball player likes to release his foul shots at an angle $\theta = 50^\circ$ to the horizontal, as shown in Fig. 12.E1. What initial speed v_0 will cause the ball to pass through the center of the rim?

[***Ans.*** $v_0 = 7$ m/s]

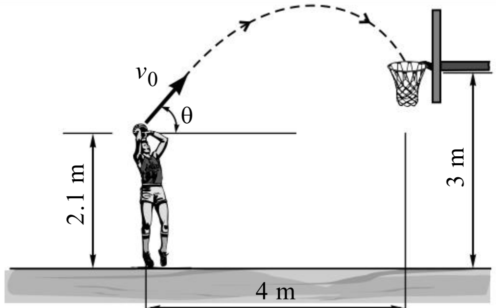


Fig. 12.E1

2. The drinking fountain is designed such that the nozzle is located from the edge of the basin, as shown in Fig. 12.E2. Determine the maximum and the minimum speed at which the water can be ejected from the nozzle so that it does not splash over the sides of the basin at B and C, which are at same level.

[***Ans.*** $u_{min} = 0.838$ m/s and $u_{max} = 1.764$ m/s.]

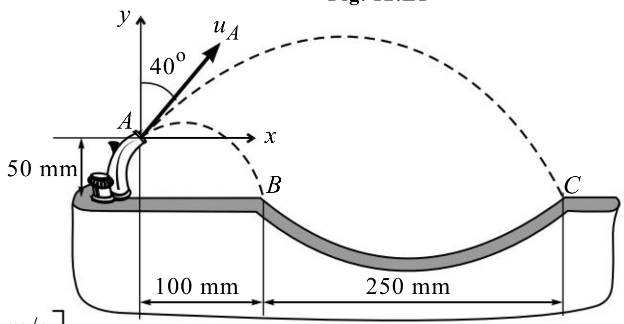


Fig. 12.E2

3. The snowmobile is traveling at 10 m/s, as shown in Fig. 12.E3. When it leaves embankment at A, determine **(a)** the time of flight from A to B, **(b)** the speed at which it strikes the ground at B and **(c)** Range 'R'.

[***Ans.*** **(a)** 2.48 seconds, **(b)** 19.5 m/s and **(c)** 19 m.]

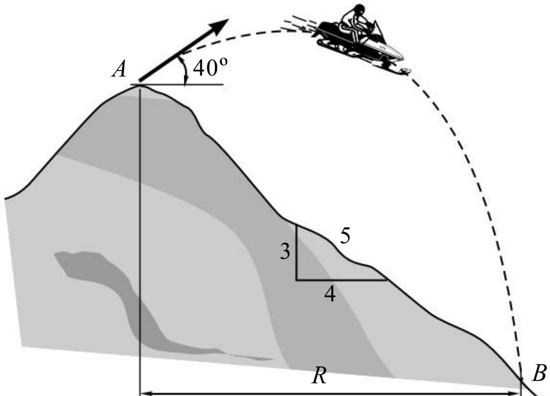


Fig. 12.E3

4. A projectile is launched with an initial speed of 200 m/s at an angle of 60º with respect to the horizontal, as shown in Fig. 12.E4. Compute the range R as measured up the incline.

[***Ans.*** $R = 2970$ m]

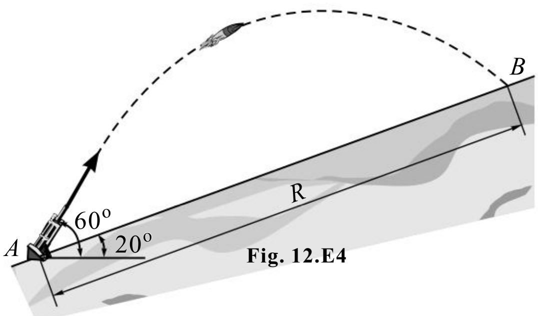


Fig. 12.E4

5. A projectile is fired at an angle of 60°, as shown in Fig. 12.E5. At what elevation y does it strike the hill whose equation has been estimated as $y = 10^{-5}\, x^2$ m ?

Neglect air friction and take the muzzle velocity as 1000 m/s.

[**Ans.** $y = 34.19$ m]

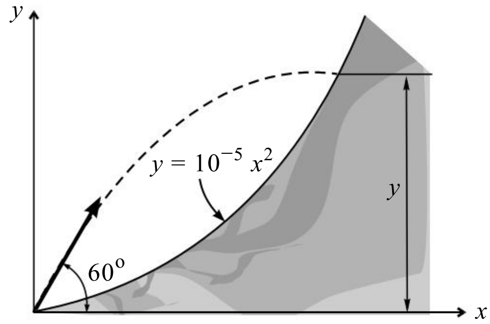


Fig. 12.E5

6. A projectile is launched from point A with the initial conditions shown in Fig. 12.E6. Determine the slant distance 's' which locates the point B of impact. Calculate the time of flight t.

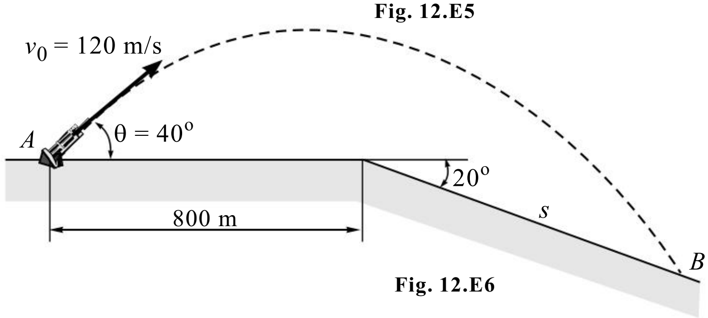


Fig. 12.E6

7. A box released from a helicopter moving horizontally with constant velocity 'u' from a certain height 'h' from the ground takes 5 seconds to reach the ground hitting it at an angle of 75°, as shown in Fig. 12.E7. Determine **(a)** the horizontal distance 'x', **(b)** the height 'h' and **(c)** the velocity 'u'.

[**Ans.** **(a)** $x = 65.715$ m, **(b)** $h = 122.625$ m and **(c)** $u = 13.143$ m/s.]

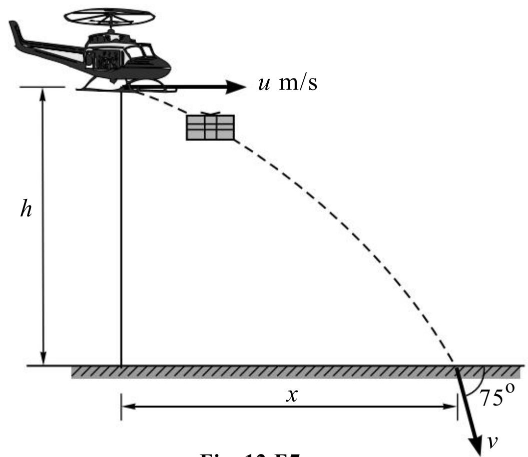


Fig. 12.E7

8. A motorist wants to jump over a ditch as shown in Fig. 12.E8. Find the necessary minimum velocity at A in m/s of the motorcycle. Also, find the direction and the magnitude of the velocity of the motorcycle when it just clears the ditch.

[**Ans.** 6.26 m/s and 8.85 m/s at 45° to horizontal.]

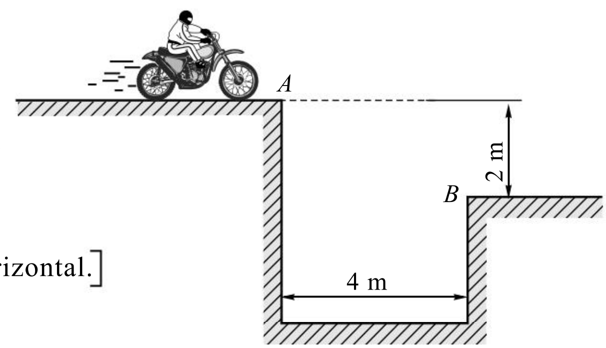


Fig. 12.E8

9. Calculate the minimum speed with which a motorcycle stunt driver must leave the 20º ramp at *B* in order to clear the ditch at *C*, as shown in Fig. 12.E9.

[*Ans.* 4.31 m/s]

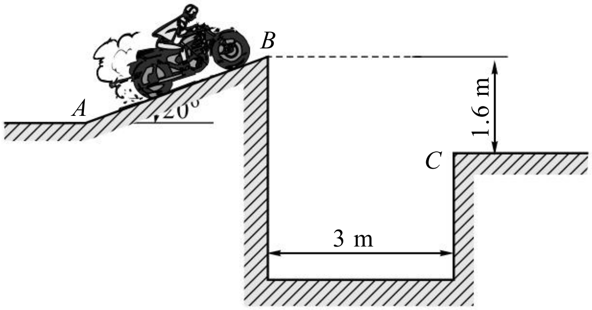


Fig. 12.E9

10. A ball is dropped vertically on a 20º incline at *A*, the direction of rebound forms an angle of 40º with the vertical as shown in Fig. 12.E10. Knowing the ball next strikes the incline at *B,* determine **(a)** the velocity of rebound at *A* and **(b)** the time required for the ball to travel

[*Ans.* 4.78 m/s $\measuredangle 50^\circ$ and t = 0.976 seconds.]

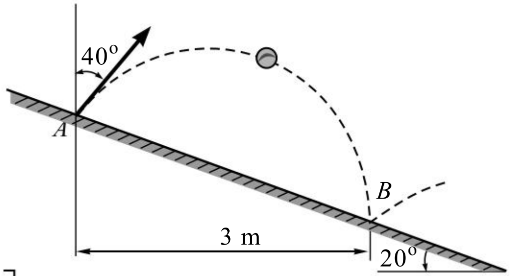


Fig. 12.E10

11. The muzzle velocity of a long-range rifle at *A* is u = 400 m/s. Determine the two angles of elevation θ_1 and θ_2 which will permit the projectile to hit the mountain target *B*, as shown in Fig. 12.E11.

[*Ans.* $\theta_1 = 26.1^\circ$ and $\theta_2 = 80.6^\circ$.]

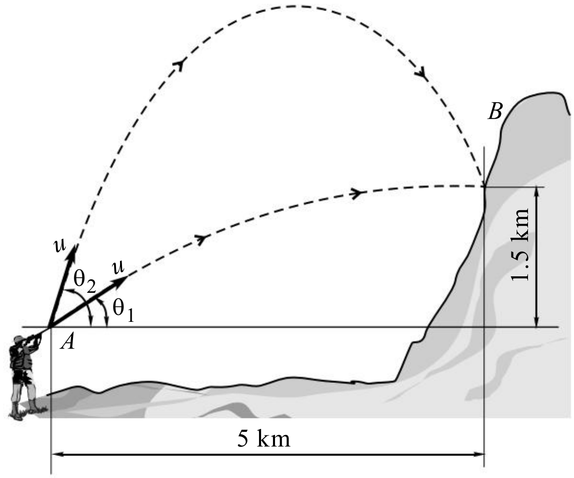


Fig. 12.E11

Based on Curvilinear Motion

12. The curvilinear motion of a particle is given by $v_x = 50 - 16t$ m/s and $y = 100 - 4t^2$ m. Determine the velocity and acceleration when the position $y = 0$ is reached.

[*Ans.* $\boldsymbol{v} = -30\boldsymbol{i} - 40\boldsymbol{j}$ m/s and $\boldsymbol{a} = -16\boldsymbol{i} - 8\boldsymbol{j}$ m/s^2.]

13. In the curvilinear motion, particle *P* moves along the fixed path $9y = x^2$ where x and y are expressed in centimetres. At any instant t, the x-coordinate of *P* is given by $x = t^2 - 14t$. Determine the y-component of the velocity and acceleration of *P* when t = 15 seconds.

[*Ans.* v = 53.33 cm/s and a = 63.56 cm/s^2.]

14. For the curvilinear motion $y = 4t^3 - 3t$ m and $ax = 12\ t$ m/s^2. If $v_x = 4$ m/s, when $t = 0$. Calculate magnitude of velocity and acceleration for $t = 1$ second.

[***Ans.*** $v = 13.45$ m/s and $a = 26.83$ m/s^2.]

15. At any instant the horizontal position of the weather balloon in Fig. 12.E15 is designed by $x = (9t)$ m where t is given in seconds. If the equation of the path is $y = x^2/30$, determine **(a)** the distance of the balloon from the station at A when $t = 2$ sec, **(b)** the magnitude and direction of the velocity when $t = 2$ seconds and **(c)** the magnitude and direction of the acceleration when $t = 2$ seconds.

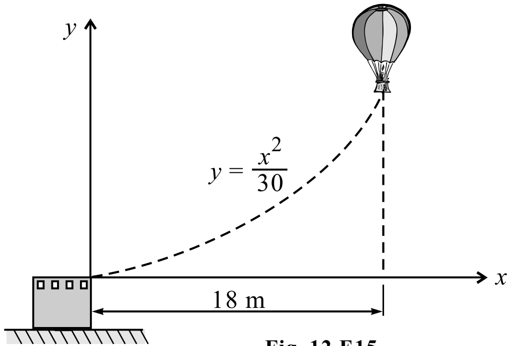


Fig. 12.E15

[***Ans.*** **(a)** 21 m, **(b)** 14.1 m/s $\angle 50.2^\circ$ and **(c)** 5.4 m/s^2.]

16. A particle moves in a circular path of 0.3 m radius. Calculate acceleration if **(a)** speed is constant at 0.6 m/s and **(b)** speed is 0.6 m/s but increasing at the rate of 0.9 m/s^2 each second.

[***Ans.*** **(a)** 1.2 m/s^2 and **(b)** 1.5 m/s^2.]

17. A car is travelling along a circular curve that has a radius of 50 m. If its speed is 16 m/s and is increasing uniformly at 8 m/s^2, determine the magnitude of its acceleration at this instant.

[***Ans.*** 9.498 m/s^2]

18. A particle moves in a circular path of 4 m radius. Calculate 4 seconds later the particles total acceleration and distance travelled if **(a)** the speed is constant at 2 m/s and **(b)** speed is 2 m/s at the instant and is increasing at a rate of 0.7 m/s^2.

[***Ans.*** **(a)** $a = 1$ m/s^2; $s = 8$ m and **(b)** $a = 5.802$ m/s^2; $s = 13.6$ m .]

Based on Relative Motion

19. Two ships A and B leave a port at the same time. The ship A is travelling N-W at 32 kmph and ship B, 40° South of West at 24 km/hr. Determine **(a)** the speed of ship B relative to ship A and **(b)** at what time, they will be 150 km apart.

[***Ans.*** **(a)** $v_{B/A} = 38.3$ km/hr 83.64° and **(b)** $t = 3.92$ hrs.]

20. The velocities of commuter trains A and B are as shown in Fig. 12.E20. Knowing that the speed of each train is constant and that B reaches the crossing 10 min after A has passed through the same crossing, determine **(a)** the relative velocity of B with respect to A and **(b)** the distance between the fronts of the engines 3 min after A passed through the crossing.

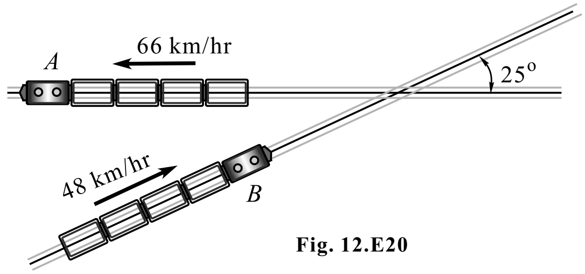


Fig. 12.E20

[***Ans.*** **(a)** $v_{B/A} = 30.93$ m/s $\angle 10.5^\circ$ and **(b)** 2.957 km.]

21. Automobile A is travelling East at the constant speed of 36 km/hr. As automobile A crosses the intersection shown in Fig. 12.E21, automobile B starts from rest, 35 m North of the intersection and moves South with a constant acceleration of 1.2 m/s^2. Determine the position, velocity and acceleration of B relative to A 5 seconds after A crosses the intersection.

[***Ans.*** $s_{B/A}$ = 53.85 m; 21.8^o,
$v_{B/A}$ = 11.66 m/s; 30.96^o and $a_{B/A}$ = 1.2 m/s^2 (↓).]

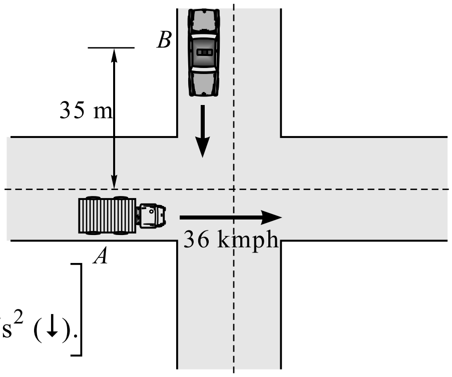


Fig. 12.E21

22. Two cars P and Q pass through the intersection at the same instant and travel with velocities of 36 km/hr and 72 km/hr along the roads AB and DC respectively (Refer to Fig. 12.E22). Find the velocity of the car Q w.r.t the car P and the distance between the two cars after 2 seconds.

[***Ans.*** $v_{Q/P}$ = 26.46 m/s ; 70.89^o
and s = 52.92 m.]

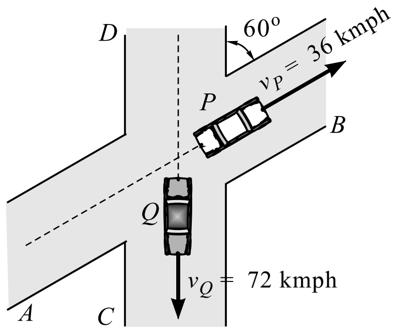


Fig. 12.E22

23. A jet of water is discharged at A with a velocity of 20 m/s to strike a moving plate B, as shown in Fig. 12.E23. If the plate is moving downward with velocity of 1 m/s. determine the relative velocity of water w.r.t. the plate just before it strikes.

[***Ans.*** $v_{W/P}$ = 20.094 m/s ; 5.549^o]

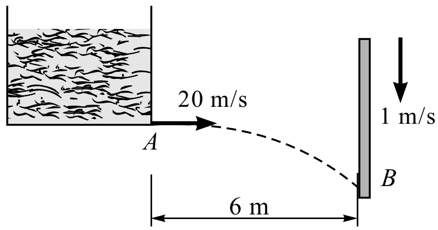


Fig. 12.E23

[II] Review Questions

1. Explain the following :

(i) Curvilinear motion by rectangular component method.

(ii) Curvilinear motion by tangential and normal component method.

2. Derive velocity by rectangular component method.

3. Derive acceleration by rectangular component method.

4. Derive the relation for component of normal acceleration $a_n = \dfrac{v^2}{\rho}$.

5. Show the relation between rectangular components and tangential and normal components of acceleration.

6. What is projectile motion ?

7. Derive the general equation of projectile motion.

8. Derive the expression for maximum height for projectile motion.

[III] Fill in the Blanks

1. If a particle is moving along curved path then it is said to perform ________ motion.
2. In __________ motion, displacement, velocity and acceleration are always directed along the path of a particle.
3. The component of normal acceleration (a_n) is ________ acceleration.
4. The component of tangential acceleration (a_t) is equal to the rate of change of ________.
5. For uniform speed, a_t = ____ (uniform circular motion).
6. If any object is thrown obliquely in air, it follows ________ path and such a motion is called projectile motion.
7. Projectile motion is the combination of _________ motion with constant velocity and _________ motion under gravity, happening simultaneously.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. In curvilinear motion velocity of a particle is always ________ to the curved path at every instant.

   (a) tangential **(b)** normal **(c)** horizontal **(d)** vertical

2. In curvilinear motion direction of acceleration _________.

   (a) remains constant **(b)** changes at every instant
   (c) is always tangential **(d)** is always normal

3. If data is given in rectangular component form then radius of curvature is calculated by the relation _______.

   (a) $\rho = \left| \dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} \right|$ **(b)** $\rho = \left| \dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{2/3}}{\dfrac{d^2y}{dx^2}} \right|$

   (c) $\rho = \left| \dfrac{(v_x^2 + v_y^2)^{3/2}}{v_x a_y - v_y a_x} \right|$ **(d)** $\rho = \left| \dfrac{(v_x^2 - v_y^2)^{3/2}}{v_x a_y + v_y a_x} \right|$

4. Component of tangential acceleration a_t is calculated by the relation _______.

   (a) $a_t = (a)(e_t)$ **(b)** $a_t = \bar{a} \cdot \bar{e}_t$ **(c)** $a_t = \bar{a} \times \bar{e}_t$ **(d)** $a_t = \bar{a} \cdot \bar{v}_t$

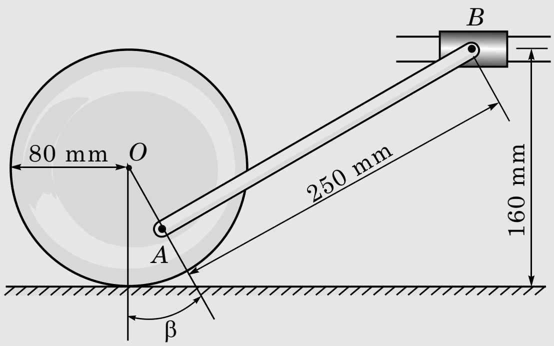
5. Maximum height of projectile from a point of projection is given by the relation _______.

   (a) $H = \dfrac{2u \sin \theta}{g}$ **(b)** $H = \dfrac{2u^2 \sin 2\theta}{g}$ **(c)** $H = \dfrac{u^2 \sin^2 \theta}{g}$ **(d)** $H = \dfrac{u^2 \sin^2 \theta}{2g}$

6. If two projectiles having same velocity of projection, but complementary angle of projection, then the range of both the projectiles will be _______.

   (a) same **(b)** different **(c)** zero **(d)** None of these

13



KINEMATICS OF RIGID BODIES

13.1 Introduction

In kinematics of particle, we have discussed the relationship of displacement, velocity and acceleration w.r.t. time. Kinematics means there is no involvement of force and mass. Particle means there is no involvement of dimension in analysis.

Now, in *kinematics of rigid bodies* we have to consider the dimension of body but still force and mass are not involved.

13.2 Types of Motion

A particle can perform only translation motion but a rigid body can perform any motion among the three basic motions.

1. Translation motion,
2. Rotational motion and
3. Plane motion.

13.2.1 Translation Motion

A body is said to perform ***translation motion*** *if an imaginary straight line drawn on the body remains parallel to original position during its motion at any instant.*

It is observed that all the particles of the body move along parallel paths in a translation motion.

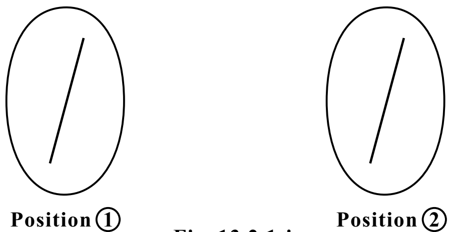


Fig. 13.2.1-i

Translation motion can happen in rectilinear form or curvilinear form.

1. **Rectilinear Translation Motion** *A body is performing* ***rectilinear translation motion****, means the body is shifting its position from position 1 to 2 along a* ***straight*** *path.* Hence, we can observe path traced by different particles of the body, say point A, G and B move along parallel paths. Therefore, displacement, velocity and acceleration of each and every particle at any instant is same.

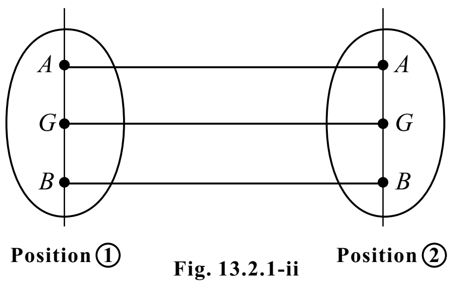


Fig. 13.2.1-ii

2. **Curvilinear Translation Motion :** *A body is performing* ***curvilinear translation motion****, means the body is shifting its position from position ① to ② along a* ***curved*** *path.* Hence, we can observe path traced by different particles of the body, say point A, G and B move along parallel paths. Therefore, displacement, velocity and acceleration of each and every particle at any instant is same.

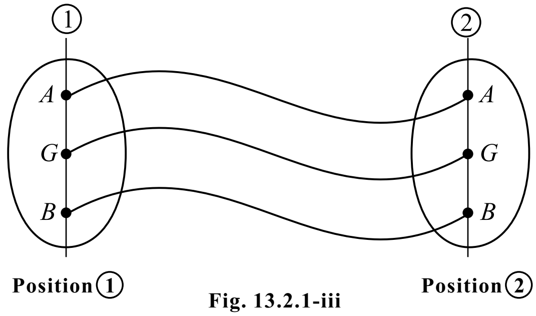


Fig. 13.2.1-iii

Note : In translation motion (Rectilinear as well as curvilinear) displacement, velocity and acceleration of each and every particle at any instant is same. Therefore, considering all the above effect at G (C*entre of gravity*) we can assume the rigid body is similar to particle in translation motion.

13.2.2 Fixed Axis Rotational Motion

Consider the body rotating about an axis perpendicular to the plane of motion. The different points on the body move along the concentric circular path. Here the point O is acting as the *centre of rotation* and axis perpendicular to the plane of motion and passing through the centre of rotation is called as *axis of rotation*. This axis is stationary therefore such motion is called *fixed axis rotational motion*.

For rotational motion, terms like displacement, velocity and acceleration are specified as angular displacement, angular velocity and angular acceleration.

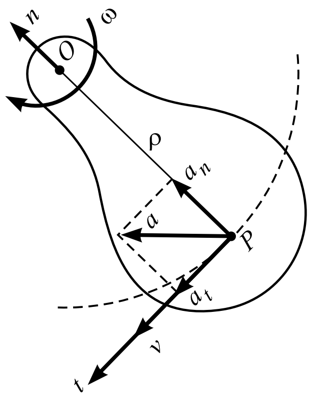


Fig. 13.2.2-i

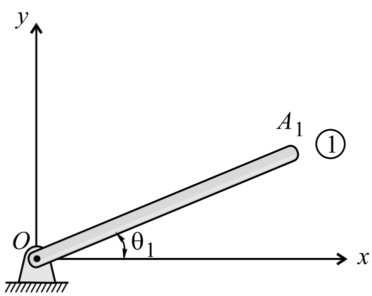


Fig. 13.2.2-ii(a)

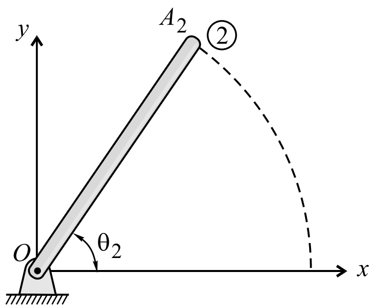


Fig. 13.2.2-ii(b)

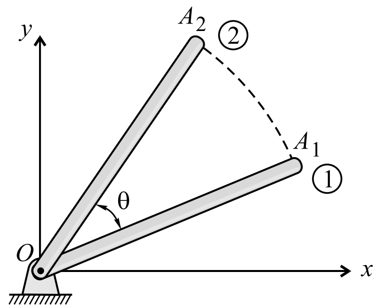


Fig. 13.2.2-ii(c)

Rod OA is hinged at O. Initial position ① of rod OA is inclined at θ_1 with x-axis. Final position ② of rod OA is inclined at θ_2 with the x-axis.

Angular Displacement (θ - theta)

If θ_1 is the angular position of the body in position ① and it changes to θ_2 at position ② then ***angular displacement*** of the body θ is given as follows.

$$\text{Angular displacement} = \text{Final angular position} - \text{Initial angular position}$$

$$\theta = \theta_2 - \theta_1$$

Angular displacement is measured in unit radian.

$$1 \text{ revolution} = 2\pi \text{ radian} = 360^o$$

Angular Velocity (ω - omega)

*The rate of change of angular position with respect to time is called the **angular velocity** of the rotating body.*

$$\frac{d\theta}{dt} = \omega$$

The direction of angular velocity is perpendicular to the plane of motion and it is decided by the right-hand-thumb rule. For anticlockwise sense of rotation, the direction of stretched right-hand- thumb is towards the observer, therefore, it is positive and for clockwise it is negative.

$$\omega \ (\circlearrowleft) \ +\text{ve} \quad \text{and} \quad \omega \ (\circlearrowright) \ -\text{ve}$$

Angular velocity is measured in radian/second.

$$1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

Angular Acceleration (α - alpha)

*The rate of change of angular velocity with respect to time is called the **angular acceleration** of the rotating body.*

$$\frac{d\omega}{dt} = \alpha$$

The direction of angular acceleration acts along the axis of rotation, i.e., perpendicular to the plane of motion and is similarly decided by the right-hand-thumb rule.

The sense of angular acceleration is same as the sense of angular velocity, if the angular velocity increases with time and is opposite of the sense of angular velocity, if the angular velocity decreases with time.

Angular acceleration is measured in radian/second^2.

Types of Fixed Axis Rotational Motion

1. Motion with Uniform (Constant) Angular Velocity

$$\text{Angular displacement} = \text{Angular velocity} \times \text{Time}$$

$$\theta = \omega \times t$$

2. Motion with Uniform (Constant) Angular Acceleration

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

where ω_0 = Initial angular velocity
ω = Final angular velocity
α = Angular acceleration
θ = Angular displacement
t = Time interval

3. Motion with Variable Angular Acceleration

$$\alpha = \frac{d\omega}{dt} \text{ or } \alpha = \omega\frac{d\omega}{d\theta} \quad \therefore \ \omega = \frac{d\theta}{dt}$$

Derivation of Rotational Motion with Uniform (Constant) Angular Acceleration

(i) $\alpha = \dfrac{d\omega}{dt}$

$\therefore \ d\omega = \alpha\, dt$

Integrating both sides, we get

$$\int_{\omega_0}^{\omega} d\omega = \alpha\int_0^t dt$$

$\therefore \ \omega - \omega_0 = \alpha t$

$\therefore \ \omega = \omega_0 + \alpha t$

(ii) $\omega = \dfrac{d\theta}{dt}$

$\therefore \ d\theta = \omega\, dt$

Integrating both sides, we get

$$\int_0^{\theta} d\theta = \int_0^t \omega\, dt \quad \Rightarrow \quad \int_0^{\theta} d\theta = \int_0^t (\omega_0 + \alpha t)\, dt$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

(ii) $\alpha = \dfrac{d\omega}{dt} \quad \therefore \ \alpha = \dfrac{d\omega}{d\theta}\cdot\dfrac{d\theta}{dt}$

$\therefore \ \alpha = \omega\dfrac{d\omega}{d\theta}$

$\therefore \ \omega\, d\omega = \alpha\, d\theta$

Integrating both sides, we get

$$\int_{\omega_0}^{\omega} \omega\, d\omega = \int_0^{\theta} \alpha\, d\theta$$

$\therefore \ \dfrac{1}{2}(\omega^2 - \omega_0^2) = \alpha\theta$

$\therefore \ \omega^2 = \omega_0^2 + 2\alpha\theta$

Comparison Between Translation and Rotational Motion

Translation Motion	Rotational Motion
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

13.3 Relationship Between Rope, Pulley and Block

Figure 13.3-i shows a pulley of radius r mounted on a pin. Block A is connected by rope which is wound around the pulley. The rotational motion of pulley will relate to translation motion.

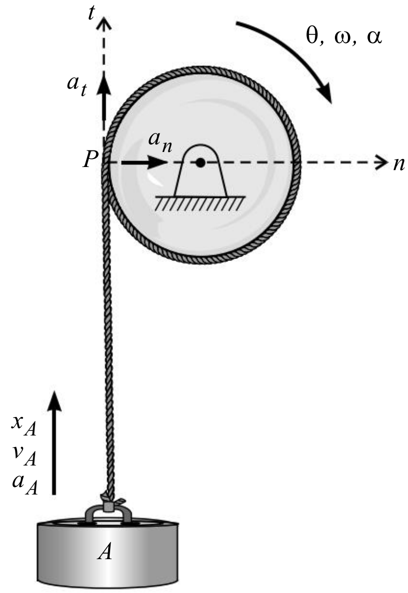


Fig. 13.3-i

Consider at given instant the pulley has an angular position (θ), angular velocity (ω) and angular acceleration (α).

The block A will have corresponding position (x_A), velocity (v_A) and acceleration (a_A) at this instant.

Let P be the common point between pulley and rope. Here pulley is performing rotational motion and block is performing translation motion. Since P is the common point of contact between both pulley and rope, we have

$$x_P = x_A = r\theta$$

$$v_P = v_A = r\omega$$

$$a_P = a_A = r\alpha$$

{a_A is the component of acceleration along tangential direction a_t}

$$a_n = \frac{v^2}{r}$$ {a_n is the component of acceleration along normal direction a_n}

13.4 Relationship Between Two Contact Pulleys Rotating without Slipping

Consider pulley ① of radius r_1 and pulley ② of radius r_2 mounted on pins rotating as shown in Fig. 13.4-i without slipping.

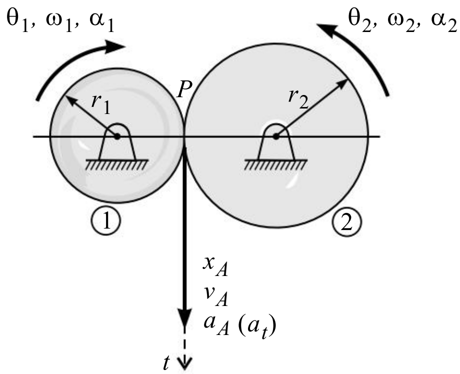


Fig. 13.4-i

At a given instant, let pulley ① have angular position θ_1, angular velocity ω_1 and angular acceleration α_1. Let pulley ② have angular position θ_2, angular velocity ω_2 and angular acceleration α_2.

Let P be the common point of contact. If pulley ① rotates clockwise, pulley ② will rotate anticlockwise.

Assuming point P on pulley ①, we have the following relationship

$$\left.\begin{aligned} x_1 &= r_1\theta_1 = x_P \\ v_1 &= r_1\omega_1 = v_P \\ a_1 &= r_1\alpha_1 = a_P \ (a_t) \end{aligned}\right\} \qquad \dots (13.1)$$

Assuming point P on pulley ②, we have the following relationship

$$\left.\begin{aligned} x_2 &= r_2\theta_2 = x_P \\ v_2 &= r_2\omega_2 = v_P \\ a_2 &= r_2\alpha_2 = a_P \;(a_t) \end{aligned}\right\} \quad \ldots (13.2)$$

From relations (13.1) and (13.2), we have

$$\begin{aligned} x_P &= r_1\theta_1 = r_2\theta_2 \\ v_P &= r_1\omega_1 = r_2\omega_2 \\ a_P &= r_1\alpha_1 = r_2\alpha_2 \end{aligned}$$

13.5 General Plane Motion

General plane motion *is the combination of translation motion and rotational motion happening simultaneously.*

Example

Consider a rod AB having one end A on vertical wall and other end B on floor is sliding. Therefore, velocity of point A (v_A) will be vertically down and that of point B (v_B) will be horizontally towards right [Fig.13.5-i(a)].

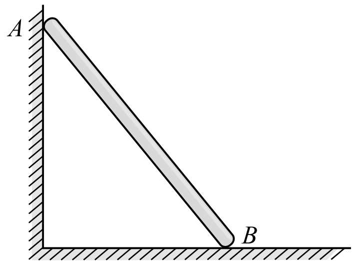


Fig. 13.5-i(a)

Drawing perpendicular to direction of v_A and v_B we get centre I. Hence, I is the ***Instantaneous Centre of Rotation (ICR)*** [see Fig. 13.5-i(b)].

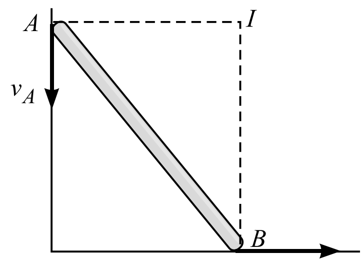


Fig. 13.5-i(b)

Why *I* is called ICR ?

I is called Instantaneous Centre of Rotation because the velocity of point A and B at an instance is giving I. It means at some other instant rod AB will have position $A'B'$ and velocities v_A' and v_B'. Therefore, its ICR will be I' [see Fig. 13.5-i(c)].

How to identify General Plane Motion ?

In this example, we have a rod AB slipping against vertical wall and horizontal floor. AB is the instant position, after some time, next instant $A'B'$ is another position. Here, we observe that the rod had shifted its position, which means there is translation motion involved, also the angle of rod had changed, which means there is rotational motion. This is happening simultaneously. Therefore, rod is performing general plane motion.

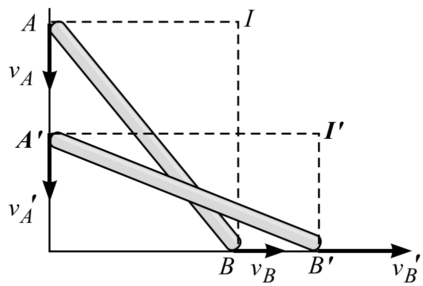


Fig. 13.5-i(c)

Note : When a body is performing general plane motion, it is assumed to perform fixed axis rotation about some centre say I. This centre of roatation changes its position instant to instant. Therefore, it is called as *Instantaneous Centre of Rotation (ICR)*.

13.5.1 Link Mechanism

Bar AB is linked to bar BC. At C there is a slider, which is free to slide along the horizontal slot. Link AB is performing fixed axis rotation about point A, means A is the centre of rotation and AB is the radius. If ω_{AB} is the angular velocity given to bar AB then v_B is the linear velocity of end point B. The slide C is sliding horizontally. Therefore, v_C is the linear velocity of slider. Here link BC is performing general plane motion. So, it must have instantaneous centre of rotation. By drawing perpendicular to direction of v_B and v_C we can locate the intersection point as I (ICR).

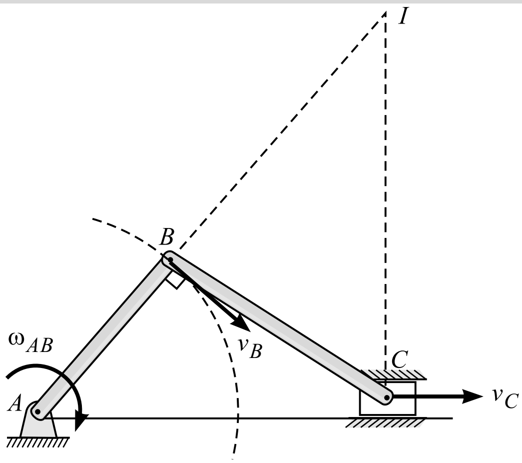


Fig. 13.5-1-i

13.5.2 Rolling of a Body Without Slipping

A rolling body is performing general phase motion. If a body is rolling without slipping on stationary surface then the point of contact with stationary surface is the *Instantaneous Centre of Rotation (ICR)*. In Fig. 13.5.2-i I is the ICR.

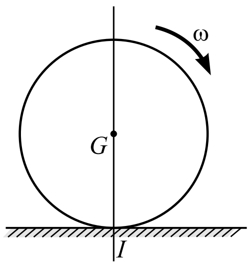


Fig. 13.5.2-i

Figure 13.5.2-ii shows if body with various shapes rolls, it has to roll about point I. Triangle, square, pentagon, hexagon and polygon with infinite side (circle) will have I as the ICR.

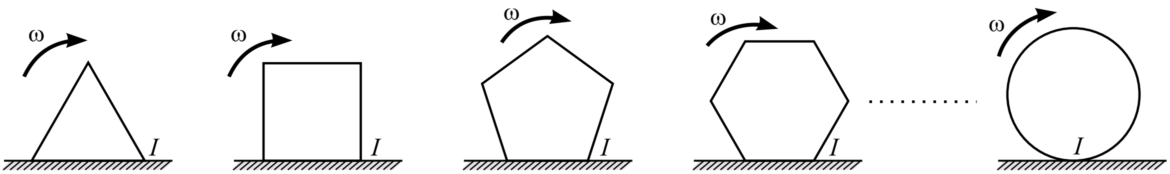


Fig. 13.5.2-ii

13.5.3 How to Locate the Instantaneous Centre of Rotation (ICR)

Let us consider examples for better understanding.

Example 1

If rod AB is performing plane motion and velocity of two points A and B are known in direction then perpendicular line drawn to the direction of velocity v_A and v_B will intersect at some point I. Here I is the ICR.

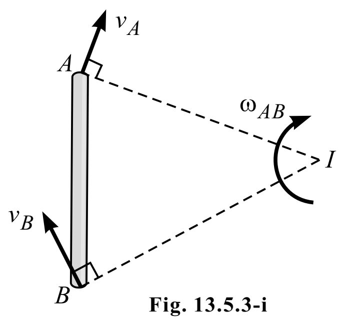


Fig. 13.5.3-i

Relation between linear velocity and angular velocity

In general, we know $v = r\omega$

From Fig. 13.5.3-i, we have

$v_A = (IA)\omega_{AB}$

$v_B = (IB)\omega_{AB}$

Or $\omega_{AB} = \dfrac{v_A}{IA} = \dfrac{v_B}{IB}$

Example 2

Consider the rod AB performing plane motion. The velocity of end point A is v_A = 8 m/s and velocity of end point B is v_B = 3 m/s. v_A is parallel to v_B.

Draw v_A and v_B in proportion to their magnitude (i.e., v_A = 8 m/s is more than v_B = 3 m/s). Then draw line from tip of v_A to tip of v_B and extend till it intersects the AB extension at I. Here I is the ICR.

We have the following relation :

$v_A = (IA)\omega_{AB}$

$v_B = (IB)\omega_{AB}$

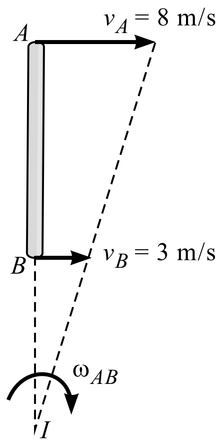


Fig. 13.5.3-ii

Example 3

Consider a rod AB performing plane motion. The velocity of end point A is v_A = 8 m/s (→) and velocity of end point B is v_B = 3 m/s(←). v_A is parallel to v_B.

Draw v_A and v_B in proportion to their magnitude [i.e., v_A = 8 m/s (→) is greater than v_B = 3 m/s (←)]. Then draw line from tip of v_A to tip of v_B and extend till it intersects the AB extension at I. Here I is the ICR.

We have the following relation :

$v_A = (IA)\omega_{AB}$

$v_B = (IB)\omega_{AB}$

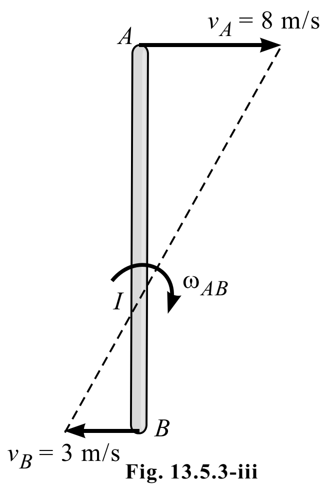


Fig. 13.5.3-iii

13.6 Solved Problems

Problem 1

A fly wheel starts from rest and after half a minute rotates at 2000 rpm. Calculate the **(i)** angular acceleration and **(ii)** number of revolution made by the wheel within this period.

Solution

$$\omega_0 = 0 \qquad \omega = \frac{2\pi \times 2000}{60}$$

$$t = 30 \text{ sec} \qquad \omega = 209.44$$

(i)

$$\omega = \omega_0 + \alpha t$$

$$209.44 = 0 + \alpha \times 30$$

$$\alpha = 6.98 \text{ rad/s}^2$$

(ii)

$$\theta = \omega_0 t + \frac{1}{2}\alpha \times t^2$$

$$\theta = 0 + \frac{1}{2} \times 6.98 \times 30^2$$

$$\theta = 3141 \text{ rad}$$

$$\text{Number of revolution } n = \frac{\theta}{2\pi} = \frac{3141}{2\pi}$$

$n = 500$ ***Ans.***

Problem 2

A rotor of turbine has an initial angular velocity of 1800 rpm. Accelerating uniformly, it doubled its velocity in 12 seconds. Find the revolutions performed by it in this interval.

Solution

$$\omega_0 = \frac{2\pi N}{60} = \frac{2\pi \times 1800}{60}$$

$$\omega_0 = 60\pi \text{ rad/s} \text{ and } \omega = 2\omega_0 = 120\pi \text{ r/s}$$

$$t = 12 \text{ seconds}$$

$$\omega = \omega_0 + \alpha t$$

$$120\pi = 60\pi + \alpha(12)$$

$$\alpha = 5\pi \text{ r/s}^2$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = 60\pi \times 12 + \frac{1}{2} \times 5\pi \times 12^2$$

$$\theta = 1080\pi \text{ rad}$$

$$\text{Number of revolutions } n = \frac{\theta}{2\pi} = \frac{1080\pi}{2\pi}$$

$n = 540$ ***Ans.***

Problem 3

A fly wheel starting from rest and accelerating uniformly performs 25 revolutions in 5 seconds. Find its angular acceleration and its angular velocity after 10 seconds.

Solution

$1 \text{ rev} = 2\pi \quad ; t = 5 \text{ seconds} \quad ; \quad \omega_0 = 0$

$$\theta = 25 \times 2\pi$$

$$\theta = 50\pi \text{ rad}$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$50\pi = 0 + \frac{1}{2} \times \alpha \times 5^2$$

$$\alpha = 4\pi \text{ rad/s}^2$$

α is constant, at $t = 10$ seconds

$$\omega = \omega_0 + \alpha t$$

$$\omega = 0 + 4\pi \times 10$$

$\omega = 40\pi$ rad/s ***Ans.***

Problem 4

A table fan rotating at a speed of 2400 rpm is switched off and the resulting variation of rpm with time is, as shown in Fig. 13.4. Determine the total number of revolutions the fan has made in 25 seconds, when it finally comes to rest.

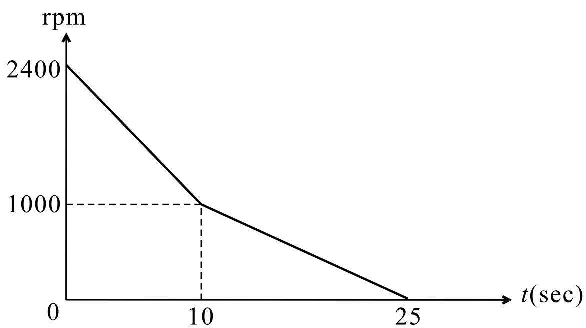


Fig. 13.4

Solution

We know, $\omega = \frac{2\pi N}{60}$ (where N is rpm)

The above graph can be observed as ω-t diagram and can be compared with v-t diagram. In v-t diagram, area under v-t diagram is linear displacement.

Similarly ω-t diagram, area under ω-t diagram will be angular displacement.

$$\therefore \theta = \left(\frac{1}{2} \times 10 \times 1400 + 10 \times 1000 + \frac{1}{2} \times 15 \times 1000\right)\frac{2\pi}{60}$$

$$\therefore \theta = 816.67\,\pi \text{ rad}$$

$$\therefore \text{Number of revolutions } n = \frac{\theta}{2\pi} = \frac{816.67\pi}{2\pi}$$

$n = 408.33$ ***Ans.***

Problem 5

A motor gives disk A an angular acceleration of $\alpha_A = (0.6t^2 + 0.75)$ rad/s^2, where t is in seconds. If the initial angular velocity of the disk is $\omega_0 = 6$ rad/s, determine the magnitude of the velocity and acceleration of block B when $t = 2$ seconds.

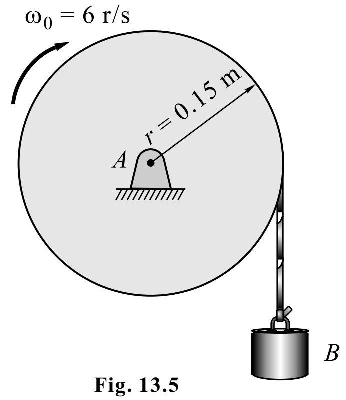


Fig. 13.5

Solution

At $t = 2$ seconds

$\alpha = 0.6t^2 + 0.75 = 0.6 \times 2^2 + 0.75$

$\alpha = 3.15$ rad/s^2

$a_B = r\alpha = 0.15 \times 3.15$

$a_B = 0.4725$ m/s^2

$\alpha = 0.6t^2 + 0.75$

$$\frac{d\omega}{dt} = 0.6t^2 + 0.75$$

$d\omega = (0.6t^2 + 0.75)\, dt$

Integrating both sides, we have

$$\int d\omega = \int (0.6t^2 + 0.75)\, dt$$

$$\omega = \frac{0.6t^3}{3} + 0.75t + c_1$$

At $t = 0$, $\omega = 6$ r/s $\therefore c_1 = 6$

$\omega = 0.2\, t^3 + 0.75 \times t + 6$

At $t = 2$ seconds

$\omega = 0.2 \times 2^3 + 0.75 \times 2 + 6 = 9.1$ rad/s **Ans.**

$v = r\omega$

$v_B = 0.15 \times 9.1 = 1.365$ m/s **Ans.**

Problem 6

Pulley A starts from rest and rotates with a constant angular acceleration of 2 r/s^2 anticlockwise. Pulley A causes double pulley B to rotate without slipping. Block C hangs by a rope wound on B, refer to Fig. 13.6(a). Determine at $t = 3$ seconds.

(i) Acceleration, velocity and position of block C.

(ii) Acceleration of point P on pulley B.

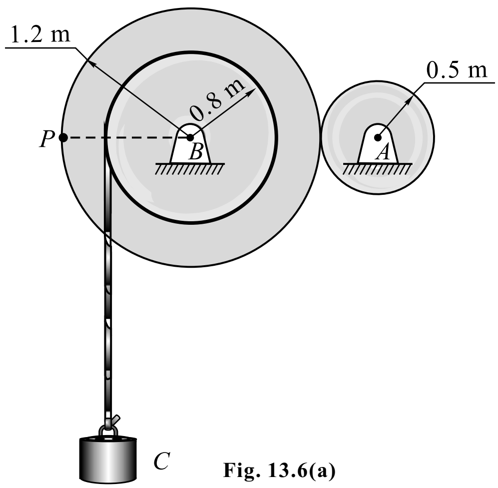


Fig. 13.6(a)

Solution

(i) The common contact point between double pulley B and pulley A will have the following relation :

$$s = r_A \theta_A = r_B \theta_B$$

$$v = r_A \omega_A = r_B \omega_B$$

$$a = r_A \alpha_A = r_B \alpha_B$$

$$\alpha_A = 2 \text{ r/s}^2, \; w_0 = 0, \quad t = 3 \text{ sec}$$

$$\omega_A = \omega_0 + \alpha t \qquad \theta_A = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega_A = 0 + 2 \times 3 \qquad \theta_A = 0 + \frac{1}{2} \times 2 \times 3^2$$

$$\omega_A = 6 \text{ r/s} \qquad \theta_A = 9 \text{ rad}$$

$$\therefore \; \omega_B = \frac{r_A \omega_A}{r_B} = \frac{0.5 \times 6}{1.2} \qquad \therefore \; \theta_B = \frac{r_A \theta_A}{r_B} = \frac{0.5 \times 9}{1.2}$$

$$\therefore \; \omega_B = 2.5 \text{ r/s} \qquad \therefore \; \theta_B = 3.75 \text{ rad}$$

$$\therefore \; \alpha_B = \frac{r_A \alpha_A}{r_B} = \frac{0.5 \times 2}{1.2}$$

$$\therefore \; \alpha_B = 0.833 \text{ r/s}^2$$

The common contact point between double pulley B (inner radius 0.8 m) and rope connected to block C will have the following relations :

$$s_C = r_B \theta_B \; ; \qquad v_C = r_B \omega_B \; ; \qquad a_C = r_B \alpha_B$$

$$s_C = 0.8 \times 3.75 \qquad v_C = 0.8 \times 2.5 \qquad a_C = 0.8 \times 0.833$$

$$s_C = 3 \text{ m} \qquad v_C = 2 \text{ m/s} \qquad a_C = 0.67 \text{ m/s}^2$$ ***Ans.***

(ii) Acceleration of point P

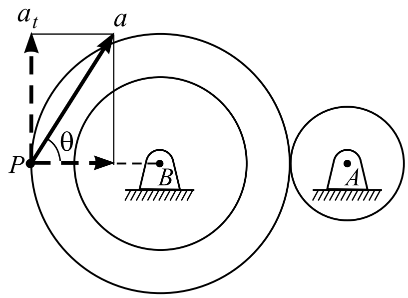


Fig. 13.6(b)

$$a_n = r_B \omega_B^2$$

$$a_n = 1.2 \times 2.5^2$$

$$a_n = 7.5 \text{ m/s}^2$$

$$a_t = r_B \alpha_B$$

$$a_t = 1.2 \times 0.833$$

$$a_t = 1 \text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2}$$

$$= \sqrt{(1)^2 + (7.5)^2}$$

$$a = 7.566 \text{ m/s}^2 \; (\measuredangle \theta)$$ ***Ans.***

$$\tan\theta = \frac{a_t}{a_n} \qquad \therefore \; \theta = 7.595^\circ$$ ***Ans.***

13.7 Solved Problems on Instantaneous Centre of Rotation (ICR)

Problem 7

The crank *BC* of a slider crank mechanism is rotating at constant speed of 30 rpm, as shown in Fig. 13.7(a) clockwise. Determine the velocity of the cross head *A* at the given instant.

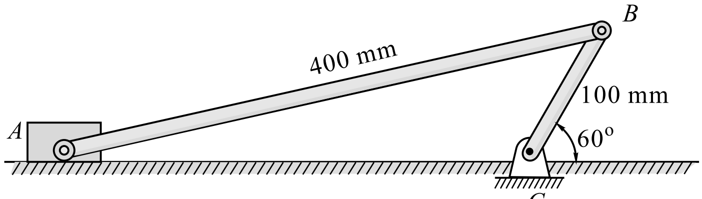


Fig. 13.7(a)

Solution

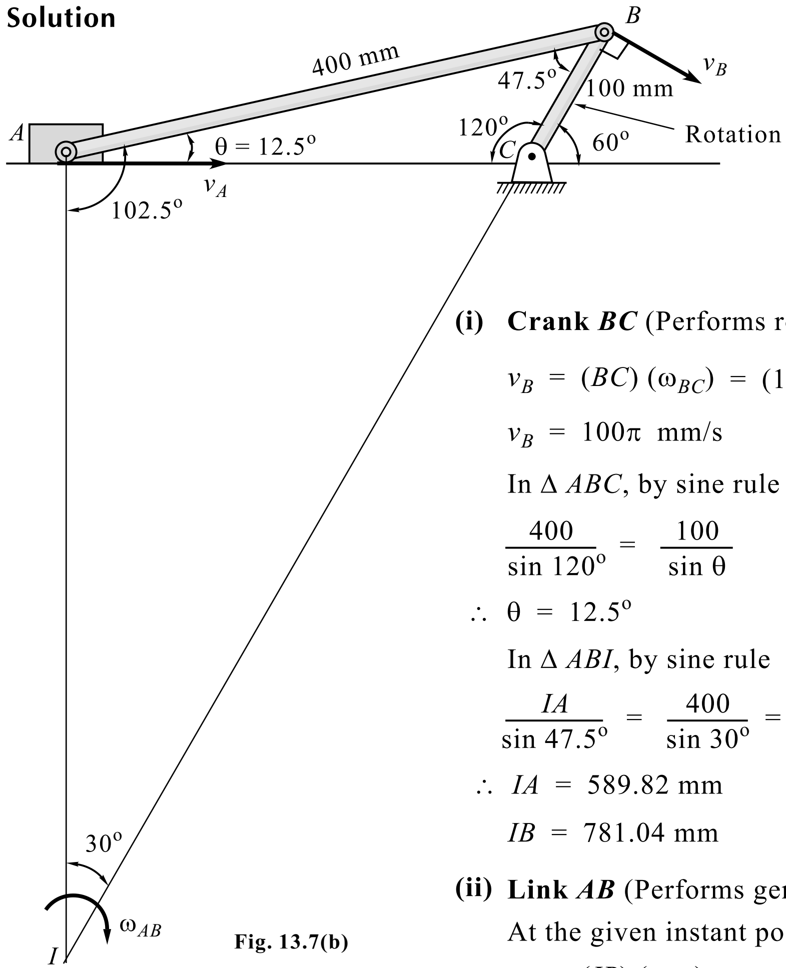


Fig. 13.7(b)

(i) Crank *BC* (Performs rotational motion about point *C*)

$$v_B = (BC)(\omega_{BC}) = (100)(30)\left(\frac{2\pi}{60}\right)$$

$$v_B = 100\pi \text{ mm/s}$$

In Δ *ABC*, by sine rule

$$\frac{400}{\sin 120^\circ} = \frac{100}{\sin \theta}$$

$$\therefore \theta = 12.5^\circ$$

In Δ *ABI*, by sine rule

$$\frac{IA}{\sin 47.5^\circ} = \frac{400}{\sin 30^\circ} = \frac{IB}{\sin 102.5^\circ}$$

$$\therefore IA = 589.82 \text{ mm}$$

$$IB = 781.04 \text{ mm}$$

(ii) Link *AB* (Performs general plane motion)

At the given instant point *I* is the ICR

$$v_B = (IB)(\omega_{AB})$$

$$\omega_{AB} = \frac{100\pi}{781.04}$$

$$\omega_{AB} = 0.402 \text{ r/s } (\circlearrowright)$$

$$v_A = (IA)(\omega_{AB}) = 589.82 \times 0.402$$

$$v_A = 237.12 \text{ mm/s} \quad \textit{\textbf{Ans.}}$$

Problem 8

Figure 13.8(a) shows a collar B which moves upwards with a constant velocity of 1.5 m/s. At the instant when $\theta = 50^\circ$, determine **(i)** the angular velocity of rod AB which is pinned at B and freely resting at A against 25° sloping ground, **(ii)** the velocity of end A of the rod and **(iii)** the velocity of mid point C of rod AB.

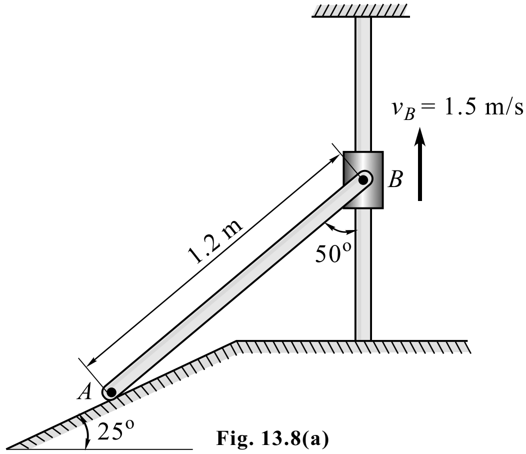


Fig. 13.8(a)

Solution

(i) In $\Delta\, IAB$, using sine rule

$$\frac{1.2}{\sin 65^\circ} = \frac{IA}{\sin 40^\circ} = \frac{IB}{\sin 75^\circ}$$

$$\therefore\; IA = 0.851 \text{ m} \quad IB = 1.28 \text{ m}$$

(ii) **Rod *AB*** (Performs general plane motion)

At the given instant point I is the ICR

$$v_B = (IB)\,(\omega_{AB})$$

$$\omega_{AB} = \frac{1.5}{1.28}$$

$$\omega_{AB} = 1.172 \text{ r/s } (\circlearrowleft) \quad \textbf{\textit{Ans.}}$$

$$v_A = (IA)\,(\omega_{AB}) = 0.851 \times 1.172$$

$$v_A = 1 \text{ m/s } (\measuredangle 25^\circ) \quad \textbf{\textit{Ans.}}$$

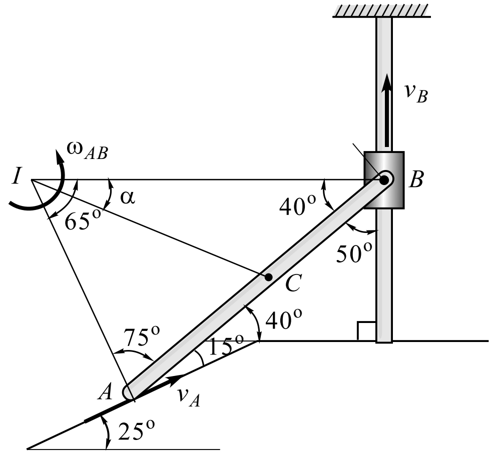


Fig. 13.8(b)

In $\Delta\, ICB$,

$$(IC)^2 = (IB)^2 + (CB)^2 - 2(IB)(CB)\cos 40^\circ = (0.851)^2 + (0.6)^2 - 2(1.28)(0.6)\cos 40^\circ$$

$$IC = 0.906 \text{ m}$$

$$\therefore\; v_C = (IC)\,\omega_{AB} = (0.906) \times 1.17$$

$$v_C = 1.06 \text{ m/s} \qquad \textbf{\textit{Ans.}}$$

Problem 9

In Fig. 13.9(a) collar C slides on a horizontal rod. In the position shown rod AB is horizontal and has angular velocity of 0.6 rad/s clockwise. Determine angular velocity of BC and velocity of the collar C.

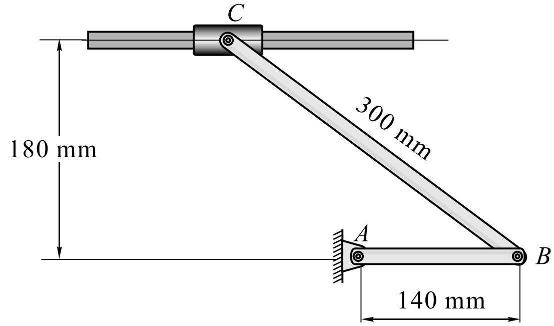


Fig. 13.9(a)

Solution

$$IB = \sqrt{300^2 - 180^2} = 240 \text{ mm}$$

(i) Rod *AB* (Performing rotational motion about A)

$$v_B = (AB).\ \omega_{AB} = 140 \times 0.6 = 84 \text{ mm/s}$$

$$\therefore\ v_B = 84 \text{ mm/s } (\downarrow)$$

(ii) Rod *BC* (Performs general plane motion)

At the given instant point I is the ICR (Instantaneous centre of rotation)

$$v_B = (IB)\,(\omega_{BC})$$

$$\omega_{BC} = \frac{84}{240}$$

$$\omega_{BC} = 0.35 \text{ rad/s } (\circlearrowleft) \quad \textbf{\textit{Ans.}}$$

$$v_C = (IC)\,(\omega_{BC}) = 180 \times 0.35$$

$$v_C = 63 \text{ mm/s } (\leftarrow) \quad \textbf{\textit{Ans.}}$$

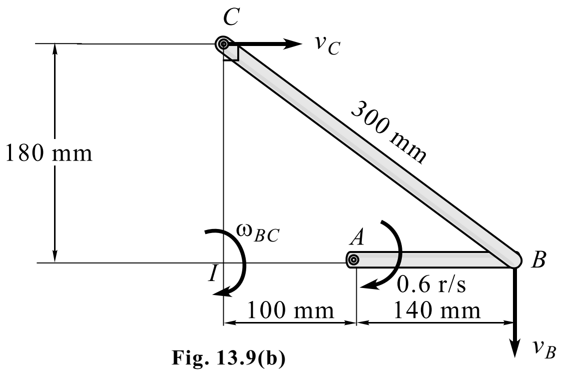


Fig. 13.9(b)

Problem 10

Crank OA rotates at 60 r.p.m. in clockwise sense. In the position shown $\theta = 40^\circ$ shown in Fig. 13.10(a), determine angular velocity of AB and velocity of B which is constrained to move in a horizontal cylinder.

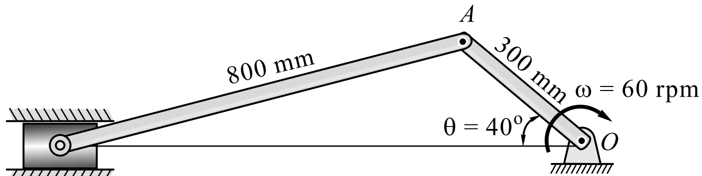


Fig. 13.10(a)

Solution

(i) $$\omega_{OA} = 60 \text{ rpm} = 60 \times \frac{2\pi}{60} \text{ rad/s}$$

$$\omega_{OA} = 2\pi \text{ rad/s}$$

$$\therefore\ \sin 40^\circ = \frac{h}{300}$$

$$\therefore\ h = 300 \times \sin 40^\circ = 192.84 \text{ mm}$$

$$\therefore\ \sin\theta = \frac{h}{800} = \frac{192.84}{800}$$

$$\theta = 13.95^\circ$$

Using sine rule, we get

$$\frac{AB}{\sin 50^\circ} = \frac{IA}{\sin 76.05^\circ} = \frac{IB}{\sin 53.95^\circ}$$

$$\frac{800}{\sin 50^\circ} = \frac{IA}{\sin 76.05^\circ} = \frac{IB}{\sin 53.95^\circ}$$

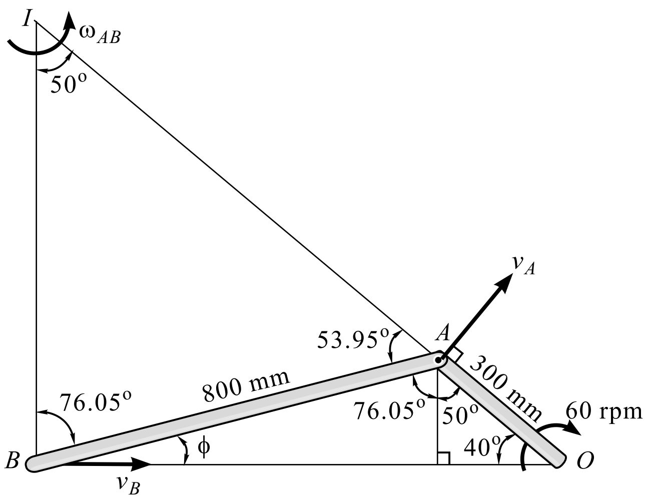


Fig. 13.10(b)

$$\therefore\ IA = 800 \times \frac{\sin 76.05^\circ}{\sin 50^\circ} = 1013.53 \text{ mm} = 1.04 \text{ m}$$

$$\therefore\ IB = 800 \times \frac{\sin 53.95^\circ}{\sin 50^\circ} = 844.34 \text{ mm} = 0.844 \text{ m}$$

(ii) Rod *OA* (Performs rotational motion about O)

$v_A = (OA)(\omega_{OA}) = 0.3 \times 2\pi = 1.89$ m/s

(iii) Rod *AB* (Performs general plane motion)

At the given instant point I is the ICR

$v_A = (IA)(\omega_{AB})$

$\omega_{AB} = \dfrac{1.89}{1.04}$

$\omega_{AB} = 1.817$ r/s (↺) ***Ans.***

$v_B = (IB)(\omega_{AB}) = 0.844 \times 1.817$

$v_B = 1.53$ m/s (→) ***Ans.***

Problem 11

Locate the instantaneous centre of rotation of link AB. Find also the angular velocity of link OA. Take velocity of slider at B = 2500 mm/s. The link and slider mechanism is as shown in Fig. 13.11(a).

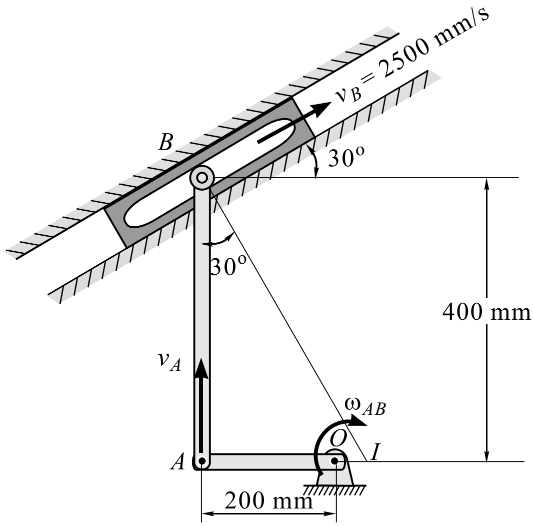


Fig. 13.11(a)

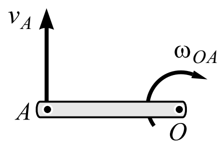


Fig. 13.11(b) : Link *OA* alone

Solution

(i) $\tan 30^\circ = \dfrac{IA}{AB} = \dfrac{IA}{400}$

$\therefore\ IA = 230.94$ mm ***Ans.***

$\cos 30^\circ = \dfrac{400}{IB}$

$IB = 461.88$ mm ***Ans.***

(ii) Link *AB* (Performs general plane motion)

At the given instant point I is the ICR

$v_B = (IB)(\omega_{AB})$

$\omega_{AB} = \dfrac{2500}{461.88}$

$\omega_{AB} = 5.413$ rad/s (↻)

(iii) Link *OA* (Performs rotational motion about point O)

$v_A = (OA)(\omega_{OA})$

$1250 = 200\,(\omega_{OA})$

$\omega_{OA} = 6.25$ rad/s (↻) ***Ans.***

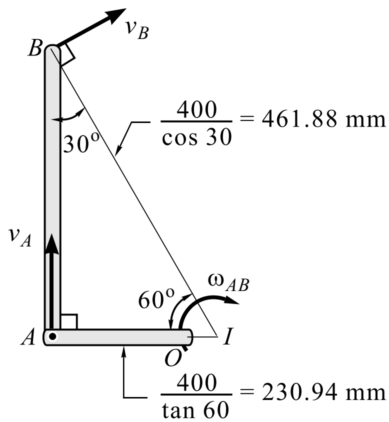


Fig. 13.11(c) : Link *AB* alone

Problem 12

A bar AB, 24 cm long, is hinged to a wall at A as shown in Fig. 13.12(a). Another bar CD 32 cm long is connected to it by a pin at B such that CB = 12 cm and BD = 20 cm. At the instant shown, ($AB \perp CD$) the angular velocities of the bars are ω_{AB} = 4 rad/s and ω_{CD} = 6 rad/s. Determine the linear velocities of C and D. (**Hint:** Bar CD is in plane motion.)

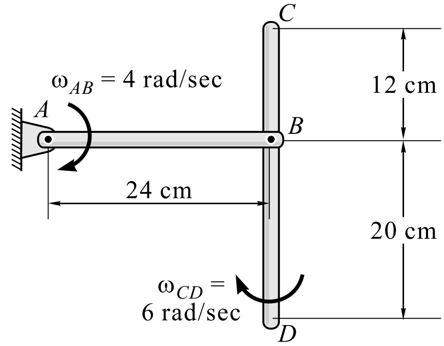


Fig. 13.12(a)

Solution

(i) Rod *AB* (Performs rotational motion about point A)

$$\therefore\ v_B = (AB)\,\omega_{AB} = (24)\,(4)$$

$$v_B = 96 \text{ cm/s } (\downarrow)$$

(ii) Rod *CD* (Performs general plane motion)

Let us assume the point I to be ICR

$$v_B = (IB)\,\omega_{CD}$$

$$IB = \frac{96}{6} = 16 \text{ cm}$$

$$IC = \sqrt{16^2 + 12^2} = 20 \text{ cm}$$

$$v_C = (IC)\,(\omega_{CD}) = 20 \times 6$$

$$v_C = 120 \text{ cm/s } (\theta) \quad \textbf{\textit{Ans.}}$$

$$v_D = (ID)\,(\omega_{CD}) = \left(\sqrt{16^2 + 20^2}\right)(6)$$

$$v_D = 153.67 \text{ cm/s } (\alpha) \quad \textbf{\textit{Ans.}}$$

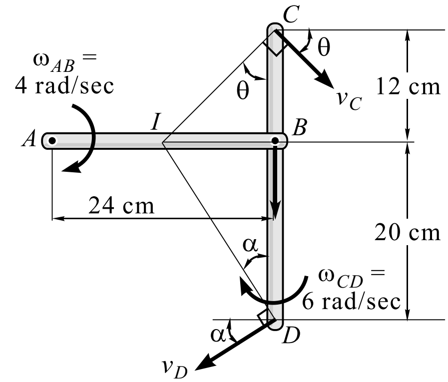


Fig. 13.12(b)

$$\tan\theta = \frac{16}{12} \qquad \therefore\ \theta = 53.13^{\circ}$$

$$\tan\alpha = \frac{16}{12} \qquad \therefore\ \alpha = 38.66^{\circ}$$

Problem 13

In the device shown in Fig. 13.13(a). Find the velocity of point B and angular velocity of both the rods. The wheel is rotating at 2 rad/s anticlockwise.

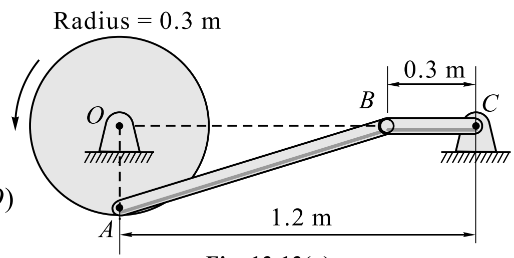


Fig. 13.13(a)

Solution

(i) Wheel (Performs rotational motion about point O)

$$v_A = (OA)\,\omega = 0.3 \times 2 \qquad \therefore\ v_A = 0.6 \text{ m/s}$$

(ii) Rod *AB* (Performs general plane motion)

Let us assume the point I to be ICR

$$IA = 0.3 \text{ m}$$

$$IB = 1.2 - 0.3 = 0.9 \text{ m}$$

$$v_A = (IA)\,(\omega_{AB})$$

$$\omega_{AB} = \frac{0.6}{0.3} \qquad \therefore\ \omega_{AB} = 2 \text{ rad/s } (\circlearrowleft) \quad \textbf{\textit{Ans.}}$$

$$v_B = (IB)\,(\omega_{AB}) = 0.9 \times 2 \qquad \therefore\ v_B = 1.8 \text{ m/s } (\ -\ ) \quad \textbf{\textit{Ans.}}$$

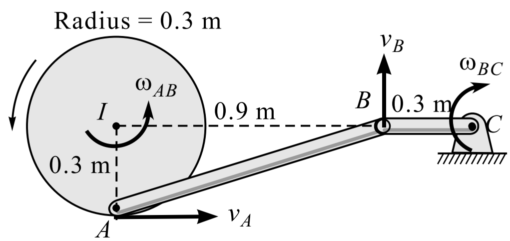


Fig. 13.13(b)

(iii) Rod *BC* (Performs rotational motion about point *B*)

$$v_B = (BC)\,\omega_{BC}$$

$$1.8 = 0.3 \times \omega_{BC}$$

$$\therefore\ \omega_{BC} = 6 \text{ rad/s } (\circlearrowleft) \quad \textit{\textbf{Ans.}}$$

Problem 14

In Fig. 13.14(a), *C* is constrained to move in a vertical slot. *A* and *B* move on horizontal floor. Rod *CA* and *CB* are connected with smooth hinges. If $v_A = 0.45$ m/s to the right. Find velocity of *C* and *B*. Also find the angular velocity of the two rods.

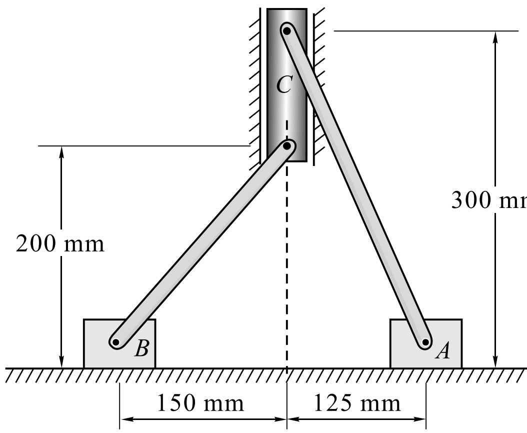


Fig. 13.14(a)

Solution

$$I_1A = 0.3 \text{ m}\ ;\ I_1C = 0.125 \text{ m}\ ;$$

$$I_2B = 0.2 \text{ m}\ ;\ I_2C = 0.150 \text{ m}$$

(i) Rod *AC*

(Performs general plane motion) I_1 (ICR)

$$\frac{v_A}{I_1A} = \frac{v_C}{I_1C} = \omega_{AC}$$

$$\frac{0.45}{0.3} = \frac{v_C}{0.125} = \omega_{AC}$$

$$\therefore\ v_C = 0.1875 \text{ m/s } (\downarrow) \quad \textit{\textbf{Ans.}}$$

$$\omega_{AC} = 1.5 \text{ rad/s } (\circlearrowleft) \quad \textit{\textbf{Ans.}}$$

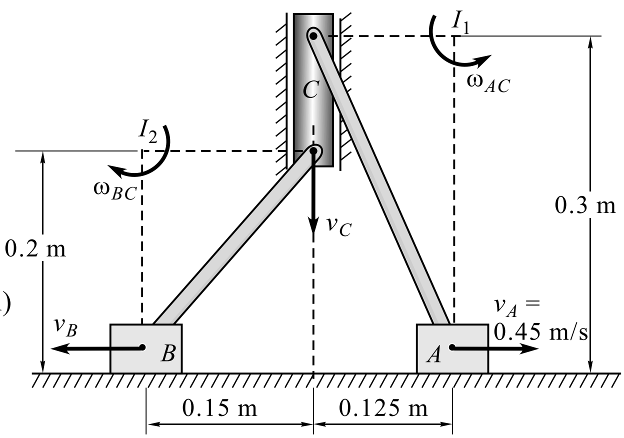


Fig. 13.14(b)

(ii) Rod *BC*

(Performs general plane motion) I_2 (ICR)

$$\frac{v_C}{I_2C} = \frac{v_B}{I_2B} = \omega_{BC}$$

$$\frac{0.1875}{0.150} = \frac{v_B}{0.2} = \omega_{BC}$$

$$\therefore\ v_B = 0.25 \text{ m/s } (\leftarrow) \quad \textit{\textbf{Ans.}}$$

$$\omega_{BC} = 1.25 \text{ rad/s } (\circlearrowright) \quad \textit{\textbf{Ans.}}$$

Problem 15

C is a uniform cylinder to which a rod *AB* is pinned at *A* and the other end of the rod *B* is moving along a vertical wall as shown in Fig. 13.15(a).

If the end *B* of the rod is moving upward along the wall at a speed of 3.3 m/s, find the angular velocity of the cylinder assuming that the cylinder is rolling without slipping.

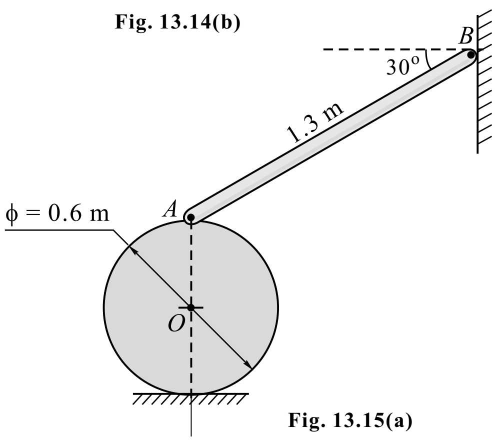


Fig. 13.15(a)

Solution

(i) Rod *AB* (Performs general plane motion)

At the given instant point I_1 is the ICR

$$v_B = (I_1B)(\omega_{AB})$$

$$\omega_{AB} = \frac{3.3}{1.3 \cos 30^o} \qquad \therefore \omega_{AB} = 2.931 \text{ rad/s } (\circlearrowleft)$$

$$v_A = (I_1A)(\omega_{AB}) = (1.3 \sin 30^o) \times (2.931)$$

$$v_A = 1.9$$

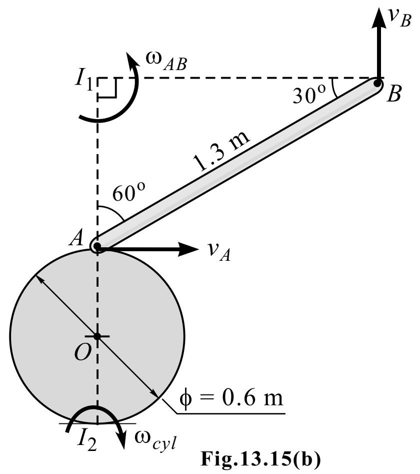


Fig.13.15(b)

(ii) Cylinder (Performs general plane motion)

At the given instant point I_2 is the ICR

$$v_A = (I_2A)(\omega_{cyl.})$$

$$\omega_{cyl.} = \frac{1.9}{0.6} \qquad \therefore \omega_{cyl.} = 3.167 \text{ r/s } (\curvearrowright) \textbf{ Ans.}$$

Problem 16

In the position shown in Fig. 13.16(a) rod *AB* is horizontal and has angular velocity 1.8 rad/s in clockwise sense. Determine angular velocities of *BC* and *CD*.

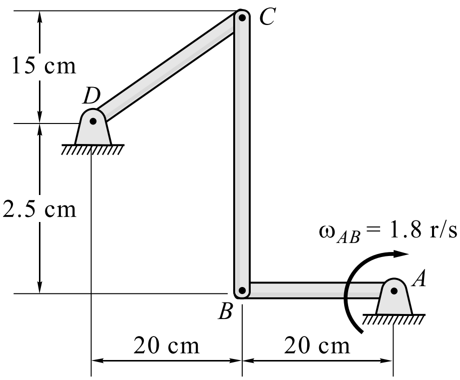


Fig. 13.16(a)

Solution

$$\tan\theta = \frac{15}{20} \qquad \therefore \theta = 36.87^o$$

$$\tan 36.87^o = \frac{BC}{IB} = \frac{37.5}{IB} \qquad \therefore IB = 50 \text{ cm}$$

$$IC = \sqrt{(IB)^2 + (BC)^2} = 62.5 \text{ cm}$$

(i) Rod *AB* (Performs rotational motion about *A*)

$$v_B = (AB)(\omega_{AB}) = 20 \times 1.8 = 36 \text{ cm/s}$$

(ii) Rod *BC* (Performs general plane motion)

At the given instant point *I* is the ICR

$$v_B = (IB)(\omega_{BC})$$

$$36 = 50 \times \omega_{BC}$$

$$\omega_{BC} = 0.72 \text{ r/s } (\circlearrowleft) \quad \textbf{Ans.}$$

$$v_C = (IC)(\omega_{BC}) = 62.5 \times 0.72$$

$$v_C = 45 \text{ cm/s}$$

(iii) Consider Rod *CD*

$$v_C = (CD)(\omega_{CD})$$

$$45 = 25 \times \omega_{CD}$$

$$\omega_{CD} = 1.8 \text{ r/s } (\circlearrowleft) \quad \textbf{Ans.}$$

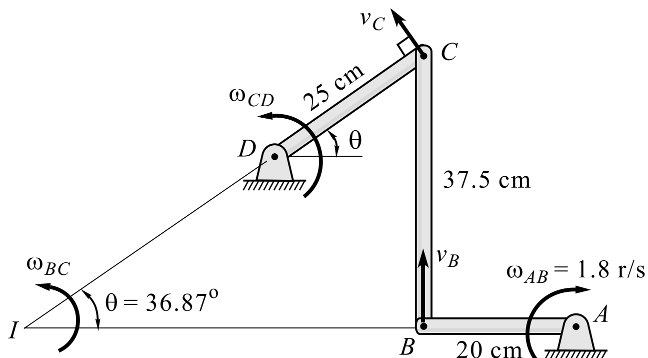


Fig. 13.16(b)

Problem 17

A roller of radius 8 cm rides between two horizontal bars moving in the opposite directions as shown in Fig. 13.17(a). Locate the instantaneous centre of velocity and give its distance from B. Assume no slip conditions at the points A and B. Locate the position of the instantaneous centre where both the bars are moving in the same direction.

Solution

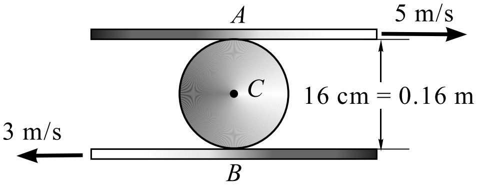


Fig. 13.17(a)

Roller is performing general plane motion.

At the given instant point I is the ICR

Method I

$$v_A = (IA)\,\omega$$

$$\omega = \frac{5}{0.16 - h} \quad \ldots\ldots \text{(I)}$$

$$v_B = (IB)\,\omega$$

$$\therefore\ \omega = \frac{3}{h} \quad \ldots\ldots \text{(II)}$$

$$\text{(I)} = \text{(II)}$$

$$\frac{5}{0.16 - h} = \frac{3}{h}$$

$$5h = 3(0.16 - h)$$

$$5h = 0.48 - 3h$$

$$8h = 0.48$$

$$h = 0.06 \text{ m}$$

$$IB = 0.06 \text{ m} \qquad \textbf{\textit{Ans.}}$$

Method II

$$\frac{v_A}{IA} = \frac{v_B}{IB}$$

$$\frac{5}{IA} = \frac{3}{IB}$$

$$\therefore\ 5IB = 3IA \quad \ldots\ldots \text{(I)}$$

$$\therefore\ IB = \frac{3}{5}\,IA$$

$$\therefore\ IA + IB = 0.16$$

$$\therefore\ IA + \left(\frac{3}{5}IA\right) = 0.16$$

$$\therefore\ IA = 0.1 \text{ m}$$

$$\therefore\ IB = \frac{3 \times 0.1}{5}$$

$$IB = 0.06 \text{ m} \qquad \textbf{\textit{Ans.}}$$

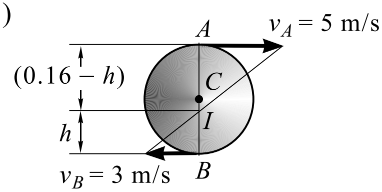


Fig. 13.17(b)

Method I

$$v_A = (IA)\,\omega$$

$$\omega = \frac{5}{(h + 0.16)} \quad \ldots\ldots \text{(I)}$$

$$v_B = (IB)\,\omega$$

$$\therefore\ \omega = \frac{3}{h} \quad \ldots\ldots \text{(II)}$$

$$\text{(I)} = \text{(II)}$$

$$\frac{5}{h + 0.16} = \frac{3}{h}$$

$$5h = 3(h + 0.16)$$

$$5h - 3h = 0.48$$

$$2h = 0.48$$

$$h = 0.24 \text{ m}$$

$$IB = 0.24 \text{ m} \qquad \textbf{\textit{Ans.}}$$

Method II

$$\frac{v_A}{IA} = \frac{v_B}{IB}$$

$$\therefore\ \frac{5}{IA} = \frac{3}{IB}$$

$$\therefore\ IB = \frac{3}{5}\,IA$$

But

$$IA - IB = 16$$

$$\therefore\ IA - \left(\frac{3}{5}\,IA\right) = 0.16$$

$$IA = 0.4 \text{ m}$$

$$IB = 0.24 \text{ m} \qquad \textbf{\textit{Ans.}}$$

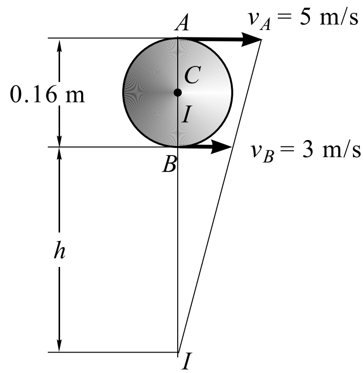


Fig. 13.17(c)

Exercises

[I] Problems

1. A wheel has an initial clockwise angular velocity 8 rad/s and a constant angular acceleration of 2 rad/s^2. Determine the number of revolutions the wheel must undergo to acquire a clockwise angular velocity of 15 rad/s. What time is required ?

[***Ans.*** n = 6.41 rev. and t = 3.5 seconds.]

2. A wheel accelerates uniformly from rest to a speed of 200 rpm in 1/2 seconds. It then rotates at that speed for 2 seconds before decelerating to rest in 1/3 seconds. How many revolutions does it make during the entire time interval ?

[***Ans.*** 8.06 rev.]

3. The angular displacement of a rotating can is defined by the relation : $\theta = t^3 - 3t^2 + 6$

where θ is expressed in radians, determine the angular displacement, angular velocity and angular acceleration of the can when t = 3 seconds.

[***Ans.*** 6 rad, 9 rad/s and 12 rad/s^2.]

4. A small grinding wheel is run by electric motor with a rated speed of 3000 r.p.m. When the power is put on the motor reaches its rated speed in 4 seconds and when the power is put off the unit comes to rest in 60 seconds. Assuming uniform acceleration and uniform retardation determine **(a)** the angular acceleration of the wheel, **(b)** the angular retardation of the wheel and **(c)** the total number of revolutions made by the wheel in reaching its rated speed and in coming to rest.

[***Ans.*** (a) α = 78.54 rad/s^2, (b) α = –5.236 r/s^2, (c) n = 1000 rev. and n = 1500 rev.]

5. The relation of the rigid body is defined as follows where θ is angular displacement in radians and t seconds : **(a)** $\theta = 3t^2 - 2t$ **(b)** $\theta = t^3 - 1.5t^2$ and **(c)** $\theta = 2 \sin (\pi t/4)$. Determine angular velocity and acceleration in each case at t = 2 seconds.

[***Ans.*** **(a)** ω = 10 rad/s, α = 6 rad/s^2, **(b)** ω = 6 rad/s, α = 9 rad/s^2 and
(c) ω = 0, $\alpha = -\pi^2/8$ rad/s^2.]

6. The rotation of the rigid body is defined as follows :

(a) ϕ = 3t2 – 2t **(b)** ϕ = 2 sin ($\pi t/4$)

where ϕ is the displacement in radians and t is seconds.

Determine angular velocity and acceleration in each case after 2 seconds.

[***Ans.*** **(a)** ω = 10 rad/s, α = 6 rad/s^2 and **(b)** ω = 0, $\alpha = -\pi^2/8$ rad/s^2.]

7. The motion of a flywheel around its geometrical axis is described by the equation :

$\omega = 15t^2 + 3t + 2$ rad/s and angular displacement is 160 radians at t = 3 seconds. Find the angular acceleration, velocity and displacement at t = 1 second.

[***Ans.*** θ = 14 rad. ω = 20 rad/s and α = 33 rad/s^2.]

8. A bar, 3 m long slides down the plane shown in Fig. 13.E8. The velocity of end A is 3.6 m/s to the right. Determine the angular velocity of AB and velocity of end B and centre C at the instant shown.

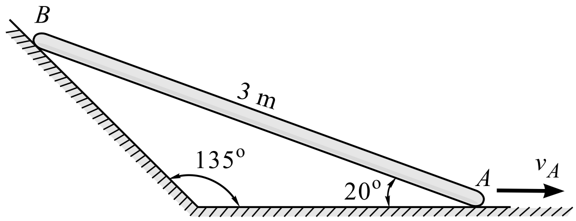


Fig. 13.E8

[**Ans.** ω_{AB} = 0.9363 rad/s (↺), v_B = 3.733 m/s and v_C = 3.387 m/s.]

9. Arm AB rotates anticlockwise with uniform angular velocity 10 rad/s, as shown in Fig. 13.E9. Point C is constrained to move along the x-axis. Calculate the angular velocity of bar BC. Also determine the velocity of C.

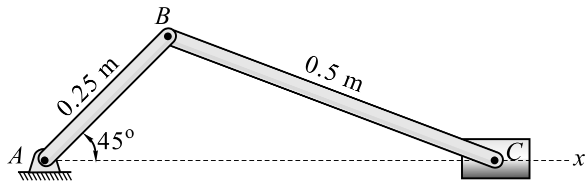


Fig. 13.E9

[**Ans.** ω_{BC} = 3.78 rad/s (↻) and v_C = 2.434 m/s (←).]

10. For the crank and connecting rod mechanism shown in Fig. 13.E10, determine the velocity of the crosshead P and angular velocity of connecting rod AP either by using kinematics relationship or by using instantaneous center method.

Given OA = 100 mm, AP = 400 mm, O is hinged point, P is considered to move in vertical direction.

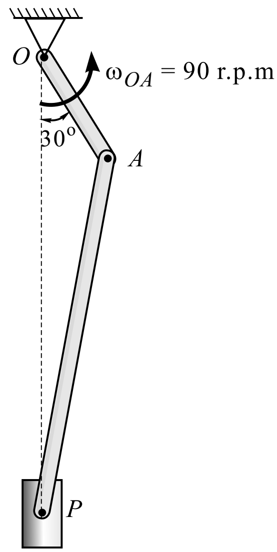


Fig. 13.E10

[**Ans.** v_P = 0.574 m/s (↑) and ω_{AP} = 2.057 rad/s (↻).]

11. Block D shown in Fig. 13.E11 moves with a speed of 3 m/s. Determine velocity of links BD and AB and velocity of point B at the instant shown. Take length of link AB and BD as 0.4 m.

[**Ans.** ω_{BD} = 5.3 rad/s (↺), ω_{AB} = 5.3 rad/s (↻) and v_B = 2.12 m/s.]

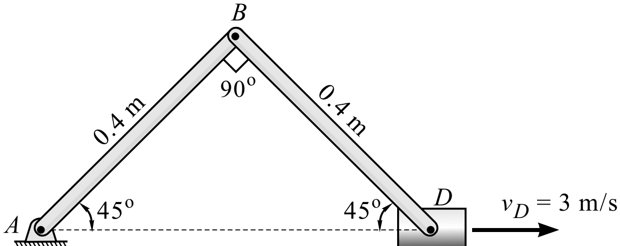


Fig. 13.E11

12. The bar AB has an angular velocity of 6 rad/s clockwise when $\theta = 50°$. Determine the corresponding angular velocities of bars BC and CD at this instant in Fig. 13.E12.

[**Ans.** ω_{BC} = 4.692 rad/s (↺) and
ω_{CD} = 9.19 rad/s (↺).]

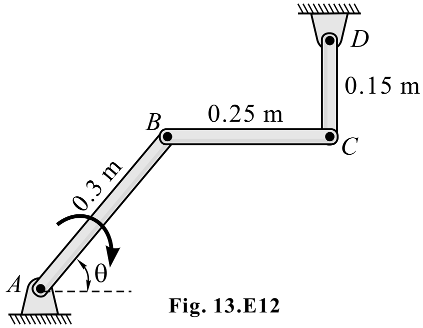


Fig. 13.E12

13. For the position shown in Fig. 13.E13, the angular velocity of bar AB is 10 rad/s anticlockwise. Determine the angular velocities of bars BC and CD.

[**Ans.** ω_{BC} = 18 rad/s (↺) and
ω_{CD} = 30 rad/s (↺).]

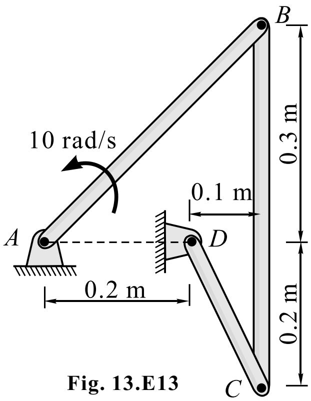


Fig. 13.E13

14. The bar AB of the linkage shown in Fig. 13.E14 has a clockwise angular velocity of 30 rad/s when $\theta = 60°$. Compute the angular velocities of member BC and the wheel at this instant.

[**Ans.** ω_{BC} = 15 rad/s (↺) and
ω_{CD} = 52 rad/s (↺).]

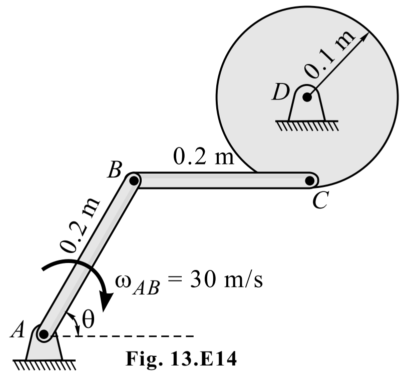


Fig. 13.E14

15. In the position shown in the Fig. 13.E15, bar AB has constant angular velocity of 3 rad/s anticlockwise. Determine the angular velocity of bars BD and DE.

[**Ans.** ω_{BD} = 0 and
ω_{ED} = 1.6 rad/s (↻).]

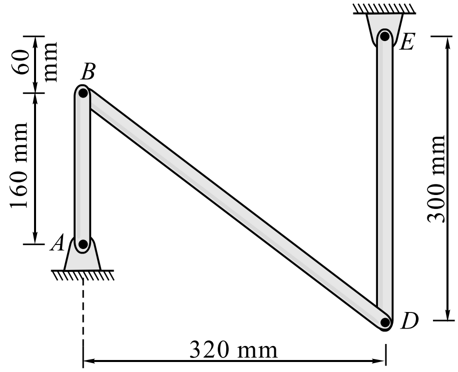


Fig. 13.E15

16. For the mechanism shown in Fig. 13.E16, bar *AB* has a constant angular velocity of 12 rad/s counterclockwise. Determine the angular velocity of the bar *BC* and *CD* at the instant shown.

[***Ans.*** $\omega_{BC} = 0$ and
$\omega_{CD} = 6.4$ rad/s (↻).]

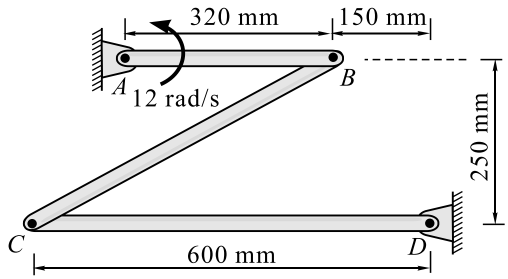


Fig. 13.E16

17. A wheel moves on a surface without slipping such that its centre has a velocity of 4 m/s towards horizontally. The angular velocities of the wheel is 4 rad/s clockwise. Determine the velocities of points *P*, *Q* and *R* shown on the wheel in Fig. 13.E17.

Given : Diameter of wheel = 2 m,

Distance *CP* = 600 mm,

v_C = 4 m/s (→) and ω = 4 rad/s (↻).

[***Ans.*** $v_P = 5.322$ m/s, $v_Q = 5.657$ m/s and $v_R = 0$.]

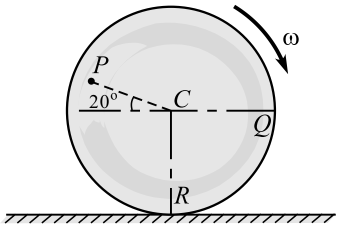


Fig. 13.E17

18. A compound wheel rolls without slipping as shown in Fig. 13.E18. The velocity of centre *C* is 1 m/s. Find the velocities of the point *A*, *B* and *D*.

[***Ans.*** $v_A = 3$ m/s (→), $v_B = 5.657$ m/s (←)
and $v_D = 2.236$ m/s.]

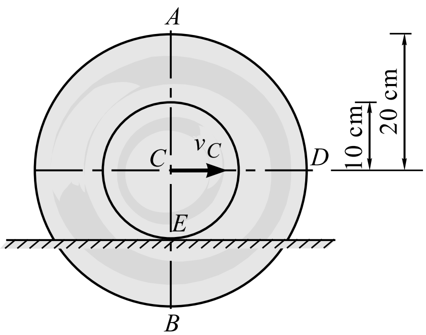


Fig. 13.E18

19. Due to slipping, points *A* and *B* on the rim of the disk have the velocities as shown in Fig. 13.E19. Determine the velocities of the centre point *C* and point *D* at this instant. Take radius of disk as 0.24 m.

[***Ans.*** $v_C = 0.75$ m/s (←) and $v_D = 2.83$ m/s.]

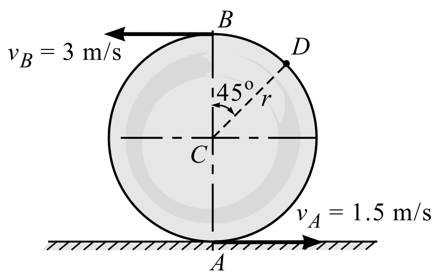


Fig. 13.E19

[II] Review Questions

1. What are the different types of rigid body motions ?
2. Distinguish between translation motion and rotational motion.
3. Explain the following terms for rotating rigid body :

   (i) Angular displacement, (ii) Angular velocity and (iii) Angular acceleration.
4. Compare the equation of translation motion and rotational motion.
5. Explain instantaneous centre of rotation (ICR).
6. General plane motion is the combination of translation motion and rotational motion. Justify.
7. Explain the behaviour of a rolling body.

[III] Fill in the Blanks

1. Translation motion can happen in ________ form and curvilinear form.
2. In translation motion, displacement, velocity and acceleration of each and every particle of a rigid body at any instant is ________.
3. The rate of change of angular position w.r.t. time is called as angular ________ of a rotating body.
4. General plane motion is the combination of ________ motion and ________ motion.
5. If a body is rolling without slipping on stationary surface then the point of contact with the stationary surface is called ________.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. In kinematics of rigid body, we have to consider the ________ of the body.

   (a) dimension **(b)** mass **(c)** force **(d)** density
2. Angular displacement is measured in unit _________.

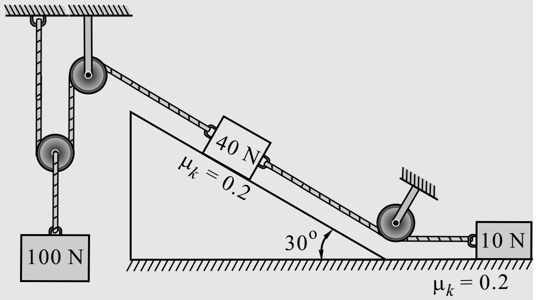
   (a) degree **(b)** radian **(c)** radian/s **(d)** radian/s^2
3. Angular velocity is measured in unit _________.

   (a) degree **(b)** radian **(c)** radian/s **(d)** radian/s^2
4. Angular acceleration is measured in unit _________.

   (a) degree **(b)** radian **(c)** radian/s **(d)** radian/s^2

14

KINETICS OF PARTICLES - I NEWTON'S SECOND LAW/ D'ALEMBERT'S PRINCIPLE



14.1 Introduction to Kinetics

- **Kinetics :** *It is the study of geometry of motion with reference to the cause of motion*. Here we consider force and mass.

Basic Concepts

- **Particle :** *It is a matter with considerable mass but negligible dimension,* i.e., any object whose mass is considered but dimension is not considered.
- **Force :** *An external agency which changes or tends to change the state of rest or of uniform motion of a body upon which it acts is known as force*.
- **Mass :** *It is the quantity of matter contained in a body.*

  The quantity does not change on account of the position occupied by the body. The force of attraction exerted by the earth on two different bodies with the same mass will be same. *Mass is the property of body which measures its resistance to a change of motion*. Its SI unit is kg.
- **Weight :** *The gravitational force of attraction exerted by the earth on a body is known as the weight of the body*. This force exists whether the body is at rest or in motion. Since this attraction is a force, the weight of body should be expressed in Newton (N) in SI units.

14.2 Newton's Second Law of Motion

- **Newton's Second Law of Motion :** *If an external unbalanced force acts on a body, the momentum of the body changes. The rate of change of momentum is directly proportional to the force and takes place in the direction of motion.*

- **Momentum :** *The quantity of motion possessed by the body.* Linear momentum of a body is calculated as the product of mass and velocity of the body.

  Rate of change of momentum is directly proportional to the force, i.e.,

$$\frac{d}{dt}(m\overline{v}) \propto \overline{F}$$
$$\frac{d}{dt}(m\overline{v}) = k\overline{F}$$
$$m\frac{d\overline{v}}{dt} = k\overline{F}$$
$$m\overline{a} = k\overline{F}$$

when $m = 1$, $a = 1$, $F = 1$ then $k = 1$

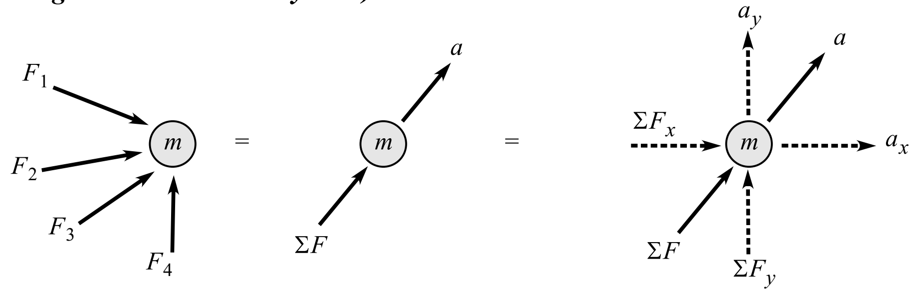
$$\therefore \overline{F} = m\overline{a}$$

In other words, we can also say that *if the resultant force acting on the particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of the resultant force.*

$$\overline{F} = m\overline{a}$$

For Rectilinear Motion

(Rectangular Coordinate System)



$$\sum F = ma$$

$$\sum F_x = ma_x$$
$$\sum F_y = ma_y$$

Fig. 14.2-i

For Curvilinear Motion

(Tangent and Normal Coordinate System)

a_t = Tangential component of acceleration

a_n = Normal component of acceleration

$$\sum F_t = ma_t = m\frac{dv}{dt}$$
$$\sum F_n = ma_n = m\frac{v^2}{\rho}$$

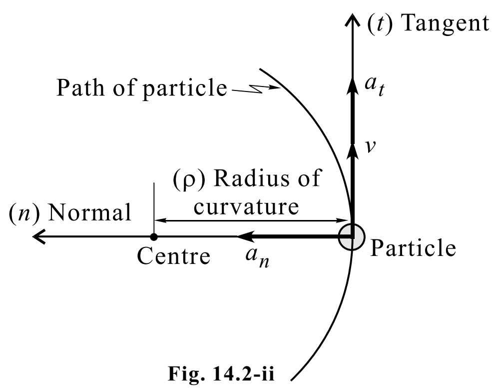


Fig. 14.2-ii

14.3 D'Alemberts' Principle (Dynamic Equilibrium)

- **Dynamic Equilibrium :** The force system consisting of external forces and inertia force can be considered to keep the particle in equilibrium. Since the resultant force externally acting on the particle is not zero, the particle is said to be in *dynamic equilibrium*.

- **D'Alemberts' Principle :** *The algebraic sum of external force (ΣF) and inertia force (–ma) is equal to zero.*

$$\sum F + (-ma) = 0$$

- **For Rectilinear Motion**

$$\sum F_x + (-ma_x) = 0 \quad \text{and} \quad \sum F_y + (-ma_y) = 0$$

- **For Curvilinear Motion**

$$\sum F_t + (-ma_t) = 0 \quad \text{and} \quad \sum F_n + (-ma_n) = 0$$

Note : Comparing D'Alemberts' Principle with Newton's Second Law

We understand Newton's Law as the original and D'Alembert had expressed the same concept in a different wording with adjustment of mathematical expression. So in this book we have solved problems considering Newton's Second Law.

How to Analyse a Problem ?

1. Draw a FBD of a particle showing all active and reactive force with all known and unknown values by considering geometrical angles if any. (*Similar to FBD in Statics*)
2. Show direction of acceleration and consider +ve sign convention along the direction of acceleration.
3. Assumption for direction of acceleration.
   - **a.** If the friction is not given then one can assume any direction for acceleration. Positive answer means assumed direction is correct.
   - **b.** If the friction is given then one has to carefully analyse the problem and assume the direction of acceleration. Here we must get +ve answer. In case if answer is negative then one should resolve the whole problem with change in the direction opposite to the assumed direction.
4. If more than one particles are involved in a system then find the kinematic relation between the particles (i.e., relation of displacement, velocity and acceleration).
5. For finding the kinematic relation of connected particles introduce the tension in each cord. Apply the Virtual Work Principle which says - *total virtual work done by internal force (tension) is zero*. Consider work done to be +ve if displacement and tension are in same direction and work done to be –ve if displacement and tension are in one direction.

14.4 Solved Problems Based on Rectilinear Motion

Problem 1

A crate of mass 20 kg is pulled up the inclined 20^o by force F which varies as per the graph shown in Fig. 14.1(a). Find the acceleration and velocity of the crate at $t = 5$ seconds, knowing that its velocity was 4 m/s at $t = 0$. Take $\mu_k = 0.2$.

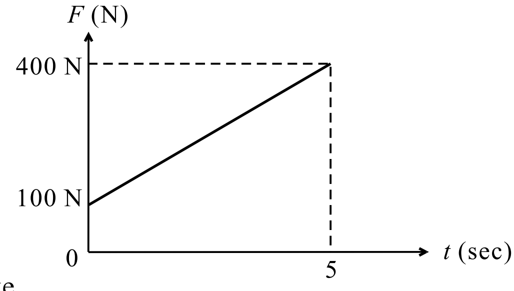


Fig. 14.1(a)

Solution

(i) From the graph F-t shown in Fig. 14.1(a), we have

$$y = mx + c$$

$$m = \frac{400 - 100}{5} = 60 \quad \text{and} \quad c = 100$$

$$\therefore \; F = 60t + 100$$

(ii) Consider the FBD of the crate

Refer to Fig. 14.1(b)

By Newton's second law, we have

$$\sum F_x = ma_x$$

$$F - 0.2N - 20 \times 9.81 \sin 20^o = 20 \times a$$

$$60t + 100 - 0.2 \times 20 \times 9.81 \cos 20^o - 20 \times 9.81 \sin 20^o = 20 \times a$$

$$60t - 4 = 20a$$

$$a = 3t - 0.2 \quad \text{(Variable acceleration)}$$

$$a = 3 \times 5 - 0.2$$

$$a = 14.8 \text{ m/s}^2$$ ***Ans.***

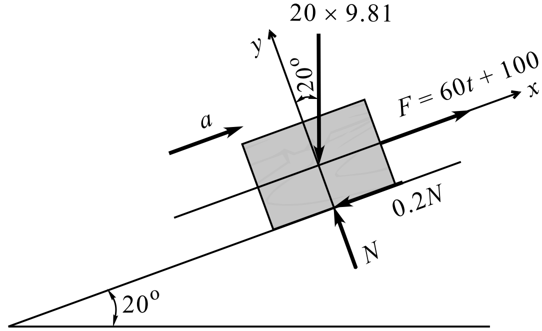


Fig. 14.1(b)

(iii) $a = \dfrac{dv}{dt}$

$$\therefore \; dv = a\,dt$$

$$\therefore \; dv = (3t - 0.2)\,dt$$

Integrating both sides

$$\int_{v_1 = 4 \text{ m/s}}^{v_2} dv = \int_{t_1 = 0}^{t_2 = 5} (3t - 0.2)\,dt$$

$$\left[v\right]_4^{v_2 = ?} = \left[\frac{3 \times t^2}{2} - 0.2t\right]_0^5$$

$$v_2 - 4 = \left[\frac{3 \times 5^2}{2} - 0.2 \times 5\right]$$

$$v_2 = 32.5 \text{ m/s}$$ ***Ans.***

Problem 2

A 50 kg block kept on the top of a 15^o sloping surface is pushed down the plane with an initial velocity of 20 m/s. If $\mu_k = 0.4$, determine the distance travelled by the block and the time it will take as it comes to rest.

Solution

(i) Consider the FBD of 50 kg block (Refer to Fig. 14.2)

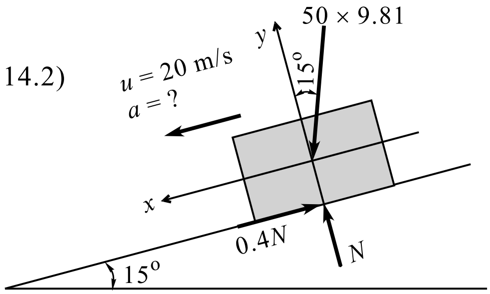


Fig. 14.2

(ii) By Newton's second law, we have

$$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$$

$$N - 50 \times 9.81 \cos 15^o = 0$$

$$N = 50 \times 9.81 \cos 15^o$$

$$\sum F_x = ma_x$$

$$50 \times 9.81 \sin 15^o - 0.4 \times 50 \times 9.81 \cos 15^o = 50a$$

$\therefore\ a = -1.25 \text{ m/s}^2$ (Retardation)

(iii) $u = 20$ m/s ; $v = 0$; $a = -1.25 \text{ m/s}^2$; $s = ?$; $t = ?$

$$v = u + at$$

$$0 = 20 + (-1.25)t$$

$\therefore\ t = 16$ seconds.

$$s = ut + \frac{1}{2}at^2$$

$$s = 20 \times 16 + \frac{1}{2}(-1.25) \times (16)^2$$

$s = 160$ m ***Ans.***

Problem 3

An aeroplane has a mass of 25000 kg and its engines develop a total thrust of 40 kN along the run way. The force of air resistance to motion of aeroplane is given by $R = 2.25v^2$ where v is m/s and R is in Newton. Determine the length of runway required if the plane takes off and becomes airborne at a speed of 240 km/hr.

Solution

(i) Consider the FBD of the plane (Refer to Fig. 14.3)

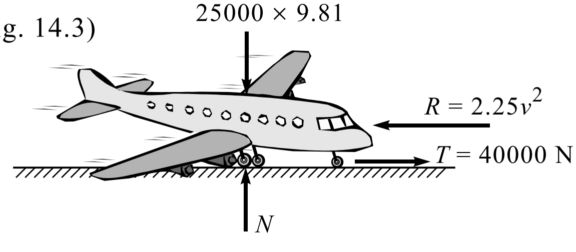


Fig. 14.3 : FBD of Plane

By Newton's second law, we have

$$\sum F_x = ma_x$$

$$40000 - 2.25v^2 = 25000a$$

$$40000 - 2.25v^2 = 25000 \times v\frac{dv}{ds}$$

$$\therefore\ ds = 25000\left(\frac{v\,dv}{40000 - 2.25v^2}\right)$$

Integrating both the sides, we get

$$\int_0^s ds = 25000\int_0^{66.67}\left(\frac{v\,dv}{40000 - 2.25v^2}\right) \qquad \left[v = 240 \times \frac{5}{18} = 66.67 \text{ m/s}\right]$$

$$s = \frac{25000}{-2.25 \times 2}\left[\log_e (40000 - 2.25\,v^2)\right]_0^{66.67}$$

$\therefore\ s = 1598.3$ m (Runway length) ***Ans.***

Problem 4

Two blocks A (mass 10 kg), B (mass 28 kg) are separated by 12 m, as shown in Fig. 14.4(a). If the blocks start moving, find the time 't' when the blocks collide. Assume μ = 0.25 for block A and plane and μ = 0.10 for block B and plane.

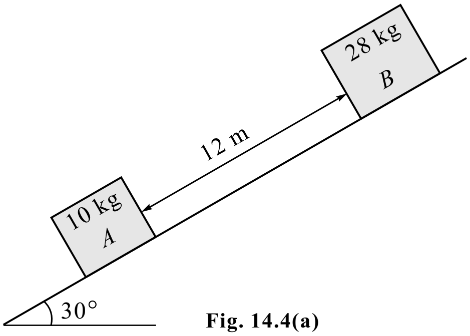


Fig. 14.4(a)

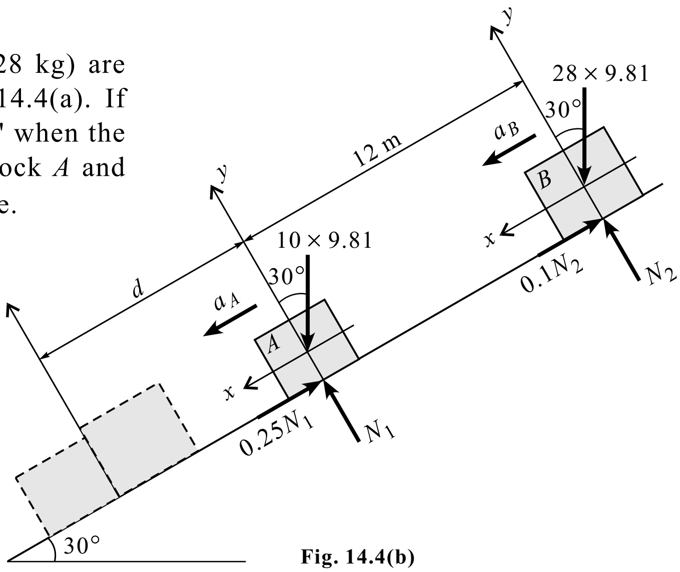


Fig. 14.4(b)

Solution

Refer to Fig. 14.4(b)

(i) Consider the FBD of Block *A*

By Newton's second law, we have

$$\sum F_x = ma_x$$

$$10 \times 9.81 \sin 30^\circ - 0.25 \times 10 \times 9.81 \cos 30^\circ = 10a_A$$

$$a_A = 2.781 \text{ m/s}^2 \ (30^\circ)$$

(ii) Consider the FBD of Block *B*

By Newton's second law, we have

$$\sum F_x = ma_x$$

$$29 \times 9.81 \sin 30^\circ - 0.1 \times 28 \times 9.81 \cos 30^\circ = 28a_B$$

$$a_B = 4.055 \text{ m/s}^2 \ (30^\circ)$$

(iii) Motion of Block *A*

$$d = 0 + \frac{1}{2} a_A t^2 \qquad \text{... (I)}$$

(iv) Motion of Block *B*

$$d + 12 = 0 + \frac{1}{2} a_B t^2 \qquad \text{... (II)}$$

(v) From Eqs. (I) and (II), we get

$$\frac{1}{2} \times 2.781 \times t^2 + 12 = \frac{1}{2} \times 4.055 \times t^2$$

$\therefore$ t = 4.34 seconds (Time when the blocks collide) ***Ans.***

Problem 5

An elevator being lowered into mine shaft starts from rest and attains a speed of 10 m/s within a distance of 15 metres. The elevator alone has a mass of 500 kg and it carries a box of mass 600 kg in it. Find the total tension in cables supporting the elevator, during this accelerated motion. Also find the total force between the box and the floor of the elevator.

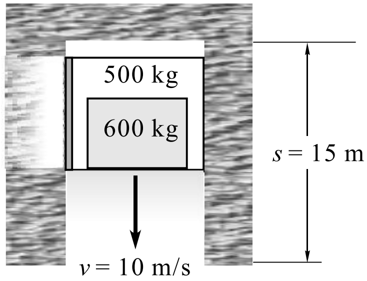


Fig. 14.5(a)

Solution

(i) Considering uniform acceleration of elevator

we have

$$v^2 = u^2 + 2as$$

$$10^2 = 0^2 + 2a \times 15$$

$$a = 3.33 \text{ m/s}^2$$ ***Ans.***

(ii) Considering the FBD of elevator with box

By Newton's second law, we have

$$\sum F_y = ma_y$$

$$(500 + 600)9.81 - T = (500 + 600)a$$

$$T = 1100 \times 9.81 - 1100 \times 3.33$$

$$T = 7128 \text{ N}$$ ***Ans.***

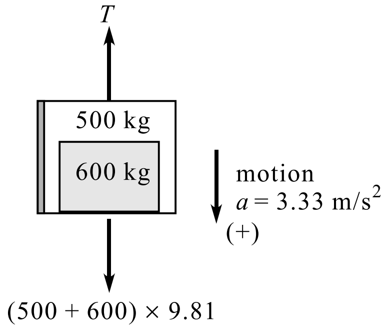


Fig. 14.5(b) : FBD of Elevator with Box

(iii) Consider the FBD of the box

Let N be the normal reaction exerted between the box and floor of the elevator.

$$\sum F_y = ma_y$$

$$600 \times 9.81 - N = 600 \times 3.33$$

$$N = 3888 \text{ N}$$ ***Ans.***

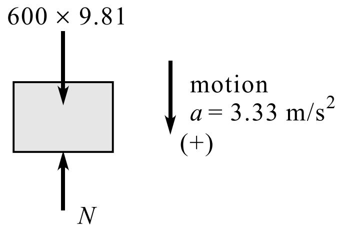


Fig. 14.5(c) : FBD of Box

Problem 6

Two weights W_1 = 400 N and W_2 = 100 N are connected by a string and move along a horizontal plane under the action of force P = 200 N applied horizontally to the weight W_1. The coefficient of friction between the weights and the plane is 0.25. Determine the acceleration of the weights and the tension in the string. Will the acceleration and tension in the string remain the same if the weights are interchanged ?

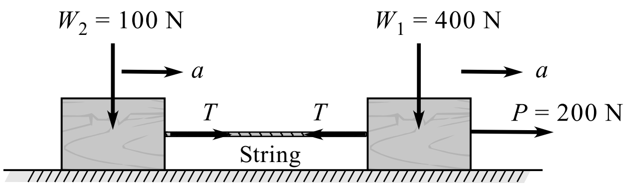


Fig. 14.6(a)

Solution

Note : Since both the block are connected by a single string, therefore, acceleration will remain same.

Case I

(i) Consider the FBD of Block W_1

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$$

$$N_1 - 400 = 0$$

$$N_1 = 400 \text{ N}$$

$$\sum F_x = ma_x$$

$$200 - T - \mu N_1 = \frac{400}{9.81} \times a$$

$$200 - T - 0.25 \times 400 = 40.78a$$

$$100 - T = 40.78a \qquad \text{..... (I)}$$

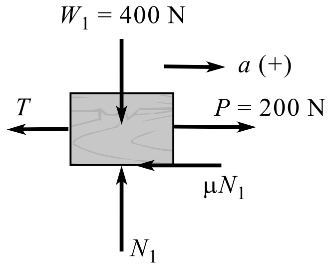


Fig. 14.6(b) : FBD of Block W_1

(ii) Consider the FBD of Block W_2

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$$

$$N_2 - 100 = 0$$

$$N_2 = 100 \text{ N}$$

$$\sum F_x = ma_x$$

$$T - \mu N_2 = \frac{100}{9.81} a$$

$$T - 0.25 \times 100 = 10.19a$$

$$T - 25 = 10.19a \qquad \text{..... (II)}$$

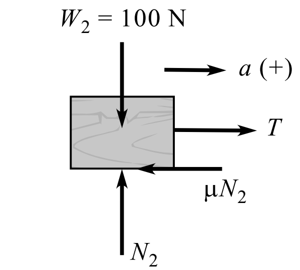


Fig. 14.6(c) : FBD of Block W_2

Solving Eqs. (I) and (II)

$T = 39.98\text{N}$ and $a = 1.47 \text{ m/s}^2$ ***Ans.***

Case II : Weights Interchanged

Refer to Fig. 14.6(d).

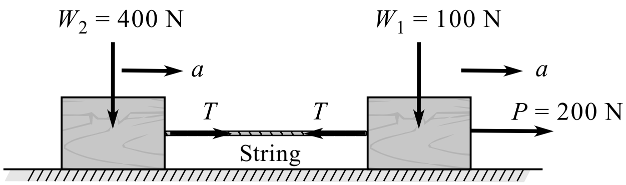


Fig. 14.6(d)

(i) Consider the FBD of Block W_1

By Newton's second law, we have

$$\sum F_y = a_y = 0 \quad (\because a_y = 0)$$

$$N_1 - 100 = 0$$

$$N_1 = 100 \text{ N}$$

$$\sum F_x = ma_x$$

$$200 - T - \mu N = \frac{100}{9.81} a$$

$$175 - T = 10.19a \qquad \text{..... (III)}$$

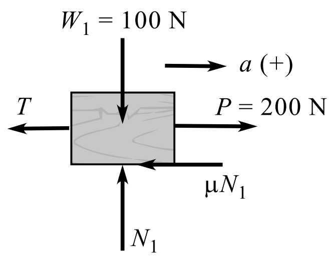


Fig. 14.6(e) : FBD of Block W_1

(ii) Consider the FBD of Block W_2

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$$

$$N_2 - 400 = 0 \quad \therefore N_2 = 400 \text{ N}$$

$$\sum F_x = ma_x$$

$$T - \mu N_2 = \frac{400}{9.81} a$$

$$T - 100 = 40.78a \quad ..... \text{ (IV)}$$

Solving Eqs. (III) and (IV)

$T = 160$ N and $a = 1.47$ m/s^2 ***Ans.***

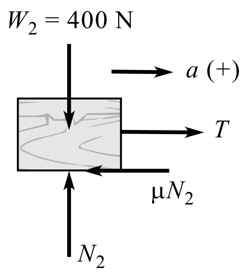


Fig. 14.6(f) : FBD of Block W_2

Referring to both the cases we can conclude that tension in string changes, if the position of weights is interchanged whereas acceleration remains same for both the cases.

Problem 7

The 100 kg crate shown in Fig. 14.7(a) is hoisted up by the incline using the cable and motor M. For a short time, the force in the cable is $F = 800\ t^2$ N where t is in seconds. If the crate has an initial velocity $v_1 = 2$ m/s when $t = 0$ s, determine the velocity when $t = 2$ s. The coefficient of kinetic friction between the crate and the incline is $\mu_k = 0.3$.

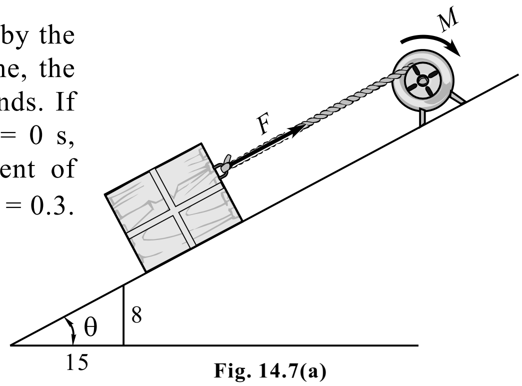


Fig. 14.7(a)

Solution

(i) Consider the FBD of the crate

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \qquad (\because a_y = 0)$$

$$N - 100 \times 9.81 \cos\theta = 0$$

$$N = 865.53 \text{ N} \qquad \left(\tan\theta = \frac{8}{15} \quad \therefore \theta = 28.07^o\right)$$

$$\sum F_x = ma_x$$

$$F - 100 \times 9.81 \sin\theta - \mu_k N = 100 \times a$$

$$800\ t^2 - 100 \times 9.81 \times \sin\theta - 0.3 \times 865.53 = 100\ a$$

$$a = 8t^2 - 7.213$$

$$\frac{dv}{dt} = 8t^2 - 7.213 = 8t^2 - 7.213$$

$$\int_{v_1 = 2 \text{ m/s}}^{v_2} dv = \int_{t_1 = 0}^{t_2 = 2 \text{ s}} (8t^2 - 7.213)\, dt$$

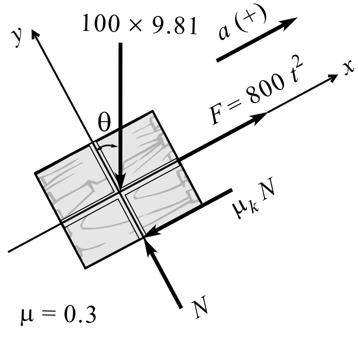


Fig. 14.7(b) : FBD of Crate

$$\Big[v\Big]_2^{v_2} = \left[\frac{8t^3}{3} - 7.213\right]_0^2 \quad \Rightarrow \quad v_2 - 2 = \left[\frac{8(2)^3}{3} - 7.213(2)\right] - \Big[0\Big]$$

$v_2 = 8.91$ m/s ***Ans.***

Problem 8

A body of mass 25 kg resting on a horizontal table is connected by string passing over a smooth pulley at the edge of the table to another body of mass 3.75 kg and hanging vertically, as shown in Fig. 14.8(a). Initially, the friction between 25 kg mass and the table is just sufficient to prevent the motion. If an additional 1.25 kg is added to the 3.75 kg mass, find the acceleration of the masses.

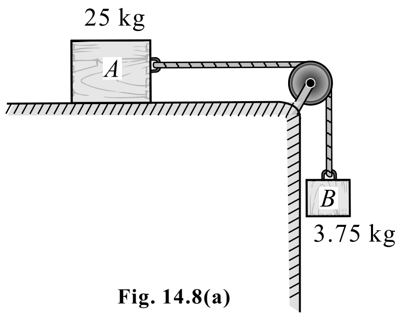


Fig. 14.8(a)

Solution

(i) Static equilibrium analysis

Consider the FBD of block *A*

$$\sum F_y = 0$$

$$N - 25 \times 9.81 = 0$$

$$N = 245.25\text{N}$$

$$\sum F_x = 0$$

$$T - \mu N = 0$$

$$3.75 \times 9.81 - \mu \times 245.25 = 0$$

$$\mu = 0.15$$

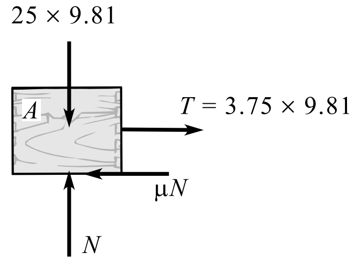


Fig. 14.8(b) : FBD of Block *A*

(ii) Dynamic equilibrium analysis

Assume $\mu_s = \mu_k = 0.15$

By Newton's second law, we have

Consider the *FBD* of block *A*

$$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$$

$$N - 25 \times 9.81 = 0$$

$$N = 245.25\text{N}$$

$$\sum F_x = ma_x$$

$$T - \mu N = 25a$$

$$T - 0.15 \times 245.25 = 25a$$

$$T = 36.79 + 25a \qquad \text{..... (I)}$$

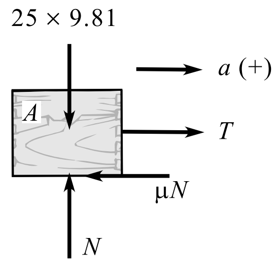


Fig. 14.8(c) : FBD of Block *A*

(iii) Consider the FBD of Block *B*

By Newton's second law, we have

$$\sum F_y = ma_y$$

$$5 \times 9.81 - T = 5 \times a$$

$$T = 49.05 - 5a \qquad \text{..... (II)}$$

Equating Eqs. (I) and (II), we get

$a = 0.409\ \text{m/s}^2$ and $T = 47.005$ N ***Ans.***

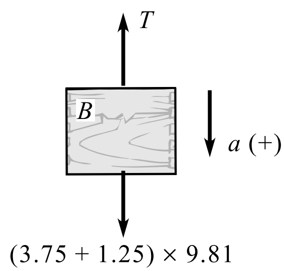


Fig. 14.8(d) : FBD of Block *B*

Problem 9

A horizontal force P = 600 N is exerted on block A of mass 120 kg, as shown in Fig. 14.9(a). The coefficient of friction between block A and the horizontal plane is 0.25. Block B has a mass of 30 kg and the coefficient of friction between it and the plane is 0.4. The wire between the two blocks makes 30° with the horizontal. Calculate the tension in the wire.

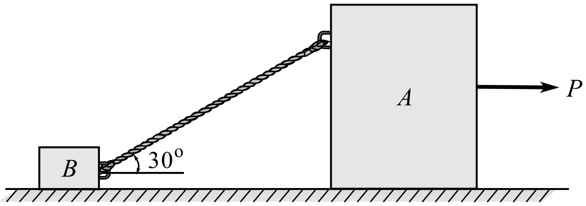


Fig. 14.9(a)

Solution

As both the blocks are connected by a single wire, the acceleration of both the blocks will be the same.

(i) Consider the FBD of Block B

By Newton's second law, we have

$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$

$N_B + T \sin 30^\circ - 30 \times 9.81 = 0$

$N_B = 294.3 - T \sin 30^\circ$

$\sum F_x = ma_x$

$T \cos 30^\circ - \mu_B N_B = 30 \times a$

$T \cos 30^\circ - 0.4\,(294.3 - T \sin 30^\circ) = 30a$

$T - 28.14a = 110.43$ ..... (I)

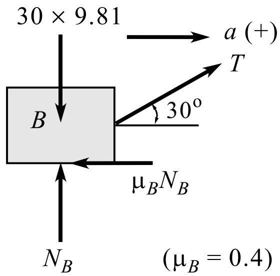


Fig. 14.9(b) : FBD of Block B

(ii) Consider the FBD of Block A

By Newton's second law, we have

$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$

$N_A - T \sin 30^\circ - 120 \times 9.81 = 0$

$N_A = 1177.2 + T \sin 30^\circ$

$\sum F_x = ma_x$

$600 - \mu_A N_A - T \cos 30^\circ = 120 \times a$

$600 - 0.25(1177.2 + T \sin 30^\circ) - T \cos 30^\circ = 120\,a$

$T + 121.09a = 308.476$ ..... (II)

Solving Eqs. (I) and (II), we get

$T = 147.78$ N and $a = 1.327$ m/s^2 ***Ans.***

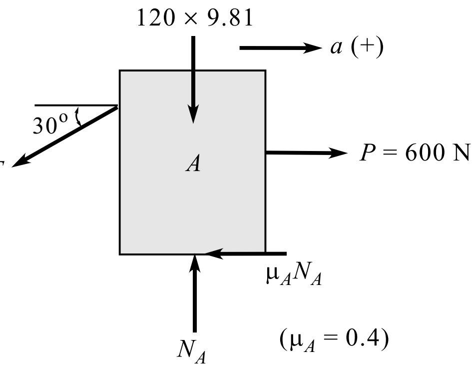


Fig. 14.9(c) : FBD of Block A

Problem 10

Masses A and B are 7.5 kg and 27.5 kg respectively as shown in Fig. 14.10(a). The coefficient of friction between A and the plane is 0.25 and between B and the plane is 0.1. What is the force between the two as they slide down the incline?

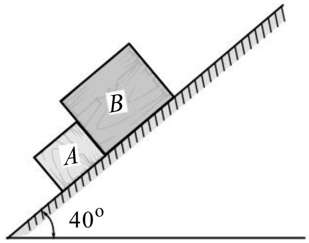


Fig. 14.10(a)

Solution

(i) Consider the FBD of Block *A*

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$$

$$N_A - 7.5 \times 9.81 \cos 40^\circ = 0$$

$$N_A = 56.36 \text{ N}$$

$$\sum F_x = ma_x$$

$$P + 7.5 \times 9.81 \sin 40^\circ - 0.25 \times 56.36 = 7.5a$$

$$33.2 + P = 7.5a \quad \text{..... (I)}$$

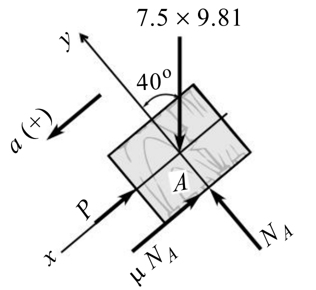


Fig. 14.10(b) : FBD of Block *A*

(ii) Consider the FBD of Block *B*

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$$

$$N_B - 27.5 \times 9.81 \cos 40^\circ = 0$$

$$N_B = 206.66 \text{ N}$$

$$\sum F_x = ma_x$$

$$27.5 \times 9.81 \sin 40^\circ - P - 0.1 \times 206.66 = 27.5a$$

$$152.74 - P = 27.5a \quad \text{..... (II)}$$

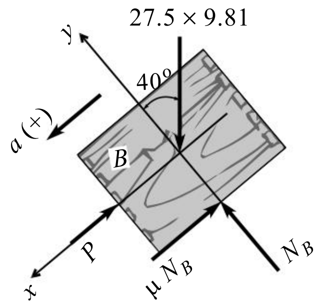


Fig. 14.10(c) : FBD of Block *B*

Solving Eqs. (I) and (II)

$P = 6.625$ and $a = 5.31 \text{ m/s}^2$ ***Ans.***

Problem 11

In the system of pulleys, the pulleys are massless and the strings are inextensible. Mass of A = 2 kg, mass of B = 4 kg and mass = 6 kg as shown in Fig. 14.11(a). If the system is released from rest, find **(i)** tension in each of the three string and **(ii)** acceleration of each of the three masses.

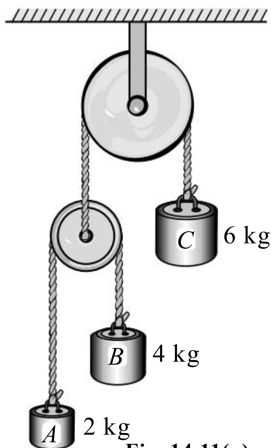


Fig. 14.11(a)

Solution

Assume the direction of motion of all block as above.

(i) Kinematic relation

$$Tx_A + Tx_B + 2Tx_C = 0$$

$$x_A + x_B + 2x_C = 0$$

Differentiating w.r.t. t

$$v_A + v_B + 2v_C = 0$$

Differentiating w.r.t. t again,

$$a_A + a_B + 2a_C = 0 \qquad \text{..... (I)}$$

(ii) Consider the FBD of Block *A*

$$\sum F_y = ma_y$$

$$T = 2 \times 9.81 = 2a_A$$

$$a_A = 0.5T - 9.81 \qquad \text{..... (II)}$$

(iii) Consider the FBD of Block *B*

$$\sum F_y = ma_y$$

$$T - 4 \times 9.81 = 4a_B$$

$$a_B = 0.25T - 9.81 \qquad \text{..... (III)}$$

(iv) Consider the FBD of Block *C*

$$\sum F_y = ma_y$$

$$2T - 6 \times 9.81 = 6a_C$$

$$a_C = 0.33T - 9.81 \qquad \text{..... (IV)}$$

(v) Putting Eqs. (II), (III) and (IV) in Eq. (I)

$$a_A + a_B + a_C = 0$$

$$(0.5T - 9.81) + (0.25T - 9.81) + (0.33T - 9.81) = 0$$

$$0.5T + 0.25T + 0.33T - 9.81 - 9.81 - 9.81 = 0$$

$$1.08T - 29.43 = 0$$

$$T = 27.25 \text{ N} \quad \textit{\textbf{Ans.}}$$

(vi) From Eq. (I)

$$a_A = 0.5 \times 27.25 - 9.81$$

$$a_A = 3.82 \text{ m/s}^2 \ (\uparrow) \quad \textit{\textbf{Ans.}}$$

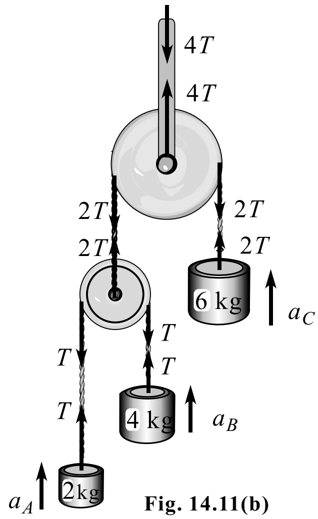


Fig. 14.11(b)

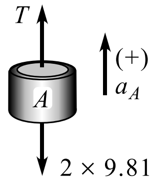


Fig. 14.11(c) : FBD of Block *A*

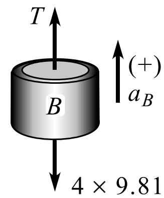


Fig. 14.11(d) : FBD of Block *B*

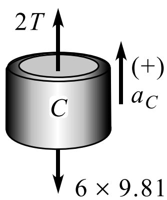


Fig. 14.11(e) : FBD of Block *C*

(vii) From Eq. (II)

$$a_B = 0.25 \times 27.25 - 9.81$$

$$a_B = -3 \text{ m/s}^2 \text{ (Wrong assumed direction)}$$

$$a_B = 3 \text{ m/s}^2 \ (\downarrow) \quad \textit{\textbf{Ans.}}$$

(viii) From Eq. (III)

$$a_C = 0.33 \times 27.25 - 9.81$$

$$a_C = -0.82 \text{ m/s}^2 \text{ (Wrong assumed direction)}$$

$$a_C = 0.82 \text{ m/s}^2 \ (\downarrow) \quad \textit{\textbf{Ans.}}$$

Problem 12

Determine the tension developed in chords attached to each block and the accelerations of the blocks when the system, shown in Fig. 14.12(a), is released from rest. Neglect the mass of the pulleys and chords.

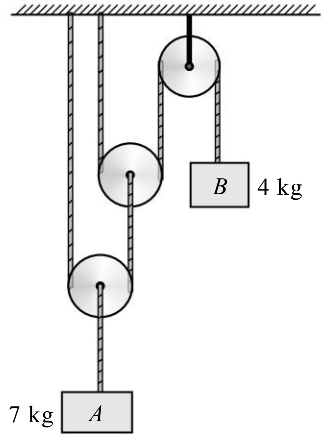


Fig. 14.12(a)

Solution

(i) Kinematic relation

Work done by internal forces = 0

$4Tx_A - Tx_B = 0$

$4x_A - x_B = 0$

Differentiating w.r.t. t

$4v_A - v_B = 0$

Differentiating w.r.t. t

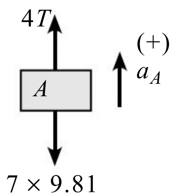
$4a_A - a_B = 0$ ..... (I)

(ii) Consider the FBD of Block *A*

$\sum F_y = ma_y$

$4T - 7 \times 9.81 = 7a_A$

$a_A = 0.5714T - 9.81$ ..... (II)



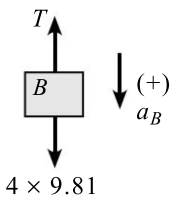
FBD of Block *A*

(iii) Consider the FBD of Block *B*

$\sum F_y = ma_y$

$4 \times 9.81 - T = 4a_B$

$a_B = 9.81 - 0.25T$ ..... (III)



FBD of Block *B*

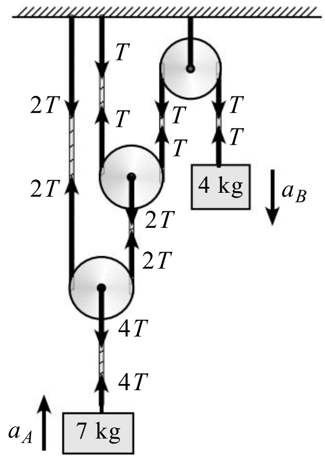


Fig. 14.12(b)

(v) Putting Eqs. (II) and (III) in Eq. (I)

$4(0.5714T - 9.81) - (9.81 - 0.25T) = 0$

$2.286T - 39.24 - 9.81 + 0.25T = 0$

$T = 19.34$ N (Tension in cord attached to block A) ***Ans.***

$4T = 77.36$ N (Tension in cord attached to block B) ***Ans.***

(vi) From Eqs. (II) and (III), we get

$a_A = 0.5714 \times 19.34 - 9.81$

$a_A = 1.241 \text{ m/s}^2$ $(\uparrow)$ ***Ans.***

$a_B = 9.81 - 0.25 \times 19.34$

$a_B = 4.975 \text{ m/s}^2$ $(\downarrow)$ ***Ans.***

Problem 13

Block A = 100 kg, shown in Fig. 14.13(a), is observed to move upward with an acceleration of 1.8 m/s^2. Determine **(i)** mass of block B and **(ii)** the corresponding tension in the cable.

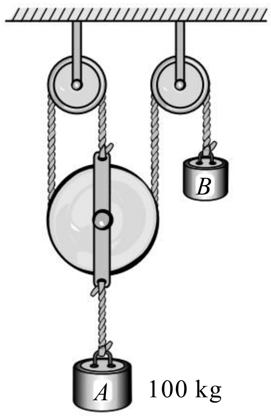


Fig. 14.13(a)

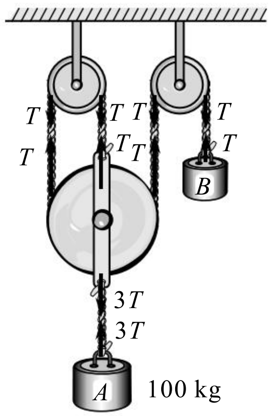


Fig. 14.13(b)

Solution

(i) Kinematic relation

Work done by internal forces $= 0$

$$3Tx_A - Tx_B = 0$$

$$3x_A = x_B$$

Differentiating w.r.t. t

$$3v_A = v_B$$

Differentiating w.r.t. t

$$3a_A = a_B$$

$$\therefore \; a_B = 3 \times 1.8 \qquad (\because \; a_A = 1.8 \text{ m/s}^2)$$

$$a_B = 5.4 \text{ m/s}^2$$

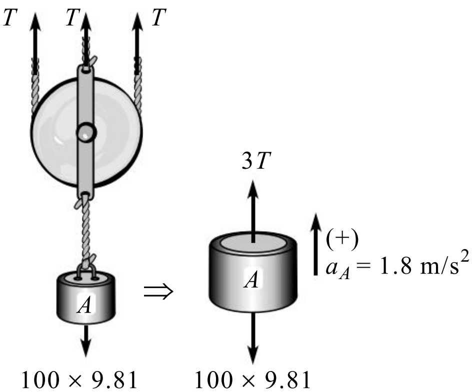


Fig. 14.13(c) : FBD of Block *A*

(ii) Consider the FBD of Block *A*

By Newton's second law, we have

$$\sum F_y = ma_y$$

$$3T - 100 \times 9.81 = 100 \times 1.8$$

$T = 387$ N ***Ans.***

(iii) Consider the FBD of Block *B*

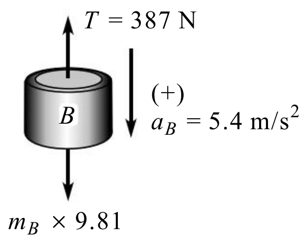


Fig. 14.13(d) : FBD of Block *B*

By Newton's second law, we have

$$\sum F_y = ma_y$$

$$m_B \times 9.81 - T = m_B \times a_B$$

$$m_B\,(9.81 - 5.4) = 387$$

$m_B = 87.76$ kg ***Ans.***

Problem 14

At a given instant the 50 N block *A* is moving downward with a speed of 1.8 m/s. Determine its speed 2 s later. Block *B* has a weight 20 N, and the coefficient of kinetic friction between it and the horizontal plane is $\mu_k = 0.2$. Neglect the mass of pulley's and chord. Use D'Alemberts principle.

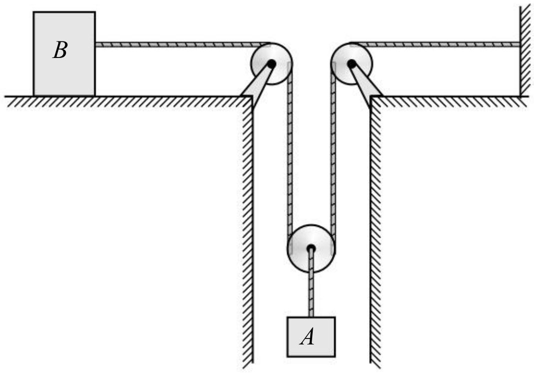


Fig. 14.14(a)

Solution

(i) Kinematic relation

$$Tx_B - 2Tx_A = 0$$

$$x_B - 2x_A = 0$$

Differentiating w.r.t. t

$$v_B - 2v_A = 0$$

Differentiating w.r.t. t again

$$a_B - 2a_A = 0$$

$$a_B = 2a_A \qquad \text{..... (I)}$$

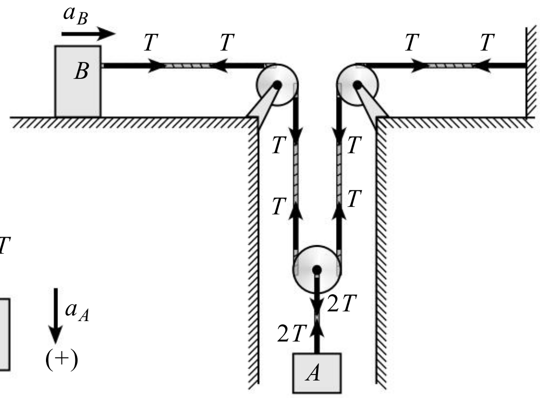


Fig. 14.14(b)

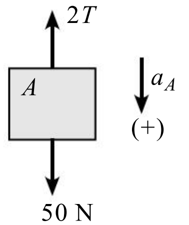
(ii) Consider the FBD of Block *A*

By D'Alembert's principle, we have

$$\sum F_y + (-ma_y) = 0$$

$$50 - 2T - \frac{50}{9.81} a_A = 0$$

$$T = 25 - 2.548\, a_A \qquad \text{..... (II)}$$



FBD of Block *A*

(iii) Consider the FBD of Block *B*

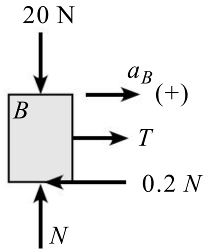
By D'Alembert's principle, we have

$$\sum F_x + (-ma_x) = 0$$

$$T - 0.2N - \frac{20}{9.81} a_B = 0$$

$$T - 0.2 \times 20 - \frac{20}{9.81} a_B = 0$$

$$T = 4 + 2.039\, a_B \qquad \text{..... (III)}$$



FBD of Block *B*

(iv) Equating Eqs. (II) and (III)

$$25 - 2.548\, a_A = 4 + 2.039\, a_B$$

$$2.548\, a_A + 2.039(2a_A) = 25 - 4$$

$$6.626\, a_A = 21$$

$$a_A = 3.169 \text{ m/s}^2\ (\downarrow)$$

(v) Speed = ? after 2 sec.

$$v = u + at$$

$$v_A = 1.8 + 3.169 \times 2 = 8.138 \text{ m/s}\ (\downarrow) \qquad \textbf{\textit{Ans.}}$$

Problem 15

Two blocks, shown in Fig. 14.15(a), start from rest. If the cord is inextensible, friction and inertia of pulley are negligible, calculate acceleration of each block and tension in each cord. Consider coefficient of friction as 0.25.

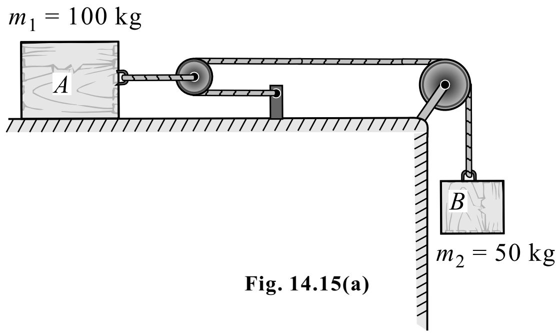


Fig. 14.15(a)

Solution

(i) Kinematic relation

Work done by internal forces $= 0$

$$2T \times x_1 - Tx_2 = 0$$

$$2x_1 = x_2$$

Differentiating w.r.t. t

$$2v_1 = v_2$$

Differentiating w.r.t. t again,

$$2a_1 = a_2$$

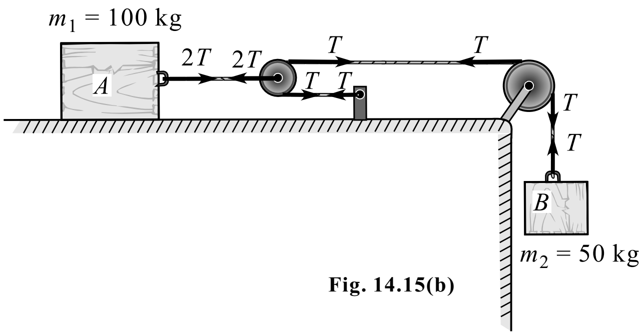


Fig. 14.15(b)

(ii) Consider the FBD of Block m_1

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \qquad (\because\ a_y = 0)$$

$$N_1 - 100 \times 9.81 = 0$$

$$N_1 = 981 \text{ N}$$

$$\sum F_x = ma_x$$

$$2T - \mu N_1 = 100a_1$$

$$2T - 0.25 \times 981 = 100a_1$$

$$T = 122.625 + 50a_1 \qquad \text{..... (I)}$$

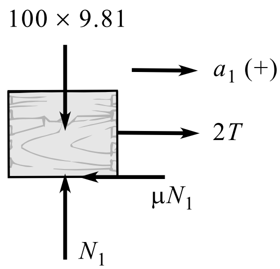


Fig. 14.15(c) : FBD of Block m_1

(iii) Consider the FBD of Block m_2

By Newton's second law, we have

$$\sum F_y = ma_y$$

$$50 \times 9.81 - T = 50a_2$$

$$T = 490.5 - 100a_2 \qquad \text{..... (II)}$$

Solving Eqs. (I) and (II), we get

$T = 245.25$ N; $\quad a_1 = 2.45$ m/s^2 ***Ans.***

$2T = 490.5$ N; $\quad a_2 = 4.9$ m/s^2 ***Ans.***

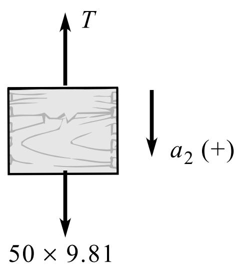


Fig. 14.15(d) : FBD of Block m_2

Problem 16

A system shown in Fig. 14.16(a) is at rest initially. Neglecting friction determine velocity of block A after it has moved 2.7 m when pulled by a force of 90 N.

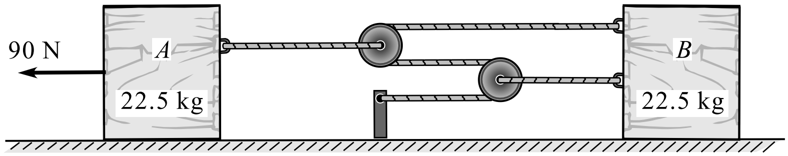


Fig. 14.16(a)

Solution

(i) Kinematic relation

Work done by internal forces = 0

$$\therefore\ 3Tx_B - 2Tx_A = 0$$

$$3x_B = 2x_A$$

Differentiating w.r.t. t

$$3v_B = 2v_A$$

Differentiating w.r.t. t

$$3a_B = 2a_A$$

$$a_B = \frac{2}{3}\,a_A$$

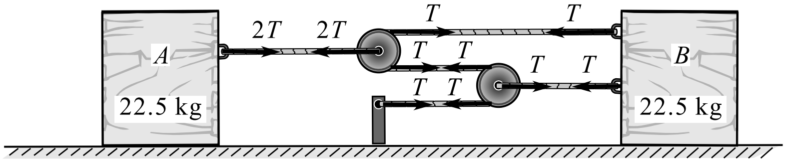


Fig. 14.16(b)

(ii) Consider the FBD of Block A

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \qquad (\because\ a_y = 0)$$

$$N_A - 22.5 \times 9.81 = 0$$

$$N_A = 22.5 \times 9.81 = 220.725 \text{ N}$$

$$\sum F_x = ma_x$$

$$90 - 2T = 22.5 \times a_A \qquad \text{..... (I)}$$

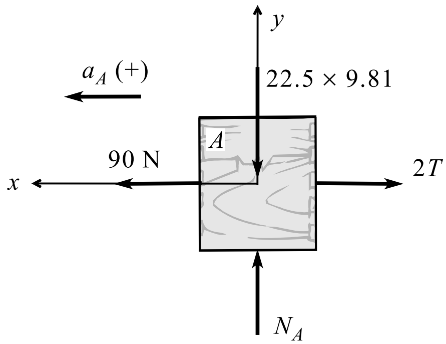


Fig. 14.16(c) : FBD of Block A

(iii) Consider the FBD of Block B

By Newton's second law, we have

$$\sum F_y = ma_y \qquad (\because\ a_y = 0)$$

$$N_B - 22.5 \times 9.81 = 0 \quad \therefore\ N_B = 220.725 \text{ N}$$

$$\sum F_x = ma_x$$

$$T + 2T = 22.5 \times a_B$$

$$3T = 22.5a_B$$

$$T = \frac{22.5a_B}{3} \qquad \text{..... (II)}$$

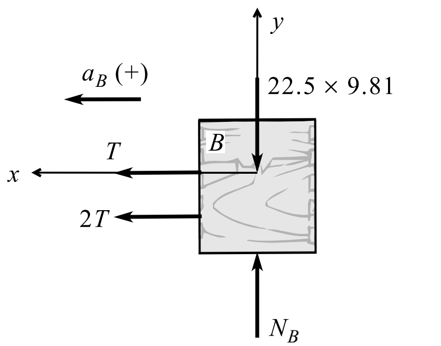


Fig. 14.16(d) : FBD of Block B

Putting Eq. (II) in Eq. (I)

$$90 - 2\left(\frac{22.5a_B}{3}\right) = 22.5a_A$$

From kinematic relation $a_B = \dfrac{2}{3}a_A$

$$90 - 2 \times \frac{22.5}{3} \times \frac{2}{3} a_A = 22.5a_A$$

$$90 = 32.5a_A$$

$$a_A = 2.77 \text{ m/s}^2$$

$$a_B = \frac{2}{3} a_A = \frac{2}{3} \times 2.77 \quad \therefore \; a_B = 1.85 \text{ m/s}^2$$

(iv) $x_A = 2.7$ m; $a_A = 2.77$ m/s^2, $u = 0$

$$v^2 = u^2 + 2as = 0 + 2 \times 2.77 \times 2.7$$

$v = 3.87$ m/s ***Ans.***

Problem 17

Masses $A = 5$ kg, $B = 10$ kg and $C = 20$ kg are connected as shown in Fig. 14.17(a) by inextensible cord passing over massless and frictionless pulleys. The coefficients of friction for mass A and B and ground is 0.2. If the systems is released from rest, find the acceleration. a_A, a_B and a_C and tension T in the cord. Present your answer in tabular form.

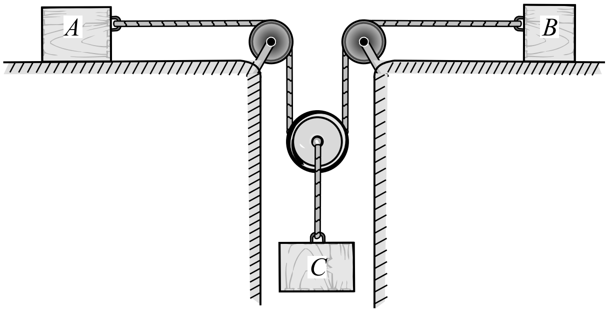


Fig. 14.17(a)

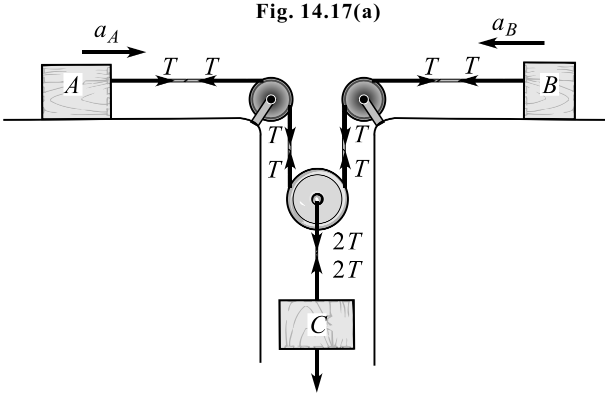


Fig. 14.17(b)

Solution

(i) Kinematic relation

$$Tx_A + Tx_B - 2Tx_C = 0$$

$$x_A + x_B - 2x_C = 0$$

Differentiating w.r.t. t

$$v_A + v_B - 2v_C = 0$$

Differentiating w.r.t. t again

$$a_A + a_B - 2a_C = 0$$

$$a_C = \frac{a_A + a_B}{2}$$

(ii) Consider the FBD of Block A

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \qquad (\because a_y = 0)$$

$$N_A - 5 \times 9.81 = 0 \qquad \therefore N_A = 49.05 \text{ N}$$

$$\sum F_x = ma_x$$

$$T - \mu N_A = 5 \times a_A$$

$$T - 0.2 \times 49.05 = 5a_A$$

$$T - 9.81 = 5a_A$$

$$a_A = \frac{T - 9.81}{5} \qquad \ldots\ldots \text{(I)}$$

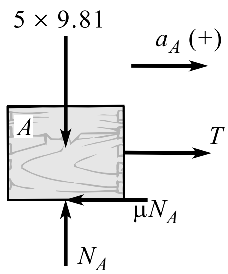


Fig. 14.17(c) : FBD of Block *A*

(iii) Consider the FBD of Block *B*

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \qquad (\because\ a_y = 0)$$

$$N_B - 10 \times 9.81 = 0$$

$$N_B = 98.1 \text{ N}$$

$$\sum F_x = ma_x$$

$$T - \mu N_B = 10a_B$$

$$T - 0.2 \times 98.1 = 10a_B$$

$$T - 19.62 = 10a_B$$

$$a_B = \frac{T - 19.62}{10} \qquad \text{..... (II)}$$

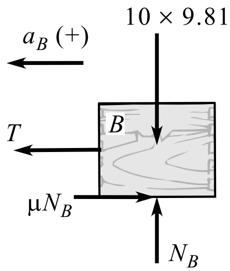


Fig. 14.17(d) : FBD of Block *B*

(iv) Consider the FBD of Block *C*

By Newton's second law, we have

$$\sum F_y = ma_y$$

$$20 \times 9.81 - 2T = 20a_C$$

$$20 \times 9.81 - 2T = 20\left[\frac{a_A + a_B}{2}\right]$$

$$196.2 - 2T = 10(a_A + a_B) \qquad \text{..... (III)}$$

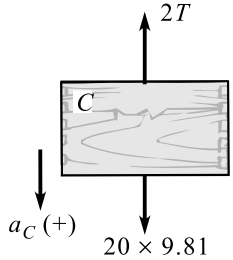


Fig. 14.17(e) : FBD of Block *C*

Putting Eqs. (I) and (II) in (III)

$$196.2 - 2T = 10\left[\left(\frac{T - 9.81}{5}\right) + \left(\frac{T - 19.62}{10}\right)\right]$$

$$196.2 - 2T = 2(T - 9.81) + (T - 19.62)$$

$$5T = 235.44$$

$$T = 47.09 \text{ N} \quad \textbf{\textit{Ans.}}$$

From Eq. (I)

$$a_A = \frac{47.09 - 9.81}{5} = 7.456 \text{ m/s}^2$$

From Eq. (II)

$$a_B = \frac{47.09 - 19.62}{10} = 2.75 \text{ m/s}^2$$

From kinematic relation

$$a_C = \frac{a_A + a_B}{2} = \frac{7.456 + 2.75}{2} = 5.103 \text{ m/s}^2$$

a_A	a_B	a_C	T
7.456 m/s^2	2.75 m/s^2	5.103 m/s^2	47.09 N

Ans.

Problem 18

Masses A (5 kg), B (10 kg), C (20 kg) are connected, as shown in the Fig. 14.18(a) by inextensible cord passing over massless and frictionless pulleys. The coefficient of friction for masses A and B with ground is 0.2. If the system is released from rest, find the acceleration of the blocks and tension in the cords.

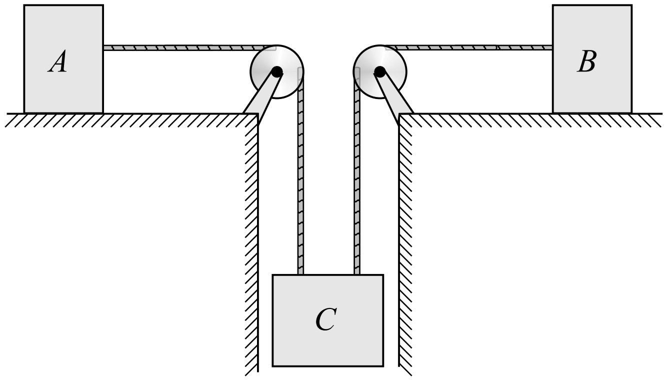


Fig. 14.18(a)

Solution

(i) Kinematic relation

All the three blocks are connected directly to each other.

$\therefore$ Acceleration of all the three blocks will be same.

$a_A = a_B = a_C = a$ ..... (I)

(ii) Consider the FBD of Block A

By Newton's second law, we have

$\sum F_x = ma_x$

$T_1 - 0.2N_A = 5a_A$

$T_1 - 0.2 \times 5 \times 9.81 = 5a_A$

$T_1 = 5a_A + 9.81$ ..... (II)

(iii) Consider the FBD of Block B

By Newton's second law, we have

$\sum F_x = ma_x$

$T_2 - 0.2N_B = 10a_B$

$T_2 = 0.2 \times 10 \times 9.81 + 10a_B$

$T_2 = 19.62 + 10a_B$ ..... (III)

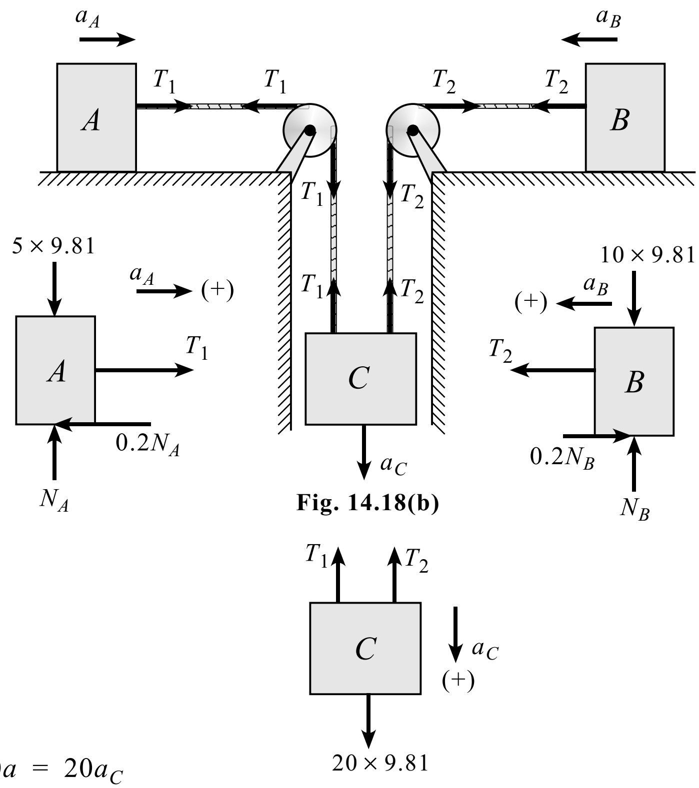


Fig. 14.18(b)

(iv) Consider the FBD of Block C

By Newton's second law, we have

$20 \times 9.81 - T_1 - T_2 = 20a_C$

From Eq. (II) and (III)

$20 \times 9.81 - 5a - 9.81 - 19.62 - 10a = 20a_C$

$35a = 166.77$

$a = 4.765 \text{ m/s}^2$

(v) Substituting the value of a in Eqs. (II) and (III), we get

$T_1 = 5 \times 4.765 + 9.81$ $T_2 = 19.62 + 10 \times 4.765$

$T_1 = 33.63$ N ***Ans.*** $T_2 = 67.27$ N ***Ans.***

14.5 Solved Problems Based on Curvilinear Motion

Problem 19

The pendulum bob has a mass m and is released from rest when $\theta = 0°$, as shown in Fig. 14.19(a). For any position B of the pendulum, determine **(i)** the tangential component of acceleration a_t and obtain its velocity v by integration and **(ii)** the value of θ at which the cord will break knowing that it can withstand a maximum tension equal to twice the weight of the pendulum bob. Take length of cord l and neglect the size of the bob.

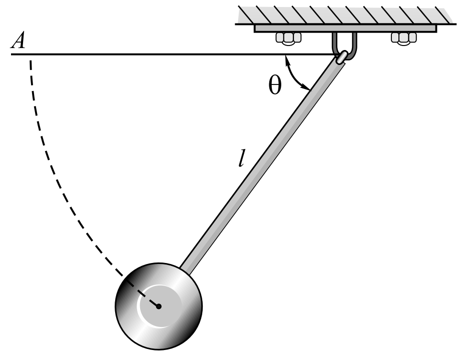


Fig. 14.19(a)

Solution

Consider the FBD of bob at Position *B*

$$\sum F_y = ma_t$$

$$mg \cos\theta = ma_t$$

$$a_t = g\cos\theta \quad \textit{\textbf{Ans.}}$$

$$v\frac{dv}{ds} = g\cos\theta$$

$$v\,dv = g\cos\theta\,ds$$

Integrating $v\,dv = g\cos\theta\,(l\,d\theta) \quad (\because\ ds = l\,d\theta)$

$$\int v\,dv = \int gl\cos\theta\,d\theta$$

$$\frac{v^2}{2} = gl\sin\theta + c$$

At $v = 0,\ u = 0 \quad \therefore\ c = 0$

$$v^2 = 2gl\sin\theta$$

$$v = \sqrt{2gl\sin\theta} \quad \textit{\textbf{Ans.}}$$

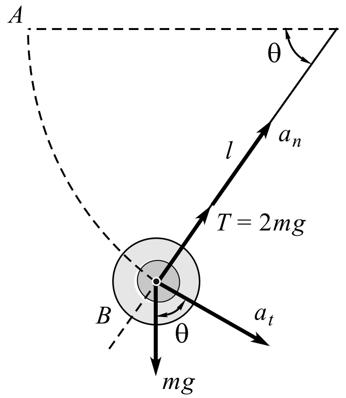


Fig. 14.19(b) : FBD of Bob at Position *B*

$$\sum F_n = ma_n = m\frac{v^2}{l} \qquad (\because\ l = \text{radius of curvature})$$

$$T - mg\sin\theta = m\frac{v^2}{l}$$

$$2mg - mg\sin\theta = \frac{m}{l}(2gl\sin\theta)$$

$$2 - \sin\theta = 2\sin\theta$$

$$2 = 3\sin\theta$$

$$\sin\theta = \frac{2}{3} \qquad \therefore\ \theta = 41.81°$$

Thus at $\theta = 41.81°$ the cord will break. ***Ans.***

Problem 20

Two wires AC and BC are tied at C to a sphere of mass 5 kg which revolves at a constant speed v in the horizontal circle of radius 1.5 m, as shown in Fig. 14.20(a). Determine the minimum and maximum value of v if both the wires are to be remain taut and tension in either of the wires is not to exceed 70 N.

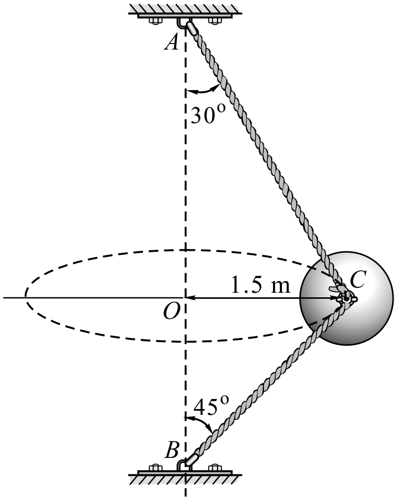


Fig. 14.20(a)

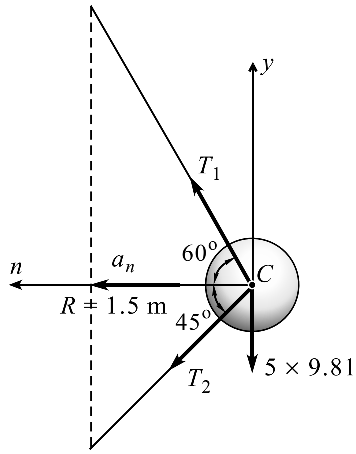


Fig. 14.20(b) : FBD of Sphere *C*

Solution

Consider the FBD of the sphere

$\sum F_y = ma_y = 0 \qquad (\because a_y = 0)$

$T_1 \sin 60^\circ - T_2 \sin 45^\circ - 5 \times 9.81 = 0 \qquad \text{..... (I)}$

$\sum F_n = ma_n = m \dfrac{v^2}{R}$

$T_1 \cos 60^\circ + T_2 \cos 45^\circ = \dfrac{5 \times v^2}{1.5} \qquad \text{..... (II)}$

The arrangement is such that T_1 is always greater than T_2

For minimum velocity $T_2 = 0$

From Eq. (I), we get

$T_1 = 56.64$ N

From Eq. (II), we get

$v_{min} = 2.914$ m/s ***Ans.***

For maximum velocity $T_1 = 70$ N (given)

From Eq. (I), we get

$T_2 = 16.36$ N

From Eq. (II), we get

$v_{max} = 3.74$ m/s ***Ans.***

Problem 21

An automobile weighing 12 kN is moving with uniform speed of 72 kmph. Over a vertical curve *ABC* of parabolic shape. Determine the total pressure exerted by the wheels of the automobile as it passes the topmost point *B*, as shown in Fig. 14.21(a).

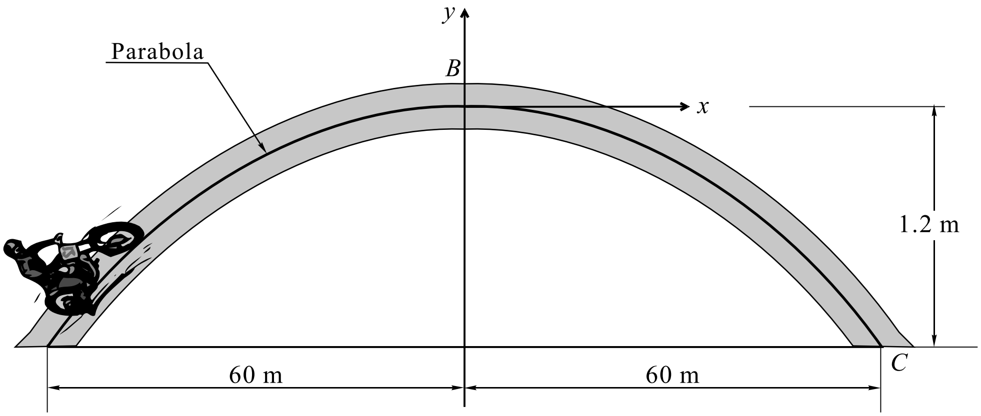


Fig. 14.21(a)

Solution

We know, equation of parabola is $x^2 = -ky \quad \therefore \; y = -\dfrac{x^2}{k}$

Coordinate of point C (60, −1.5)

$$-1.5 = -\frac{60^2}{k}$$

$$k = 3000$$

$$\therefore \; y = -\frac{x^2}{3000}$$

$$\frac{dy}{dx} = \frac{-2x}{3000} = \frac{-2}{3000} = \frac{-1}{1500} \quad \Bigg| \quad \text{At point } B\text{, slope, i.e., } \frac{dy}{dx} = 0$$

$$\text{Radius of curvature } R = \left| \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} \right| = \left| \frac{\left[1 + (0)^2\right]^{3/2}}{-\dfrac{1}{1500}} \right|$$

$$R = 1500 \text{ m}$$

Consider the FBD of automobile at Point *B*

$$\sum F_n = ma_n = \frac{mv^2}{R}$$

$$12 \times 10^3 - N = \frac{12 \times 10^3}{9.81} \times \frac{20^2}{1500}$$

$$N = 326.2 \text{ N} \quad \textbf{\textit{Ans.}}$$

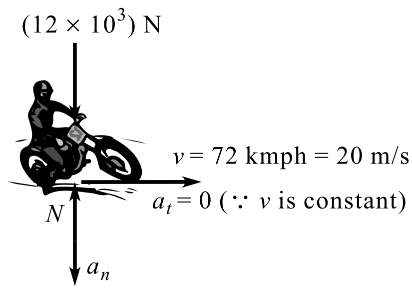


Fig. 14.21(b) : FBD of Automobile at Point *B*

Problem 22

A van of mass 1000 kg travels at constant speed along a vertical curve as shown in Fig. 14.22(a). Find the maximum speed at which the van may travel so that it would remain in contact with the road at all time. At this speed, find the reaction from the ground when the van reaches point *B*.

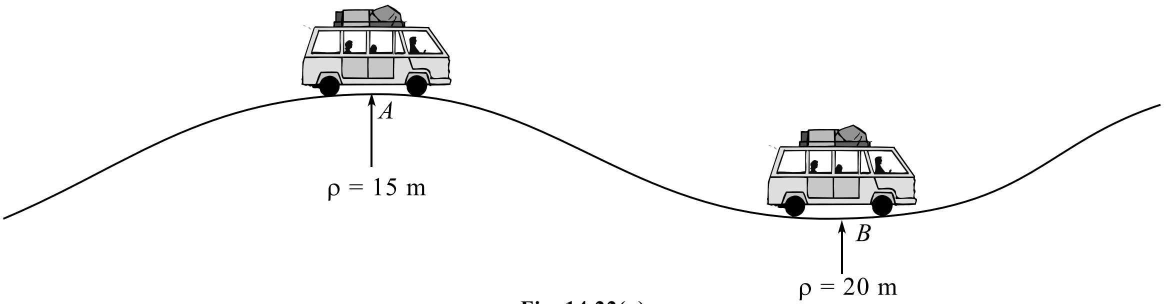


Fig. 14.22(a)

Solution

(i) Van is performing curvilinear motion

At position *A*, draw FBD [Refer to Fig. 14.22(b)]

By Newton's second law, we have

$$\sum F_x = ma_x = \frac{mv^2}{\rho}$$

$$1000 \times 9.81 - N = \frac{1000 \times v^2}{15}$$

[$N = 0$ because van is just in contact and about to jump the limiting condition]

$$v^2 = 9.81 \times 15$$

$$v = 12.13 \text{ m/s} \quad \textbf{\textit{Ans.}}$$

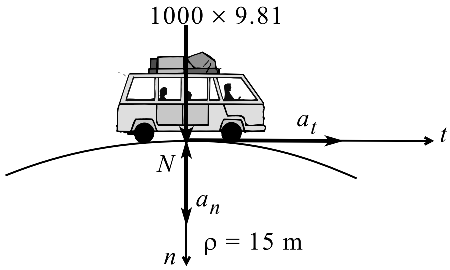


Fig. 14.22(b) : FBD at Position *A*

(ii) Consider the FBD at Position *B* [Refer to Fig. 14.22(c)]

By Newton's second law, we have

$$\sum F_x = ma_x = \frac{mv^2}{\rho}$$

$$N - 1000 \times 9.81 = \frac{1000 \times 12.13^2}{20}$$

$$N = 17166.85 \text{ N} \quad \textbf{\textit{Ans.}}$$

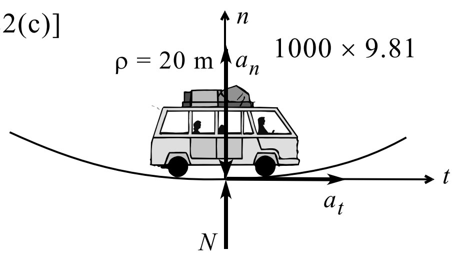


Fig. 14.22(c) : FBD at Position *B*

Problem 23

A 10 kg sphere is connected to two strings, as shown in Fig. 14.23(a). The mass is revolving in the horizontal plane around a vertical axis. Find the range of speeds that the mass can have if both the strings must remain taut. $AC = 2$ m.

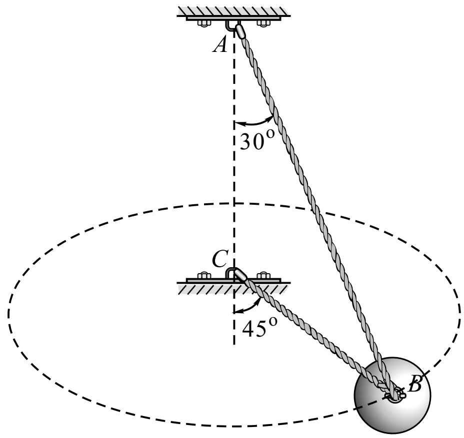


Fig. 14.23(a)

Solution

(i) In ΔABC, by sine rule

$$\frac{2}{\sin 15^\circ} = \frac{AB}{\sin 135^\circ} = \frac{BC}{\sin 30^\circ}$$

$$AB = 5.464 \text{ m and } BC = 3.864 \text{ m}$$

For radius of rotation ρ

$$\cos 45^\circ = \frac{\rho}{BC} \quad \therefore \; \rho = 2.732 \text{ m}$$

(ii) Consider the FBD of the sphere [Refer to Fig. 14.23(b)]

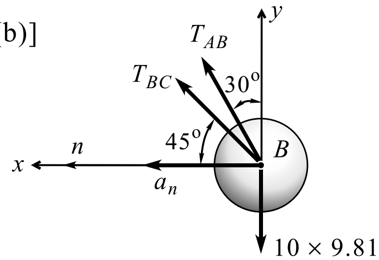


Fig. 14.23(b) : FBD of Sphere

By Newton's second law, we have

$$\sum F_y = ma_y = 0 \quad (\because a_y = 0)$$

$$T_{AB} \cos 30^\circ + T_{BC} \sin 45^\circ = 10 \times 9.81 \qquad \text{..... (I)}$$

$$\sum F_n = ma_n = \frac{mv^2}{\rho}$$

$$T_{BC} \cos 45^\circ + T_{AB} \sin 30^\circ = \frac{10 \times v^2}{2.732} \qquad \text{..... (II)}$$

(iii) For maximum speed (v_{max}) tension in string AB would become slack,

i.e., at $T_{AB} = 0 \quad v \Rightarrow v_{max}$

For minimum speed (v_{min}) tension in string BC would become slack,

i.e., at $T_{BC} = 0 \quad v \Rightarrow v_{min}$

(iv) For v_{max}, from Eq. (I)

$$0 + T_{BC} \sin 45^\circ = 10 \times 9.81 \quad \therefore \; T_{BC} = 138.73 \text{ N}$$

From Eq. (II), we get

$$138.73 \cos 45^\circ + 0 = \frac{10 \times (v_{max})^2}{2.732}$$

$\therefore \; v_{max} = 5.177$ m/s ***Ans.***

(v) For v_{min}, from Eq. (I)

$$T_{AB} \cos 30^\circ + 0 = 10 \times 9.81 \quad \therefore \; T_{AB} = 113.28 \text{ N}$$

From Eq. (II), we get

$$0 + 113.28 \sin 30^\circ = \frac{10 \times (v_{min})^2}{2.732}$$

$\therefore \; v_{min} = 3.934$ m/s ***Ans.***

Exercises

[I] Problems

1. A 80 kg block shown in Fig. 14.E1 rests on a horizontal plane. Find the magnitude of force P required to give the block an acceleration of 2.5 m/s^2 to the right. Take $\mu_k = 0.25$.

[***Ans.*** P = 535 N]

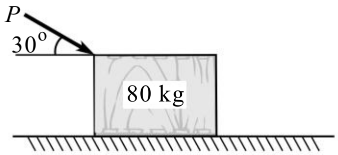


Fig. 14.E1

2. A car travelling at a speed of v = 60 kmph is braked and comes to rest in 6 seconds after the brakes are applied. Find the minimum coefficient of friction between the wheels and the road.

[***Ans.*** μ = 0.278]

3. Three blocks m_1, m_2 and m_3 of masses 1.5 kg, 2 kg and 1 kg respectively are placed on a rough surface (μ = 0.2) as shown in Fig. 14.E3. If a force F is applied so as to give the blocks acceleration of 3 m/s^2, then what will be the force that 1.5 kg block exerts on the 2 kg block. Also find force F.

[***Ans.*** 14.89 N and 22.33 N.]

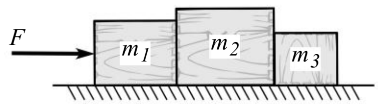


Fig. 14.E3

4. Block A of weight 4 N is connected to block B of weight 8 N by an inextensible string as shown in Fig. 14.E4. Find the velocity of block A if it falls by 0.6 m starting from rest. $\mu_k = 0.2$. Also find the tension in the string.

[***Ans.*** v = 1.53 m/sec and T = 3.2 N.]

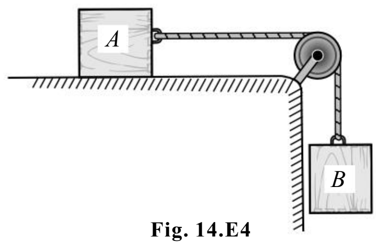


Fig. 14.E4

5. Figure 14.E5 shows a 4 kg mass resting on a smooth plane inclined 30° with the horizontal. A cord passes from this mass over a frictionless, massless pulley to an 8 kg mass which when released will drop vertically down. What will be the velocity of 8 kg mass 3 seconds after it is released from rest.

[***Ans.*** 14.7 m/s]

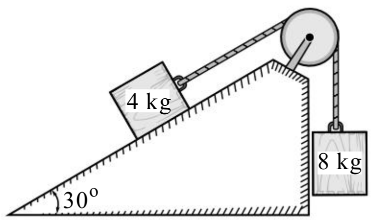


Fig. 14.E5

6. A horizontal force P = 70 N is exerted on mass A = 16 kg as shown in Fig. 14.E6. The μ between A and the horizontal plane is 0.25. B has a mass of 4 kg and coefficient of friction between it and the plane is 0.50. The cord between the two masses makes an angle of 10° with the horizontal. What is the tension in the cord ?

[***Ans.*** T = 20.5 N]

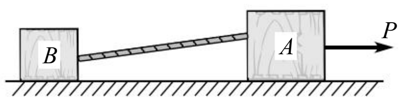


Fig. 14.E6

7. Two masses of 5 kg and 2 kg are positioned over frictionless and massless pulley as shown in Fig. 14.E7. If the 5 kg mass is released from rest, determine the speed at which the 5 kg mass will hit the ground.

[***Ans.*** $v = 4.1$ m/s]

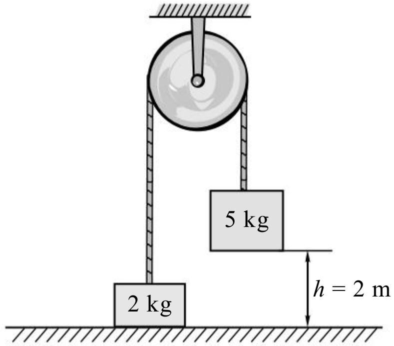


Fig. 14.E7

8. The 25 N block B rests on the smooth surface as shown in Fig. 14.E8. Determine its acceleration when the 15 N block A is released from rest. What would be the acceleration of B if the block of A was replaced by a 15 N vertical force acting on the attached cord?

[***Ans.*** 2.56 m/s^2 and 2.94 m/s^2.]

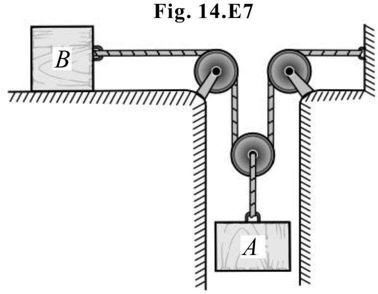


Fig. 14.E8

9. The block shown in Fig. 14.E9 has a weight of 500 N and is acted upon by a variable force having magnitude $P = 200t$. Compute the block velocity 2 seconds after P has been applied. The blocks initial velocity is 3 m/s down the plane. Take $\mu_k = 0.3$ Also find the distance traveled at the end of 2 seconds.

[***Ans.*** 15.6 m/s and 28.4 m.]

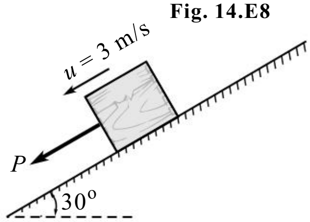


Fig. 14.E9

10. Three blocks A, B and C are connected as shown in Fig. 14.E10. Find acceleration of the masses and the tension T_1 and T_2 in the strings.

[***Ans.*** $T_1 = 32.8$ N, $T_2 = 103.8$ N and $a = 4.6$ m/s^2.]

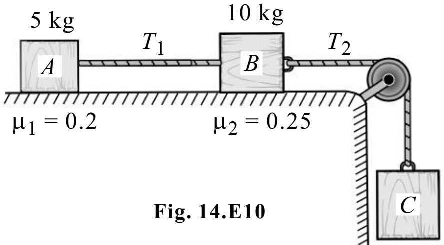


Fig. 14.E10

11. Three bodies A, B and C of weights 100 N, 200 N and 300 N respectively are connected by inextensible string passing over a smooth pulley as shown in Fig. 14.E11. The coefficient of friction between block A and plane is 0.1 and that between block B and plane is 0.2. Find the acceleration of the bodies A, B and C if the system starts from rest. Neglect the weight of the pulley. Also find the tensions in the string between **(a)** A and B and **(b)** B and C. Also find the time taken by body C to travel a distance of 10 m.

[***Ans.*** $a = 0.858$ m/s^2, $T_1 = 273.74$ N, $T_2 = 86.53$ N and $t = 4.83$ seconds.]

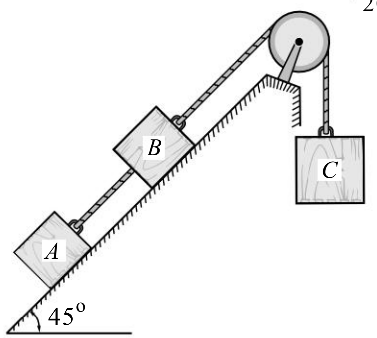


Fig. 14.E11

12. Block A has a mass of 25 kg and block B a mass of 15 kg, as shown in Fig. 14.E12. The coefficient of friction between all surface of contact are $\mu_s = 0.20$ and $\mu_k = 0.15$ knowing that $\theta = 25°$ and magnitude of force P applied to block B is 250 N, determine **(a)** the acceleration of block A and B and **(b)** tension in the cord.

[***Ans.*** **(a)** $a_A = a_B = 2.213$ m/s^2 and **(b)** $T = 192.31$ N.]

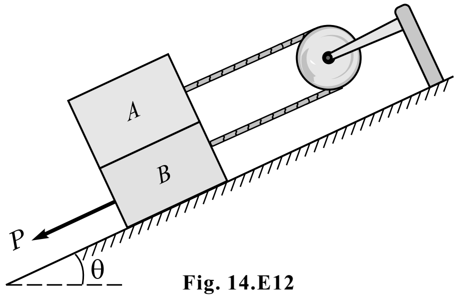


Fig. 14.E12

13. Determine the least coefficient of friction between A and B as shown in Fig. 14.E13 so that slip will not occur. A is a 40 kg mass, B is a 15 kg mass and F is 500 N parallel to the plane which is smooth.

[***Ans.*** 0.30]

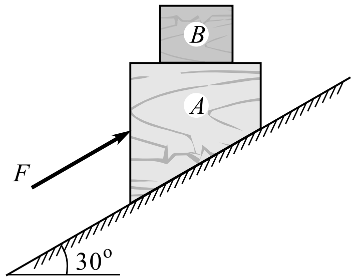


Fig. 14.E13

14. The two blocks shown in Fig. 14.E14 are originally at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys and between block A and the horizontal surface, determine **(a)** the acceleration of each block and **(b)** the tension in the cable.

[***Ans.*** $a_A = 2.49$ m/s^2 (→), $a_B = 0.831$ m/s^2 (↓) and $T = 74.8$ N.]

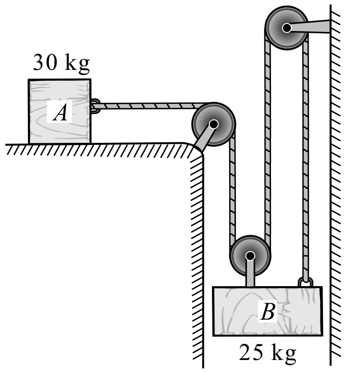


Fig. 14.E14

15. In Fig. 14.E15, the two blocks are originally at rest. Neglecting the masses of the pulleys and considering the coefficient of friction between the block A and inclined plane as 0.25, determine **(a)** the acceleration of each block and **(b)** the tension in the cable.

[***Ans.*** $a_A = 1.854$ m/s^2, $a_B = 0.927$ m/s^2 and $T = 7244$ N, $T_1 = 14488$ N.]

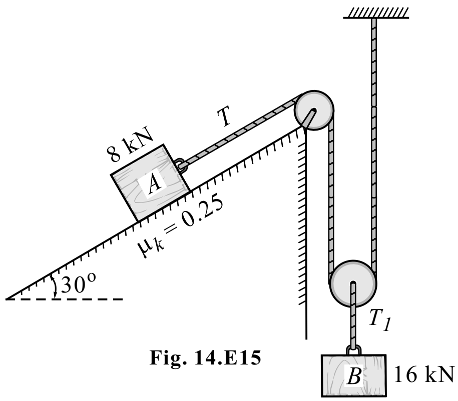


Fig. 14.E15

16. The blocks A and B shown in Fig. 14.E16 have a mass of 10 kg and 100 kg, respectively. Determine the distance B travels from the point where it is released from rest to the point where its speed becomes 2 m/s.

[***Ans.*** 0.883 m]

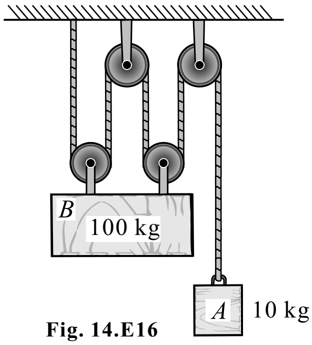


Fig. 14.E16

17. A force P applied to a system of light pulleys to pull body A of mass 4000 kg, as shown in Fig. 14.E17. What is the speed of A after 2 seconds starting from rest?

[***Ans.*** 1 m/s]

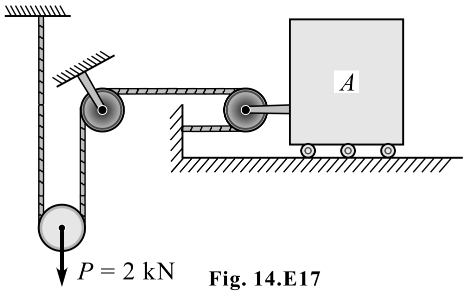


Fig. 14.E17

18. Determine the acceleration of the 5 kg cylinder A as shown in Fig. 14.E18. Neglect the mass of the pulleys and cords. The block at B has a mass of 10 kg. The coefficient of kinetic friction between block B and the surface is 0.1.

[***Ans.*** 0.0595 m/s]

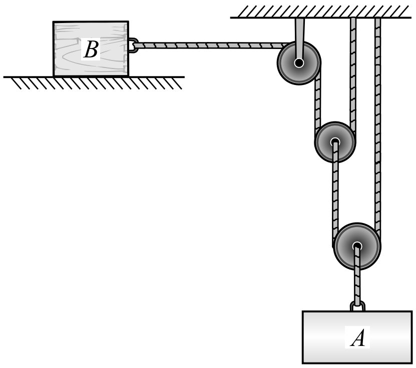


Fig. 14.E18

19. Determine the tension developed in the cords attached to each block and the acceleration of the blocks as shown in Fig. 14.E19. Neglect the mass of the pulley and cords.

[***Ans.*** $a_A = 1.51$ m/s^2 (↑), $a_B = 6.04$ m/s^2 (↓), $T_A = 22.6$ N and $T_B = 90.6$ N.]

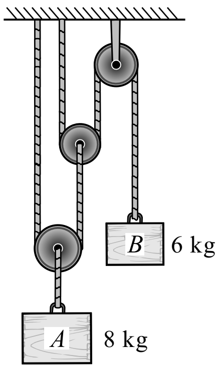


Fig. 14.E19

20. The slider block A shown in Fig. 14.E20 starts from rest and moves to the left with constant acceleration. Knowing that velocity of B is 60 cm/s, after moving through a distance of 100 cm, calculate acceleration of A and B.

[***Ans.*** a_A = 27 m/s^2 and a_B = 18 cm/s^2.]

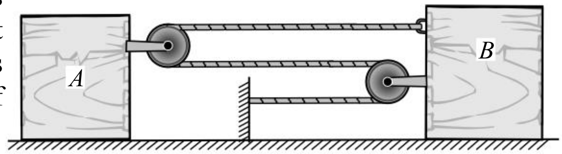


Fig. 14.E20

21. Determine the accelerations of bodies A and B and the tension in the cable due to the application of the 300 N force in Fig. 14.E21. Neglect all friction and the masses of the pulleys.

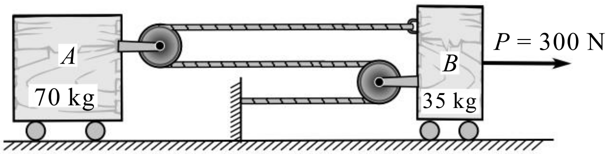


Fig. 14.E21

[***Ans.*** a_A = 2.34 m/s^2, a_B = 1.558 m/s^2 and T = 81.8 N.]

22. A car while travelling at a speed of 50 kmph passes through a curved portion of road in the form of an arc of a circle of radius 10 m in the vertical plane. The reaction offered by the lowest point of the arc on the car is **(a)** 14.55 kN, **(b)** 29.10 kN, **(c)** 7.25 kN and **(d)** 145.4 kN. Take g = 9.81 m/s^2.

[***Ans.*** None, R = 29.1 m where m = mass of car in kg.]

23. Two wires AC and BC are tied at C to a sphere which revolves at constant speed v in the horizontal circle, as shown in Fig. 14.E23. Determine minimum velocity v at which tension in either wire does not exceed 35 N.

[***Ans.*** 3.193 m/s]

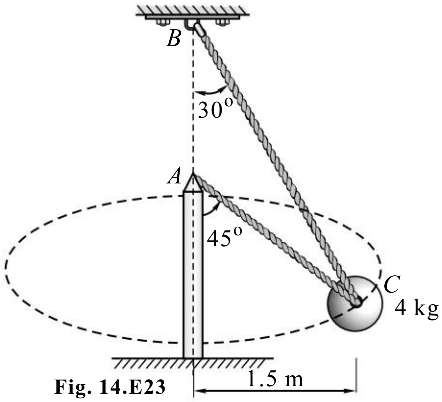


Fig. 14.E23

24. Two wires AC and BC are tied at C to a sphere which revolves at a constant speed v in the horizontal circle, as shown in Fig. 14.E24. Determine the range of values of v for which both wires remain taut.

[***Ans.*** 3.01 to 3.96 m/s.]

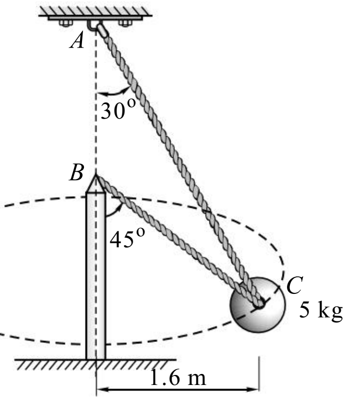


Fig. 14.E24

25. A bus moving with a velocity 12 m/s suddenly turns round a curve of radius of 8 m. Find the force acting on a passenger of 70 kg due to this circular motion.

[***Ans.*** 1260 N.]

26. Determine the rated speed of a highway curve of radius ρ = 125 m banked through an angle θ = 18°. The rated speed of a banked curved road is the speed at which a car should travel if no lateral friction force is to be exerted on its wheels. Take g = 9.81 m/s^2.

[***Ans.*** 19.96 m/s.]

[II] Review Questions

1. State Newton's second law of motion.
2. Derive the expression $F = ma$.
3. State D'Alembert's principle.
4. Explain the concept of virtual work done by internal force tension.
5. Compare Newton's second law with D'Alembert's principle.

[III] Fill in the Blanks

1. Mass is the quantity of ________ contained in a body.
2. ________ is the property of a body which measures its resistance to ta change of motion.
3. Rate of change of ________ is directly proportional to the force.
4. As per D'Alembert's principle, the algebraic sum of external force and ________ force is equal to zero.
5. Total virtual work done by internal force (tension) is equal to ________.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. In kinetics of rigid body we consider the ________.

   (a) force **(b)** mass **(c)** acceleration **(d)** All of these

2. The direction of acceleration and direction of particle motion can be _________.

   (a) same **(b)** opposite **(c)** same or opposite **(d)** None of these

3. State true or false : If there is no friction in the system, one can assume any direction for acceleration.

   (a) true **(b)** false

4. Work done by frictional force is always _________.

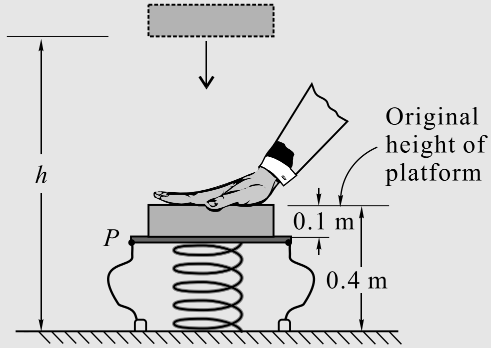
   (a) positive **(b)** negative **(c)** zero **(d)** None of these

5. If a block is sliding down on frictional surface inclined at θ with horizontal, then the component of acceleration due to gravity along inclined is equal to _________ m/s^2.

   (a) 9.81 **(b)** −9.81 **(c)** $9.81 \sin \theta$ **(d)** $9.81 \cos \theta$

15

KINETICS OF PARTICLES - II WORK N ENERGY PRINCIPLE



15.1 Introduction

In the previous chapter, problems were solved using *Newton's Second Law*. In this chapter, we are going to approach by *Work Energy Principle*. This method is advantageous over Newton's second law when the problem involves *force*, *velocity* and *displacement*, rather than *acceleration*. Also when spring force is involved one should prefer work energy principle to solve the problem.

15.2 Work Done by a Force

If a particle is subjected to a force F and particle is displaced by s from position ① to position ② then work done U is the product of force and displacement.

Work done = Force × Displacement

$U = F \times s$

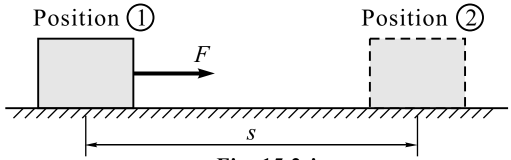


Fig. 15.2-i

If a particle is subjected to a force F at an angle θ with horizontal and the particle is displaced by s from position ① to position ② then work done U is the product of force component in the direction of displacement $F \cos \theta$ and displacement s.

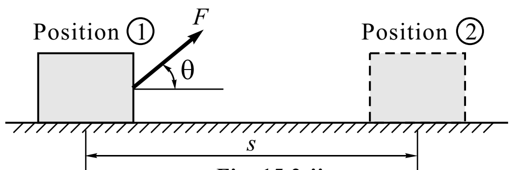


Fig. 15.2-ii

Work done = Component of force in direction of displacement × Displacement

$U = F \cos \theta \times s$

Important Points About Work Done

1. *Work done by a force is positive if the direction of force and the direction of displacement both are in same direction.* **Example :** Work done by force of gravity is positive when a body moves from an upper position to lower position.
2. *Work done by a force is negative if the direction of force and the direction of displacement both are in opposite direction.* **Example :** Work done by force of gravity is negative when a body moves from a lower position to a higher position.
3. *Work done by a force is zero if either the displacement is zero or the force acts normal to the displacement.* **Example :** Gravity force does not work when body moves horizontally.
4. *Work is a scalar quantity.*
5. *Unit of work is N.m or Joule (J).*

15.3 Work Done by Weight Force

In Fig. 15.3-i, a particle of mass m is displaced from position ① to position ② then work done is given by

Work done = Component of weight in the direction of displacement × Displacement

$$U = mg \sin\theta \times s$$

Work done = Weight force × Displacement in the direction of weight force

$$U = mg \times s \sin\theta$$

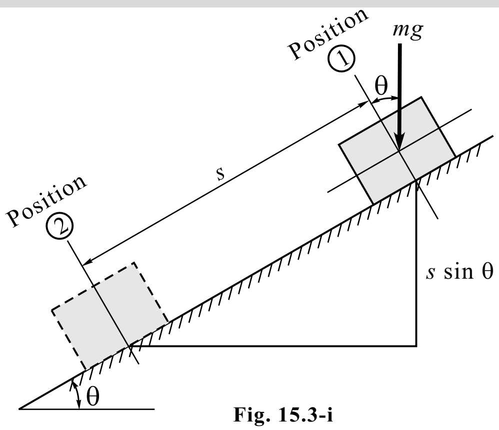


Fig. 15.3-i

15.4 Work Done by Frictional Force

In Fig. 15.4-i, a particle of mass m is moves from position ① to position ②, then work done on frictional force is given by

Work done = –Frictional force × Displacement

$$U = -\mu N \times s$$

Note : (i) Work done by friction force is –ve because direction of frictional force and displacement are opposite.

(ii) Work done by normal reaction (N) and component of weight force perpendicular to inclined plane ($mg \cos\theta$) is zero.

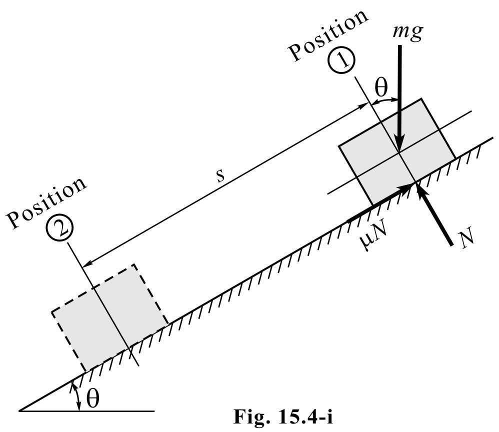


Fig. 15.4-i

15.5 Work Done by Spring Force

Consider a spring of stiffness k as shown in Fig. 15.5-i, a undeformed (free/original) length.

Let x_1 be the deformation of spring at position ①.

Let x_2 be the deformation of spring at position ②.

$\therefore$ Spring force $F = -k \times x$

where k is the spring stiffness (N/m)

x is the deformation of spring (m)

–ve sign indicates direction of spring force acts towards original position.

Work done = Spring force × Deformation

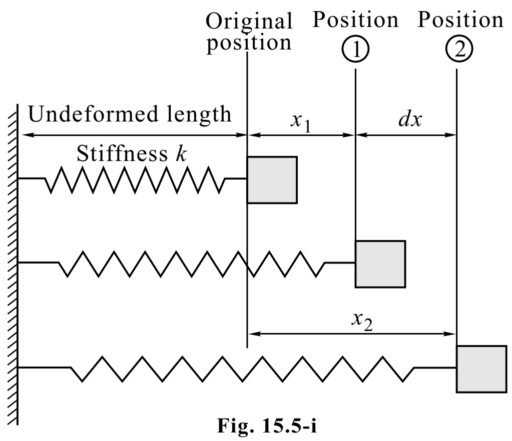


Fig. 15.5-i

$$U = \int_{x_1}^{x_2} -kx\, dx$$

$$\therefore \quad U = -\frac{1}{2} k\left(x_2^2 - x_1^2\right)$$

$$\therefore \quad U = \frac{1}{2} k\left(x_1^2 - x_2^2\right)$$

15.6 Work - Energy Principle

Work done by the forces acting on a particle during some displacement is equal to the change in kinetic energy during that displacement.

Proof

Consider the particle having mass m is acted upon by a force F and moving along a path which can be rectilinear or curvilinear as shown in Fig. 15.6-i.

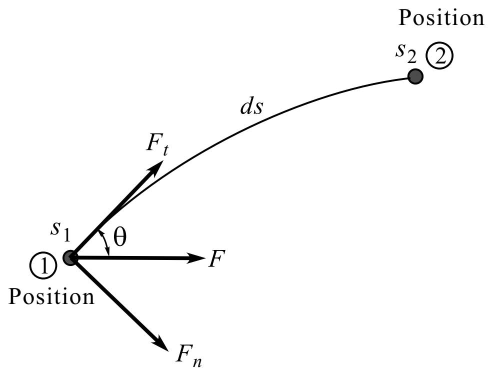


Fig. 15.6-i

Let v_1 and v_2 be the velocities of the particle at position 1 and position 2 and the corresponding displacement s_1 and s_2 respectively.

By Newton's second law, we have

$$\sum F_t = ma_t$$

$$F \cos\theta = ma_t = m\frac{dv}{dt}$$

$$F \cos\theta = m\frac{dv}{ds} \times \frac{ds}{dt}$$

$$F \cos\theta = mv \times \frac{dv}{ds}$$

$$F \cos\theta\, ds = mv\, dv$$

Integrating both sides, we have

$$\int_{s_1}^{s_2} F\cos\theta\, ds = \int_{v_1}^{v_2} mv\, dv$$

$$\therefore \quad U_{1\text{-}2} = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2$$

Work done = Change in Kinetic Energy

Kinetic Energy of a Particle

It is the energy possessed by a particle by virtue of its motion.

If a particle of mass m is moving with the velocity v, its kinetic energy is given by

$$\text{KE} = \frac{1}{2}mv^2 \qquad \text{Unit of KE is N.m or Joule (J)}$$

Potential Energy of a Particle

It is the energy possessed by a particle by virtue of its position.

If a particle of mass m is moving from position ① to position ②, then

Work done = Weight force × Displacement

$U = mgh = \text{PE}$

Unit of PE is N.m or Joule (J)

Note : Work done by weight force will be negative if moved from lower position to upper position.

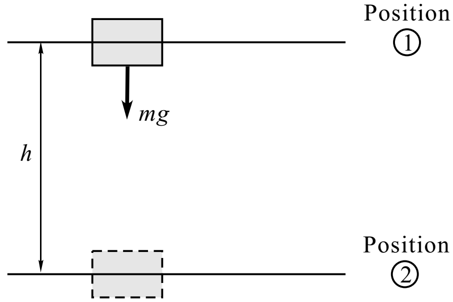


Fig. 15.6-ii

Conservative Forces

If the work of a force is moving a particle between two positions is independent of the path followed by the particle and can be expressed as a change in its potential energy, then such forces is called as ***conservative forces****.*

Example : Weight force of particle (gravity force), spring force and elastic force.

Non-Conservative Forces

The forces in which the work is dependent upon the path followed by the particles is known as ***non-conservative forces****.*

Example : Frictional force, viscous force.

Principle of Conservation of Energy

When a particle is moving from position ① to position ② under the action of only conservative forces (i.e., when frictional force does not exist) then by ***energy conservation principle*** *we say that the total energy remains constant.*

Total energy = Kinetic energy + Potential energy + Strain energy of spring

$$\text{Total energy} = \frac{1}{2} mv^2 \pm mgh + \frac{1}{2} kx^2$$

Power

It is defined as the rate of doing work.

$$\text{Power} = \frac{\text{Work}}{\text{Time}}$$

$$\text{Power} = \frac{\text{Force} \times \text{Displacement}}{\text{Time}}$$

Power = Force × Velocity

Unit of power is Joule/second (J/s) or watt (W)

1 W = 1 J/s = 1 Nm/s

One metric horse power = 735.5 watt

15.7 Solved Problems

Problem 1

A force of 500 N is acting on a block of mass 50 kg resting on a horizontal surface as shown in Fig. 15.1(a). Determine its velocity after the block has travelled a distance of 10 m. Coefficient of kinetic friction $\mu_k = 0.5$.

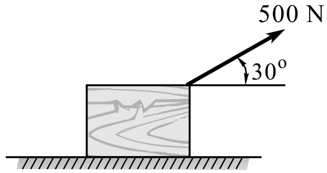


Fig. 15.1(a)

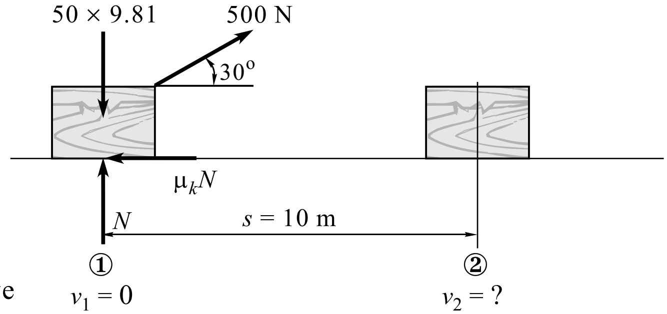


Fig. 15.1(b)

Solution

(i) $\Sigma F_y = ma_y = 0 \ (\because a_y = 0)$

$N_1 - 50 \times 9.81 + 500 \sin 30^{\circ} = 0$

$N = 240.5$ N

(ii) By principle of work energy, we have

Work done = Change in KE

$$500 \cos 30^{\circ} \times 10 - \mu_k N \times 10 = \frac{1}{2} \times 50 \times v_2^2 - 0$$

$$500 \cos 30^{\circ} \times 10 - 0.5 \times 240.5 \times 10 = 25 \times v_2^2$$

$v_2 = 11.185$ m/s ***Ans.***

Problem 2

Block A has a mass of 2 kg and has a velocity of 5 m/s up the plane shown in Fig. 15.E2. Use the principle of work energy; locate the rest position of the block.

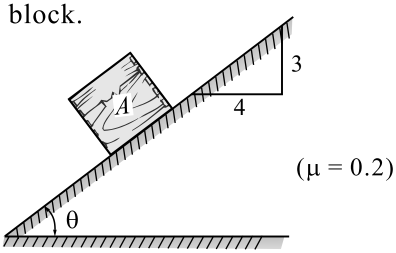


Fig. 15.2(a)

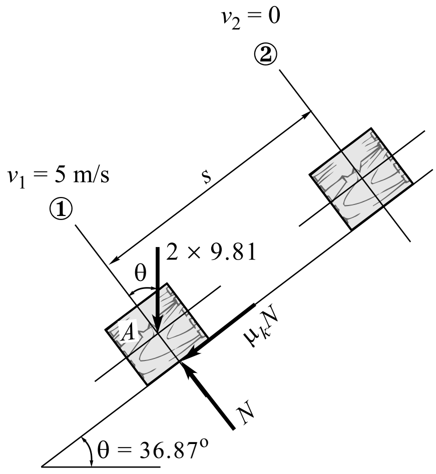


Fig. 15.2(b)

Solution

(i) By principle of work energy, we have

Work done = Change in KE

$$-2 \times 9.81 \sin \theta \times s - \mu N \times s = 0 - \frac{1}{2} \times 2 \times 5^2$$

$$-2 \times 9.81 \sin 36.87^{\circ} \times s - 0.2 \times 2 \times 9.81 \cos 36.87^{\circ} \times s = -\frac{1}{2} \times 2 \times 5^2$$

$s = 1.68$ m ***Ans.***

Problem 3

A mass of 20 kg is projected up an inclined of 26° with velocity of 4 m/s, as shown in Fig. 15.E3. If $\mu = 0.2$, **(i)** find maximum distance that the package will move along the plane and **(ii)** What will be the velocity of the package when it comes back to initial position ?

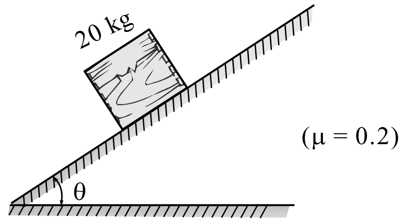


Fig. 15.3(a)

Solution

Case (i) Upward motion, from ① to ②

Refer to Fig. 15.3(b).

At position ①

$v_1 = 4$ m/s

At position ②

$v_2 = 0$

By work energy principle, we have

Work done = Change in KE

$$-20 \times 9.81 \sin 26^\circ \times d - 0.2 \times 20 \times 9.81 \cos 26^\circ \times d = 0 - \frac{1}{2} \times 20 \times 4^2$$

$$d(86 + 35.37) = 160$$

$d = 1.32$ m ***Ans.***

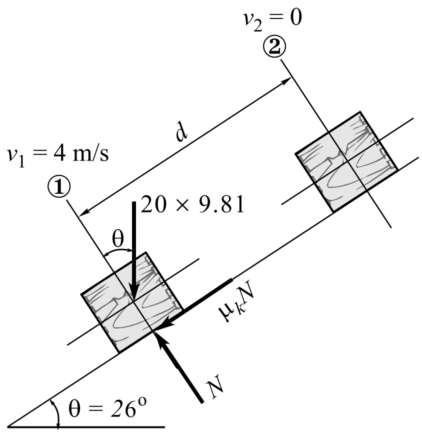


Fig. 15.3(b)

Case (ii) Downward motion, from ② to ①

Refer to Fig. 15.3(c).

At position ①

$v_1 = 0$

At position ②

$v_2 = ?$

Displacement $d = 1.32$ m

By work energy principle, we have

Work done = Change in KE

$$20 \times 9.81 \sin 26^\circ \times 1.32 - 0.2 \times 20 \times 9.81 \cos 26^\circ \times 1.32 = \frac{1}{2} \times 20 v_2^2 - 0$$

$v_2 = 2.59$ m/s ***Ans.***

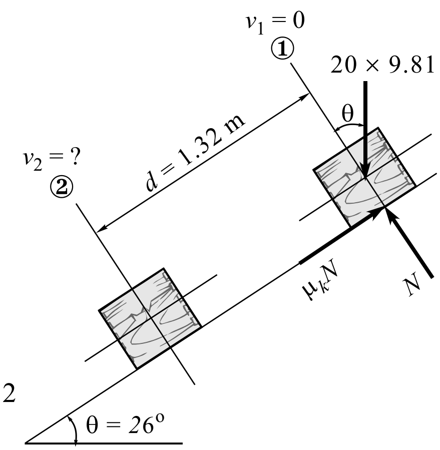


Fig. 15.3(c)

Problem 4

The 0.8 kg collar slides with negligible friction on the fixed rod in the vertical plane as shown in Fig. 15.E4. If the collar starts from the rest at A under the action of constant 8 N horizontal force. Calculate the velocity as it hits the stop at B.

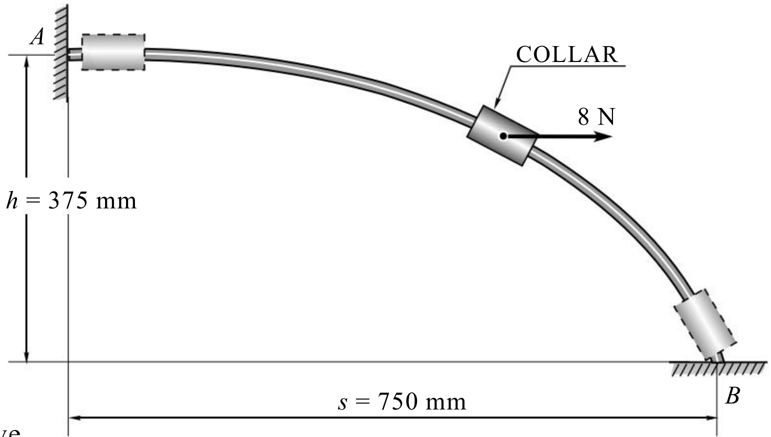


Fig. 15.4

Solution

At position A : $v_A = 0$

At position B : $v_B = ?$

By work energy principle, we have

Work done = Change in KE

$$mgh + 8 \times s = \frac{1}{2} \times 0.8 \times v_B^2 - 0$$

$$0.8 \times 9.81 \times 0.375 + 8 \times 0.75 = 0.4\, v_B^2$$

$$v_B = 4.728 \text{ m/s} \quad \textit{Ans.}$$

Problem 5

(i) Determine the distance in which a car moving at 90 kmph can come to rest after the power is switched off if μ between tyres and road is 0.8.

(ii) Determine also the maximum allowable speed of a car, if it is to stop in the same distance as above on ice road where the coefficient of friction between tyres and road is 0.08.

Solution

(i) μ = 0.8

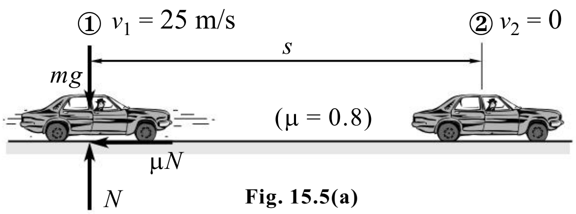


Fig. 15.5(a)

At position ①

$$v_1 = 90 \times \frac{1000}{3600} = 25 \text{ m/s}$$

At position ② $v_2 = 0$

By work energy principle, we have

Work done = Change in KE

$$-\mu N \times s = 0 - \frac{1}{2} \times m \times (25)^2$$

$$-0.8 \times mg \times s = -\frac{1}{2}\, m \times 625$$

$$s = 39.82 \text{ m} \quad \textit{Ans.}$$

(ii) For Ice Road

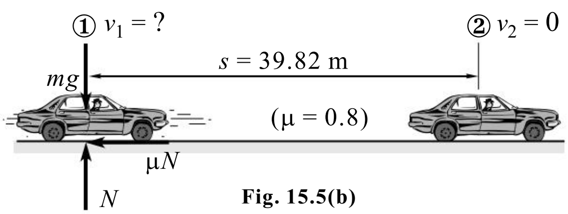


Fig. 15.5(b)

$\mu = 0.08$, $v_1 = ?$ $v_2 = 0$, $s = 39.82$ m

By work energy principle, we have

Work done = Change in KE

$$-\mu N \times s = 0 - \frac{1}{2} \times m \times v_1^2$$

$$-0.08 \times mg \times 39.82 = -\frac{1}{2}\, m \times v_1^2$$

$$v_1 = 7.91 \text{ m/s} \quad \textit{Ans.}$$

Problem 6

A block of mass 8 kg slides freely on a smooth vertical rod as shown in Fig. 15.6(a). The mass is released from rest at a distance of 500 mm from the top of the spring. The spring constant is 60 N/mm. Determine the velocity of block when the spring has compressed through 20 mm. The free length of the spring is 400 mm.

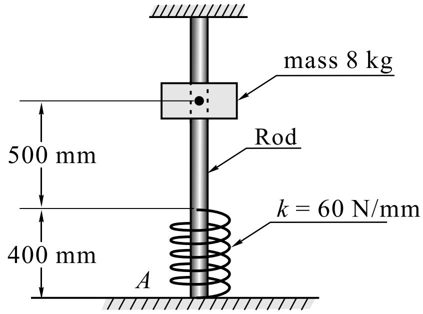


Fig. 15.6(a)

Solution

Given : $k = 60 \text{ N/mm} = 60000 \text{ N/m}$

At position ① : $v_1 = 0$, $x_1 = 0$

At position ② : $v_2 = ?$, $x_2 = 0.02$ m

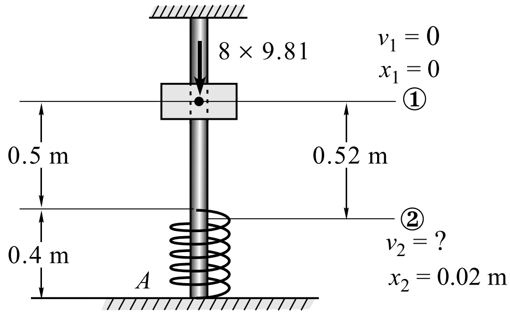


Fig. 15.6(b)

Total displacement = 500 + 20 = 520 mm

$s = 0.52$ m

By work energy principle, we have

Work done = Change in kinetic energy

$$8 \times 9.81 \times 0.52 + \frac{1}{2} \times 60000 \times (0 - 0.02^2) = \frac{1}{2} \times 8 \times v_2^2 - 0$$

$$28.8096 = 4\, v_2^2$$

$v_2 = 2.684$ m/s ***Ans.***

Problem 7

Collar of mass 15 kg is at rest at '*A*'. It can freely slide on a vertical smooth rod *AB*. The collar is pulled up with a constant force *F* = 600 *N* applied as shown in Fig. 15.E7. Unstretched length of spring is 1 m. Calculate velocity of the collar when it reaches position *B*. *Given* : Spring constant $k = 3$ N/mm. *AC* is horizontal.

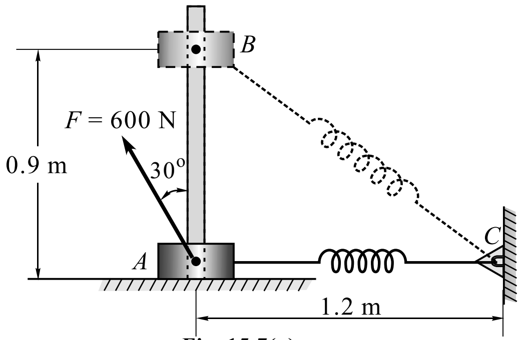


Fig. 15.7(a)

Solution

Given : $k = 3 \text{ N/mm} = 3000 \text{ N/m}$

At position ①

$v_1 = 0$

$x_1 = 1.2 - 1 = 0.2$ m

At position ②

$v_2 = 0$

$x_2 = (BC) - 1 = 1.5 - 1$

$\quad = 0.5$ m

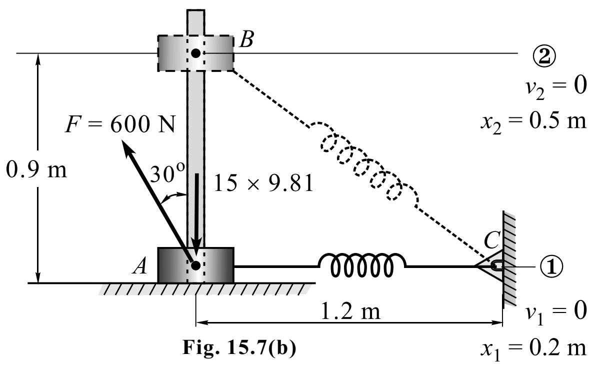


Fig. 15.7(b)

By work energy principle

Work done = Change in kinetic energy

$$600 \cos 30^\circ \times 0.9 - 15 \times 9.81 \times 0.9 + \frac{1}{2} \times 3000(0.2^2 - 0.5^2) = \frac{1}{2} \times 15 \times v_2^2 - 0$$

$$20.22 = 7.5\, v_2^2$$

$v_2 = 1.64$ m/s (↑) ***Ans.***

Problem 8

A collar A of mass 10 kg moves in vertical guide as shown in Fig. 15.8(a). Neglecting the friction between the guide and the collar, find its velocity when it passes through position ② after starting from rest in position ①. The spring constant is 200 N/m and the free length of the spring is 200 mm.

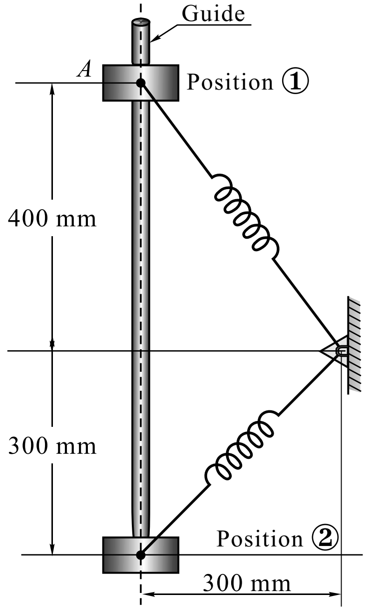


Fig. 15.8(a)

Solution

Given : $k = 200$ N/m

Free length of spring = 200 mm = 0.2 m

At position ①

$x_1 = 500 - 200 = 300$ mm

$x_1 = 0.3$ m

$v_1 = 0$

At position ②

$x_2 = 424.26 - 200 = 224.26$ mm

$x_2 = 0.224$ m

$v_2 = ?$

By work energy principle, we have

Work done = Change in kinetic energy

$$10 \times 9.81 \times 0.7 + \frac{1}{2} \times 200 \times (0.3^2 - 0.224^2) = \frac{1}{2} \times 10 \times v_B^2 - 0$$

$$72.65 = 5\, v_B^2$$

$v_2 = 3.81$ m/s ***Ans.***

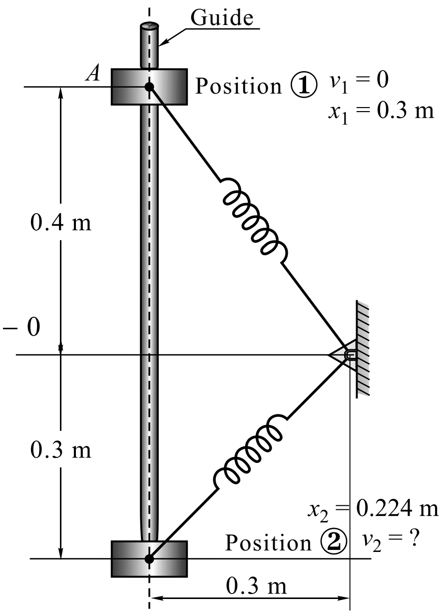


Fig. 15.8(b)

Problem 9

The mass m = 1.8 kg slides from rest at A along the frictionless rod bent into a quarter circle. The spring with modulus k = 16 N/m has an unstretched length of 400 mm.

(i) Determine the speed of m at B.

(ii) If the path is elliptical, what is the speed at B.

Solution

(i) At position ①

$v_1 = 0$

$x_1 = (600 - 400) = 200$ mm

$x_1 = 0.2$ m

At position ②

$v_2 = ?$

$x_2 = (600 - 400) = 200$ mm

$x_2 = 0.2$ m

By work energy principal, we have

Work done = Change in kinetic energy

$$mgh + \frac{1}{2}k(x_1^2 - x_2^2) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$1.8 \times 9.81 \times 0.6 + \frac{1}{2} \times 16 \times (0.2^2 - 0.2^2) = \frac{1}{2} \times 1.8 \times v_2^2 - 0$$

$v_2 = 3.43$ m/s

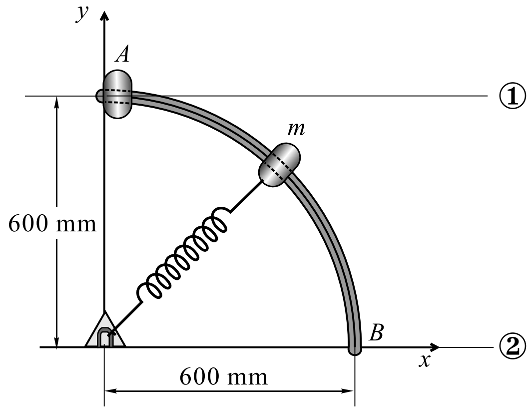


Fig. 15.9(a)

(ii) At position ①

$v_1 = 0$

$x_1 = (600 - 400) = 200$ mm

$x_1 = 0.2$ m

At position ②

$v_2 = ?$

$x_2 = (900 - 400) = 500$ mm

$x_2 = 0.5$ m

By work energy principal, we have

Work done = Change in kinetic energy

$$mgh + \frac{1}{2}k(x_1^2 - x_2^2) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$1.8 \times 9.81 \times 0.6 + \frac{1}{2} \times 16 \times (0.2^2 - 0.5^2) = \frac{1}{2} \times 1.8 \times v_2^2 - 0$$

$v_2 = 3.15$ m/s ***Ans.***

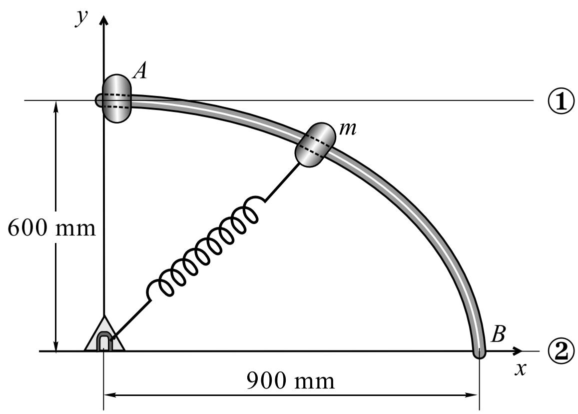


Fig. 15.9(b)

Problem 10

The slider of mass 1 kg attached to a spring of stiffness 400 N/m and unstretched length 0.5 m is released from A as shown in Fig. 15.10(a). Determine the velocity of the slider as it passes through B and C. Also compute the distance beyond C where the slider will come to the rest.

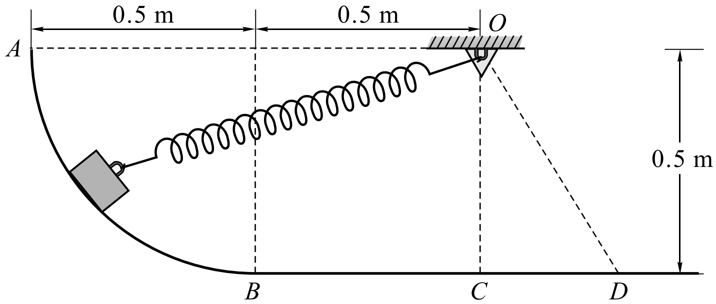


Fig. 15.10(a)

Solution

Redrawing the given Fig. 15.10(b).

Method I

(i) From *A* to *B*

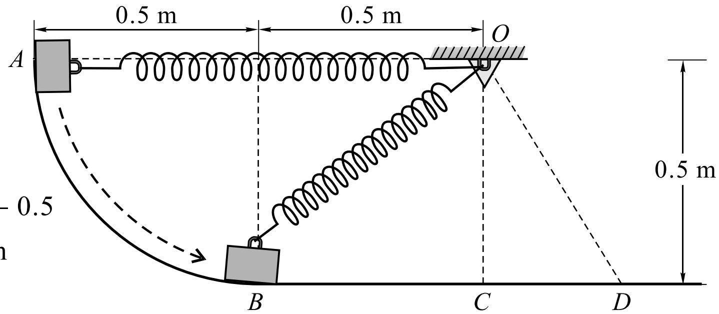


Fig. 15.10(b)

At position *A*	**At position *B***
$v_A = 0$	$v_B = ?$
$x_A = 0.5$ m	$x_B = 0.707 - 0.5$
	$x_B = 0.207$m

Displacement $s = 0.5$ m

By work energy principal, we have

Work done = Change in kinetic energy

$$1 \times 9.81 \times 0.5 + \frac{1}{2} \times 400\,(0.5^2 - 0.207^2) = \frac{1}{2} \times 1 \times v_B^2 - 0$$

$v_B = 9.63$ m/s ***Ans.***

(ii) From *A* to *C*

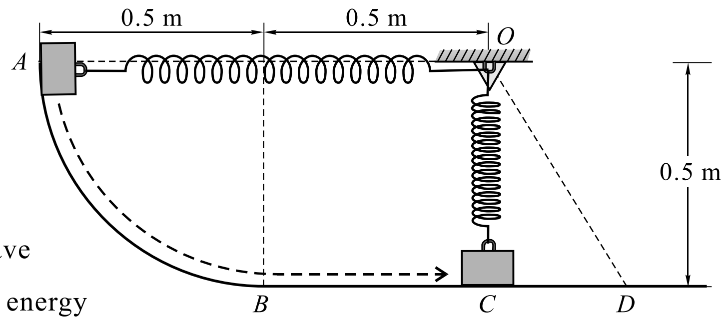


Fig. 15.10(c)

At position *A*	**At position *C***
$v_A = 0$	$v_C = ?$
$x_A = 0.5$ m	$x_C = 0$

Displacement $s = 0.5$ m

By work energy principal, we have

Work done = Change in kinetic energy

$$1 \times 9.81 \times 0.5 + \frac{1}{2} \times 400\,(0.5^2 - 0^2) = \frac{1}{2} \times 1 \times v_C^2 - 0$$

$v_C = 10.48$ m/s ***Ans.***

(iii) From *A* to *D*

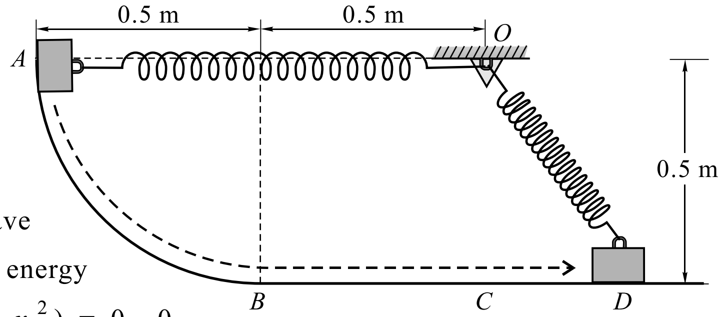


Fig. 15.10(d)

At position *A*	**At position *D***
$v_A = 0$	$v_D = 0$
$x_A = 0.5$ m	$x_D = ?$

Displacement $s = 0.5$ m

By work energy principal, we have

Work done = Change in kinetic energy

$$1 \times 9.81 \times 0.5 + \frac{1}{2} \times 400\,(0.5^2 - x_D^2) = 0 - 0$$

$x_D = 0.524$ ***Ans.***

OD = Unstretched length of spring + x_D = 0.5 + 0.524 = 1.024 m ***Ans.***

Distance beyond C to D

$$CD = \sqrt{(OD)^2 - (OC)^2} = \sqrt{(1.024)^2 - (0.5)^2} = 0.89 \text{ m} \quad \textbf{\textit{Ans.}}$$

Method II

Total Energy = KE ± PE + SE

$$TE = \frac{1}{2} mv^2 \pm mgh + \frac{1}{2} kx^2$$

(i) At position *A* ($v_A = 0,\ x_A = 0.5,\ h = 0$)

$$TE = 0 + 0 + \frac{1}{2} \times 400 \times 0.5^2 = 50 \text{ J}$$

(ii) At position *B* ($v_B = ?,\ x_B = 0.207,\ h = -0.5$m)

TE = KE ± PE + SE

$$50 = \frac{1}{2} \times 1 \times v_B^2 - 1 \times 9.81 \times 0.5 + \frac{1}{2} \times 400 \times 0.207^2$$

$v_B = 9.63$ m/s ***Ans.***

(iii) At position *C* ($v_C = ?,\ x_C = 0,\ h = -0.5$m)

TE = KE ± PE + SE

$$50 = \frac{1}{2} \times 1 \times v_C^2 - 1 \times 9.81 \times 0.5 + \frac{1}{2} \times 400 \times 0$$

$v_C = 10.48$ m/s ***Ans.***

(iv) At position *D* ($v_D = ?,\ x_D = ?,\ h = -0.5$m)

TE = KE ± PE + SE

$$50 = 0 - 1 \times 9.81 \times 0.5 + \frac{1}{2} \times 400\, x_D^2$$

$x_D = 0.524$ m ***Ans.***

OD = Unstrected length of spring + x_D^2 = 0.5 + 0.524 = 1.024 m ***Ans.***

Distance beyond C to D

$$CD = \sqrt{(OD)^2 - (OC)^2} = \sqrt{(1.024)^2 - (0.5)^2} = 0.89 \text{ m} \quad \textbf{\textit{Ans.}}$$

Problem 11

A 1 kg collar is attached to a spring and slides without friction along a circular rod which lies in horizontal plane as shown in Fig. 15.E11. The spring has a constant $k = 250$ N/m and is undeformed when collar is at B. Knowing that collar passes through point D with a speed of 1.6 m/s, determine the speed of the collar when it passes through point C and point B.

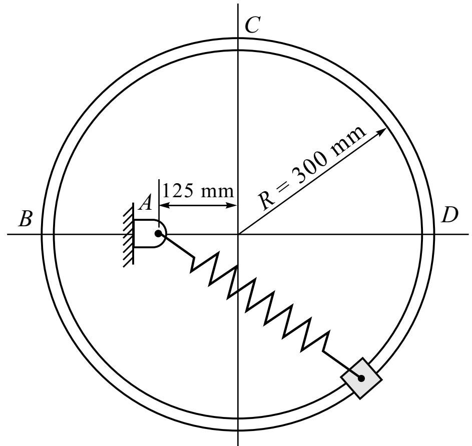


Fig. 15.11(a)

Solution

Undeformed length of spring is at B,

$$= 300 - 125 = 175 \text{ mm}$$
$$= 0.175 \text{ m}$$

Deformation of spring at position D,

$$= 125 + 300 - 175 = 250 \text{ mm}$$
$$= 0.25 \text{ m}$$

Deformation of spring at position C,

$$= \sqrt{125^2 + 300^2} - 175$$
$$= 325 - 175 = 150 \text{ mm}$$
$$= 0.15 \text{ m}$$

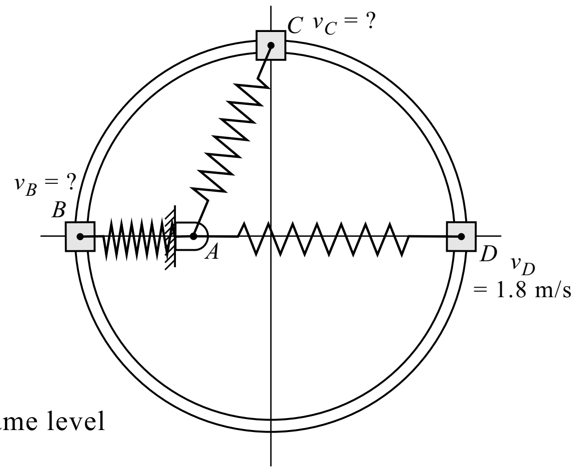


Fig. 15.11(b)

By principle of conservation of energy, we have

Total energy at any position remains constant.

P.E. throughout the ring is zero because it is at same level (horizontal).

Total energy at position D = Total energy at position B

(KE + PE + SE) at D = (KE + PE + Spring energy) at B

$$\frac{1}{2} \times 1 \times 1.8^2 + 0 + \frac{1}{2} \times 250 \times 0.25^2 = \frac{1}{2} \times 1 \times v_B^2 + 0 + \frac{1}{2} \times 250 \times 0^2$$

$$9.43 = 0.5 v_B^2$$

$v_B = 4.343$ m/s ***Ans.***

Total energy at position D = Total energy at position C

$$9.43 = \frac{1}{2} \times 1 \times v_C^2 + 0 + \frac{1}{2} \times 250 \times 0.15^2$$

$$9.43 = 0.5 v_C^2 + 2.8125$$

$\therefore\ v_C = 3.638$ m/s ***Ans.***

Problem 12

A 20 N block is released from rest. It slides down the inclined having $\mu = 0.2$ as shown in Fig. 15.12(a). Determine the maximum compression of the spring and the distance moved by the block when the energy is released from compressed spring. Springs constant $k = 1000$ N/m.

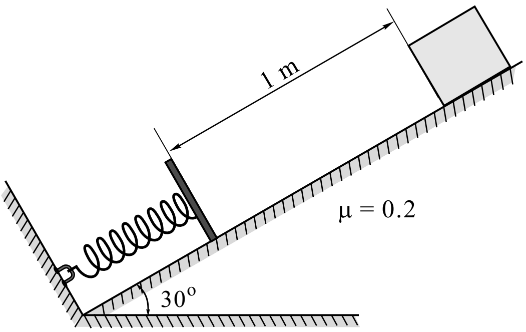


Fig. 15.12(a)

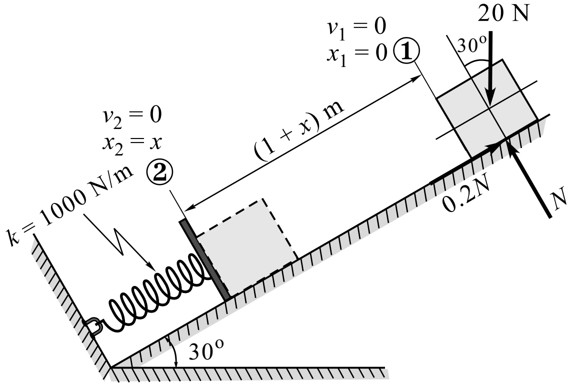


Fig. 15.12(b)

Solution

Part (i) Maximum compression of the spring

Let x be the maximum deformation of spring at position ② where the block comes to rest ($v_2 = 0$).

By work energy principal, we have

Work done = Change in kinetic energy

$$\frac{1}{2} \times 1000(0^2 - x^2) + 20 \sin 30^\circ (1 + x) - 0.2 \times 20 \cos 30^\circ (1 + x) = 0 - 0$$

$\therefore$ $x = 0.121$ m ***Ans.***

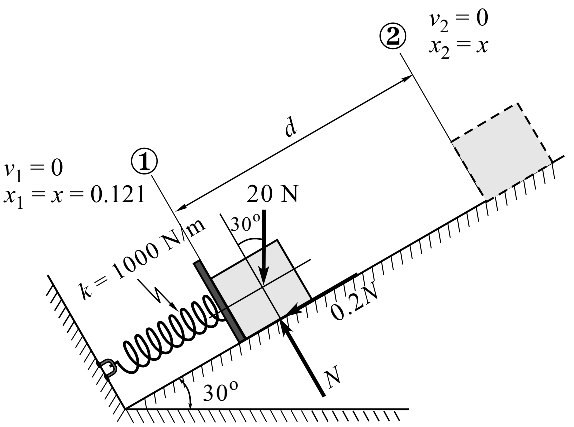


Fig. 15.12(c)

Part (ii) Distance moved by the block

By work energy principal, we have

Work done = Change in kinetic energy

$$\frac{1}{2} \times 1000(0.121^2 - 0^2) - 20 \sin 30^\circ \times d - 0.2 \times 20 \cos 30^\circ \times d = 0 - 0$$

$\therefore$ $d = 0.5437$ m ***Ans.***

Problem 13

A spring is used to stop a 100 kg package which is moving down a 30° incline. The spring has a constant $k = 30$ kN/m and is held by cables so that it is initially compressed 90 mm. If the velocity of package is 5 m/s when it is 9 m from the spring, determine the maximum additional deformation of spring in bringing the package to rest. Assume coefficient of friction as 0.2.

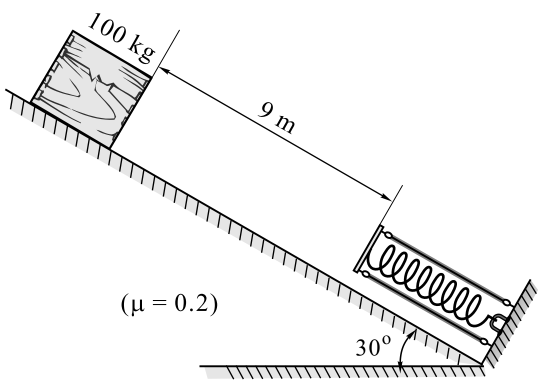


Fig. 15.13(a)

Solution

Let x be the maximum additional deformation of spring in bringing the package to rest.

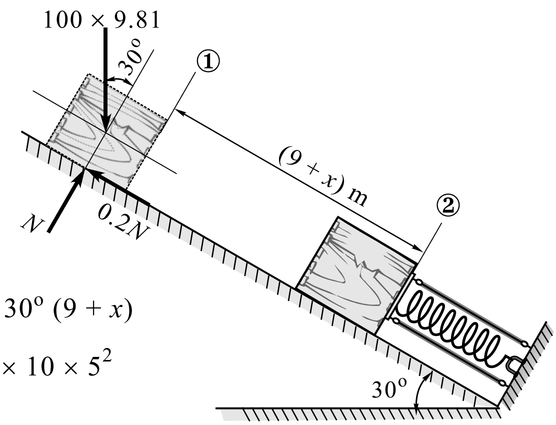


Fig. 15.13(b)

At position ①	At position ②
$v_1 = 5$ m/s	$v_2 = 0$
$x_1 = 0.09$ m	$x_2 = (x + 0.09)$ m

Displacement $s = (9 + x)$ m, $k = 30{,}000$ N/m

By work energy principal, we have

Work done = Change in kinetic energy

$$100 \times 9.81 \sin 30^o \times (9 + x) - 0.2 \times 100 \times 9.81 \cos 30^o\,(9 + x)$$

$$+ \frac{1}{2} \times 30000[(0.09)^2 - (x + 0.09)^2] = 0 - \frac{1}{2} \times 10 \times 5^2$$

$$15000x^2 + 2379.41x - 2885.31 = -1250$$

$$15000x^2 + 2379.41x - 1635.31 = 0$$

Solving quadratic equation, we get

$x = 0.4517$ m ***Ans.***

Problem 14

Two springs each having stiffness of 0.5 N/cm are connected to ball B having a mass of 5 kg in a horizontal position producing initial tension of 1.5 m in each spring as shown in Fig. 15.E14(a). If the ball is allowed to fall from rest what will be its velocity after it has fallen through a height of 15 cm.

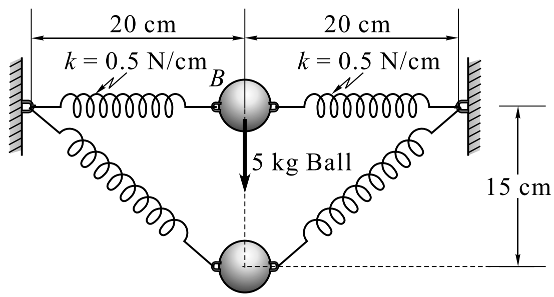


Fig. 15.14(a)

Solution

Method I

Initial position Tension = 1.5 N

$T = kx$

$1.5 = (0.5)\,(x)$

$x = 3$ cm (Deformation in initial position)

$\therefore$ Free length of spring $= 20 - 3 = 17$ cm

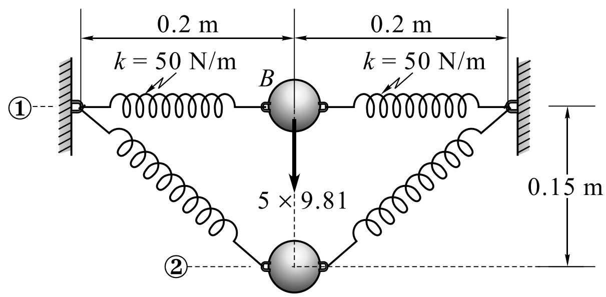


Fig. 15.14(b)

At position ①	At position ②
$v_1 = 0$	$v_2 = ?$
$x_1 = 3$ cm	$x_2 = (25 - 17) = 8$ cm
$\therefore\ x_1 = 0.03$ m	$x_2 = 0.08$ m
Displacement $h = 15$ cm	Spring constant $k = 0.5$ N/cm
$\therefore\ h = 0.15$ m	$\therefore\ k = 50$ N/m

By principle of work energy, we have

Work done = Change in kinetic energy

$$5 \times 9.81 \times 0.15 + \left[\frac{1}{2} \times 50(0.03^2 - 0.08^2)\right] \times 2 = \frac{1}{2} \times 5 \times v_2^2 - 0$$

$v_2 = 1.68$ m/s ***Ans.***

Method II

By principle of conservation of energy

Total energy = KE + P.E. + S.E.

Total energy remains constant at any position.

Total energy at position ① = Total energy at position ②

(KE + P.E. + S.E.) at position ① = (KE + P.E. + S.E.) at position ②

$$\frac{1}{2} \times 5 \times 0^2 + 5 \times 9.81 \times 0 + \frac{1}{2} \times 50 \times 0.03^2 = \frac{1}{2} \times 5 \times v_2^2 - 5 \times 9.81 \times 0.15 + \frac{1}{2} \times 50 \times 0.08^2$$

$$0.0225 = 2.5v_2^2 - 7.3575 + 0.16$$

$v_2 = 1.69$ m/s ***Ans.***

Problem 15

An 8 kg plunger is released from rest in the position shown in Fig. 15.15(a) and is stopped by two nested spring. The constant of the outer spring is k_o = 3 k N/m and constant of the inner spring k_i = 10 kN/m. Determine the maximum deflection of the outer spring.

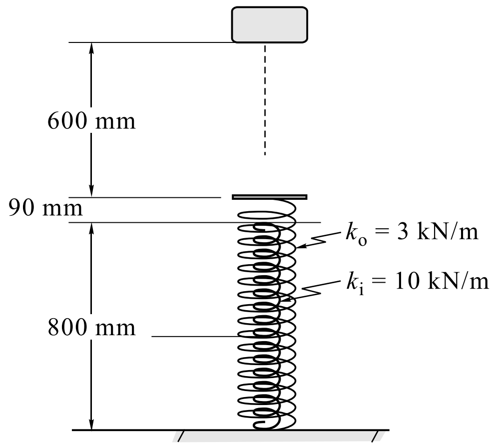


Fig. 15.15(a)

Solution

Let x be the deformation of inner spring.

At position ①	At position ②
$v_1 = 0$	$v_2 = 0$
$x_{i1} = 0$	$x_{i2} = x$
$x_{o1} = 0$	$x_{o2} = (0.09 + x)$

Displacement $s = (0.69 + x)$

By work energy principle, we have

Work done = Change in kinetic energy

$$8 \times 9.81 \times (0.69 + x) + \frac{1}{2} \times 3000[0^2 - (0.09 + x)^2] + \frac{1}{2} \times 10000(0^2 - x^2) = 0 - 0$$

$x = 0.067$ m = 67 mm

∴ The maximum additional deformation of outer spring will be

$d = x + 90$

$d = 157$ mm ***Ans.***

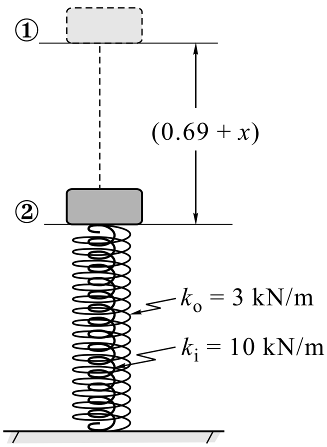


Fig. 15.15(b)

Exercises

[I] Problems

1. The 17.5 kN automobile shown in Fig. 15.E1, is travelling down the 10º inclined road at a speed of 6 m/s. If the driver wishes to stop his car, determine how far 's' his tyres skid on the road if he jams on the brakes, causing his wheels to lock. the coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.5$.

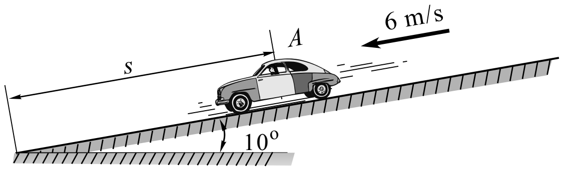


Fig. 15.E1

[***Ans.*** $s = 5.75$ m]

2. Packages are thrown down on incline at A with a velocity of 1.2 m/s as shown in Fig. 15.E2. The package slide along the surface ABC to a conveyer belt which moves with a velocity of 2.4 m/s. Knowing that $\mu_k = 0.25$ between the packages and the surface ABC, determine the distance 'd' if the packages are to arrive at C with a velocity of 2.4 m/s.

[***Ans.*** $d = 6.08$ m]

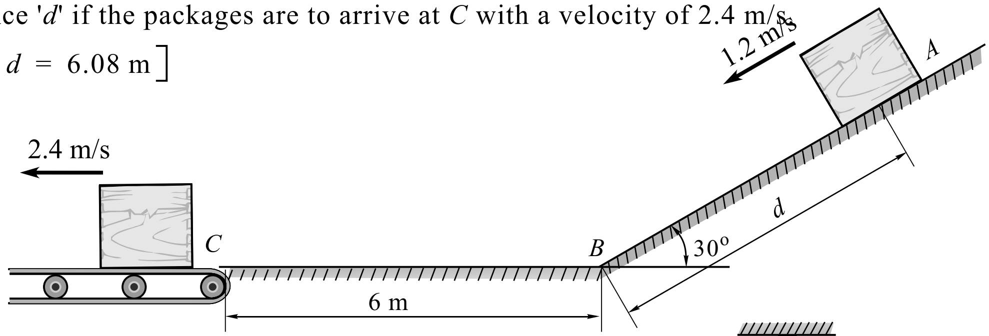


Fig. 15.E2

3. The 6 kg cylindrical collar is released from rest in the position shown in Fig. 15.E3 and drops onto the spring. Calculate the velocity v of the cylinder when the spring has been compressed 50 mm.

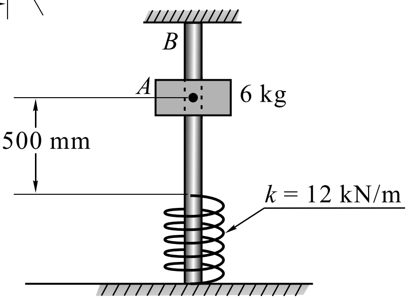


Fig. 15.E3

[***Ans.*** $v = 2.41$ m/s]

4. A 24 N package is placed with no initial velocity at the top of a incline shown in Fig. 15.E4. Knowing that μ_k between the package and the surface is 0.25. Determine **(a)** how far the package will slide on the horizontal portion, **(b)** the maximum velocity reached by the package and **(c)** the amount of the energy dissipated due to friction between A and B.

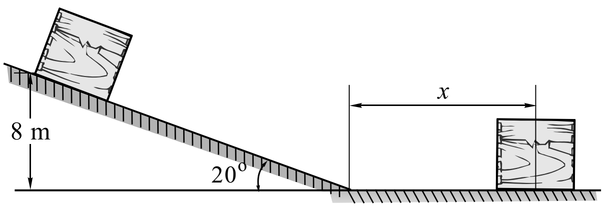


Fig. 15.E4

[***Ans.*** **(a)** 10.02 m, **(b)** 7.02 m/s and **(c)** 131.97 J.]

5. A 20 N block slides with initial velocity of 2 m/s down an inclined plane on to a spring of modulus 1000 N/m for a distance of 1 m as shown in Fig. 15.E5. Find maximum compression of the spring neglecting friction.

[***Ans.*** 177.63 mm]

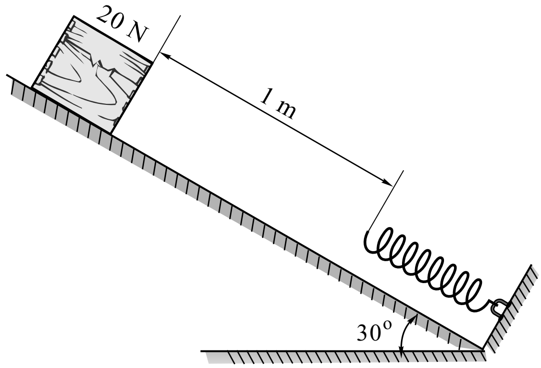


Fig. 15.E5

6. A collar of mass 5 kg can slide along a vertical bar as shown in Fig. 15.E6. The spring attached to the collar is in undeformed state of length 20 cm and stiffness 500 N/m. If the collar is suddenly released, find the velocity of the collar if it moves 15 cm down as shown in the figure.

[***Ans.*** 1.66 m/s]

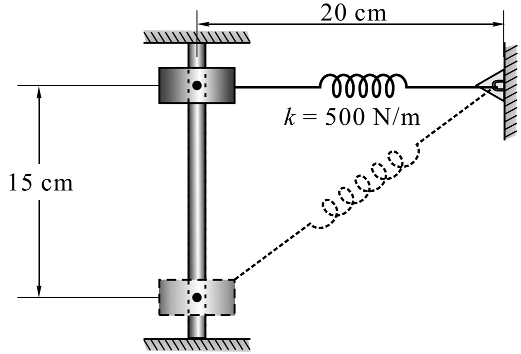


Fig. 15.E6

7. A 10 kg collar slides without friction along a vertical road as shown in Fig. 15.E7. The spring attached to the collar has an undeformed length of 100 mm and a constant of 500 N/m. If the collar is released from rest in position ①, determine its velocity after it has moved 150 mm to position ②.

[***Ans.*** $v = 1.54$ m/s]

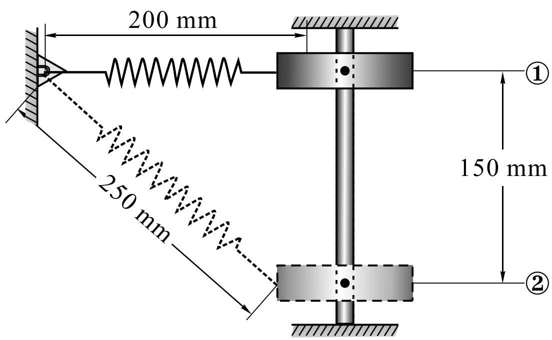


Fig. 15.E7

8. The collar has a mass of 20 kg and slides along the smooth rod. Two springs are attached to it and the ends of the rod as shown in Fig. 15.E8. If each spring has an uncompressed length of 1 m and the collar has a speed of 2 m/s when $s = 0$, determine the maximum compression of each spring due to the back-and-forth (oscillating) motion of the collar.

[***Ans.*** $s = 0.73$ m]

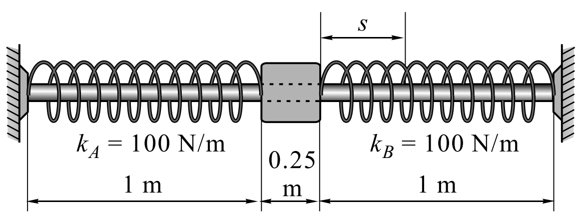


Fig. 15.E8

9. A 10 kg block rests on the horizontal surface shown in Fig. 15.E9. The spring, which is not attached to the block, has a stiffness k = 500 N/m and is initially compressed 0.2 m from C to A. After the block is released from rest at A, determine its velocity when it passes point D. The coefficient of kinetic friction between the block and the plane is μ_k = 0.2

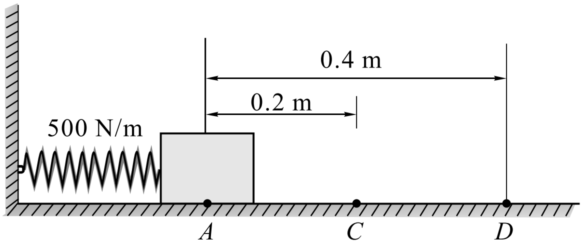


Fig. 15.E9

[***Ans.*** 0.656 m/s]

10. Figure 15.E10 shows a wagon weighing 500 kN starts from rest, runs 30 m down one per cent grade and strikes the bumper post. If the rolling resistance of the track is 5 N/km. Find the velocity of the wagon when it strikes the post.

If the bumper spring which compresses 1 mm for every 15 kN, determine by how much this spring will be compressed.

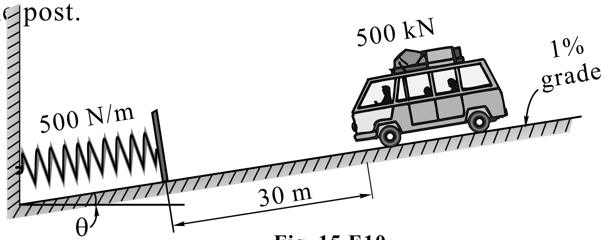


Fig. 15.E10

[***Ans.*** 1.716 m/s; 100.2 mm]

11. In Fig. 15.E11, the block P of weight 50 N is pulled so that the extension in the spring is 10 cm. The stiffness of the spring is 4 N/cm and the coefficient of friction between the block and the plane O-X is μ = 0.3.

Find the **(a)** velocity of the block as the spring returns to its undeformed state and **(b)** maximum compression in the spring.

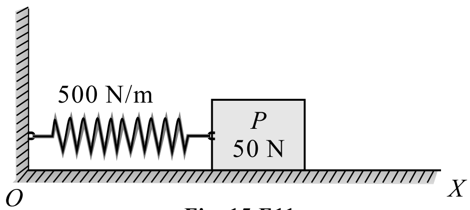


Fig. 15.E11

[***Ans.*** v = 0.443 m/s and x = 2.5 cm.]

12. Figure 15.E12 shows a collar A having mass of 5 kg that can slide without friction on a pipe. If it is released from rest at the position shown, where the spring is unstretched, what speed will the collar have after moving 50 mm? Take spring constant as 2000 N/m.

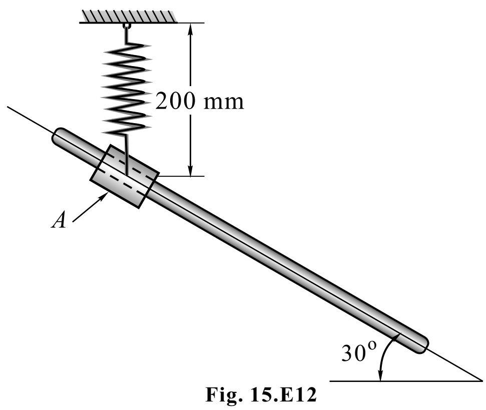


Fig. 15.E12

[***Ans.*** 0.5 m/s]

13. The 25 N cylinder is falling A with a speed v_A = 3 m/s on to the platform as shown in Fig. 15.E13. Determine the maximum displacement of the platform, caused by the collision. The spring has an unstretched length of 0.53 m and is originally kept in compression by the 0.3 m long cables attached to the platform. Neglect the mass of the platform and spring and any energy lost during the collision.

[**Ans.** 0.022 m]

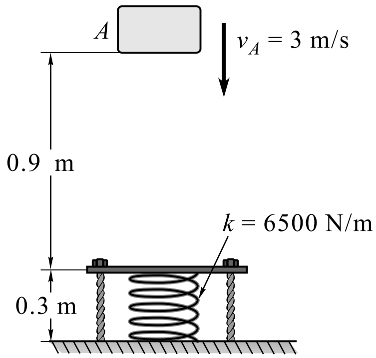


Fig. 15.E13

14. The platform P, shown in Fig. 15.E14 has negligible mass and it tied down so that the 0.4 m long cords keep the spring compressed 0.6 m when nothing is on the platform. If a 2 kg block is placed on the platform and released from rest after the platform is pushed down 0.1 m, determine the maximum height 'h' the block rises in the air, measured from the ground.

[**Ans.** 0.963 m]

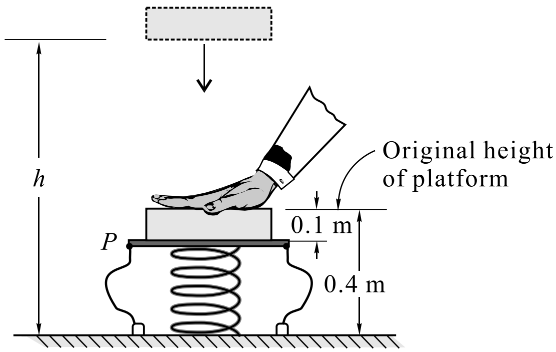


Fig. 15.E14

15. The 30 N ball is fired from a tube by a spring having a stiffness K = 4000 N/m as shown in Fig. 15.E15. Determine how, far the spring must be compressed to fire the ball from the compressed position to a height of 2.4 m, at which point it has a velocity of 1.8 m/s.

[**Ans.** 0.196 m]

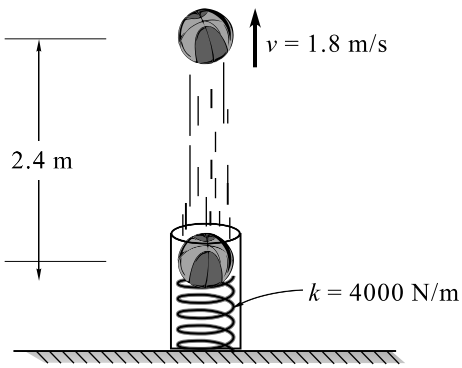


Fig. 15.E15

16. If a mass 'm' hangs freely stretches the spring a distance 'C' m, as shown in Fig. 15.E16. Show that if the mass is suddenly released the spring stretches a distance '$2C$' before mass starts to return upward (To show $S = 2C$).

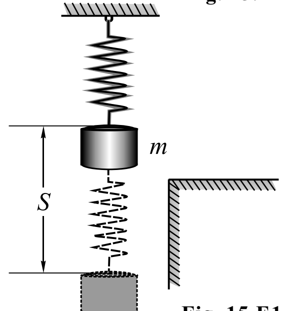


Fig. 15.E16

17. A 3 kg collar is attached to a spring and slides without friction in a vertical plane along the curved rod *ABC*, as shown in Fig. 15.E17. The spring is undeformed when collar is at *C* and its constant is 600 N/m. If the collar is released at *A* with no initial velocity, determine the velocity **(a)** as it passes through *B* and **(b)** as it releases *C*.

[**Ans.** $v_B = 2.33$ m/s and $v_C = 1.23$ m/s.]

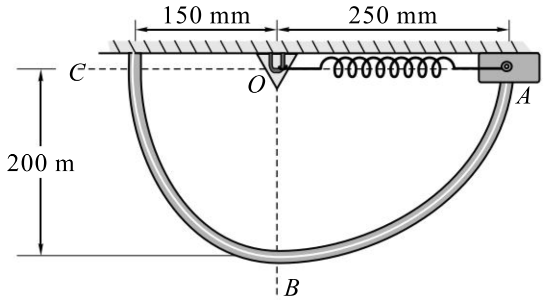


Fig. 15.E17

18. A 1.5 kg collar is attached to a spring and slides without friction along a circular rod in a horizontal plane as shown in Fig. 15.E18. The spring has an undeformed length of 150 mm and a constant k = 400 N/m. Knowing that the collar is in equilibrium at *A* and is given a slight push to get it moving, determine the velocity of the collar **(a)** as it passes through *B* and **(b)** as it passes through *C*.

[**Ans.** $v_B = 3.46$ m/s and $v_C = 4.47$ m/s.]

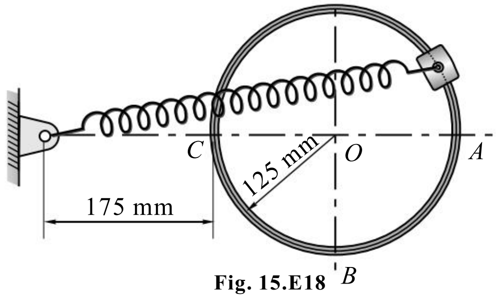


Fig. 15.E18

19. A 7 kg collar *A* slides with negligible friction on the fixed vertical shaft as shown in Fig. 15.E19. When the collar is released from rest at the bottom position shown, it moves up the shaft under the action of constant force F = 200 N applied to the cable. Calculate the stiffness k, which the spring must have if its maximum compression is to be limited to 75 mm. The position of the small pulley at *B* is fixed.

[**Ans.** $k = 8.79$ kN/m]

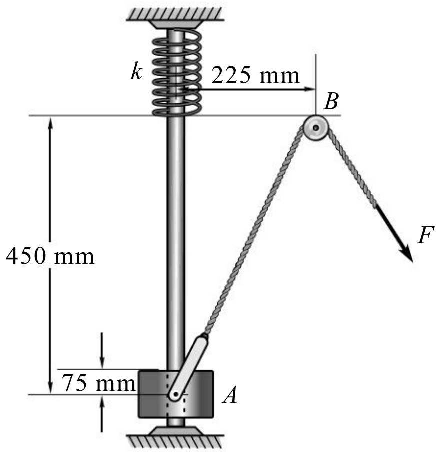


Fig. 15.E19

20. The 15 kg collar *A* is released from rest in the position shown in Fig. 15.E20 and slides with negligible friction up the fixed rod inclined 30° from the horizontal under the action of a constant force P = 200 N applied to the cable. Calculate the required stiffness k of the spring sot that its maximum deflection equals 180 mm. The position of the small pulley at *B* is fixed.

[**Ans.** $k = 1957$ kN/m]

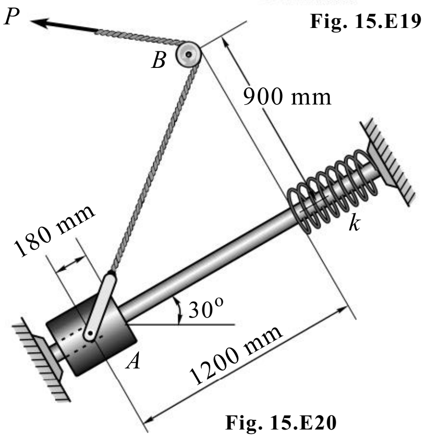


Fig. 15.E20

[II] Review Questions

1. Explain the following :
   (a) Work done by a force
   (b) Work done by a weight force
   (c) Work done by a frictional force
   (d) Work done by a spring force
2. State and prove work energy principle.
3. State the principle of conservation of energy.
4. Explain the term power.

[III] Fill in the Blanks

1. Work done by a force is ________ if the direction of force and the direction of displacement both are in opposite direction.
2. Work is a ________ quantity.
3. Work done by frictional force is always ________.
4. The energy possessed by a particle by virtue of its motion is called ________.
5. Work done by weight force will be ________ if moved from lower position to upper position.

[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. Work done by normal reaction and component of weight force perpendicular to inclined plane is ________.

   (a) positive **(b)** negative **(c)** zero **(d)** none of these
2. The energy possessed by a particle by virtue of its position is called as _________.

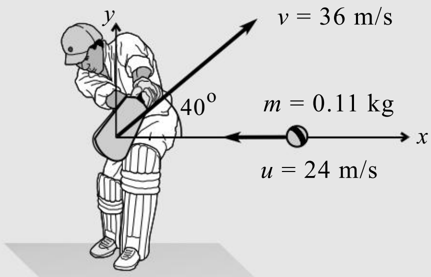
   (a) potential energy **(b)** kinetic energy **(c)** strain energy **(d)** heat energy
3. Frictional force in a system is considered as ________ force.

   (a) conservative **(b)** non-conservative **(c)** neutral **(d)** virtual
4. Work done by the forces in a system may be _________.

   (a) positive **(b)** negative **(c)** zero **(d)** any one of these

16

Kinetics of Particles - III Impulse Moment Principle and Impact



16.1 Introduction

In the first part of kinetics, we had used the term *force*, *mass* and *acceleration* to solve the problem by *Newton's Second Law* whereas in the second part of kinetics, we have used the term *force*, *velocity* and *displacement* to solve the problem by *Work Energy Principle*. In this chapter, we are going to use the term *force*, *time* and *velocity* to solve the problem by *Impulse Momentum Principle*.

16.2 Principle of Impulse and Momentum

Let F be the force acting on a particle having mass m and producing an acceleration a.

By Newton's second law of motion, we have

$$F = ma$$

$$F = m\frac{dv}{dt} \qquad \left(\because a = \frac{dv}{dt}\right)$$

$$F\,dt = m\,dv$$

Integrating both sides

$$\therefore \int_{t_1}^{t_2} F\,dt = \int_{v_1}^{v_2} m\,dv$$

$$\therefore \int_{t_1}^{t_2} F\,dt = mv_2 - mv_1$$

The term $\int_{t_1}^{t_2} F\,dt$ is called *impulse* and its unit is N-s (Newton-second).

The term mass × velocity is called *momentum*.

So, we have Impulse = Final momentum – Initial momentum

Since the velocity is a vector quantity, impulse is also a vector quantity.

Impulse of Force

When a large force acts over a small finite period the force is called as an *impulse force*.

Impulse of force F acting over a time interval from t_1 to t_2 is defined as

$$I = \int_{t_1}^{t_2} F\,dt$$

Points to be considered :

1. When impulse force acts on the system, non-impulsive force such as weight of the bodies is neglected.
2. When the impulsive forces acts for very small time, impulse due to external forces is zero.
3. The internal forces between the particles need not be considered as the sum of impulses or internal forces are zero.

In Component Form

$$\int_{t_1}^{t_2} F_x \, dt = mv_{x_2} - mv_{x_1}$$

and

$$\int_{t_1}^{t_2} F_y \, dt = mv_{y_2} - mv_{y_1}$$

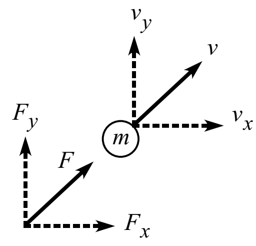


Fig. 16.2-i

The component of the resultant linear impulse along any direction is equal to change in the component of momentum in that direction.

16.3 Principle of Conservation Momentum

In a system, if the resultant force is zero, the impulse momentum equation reduces to final momentum equal to initial momentum. Such situation arises in many cases because the force system consists of only action and reaction on the elements of the system. The resultant force is zero, only when entire system is considered, but not when the free body of each element of the system is considered.

If a man jumps off a boat, the action of the man is equal and opposite to the reaction of the boat. Hence, the resultant is zero in the system.

Initial momentum = Final momentum

Similar equation holds good when we consider the system of a gun and shell.

16.4 Impact

Phenomenon of collision of two bodies, which occurs for a very small interval of time and during which two bodies exert very large force on each other, is called an *impact*.

Line of Impact

The common normal to the surfaces of two bodies in contact during the impact is called *line of impact*. Line of impact is perpendicular to common tangent.

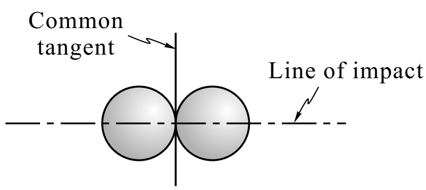


Fig. 16.4-i

Central Impact

When the mass centres C_1 and C_2 of the colliding bodies lie on the line of impact, it is called *central impact*.

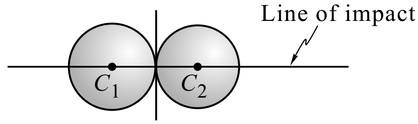


Fig. 16.4-ii

Non-Central Impact

When the mass centres C_1 and C_2 of the colliding bodies do not lie on the line of impact, it is called *non-central* or *eccentric impact*.

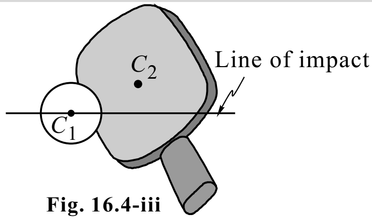


Fig. 16.4-iii

Direct Central Impact

When the direction of motion of the mass centres of two colliding bodies is along the line of impact then we say it is *direct central impact*. Here, the velocities of two bodies collision are collinear with the line of impact.

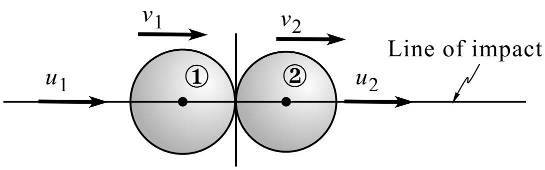


Fig. 16.4-iv

u_1 = Velocity of body ① before collision

u_2 = Velocity of body ② before collision

v_1 = Velocity of body ① after collision

v_2 = Velocity of body ② after collision

m_1 = Mass of body ①

m_2 = Mass of body ②

1. Total momentum of system is conserved along the line of impact.

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

2. Coefficient of restitution relation

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

Oblique Central Impact

When the direction of motion of the mass centres of one or two colliding bodies is not along the line of impact (i.e., at the same angle with the line of impact) then we say it is *oblique central impact*. Here the velocities of two bodies collision are not collinear with the line of impact.

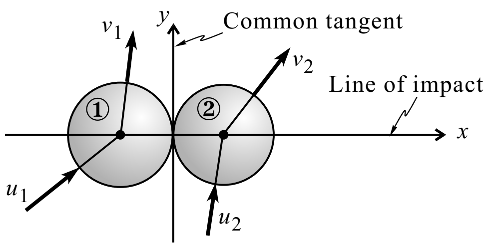


Fig. 16.4-v

1. The component of the total momentum of the two bodies along the line of impact is conserved.

$$m_1u_{1x} + m_2u_{2x} = m_1v_{1x} + m_2v_{2x}$$

2. Coefficient of restitution relation along the line of impact is

$$e = -\left[\frac{v_{2x} - v_{1x}}{u_{2x} - u_{1x}}\right]$$

3. Component of the momentum along the common tangent is conserved, which means the component of the velocities along the common tangent remains unchanged.

$$m_1 u_{1y} = m_1 v_{1y} \qquad \therefore\ u_{1y} = v_{1y}$$
$$m_2 u_{2y} = m_2 v_{2y} \qquad \therefore\ u_{2y} = v_{2y}$$

Note : Here the x-axis is line of impact which can be treated as normal (n) and y-axis is the common tangent which can be treated as tangent (t).

16.5 Coefficient of Restitution (e)

When two bodies collide for a very small interval of time, there will be phenomena of *Deformation* and *Restitution* (*Regain*) of shape.

By Impulse Momentum Principle for the process of deformation of the colliding body of mass m_1, we have

$$m_1 u_1 - \int F_D\, dt = m_1 u \qquad \text{... (16.1)}$$

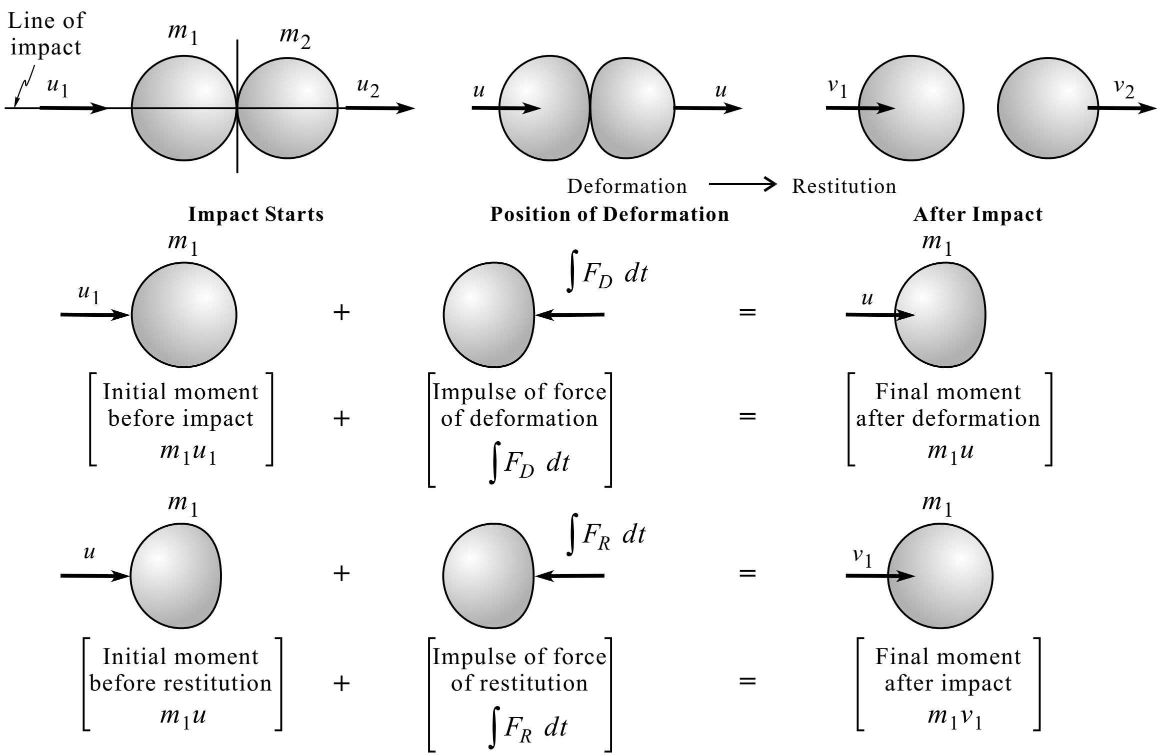


Fig. 16.5-i

Similarly, by Impulse Momentum Principle for the process of restitution of body of mass m_1, we have

$$m_1 u - \int F_R\, dt = m_1 v_1 \qquad \text{... (16.2)}$$

From Eqs. (16.1) and (16.2), we have

$$\int F_D\, dt = m_1 u_1 - m_1 u \quad \text{and} \quad \int F_R\, dt = m_1 u - m_1 v_1$$

$$\therefore \quad \frac{\int F_R\, dt}{\int F_D\, dt} = \frac{m_1(u - v_1)}{m_1(u_1 - u)} = \frac{u - v_1}{u_1 - u} = e \qquad \text{... (16.3)}$$

Similarly, by Impulse Momentum Principle for the process of deformation and restitution, we have

$$\frac{\int F_R\, dt}{\int F_D\, dt} = \frac{v_2 - u}{u - u_2} \qquad \text{... (16.4)}$$

From Eqs. (16.3) and (16.4), we get

$$\left[\frac{u - v_1 + v_2 - u}{u_1 - u + u - u_2}\right] = e$$

$$v_2 - v_1 = e(u_1 - u_2)$$

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{\text{Velocity of separation}}{\text{Velocity of approach}}$$

$$\therefore \quad e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

Classification of Impact Based on Coefficient of Restitution

1. **Perfectly Elastic Impact**
   - **(a)** Coefficient of restitution $e = 1$.
   - **(b)** Momentum is conserved along the line of impact

     $$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
   - **(c)** KE is conserved. No loss of KE

     $\therefore$ Total KE before impact = Total KE after impact

     $$\frac{1}{2} m_1u_1^2 + \frac{1}{2} m_2u_2^2 = \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2$$

2. **Perfectly Plastic Impact**
   - **(a)** Coefficient of restitution $e = 0$.
   - **(b)** After impact both the bodies collide and move together.
   - **(c)** Momentum is conserved

     $$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

     where v is the common velocity after impact.
   - **(d)** There is loss of KE during impact. Thus, KE is not conserved.

     Loss of KE = Total KE before impact – Total KE after impact

     $$\text{Loss of KE} = \left(\frac{1}{2} m_1u_1^2 + \frac{1}{2} m_2u_2^2\right) - \left(\frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2\right)$$

3. **Semi-elastic Impact**

   Coefficient of restitution $(0 < e < 1)$

16.6 Solved Problems Based on Impact

Problem 1

Two particles of masses 10 kg and 20 kg are moving along a straight line towards each other at velocities of 4 m/s and 1 m/s, respectively, as shown in Fig.16.1. If e = 0.6, determine the velocities of the particles immediately after their collision. Also find the loss of kinetic energy.

Solution

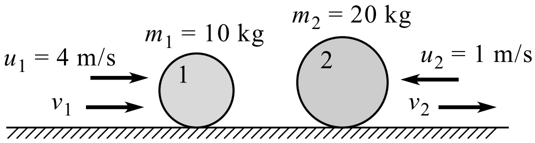
(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$10 \times 4 + 20 \times (-1) = 10v_1 + 20v_2$$

$$20 = 10v_1 + 20v_2$$

$$v_1 + 2v_2 = 2 \qquad \text{... (I)}$$



(v_1 and v_2 are the assumed directions toward right after impact)

Fig. 16.1

(ii) By coefficient of restitution, we have

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$0.6 = -\left[\frac{v_2 - v_1}{-1 - 4}\right]$$

$$v_2 - v_1 = 3 \qquad \text{... (II)}$$

Solving Eqs. (I) and (II), we get

$v_2 = 1.667$ m/s ($\rightarrow$) ***Ans.***

$v_1 = -1.333$ m/s

$\therefore\ v_1 = 1.333$ m/s ($\leftarrow$) ***Ans.***

(iii) Loss of KE = Initial KE – Final KE

$$\text{Loss of KE} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$\text{Loss of KE} = \left[\frac{1}{2}\times 10 \times 4^2 + \frac{1}{2}\times 20 \times (-1)^2\right] - \left[\frac{1}{2}\times 10 \times (-1.333)^2 + \frac{1}{2}\times 20 \times 1.667^2\right]$$

$$\text{Loss of KE} = 90 - 36.67$$

$$= 53.33 \text{ J}$$

$$\%\text{ loss in KE} = \frac{\text{Loss in KE}}{\text{Initial KE}} \times 100$$

$$= \frac{53.33}{90} \times 100$$

$\therefore$ % loss in KE = 59.27 % ***Ans.***

Problem 2

A 50 gm ball is dropped from a height of 600 mm on a small plate as shown in Fig.16.2(a). It rebounds to a height of 400 mm when the plate directly rests on the ground and to a height of 250 mm when a foam rubber mat is placed between the plate and the ground. Determine **(i)** the coefficient of restitution between the plate and the ground and **(ii)** the mass of the plate.

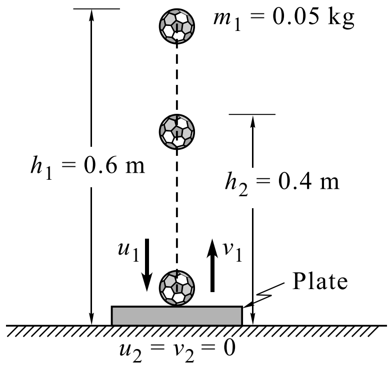


Fig. 16.2(a)

Solution

(i) The plate is kept directly on the ground

$$u_1 = \sqrt{2gh_1} \ (\downarrow) \ \text{(velocity before impact)}$$

$$u_1 = \sqrt{2 \times 9.81 \times 0.6}$$

$$u_1 = 3.43 \text{ m/s } (\downarrow)$$

$$v_1 = \sqrt{2gh_2} \ (\uparrow) \ \text{(velocity after impact)}$$

$$v_1 = \sqrt{2 \times 9.81 \times 0.4}$$

$$v_1 = 2.8 \text{ m/s } (\uparrow)$$

Coefficient of restitution

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right] = -\left[\frac{0 - 2.8}{0 - (-3.43)}\right]$$

$e = 0.816$ ***Ans.***

(ii) The plate is kept on the foam rubber mat

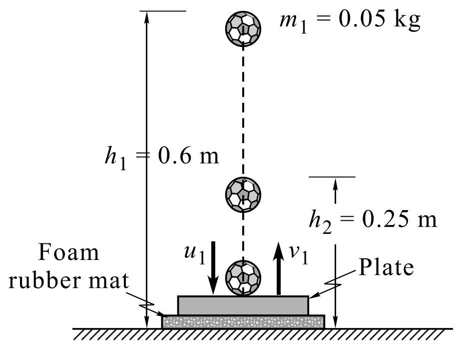


Fig. 16.2(b)

$$u_1 = \sqrt{2 \times 9.81 \times 0.6}$$

$$u_1 = 3.43 \text{ m/s } (\downarrow)$$

$$v_1 = \sqrt{2 \times 9.81 \times 0.25}$$

$$v_1 = 2.215 \text{ m/s } (\uparrow)$$

Coefficient of restitution gives

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$0.816 = -\left[\frac{-v_2 - 2.215}{0 - (-3.43)}\right]$$

$$\therefore \ v_2 = 0.584 \text{ m/s } (\downarrow)$$

(iii) By law of conservation of momentum, we have

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$0.05 \times (-3.43) + m_2 \times 0 = 0.05 \times 2.215 + m_2 \times (-0.584)$$

$\therefore \ m_2 = 0.483$ kg (mass of the plate) ***Ans.***

Problem 3

Two smooth spheres ① and ② having a mass of 2 kg and 4 kg, respectively collide with initial velocities, as shown in Fig. 16.3(a). If the coefficient of restitution for the spheres is e = 0.8, determine the velocities of each sphere after collision.

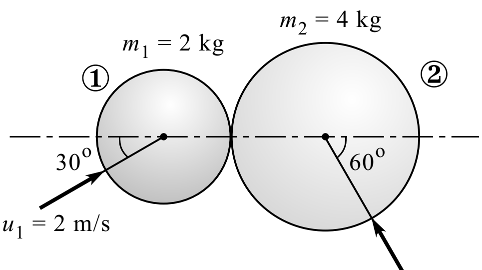


Fig. 16.3(a)

Solution

(i) By law of conservation of momentum along line of impact, we have

$$m_1u_{1x} + m_2u_{2x} = m_1v_{1x} + m_2v_{2x}$$

$$2 \times 2 \cos 30^\circ + 4 \times (-4 \cos 60^\circ) = 2(-v_{1x}) + 2v_{2x}$$

$$-v_{1x} + 2v_{2x} = -2.268 \quad \text{... (I)}$$

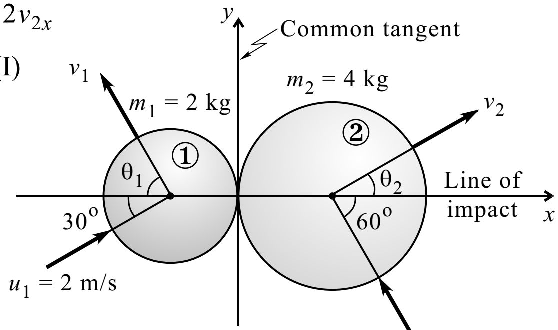


Fig. 16.3(b)

Coefficient of restitution along the line of impact gives

$$e = -\left[\frac{v_{2x} - v_{1x}}{u_{2x} - u_{1x}}\right]$$

$$0.8 = -\left[\frac{v_{2x} - (-v_{1x})}{-4 \cos 60^\circ - 2 \cos 30^\circ}\right]$$

$$v_{2x} + v_{1x} = 2.986 \text{ m/s} \quad \text{... (II)}$$

Solving Eqs. (I) and (II), we get

v_{1x} = 2.747 m/s (←) and v_{2x} = 0.239 m/s (→) ***Ans.***

(ii) Component of velocity before and after impact along common tangent is conserved

$$v_{1y} = 2 \sin 30^\circ \qquad v_{2y} = 4 \sin 60^\circ$$

$$v_{1y} = 1 \text{ m/s } (\uparrow) \qquad v_{2y} = 3.464 \text{ m/s } (\uparrow)$$

For v_1, we have

$$\tan \theta_1 = \frac{v_{1y}}{v_{1x}} = \frac{1}{2.747}$$

$$\theta_1 = 20^\circ$$

$$v_1 = \sqrt{{v_{1x}}^2 + {v_{1y}}^2}$$

$$= \sqrt{2.747^2 + 1^2}$$

v_1 = 2.923 m/s (θ_1) ***Ans.***

Velocity of sphere ①

For v_2, we have

$$\tan \theta_2 = \frac{v_{2y}}{v_{2x}} = \frac{3.464}{0.239}$$

$$\theta_2 = 86.05^\circ$$

$$v_2 = \sqrt{{v_{2x}}^2 + {v_{2y}}^2}$$

$$= \sqrt{0.239^2 + 3.464^2}$$

v_2 = 3.472 m/s (θ_2) ***Ans.***

Velocity of sphere ②

Problem 4

Two balls of same mass 0.5 kg moving with velocities as shown in Fig. 16.4(a), collide. If after collision ball ② travels along a line 30° counter clockwise from y-axis, determine the coefficient of restitution.

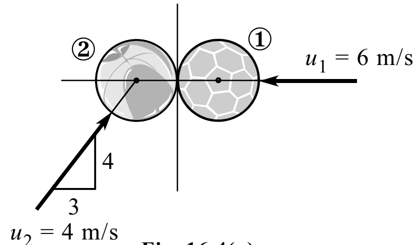


Fig. 16.4(a)

Solution

(i) By law of conservation of momentum along line of impact, we have

$$m_1 u_{1x} + m_2 u_{2x} = m_1 v_{1x} + m_2 v_{2x}$$

$$u_{1x} + u_{2x} = v_{1x} + v_{2x}$$

$$-6 + 4\cos 53.13^\circ = v_{1x} + (-v_2 \cos 60^\circ)$$

$$v_{1x} - 0.5 v_2 = -3.6$$

$$2v_{1x} - v_2 = -7.2 \qquad \text{... (I)}$$

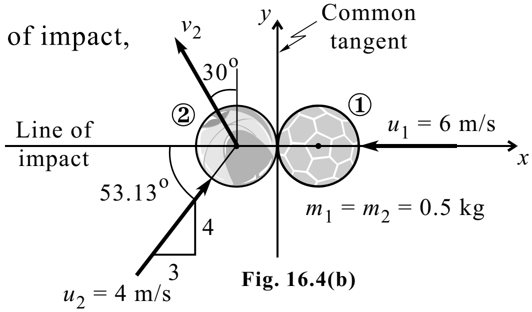


Fig. 16.4(b)

(ii) Component of velocity before and after impact along common tangent is conserved

$$v_2 \sin 60^\circ = 4 \sin 53.13^\circ$$

$$v_2 = 3.7 \text{ m/s}$$

From Eq. (I)

$$2v_{1x} - 3.7 = -7.2$$

$$v_{1x} = -1.75 \text{ m/s} \qquad \therefore v_{1x} = 1.75 \text{ m/s } (\leftarrow)$$

$$\because v_{1y} = 0 \quad \therefore v_1 = v_{1x} = 1.75 \text{ m/s } (\leftarrow)$$

Coefficient of restitution along the line of impact gives

$$e = -\left[\frac{v_{2x} - v_{1x}}{u_{2x} - u_{1x}}\right]$$

$$e = -\left[\frac{-3.7\cos 60 - (-1.75)}{4\cos 53.13 - (-6)}\right]$$

$\therefore e = 0.012$ ***Ans.***

Problem 5

A billiard ball, shown in Fig. 16.5 moving with a velocity of 5 m/s strikes a smooth horizontal plane at an angle 45° with the horizontal. If $e = 0.6$, between ball and plane what is the velocity with which the ball rebounds ?

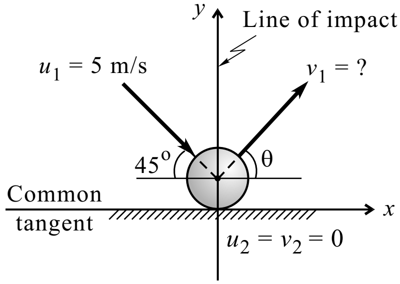


Fig. 16.5

Solution

(i) Coefficient of restitution along the line of impact gives

$$e = -\left[\frac{v_{2y} - v_{1y}}{u_{2y} - u_{1y}}\right]$$

$$0.6 = -\left[\frac{0 - v_{1y}}{0 - (-5 \sin 45^o)}\right]$$

$$v_{1y} = 2.12 \text{ m/s } (\uparrow)$$

(ii) Component of velocity before and after impact along common tangent is conserved

$$v_{1x} = 5 \cos 45^o \qquad \therefore v_{1x} = 3.54 \text{ m/s } (\rightarrow)$$

$$\tan\theta = \frac{v_{1y}}{v_{1x}} = \frac{2.12}{3.54} \qquad \therefore \theta = 30.92^o$$

$$v_1 = \sqrt{{v_{1x}}^2 + {v_{1y}}^2} = \sqrt{3.54^2 + 2.12^2}$$

$$\therefore v = 4.126 \text{ km/hr } (\angle 30.92^o)$$

The ball rebounds with a velocity 4.126 m/s at an angle of 30.92^o w.r.t. x-axis. ***Ans.***

Problem 6

A ball is thrown against a wall with a velocity u forming an angle 30^o with the horizontal as shown in Fig. 16.6(a). Assuming frictionless conditions and $e = 0.5$, determine the magnitude and direction of velocity of ball after it rebounds from the wall.

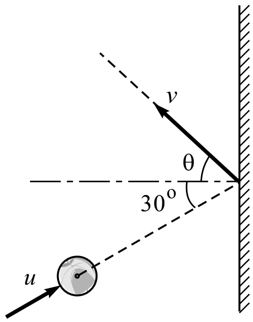


Fig. 16.6(a)

Solution

(i) Coefficient of restitution along the line of impact gives

$$e = -\left[\frac{v_{2x} - v_{1x}}{u_{2x} - u_{1x}}\right]$$

$$0.5 = -\left[\frac{0 - (-v_1 \cos\theta)}{0 - u_1 \cos 30^o}\right]$$

$$0.433u_1 = v_1 \cos\theta \qquad \text{... (I)}$$

(ii) Component of velocity before and after along common tangent is conserved

$$u_1 \sin 30 = v_1 \sin\theta \qquad \text{... (II)}$$

Dividing Eq. (I) by Eq. (II)

$$\frac{0.433u_1}{u_1 \sin 30} = \frac{v_1 \cos\theta}{v_1 \sin\theta}$$

$$\tan\theta = \frac{\sin 30^o}{0.433} \qquad \therefore \theta = 49.12^o$$

Putting the value in Eq. (I), we get

$$0.433u_1 = v_1 \cos 49.12^o$$

$$v_1 = 0.6616u$$

The ball rebounds with velocity $0.661u$ at an angle of 49.12^o. ***Ans.***

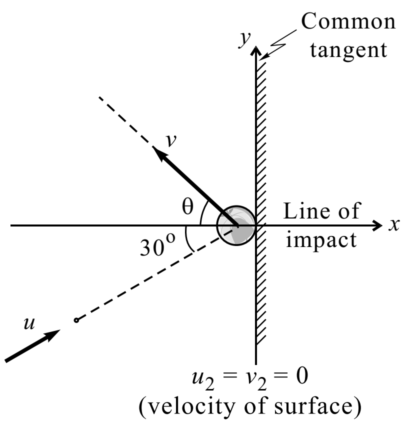


Fig. 16.6(b)

Problem 7

A ball is dropped on an inclined plane from a height of 3 m vertically down, and the ball is observed to move horizontally with a velocity v, as shown in Fig. 16.7(a). If the coefficient of restitution is $e = 0.6$, determine the inclination of the plane and the velocity of the ball after impact.

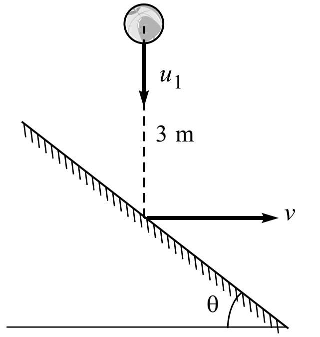


Fig. 16.7(a)

Solution

(i) Velocity before impact

$$u_1 = \sqrt{u^2 + 2gh} = \sqrt{0 + 2 \times 9.81 \times 3}$$

$$u_1 = 7.672 \text{ m/s} \quad (\downarrow)$$

(ii) Coefficient of restitution along the line of impact gives

$$e = -\left[\frac{v_{2y} - v_{1y}}{u_{2y} - u_{1y}}\right]$$

$$0.6 = -\left[\frac{0 - v_1 \sin\theta}{0 - (-7.672 \cos\theta)}\right]$$

$$v_1 \sin\theta = 4.603 \cos\theta$$

$$v_1 = 4.603 \cot\theta \quad \text{... (I)}$$

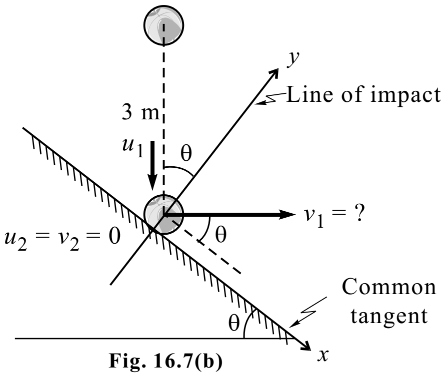


Fig. 16.7(b)

(iii) Component of velocity along common tangent before and after impact is conserved

$$v_1 \cos\theta = 7.672 \sin\theta$$

$$v_1 = 7.672 \tan\theta \quad \text{... (II)}$$

From Eqs. (I) and (II), we get

$$4.603 \cot\theta = 7.672 \tan\theta$$

$$\tan^2\theta = \frac{1}{1.667} \quad \therefore \ \theta = 37.76^\circ$$ (Inclination of the plane) ***Ans.***

From Eq. (II), we get

$$v_1 = 7.672 \tan 37.76^\circ \quad \therefore \ v_1 = 5.942 \text{ m/s } (\rightarrow)$$ (Velocity of the ball after impact) ***Ans.***

Problem 8

Ball A of mass m is released from rest and slides down a smooth bowl and strikes another ball B of mass $m/4$, which is resting at bottom of the bowl, as shown in Fig. 16.8. Determine height h from which ball A should be released, so that after direct central impact, ball B just leaves the bowls surface. Take coefficient of restitution $e = 0.8$.

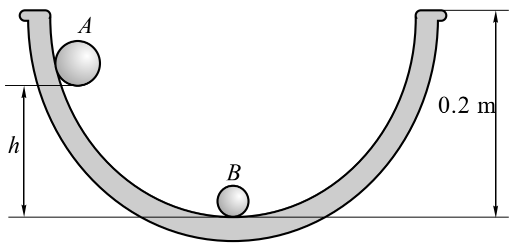


Fig. 16.8

Solution

$$m_A = m_1 = m \ ; \ u_A = u_1 = \sqrt{2gh} \ ; \ v_A = v_1 = ?$$

$$m_B = m_2 = 0.25m \ ; \ u_B = u_2 = 0 \ ; \ v_B = v_2 = \sqrt{2g \times 0.2} = 1.981 \text{ m/s}$$

(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$m\sqrt{2gh} + 0 = mv_1 + 0.25mv_2$$

$$\sqrt{2gh} = v_1 + 0.25 \times 1.981$$

$$\sqrt{2gh} = v_1 + 0.495 \quad \text{... (I)}$$

(ii) Coefficient of restitution gives

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right] \Rightarrow 0.8 = -\left[\frac{1.981 - v_1}{0 - \sqrt{2gh}}\right]$$

$$0.8\sqrt{2gh} = 1.981 - v_1$$

$$\sqrt{2gh} = 2.476 - 1.25v_1 \quad \text{... (II)}$$

Comparing Eqs. (I) and (II), we get

$$v_1 + 0.495 = 2.476 - 1.25v_1$$

$v_1 = 0.88$ m/s ***Ans.***

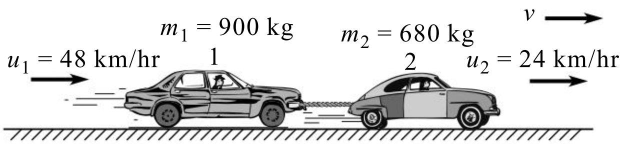
Substituting in Eq. (I), we get

$$\sqrt{2gh} = 0.88 + 0.495$$

$\therefore\ h = 0.096$ m ***Ans.***

Problem 9

A 900 kg car is travelling 48 km/hr couples to 680 kg car travelling 24 km/hr in the same directions, as shown in Fig. 16.9. What is their common speed after coupling.



(v is the common speed after coupling)

Fig. 16.9

Solution

By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

$$900 \times 48 + 680 \times 24 = (900 + 680)v$$

$v = 37.67$ km/hr (→) ***Ans.***

Problem 10

A 20 gm bullet is fired with a velocity of magnitude 600 m/s into a 4.5 kg block of wood, which is stationary as shown in Fig. 16.10. Knowing that the coefficient of kinetic friction between the block and the floor is 0.4. Determine **(i)** how far the block will move and **(ii)** the percentage of the initial energy lost in friction between the block and the floor.

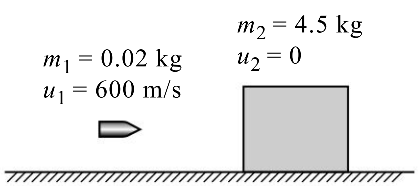


Fig. 16.10(a)

Solution

(i) By law of conservation of energy, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v_1$$

$$0.02 \times 600 + 0 = (0.02 + 4.5)v_1$$

$$v_1 = 2.655 \text{ m/s } (\rightarrow)$$

(velocity of bullet and block together after impact)

(ii) By work energy principal, we have

Work done = Change in kinetic energy

$$-0.4 \times 4.52 \times 9.81 \times d = 0 - \frac{1}{2} \times 4.52 \times 2.655^2$$

$\therefore$ $d = 0.9$ m ***Ans.***

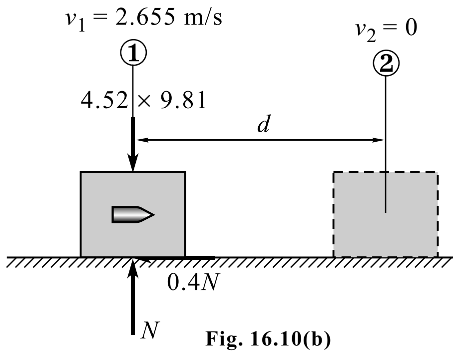


Fig. 16.10(b)

(iii) Initial KE = $\frac{1}{2} \times 0.02 \times 600^2 = 3600$ J

Energy lost in friction = $0.4 \times 4.52 \times 9.81 \times 0.9 = 15.93$ J

$\therefore$ percentage loss = $\frac{15.93}{3600} = 0.44$ % ***Ans.***

Problem 11

A 750 kg hammer of a drop hammer pile driver falls from a height of 1.2 m onto the top of a pile, as shown in Fig. 16.11. The pile is driven 100 mm into the ground. Assume perfectly plastic impact, determine the average resistance of the ground to penetration. Assume mass of pile as 2250 kg.

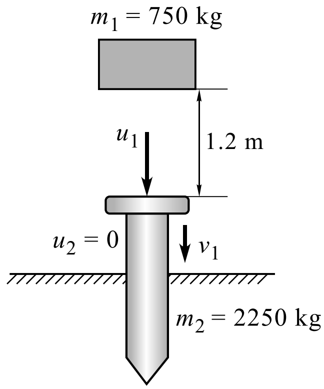


Fig. 16.11(a)

Solution

(i) Velocity before impact

$$m_1 = 750 \text{ kg}$$

$$u_1 = \sqrt{u^2 + 2gh} = \sqrt{0 + 2 \times 9.81 \times 1.2}$$

$$u_1 = 4.852 \text{ m/s } (\downarrow)$$

$$m_2 = 2250 \text{ kg} ; \quad u_2 = 0$$

(ii) For perfectly plastic impact

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v_1$$

$$750 \times 4.852 + 2250 \times 0 = (750 + 2250)v_1$$

$$v_1 = 1.213 \text{ m/s } (\downarrow)$$

After impact, both hammer and pile will move together with velocity $v_1 = 1.213$ m/s $(\downarrow)$

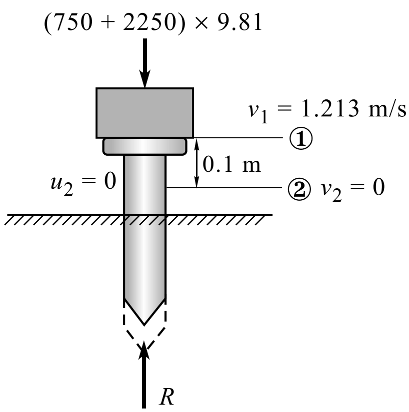


Fig. 16.11(b)

(iii) By work energy principal, we have

Work done = Change in kinetic energy

$$(750 + 2250) \times 9.81 \times 0.1 - R \times 0.1$$

$$= 0 - \frac{1}{2} \times (750 + 2250) \times 1.213^2$$

$$R = 51482.3 \text{ N } (\uparrow)$$

Average resistance force of the ground $R = 51482.3$ N $(\uparrow)$ ***Ans.***

Problem 12

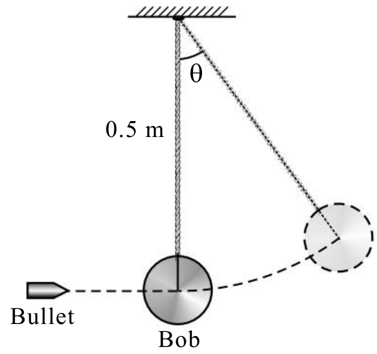


Fig. 16.12(a)

A bullet of mass 10 gm is moving with a velocity of 100 m/s and hits a 2 kg bob of a simple pendulum, horizontally, as shown in Fig. 16.12(a). Determine the maximum angle θ through which the pendulum string 0.5 m long may swing if **(i)** the bullet gets embedded in the bob and **(ii)** the bullet escapes from the other end of the bob with a velocity 10 m/s.

Solution

Case I : Find θ when the bullet gets embedded in the bob (Perfectly plastic impact)

(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v_1$$

$$0.01 \times 100 + 2 \times 0 = (0.01 + 2)v_1$$

$$v_1 = 0.4975 \text{ m/s } (\rightarrow)$$

Velocity of bullet and bob together after impact $v_1 = 0.4975$ m/s $(\rightarrow)$.

(ii) By work energy principal, we have

Work done = Change in kinetic energy

$$-2.01 \times 9.81 \times h = 0 - \frac{1}{2} \times 2.01 \times 0.4975^2$$

$$\therefore\ h = 0.0127 \text{ m}$$

(iii) $\cos\theta = \dfrac{0.5 - h}{0.5} = \dfrac{0.5 - 0.0127}{0.5}$

$\therefore\ \theta = 12.94^{\circ}$ ***Ans.***

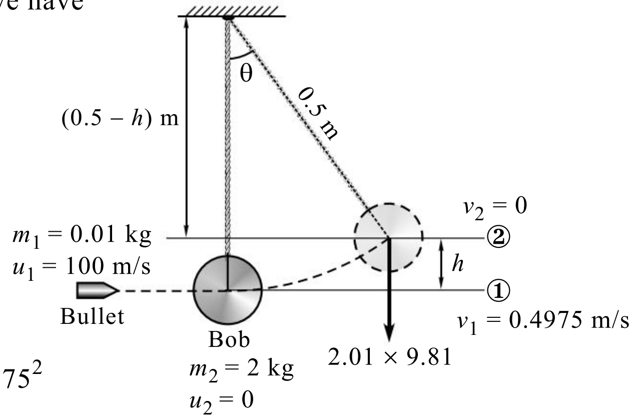


Fig. 16.12(b)

Case II : Find θ when the bullet escapes from the other end of the bob with velocity 10 m/s

(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$0.01 \times 100 + 2 \times 0 = 0.01 \times 10 + 2 \times v_2$$

$$v_2 = 0.45 \text{ m/s } (\rightarrow)$$

(ii) By work energy principal, we have

Work done = Change in kinetic energy

$$-2 \times 9.81 \times h = 0 - \frac{1}{2} \times 2 \times 0.45^2$$

$$\therefore\ h = 0.01032 \text{ m}$$

(iii) $\cos\theta = \dfrac{0.5 - h}{0.5} = \dfrac{0.5 - 0.01032}{0.5}$

$\therefore\ \theta = 11.66^{\circ}$ ***Ans.***

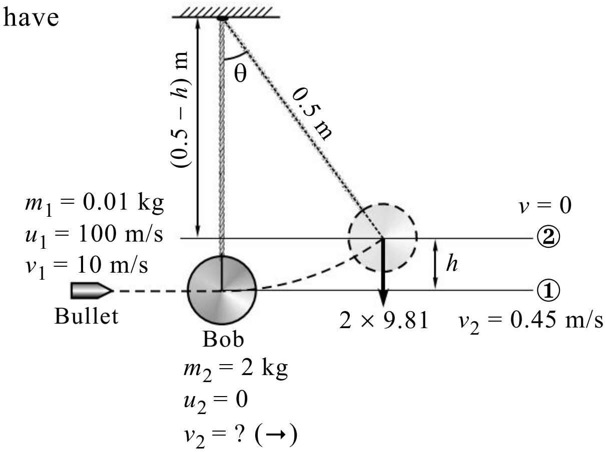


Fig. 16.12(c)

Problem 13

Determine the horizontal velocity u_A shown in Fig. 16.13(a) at which we must throw the ball so that it bounces once on the surface and then lands into the cup at C. Take the coefficient of restitution between ball and surface as $e = 0.6$ and neglect the size of the cup.

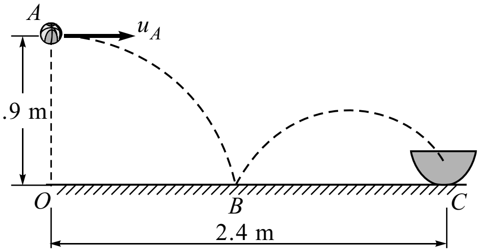


Fig. 16.13(a)

Solution

(i) Consider motion from *A* to *B*

Vertical motion

$$h = u\,t + \frac{1}{2}gt^2$$

$$0.9 = 0 + \frac{1}{2} \times 9.81 \times t_1^2$$

$$t_1 = 0.428 \text{ seconds}$$

$$v^2 = u^2 + 2gh$$

$$v = \sqrt{2 \times 9.81 \times 0.9} = 4.202 \text{ m/s}$$

$\therefore\ u_1 = 4.202$ m/s (↓) (velocity before impact at B)

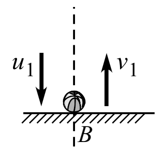
(ii) Consider the impact at *B*

Coefficient of restitution gives

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$0.6 = -\left[\frac{0 - v_1}{0 - (-4.202)}\right]$$

$\therefore\ v_1 = 2.52$ m/s (↑) (velocity of ball after impact)



$u_2 = v_2 = 0$
(velocity of surface before and after impact)
Fig. 16.13(b)

(iii) Consider motion from *B* to *C*

Vertical motion

$$h = u\,t + \frac{1}{2}gt^2$$

$$0 = 2.52 \times t_2 - \frac{1}{2} \times 9.81 \times t_2^2$$

$$t_2 = 0.514 \text{ seconds}$$

(iv) Consider projectile motion from *A* to *B* and *B* to *C*

Total time $t = t_1 + t_2 = 0.428 + 0.514$

$t = 0.942$ seconds

In projectile motion, horizontal motion happens with constant velocity

$\therefore$ Displacement = Velocity × Time

$2.4 = u_A \times 0.942$

$\therefore\ u_A = 2.548$ m/s (→) ***Ans.***

Problem 14

A small steel ball is to be projected horizontally such that it bounces twice on the surface and lands into a cup placed at a distance of 8 m, as shown in Fig. 16.14. If the coefficient of restitution for each impact is 0.8, determine the velocity of projection 'u' of the ball.

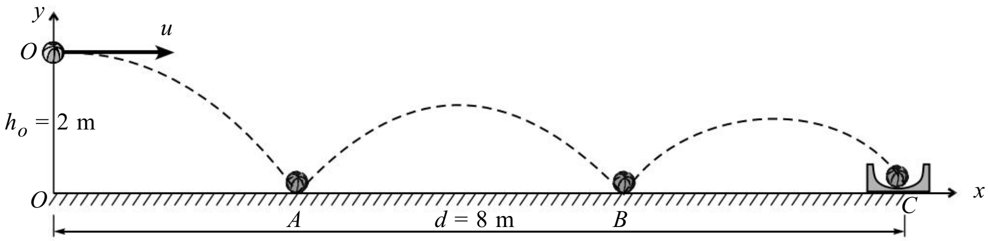


Fig. 16.14

Solution

(i) Consider motion from *O* to *A*

$$u_{1y} = \sqrt{2 \times 9.81 \times 2} \qquad \therefore\ u_{1y} = 6.264 \text{ m/s} \quad (\downarrow)$$

v_{1yA} = ? (↑) ; $u_{2y} = v_{2y} = 0$ (velocity of flow before and after impact)

(ii) Impact at *A*

$$e = -\left[\frac{v_{2y} - v_{1y}}{u_{2y} - u_{1y}}\right] = -\left[\frac{0 - v_{1yA}}{0 - (-6.264)}\right] = 0.8$$

$$v_{1yA} = 5.01 \text{ m/s } (\uparrow)$$

(iii) Impact at *B*

$$e = -\left[\frac{v_{2y} - v_{1y}}{u_{2y} - u_{1y}}\right] = -\left[\frac{0 - v_{1yB}}{0 - (-5.01)}\right] = 0.8$$

$$v_{1yB} = 4 \text{ m/s } (\uparrow)$$

(iv) Time from *O* to *A*

$$h = u\,t + \frac{1}{2}gt^2$$

$$2 = 0 + \frac{1}{2} \times 9.81 \times t_1^2$$

$$t_1 = 0.6386 \text{ seconds}$$

(v) Time from *A* to *B*

$$h = u\,t + \frac{1}{2}gt^2$$

$$0 = 5.01 \times t_2 - \frac{1}{2} \times 9.81 \times t_2^2$$

$$t_2 = 1.021 \text{ seconds}$$

(vi) Time from *B* to *C*

$$h = u\,t + \frac{1}{2}gt^2$$

$$0 = 4 \times t_3 - \frac{1}{2} \times 9.81 \times t_3^2$$

$$t_3 = 0.8155 \text{ seconds}$$

(vii) Total time

$$t = t_1 + t_2 + t_3 = 0.6386 + 1.021 + 0.8155$$

$$t = 2.475 \text{ seconds}$$

(viii) For projectile motion and oblique impact component of velocity in horizontal direction through the motion remains constant.

$\therefore$ Displacement = Velocity × Time

$$8 = u \times 2.475$$

$\therefore$ $u = 3.232$ m/s (→) ***Ans.***

16.7 Solved Problems Based on Impulse and Momentum

Problem 15

A cannon gun is nested by three springs each of stiffness 250 kN/cm, as shown in Fig. 16.15(a). The gun fires a 500 kg shell with a muzzle velocity of 1000 m/s. Calculate the total recoil and the maximum force developed in each spring if the gun has a mass of 80,000 kg.

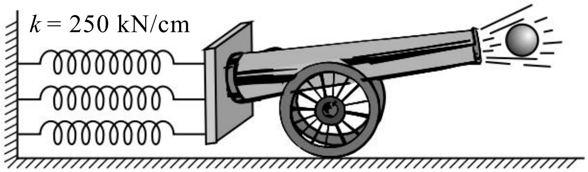


Fig. 16.15(a)

Solution

(i) By law of conservation of momentum, we have

Initial momentum = Final momentum

$$0 = 500 \times 1000 + 80000 \times v_{gun}$$

$$v_{gun} = -6.25 \text{ m/s}$$

$$\therefore \; v_{gun} = 6.25 \text{ m/s } (\leftarrow) \quad \textbf{Ans.}$$

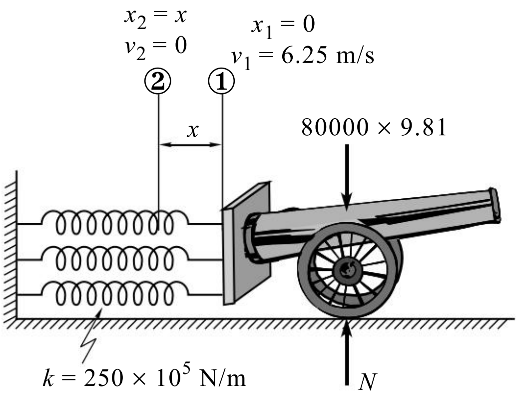


Fig. 16.15(b)

(iii) By work energy principal, we have

Work done = Change in kinetic energy

$$3\left[\frac{1}{2} \times 250 \times 10^5 (0^2 - x^2)\right] = 0 - \frac{1}{2} \times 80000 \times 6.25^2$$

$x = 0.204$ m (maximum compression of spring)

Spring force $F = kx$

$$\therefore \; F = 250 \times 10^5 \times 0.204$$

$$\therefore \; F = 51.025 \times 10^5 \text{ N} \quad \textbf{Ans.}$$

Problem 16

Figure 16.16(a) shows a block of weight $W = 10$ N sliding down from rest on a rough inclined plane. The coefficient of friction $\mu = 0.2$ and $\theta = 30^o$. Calculate **(i)** the impulse of the forces acting in the interval $t = 0$ to $t = 5$ seconds, **(ii)** the velocity at the end of 5 sec and **(iii)** Distance covered by the block.

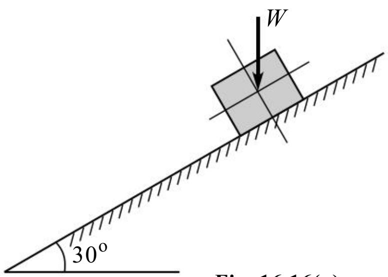


Fig. 16.16(a)

Solution

(i) Impulse = Net force × Time interval

$$\text{Impulse} = (10 \sin 30^o - 0.2 \times 10 \cos 30^o) \times 5$$

$$\text{Impulse} = 16.34 \text{ N-s} \quad \textbf{Ans.}$$

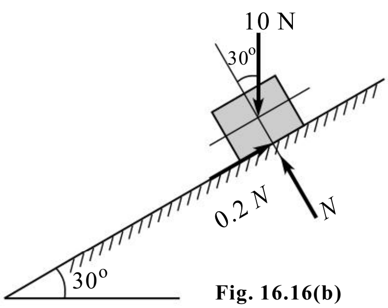


Fig. 16.16(b)

(ii) By impulse momentum principal, we have

Impulse = Change in momentum

$$16.34 = \frac{10}{9.81}(v - 0)$$

$$v = 16.03 \text{ m/s} \quad (30^\circ) \quad \textit{Ans.}$$

(iii) $d = \left(\frac{u+v}{2}\right) t$

$$d = \frac{0 + 16.03}{2} \times 5$$

$$d = 40.075 \text{ m} \quad \textit{Ans.}$$

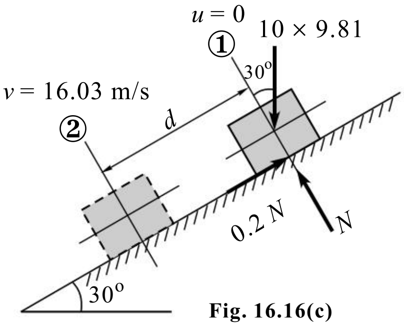


Fig. 16.16(c)

Problem 17

A ball of mass 110 gm is moving towards a batsman with a velocity of 24 m/s as shown in Fig. 16.17(a). The batsman hits the ball by the bat, the ball attains a velocity of 36 m/s. If ball and bat are in contact for a period of 0.015 seconds, determine the average impulsive force exerted on the ball during the impact.

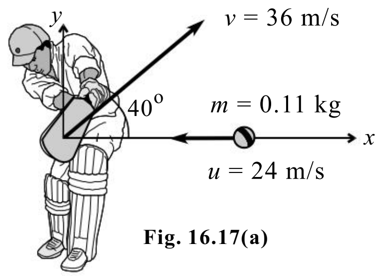


Fig. 16.17(a)

Solution

(i) Velocities before impact and after impact

$$u_x = 24 \text{ m/s } (\leftarrow)$$

$$u_y = 0$$

$$v_x = 36 \cos 40^\circ \text{ m/s } (\rightarrow)$$

$$v_y = 36 \sin 40^\circ \text{ m/s } (\uparrow)$$

(ii) By impulse momentum principle, we have

$$F_x \times t = m\,(v_x - u_x)$$

$$F_x \times 0.015 = 0.11 \times [36 \cos 40^\circ - (-24)]$$

$$\therefore \; F_x = 378.24 \text{ N } (\rightarrow)$$

$$F_y \times t = m\,(v_y - u_y)$$

$$F_y \times 0.015 = 0.11 \times [36 \sin 40^\circ - 0]$$

$$\therefore \; F_y = 169.7 \text{ N } (\uparrow)$$

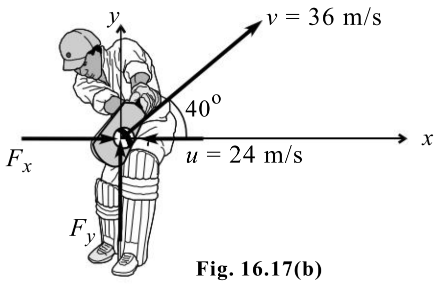


Fig. 16.17(b)

(iii) Resultant force exerted

$$\tan\theta = \frac{F_y}{F_x} = \frac{169.7}{378.24}$$

$$\theta = 24.16^\circ$$

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{378.24^2 + 169.7^2}$$

$$F = 414.56 \text{ N} \quad (24.16^\circ) \quad \textit{Ans.}$$

Problem 18

A boy of mass 60 kg and a girl of mass 50 kg dive off the end of a boat of mass 160 kg with a horizontal velocity of 2 m/s relative to the boat, as shown in Fig. 16.18. Considering the boat to be initially at rest, find its velocity just after **(i)** both the boy and girls dive off simultaneously and **(ii)** the boy dives first followed by the girl.

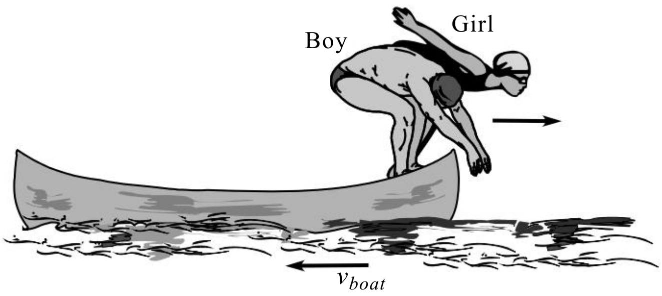


Fig. 16.18

Solution

(i) Both boy and girl dive off simultaneously

When boy and girl will jump together towards the right, the boat will move in opposite direction, i.e., towards left.

Here, velocity of boy and girl is 2 m/s relative to the boat

$\therefore\ v_{boy/boat} = v_{boy} - v_{boat}$

$2 = v_{boy} - (-v_{boat})$

$v_{boy} = 2 - v_{boat}$

and $v_{girl/boat} = v_{girl} - v_{boat}$

$2 = v_{girl} - (-v_{boat})$

$v_{girl} = 2 - v_{boat}$

By conservation of momentum principle to the system of boy, girl and boat

Initial momentum = Final momentum

$0 = (\text{mass} \times \text{velocity})_{boy} + (\text{mass} \times \text{velocity})_{girl} + (\text{mass} \times \text{velocity})_{boat}$

$0 = 60(2 - v_{boat}) + 50(2 - v_{boat}) + 160(-v_{boat})$

$0 = 120 - 60v_{boat} + 100 - 50v_{boat} - 160v_{boat}$

$-220 = -270v_{boat}$

$v_{boat} = 1.227 \text{ m/s } (\leftarrow)$ ***Ans.***

(ii) The boy dives first followed by the girl

Here, if boy is jumping first and girl is still on boat.

By conservation of momentum principle

Initial momentum = Final momentum

$0 = (\text{Mass} \times \text{Velocity})_{boy} + (\text{Mass} \times \text{Velocity})_{boat}$

$0 = 60(2 - v_{boat}) + 160(-v_{boat})$

$0 = 120 - 60v_{boat} - 160v_{boat}$

$-220v_{boat} = -120$

$v_{boat} = 0.5455 \text{ m/s } (\leftarrow)$ ***Ans.***

Later, the girl jumps from the boat when the boat is moving back with velocity 0.5455 m/s.

By conservation of momentum principle

$$\text{Initial momentum} = \text{Final momentum}$$

$$(\text{mass} \times \text{velocity})_{boat} = (\text{mass} \times \text{velocity})_{girl} + (\text{mass} \times \text{velocity})_{boat}$$

$$160(-0.5455) = 50(2 - v_{boat}) + 160(-v_{boat})$$

$$-87.28 = 100 - 50v_{boat} - 160v_{boat}$$

$$-210v_{boat} = -187.28$$

$$v_{boat} = 0.8918 \text{ m/s } (\leftarrow) \quad \textbf{\textit{Ans.}}$$

Problem 19

A boy having a mass of 60 kg and a girl having a mass of 50 kg stand motionless at the end of a boat, which has a mass of 30 kg, as shown in Fig. 16.19. If they exchange their positions, determine the final positions of boat. Neglect friction.

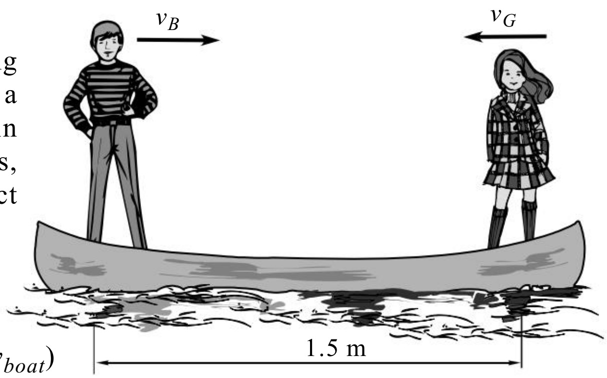


Fig. 16.19

Solution

$$\text{Initial momentum} = \text{Final momentum}$$

$$0 = m_Bv_B + m_G(-v_G) + (m_B + m_G + m_{boat})(v_{boat})$$

$$0 = 60 \times \left(\frac{1.5}{t}\right) - 50 \times \left(\frac{1.5}{t}\right) + (60 + 50 + 30) \times \left(\frac{x}{t}\right)$$

$$x = -0.107 \text{ m}$$

$$\therefore \; x = 0.107 \text{ m } (\leftarrow) \text{ (Backward displacement of boat)} \quad \textbf{\textit{Ans.}}$$

Problem 20

A particle of mass 1 kg is acted upon by a force F which varies, as shown in Fig. 16.20(a). If initial velocity of the particle is 10 m/s determine **(i)** what is the maximum velocity attained by the particle and **(ii)** the time when particle will be at point of reversal.

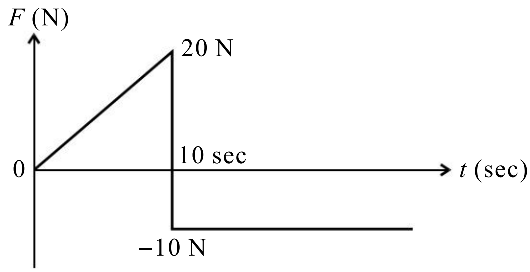


Fig. 16.20(a)

Solution

(i) For v_{max}

At force F = 20 N the velocity of particle will be maximum

$$\text{Change in velocity} = \frac{\text{Area under } F\text{-}t \text{ diagram}}{\text{Mass}}$$

$$v_{max} - 10 = \frac{\frac{1}{2} \times 10 \times 20}{1}$$

$$\therefore \; v_{max} = 110 \text{ m/s} \quad \textbf{\textit{Ans.}}$$

(ii) Let t be the time when particle will be at the point of reversal, where velocity $v = 0$

$$\text{Change in velocity} = \frac{\text{Area under } F\text{-}t \text{ diagram}}{\text{Mass}}$$

$$0 - 10 = \frac{\frac{1}{2} \times 10 \times 20 - 10(t - 10)}{1}$$

$$-10 = 100 - 10t + 100 \Rightarrow 10t = 210$$

$t = 21$ seconds ***Ans.***

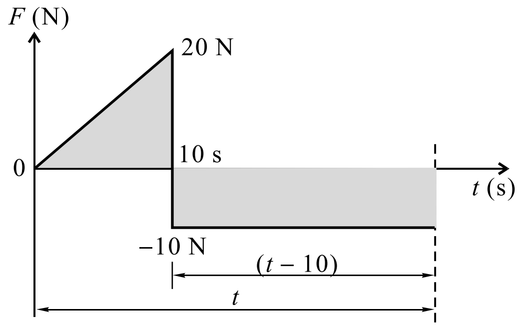


Fig. 16.20(b)

Problem 21

A body which is initially at rest, at the origin is subjected to a force varying with time, as shown in Fig. 16.21(a). Find the time **(i)** when the body again comes to rest and **(ii)** when it comes again to its original position.

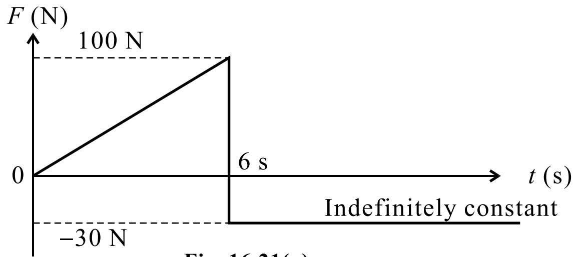


Fig. 16.21(a)

Solution

Let t be the time taken by the body to come to rest

(i) $\int F\,dt = 0$

Area under F-t curve

$$\frac{1}{2} \times 6 \times 100 + (-30)(t - 6) = 0$$

$t = 16$ seconds ***Ans.***

At $t = 16$ seconds body will again come to rest.

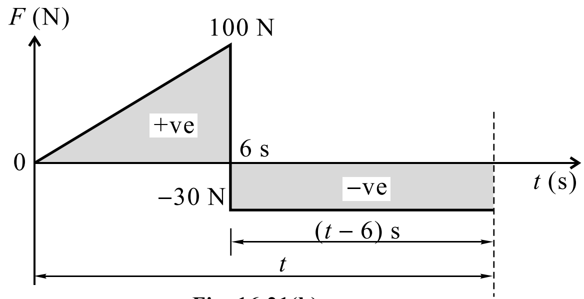


Fig. 16.21(b)

(ii) Let T be the time when body again comes to its original position (i.e., $s = 0$)

Displacement = Moment of area of F-t curve about P

$$0 = \left(\frac{1}{2} \times 6 \times 100\right)(T - 4) - 30(T - 6)\left(\frac{T - 6}{2}\right)$$

$T = 27.83$ seconds ***Ans.***

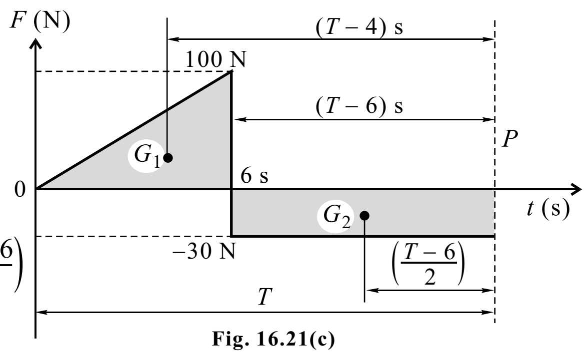


Fig. 16.21(c)

Exercises

[I] Problems

Problems on Impact

1. Two spheres A and B of radius 25 mm are travelling along the same straight path in the opposite direction. Sphere A of mass 1.2 kg is moving with 5 m/s towards right and sphere B of mass 2.4 kg is moving with a 2.5 m/s towards left. If the coefficient of restitution is 0.8, determine their final velocities.

   [***Ans.*** $v_1 = 4$ m/s $(\leftarrow)$ and $v_2 = 2$ m/s $(\rightarrow)$.]

2. A 20 mg rail wagon moving at 0.5 m/s to the right collides with a 35 mg wagon at rest. If after the collision, the 35 mg wagon is observed to move to the right at 0.3 m/s, determine the value of e after impact.

   [***Ans.*** $e = 0.65$]

3. A ball is dropped from a height of 9 m upon a horizontal slab. If it rebounds to a height of 5.76 m, show that the coefficient the restitution is 0.8.

4. A ball drops from the ceiling of a room and after rebounding twice from the floor reaches a height equal to one-fourth of the height of the ceiling. Show that the coefficient of restitution is 0.707.

5. A smooth spherical ball A of mass 120 gm is moving from left to right with 2 m/s in horizontal plane. Another identical ball B is travelling in a perpendicular direction with a velocity of. 6 m/s collides with A, as shown in Fig. 16.E20. Determine the velocity of the balls A and B after the impact. Assume $e = 0.8$.

   $\left[\textbf{Ans. } v_A = 0.2 \text{ m/s } (\rightarrow) \text{ and } v_B = 6.26 \text{ m/s } \left(\measuredangle 73.3^\circ\right)\right]$

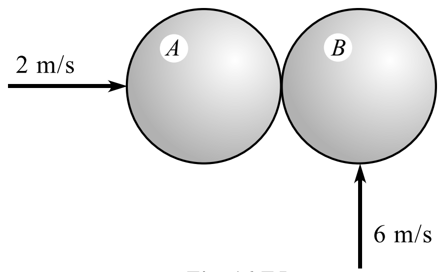


Fig. 16.E5

6. The magnitude and direction of the velocities of two identical frictionless balls before they strike each other, is shown in Fig. 16.E21. Assume $e = 0.9$, determine the magnitude and direction of the velocity of each ball after the impact.

   $\left[\textbf{Ans. } v_A = 2.32 \text{ m/s } \left(40.3^\circ \measuredangle\right) \text{ and } v_B = 4.19 \text{ m/s } \left(\measuredangle 55.6^\circ\right)\right]$

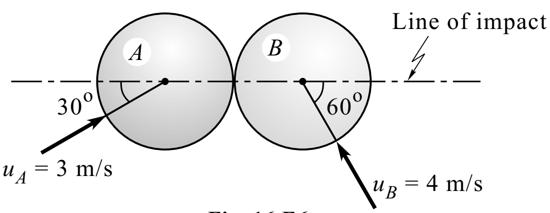


Fig. 16.E6

7. A ball is thrown downwards with a velocity of 12 m/s and at an angle of 30° with the horizontal from the top of the building 12 m high. Find where the ball will hit the ground the second time if the coefficient of restitution between the ball and ground is 0.75.

   [***Ans.*** 37.28 m]

8. Two billiard balls of equal mass collide with velocities u_1 = 1.5 m/s and u_2 = 2 m/s, as shown in Fig. 16.E8. Find velocity of balls after impact and percentage loss in KE. Take e = 0.9.

[***Ans.*** v_1 = 0.875 m/s (←), v_2 = 2.21 m/s $(\angle 51.5^\circ)$
Percentage loss = 9.6 %]

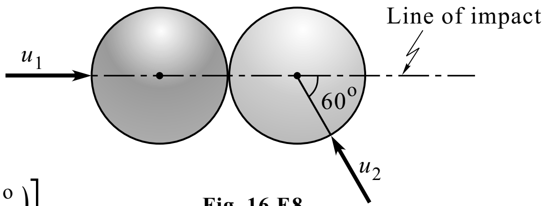


Fig. 16.E8

9. Two smooth billiards balls A and B have an equal mass of m = 0.2 kg. If A strike B with a velocity of v_A = 2 m/s, as shown in Fig. 16.E9, determine their final velocities just after collision. Ball B is originally at rest and the coefficient of restitution is e = 0.75.

[***Ans.*** v_A = 1.3 m/s (81.6°) and v_B = 1.34 m/s (←).]

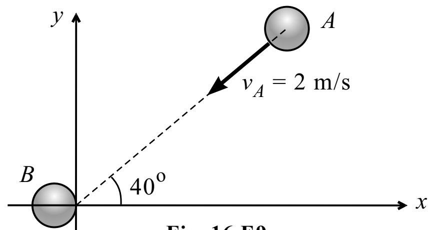


Fig. 16.E9

10. A ball is thrown against a frictionless wall. Its velocity before striking the wall is shown in Fig. 16.E10. Knowing e = 0.9. Determine the velocity after impact.

[***Ans.*** 9.26 m/s (32.7°)]

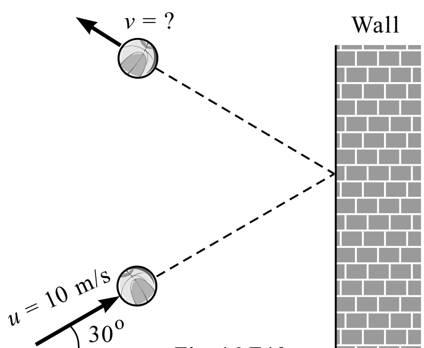


Fig. 16.E10

11. A 100 N ball shown in Fig. 16.E11 is released from the position shown by continuous line. It strikes a freely suspended 75 N ball. After impact, 75 N ball is raised by an angle θ = 48°. Determine the coefficient of restitution.

[***Ans.*** e = 0.162]

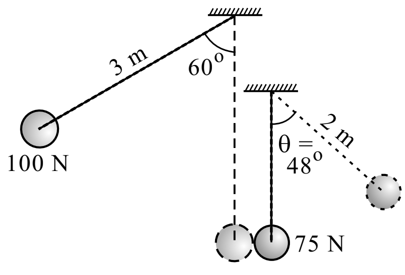


Fig. 16.E11

12. A 2 kg sphere A is released from rest when $\theta_A = 60^\circ$, as shown in Fig. 16.E12 and strike a sphere B of mass 4 kg which is at rest. If the impact is assumed to be perfectly elastic, determine the value of θ_B corresponding to highest position to which sphere B will rise.

[***Ans.*** $\theta_B = 38.94^\circ$]

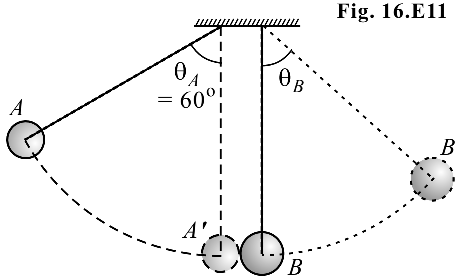


Fig. 16.E12

13. A bullet of mass 0.01 kg moving with a velocity of 100 m/s hits a 1 kg bob of a simple pendulum horizontally as shown in Fig. 16.E13. Find the maximum angle through which the pendulum swings when **(a)** the bullet gets embedded in the bob, **(b)** the bullet rebounds from surface of the bob with a velocity of 20 m/s and **(c)** The bullet escapes from the other end of bob with a velocity of 20 m/s Given length of pendulum as 1 m. Take $g = 10$ m/s^2.

[***Ans.*** **(a)** $\theta = 18.01^o$, **(b)** $\theta = 21.87^o$ and **(c)** $\theta = 14.53^o$.]

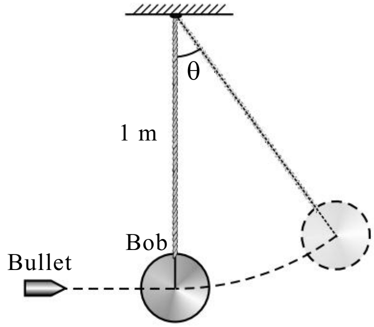


Fig. 16.E13

14. The bullet travelling horizontally with a velocity of 600 m/s and weighing 0.25 N strikes a wooden block weighing 50 N resting on a rough horizontal floor, as shown in Fig. 16.E14. The $\mu_k = 0.5$, find the distance through which the block is displaced from its initial position. Assume : Bullet after striking remains buried in the block. [***Ans.*** 0.89 m]

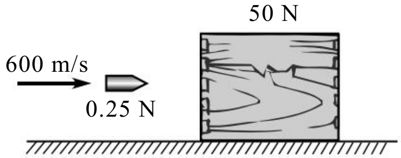


Fig. 16.E14

15. The bag A having a weight of 30 N is released from rest at a position $\theta = 0^o$, as shown in Fig. 16.E15. After falling, $\theta = 90^o$ it strike a 90 N box B. If the coefficient of restitution between the bag and the box is $e = 0.5$. Determine **(a)** velocities of the bag and box just after impact, **(b)** maximum compression of the spring, **(c)** maximum and minimum tension in the cord after impact and **(d)** loss of energy during collision.

$\left[\begin{array}{l}\textbf{\textit{Ans.}}\ \textbf{(a)}\ v_A = 0.55 \text{ m/s } (\rightarrow),\ v_B = 1.66 \text{ m/s } (\leftarrow), \\ \textbf{(b)}\ x = 0.5 \text{ m, } \textbf{(c)}\ 30.93 \text{ N; } 29.54 \text{ N and } \textbf{(d)}\ 16.9 \text{ J.}\end{array}\right]$

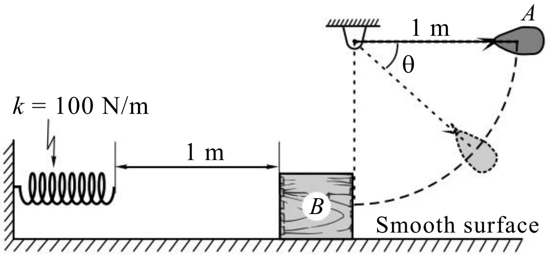


Fig. 16.E15

16. A 2 kg sphere is released from rest, when $\theta = 60^o$. It strikes 2.5 kg block B which is at rest as shown in Fig. 16.E16. The velocity of the sphere is zero after impact. Block moves through a distance of 1.5 m before coming to rest. Find **(a)** the coefficient of restitution, and **(b)** coefficient of friction.

[***Ans.*** **(a)** $e = 0.8$ and **(b)** $\mu_k = 0.256$.]

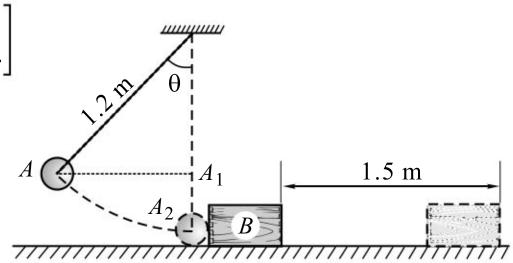


Fig. 16.E16

17. A 30 kg block is dropped from a height of 2 m on to the 10 kg pan of a spring scale. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. $K = 20$ kN/m.

[***Ans.*** 225 mm]

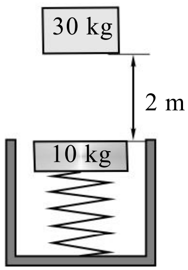


Fig. 16.E17

18. A 2 kg sphere A is moving to the left with a velocity of 15 m/s when it strikes the vertical face of a 4 kg block, which is at rest. The block B is supported on rollers and is attached to spring of constant K = 5000 N/m, as shown in Fig. 16.E18. If coefficient of restitution for the block and the sphere e = 0.75, determine the maximum compression (shortening) of the spring due to the impact. Neglect friction.

[***Ans.*** 0.2475 m]

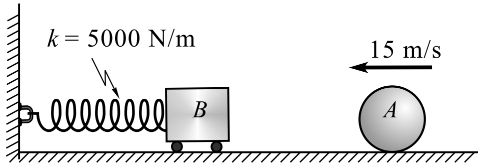


Fig. 16.E18

19. A 1 kg ball transverses a frictionless tube, as shown. Falling through a height of 1.5 m it then strikes a 2 kg ball hung on a rope 1.5 m long. Determine the height to which the hung ball will rise if **(a)** the collision is perfectly elastic and **(b)** the coefficient of restitution is 0.7.

[***Ans.*** **(a)** 0.666 m and **(b)** 0.481 m.]

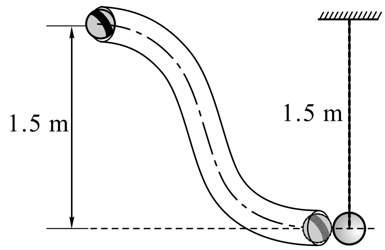


Fig. 16.E19

20. A 600 gm block rests on the edge of a table, as shown in Fig. 16.E20. A second block, weighing 400 gm and moving with velocity u_A, strikes the first block and causes the trajectory shown in the figure. The impact is assumed to be "nearly elastic" with an assumed value of the coefficient of restitution of 0.95. Find the initial velocity u_A and the final velocity v_A of the striking block.

[***Ans.*** u_A = 7.26 m/s (→) and v_A = 1.23 m/s (←).]

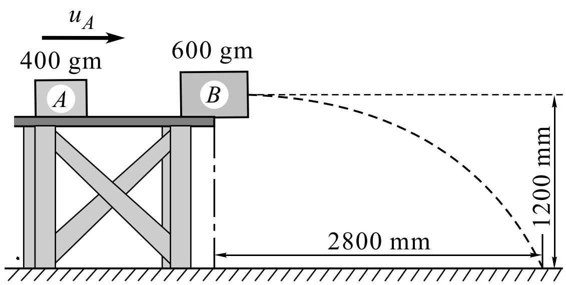


Fig. 16.E20

Problems on Impulse and Momentum

21. A block of mass 50 kg is resting on the horizontal surface is acted upon by a force F which varies as shown in Fig. 16.E21. If the coefficient of friction between the block and the surface is 0.2, find the velocity of the block at t = 5 s and 10 s. Also determine the time when the block will come to rest.

[***Ans.*** 15.19 m/s, 17.88 m/s and 19.11 s.]

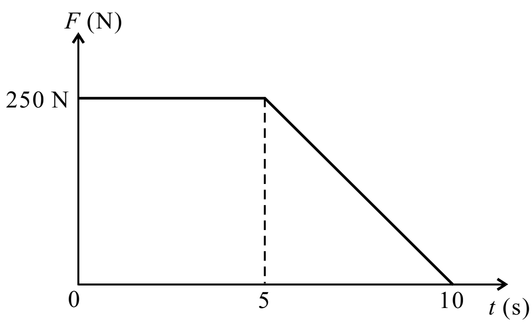


Fig. 16.E21

22. 20 kg block shown in Fig. 16.E22 is originally at rest on a horizontal surface for which the coefficient of static friction is 0.6 and the coefficient of kinetic friction is 0.5. If the horizontal force F is applied such that it varies with the time as shown, determine the speed of the block in 10 sec.

[***Ans.*** $v = 31.7$ m/s]

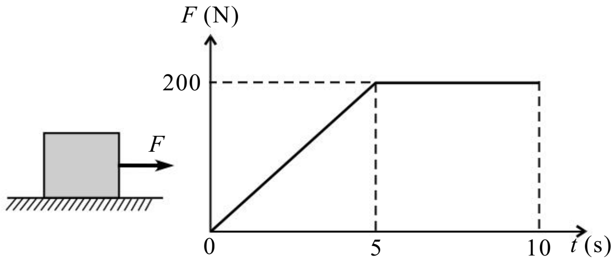


Fig. 16.E22

23. The cart is moving down the incline with a velocity $v_0 = 20$ m/s at $t = 0$, at which time the force F begins to act, as shown in Fig. 16.E23. After 5 seconds, the force continues at the 50 N level. Determine the velocity of the cart at time $t = 8$ seconds and calculate the time t at which the cart velocity is zero.

[***Ans.*** $v_2 = 1.423$ m/s (15°)
and $t = 8.25$ sec.]

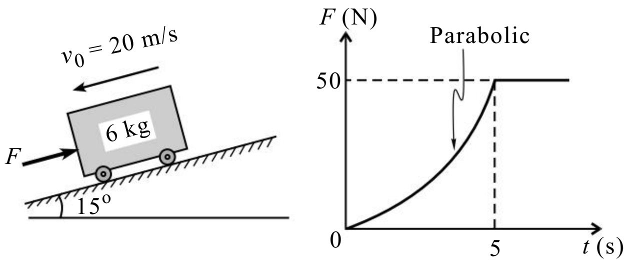


Fig. 16.E23

24. The 150 kg car A is coasting freely at 1.5 m/s on the horizontal track when it encounters a car B having a mass of 120 kg and coasting at 0.75 m/s towards it, as shown in Fig. 16.E24. If the cars meet and couple together, determine the speed of both the cars just after the coupling.

[***Ans.*** 0.5 m/s]

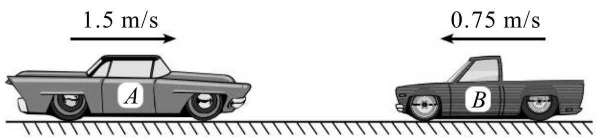


Fig. 16.E24

25. The loaded mine skip has a mass of 3000 kg, as shown in Fig. 16.E25. The hoisting drum produces a tension T in the cable according to the time schedule shown. If the skip is at rest against A when the drum is activated, determine the speed v of the skip when $t = 6$ seconds. Friction loss may be neglected.

[**Hint :** Tension on block is $2T$, i.e., Impulse by $T = 2$ (Area under the graph)]

[***Ans.*** 9.1 m/s]

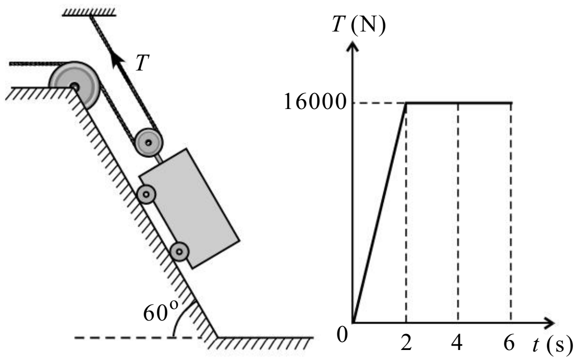


Fig. 16.E25

26. A 10 kg package drops from the chute into a 25 kg cart with a velocity of 3 m/s as shown in Fig. 16.E26. Knowing that the cart is initially at rest and can roll freely, determine **(a)** the final velocity of the cart, **(b)** the impulse exerted by the cart on the package and **(c)** the percentage energy lost in the impact.

[***Ans.*** 0.742 m/s; 23.86 N-sec and 78.6 %.]

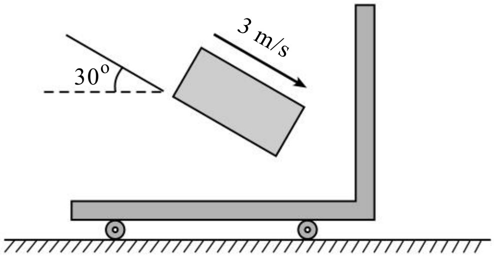


Fig. 16.E26

27. A 68 kg man sitting in a 79.8 kg Canoe fires a gun, discharging a 57 gm bullet into 45 kg sandbag suspended on a rope 91 cm long on a bank of river, as shown in Fig. 16.E27. It was calculated from the observation of the angle of the swing that the bag with the bullet embedded in it swings through a height of 30.5 mm. What is the velocity of the canoe?

[***Ans.*** 0.236 m/s (←)]

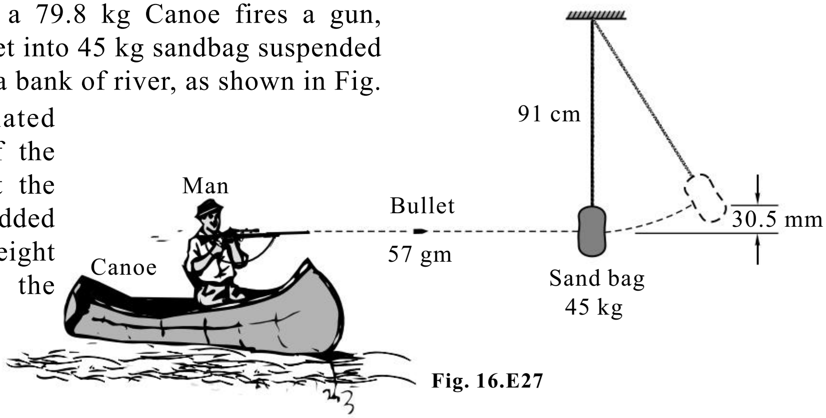


Fig. 16.E27

28. A 45 kg boy runs and jumps on his 10 kg sled with a horizontal velocity of 4.6 m/s, as shown in Fig. 16.E28. If the sled and the boy coast 25 m on the level snow before coming to rest, compute the coefficient of kinetic friction μ_k between the snow and the runners of the sled.

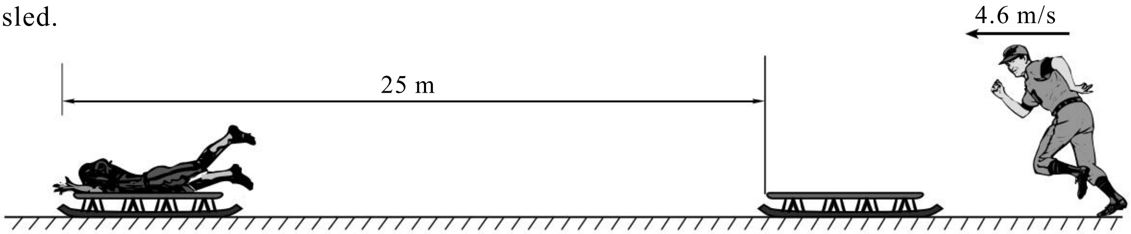


Fig. 16.E28

29. The man weights 750 N and jumps onto the boat which has a weight of 1000 N, as shown in Fig. 16.E29. If he has a horizontal component of velocity relative to the boat of 0.9 m/s, just before he enters the boat, and the boat is travelling $v_B = 0.6$ m/s away from the pier when he makes the jump, determine the resulting velocity of the man and the boat.

[***Ans.*** 0.986 m/s]

Fig. 16.E29

30. A block having mass of 50 kg rests on the surface of the cart having a mass of 75 kg, as shown in Fig. 16.E30. If the spring, which is attached to the cart, and not the block, is compressed 0.2 m and the system is released from rest, determine **(a)** the speed of the block after the spring becomes undeformed and **(b)** the speed of the block with respect to the cart after the spring becomes undeformed. Take k = 300 N/m.

[***Ans.*** **(a)** 0.379 m/s and **(b)** 0.632 m/s.]

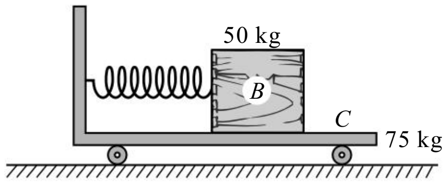


Fig. 16.E30

31. A spring normally 150 mm long is connected to the two masses, as shown in Fig. 16.E31 and compressed 50 mm. If the system is released on a smooth horizontal plane, what will be the speed of each block when the spring again comes to its normal length? The spring constant is 2100 N/m.

[***Ans.*** v_1 = 1.77 m/s (←) and v_2 = 1.18 m/s (→).]

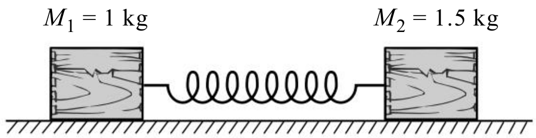


Fig. 16.E31

32. A man of weight 500 N is standing in a boat of weight 2000 N. The boat is initially at rest. The man moves from end A to the right a distance equal to 2.4 m, as shown in Fig. 16.E32 and stops. What is the corresponding displacement of boat? Neglect resistance of water. Assume uniform motion.

[***Ans.*** 0.48 m]

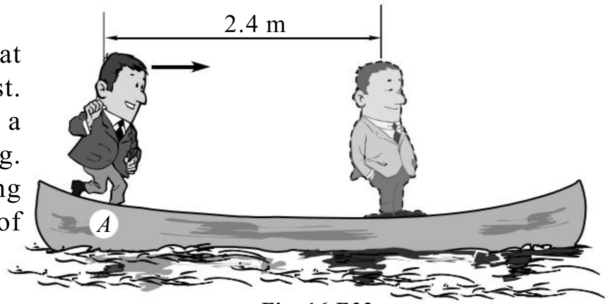


Fig. 16.E32

33. A rigid pile P shown in Fig. 16.E33 has a mass of 800 kg and is driven in to the ground using a hammer H that has a mass of 300 kg. The hammer falls from rest from a height of h = 0.5 m and strikes the top of the pile. Determine the impulse, which the hammer imparts on the pile, if the pile is surrounded entirely by loose sand so that after striking the hammer does not rebound off the pile.

[***Ans.*** 683 N-sec]

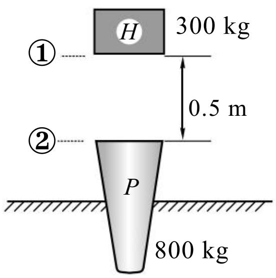


Fig. 16.E33

34. A shell of weight 200 N is fired from a gun with a velocity of 350 m/s, as shown in Fig. 16.E34. The gun and its carriage have a total weight of 12 kN. Find the stiffness of each spring which is required, to bring the gun to a halt within 300 mm of the spring compression.

[***Ans.*** 115.63 N/m]

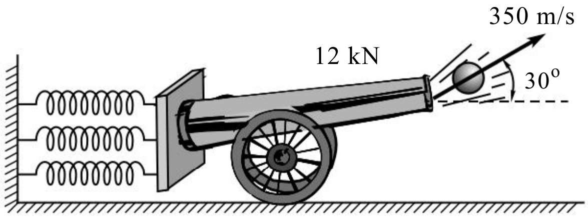


Fig. 16.E34

[II] Review Questions

1. What is impulsive force ?
2. State and derive the impulse momentum principle.
3. State the principle of conservation of momentum.
4. Explain the following terms :
   (a) Impulse (b) Line of impact (c) Central impact
   (d) Non-central impact (e) Direct central impact (f) Oblique central impact
   (g) Coefficient of restitution

[III] Fill in the Blanks

1. The product of mass and velocity is termed as ________.
2. Phenomenon of collision of two bodies, which occurs for a very small interval of time and during which two bodies exert very large force on each other, is called ________.
3. Line of impact is perpendicular to the ________.
4. If mass centres lies along the line of impact then such impact is called ________ impact.
5. Rebound stroke of a carom board is the example of ________ central impact.
6. The ratio of velocity of separation to velocity of approach is called ________.
7. For a perfectly plastic impact, value of e is equal to ________.

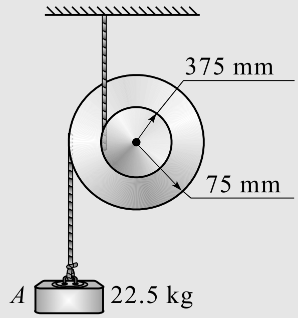
[IV] Multiple-choice Questions

Select the appropriate answer from the given options.

1. When a large force acts over a small finite period the force is called as an ________ force.
   (a) impulse **(b)** frictional **(c)** resultant **(d)** equilibrant
2. When the impulsive forces act for a very small interval of time, impulse due to external forces is _________.
   (a) large **(b)** small **(c)** zero **(d)** infinite
3. If a man jumps off a boat, the action of the man is equal and opposite to the reaction of the boat. Hence the ________ is zero in the system.
   (a) impulse **(b)** resultant **(c)** equilibrant **(d)** impact
4. The common normal to the surface of two bodies in contact during the impact is called as _________.
   (a) line of impact **(b)** common tangent **(c)** central impact **(d)** oblique impact
5. The relation total momentum is conserved before and after impact is always along _________.
   (a) common tangent **(b)** line of impact **(c)** vertical **(d)** horizontal
6. In a perfectly plastic impact after impact both the bodies move _________.
   (a) separately **(b)** tangentially **(c)** together **(d)** horizontally

17

KINETICS OF RIGID BODIES



17.1 Introduction

Kinetics : It is the study of geometry of motion with reference to the cause of motion.(Force and mass are considered.)

Kinetics of Particle : It is the study of geometry of translation motion with reference to the cause of motion.

Kinetics of Rigid Bodies : It is the study of geometry of translation or rotational or general plane motion with reference to the cause of motion.

Translation motion, rotational motion and general plane motion are already discussed in Chapter 13. In this chapter, we shall learn motion analysis of rigid bodies whose dimension is given. The force system acting on rigid body may cause translation motion or rotational motion or both simultaneously, i.e., general plane motion. Therefore, the body will have rectilinear acceleration and/or angular acceleration, we can do the motion analysis by using Newton's second law.

For rectilinear acceleration mass is a measure of the body's resistance. Similarly, for angular acceleration mass moment of inertia is a measure of the body's resistance. The concept of mass moment of inertia is necessary for the motion analysis of rigid bodies, which we have discussed in Chapter 5.

Newton's Second Law of Motion : *The acceleration of a body is proportional to the resultant force acting on it and is in the direction of this force* ($F = ma$). This law forms the basis for motion analysis of rigid bodies.

1. Translation Motion

(a) Rectilinear Motion

(i) $\sum F_x = ma_x$

(ii) $\sum F_y = ma_y$

(iii) $\sum M = 0$

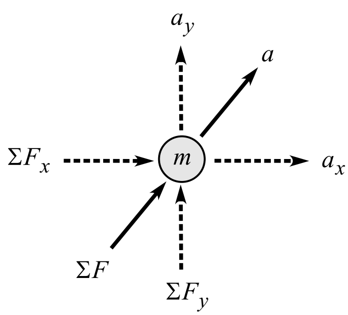


Fig. 17.1-i

(b) Curvilinear Motion

(i) $\sum F_t = ma_t$

(ii) $\sum F_n = ma_n$

(iii) $\sum M = 0$

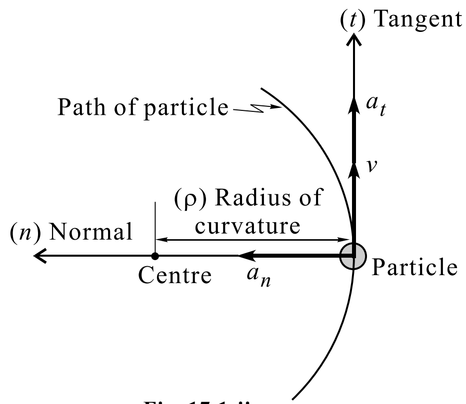


Fig. 17.1-ii

2. Fixed Axis Rotation

(a) Centroidal Rotational Motion : If the axis of rotation passes through the centre of gravity of the body then it is said to perform centroidal rotational motion

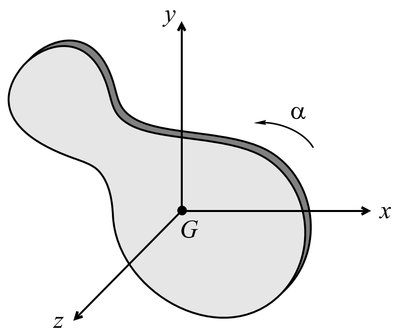


Fig. 17.1-iii

(i) $\sum F_x = 0$

(ii) $\sum F_y = 0$

(iii) $\sum M_G = I_G \alpha$

where

$\sum M_G$ = Algebraic sum of moment of all the forces and couples acting on the rigid body about the centre of rotation.

I_G = Mass moment of inertia of the body about the axis of rotation.

α = Angular acceleration.

$I_G \alpha$ = Inertia couple.

Sign Convention : Moment of force in the direction of α is considered positive.

(b) Non-centroidal Rotational Motion : If the axis of rotation does not pass through the centre of gravity of the body then it is said to perform non-centroidal rotational motion

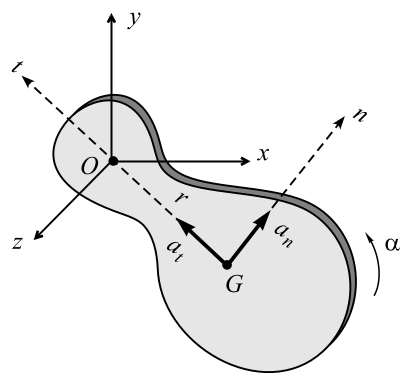


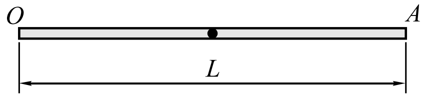
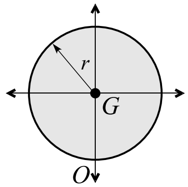
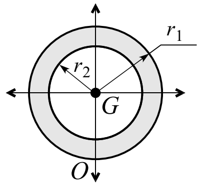
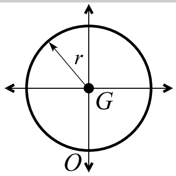
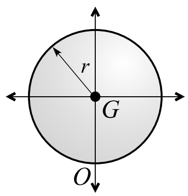
Fig. 17.1-iv

(i) $\sum F_t = ma_t = mr\alpha$

(ii) $\sum F_n = ma_n = mr\omega^2$

(iii) $\sum M_O = I_O \alpha$

17.2 Formulae for Mass Moment of Inertia

	Figure	Mass Moment of Inertia
1. Straight Rod		$I_G = \frac{1}{12} mL^2$ $I_O = \frac{1}{3} mL^2$
2. Solid Cylinder or Disc		$I_G = \frac{1}{2} mr^2$ $I_O = \frac{3}{2} mr^2$
3. Hollow Cylinder		$I_G = \frac{1}{2} m(r_1^2 - r_2^2)$ $I_O = I_G + mr_1^2$ $I_O = \frac{1}{2} m(r_1^2 - r_2^2) + mr_1^2$
4. Ring or Hoop (Thin Drum)		$I_G = mr^2$ $I_O = 2mr^2$
5. Sphere		$I_G = \frac{2}{5} mr^2$ $I_O = \frac{7}{5} mr^2$

6. For non-uniform or non-homogeneous rigid bodies

$$I = mK^2 \qquad \text{where } K = \text{radius of gyration.}$$

7. Compound Body

If two or more than two regular rigid bodies are rigidly connected to each other than

$$I = I_1 + I_2 + I_3 + I_4 + \ldots\ldots + I_n$$

Procedure to Solve Problems

1. Identify the number of particles and rigid bodies in the given system and their direction of motion.

2. Obtain the kinematic relation (i.e., relationship between acceleration of different particles and the rigid bodies).

3. Consider the FBD of each particle and the rigid body, then by applying Newton's second law, find the required unknowns.

17.3 Solved Problems Based on Centroidal Rotation

Problem 1

Two bodies of 9 kg and 13.5 kg are suspended on two ends of a string passing over a pulley of radius 275 mm and M.I. = 16.5 kgm^2 as shown in Fig. 17.1. Determine the tensions in the strings and the angular acceleration of the pulley.

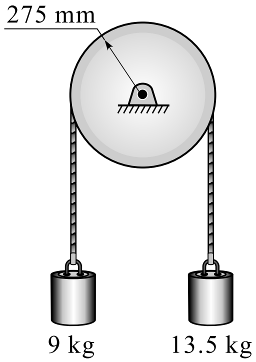


Fig. 17.1

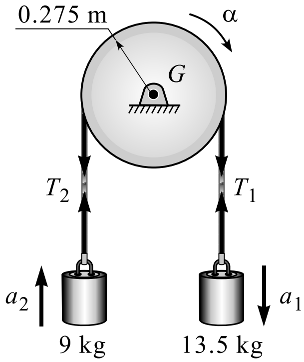


Fig. 17.1(a)

Solution

(i) FBD of the entire system is as shown in Fig. 17.1(a). Tension on each side differs say T_1 and T_2.

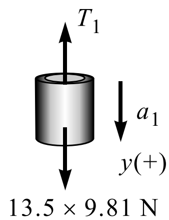
Kinematic relation is give as

$$a_1 = 0.275\alpha \text{ and}$$
$$a_2 = 0.275\alpha$$

(ii) Consider the FBD of 13.5 kg Block

By Newton's second law,

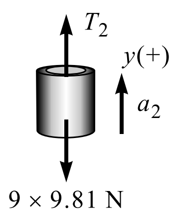
$$\Sigma F_y = ma_y$$
$$13.5 \times 9.81 - T_1 = 13.5 \times a_1$$
$$T_1 = 132.44 - 13.5a_1 \qquad \text{..... (I)}$$



(ii) Consider the FBD of 9 kg Block

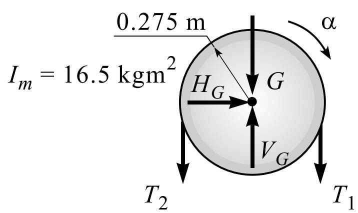
By Newton's second law,

$$\Sigma F_y = ma_y$$
$$T_2 - 9 \times 9.81 = 9 \times a_2$$
$$T_2 = 88.29 + 9a_2 \qquad \text{..... (II)}$$



(iii) Consider the FBD of Pulley

$$\Sigma M_G = I_G\alpha$$
$$T_1 \times 0.275 - T_2 \times 0.275 = 16.5 \times \alpha$$
$$T_1 - T_2 = 60\alpha \qquad \text{..... (III)}$$



(iv) Putting values of T_1 and T_2 in Eq. (III), we get

$$[132.44 - 13.5 \times 0.275\alpha] - [88.29 + 9 \times 0.275\alpha] = 60\alpha$$
$$\alpha = 0.67 \text{ r/s}^2$$
$$\therefore\ T_1 = 129.95 \text{ N and } T_2 = 89.95 \text{ N}$$ ***Ans.***

Problem 2

A two-step pulley shown in Fig. 17.2 has a total weight of 280 N and radius of gyration of 200 mm.

(a) Determine the resulting motion, assuming no slip between the pulleys and rope and state the linear acceleration of weight W_2.

(b) If the above system starts from rest and then the weight W_1 is detached after a time interval of 10 s, determine how long the motion of weight W_2 will continue in the same direction.

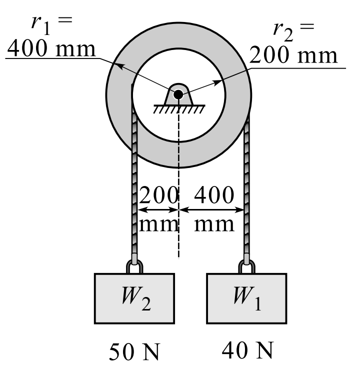


Fig. 17.2

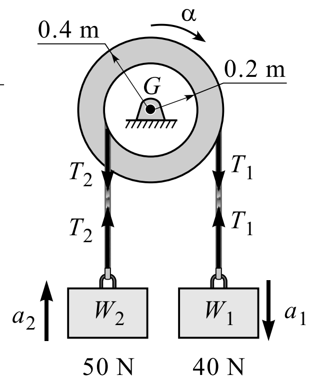


Fig. 17.2(a)

Solution

(a) A rope is passed through a solid cylinder. Tension on each side are T_1 and T_2. FBD is drawn as shown in Fig. 17.2(a).

Kinematic relation is give as

$a_1 = 0.4\alpha$ and $a_2 = 0.275\alpha$

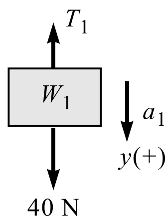
(i) Consider the FBD of W_1

By Newton's second law,

$$\Sigma F_y = ma_y$$

$$40 - T_1 = \frac{40}{9.81} \times a_1$$

$$T_1 = 40 - 4.08a_1 \qquad \text{..... (I)}$$



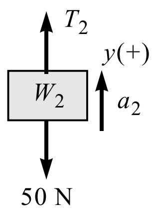
(ii) Consider the FBD of W_2

By Newton's second law,

$$\Sigma F_y = ma_y$$

$$T_2 - 50 = \frac{50}{9.81} \times a_2$$

$$T_2 = 50 + 5.1a_2 \qquad \text{..... (II)}$$



(iii) Consider the FBD of Pulley

$$K_G = 0.2 \text{ m}$$

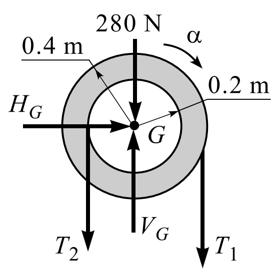
$$I_G = mK_G^2 = \frac{280}{9.81}(0.2)^2$$

$$I_G = 1.14 \text{ kgm}^2$$

$$\Sigma M_G = I_G\alpha$$

$$T_1 \times 0.4 - T_2 \times 0.2 = 1.14 \times \alpha$$

$$2T_1 - T_2 = 5.7\alpha \qquad \text{..... (III)}$$



(iv) Putting values of T_1 and T_2 in Eq. (III), we get

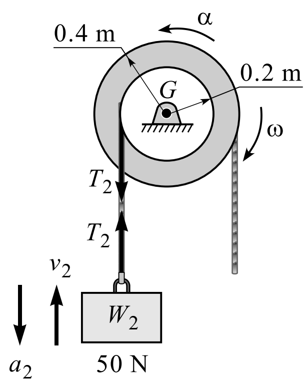
$$2[40 - 4.08 \times 0.4\alpha] - [50 + 5.1 \times 0.2\alpha] = 5.7\alpha$$

$$\therefore \alpha = 3 \text{ r/s}^2$$

Thus, we get $a_2 = 0.6 \text{ m/s}^2$, $T_1 = 35.1$ N and $T_2 = 53.06$ N ***Ans.***

(b) The instant at which W_2 is detached, the remaining system will be subjected to angular acceleration, i.e., α is in anticlockwise direction (↺).

Velocity of W_2 after 10 seconds (i.e., just before detachment of W_1).



$$v = u + at$$

$$v = 0 + 0.6 \times 10$$

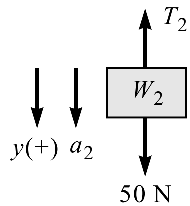
$$\therefore v = 6 \text{ m/s}$$

Kinematic relation is give as

$$a_2 = 0.2\alpha$$

(i) Consider the FBD of W_2

By Newton's second law,

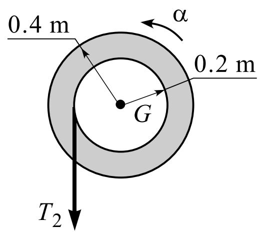


$$\Sigma F_y = ma_y$$

$$50 - T_2 = \frac{50}{9.81} \times a_2$$

$$T_2 = 50 - 5.1a_2 \quad \text{..... (I)}$$

(ii) Consider the FBD of Pulley



$$\Sigma M_G = I_G\,\alpha$$

$$T_2 \times 0.2 = 1.14 \times \alpha$$

$$\therefore T_2 = 5.7\alpha \quad \text{..... (II)}$$

(iii) From Eqs. (I) and (II), we get

$$50 - 5.1a_2 = 5.7\alpha$$

$$\therefore \alpha = 7.44 \text{ r/s}^2$$

Thus, we get $a_2 = 1.49 \text{ m/s}^2$ ***Ans.***

Let t be the time interval for which W_2 will continue its initial direction of motion and at this instant its velocity is 0.

$$v = u + at$$

$$0 = 6 + (-1.49) \times t$$

$$\therefore t = 4.03 \text{ seconds}$$ ***Ans.***

Problem 3

Two double pulleys shown in Fig. 17.3 are connected to each other by a string. If each pulley mounted on frictionless bearings, mass moment of inertia of each pulley is 0.2 kgm^2 and inner and outer radii of each pulley are 80 mm and 120 mm respectively; then calculate acceleration of the mass 6 kg and tension in the string connecting the two pulleys.

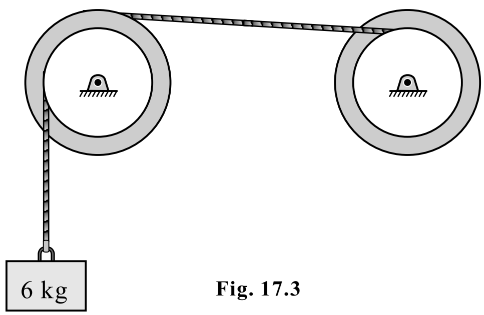


Fig. 17.3

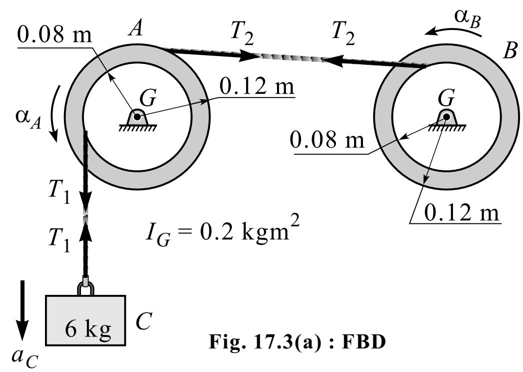


Fig. 17.3(a) : FBD

Solution

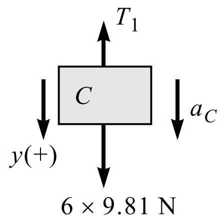
FBD of the system is as shown in Fig. 17.3(a).

(i) Kinematic relation is give as

$a_C = 0.08\alpha_A$

$0.12\alpha_A = 0.08\alpha_B \quad \therefore \; \alpha_B = 1.5\alpha_A$

(ii) Consider the FBD of Block *C*

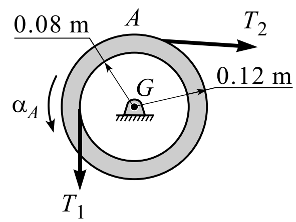


$\Sigma F_y = ma_y$

$6 \times 9.81 - T_1 = 6 \times a_C$

$T_1 = 58.86 - 6a_C$ ..... (I)

(iii) Consider the FBD of Pulley *A*

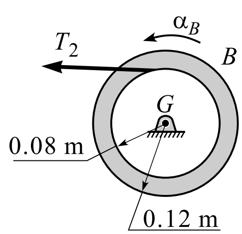


$\Sigma M_G = I_G\alpha$

$T_1 \times 0.08 - T_2 \times 0.12 = 0.2 \times \alpha_A$

$\therefore \; T_1 - 1.5T_2 = 2.5\alpha_A$ ..... (II)

(iv) Consider the FBD of Pulley *B*



$\Sigma M_G = I_G\alpha$

$T_2 \times 0.08 = 0.2 \times \alpha_B$

$\therefore \; T_2 = 2.5\alpha_B$ ..... (III)

(v) From Eqs. (I), (II) and (III), we get

$58.86 - 6 \times 0.08\alpha_A - 1.5(2.5 \times 1.5\alpha_A) = 2.5\alpha_A$

$\therefore \; \alpha_A = 6.84$ r/s^2

Thus, we get $a_C = 0.08 \times 6.84 = 0.55$ m/s^2 ***Ans.***

$T_2 = 2.5 \times 1.5 \times 6.84$

$\therefore T_2 = 25.65$ N ***Ans.***

17.4 Solved Problems Based on Non-centroidal Rotation

Problem 4

A cylinder of mass m and radius 0.2 m is rolling off a loading platform. Find its angular acceleration when $\theta = 30^o$. Also find horizontal and vertical reactions given by the platform, as shown in Fig. 17.4.

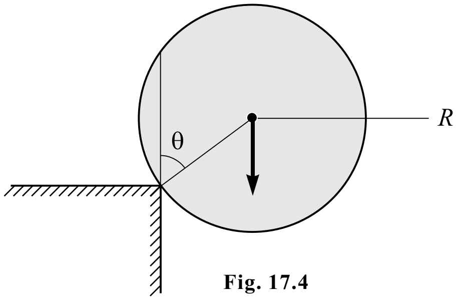


Fig. 17.4

Solution

FBD is drawn, as shown in Fig. 17.4(a).

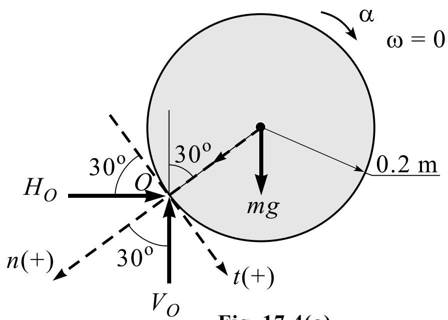


Fig. 17.4(a)

(i) $\Sigma M_O = I_O \alpha$

$$mg \times 0.2 \sin 30^o = \frac{3}{2} \times m \times 0.2^2 \times \alpha$$

$$\therefore \ \alpha = 16.35 \text{ r/s}^2$$

(ii) $\Sigma F_t = m r \alpha$

$$H_O \cos 30^o - V_O \sin 30^o + mg \sin 30^o = m \times 0.2 \times 16.35$$

$$0.87 H_O - 0.5 V_O = (-1.635)\, m \quad \text{... (I)}$$

(iii) $\Sigma F_n = m r \omega^2 = 0 \qquad [\because \omega = 0]$

$$-H_O \sin 30^o - V_O \cos 30^o + mg \cos 30^o = m \times 0.2 \times 0^2$$

$$\therefore \ 0.5 H_O + 0.87 V_O = 8.5 m \quad \text{... (II)}$$

Solving Eqs. (I) and (II), we get

$H_O = (2.81m)$ N $(\rightarrow)$ and $V_O = (8.16m)$ N $(\uparrow)$ ***Ans.***

Problem 5

A uniform slender rod of length 900 mm and mass 2.5 kg hangs freely from a hinge at A in a vertical plane. At what distance from A, a force of 15 N be applied, so that the horizontal component of the reaction at A is zero ? Find the corresponding angular acceleration of the rod. Use preferably D'Alembert's principle.

Solution

Figure 17.5 can be drawn from the given data.

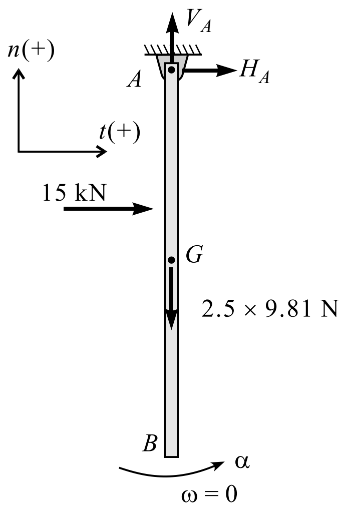


Fig. 17.5

Note that force 15 kN is not vertical because moment of the force applied and weight will be 0 of A and then the rod will not perform rotation motion. Also, force of 15 kN is not inclined as angle is not given.

(i) $\Sigma F_t = m r \alpha$

$$15 = 2.5 \times 0.45 \times \alpha$$

$$\therefore \ \alpha = 13.33 \text{ r/s}^2 \quad \textbf{\textit{Ans.}}$$

(ii) $\Sigma M_A = I_A \alpha$

$$15 \times y = \frac{1}{3} \times 2.5 \times 0.9^2 \times 13.33$$

$$\therefore \ y = 0.6 \text{ m} \quad \textbf{\textit{Ans.}}$$

Problem 6

A pendulum consists of a 20 kg sphere and 5 kg uniform bar, as shown in Fig. 17.6. Compute the reaction at the pin C just after the cord AB is cut.

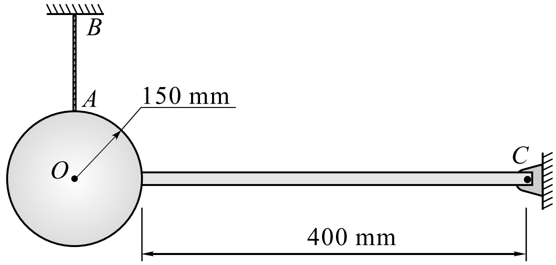


Fig. 17.6

Solution

Consider the FBD of compound body as shown in Fig. 17.6(a).

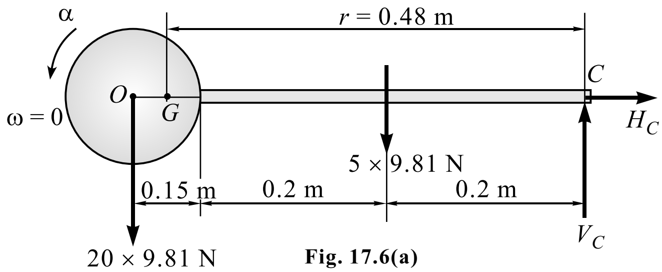


Fig. 17.6(a)

(i) $$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{(20)(0.55) + (5)(0.2)}{20 + 5}$$

$$\therefore \bar{x} = 0.48 \text{ m}$$

(ii) $$I_C = I_{C_1} + I_{C_2}$$

$$= \left(\frac{2}{5} \times 20 \times 0.15^2\right) + (20 \times 0.55^2) + \left(\frac{1}{3} \times 5 \times 0.4^2\right)$$

$$\therefore I_C = 6.5 \text{ kgm}^2$$

(iii) $$\Sigma M_C = I_C \alpha$$

$$5 \times 9.81 \times 0.2 + 20 \times 9.81 \times 0.55 = 6.5 \times \alpha$$

$$\therefore \alpha = 18.11 \text{ r/s}^2$$

(iv) $$\Sigma F_t = m\, r\, \alpha$$

$$-V_C + 5 \times 9.81 + 20 \times 9.81 = 25 \times 0.48 \times 18.11$$

$$\therefore V_C = 27.93 \text{ N } (\uparrow)$$ ***Ans.***

(v) $$\Sigma F_n = m\, r\, \omega^2 = 0 \qquad [\because \omega = 0]$$

$$\therefore H_C = 0$$ ***Ans.***

Problem 7

A constant torque T Nm is applied to produce an angular acceleration of 2 rad/s^2 about the vertical axis through the pin at A for the rigidly connected system of disc P (6 kg mass), disc Q (4 kg mass) and rod AB (3 kg mass) as shown in Fig. 17.7. Find the **(i)** moment of inertia of the system about the axis through A, **(ii)** torque T Nm and **(iii)** angular velocity of the system and the angle turned after 10 seconds. Consider the system in horizontal plane and at rest before T is applied.

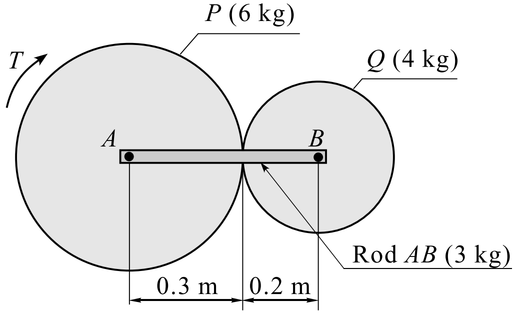


Fig. 17.7

Solution

From Fig. 17.7 and given data, we have

(i) $I_A = I_P + I_Q + I_{AB}$

$$= \left(\frac{1}{2} \times 6 \times 0.3^2\right) + \left(\frac{1}{2} \times 4 \times 0.2^2 + 4 \times 0.5^2\right) + \left(\frac{1}{3} \times 3 \times 0.5^2\right)$$

$\therefore$ $I_A = 1.6 \text{ kgm}^2$ ***Ans.***

(ii) $\Sigma M_A = I_A \alpha$

$\therefore$ $T = 1.6 \times 2$

$\therefore$ $T = 3.2$ Nm ***Ans.***

(iii) Since α is constant (As α is due to T and T is constant), we have

$\omega = \omega_0 + \alpha t$

$\therefore$ $\omega = 0 + 2 \times 10$

$\therefore$ $\omega = 20$ r/s ***Ans.***

Also,

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= 0 + \frac{1}{2} \times 2 \times 10^2$$

$\therefore$ $\theta = 100$ rad ***Ans.***

17.5 Plane Motion

Rolling of bodies with or without slipping on a stationary surface is a *plane motion*.

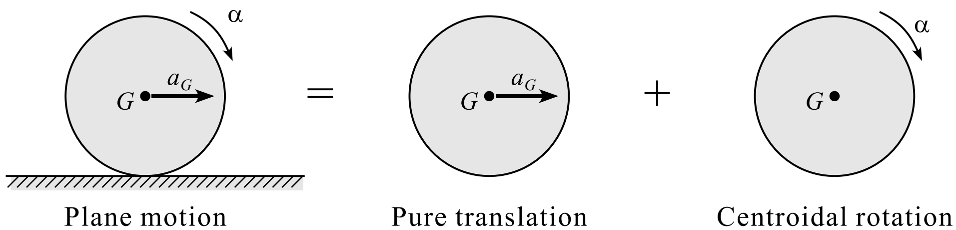


Fig. 17.5-i

(i) $\Sigma F_x = m\, a_x$

(ii) $\Sigma F_y = m\, a_y$

(iii) $\Sigma M_G = I_G \alpha$

Note : If the rigid body rolls without slipping on a stationary surface then consider its non-centroidal rotation at the point of contact with the stationary surface.

17.6 Solved Problems Based on Plane Motion

Problem 8

A solid cylinder of weight $W = 100$ N and radius $r = 0.5$ m is pulled along a horizontal plane by a horizontal force $P = 20$ N applied to the end of the string wound round the cylinder, as shown in Fig. 17.8. Assume no slip, determine the **(i)** acceleration of centre G and **(ii)** acceleration of D.

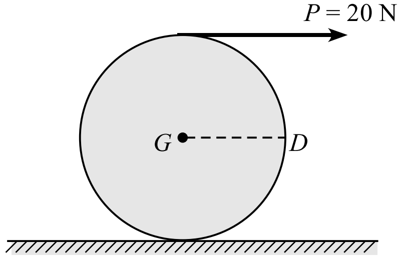


Fig. 17.8

Solution

Refer to Fig. 17.8(a), we get

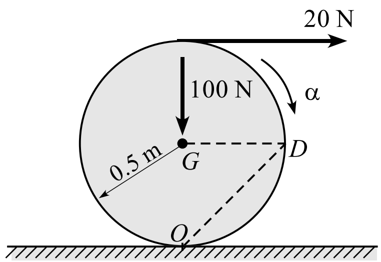


Fig. 17.8(a)

(i) $\Sigma M_O = I_O \alpha$

$$20 \times 1 = \frac{3}{2} \times \frac{100}{9.81} \times 0.5^2 \times \alpha$$

$$\therefore \ \alpha = 5.23 \text{ r/s}^2$$

(ii) For point G

$$a_G = r\alpha = 0.5 \times 5.23$$

$$\therefore \ a_G = 2.62 \text{ m/s}^2$$ ***Ans.***

(ii) For point D

$$a_D = r\alpha = 0.71 \times 5.23$$

$$\therefore \ a_D = 3.7 \text{ m/s}^2$$ ***Ans.***

Problem 9

A roller of mass 20 kg (weight 200 N) with its centre G, as shown in Fig. 17.9 is wound round on its circumference by means of a rope which is taken along line PAB over a smooth pulley at A. Determine the acceleration of the roller when the mass of the block attached at end B is 10 kg (weight 100 N).

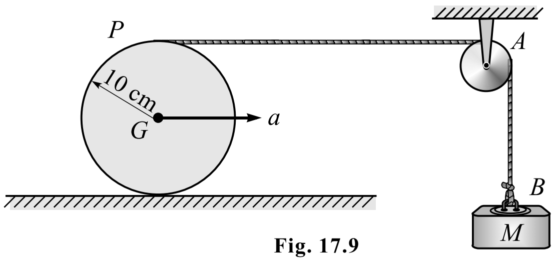


Fig. 17.9

Solution

Refer to Fig. 17.9(a), we get

(i) Kinematic relation

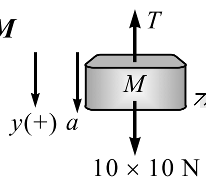
$$a = 0.2 \times \alpha$$

(ii) Consider the FBD of Block *M*

$$\Sigma F_y = m a_y$$

$$\therefore \ 100 - T = 10 \times a$$

$$\therefore \ T = 100 - 10a \quad \text{..... (I)}$$



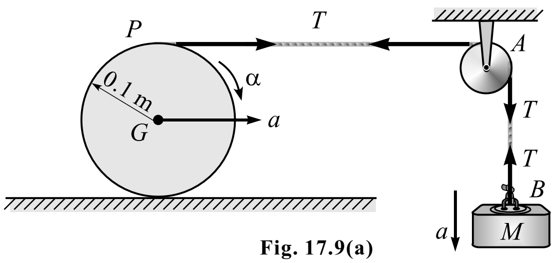
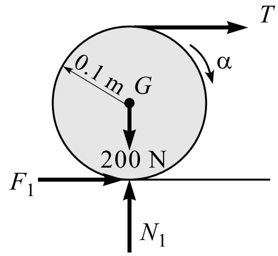


Fig. 17.9(a)

(iii) Consider the FBD of Roller



$$\Sigma M_O = I_O \alpha$$

$$T \times 0.2 = \frac{3}{2} \times 20 \times 0.1^2 \times \alpha$$

$$T = 1.5\,\alpha \qquad \text{..... (II)}$$

(iv) From Eqs. (I) and (II), we get

$$1.5\,\alpha = 100 - 10a$$

$$\therefore \alpha = 28.57 \text{ r/s}^2$$

$$a_G = r\alpha = 0.1 \times 28.57$$

$$\therefore a_G = 2.857 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

Problem 10

A solid sphere and a thin hoop of equal masses m and radii R are connected together by a frame and are free to roll without slipping down the inclined plane as shown in Fig. 17.10. Neglecting the mass of the frame determine the acceleration of the system. Also calculate the force in the frame. Assume frictionless bearings.

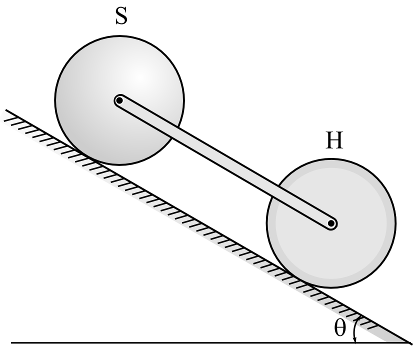
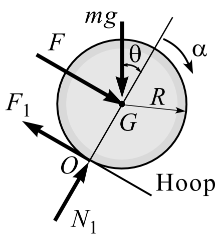


Fig. 17.10

Solution

(i) Consider the FBD of Hoop



$$\Sigma M_O = I_O \alpha$$

$$F \times R + mg \sin\theta \times R = 2mR^2\alpha$$

$$F = 2mR\alpha - mg \sin\theta \qquad \text{..... (I)}$$

(ii) Consider the FBD of Sphere

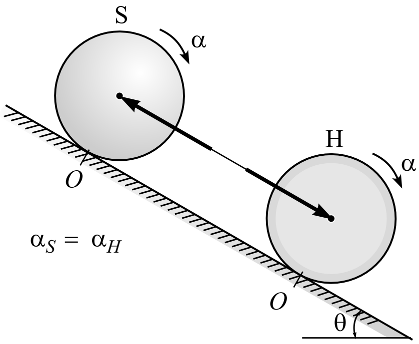
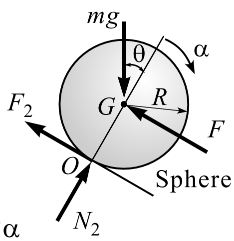


Fig. 17.10(a)



$$\Sigma M_O = I_O \alpha$$

$$-F \times R + mg \sin\theta \times R = \frac{7}{5}\, mR^2\alpha$$

$$F = mg \sin\theta - \frac{7}{5}\, mR\alpha \qquad \text{..... (II)}$$

(iii) From Eqs. (I) and (II), we get

$$2mR\alpha - mg \sin\theta = mg \sin\theta - \frac{7}{5}\, mR\alpha$$

$$\therefore 2g \sin\theta = \frac{17}{5} R\alpha$$

$$\therefore \alpha = \frac{10}{17} \times \frac{g}{R} \times \sin\theta$$

$$\therefore a_G = r\alpha = \frac{10}{17}\, g \sin\theta \quad \textbf{\textit{Ans.}}$$

From Eq. (I) , we get

$$F = 2mR \times \frac{10}{17} \times \frac{g}{R} \sin\theta - mg \sin\theta$$

$$\therefore F = \frac{3}{17}\, mg \sin\theta \quad \textbf{\textit{Ans.}}$$

Problem 11

For the system shown Fig. 17.11, calculate the linear accelerations of the two bodies. Radius of gyration of the composite pulley about its centre = 8 cm. Radius of pulley : r_1 = 15 cm and r_2 = 10 cm.

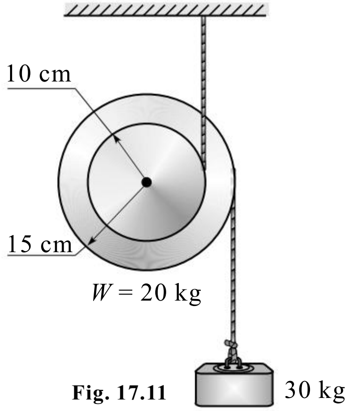


Fig. 17.11

Solution

Refer to Fig. 17.11(a), we get

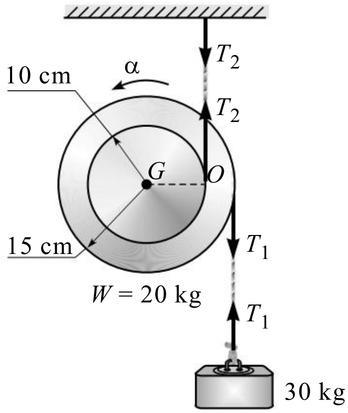


Fig. 17.11(a)

(i) $K = 0.08$ m

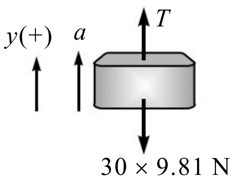
$m_1 = (0.05)(30) = 1.5\ (\circlearrowright)$

$m_2 = (0.1)(20) = 2\ (\circlearrowleft)$

Kinematic relation

$a = 0.05 \times \alpha$

(ii) Consider the FBD of 30 kg Block

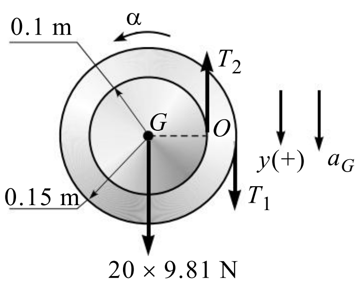


$\Sigma F_y = m\, a_y$

$T_1 - 30 \times 9.81 = 30 \times a$

$\therefore\ T_1 = 30a + 294.3$ ..... (I)

(iii) Consider the FBD of Pulley



$\Sigma M_O = I_O\, \alpha$

$$20 \times 9.81 \times 0.1 - T_1 \times 0.05 = [20 \times 0.08^2 + 20 \times 0.1^2] \times \alpha$$

$\therefore\ T_1 = -6.56\alpha + 392.4$ ..... (II)

(iv) From Eqs. (I) and (II), we get

$30 \times 0.05\alpha + 294.3 = 392.4 - 6.56\alpha$

$\therefore\ \alpha = 12.17\ \text{r/s}^2$

$\therefore\ a = 0.05 \times 12.17 = 0.61\ \text{m/s}^2$ ***Ans.***

$\therefore\ a_G = 0.1 \times 12.17 = 1.22\ \text{m/s}^2$ ***Ans.***

$\therefore\ T_1 = 312.56$ N ***Ans.***

Also,

$\Sigma F_y = m\, a_y$

$20 \times 9.81 - T_2 + 312.56 = 20 \times 1.22$

$\therefore\ T_2 = 484.36$ N ***Ans.***

Problem 12

The wheel has a mass of 25 kg and radius of gyration k_G = 0.20 m. If 50 N.m. couple moment is applied to the wheel, determine the acceleration of its mass centre. The coefficient of static and dynamic friction between the wheel and plane at A are μ_{Static} = 0.3 and $\mu_{Dynamic}$ = 0.25, respectively.

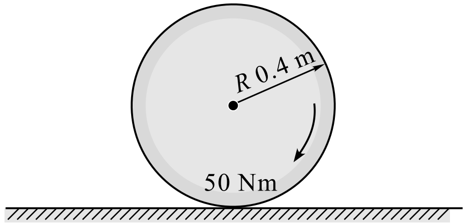


Fig. 17.12

Solution

(i) Assuming the sphere rolls without slipping. Consider the FBD of the wheel shown in Fig. 17.12(a).

$$m = 25 \text{ kg}$$
$$I_G = 1 \text{ kgm}^2$$
$$I_O = 5 \text{ kgm}^2$$

Let F_1 be the friction force required to roll the wheel without slipping.

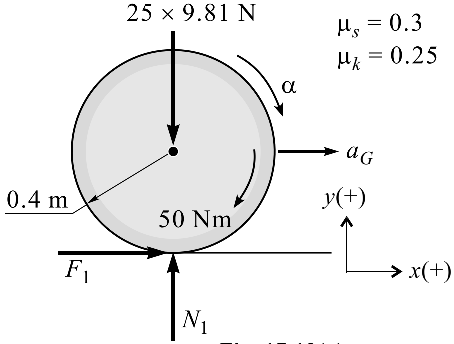


Fig. 17.12(a)

(ii) $\Sigma M_O = I_O \alpha$

$$50 = 5 \times \alpha$$
$$\therefore \alpha = 10 \text{ r/s}^2$$
$$\Sigma M_G = I_G \alpha$$
$$-F_1 \times 0.4 + 50 = 1 \times 10$$
$$\therefore F_1 = 100 \text{ N}$$

(iii) $\Sigma F_y = m a_y = 0 \quad (\because a_y = 0)$

$$\therefore N_1 = 25 \times 9.81 = 245.25 \text{ N}$$
$$F_{max} = \mu_s \times N_1 = 0.3 \times 245.25$$
$$\therefore F_{max} = 73.58 \text{ N}$$

From above results, $F_1 > F_{max}$

The wheel rolls with slipping.

$\therefore$ $\Sigma M_O = I_O \alpha$ equation is not applicable.

$$\therefore \alpha \neq 10 \text{ r/s}^2$$
$$F_{\text{actual}} = \mu_k \times N_1 = 0.25 \times 245.25 = 61.31 \text{ N}$$

To calculate the value of a_G, we have

$$\Sigma F_x = m a_x$$
$$61.31 = 25 \times a_G$$

$\therefore a_G = 2.45 \text{ m/s}^2$ ***Ans.***

Problem 13

A uniform rod 4 m long weighing 400 N is rigidly connected to the centre of a cylinder of mass 30 kg, as shown in Fig. 17.13. The diameter of cylinder is 2 m. Find the linear acceleration of block weighing 2000 N conneced to the cylinder by an inextensible string.

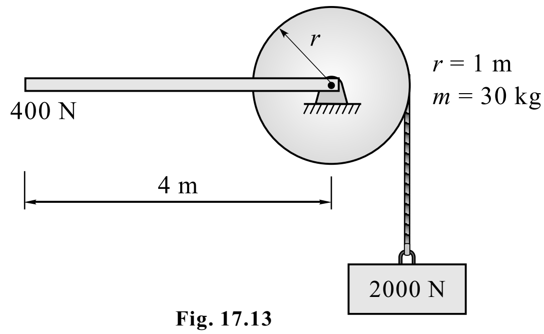


Fig. 17.13

Solution

Refer to Fig. 17.13(a).

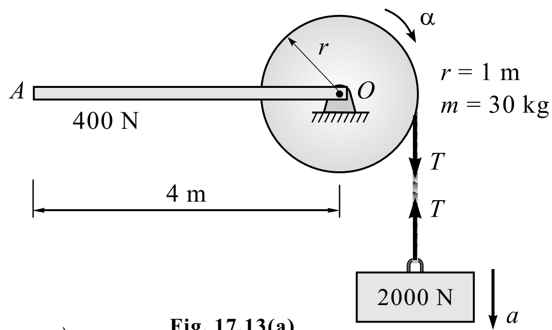


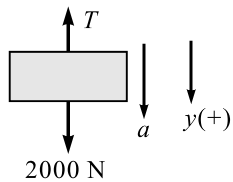
Fig. 17.13(a)

(i) Kinematic relation is give as

$$a = 1 \times \alpha$$

$$\therefore\ a = \alpha$$

(ii) Consider the FBD of 2000 N Block



$$\Sigma F_y = m\,a_y$$

$$2000 - T = \frac{2000}{9.81} \times a$$

$$\therefore\ T = 2000 - 203.87a \qquad \text{..... (I)}$$

(iii) $\Sigma M_O = I_O\,\alpha$

$$T \times 1 - 400 \times 2 = \left(\frac{1}{3} \times \frac{400}{9.81} \times 4^2 + \frac{1}{2} \times 30 \times 1^2\right) \times \alpha$$

$$\therefore\ T = 232.47\alpha + 800 \qquad \text{..... (II)}$$

(iv) From Eqs. (I) and (II), we get

$$2000 - 203.87\alpha = 232.47\alpha + 800$$

$$\therefore\ \alpha = 2.75 \text{ r/s}^2$$

$$\therefore\ a = 2.75 \text{ m/s}^2$$ ***Ans.***

Problem 14

At the position, a uniform disc of mass 15 kg and diameter 1.8 m is supported with 2 pins at A and B, as shown in Fig. 17.14. If the pin B is suddenly removed, find immediately after that **(i)** the angular acceleration of the disc and **(ii)** the reaction at pin A.

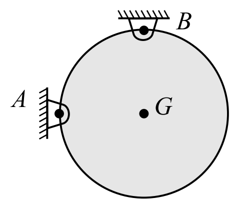


Fig. 17.14

Solution

(i) $\Sigma M_A = I_A\,\alpha$

$$15 \times 9.81 \times 0.9 = \frac{3}{2} \times 15 \times 0.9^2 \times \alpha$$

$$\therefore\ \alpha = 7.27 \text{ r/s}^2$$

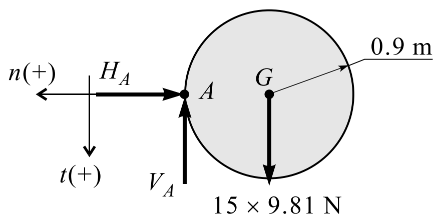


Fig. 17.14(a)

(ii) $\Sigma F_t = m\,r\,\alpha$

$$-V_A + 15 \times 9.81 = 15 \times 0.9 \times 7.27$$

$$\therefore\ V_A = 49.01 \text{ N } (\uparrow)$$ ***Ans.***

(iii) $\Sigma F_n = m\,r\,\omega^2 = 0 \qquad [\because\ \omega = 0]$

$$\therefore\ H_A = 0$$ ***Ans.***

17.7 Work Energy Principle

1. Work Energy Principle

Total work done = Final KE – Initial KE

$$U = T_2 - T_1$$

2. Force in the Spring

$$F = k\,x$$

3. Work Done by Spring Force

$$W = \frac{1}{2} k\,[x_1^2 - x_2^2]$$

4. K.E. of Rigid Body in Translation Motion

$$T = \frac{1}{2} m\,v_G^2$$

where v_G = Velocity of mass centre.

5. K.E. of Rigid Body in Rotational Motion

$$T = \frac{1}{2} I\,\omega^2$$

For centroidal rotation

$$T = \frac{1}{2} I_G\,\omega^2$$

For non-centroidal rotation

$$T = \frac{1}{2} I_O\,\omega^2$$

6. K.E. of Rigid Body in Plane Motion

T = KE due to translation of CG + KE due to rotation at the CG

$$\therefore\ T = \frac{1}{2} m\,v_G^2 + \frac{1}{2} I_G\,\omega^2$$

Note : If the body rolls without slipping then consider its non-centroidal rotation at the point of contact with the stationary surface.

$\therefore$ KE is given by

$$T = \frac{1}{2} I_O\,\omega^2$$

17.8 Solved Problems on Work Energy Principle

Problem 15

A slender rod AB, of length $L = 3$ m weighs 100 N. It rotates in a vertical plane about end A where its is hinged. When in the horizontal position, its angular velocity is 20 rpm. Find its angular velocity in rpm when it has moved an angle of 120° from the horizontal position, as shown in Fig. 17.15. Also calculate the linear velocity of a point C at a distance 2 m from the point A in the second position.

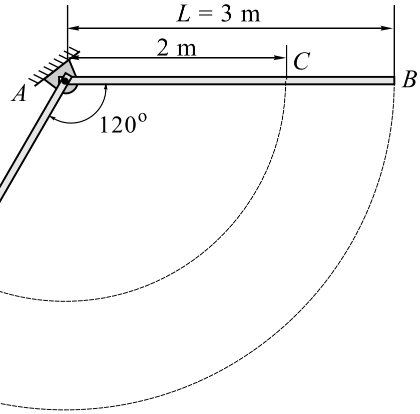


Fig. 17.15

Solution

(i) $I_A = \frac{1}{3} \times ml^2 = \frac{1}{3} \times \frac{100}{9.81} \times 3^2$

$\therefore\ I_A = 30.58 \text{ kgm}^2$

(ii) Applying work energy principle

$U = T_2 - T_1$

$100 \times 1.3 = \frac{1}{2} \times 30.58 \times \left(\omega_2^2 - 2.09^2\right)$

$\therefore\ \omega_2 = 3.59 \text{ r/s}$ ***Ans.***

(ii) $N_2 = \frac{60 \times 3.59}{2\pi}$

$N_2 = 34.26 \text{ rpm}$

To calculate the linear velocity

$v_C = AC\,\omega = 2 \times 3.59$

$\therefore\ v_C = 7.18 \text{ m/s}$ ***Ans.***

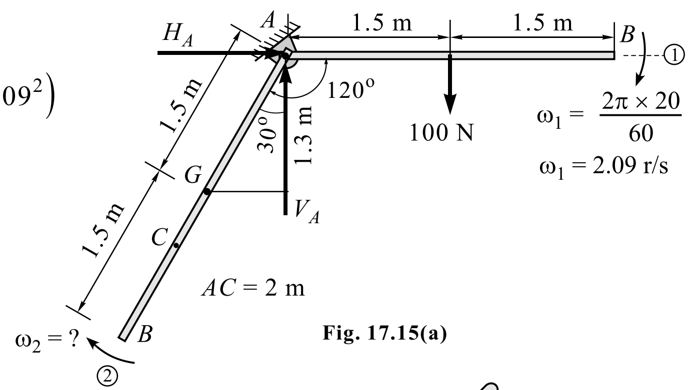


Fig. 17.15(a)

Problem 16

A uniform 5 kg bar is acted upon by a 30 N force which always acts perpendicular to the bar, as shown in Fig. 17.16. If the bar has an initial angular velocity in clockwise direction, $\omega_1 = 10$ r/s when $\theta = 0°$, determine its angular velocity at the instant $\theta = 90°$. Use work energy principle. Take length of the bar as 0.6 m.

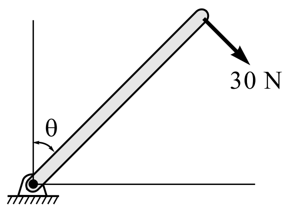


Fig. 17.16

Solution

(i) $I_A = \frac{1}{3} \times ml^2 = \frac{1}{3} \times 5 \times 0.6^2$

$\therefore\ I_A = 0.6 \text{ kgm}^2$

(ii) Applying work energy principle

$U = T_2 - T_1$

$5 \times 9.81 \times 0.3 + 18 \times \frac{\pi}{2} = \frac{1}{2} \times 0.6\left(\omega_2^2 - 10^2\right)$

$\therefore\ \omega_2 = 15.6 \text{ r/s}$ ***Ans.***

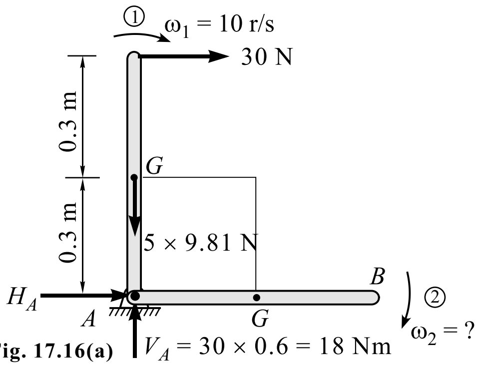


Fig. 17.16(a)

Problem 17

A uniform rod 0.9 m long and weighing 67.5 N is welded to the rim of a 1.8 m diameter hoop (thin cylinder) weighing 45 N. At the position shown in Fig. 17.17, the hoop has a clockwise angular velocity ω = 4 rad/s. Compute the angular velocity of the hoop after it has rolled without slipping through a half cycle.

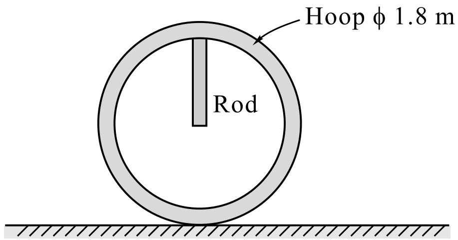


Fig. 17.17

Solution

$$m_1 = 6.88 \text{ kg}$$
$$m_2 = 4.59 \text{ kg}$$

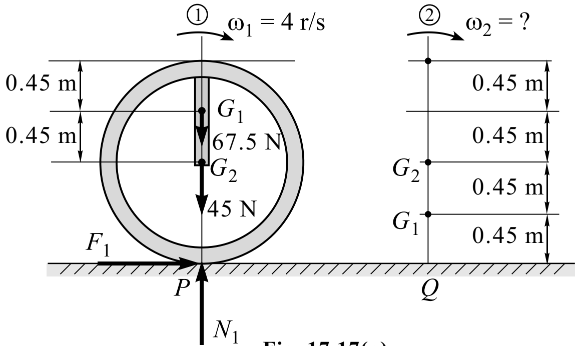


Fig. 17.17(a)

(i) $I_P = I_{rod} + I_{hoop}$

$$= \left(\frac{1}{12}\times 6.88\times 0.9^2 + 6.88\times 1.35^2\right) + \left(2\times 4.59\times 0.9^2\right)$$

$$\therefore\ I_P = 20.44 \text{ kgm}^2$$

(ii) $I_Q = I_{rod} + I_{hoop}$

$$= \left(\frac{1}{3}\times 6.88\times 0.9^2 + 2\times 4.59\times 0.9^2\right)$$

$$\therefore\ I_Q = 9.29 \text{ kgm}^2$$

(iii) Applying work energy principle

$$U = T_2 - T_1$$

$$67.5\times 0.9 = \frac{1}{2}\times 9.29\times \omega_2^2 - \frac{1}{2}\times 20.44\times 4^2$$

$\therefore\ \omega_2 = 6.95$ r/s ***Ans.***

Problem 18

A slender rod *AB* of length 1.5 m and mass 15 kg and pinned at point *O* which is 0.3 m from end *B*, as shown in Fig. 17.18. The other end is pressed against a spring of constant k = 300 kN/m until the spring is compressed 25 mm. The rod is then in a horizontal position. If the rod is released from this position, determine the angular velocity when the rod has rotated through **(i)** 60° and **(ii)** 90°.

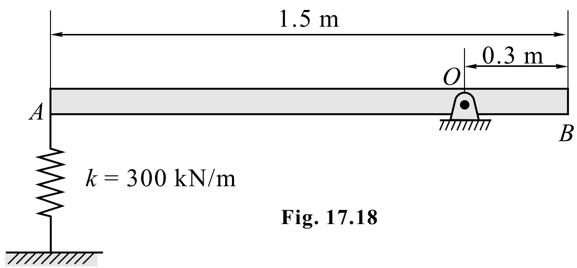


Fig. 17.18

Solution

Refer to Fig. 17.18(a).

$$I_O = \frac{1}{12}\times 15\times 1.5^2 + 15\times 0.45^2$$

$\therefore\ I_O = 5.85 \text{ kgm}^2$

Applying work energy principle

$$U = T_2 - T_1$$

$$-15 \times 9.81 \times 0.45 \sin\theta + \frac{1}{2} \times 300 \times 10^3 \times (0.025^2 - 0) = \frac{1}{2} \times 5.85 \times (\omega_2^2 - 0)$$

$$-22.64 \sin\theta + 32.05 = \omega_2^2$$

$$\omega_2^2 = 32.05 - 22.64 \sin\theta$$

(i) When $\theta = 60^\circ$

$\therefore\ \omega_2 = 3.53$ r/s ***Ans.***

(ii) When $\theta = 90^\circ$

$\therefore\ \omega_2 = 3.07$ r/s ***Ans.***

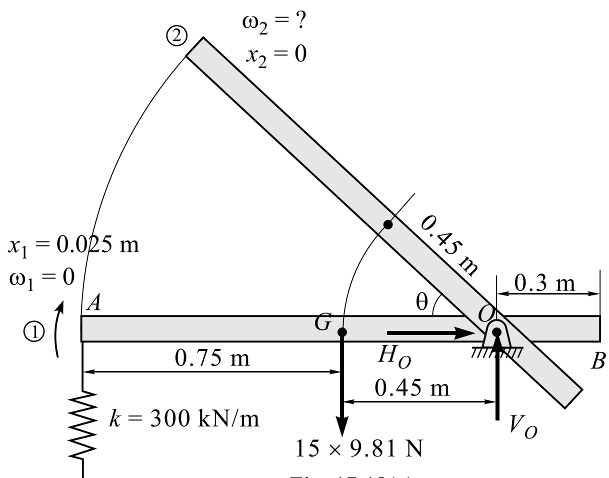


Fig. 17.18(a)

Problem 19

A 5 kg slender rod AB is welded to a 3 kg uniform disc which rotates about a point A. A spring of stiffness 80 N/m is attached to the disc and is unstretched when rod AB is horizontal. If the assembly, shown in Fig. 17.19, is released from rest determine the angular velocity of rod after it has rotated through 90°.

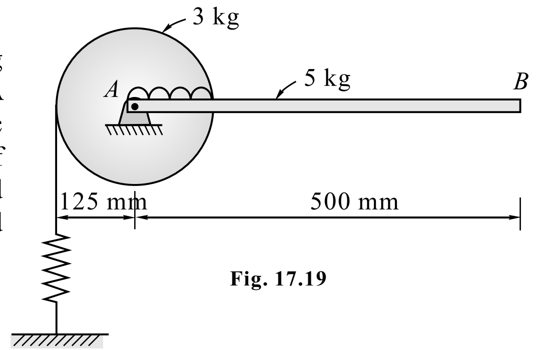


Fig. 17.19

Solution

Refer to Fig. 17.19(a).

(i) $I_A = I_{disk} + I_{rod}$

$$= \frac{1}{2} \times 3 \times 0.125^2 + \frac{1}{3} \times 5 \times 0.5^2$$

$\therefore\ I_A = 0.44 \text{ kgm}^2$

(ii) Applying work energy principle

$$U = T_2 - T_1$$

$$5 \times 9.81 \times 0.25 + \frac{1}{2} \times 80 \times (0 - 0.196^2) = \frac{1}{2} \times 0.44 \times (\omega_2^2 - 0)$$

$\therefore\ \omega_2 = 6.98$ r/s ***Ans.***

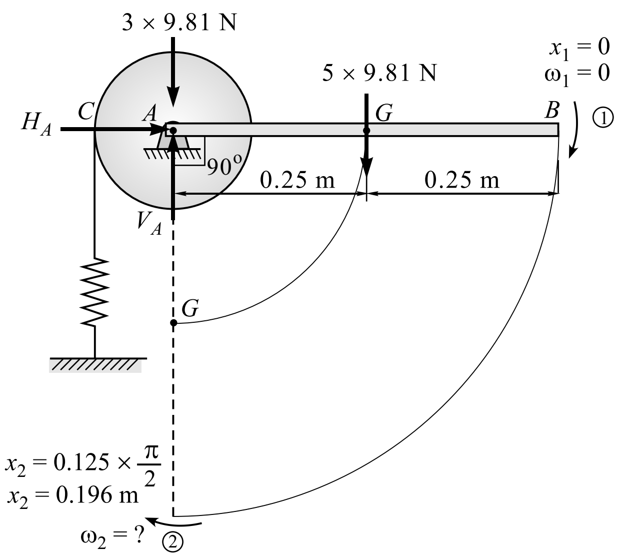


Fig. 17.19(a)

Exercises

[I] Problems

1. A thin rectangular plate of mass 70 kg is suspended by two hinges at *A* and *B,* as shown in Fig. 17.E1. If hinge *A* is suddenly removed, determine the angular acceleration of the plate and the components of the reaction at hinge *B* at this instant.

[***Ans.*** B_x = 142 N, B_y = 213 N (↑) and $\alpha = 6.788$ r/s^2 (↺).]

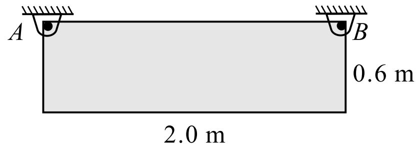


Fig. 17.E1

2. At what point should the horizontal force *F* be applied to the slender bar, shown in Fig. 17.E2, which hangs from a hinge at *A* so that the horizontal component of the reaction at *A* is 0.

[***Ans.*** $y = 2l/3$]

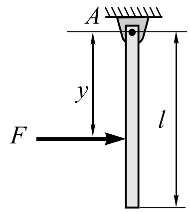


Fig. 17.E2

3. Two blocks *A* and *B* having mass of 8 kg and 15 kg are attached to the ends of a cord which passes over a 4 kg pulley, as shown in Fig. 17.E3. If the blocks are released from rest, determine the speed of the block *A* after 3 seconds. Assume the pulley to be a disk and no slipping of the cord over the pulley.

[***Ans.*** a = 2.746 m/s^2 and v_A = 8.24 m/s (↑).]

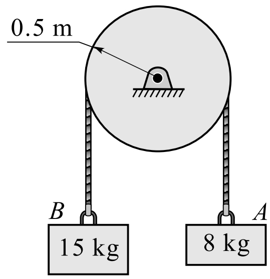


Fig. 17.E3

4. A 10 kg wheel has a radius of gyration of *K* = 225 mm and is subjected to a moment of *M* = 4 Nm, as shown in Fig. 17.E4. Determine the angular velocity at *t* = 2 seconds starting from rest and compute reactions at the pin *A*.

[***Ans.*** ω = 15.8 r/s, A_x = 0 and A_y = 98.1 N (↑).]

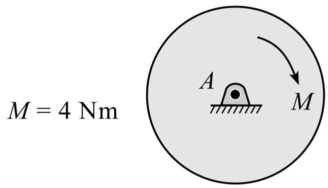


Fig. 17.E4

5. A compound pulley system, shown in Fig. 17.E5, has a mass of 30 kg, radius of gyration of 450 mm. Determine the tension in each cord and angular acceleration of the pulleys when the masses of 50 kg and 150 kg which the pulley supports are released.

[***Ans.*** α = 3.916 r/s^2 (↻), 608 N and 1295 N.]

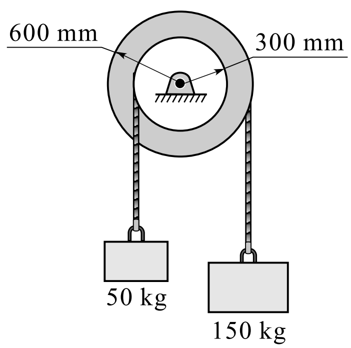


Fig. 17.E5

6. The coefficient of friction between a solid sphere whose mass is 7.5 kg and the plane is 0.1. The radius of the sphere is 150 mm, as shown in Fig. 17.E6. Determine the angular acceleration of the sphere and the linear acceleration of its mass centre.

[***Ans.*** $a = 3.5$ r/s^2 and $\alpha = 23.57$ r/s^2.]

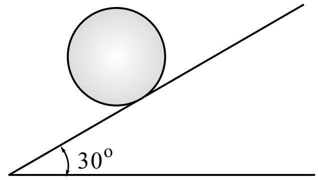


Fig. 17.E6

7. A cylindrical hollow drum of 2 kg mass and 400 mm ϕ is completely filled with solidified coal tar and is rolling down a rough inclined road making an angle of 5° to the horizontal, as shown in Fig. 17.E7. Determine **(a)** the acceleration of the centre of mass of drum after it has rolled down 5 m and **(b)** the time taken to roll down a distance of 5 m.

[***Ans.*** $a = 0.57$ r/s^2 and $t = 4.18$ sec.]

8. Determine the acceleration of lowering block *A* and the tension in the cord supporting it. Note, there is another hub and vertical string on the other side of the pulley to make the vertical plane of Fig. 17.E8, a plane of symmetry.

[***Ans.*** $a = 4.147$ r/s^2 and $T = 87.05$ N.]

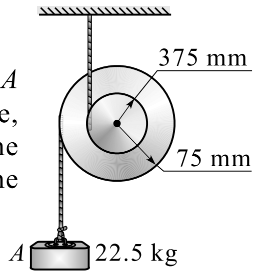


Fig. 17.E8

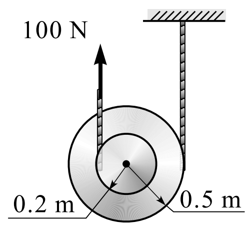


Fig. 17.E9

9. The spool has a mass of 8 kg and radius of gyration $k = 0.35$ m. If the cords of negligible mass are wrapped around the inner hub and outer rim, as shown in Fig. 17.E9. Determine the spools angular velocity 3 seconds after it is released from rest.

[***Ans.*** 30.96 r/s (↻)]

10. The wheel shown in Fig. 17.E10 has a mass of 25 kg and radius of gyration $k = 0.2$ m. If 50 NM couple is applied to the wheel. Determine the acceleration of its mass centre $\mu_s = 0.3$ and $\mu_k = 0.25$ between the plane and the wheel at *A*.

[***Ans.*** 2.45 m/s^2 (→)]

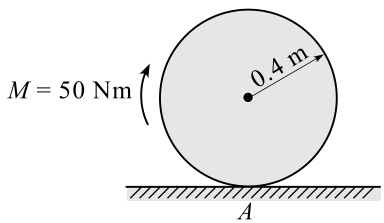


Fig. 17.E10

11. A rectangular plate 2 m × 3 m of mass 100 kg/m^2 is supported by hinges at *A* and *B,* as shown in Fig. 17.E11. If the support *A* is removed, determine the reaction at *B* and the angular acceleration of the plate at this instant.

[***Ans.*** $\alpha = 2.26$ r/s^2 (↺)]

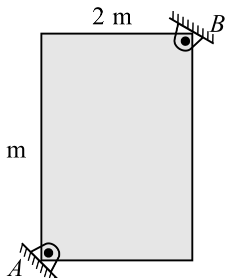


Fig. 17.E11

12. A 4 kg homogeneous slender rod is 1 m long. It pivots about one end. When released from rest in the horizontal position, if falls under the action of gravity and a constant retarding torque of 5 NM. What will be its angular speed as it passes through its lowest position.

13. A uniform bas AB of mass 40 kg and length 4 m is hinged at C. A homogeneous sphere of mass 20 kg and diameter 1.6 m is attached to the bar AB, as shown in Fig. 17.E13. Determine the MI of this mass system about pin axis C.

[***Ans.*** 387.2 kgm^2]

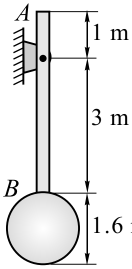


Fig. 17.E13

14. A uniform bas AB of mass M = 8 kg and length L = 6 m is supported by wire CD, as shown in Fig. 17.E14. Determine the **(a)** angular acceleration of bar AB, **(b)** linear acceleration of free end B and **(c)** reaction at the hinged support A. At the instant the wire is cut suddenly.

[***Ans.*** **(a)** 2.45 r/s^2, **(b)** 14.715 m/s^2 and **(c)** 19.62 N (↑).]

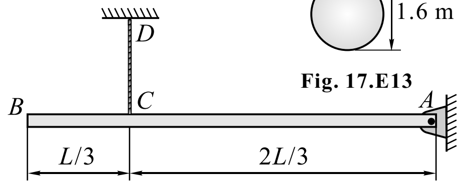


Fig. 17.E14

15. A 30 kg disc, shown in Fig. 17.E15, is pin supported at its centre. Determine the number of revolutions it must make to attain an angular velocity of 20 r/s starting from rest. It is acted upon by a constant force of F = 10 N which is applied to a cord wrapped around its periphery and a constant couple M = 5 NM.

[***Ans.*** 2.73 rev]

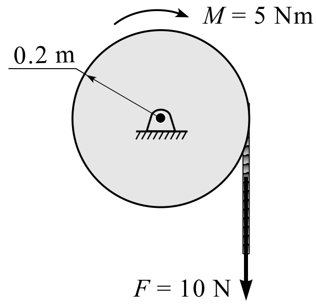


Fig. 17.E15

16. A motor supplies a constant torque of M = 5 NM. It the drum shown in Fig. 17.E16 has a weight of 300 N and I_o = 1.2 kgm^2. Determine the speed of the 100 N crate A after it rises by 3 m starting from rest.

[***Ans.*** 6.6 m/s]

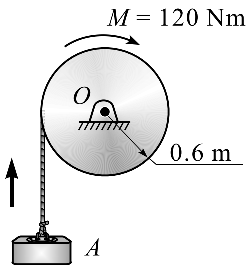


Fig. 17.E16

17. A rope is wrapped around a 28 kg solid cylinder B as shown in Fig. 17.E17. Find the speed of its centre G, after it has dropped 2 m from rest.

[***Ans.*** 5.11 m/s]

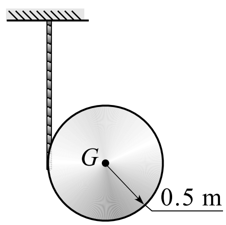


Fig. 17.E17

18. A disc, shown in Fig. 17.E18, is acted upon by a couple M = 10 NM. If the mass of the disk is 5 kg. Find its angular velocity after two revolutions starting from rest.

[***Ans.*** 66.9 r/s]

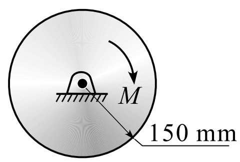


Fig. 17.E18

[II] Review Questions

1. What are the effects of translation motion, rotational motion and general plane motion on rigid bodies ?
2. How Newton's second law is applicable on a rigid body in motion ?
3. How to deal with centroidal and non-centroidal rotational motion ?
4. How to analyse rolling without slipping and rolling with slipping motion ?
5. How to use work energy principle for system of rigid bodies in motion ?
6. How is kinetic energy expressed for translation motion, rotational motion and general plane motion ?
7. How is work done related by frictional force of a rolling body without slipping on rough surface ?

[III] Fill in the Blanks

1. _________ is the study of geometry of motion with reference to the cause of motion.
2. Kinetics of particle is the study of geometry of ________ motion with reference to the cause of motion.
3. The _________ of a body is proportional to the resultant force acting on it and is in the direction of the force.
4. The concept of _______ moment of inertia is necessary for the motion analysis of rigid bodies.
5. If the axis of rotation passes through the centre of gravity of the body then it is called _______ rotational motion.

[IV] Multiple-choice Questions

Select the appropriate answers from the given options.

1. Kinetics of rigid bodies deals with _________.
   (a) translation motion **(b)** rotational motion **(c)** plane motion **(d)** All of these
2. If axis of rotation is not passing through centre of gravity then it is called as _________ motion.
   (a) centroidal **(b)** non-centroidal **(c)** plane **(d)** None of these
3. A rolling body is identified as _______ motion.
   (a) centroidal **(b)** non-centroidal **(c)** plane **(d)** All of these
4. If a body rolls ________ slipping then point of contact with stationary rough surface is identified as ICR.
   (a) with **(b)** without **(c)** 50% **(d)** 100%
5. Generally for a rolling body on rough surface without slipping on horizontal surface the direction of friction and direction of motion is ________.
   (a) same **(b)** upwards **(c)** opposite **(d)** downwards
6. Mass moment of inertia about centre of gravity of hoop is given by ______ .
   (a) $I_G = \frac{1}{2}mr^2$ **(b)** $I_G = 2mr^2$ **(c)** $I_G = mr^2$ **(d)** $I_G = \frac{2}{5}mr^2$

18

MECHANICAL VIBRATION

18.1 Introduction

If any rigid body within elastic limit structure is in equilibrium and is disturbed then it tries to maintain its equilibrium position. While doing so the body starts vibrating. The disturbed stored energy sets in oscillations about equilibrium mean position.

Vibration of a rigid body in a to and fro manner occurs on both the sides of the mean position again and again after certain interval of time. If further disturbing force does not act while vibrating, then it is called *Free Vibration*. It means now vibration motion is maintained only due to the gravitational force or elastic force. Best example to understand is simple pendulum bob up and down motion.

18.2 Simple Harmonic Motion (SHM)

To and fro periodic motion that occurs on both sides of the mean position with acceleration directed towards the mean position is known as *Simple Harmonic Motion*.

Consider a particle moving along the circumference of a circle of radius r with constant angular velocity ω (rad/sec) as shown in Fig.18.2-i.

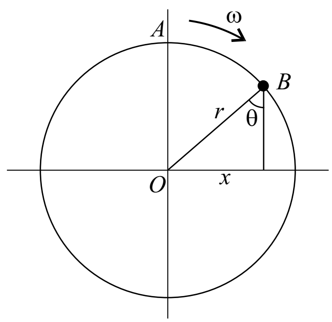


Fig. 18.2-i

$$x = r \sin \theta$$

$$x = r \sin \omega t \qquad (\because \theta = \omega t)$$

$$\frac{dx}{dt} = r\,\omega \cos \omega t$$

$$\boxed{v = r\,\omega \cos \omega t}$$

$$\frac{d^2x}{dt^2} = -r\,\omega^2 \sin \omega t$$

$$\boxed{a = -\omega^2 x} \qquad (\because r \sin \omega t = x)$$

–ve sign indicates acceleration is always opposite to displacement.

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \qquad \text{[Differential equation]}$$

$$\frac{dx}{dt} = r\,\omega \cos \omega t$$

$$\frac{dx}{dt} = r\,\omega\sqrt{1-\sin^2 \omega t}$$

$$\frac{dx}{dt} = r\,\omega\sqrt{1-\frac{x^2}{r^2}} \qquad \left(\because \sin \omega t = \sin\theta = \frac{x}{r}\right)$$

$$\boxed{v = \omega\sqrt{r^2 - x^2}}$$

Important Terms :

1. **Amplitude :** In SHM, the maximum displacement of particle from the mean position is known as amplitude. Amplitude here is equal to radius of circle r on which imaginary particle P is moving with constant angular velocity ω (rad/s).
2. **Oscillation :** In SHM, one complete periodic motion (repetitive) of the particle is known as oscillation.
3. **Beat :** In SHM, half of the oscillation is equal to beat.
4. **Time Period (T):** In SHM, the time required by the particle to complete one oscillation is known as time period or periodic time. Time in which, imaginary particle P rotates with angular velocity ω (rad/s) through one complete revolution $2\pi r$ (where r is the radius), i.e.

$$\text{Periodic time } T = \frac{2\pi}{\omega} \text{ seconds}$$

5. **Frequency (f) :** In SHM, the number of oscillations performed by the particle in unit time is known as frequency, i.e.

$$\text{Frequency } f = \frac{1}{T} = \frac{\omega}{2\pi} \text{ oscillation/s}$$

18.2.1 Solved Problems

Problem 1

A particle undergoes a SHM, the acceleration of the particle at distance of 1.5 m from centre of motion being 6 m/s^2. Find the time of an oscillation.

Solution

We have, acceleration $a = \omega^2 x$

$$\therefore\ 6 = \omega^2 \times 1.5$$

$$\therefore\ \omega = 2 \text{ rad/s}$$

Time of an oscillation is given by

$$T = \frac{2\pi}{\omega} \text{ seconds}$$

$$T = \frac{2\pi}{2} \text{ seconds}$$

$$\therefore\ T = 3.14 \text{ seconds} \quad \textbf{\textit{Ans.}}$$

Problem 2

A body performing SHM has a velocity 12 m/s when displacement is 50 mm and 3 m/s, when the displacement is 100 mm, the displacement being measured from midpoint. Calculate the frequency and amplitude of the motion. What is the acceleration when the displacement is 75 mm.

Solution

Given : $v = 12$ m/s at $x = 0.05$ m

$v = 3$ m/s at $x = 0.1$ m

We have, velocity $v = \omega\sqrt{r^2 - x^2}$

$$12 = \omega\sqrt{r^2 - 0.05^2} \quad \text{.... (i)}$$

$$3 = \omega\sqrt{r^2 - 0.1^2} \quad \text{.... (ii)}$$

Now, (i) ÷ (ii), we get

$$r = 0.1024 \text{ m}$$

and $\omega = 134.16$ rad/s

$$\text{and } \omega = 134.16 \text{ rad/s}$$

$$\text{Frequency } f = \frac{\omega}{2\pi} = 21.35 \text{ Hz}$$

$$\text{Acceleration } a = \omega^2 x$$

$$\therefore a = (134.16)^2 \times 0.075$$

$$\therefore a = 1349.92 \text{ m/s}^2$$ ***Ans.***

18.3 Simple Pendulum

A simple pendulum consists of a heavy bob suspended at the end of an inextensible, weightless string. The upper end of string is fixed. Let l be the length of the string, m be the mass of the bob.

When the pendulum is deflected by a small angle θ and released, it starts oscillating with frequency f and periodic time T. For small angle of oscillation, the bob performs simple harmonic motion (SHM).

The vertical lowest position of the bob is the mean position.

The acceleration of the bob is towards the mean position.

The acceleration is always opposite to displacement and therefore considered as negative in analysis.

At that instant t is the tangential direction along which component of acceleration in tangential direction will act (a_t).

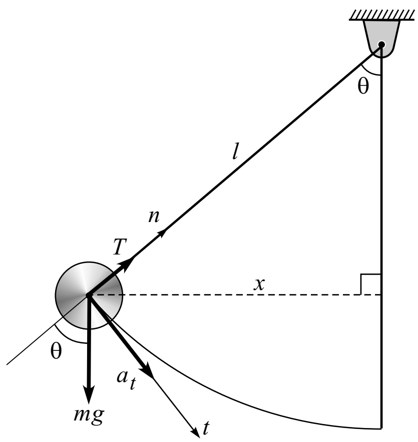


Fig. 18.3-i

By Newton's second law

$$\Sigma F_t = m\, a_t$$

$$m\, g \sin \theta = m\, a_t$$

$$a_t = g \sin \theta$$

$$\frac{dv}{dt} = g \frac{x}{l}$$

$$\frac{dv}{dt} = -a_t = \left(\frac{g}{l}\right) x$$ [−ve sign indicates acceleration is always opposite to displacement]

$$\frac{d^2x}{dt^2} + \left(\frac{g}{l}\right) x = 0$$

Comparing the differential equation, we get

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\therefore \; \omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$T = \frac{2\pi}{\omega}$$

$$T = 2\pi \sqrt{\frac{l}{g}} \quad \text{(Periodic time)}$$

18.4 Compound Pendulum

A rigid body of mass m is suspended through the point O. consider initial position making angle θ with vertical and released.

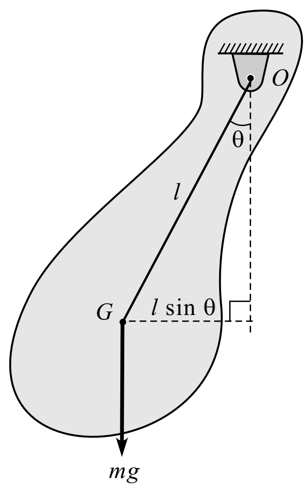


Fig. 18.4-i

$$\Sigma M_O = I_O\, \alpha$$

$$m\, g\, l \sin \theta = I_O\, \alpha$$

$$m\, g\, l\, \theta = I_O\, \alpha \quad (\because \theta << \text{ small})$$

$$\alpha = \frac{m\, g\, l\, \theta}{I_O}$$

$$-\alpha = \frac{m\, g\, l\, \theta}{I_O}$$

[−ve sign indicates angular acceleration is always opposite to angular displacement]

$$\frac{d^2\theta}{dt^2} + \left(\frac{m\, g\, l}{I_O}\right) \theta = 0$$

Comparing the differential equation, we get

$$\frac{d^2\theta}{dt^2} + \omega^2 \theta = 0$$

$$\therefore \omega^2 = \frac{m\,g\,l}{I_O}$$

$$\omega = \sqrt{\frac{m\,g\,l}{I_O}}$$

$$T = \frac{2\pi}{\omega}$$

$$T = 2\pi \sqrt{\frac{I_O}{m\,g\,l}} \qquad \text{(Periodic time)}$$

where I_O = Mass moment of inertia about point of suspension,

m = Total mass of rigid body and

l = Distance of mass centre from suspension point.

18.5 Solved Problems

Problem 3

If $a = -36x$ is the expression of SHM then find its time period.

Solution

Given : $a = -36x$

We have, acceleration $a = -\omega^2 x$

$$\therefore \omega^2 = 36$$

$$\therefore \omega = 6 \text{ rad/s}$$

Time period $T = \dfrac{2\pi}{\omega}$

$$T = \frac{2\pi}{6}$$

$\therefore T = 1.047$ seconds ***Ans.***

Problem 4

A particle performing SHM having amplitude 0.7 m and time period 1.1 seconds. Find its maximum velocity and maximum acceleration.

Solution

Given : Amplitude $r = 0.7$ m, Time period $T = 1.1$ seconds.

We know that $T = \dfrac{2\pi}{\omega}$

i.e. $1.1 = \dfrac{2\pi}{\omega}$

$$\therefore \omega = 5.712 \text{ rad/s}$$

$$x = r \sin \theta$$

$$\frac{dx}{dt} = r \cos \theta \frac{d\theta}{dt}$$

$$v = r\,\omega \cos \theta$$

$$v_{max} = r\,\omega = 0.7 \times 5.712$$

$$\therefore \; v_{max} = 4 \text{ m/s} \quad \textbf{\textit{Ans.}}$$

$$a_{max} = -r\,\omega^2 = 0.7 \times 5.712^2$$

$$\therefore \; a_{max} = 22.84 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

Problem 5

For SHM having amplitude 0.8 m and periodic time 1.3 seconds. Find displacement, velocity and acceleration after 0.6 seconds.

Solution

Given : Amplitude $r = 0.8$ m, periodic time $T = 1.3$ seconds.

We know that $T = \frac{2\pi}{\omega}$

$$\therefore \; \omega = \frac{2\pi}{T} = \frac{2\pi}{1.3}$$

$$\therefore \; \omega = 4.833 \text{ rad/s}$$

Displacement after $t = 0.6$ sec

$$\theta = \omega t = 4.833 \times 0.6$$

$$\theta = 2.9 \text{ rad}$$

$$\therefore \; \theta = 2.9 \times \frac{180}{\pi} = 166.15°$$

$$x = r \sin \theta = 0.8 \sin 166.15°$$

$$\therefore \; x = 0.192 \text{ m} \quad \textbf{\textit{Ans.}}$$

Velocity $v = r\,\omega \cos \theta = 0.8 \times 4.833 \cos 166.15°$

$$\therefore \; v = -3.754 \text{ m/s} \quad \textbf{\textit{Ans.}}$$

Acceleration $a = -\omega^2 x = 4.833^2 \times 0.192$

$$\therefore \; a = 4.485 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

Problem 6

Find the length of simple pendulum if its time period is 1.5 seconds.

Solution

Given : Periodic time $T = 1.5$ seconds.

We know that $T = 2\pi \sqrt{\frac{l}{g}}$

$$1.5 = 2\pi \sqrt{\frac{l}{9.81}}$$

$$\therefore \; l = 0.865 \text{ m} \qquad \textbf{\textit{Ans.}}$$

Problem 7

Find the period of simple pendulum for length 1.6 m.

Solution

Given : Length $l = 1.6$ m

We know that $T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{1.6}{9.81}}$

$\therefore\ T = 2.537$ seconds ***Ans.***

Problem 8

A uniform bar ABC of mass 40 kg is bent at right angle at B to form an L shape. It is then pin suspended at A, as shown in Fig. 18.8(a). If the bar is slightly displaced from its equilibrium position and released, find the time period of oscillations performed. Take $AB = 4$ m and $BC = 6$ m.

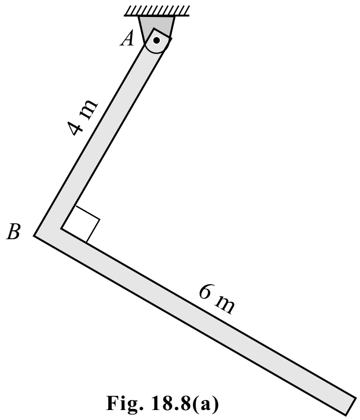


Fig. 18.8(a)

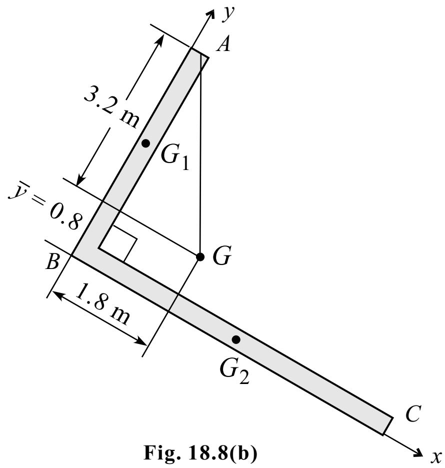


Fig. 18.8(b)

Solution

Given : Total length = 10 m

Total mass = 40 m

$$\therefore\ \text{mass per meter} = \frac{40}{10} = 4 \text{ kg/m}$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{16 \times 2 + 24 \times 0}{16 + 24}$$

$$\bar{y} = 0.8 \text{ m}$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\bar{x} = \frac{16 \times 0 + 24 \times 3}{16 + 24}$$

$$\bar{x} = 1.8 \text{ m}$$

$$AG = \sqrt{(3.2)^2 + (1.8)^2}$$

$$\therefore\ l = AG = 3.672 \text{ m}$$

Now, $I_A = I_{AB} + I_{BC}$

$$I_A = \frac{16 \times 4^2}{3} + \left(\frac{24 \times 6^2}{12} + 24 \times 5^2\right)$$

$$I_A = 757.33 \text{ kg.m}^2$$

Periodic time $T = 2\pi\sqrt{\frac{I_A}{Mgl}} = 2\pi\sqrt{\frac{757.33}{40 \times 9.81 \times 3.672}}$

$\therefore\ T = 4.556$ seconds ***Ans.***

Problem 9

Find the time period of oscillation for given arrangement of disc, rod and sphere suspended through, as shown in Fig. 18.9.

$M_{disc} = 10$ kg, $M_{rod} = 10$ kg, $M_{sphere} = 12$ kg.

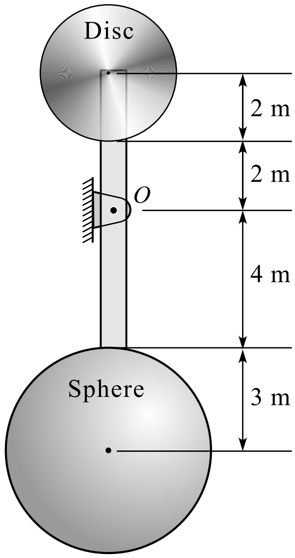


Fig. 18.9

Solution

Given : $M_{disc} = 10$ kg, $M_{rod} = 10$ kg, $M_{sphere} = 12$ kg.

(i) $$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{10 \times 4 + 10 \times (-1) + 12 \times (-7)}{10 + 10 + 12}$$

$$\bar{y} = -1.6875 \text{ m}$$

$$\therefore \; l = 1.6875 \text{ m}$$

(ii) $$I_O = I_{disc} + I_{rod} + I_{sphere}$$

$$I_O = \left(\frac{10 \times 2^2}{2} + 10 \times 4^2\right) + \left(\frac{10 \times 6^2}{12} + 10 \times 1^2\right) + \left(\frac{2}{5} \times 12 \times 3^2 + 12 \times 7^2\right)$$

$$I_O = 851.2 \text{ kg.m}^2$$

(iii) $$T = 2\pi\sqrt{\frac{I_O}{Mgl}} = 2\pi\sqrt{\frac{851.2}{32 \times 9.81 \times 1.6875}}$$

$$\therefore \; T = 7.97 \text{ sec}$$ ***Ans.***

Problem 10

A rod of length l is welded to another rod of length $2l$ to form a symmetric T section. The rods have same mass per unit length. The section is suspended, as shown in Fig. 18.10. Calculate the period of oscillation of the T section. At which point; when it is suspended, its period of oscillation will be infinite (theoretically).

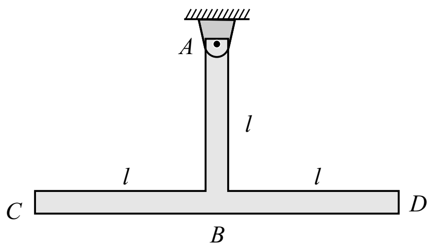


Fig. 18.10

Solution

(i) $$\bar{y} = \frac{(ml)\left(\frac{l}{2}\right) + (2ml)(l)}{ml + 2ml}$$

$$\bar{y} = \frac{5l}{6}$$

(ii) $$I_A = \frac{(ml)(l)^2}{3} + \left[\frac{(2ml)(2l)^2}{12} + (2ml)(l)^2\right]$$

$$I_A = 3ml^3 \text{ kg.m}^2$$

(iii) $T = 2\pi\sqrt{\dfrac{I_A}{Mg\bar{y}}} = 2\pi\sqrt{\dfrac{3ml^3}{(3ml)\times g\times\dfrac{5l}{6}}}$

$\therefore\ T = 2\pi\sqrt{\dfrac{6l}{5g}}$ ***Ans.***

(iv) If T section is suspended through C.G. i.e.

$\bar{y} = \dfrac{5l}{6}$ its period of oscillation will be infinite. ***Ans.***

Problem 11

A thin, homogeneous, square plate is suspended from the midpoint of one of its side, as shown in Fig. 18.11. Calculate the period of free vibration of the plate.

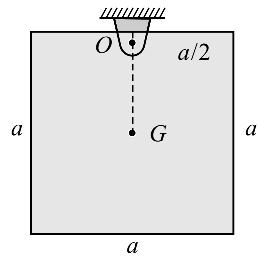


Fig. 18.11

Solution

(i) $l = OG = \dfrac{a}{2}$

(ii) $I_O = I_G + m(OG)^2$

$$I_O = \frac{1}{12}m(a^2 + a^2) + m\left(\frac{a}{2}\right)$$

$$I_O = \frac{5}{12}ma^2$$

(iii) $T = 2\pi\sqrt{\dfrac{I_O}{mgl}} = 2\pi\sqrt{\dfrac{\dfrac{5}{12}ma^2}{m\times g\times\dfrac{a}{2}}}$

$\therefore\ T = 2\pi\sqrt{\dfrac{5a}{6g}}$ ***Ans.***

Problem 12

A 3 kg uniform rod AB is attached to a spring of stiffness 1000 N/m, as shown in Fig. 18.12(a). If the end A of the rod is moved down slightly and released, determine period of vibration of the rod.

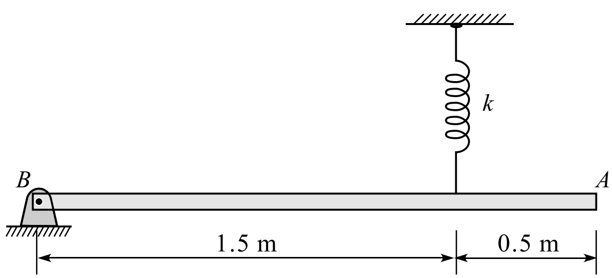


Fig. 18.12(a)

Solution

Given : Spring stiffness $k = 1000$ N/m

$$\Sigma M_B = 0$$

$$F_O \times 1.5 = 30 \times 1$$

$$\therefore\ F_O = 20\text{ N}$$

Now, $F_O = k\,x_O$

$$\therefore \; x_O = \frac{F_O}{k} = \frac{20}{1000} = 0.02 \text{ m}$$

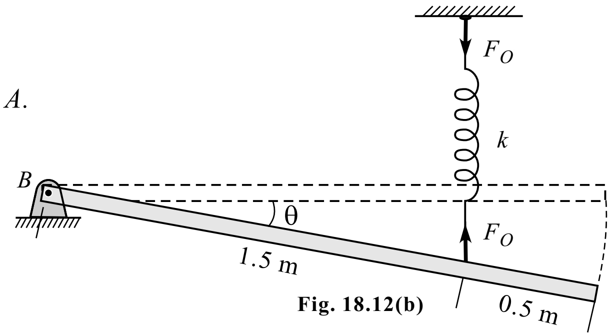


Fig. 18.12(b)

Let θ be a small angular displacement of end *A*. Refer Fig. 18.12(b).

$$\therefore \; x = 1.5\,\theta$$

$$F = k(x + x_O)$$

$$\therefore \; F = 1000(1.5\,\theta + 0.02)$$

$$I_B = \frac{ml^2}{3} = \frac{3 \times 2^2}{3}$$

$$\therefore \; I_B = 4 \text{ kg.m}^2$$

$$\Sigma M_B = I_B\,\alpha$$

$$-30 \times 1 + F \times 1.5 = 4\,\alpha$$

$$-30 + 1000(1.5\,\theta + 0.02) \times 1.5 = 4\,\alpha$$

$$-30 + 2250\,\theta + 30 = 4\,\alpha$$

$$\alpha = \frac{2250}{4}\,\theta$$

$$\alpha = 562.5\,\theta$$

$$-\frac{d^2\theta}{dt^2} = 562.5\,\theta$$

i.e. $$\frac{d^2\theta}{dt^2} + 562.5\,\theta = 0$$

Comparing with $\frac{d^2\theta}{dt^2} + \omega^2\,\theta = 0$ (Differential equation), we get

$$\omega^2 = 562.5 \qquad \therefore \; \omega = 23.72$$

Now $$T = \frac{2\pi}{\omega}$$

$$\therefore \; T = 0.265 \text{ seconds}$$ ***Ans.***

Problem 13

A 150 N block is suspended from a cord that passes over a 250 N disc, as shown in Fig. 18.13(a). The spring has stiffness k = 3750 N/m. Determine the natural period of vibration for the system.

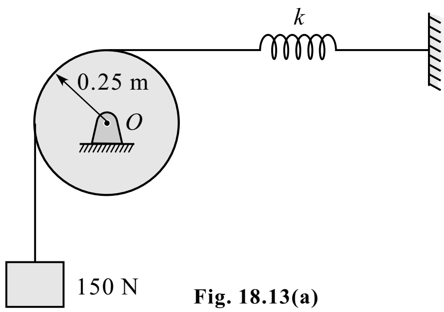


Fig. 18.13(a)

Solution

Given : Spring stiffness k = 3750 N/m

$$F_O = 150 \text{ N}$$

Now, $F_O = k\,x_O$

$$\therefore\ x_O = \frac{F_O}{k} = \frac{150}{3750} = 0.04 \text{ m}$$

$$F = k(x + x_O)$$

$$\therefore\ F = 3750(0.25\ \theta + 0.04)$$

$$I_O = \frac{250 \times 0.25^2}{9.81 \times 2}$$

$$I_O = 0.796 \text{ kg.m}^2$$

$$\Sigma\, M_O = I_O\, \alpha$$

$$-150 \times 0.25 + F \times 0.25 = 0.7964\ \alpha$$

$$-37.5 + 375.0(0.25\ \theta + 0.04) \times 0.25 = 0.7964\ \alpha$$

$$-37.5 + 234.375\ \theta + 37.5 = 0.7964\ \alpha$$

$$\alpha = \frac{234.375}{0.7964}\ \theta$$

$$\alpha = 294.3\ \theta$$

$$-\frac{d^2\theta}{dt^2} = 294.3\ \theta$$

i.e. $\dfrac{d^2\theta}{dt^2} + 294.3\ \theta = 0$

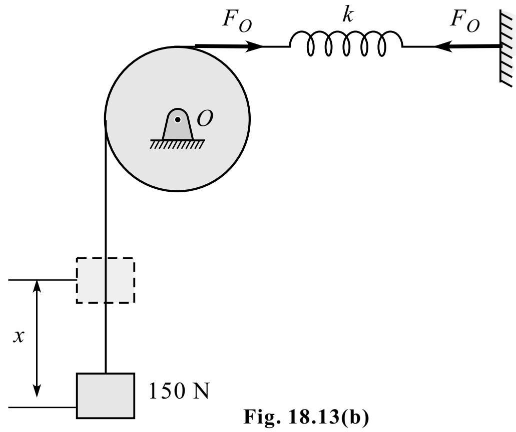


Fig. 18.13(b)

Comparing with $\dfrac{d^2\theta}{dt^2} + \omega^2\ \theta = 0$ (Differential equation), we get

$$\omega^2 = 294.3$$

$$\therefore\ \omega = 17.155$$

Now $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{17.155}$

$\therefore\ T = 0.366$ seconds ***Ans.***

Exercises

[I] Problems

1. Find the amplitude and time period of a particle moving with simple harmonic motion (SHM), which has a velocity of 8 m/s and 4 m/s at a distance of 2 m and 4 m respectively from the centre.

[***Ans.*** Amplitude $r = 4.472$ m, Time period $T = 3.142$ seconds.]

2. A particle performing SHM has a time period of 6 seconds. The velocity of the particle, when a distance of 2 cm from the extreme position is 60% of the maximum velocity. Find **(i)** Amplitude **(ii)** maximum velocity **(iii)** maximum acceleration and **(iv)** the velocity when particle is 4 cm away from the extreme position.

[***Ans.*** Amplitude $r = 0.1$ m, $v_{max} = 0.105$ m/s, $a_{max} = -0.1097$ m/s^2, $v_{4\text{ cm}} = 0.0838$ m/s.]

3. A particle moving with SHM has velocity 18 m/s and acceleration 4.2 m/s^2, when its displacement is 1 m from its mid position. Determine its periodic time, velocity and acceleration after 0.6 seconds. Also find its maximum velocity and maximum acceleration.

[***Ans.*** $T = 3.065$ seconds, $v = 6.057$ m/s, $a = -35$ m/s^2, $v_{max} = 18.12$ m/s, $a_{max} = -37.13$ m/s^2]

4. Two uniform rods each of mass M and length L are welded together to form and T shaped assembly as shown in Fig. 18.E4. Determine the time period of oscillations of the assembly.

$$\left[T = 2\pi\sqrt{\frac{17L}{18g}} \right]$$

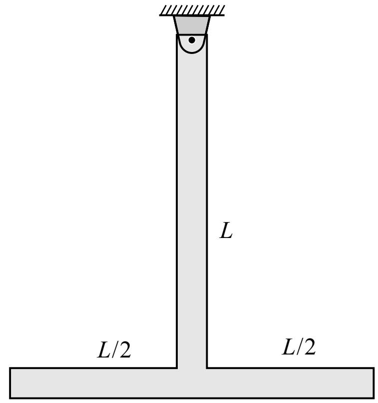


Fig. 18.E4

5. A uniform bar and a homogeneous disc are connected rigidly to each other at B and the system is suspended by a pin at A, as shown in Fig. 18.E5. Find the time period of small oscillations of the mass system. Mass of bar = Mass of disc = 40 kg.

[***Ans.*** $T = 4.37$ seconds]

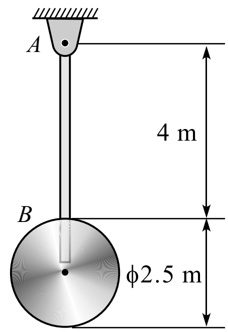


Fig. 18.E5

[II] Review Questions

1. Write a short note on 'simple pendulum' and derive its periodic time.
2. Derive the periodic time expression for a compound pendulum.

APPENDIX

IMPORTANT FORMULAE AND RESULTS

[A] Algebra

1. $a^0 = 1$; $x^0 = 1$

(Anything raised to the power zero is one)

2. $x^m \times x^n = x^{m+n}$

(If the bases are same in multiplication, the powers are added)

3. $\frac{x^m}{x^n} = x^{m-n}$

(If the bases are same in division, the powers are subtracted)

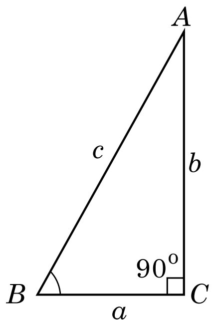
4. If $ax^2 + bx + c = 0$

then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

where a is the coefficient of x^2, b is the coefficient of x and c is the constant term.

[B] Trigonometry

1. In a right-angled triangle ABC



(a) $\frac{b}{c} = \sin\theta$ **(b)** $\frac{a}{c} = \cos\theta$ **(c)** $\frac{b}{a} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

(d) $\frac{c}{b} = \frac{1}{\sin\theta} = \text{cosec}\,\theta$ **(e)** $\frac{c}{a} = \frac{1}{\cos\theta} = \sec\theta$

(f) $\frac{a}{b} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \cot\theta$

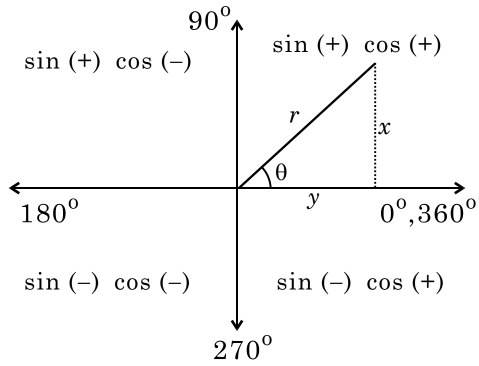
2. Table of Trigonometric Ratios

Angle	0°	(π/6) 30°	(π/4) 45°	(π/3) 60°	(π/2) 90°	(π) 180°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	–1
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	–1

3. Rules for the Change of Trigonometrical Ratios

(a)

$$\begin{aligned}
\sin(-\theta) &= -\sin\theta \\
\cos(-\theta) &= \cos\theta \\
\tan(-\theta) &= -\tan\theta \\
\cot(-\theta) &= -\cot\theta \\
\sec(-\theta) &= \sec\theta \\
\mathrm{cosec}(-\theta) &= -\mathrm{cosec}\,\theta
\end{aligned}$$



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

(b)

$$\begin{aligned}
\sin(90^\circ - \theta) &= \cos\theta \\
\cos(90^\circ - \theta) &= \sin\theta \\
\tan(90^\circ - \theta) &= \cot\theta \\
\cot(90^\circ - \theta) &= \tan\theta \\
\sec(90^\circ - \theta) &= \mathrm{cosec}\,\theta \\
\mathrm{cosec}(90^\circ - \theta) &= \sec\theta
\end{aligned}$$

(c)

$$\begin{aligned}
\sin(90^\circ + \theta) &= \cos\theta \\
\cos(90^\circ + \theta) &= -\sin\theta \\
\tan(90^\circ + \theta) &= -\cot\theta \\
\cot(90^\circ + \theta) &= -\tan\theta \\
\sec(90^\circ + \theta) &= -\mathrm{cosec}\,\theta \\
\mathrm{cosec}(90^\circ + \theta) &= \sec\theta
\end{aligned}$$

(d)

$$\begin{aligned}
\sin(180^\circ - \theta) &= \sin\theta \\
\cos(180^\circ - \theta) &= -\cos\theta \\
\tan(180^\circ - \theta) &= -\tan\theta \\
\cot(180^\circ - \theta) &= -\cot\theta \\
\sec(180^\circ - \theta) &= -\sec\theta \\
\mathrm{cosec}(180^\circ - \theta) &= \mathrm{cosec}\,\theta
\end{aligned}$$

(e)

$$\begin{aligned}
\sin(180^\circ + \theta) &= -\sin\theta \\
\cos(180^\circ + \theta) &= -\cos\theta \\
\tan(180^\circ + \theta) &= \tan\theta \\
\cot(180^\circ + \theta) &= -\cot\theta \\
\sec(180^\circ + \theta) &= -\sec\theta \\
\mathrm{cosec}(180^\circ + \theta) &= -\mathrm{cosec}\,\theta
\end{aligned}$$

4. Trigonometric Ratios of the Sum and Difference of Two Angles

In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where a, b, and c are the lengths of the three sides of a triangle. A, B and C are opposite angles of the sides a, b and c respectively.

(a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(b) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(c) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(d) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(e) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$ **(f)** $\tan(A-B) = \dfrac{\tan A - \tan B}{1 - \tan A . \tan B}$

(g) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(h) $\sin 2\theta = 2\sin\theta\cos\theta$ **(i)** $\sin^2\theta + \cos^2\theta = 1$

(j) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ **(k)** $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

(l) $1 + \tan^2\theta = \sec^2\theta$ **(m)** $1 + \cot^2\theta = \text{cosec}^2\theta$

[C] Differential Calculus

1. $\dfrac{d}{dx}$ is the sign of differentiation

2. $\dfrac{d}{dx}(x)^n = nx^{n-1}$ e.g., $\dfrac{d}{dx}(x)^6 = 6x^5$, $\dfrac{d}{dx}(x) = 1$

(To differentiate any power of x, write the power before x and subtract one from the power)

3. $\dfrac{d}{dx}(C) = 0$ e.g., $\dfrac{d}{dx}(5)^n = 0$

(Differential coefficient of a constant is zero)

4. $\dfrac{d}{dx}(u.v) = u.\dfrac{dv}{dx} + v.\dfrac{du}{dx}$

$$\left[\begin{array}{l}\text{Differential}\\ \text{coefficient of}\\ \text{product of any}\\ \text{two functions}\end{array}\right] = \left[\begin{array}{l}(1^{st}\text{ function} \times \text{Differential coefficient of second function})\\ + (2^{nd}\text{ function} \times \text{Differential coefficient of second function})\end{array}\right]$$

5. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v.\dfrac{du}{dx} - u.\dfrac{dv}{dx}}{v^2}$

$$\left[\begin{array}{l}\text{Differential}\\ \text{coefficient of two}\\ \text{functions when}\\ \text{one is divided by}\\ \text{the other}\end{array}\right] = \left[\dfrac{\begin{array}{l}(\text{Denominator} \times \text{Differential coefficient of numerator})\\ - (\text{Numerator} \times \text{Differential coefficient of denominator})\end{array}}{\text{Square of denominator}}\right]$$

6. Differential coefficient of trigonometrical functions

(a) $\dfrac{d}{dx}2(\sin x) = \cos x$; $\dfrac{d}{dx}(\cos x) = -\sin x$

(b) $\dfrac{d}{dx}(\tan x) = \sec^2 x$; $\dfrac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(c) $\frac{d}{dx}(\sec x) = \sec x \, . \tan x$; $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \, . \cot x$

(The differential coefficient whose trigonometrical function begins with co, is negative)

7. If the differential coefficient of a function is zero, the function is either maximum or minimum. Conversely, if the maximum or minimum value of a function is required, then differentiate the function and equate it to zero.

[D] Integral Calculus

1. $\int dx$ is the sign of integration

2. $\int x^n \, dx = \frac{x^{n+1}}{n+1}$ e.g., $\int x^8 = \frac{x^9}{9}$

(To integrate any power of x, add one to the power and divide by the new power)

3. $\int C \, dx = Cx$ e.g., $\int 5 \, dx = 5x$

(To integrate any constant, multiply the constant by x)

4. $\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{(n+1) \times a}$

(To integrate any bracket with power, add one to the power and divide by the new power and also divide by the coefficient of x within the bracket.)

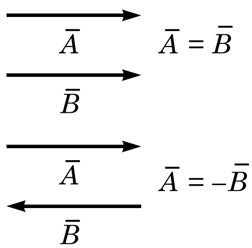
[E] Scalar Quantities

The scalar quantities (or sometimes known as scalars) are those quantities which have magnitude only such as length, mass, time, distance, volume, density, temperature, speed, etc.

[F] Vector Quantities

The vector quantities (or sometimes known as vectors) are those quantities which have magnitude and direction such as force, displacement, velocity, acceleration, momentum, etc.

Important features of vector quantities :



Equal Vector : Two vectors are said to be equal if they have the same magnitude and direction.

Negative Vector : A negative vector is defined as another vector having the same length but drawn in opposite direction.

Zero (or Null) Vector : A vector having zero magnitude is called a zero (or null) vector, it has arbitrary direction.

Unit Vector : A vector whose magnitude is unity, is known as unit vector.

[G] SI Units Used in Mechanics

Sr.	Quantity	Unit	SI Symbol
	(Base Units)		
1.	Length	meter (or *metre*)	m
2.	Mass	kilogram	kg
3.	Time	second	s
	(Derived Units)		
4.	Acceleration, linear	metre/second^2	m/s^2
5.	Acceleration, angular	radian/second^2	rad/s^2
6.	Area	metre^2	m^2
7.	Density	kilogram/metre^3	kg/m^3
8.	Force	newton	N (= kg$\cdot$m/s^2)
9.	Frequency	hertz	Hz (= 1/s)
10.	Impulse, linear	newton-second	N$\cdot$s
11.	Impulse, angular	newton-metre-second	N$\cdot$m$\cdot$s
12.	Moment of force	newton-metre	N$\cdot$m
13.	Moment of inertia, area	metre^4	m^4
14.	Moment of inertia, mass	kilogram-metre^2	kg$\cdot$m^2
15.	Momentum, linear	kilogram-metre/second	kg$\cdot$m/s (= N$\cdot$s)
16.	Momentum, angular	kilogram-metre^2/second	kg$\cdot$m^2/s (= N$\cdot$m$\cdot$s)
17.	Power	watt	W (= J/s = N$\cdot$m/s)
18.	Pressure, stress	pascal	Pa (= N'm/m^2)
19.	Product of inertia, area	metre^4	m^4
20.	Product of inertia, mass	kilogram-metre^2	kg$\cdot$m^2
21.	Spring constant	newton/metre	N/m
22.	Velocity, linear	metre/second	m/s
23.	Velocity, angular	radian/second	rad/s
24.	Volume	metre^3	m^3
25.	Work, energy	joule	J (= N$\cdot$m)
	(Supplementary and Other Acceptable Units)		
26.	Distance (navigation)	nautical mile	(= 1.852 km)
27.	Mass	ton (metric)	t (= 1000 kg)
28.	Plane angle	degrees (decimal) / radian	^{o} / -
29.	Speed	knot	(= 1.852 km/h)
30.	Time	minute / hour / day	min / h / d

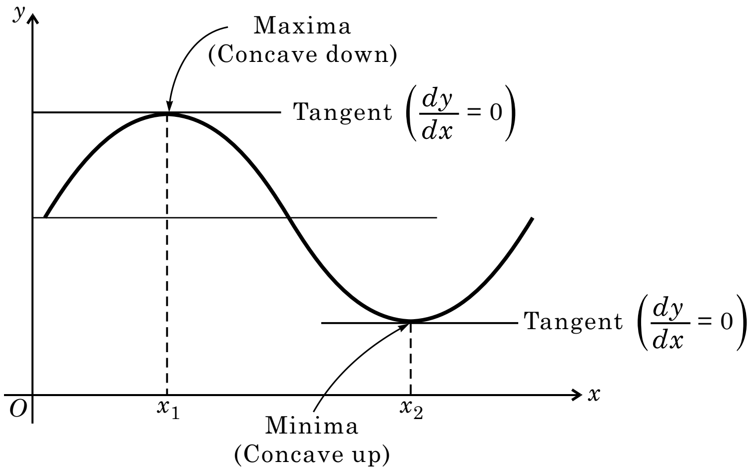
[H] SI Unit Prefixes

Multiplication Factor	Standard Form	Prefix	Symbol
1,000,000,000,000	10^{12}	tera	T
1,000,000,000	10^{9}	giga	G
1,000,000	10^{6}	mega	M
1,000	10^{3}	kilo	k
100	10^{2}	hecta	h
10	10	deca	da
0.1	10^{-1}	deci	d
0.01	10^{-2}	centi	c
0.001	10^{-3}	milli	m
0.000,001	10^{-6}	micro	μ
0.000,000,001	10^{-9}	nano	n
0.000,000,000,001	10^{-12}	pico	p

[I] Conversion Factors

Length	Area	Pressure
1 inch = 2.54 cm	1 sq.m = 10.761 sq.ft	1 psi = 1 lb/sq.inch
1 cm = 0.3937 inch	1 sq.ft = 0.0929 sq.m	= 0.0703 kg/cm^3
1 mm = 1000 microns	1 sq.mile = 2.59 sq.km	1 kg/cm^3 = 14.22 psi
1 meter = 3.281 feet	= 258.90 hectare	1 lb/sq.ft = 4.882 kg/m^2
1 foot = 0.3048 meter	= 640 acres	1 kg/m^2 = 0.205 lb/ft^2
1 mile = 1.6098 km	1 hectare = 10^4 sq.m	1 kg/sq.cm = 10 m head of water
	= 2.47 acres	1 psi = 0.0681 atmosphere
		= 2.04 inches head of mercury
		1 Pascal = 1 N/m^2
		1 MPa = 1 N/mm^2

[J] Maxima and Minima



Suppose quantity y depends on another quantity x in a manner shown in figure. It becomes maximum at x_1 and minimum at x_2. At these points, the tangent to the curve is parallel to the x-axis and hence its slope is 0.

But the slope of the curve $y = f(x)$ equals the rate of change $\frac{dy}{dx}$.

Thus, at a maximum or at a minimum $\frac{dy}{dx} = 0$. Just before the maximum, the slope is positive, at maximum it is zero and just after the maximum it is negative.

Thus, $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum, i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ at a maximum. The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$.

Thus the **condition of a maximum** is $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

Similarly, at a minimum the slope changes from negative to positive.

The slope increases at such a point and hence $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$

Thus the **condition of a minimum** is $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

Quite often it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\frac{d^2y}{dx^2}$ may be omitted.

Answer to Objective Type Questions

Chapter 2

[III] Fill in the Blanks

(1) space **(2)** time **(3)** mass **(4)** scalar **(5)** particle **(6)** resultant and equilibrant
(7) like, unlike **(8)** positive, negative **(9)** single **(10)** couple, equilibrium

[IV] Multiple-choice Questions

(1) - (a) **(2)** - (d) **(3)** - (c) **(4)** - (c) **(5)** - (b) **(6)** - (d) **(7)** - (b) **(8)** - (c) **(9)** - (c) **(10)** - (d)
(11) - (b) **(12)** - (a) **(13)** - (d) **(14)** - (b) **(15)** - (c) **(16)** - (d) **(17)** - (a) **(18)** - (c) **(19)** - (c)

Chapter 3

[III] Fill in the Blanks

(1) equal, opposite **(2)** equal, opposite
(3) uniformly distributed load, uniformly varying load **(4)** built-in **(5)** away

[IV] Multiple-choice Questions

(1) - (c) **(2)** - (d) **(3)** - (b) **(4)** - (c) **(5)** - (c) **(6)** - (c) **(7)** - (d) **(8)** - (d) **(9)** - (b) **(10)** - (a)

Chapter 4

[III] Fill in the Blanks

(1) 1 **(2)** dot **(3)** 0 **(4)** Varignon's **(5)** not

[IV] Multiple-choice Questions

(1) - (d) **(2)** - (d) **(3)** - (a) **(4)** - (b) **(5)** - (c) **(6)** - (b) **(7)** - (d)

Chapter 5

[III] Fill in the Blanks

(1) centre of gravity **(2)** centroid **(3)** $\frac{s}{2\sqrt{3}}$ **(4)** 0 **(5)** $\frac{4r}{3\pi}$

[IV] Multiple-choice Questions

(1) - (d) **(2)** - (c) **(3)** - (d) **(4)** - (c) **(5)** - (a) **(6)** - (d)

Chapter 6

[III] Fill in the Blanks

(1) Polar moment of inertia **(2)** $\frac{\pi R^4}{8}$ **(3)** cm^4 **(4)** centroid **(5)** second

[IV] Multiple-choice Questions

(1) - (c) **(2)** - (c) **(3)** - (a) **(4)** - (d) **(5)** - (c) **(6)** - (d) **(7)** - (d)

Chapter 7

[III] Fill in the Blanks

(1) joint **(2)** tension, compression **(3)** perfect **(4)** determinate **(5)** joint

[IV] Multiple-choice Questions

(1) - (c) **(2)** - (b) **(3)** - (c) **(4)** - (a) **(5)** - (b) **(6)** - (b) **(7)** - (a)

Chapter 8

[III] Fill in the Blanks

(1) small **(2)** cone of friction **(3)** angle of repose

[IV] Multiple-choice Questions

(1) - (c) **(2)** - (d) **(3)** - (c) **(4)** - (a) **(5)** - (b) **(6)** - (a) **(7)** - (c)

Chapter 9

[III] Fill in the Blanks

(1) tight

[IV] Multiple-choice Questions

(1) - (d) **(2)** - (c)

Chapter 11

[III] Fill in the Blanks

(1) position **(2)** translation **(3)** rotational **(4)** Position **(5)** speed **(6)** displacement

[IV] Multiple-choice Questions

(1) - (a) **(2)** - (d) **(3)** - (c) **(4)** - (c) **(5)** - (c)

Chapter 12

[III] Fill in the Blanks

(1) curvilinear **(2)** rectilinear **(3)** centripetal **(4)** speed **(5)** 0 **(6)** parabolic
(7) horizontal and vertical

[IV] Multiple-choice Questions

(1) - (a) **(2)** - (b) **(3)** - (c) **(4)** - (b) **(5)** - (d) **(6)** - (a)

Chapter 13

[III] Fill in the Blanks

(1) rectilinear **(2)** same **(3)** velocity **(4)** translation and rotational
(5) instantaneous centre of rotation

[IV] Multiple-choice Questions

(1) - (a) **(2)** - (b) **(3)** - (c) **(4)** - (d)

Chapter 14

[III] Fill in the Blanks

(1) matter **(2)** Mass **(3)** momentum **(4)** inertia **(5)** zero

[IV] Multiple-choice Questions

(1) - (d) **(2)** - (c) **(3)** - (a) **(4)** - (b) **(5)** - (c)

Chapter 15

[III] Fill in the Blanks

(1) negative **(2)** scalar **(3)** negative **(4)** kinetic energy **(5)** negative

[IV] Multiple-choice Questions

(1) - (c) **(2)** - (a) **(3)** - (b) **(4)** - (d)

Chapter 16

[III] Fill in the Blanks

(1) momentum **(2)** impact **(3)** common tangent **(4)** central **(5)** oblique
(6) coefficient of restitution **(7)** zero

[IV] Multiple-choice Questions

(1) - (a) **(2)** - (c) **(3)** - (b) **(4)** - (a) **(5)** - (b) **(6)** - (c)

Chapter 17

[III] Fill in the Blanks

(1) Kinetics **(2)** translation **(3)** acceleration **(4)** mass **(5)** centroidal

[IV] Multiple-choice Questions

(1) - (d) **(2)** - (b) **(3)** - (c) **(4)** - (b) **(5)** - (a) **(6)** - (c)

UNIVERSITY PAPER SOLUTIONS

Code No. : 111AC

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY, HYDERABAD

B. Tech I Year Examination, June - 2014

ENGINEERING MECHANICS

(Common to CE, ME, CHEM, MCT, MMT, MEP, AE, AME, MIE, MIM, PTE, CEE, MSNT, AGE)

Time : 3 hours **Max. Marks : 75**

N.B. (1) Question paper consists of two **Parts A** and **B**.

(2) **Part-A** is **compulsory** which carries 25 marks. Answer all questions in Part A.

(3) **Part-B** consists of 5 units. Answer any one full question from each unit. Each question carries 10 marks and may have a, b, c as sub questions.

PART - A

1.(a) State parallelogram law. **(2)**

Solution :

Law of Parallelogram of Force : If two forces acting simultaneously on a body at a point are represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of parallelogram which passes through the point of intersection of the two sides representing the forces. Refer to Fig. 1(a)-i.

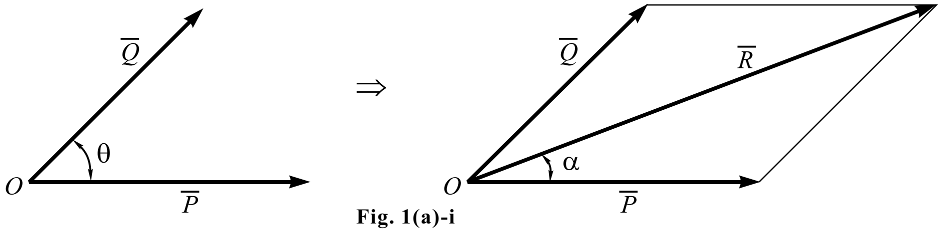


Fig. 1(a)-i

$$R = \sqrt{P^2 + Q^2 + 2\,PQ\cos\theta}$$

$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$

1.(b) Give equations of equilibrium in space. **(3)**

Solution :

If the system is in equilibrium then resultant is equal to zero.

(i) Equation of equilibrium in space for concurrent force system

$$\sum F_x = 0\,,\ \sum F_y = 0\,,\ \sum F_z = 0$$

(ii) Equation of equilibrium in space for general force system

Resultant force $\sum \bar{\boldsymbol{F}} = 0$ and

Resultant moment $\Sigma \bar{\boldsymbol{M}} = 0$

$$\sum F_x = 0\,,\ \sum F_y = 0\,,\ \sum F_z = 0$$
$$\sum M_x = 0\,,\ \sum M_y = 0\,,\ \sum M_z = 0$$

(iii) Equation of equilibrium in space for parallel force system

Forces parallel to x-axis

$$\sum F_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0$$

Forces parallel to y-axis

$$\sum F_y = 0, \quad \sum M_x = 0, \quad \sum M_z = 0$$

Forces parallel to z-axis

$$\sum F_z = 0, \quad \sum M_x = 0, \quad \sum M_y = 0$$

1.(c) Define the efficiency of screw jack. **(2)**

Solution :

Efficiency of screw jack is the ratio of mechanical advantage to velocity ratio.

$$\text{Velocity Ratio } VR = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}}$$

$$\text{Mechanical Advantage } MA = \frac{\text{Load}}{\text{Effort}}$$

$$\text{Efficiency} = \frac{MA}{VR}$$

1.(d) Draw different types of flat belt drives. **(3)**

Solution :

(i) Open-Belt Drive

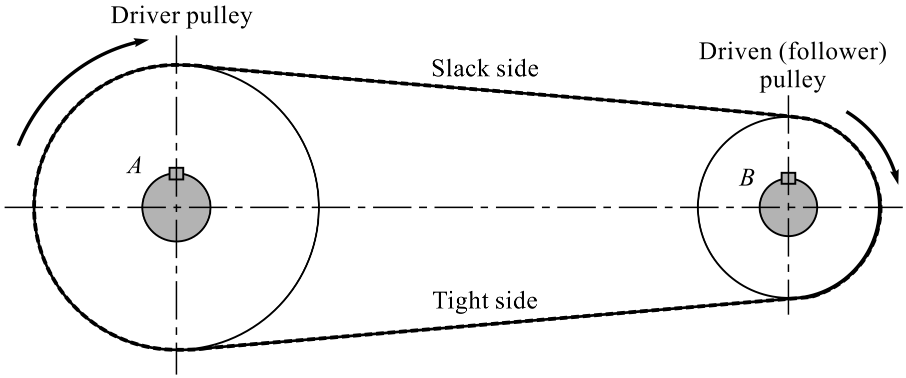


Fig. 1(d)-i

(ii) Cross-Belt Drive

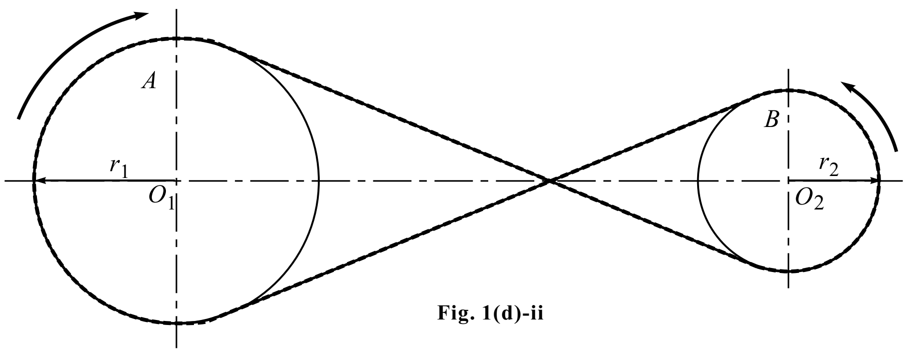


Fig. 1(d)-ii

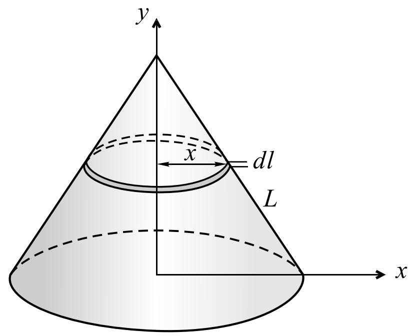
1.(e) State theorem of Pappus. **(2)**

Solution :

First Theorem

The area of a surface of revolution is equal to the length of the generating curve times the distance travelled by the centroid of the generating curve while the surface is generated.

Refer Fig. 1(e)-i, the area of the surface generated is given by the product of $2\pi\ x_G$ and the length of the surface L as if the entire length of the generating curve was concentrated at the radius x_G.

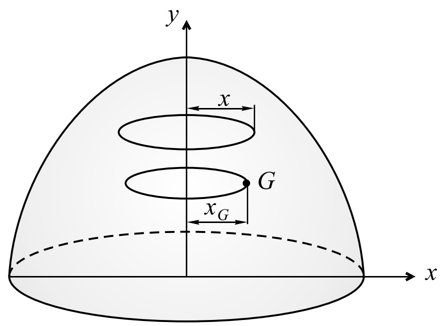


1(e) - i :Surface of Cone Generated

Second Theorem

The volume of a body of revolution is equal to the generating area times the distance travelled by the centroid of the area while the body is generated.

Refer Fig. 1(e)-ii, the volume of body generated is given by the product of $2\pi\ x_G$ and the area of the surface A as if the entire area of generating surface was concentrated at its centroid of radius x_G.



1(e) - ii

1.(f) What is radius of gyration ? **(3)**

Solution :

Radius of gyration of a body is defined as *the distance from the reference axis at which the given area is assumed to be compressed and kept as a thin strip, such that there is no change in its moment of inertia*.

Fig. 1(f)-i shows a plane figure of area A. Let I_{AB} be its moment of inertia about reference axis AB. Assume the figure to be compressed into a thin strip of the same area A at a distance k from the reference axis AB such that it has same moment of inertia (i.e., I_{AB}) as shown in Fig. 1(f)-ii.

Then from the definition

$$I_{AB} = Ak^2$$

where k is known as the *radius of gyration*.

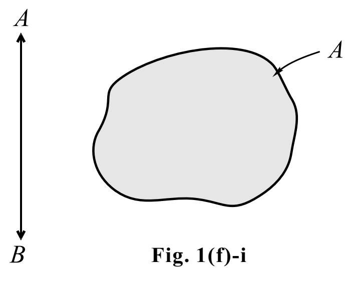


Fig. 1(f)-i

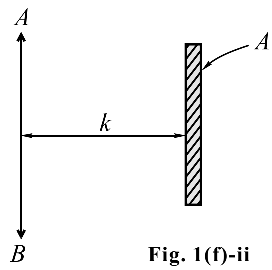


Fig. 1(f)-ii

1.(g) Derive the equation $s = ut + \frac{1}{2}at^2$, where s is the distance travelled, u is initial velocity a is acceleration and t is time. **(2)**

Solution :

$$\text{Displacement} = \text{Average velocity} \times \text{time}$$

$$s = \left(\frac{u+v}{2}\right)t \qquad \text{but } v = u + at$$

$$s = \left(\frac{u+u+at}{2}\right)t$$

$$\therefore \quad \boxed{s = ut + \frac{1}{2}at^2}$$

1.(h) Derive an equation for the acceleration of a body moving down a rough inclined plane. **(3)**

Solution :

By Newton's IInd law

$$\sum F_x = ma_x$$

$$mg\sin\theta - \mu N = ma$$

$$mg\sin\theta - \mu\, mg\cos\theta = ma$$

$$\boxed{g\sin\theta - \mu\, g\cos\theta = a}$$

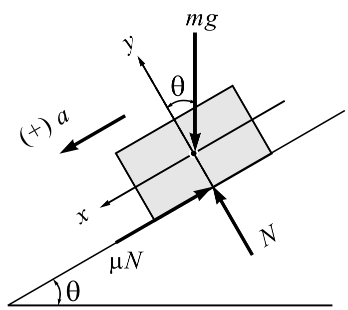


Fig. 1(h)-i

1.(i) Define frequency of simple harmonic motion. **(2)**

Solution :

Frequency of simple harmonic motion is the reciprocal of the period, i.e.

$$\boxed{f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}}$$

1.(j) Derive work - energy equation. **(3)**

Solution :

By Newton's IInd law

$$F = ma$$

$$F = m\,v\,\frac{dv}{ds}$$

$$F\,ds = m\,v\,dv$$

$$\int_{s_1}^{s_2} F\,ds = \int_{v_1}^{v_2} m\,v\,dv$$

$$F(s_2 - s_1) = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

$$F\,\Delta s = \frac{1}{2} m\, v_2^{\,2} - \frac{1}{2} m\, v_1^{\,2}$$

Work done = Final K.E. – Initial K.E.

Work done = Change in kinetic energy

PART - B

2. Two spheres, A and B rest in a vertical channel with their centres in a vertical plane. If weight of sphere A is 1000 N and that of sphere B is 400 N, radius of sphere A is 1 m and that of sphere B is 0.6 m, width of channel is 2.4 m, find the contact forces assuming all surfaces to be smooth.

Solution :

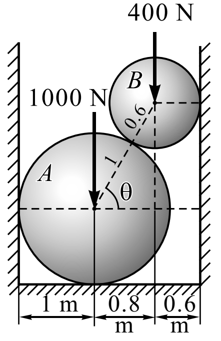


Fig. 2-i

(i) From Fig. 2-i,

$$\cos\theta = \frac{0.8}{1.6}$$

$$\therefore\ \theta = 60^o$$

(ii) Consider the F.B.D. of Sphere *B* [Fig. 2-ii]

By Lami's theorem,

$$\frac{400}{\sin 120^o} = \frac{R_1}{\sin 150^o} = \frac{R_2}{\sin 90^o}$$

$\therefore\ R_1 = 230.94 \text{ N } (\leftarrow)$ and

$R_2 = 461.88 \text{ N } (\measuredangle 60^o)$

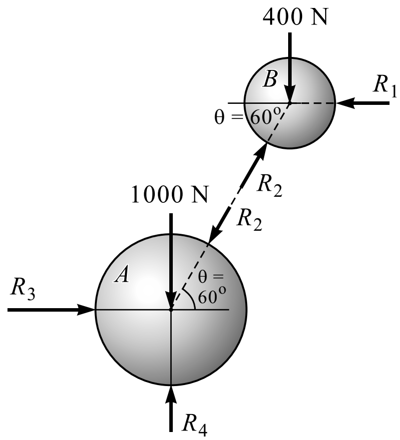


Fig. 2-ii

(iii) Consider the F.B.D. of Sphere *A* [Fig. 2-ii]

$\Sigma F_x = 0$

$R_3 - R_2 \cos 60^o = 0$

$R_3 = 461.88 \cos 60^o$

$R_3 = 230.94 \text{ N } (\rightarrow)$

$\Sigma F_y = 0$

$R_4 - 1000 - \sin 60^o = 0$

$R_4 = 1000 + 461.88 \sin 60^o$

$R_4 = 1400 \text{ N } (\uparrow)$

3. Four forces of magnitude 10 kN, 15 kN, 20 kN and 40 kN are acting at a point O as shown in Fig. 3-i. The angles made by 10 kN, 15 kN, 20 kN and 40 kN with x-axis are 30°, 45°, 60° and 90° respectively. Find the magnitude and direction of the resultant force.

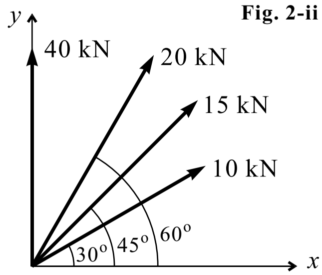


Fig. 3-i

Solution :

(i) $\Sigma F_x = 10 \cos 30° + 15 \cos 45° + 20 \cos 60°$

$\Sigma F_x = 29.27$ kN $(\rightarrow)$

(ii) $\Sigma F_y = 10 \sin 30° + 15 \sin 45° + 20 \sin 60° + 40$

$\Sigma F_y = 72.93$ N $(\uparrow)$

(iii) $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

$R = \sqrt{(29.27)^2 + (72.93)^2}$

$\therefore \boxed{R = 78.58 \text{ kN}}$

(iv) $\tan \theta = \left| \dfrac{\Sigma F_y}{\Sigma F_x} \right| = \dfrac{72.93}{29.27}$

$\boxed{\theta = 68.13°}$

Position of resultant R is as shown in figure.

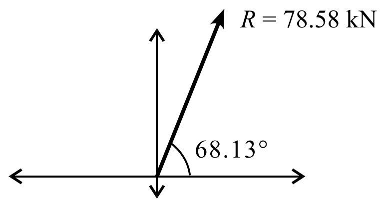


Fig. 3-ii

4. A screw jack has a square, thread of mean diameter 6 cm and pitch 0.8 cm. The coefficient of friction at the screw thread is 0.09. A load of 14 kN is to be lifted through 15 cm. Determine the torque.

Solution :

Given : $d = 6$ cm i.e. $r = 3$ cm, Pitch $p = 0.8$ cm , $\mu = 0.09$,

Load $W = 14$ kN, Axial displacement (lift) = 15 cm

Now, $\tan \phi = \mu = 0.09 \qquad \therefore \phi = 5.143°$

$\tan \theta = \dfrac{p}{2\pi r} = \dfrac{0.008}{2\pi \times 0.03} \quad \therefore \theta = 2.43°$

Torque $= P_{lift} L$

Torque $= \dfrac{r}{L} W \tan(\theta + \phi)\ L$

Torque $= 0.03 \times 14 \times 10^3 \tan(2.43° + 5.143°)$

$\boxed{T = 55.84 \text{ N.m}}$

5. Determine the maximum power that can be transmitted using a belt of 105 mm × 15 mm with an angle of tap of 160°. The density of belt is 10^3 kg/m^3 and coefficient of friction is 0.35, The tension in the belt should not exceed 1.5 N/mm^2.

Solution :

Given : Width = 105 mm, Thickness = 15 mm, Density = 10^3 kg/m^3

$\mu = 0.35, \ \sigma_{max} = 1.5$ N/mm^2,

$\theta = 160°$

i.e. $\theta = 160 \times \dfrac{\pi}{180} = 2.793$ rad

(i) Maximum permissible stress $\sigma_{max} = \dfrac{T_{max}}{\text{Area}}$

$$T_{max} = \sigma_{max} \times \text{Area}$$

$$T_{max} = 1.5 \times 10^6 \times 105 \times 10^{-3} \times 15 \times 10^{-3}$$

$$T_{max} = 2362.5 \text{ N}$$

For maximum power, we have the relation

(ii) $T_{max} = 3T_c$

$$T_c = \frac{2362.5}{3} = 787.5 \text{ N}$$

(iii) $m = \rho \times (\text{Volume of 1 m length of belt})$

$$m = 1000(105 \times 10^{-3} \times 15 \times 10^{-3} \times 1)$$

$$m = 1.575 \text{ kg/m}$$

(iv) $T_c = mv^2$

$$v = \sqrt{\frac{T_c}{m}} = \sqrt{\frac{787.5}{1.575}} \qquad \therefore\ v = 22.36 \text{ m/s}$$

(v) $T_{max} = T_1 + T_c$

$$\therefore\ T_1 = T_{max} - T_c = 2362.5 - 787.5$$

$$\therefore\ T_1 = 1575 \text{ N}$$

(vi) Power $P_{max} = T_1\left(1 - \dfrac{1}{e^{\mu\theta}}\right)v$

$$P_{max} = 1575\left(1 - \frac{1}{e^{0.35 \times 2.795}}\right)22.36 = 21976.52 \text{ watt}$$

$$\therefore\ P_{max} = 21.98 \text{ kW}$$

6. Determine the values of I_{xx} and I_{yy} for the shaded are bounded by the parabolic curve shown in Fig. 6-i.

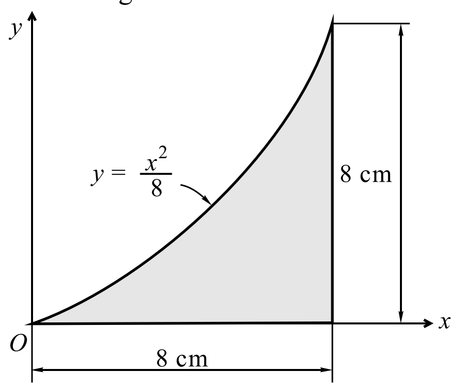


Fig. 6-i

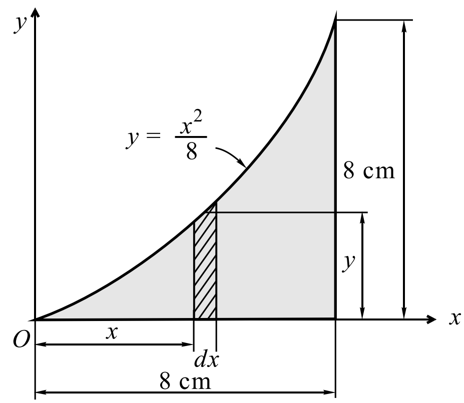


Fig. 6-ii

Solution : Refer Fig. 6-ii.

$$I_{xx} = \frac{(\text{Base})(\text{Height})^3}{3} = \frac{(dx)(y)^3}{3}$$

$$I_{xx} = \frac{dx}{3}\left(\frac{x^2}{8}\right)^3 = \frac{x^6\,dx}{3 \times 8^3}$$

$$I_{xx} = \int_0^8 \frac{x^6\,dx}{3 \times 8^3} = \frac{1}{3 \times 8^3}\left[\frac{x^7}{7}\right]_0^8$$

$$I_{xx} = \frac{8^7}{3 \times 8^3 \times 7} = \frac{8^4}{21}$$

$$\boxed{I_{xx} = 195.05 \text{ cm}^4}$$

$$I_{yy} = (\text{Area})(\text{Distance})^2 = (dx\,y)(x^2)$$

$$I_{yy} = dx\left(\frac{x^2}{8}\right) \times x^2 = \frac{x^4\,dx}{8}$$

$$I_{yy} = \int_0^8 \frac{x^4\,dx}{8} = \frac{1}{8}\left[\frac{x^5}{5}\right]_0^8 = \frac{8^5}{8 \times 5}$$

$$\boxed{I_{yy} = 819.2 \text{ cm}^4}$$

7. Find the mass moment of inertia of a solid cylindrical body of radius 'r' and height 'H' about its centroidal axes.

Solution :

Consider a thin cylindrical element of radius r and thickness dr.

Elemental mass $= dm = 2\pi r\,dr\,H\,\rho$

Mass moment of inertia of the cylindrical element about axis

$$dI = dm\,r^2$$

$$dI = (2\pi r\,dr\,H\,\rho) \times (r^2)$$

Total mass moment of the cylinder

$$I = \int_0^R (2\pi r\,dr\,H\,\rho) \times (r^2)$$

$$= \int_0^R 2\pi\,H\,\rho\,r^3\,dr = 2\pi\,H\,\rho\left[\frac{r^4}{4}\right]_0^R$$

$$= 2\pi\,H\,\rho\,\frac{R^4}{4} = (\pi\,R^2\,H\,\rho)\,\frac{R^2}{2}$$

$$\therefore \boxed{I = \frac{M\,R^2}{2}} \qquad [\because \text{Mass of cylinder } M = \pi R^2\,H\,\rho]$$

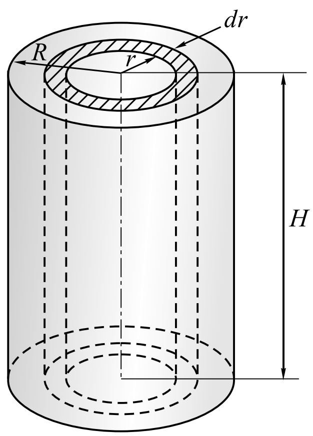


Fig. 7-i

8. Two smooth inclined planes whose inclination with horizontal are 30° and 20° are placed back to back. Two weights 100 N and 60 N are placed on the planes of 20° and 30° respectively and are connected by a cord passing over a smooth pulley. Find the acceleration of the system, tension in the string and force acting on the pulley.

Solution :

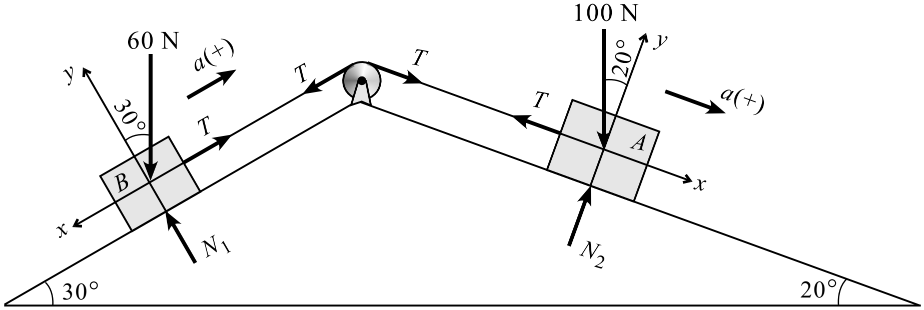


Fig. 8-i

(i) Consider the F.B.D. of 100 N Block

By Newton's IInd law

$$\sum F_x = ma_x$$

$$100 \sin 20° - T = \frac{100}{9.81} a$$

$$T = 100 \sin 20° - \frac{100}{9.81} a \qquad ..... (1)$$

(ii) Consider the F.B.D. of 60 N Block

By Newton's IInd law

$$\sum F_x = ma_x$$

$$T - 60 \sin 30° = \frac{60}{9.81} a$$

$$T = 60 \sin 30° + \frac{60}{9.81} a \qquad ..... (2)$$

(iii) Equating (i) and (2), we get

$$100 \sin 20° - \frac{100}{9.81} a = 60 \sin 30° + \frac{60}{9.81} a$$

$$16.31\, a = 4.2$$

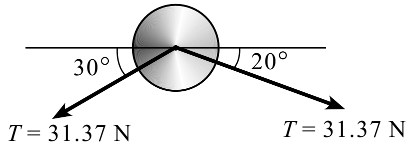
$$a = 0.2575 \text{ m/s}^2$$

From equation (1)

$$T = 100 \sin 20° - \frac{100}{9.81} \times 0.2575$$

$$T = 31.37 \text{ N}$$

(iv) Force Acting on Pulley



$$\Sigma F_x = 31.37 \cos 20^o - 31.37 \cos 30^o$$

$$\boxed{\Sigma F_x = 2.311 \text{ N} (\rightarrow)}$$

$$\Sigma F_y = -31.37 \sin 30^o - 31.37 \sin 20^o$$

$$\boxed{\Sigma F_y = -26.41 \text{ N} (\downarrow)}$$

9. The acceleration of a particle is given by $a = 6 - 2t^2$ m/sec^2 the particle starts at $t = 0$, $v_0 = 0$ and $s = 8$ m. Determine :

(i) Velocity at $t = 5$ seconds

(ii) Position at $t = 5$ seconds

(iii) Distance travelled from $t = 0$ to $t = 5$ seconds.

Solution :

Given : $a = 6 - 2t^2$ m/sec^2. At $t = 0$, $v_0 = 0$ and $s = 8$ m.

To find : $v = ?$ at $t = 5$ sec

$s = ?$ at $t = 5$ sec

$d = ?$ from $t = 0$ to $t = 5$ sec.

Integrating the given expression of a w.r.t. time t,

$$v = 6t - \frac{2t^3}{3} + c_1$$

At $t = 0$, $v_0 = 0 \;\therefore\; c_1 = 0$

$$v = 6t - \frac{2t^3}{3} \qquad \text{..... (I)}$$

Further integrating

$$s = 6\frac{t^2}{2} - \frac{2t^4}{12} + c_2$$

At $t = 0$, $s = 8$ m $\;\therefore\; c_2 = 8$

$$s = 3t^2 - \frac{t^4}{6} + 8 \qquad \text{..... (II)}$$

Put $t = 5$ s in equation (I), we get

$$v = 6 \times 5 - \frac{2 \times 5^3}{3}$$

$$\boxed{v = -53.33 \text{ m/s}}$$

Put $t = 5$ s in equation (II), we get

$$s = 3 \times 5^2 - \frac{5^4}{6} + 8$$

$$\boxed{s = -21.17 \text{ m}}$$

Distance d travelled from $t = 0$ to $t = 5$ seconds is

$$d = 8 + 21.17$$

$$\boxed{d = 29.17 \text{ m}}$$

10. A bullet fired from a gun and travelling horizontally with a velocity of 800 m/s and weighing 0.4 N strikes a wooden block of weight 100 N resting on a rough horizontal floor. The coefficient of friction between the floor and the block is 0.3. Find the distance through which the block is displaced from its initial position.

Solution :

(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v_1$$

$$\frac{0.4}{9.81} \times 800 + 0 = \frac{(100 + 0.4)}{9.81} \times v_1$$

$$32.62 = 10.23\ v_1$$

$v_1 = 3.159$ m/s (→) [velocity of block and bullet imbedded at position ①]

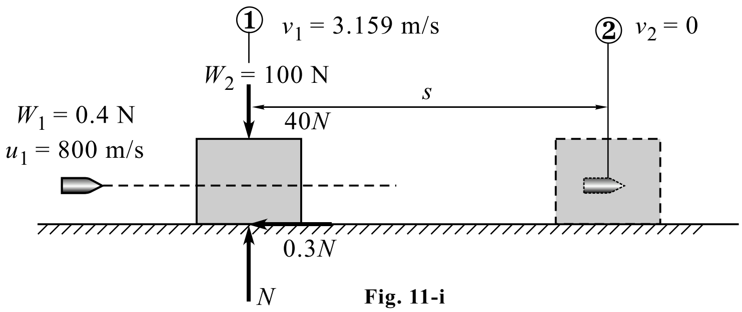


Fig. 11-i

(ii) By work energy principal, we have

$$\text{Work done} = \text{Change in kinetic energy}$$

$$-0.3N \times s = \frac{1}{2} M \times v_2^{\,2} - \frac{1}{2} M \times v_1^{\,2}$$

$$-0.3 \times 100 \times s = 0 - \frac{1}{2} \times \frac{(100 + 0.4)}{9.81} \times (3.159)^2$$

$$30\ s = 5107$$

$s = 1.7$ m (Distance moved by the block)

●●●

UNIVERSITY PAPER SOLUTIONS

Code No. : 111AC

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY, HYDERABAD

B. Tech I Year Examination, December - 2014/ January - 2015

ENGINEERING MECHANICS

(Common to CE, ME, CHEM, MCT, MMT, MEP, AE, AME, MIE, MIM, PTE, CEE, MSNT, AGE)

Time : 3 hours **Max. Marks : 75**

N.B. (1) Question paper consists of two **Parts A** and **B**.

(2) **Part-A** is **compulsory** which carries 25 marks. Answer all questions in Part A.

(3) **Part-B** consists of 5 units. Answer any one full question from each unit. Each question carries 10 marks and may have a, b, c as sub questions.

PART - A **25 Marks**

I. (a) Two forces 15 N and 12 N are acting at a point. The angle between the forces is 60°. Find the magnitude and direction of the resultant. **(2)**

Solution :

(i) We have

$$R = \sqrt{P^2 + Q^2 + 2\,PQ\cos\theta}$$

$$R = \sqrt{(12)^2 + (15)^2 + 2 \times 12 \times 15 \cos 60°}$$

$$\therefore R = 23.43 \text{ N}$$

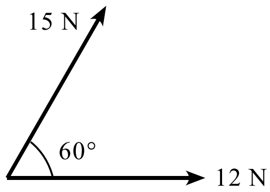


Fig. 1(a)-i

(ii)

$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$

$$\tan\alpha = \frac{15\sin 60°}{12 + 15\cos 60°}$$

$$\tan\alpha = 0.666$$

$$\alpha = 33.67°$$

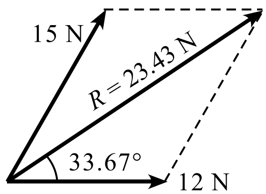


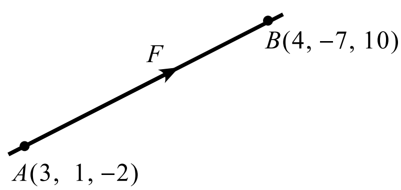
Fig. 1(a)-ii

(iii) Magnitude and direction of resultant R is as shown in Fig.1(a)-ii.

I. (b) The coordinates of initial and terminal points of a vector are (3, 1, −2) and (4, −7, 10), Specify the force vector, evaluate its magnitude and direction cosines. **(3)**

Solution :

Force vector



$$\overline{\boldsymbol{F}} = (F)(\overline{\boldsymbol{e}}_{AB})$$

$$\overline{\boldsymbol{F}} = (F)\left[\frac{(4-3)\,\mathbf{i} + (-7-1)\,\mathbf{j} + (10+2)\,\mathbf{k}}{\sqrt{(4-3)^2 + (-7-1)^2 + (10+2)^2}}\right]$$

$$\overline{\boldsymbol{F}} = (F)\frac{\mathbf{i} - 8\,\mathbf{j} + 12\,\mathbf{k}}{\sqrt{1^2 + (-8)^2 + (12)^2}}$$

$$\overline{F} = F\,(0.069\,\mathbf{i} - 0.55\,\mathbf{j} + 0.828\,\mathbf{k})$$

Note : Magnitude of force is not mentioned, so we have considered magnitude in general term, say F.

Assuming value of $F = 100$ N, we get

$$\overline{F} = 100\,(0.069\,\mathbf{i} - 0.55\,\mathbf{j} + 0.828\,\mathbf{k})$$

$$\overline{F} = 6.9\,\mathbf{i} - 55\,\mathbf{j} + 82.8\,\mathbf{k}$$

$$\therefore F_x = 6.9\,,\; F_y = -55, \text{ and } F_z = 82.8$$

$$F_x = F\cos\theta_x\,,\; F_y = F\cos\theta_y\,, \text{ and } F_z = F\cos\theta_z$$

$$\theta_x = \cos^{-1}\left(\frac{6.9}{100}\right) \qquad \theta_y = \cos^{-1}\left(\frac{-55}{100}\right) \qquad \theta_z = \cos^{-1}\left(\frac{82.8}{100}\right)$$

$$\theta_x = 86.04^\circ \qquad \theta_y = 123.37^\circ \qquad \theta_z = 34.11^\circ$$

I. **(c)** State laws of friction and angle of friction. **(2)**

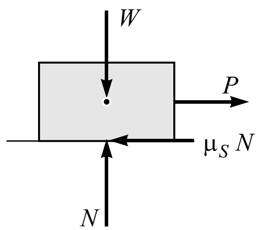
Solution :

Laws of Friction

1. Direction of frictional force is always opposite to the direction of impending motion and acts tangential at contact surface.
2. Limiting frictional force F_{max} is directly proportional to normal reactions (i.e. $F_{max} \propto N$, $\therefore\; F_{max} = \mu_S N$).
3. Coefficient of static friction μ_S is always greater than the coefficient of kinetic friction μ_K (i.e. $\mu_S > \mu_K$).
4. Frictional force depends upon the roughness of the surface and the material in contact.
5. Frictional force is independent of the area and speed.

Angle of Friction

It is the angle made by the resultant of the limiting frictional force F_{max} and the normal reaction N with the normal reaction ϕ is called as the *angle of friction*.



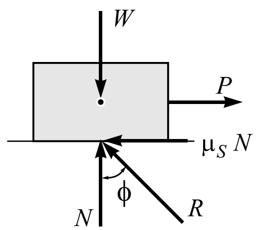
$$F_{max} = R\sin\phi$$

$$\mu_S N = R\sin\phi \qquad \ldots\text{(I)} \qquad (\because F_{max} = \mu_S N)$$

$$N = R\cos\phi \qquad \ldots\text{(II)}$$

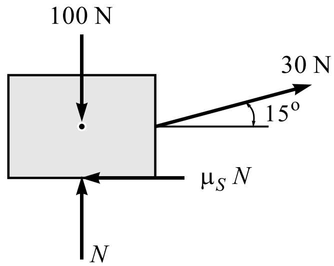
Dividing Eq. (I) by Eq. (II), we get

$$\tan\phi = \mu_S \text{ or } \phi = \tan^{-1}\mu_S$$



I. (d) The force required to pull a body of weight 100 N on a rough horizontal plane is 30 N. Determine the coefficient of friction if the force is applied at an angle of 15° with the horizontal. **(3)**

Solution :



$$\Sigma F_y = 0$$
$$N + 30 \sin 15^\circ - 100 = 0$$
$$N = 92.24 \text{ N}$$
$$\Sigma F_x = 0$$
$$30 \cos 15^\circ - \mu_S N = 0$$
$$30 \cos 15^\circ - \mu_S \times 92.24 = 0$$
$$\mu_S = 0.314$$

I. (e) Define the term centroid and centre of gravity. **(2)**

Solution :

Centroid

It is a point where the whole area of a plane lamina (figure) is assumed to act. It is a point where the entire area is supposed to be concentrated.

In other words, *centroid is the geometrical centre of a figure*. We use the term *centroid* for *two-dimensional figures*, i.e. areas, e.g., rectangle, triangle, circle, semicircle, sector, etc.

Centre of Gravity

It is a point where the whole weight of the body is assumed to act, i.e., *it is a point where entire distribution of gravitational force (weight) is suppose to be concentrated*.

The term ***centre of gravity*** is usually denoted by '*G*' for all three-dimensional rigid bodies, e.g., sphere, table, vehicle, dam, human, etc.

I. (f) Derive the expression of moment of inertia of rectangular area about their centroidal axes. **(3)**

Solution :

MI of Rectangular Area About their Centroidal Axes

Consider a rectangle of base b and height h.

AB is the reference axis through base.

Consider an elemental strip of width dy located at a distance y from the reference axis AB, as shown in Fig. 1(f)-i.

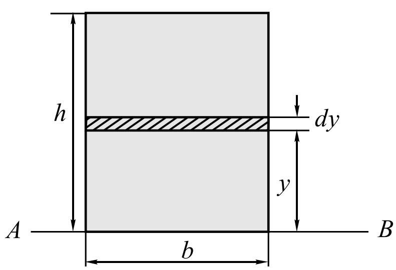


Fig. 1(f)-i

Using basic principle of MI

$$I = \int r^2 \, dA$$

Area of elemental strip $dA = b \, dy$

$$I_{AB} = \int_0^h y^2 \, b \, dy = b \int_0^h y^2 \, dy = b \left[\frac{y^3}{3}\right]_0^h$$

$$I_{AB} = \frac{bh^3}{3}$$

MI of rectangle about parallel centroidal axis to the base

By parallel axis theorem

$$I_{AB} = I_G + Ad^2$$

where d = Distance between the reference axis and parallel centroidal axis.

$$I_G = I_{AB} - Ad^2$$

$$I_G = \frac{bh^3}{3} - b \times h \times \left(\frac{h}{2}\right)^2 = \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$\boxed{I_G = \frac{bh^3}{12}}$$

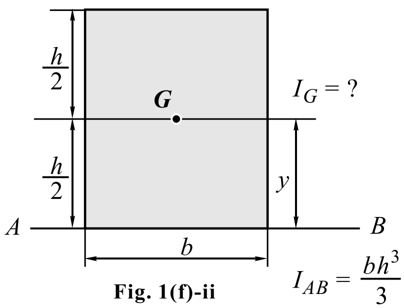


Fig. 1(f)-ii

l. **(g)** What is meant by rectilinear and curvilinear motion? How a uniform motion differs from a uniformly accelerated motion. **(2)**

Solution :

Rectilinear Motion

Motion of the particle along the straight path is called as rectilinear motion.

Curvilinear Motion

Motion of the particle along the curved path is called as curvilinear motion.

Uniformly Accelerated Motion

In uniform motion the velocity of particle throughout the motion remains constant (i.e. velocity does not change w.r.t. time), so it holds a relation

$$\text{Displacement} = \text{Velocity} \times \text{Time}$$

Whereas in uniformly accelerated motion the velocity of the particle changes uniformly w.r.t. time (i.e. velocity is not constant), so it holds the following relations

$$v = u + at$$

$$v^2 = u^2 + 2as$$

$$\text{and} \quad s = ut + \frac{1}{2}at^2$$

l. **(h)** A stone dropped in a well is heard to strike the water in 4 seconds. Find the depth of the well, assuming the velocity of sound is 335 m/s. **(3)**

Solution :

(i) Motion of stone (under gravity)

Time = t

$$h = ut + \frac{1}{2}gt^2$$

$$h = 0 + \frac{1}{2} \times 9.81\, t^2$$

$$h = 4.905\, t^2 \quad \text{... (I)}$$

(ii) Motion of sound (with constant velocity)

Time = $(4 - t)$

Displacement = Velocity × Time

$h = 335\ (4 - t)$

From equation (I)

$4.905\ t^2 = 335\ (4 - t)$

$4.905\ t^2 + 335\ t - 1340 = 0$

$t = 3.789$ seconds

(iii) Again, from equation (I)

$h = 4.905\ t^2$

$h = 4.905 \times 3.789^2$

$h = 70.42$ m

I. **(i)** Write the impulse-moment equation and mention its application. **(2)**

Solution :

Impulse Momentum Principle

$$F = ma$$

$$F = m\frac{dv}{dt} \qquad \left(\because a = \frac{dv}{dt}\right)$$

$$F\,dt = m\,dv$$

Integrating both sides

$$\therefore \int_{t_1}^{t_2} F\,dt = \int_{v_1}^{v_2} m\,dv$$

$$\therefore F(t_2 - t_1) = m(v_2 - v_1)$$

$$F\,\Delta t = mv_2 - mv_1$$

Impulse = Final momentum – Initial momentum

Impulse = Change in momentum

Whenever a heavy force act for a short interval of time change in momentum results.

Example :

(i) A nail is inserted in the wall by hammering.

(ii) A pile foundation is possible by dropping the heavy hammer.

(iii) Striking of a ball with bat.

(iv) A bomb is fired through cannon.

1. (j) A bullet weighs 0.5 N and moving with a velocity of 400 m/s hits centrally a 30 N block of wood moving away at 15 m/s and gets embedded in it. Find the velocity of the bullet after the impact and amount of kinetic energy lost. **(3)**

Solution :

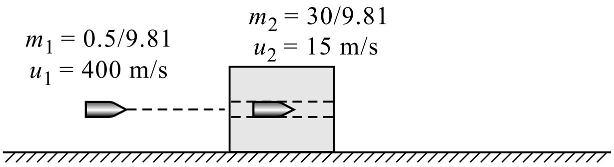


Fig. 1(j)-i

Perfectly plastic impact

(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

$$\frac{0.5}{9.81} \times 400 + \frac{30}{9.81} \times 15 = \left(\frac{0.5}{9.81} + \frac{30}{9.81}\right) v$$

$$20.39 + 45.87 = 3.109\, v$$

$v = 21.31$ m/s (velocity of block and bullet together after impact)

(ii) Kinetic energy loss = Initial KE – Final KE

$$\text{KE loss} = \left(\frac{1}{2} \times \frac{0.5}{9.81} \times 400^2 + \frac{1}{2} \times \frac{30}{9.81} \times 15^2\right) - \left(\frac{1}{2} \times \frac{30.5}{9.81} \times 21.31^2\right)$$

$$\text{KE loss} = 4421.5 - 705.94$$

$$\text{KE loss} = 3715.56 \text{ J}$$

PART - B

50 Marks

2. (a) Referring to Fig. 2(a)-i, find the magnitude and direction of the resultant and also find the point of action of the resultant.

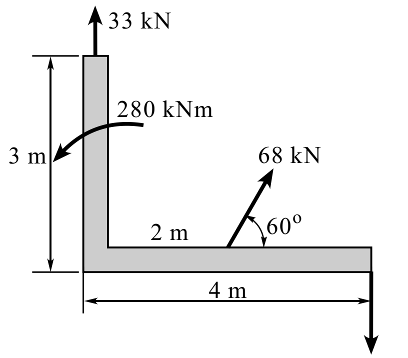


Fig. 2(a)-i

Solution :

(i) $\Sigma F_x = 68 \cos 60° = 34$ kN $(\rightarrow)$

(ii) $\Sigma F_y = 33 + 68 \sin 60° - 85 = 6.889$ kN $(\uparrow)$

(iii) $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

$R = \sqrt{(34)^2 + (6.889)^2}$

$\therefore R = 34.69 \text{ kN}$

(iv) $\tan \theta = \left|\dfrac{\Sigma F_y}{\Sigma F_x}\right| = \dfrac{6.889}{34}$

$\theta = 59.86°$

(v) $\Sigma M_O = 68 \sin 60° \times 2 - 85 \times 4 + 280 = 57.78 \text{ kN-m} (\circlearrowleft)$

(vi) By Varignon's theorem

$\Sigma M_O = R \times d$

$\therefore d = \dfrac{\Sigma M_O}{R} = \dfrac{57.78}{34.69}$

$d = 1.666 \text{ m}$

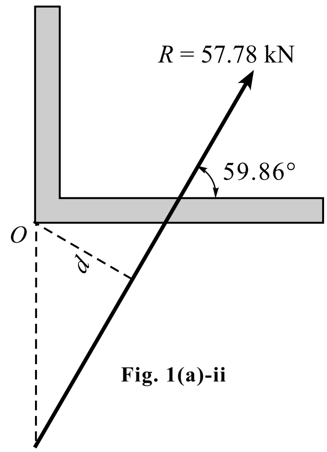


Fig. 1(a)-ii

(v) Position of resultant R w.r.t. point O is as shown in Fig.2(a)-ii.

2. **(b)** Find the axial forces in the members of the tripod loaded in the Fig. 2(b)-i, $W = 25$ kN.

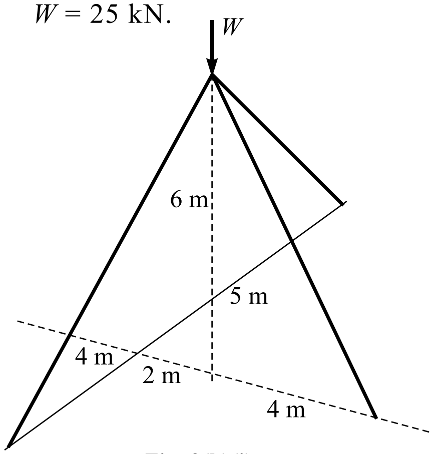


Fig. 2(b)(i)

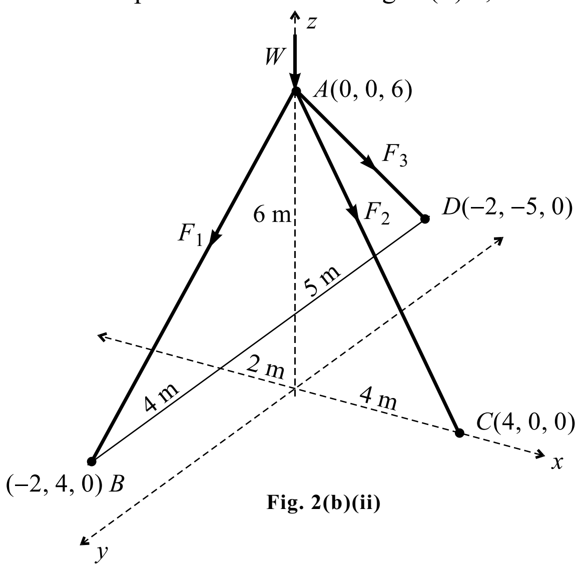


Fig. 2(b)(ii)

Solution : Refer to Fig. 2(b)-ii

(i) Force vector

$\overline{\boldsymbol{F}}_1 = (F_1)(\overline{\boldsymbol{e}}_{BA})$

$\overline{\boldsymbol{F}}_1 = (F_1)\left[\dfrac{2\,\mathbf{i} - 4\,\mathbf{j} + 6\,\mathbf{k}}{\sqrt{2^2 + 4^2 + 6^2}}\right]$

$\overline{\boldsymbol{F}}_1 = (F_1)(0.2673\,\mathbf{i} - 0.5344\,\mathbf{j} + 0.8018\,\mathbf{k})$

$$\overline{\boldsymbol{F}}_2 = (F_2)(\overline{\boldsymbol{e}}_{CA})$$

$$\overline{\boldsymbol{F}}_2 = (F_2)\left[\frac{-4\,\mathbf{i} + 0\,\mathbf{j} + 6\,\mathbf{k}}{\sqrt{4^2 + 0^2 + 6^2}}\right]$$

$$\overline{\boldsymbol{F}}_2 = (F_2)(-0.5656\,\mathbf{i} + 0\,\mathbf{j} + 0.8484\,\mathbf{k})$$

$$\overline{\boldsymbol{F}}_3 = (F_3)(\overline{\boldsymbol{e}}_{DA})$$

$$\overline{\boldsymbol{F}}_3 = (F_3)\left[\frac{2\,\mathbf{i} + 5\,\mathbf{j} + 6\,\mathbf{k}}{\sqrt{2^2 + 5^2 + 6^2}}\right]$$

$$\overline{\boldsymbol{F}}_3 = (F_3)(16.125\,\mathbf{i} + 40.311\,\mathbf{j} + 48.374\,\mathbf{k})$$

(ii) $\Sigma F_x = 0$

$$0.2673\,F_1 - 0.5656\,F_2 + 16.125\,F_3 = 0 \quad \ldots \text{(I)}$$

$\Sigma F_y = 0$

$$-0.5344\,F_1 + 0\,F_2 + 40.311\,F_3 = 0 \quad \ldots \text{(II)}$$

$\Sigma F_z = 0$

$$0.8018\,F_1 + 0.8484\,F_2 + 48.374\,F_3 + 25 = 0 \quad \ldots \text{(III)}$$

in the considered direction.

(iii) Solving Eqs. (I), (II) and (III), we get

$F_1 = -11.5489$ kN , $F_2 = -9.8229$ kN , $F_3 = -0.15310$ kN

Note : –ve sign indicates all the members are in compression.

3. **(a)** Find the reactions at supports A and B for the bracket loaded as shown in Fig. 3(a)-i.

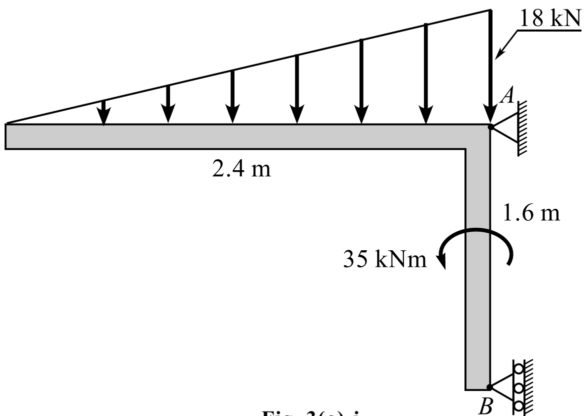


Fig. 3(a)-i

Solution : Refer to Fig. 3(a)-ii

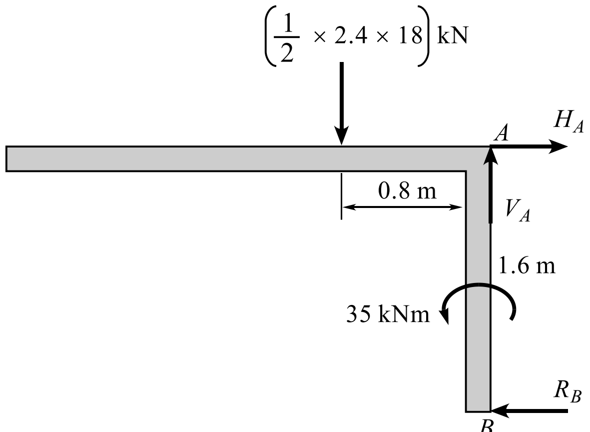


Fig. 3(a)-ii

(i) $\Sigma M_A = 0$

$$\frac{1}{2} \times 2.4 \times 18 \times 0.8 - R_B \times 1.6 + 35 = 0$$

$$R_B = 32.675 \text{ kN} \quad (\leftarrow)$$

(ii) $\Sigma F_x = 0$

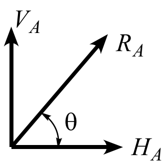
$$H_A - R_B = 0$$

$$H_A = 32.675 \text{ kN} \quad (\rightarrow)$$

(iii) $\Sigma F_y = 0$

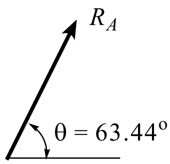
$$V_A - \frac{1}{2} \times 2.4 \times 18 = 0$$

$$V_A = 21.6 \text{ kN} \ (\uparrow)$$



(iv) $\tan\theta = \frac{V_A}{H_A} = \frac{21.6}{32.675}$

$$\theta = 33.47°$$



(v) $R_A = \sqrt{32.675^2 + 21.6^2}$

$$\therefore R_A = 39.17 \text{ kN}$$

3. **(b)** A circular plate with a radius of 3 m is suspended by three identical cables *AD*, *BE* and *CF* as shown in Fig. 3(b)-i. The plate is non homogeneous with a weight of 25 kN, which acts at a point *G*, at distance 1.5 m from origin. What is the force in cable *AD*, *BE* and *CF*?

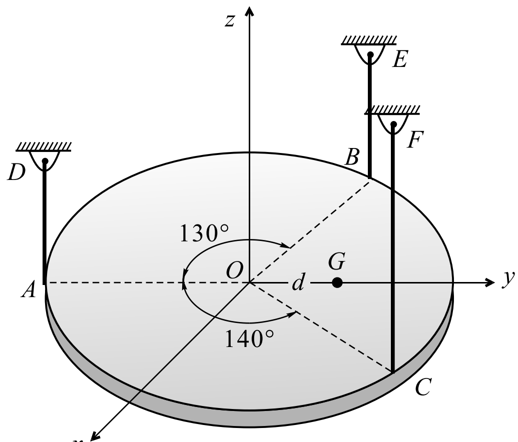


Fig. 3(b)-i

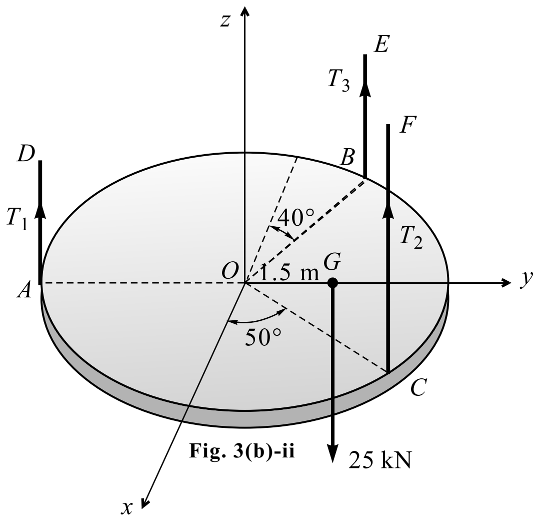


Fig. 3(b)-ii

Solution : Refer to Fig. 3(b)-ii

(i) $\Sigma F_z = 0$

$$T_1 + T_2 + T_3 = 25 \qquad \ldots \text{(I)}$$

(ii) Taking moment about y-axis

$$\Sigma M_y = 0$$

$$T_2 \times x_2 - T_3 \times x_3 = 0$$

$$T_2 \times 3 \cos 50° - T_3 \times 3 \cos 40° = 0$$

$$1.928\ T_2 - 2.298\ T_3 = 0 \qquad \text{... (II)}$$

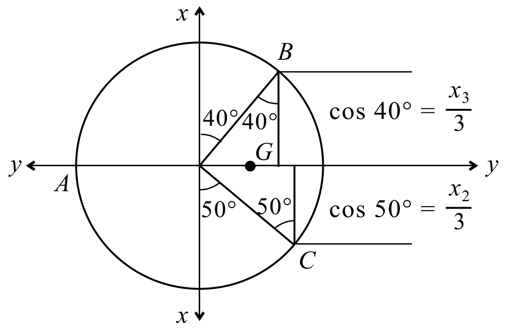


Fig. 3(b)-iii

(iii) Taking moment about x-axis

$$\Sigma M_x = 0$$

$$T_1 \times y_1 - T_2 \times y_2 - T_3 \times y_3 + 25 \times 1.5 = 0$$

$$T_1 \times 3 - T_2 \times 3 \sin 50° - T_3 \times 3 \sin 40° + 25 \times 1.5 = 0$$

$$3\ T_1 - 2.298\ T_2 - 1.928\ T_3 = -37.5 \qquad \text{... (III)}$$

Solving Eqs. (I), (II) and (III), we get

$$T_1 = 3.067 \text{ kN}$$

$$T_2 = 11.926 \text{ kN}$$

$$T_3 = 10.007 \text{ kN}$$

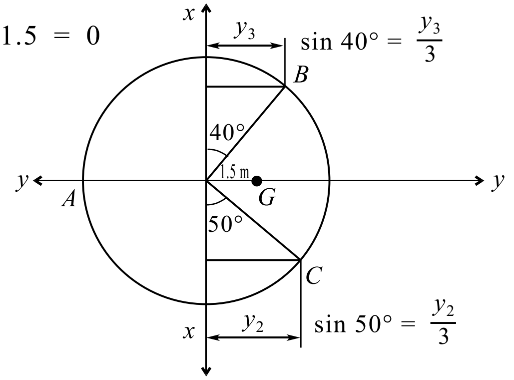


Fig. 3(b)-iv

4. (a) Consider the system, as shown in Fig. 4(a)-i, If $\theta = 70°$ and $\mu = 0.25$ at all surfaces of contact. What is the force (W) required to slide the wedge (A) in the downward direction?

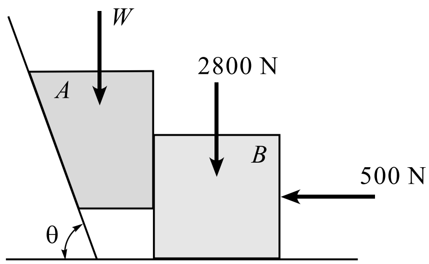


Fig. 4(a)-i

Solution :

(i) Consider FBD of Block *B*

Refer to Fig. 4(a)-ii

$$\Sigma F_y = 0$$

$$N_1 - 0.25N_2 - 2800 = 0$$

$$\therefore N_1 = 0.25N_2 + 2800$$

$$\Sigma F_x = 0,$$

$$N_2 - 0.25N_1 - 500 = 0$$

$$N_2 - 0.25(0.25N_2 + 2800) - 500 = 0$$

$$\therefore N_2 = 1280 \text{ N}$$

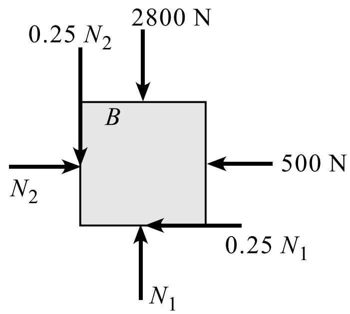


Fig. 4(a)-ii : FBD of Block *B*

(ii) Consider F.B.D. of Wedge *A*

Refer to Fig. 4(a)-iii

$\Sigma F_x = 0$

$N_3 \cos 20° - 0.25\, N_3 \cos 70° - N_2 = 0$

$0.854 N_3 - 1280 = 0$

$\therefore N_3 = 1498.83 \text{ N}$

$\Sigma F_y = 0$

$N_3 \sin 20° + 0.25\, N_3 \sin 70° + 0.25\, N_2 - W = 0$

$W = 1498.83 \sin 20° + 0.25 \times 1498.83 \sin 70° + 0.25\ 1280$

$\therefore W = 1184.74 \text{ N}$

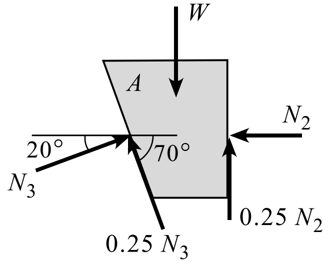


Fig. 4(a)-iii : FBD of Wedge *A*

4. (b) For the system shown, in Fig. 4(b)-i, if it is required to move the block of weight 1500 N to the right, Find the tension in the and find the reaction between the blocks. Take μ = 0.25 at all contacting surfaces.

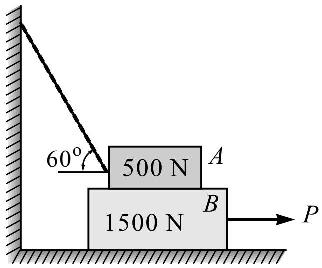


Fig. 4(b)-i

Solution :

(i) Consider the FBD of Block *A* (500 N)

$\Sigma F_y = 0$

$N_A + T \sin 60° - 500 = 0$

$N_A = 500 - T \sin 60° \qquad \ldots \text{(I)}$

$\Sigma F_x = 0$

$0.25\, N_A - T \cos 60° = 0$

$0.25(500 - T \sin 60°) - T \cos 60° = 0$

$125 - T \sin 60° \times 0.25 - T \cos 60° = 0$

$T = 174.46 \text{ N}$

From Eqn. (I)

$N_A = 500 - 174.46 \sin 60°$

$\therefore N_A = 348.91 \text{ N}$

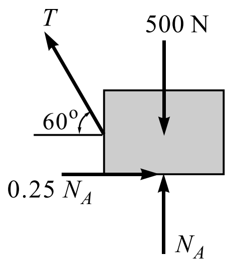


Fig. 4(b)-ii : FBD of Block *A* (500 N)

(ii) Consider the F.B.D. of Block *B* (1500 N)

$\Sigma F_y = 0$

$N_B - N_A - 1500 = 0$

$N_B = 1500 + 348.91 \qquad \therefore N_B = 1848.91 \text{ N}$

$\Sigma F_x = 0$

$P - 0.25\, N_A - 0.25\, N_B = 0$

$P = 0.25 \times 348.91 + 0.25 \times 1848.91 \quad \therefore P = 549.46 \text{ N}$

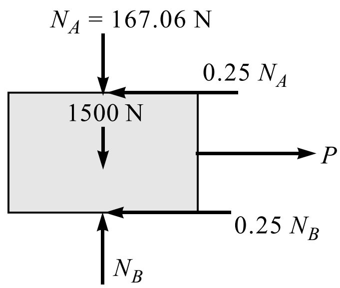


Fig. 4(b)-iii : FBD of Block *B* (1500 N)

5. (a) What is the value of P in the system as shown in Fig. 5(a)-i to the motion to impend? Assume the pulley is smooth and the coefficient of friction for all contact surfaces is 0.2.

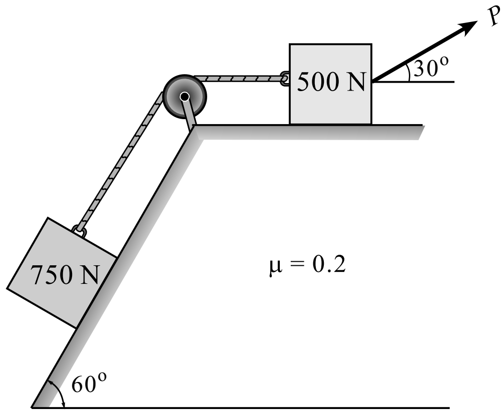


Fig. 5(a)-i

Solution :

(i) Consider FBD of Block 750 N

$\Sigma F_y = 0$

$N_1 - 750 \cos 60° = 0$

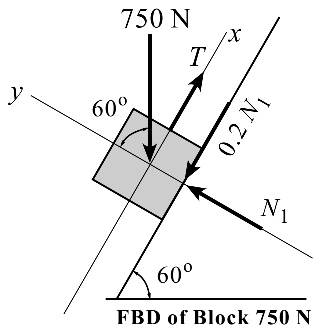
$N_1 = 375 \text{ N}$

$\Sigma F_x = 0$

$T - 0.2\, N_1 - 750 \sin 60° = 0$

$T = 0.2 \times 375 + 750 \sin 60°$

$T = 724.52 \text{ N}$



FBD of Block 750 N

(ii) Consider FBD of Block 500 N

$\Sigma F_y = 0$

$N_2 + P \sin 30° - 500 = 0$

$N_2 = 500 - P \sin 30°$

$\Sigma F_x = 0$

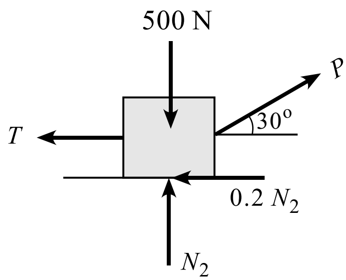
$P \cos 30° - T - 0.2\, N_2 = 0$

$P \cos 30° - 724.52 - 0.2(500 - P \sin 30°) = 0$

$P \cos 30° + 0.2\, P \sin 30° - 724.52 - 0.2 \times 500 = 0$

$0.966\, P = 824.52$

$\therefore$ $P = 853.54 \text{ N}$



FBD of Block 500 N

5. (b) Blocks A and B are placed on an 'inclined surface as shown in Fig. 5(b)-i, The mass of A is 13.5 kg and that of B is 40 kg. The coefficient of friction at all surfaces of contact is 03. Determine the angle (θ) of the surface at which the motion impends and also find the tension in the string.

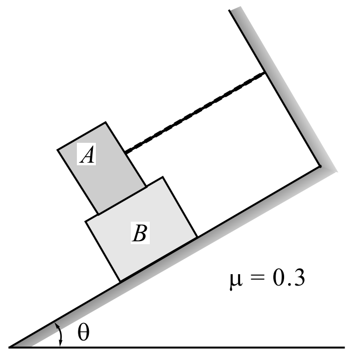
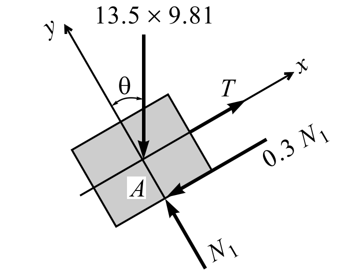


Fig. 5(b)-i

Solution :

(i) Consider the F.B.D. of Block *A*



F.B.D. of Block *A*

$$\Sigma F_y = 0$$

$$N_1 - 13.5 \times 9.81 \cos\theta = 0$$

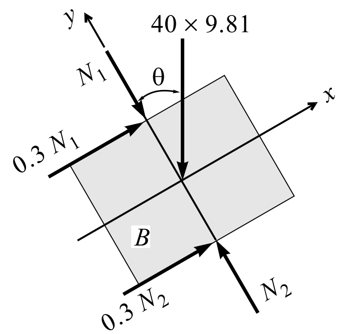
$$N_1 = 13.5 \times 9.81 \cos\theta$$

$$\Sigma F_x = 0$$

$$T - 0.3\, N_1 - 13.5 \times 9.81 \sin\theta = 0$$

$$T - 0.3\,(13.5 \times 9.81 \cos\theta) - 13.5 \times 9.81 \sin\theta = 0$$

(ii) Consider the F.B.D. of Block *B*



F.B.D. of Block *B*

$$\Sigma F_y = 0$$

$$N_2 - N_1 - 40 \times 9.81 \cos\theta = 0$$

$$N_2 = 13.5 \times 9.81 \cos\theta - 40 \times 9.81 \cos\theta$$

$$N_2 = 486.85 \cos\theta$$

$$\Sigma F_x = 0$$

$$0.3\, N_1 + 0.3 N_2 - 40 \times 9.81 \sin\theta = 0$$

$$0.3\,(13.5 \times 9.81 \cos\theta + 486.85 \cos\theta) - 40 \times 9.81 \sin\theta = 0$$

$$185.79 \cos\theta - 392.4 \sin\theta = 0$$

$$\frac{\sin\theta}{\cos\theta} = \frac{185.79}{392.4}$$

$$\tan\theta = 0.4735$$

$$\theta = 25.34°$$

6. (a) A semicircular area is removed from the trapezoid shown in Fig. 6(a)-i. Determine the *Y* coordinate of the centroid and the areal moment of inertia with respect to centroidal *X* axis for the shaded area.

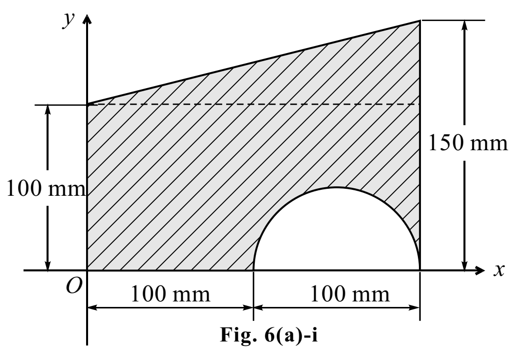


Fig. 6(a)-i

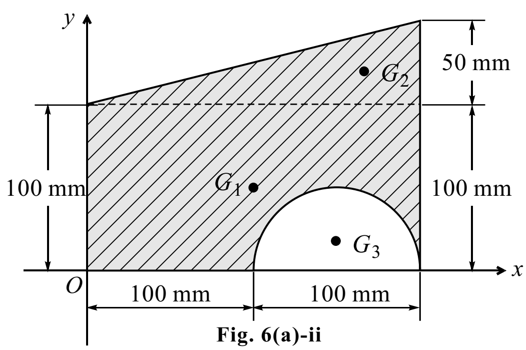


Fig. 6(a)-ii

Solution : Refer to Fig. 6(a)-ii.

(i) To find $\bar{y}$

$$\bar{y} = \frac{200 \times 100 \times 50 + \frac{1}{2} \times 200 \times 50 \times \left(100 + \frac{50}{3}\right) - \frac{\pi \times 50^2}{2} \times \frac{4 \times 50}{3\pi}}{200 \times 100 + \frac{1}{2} \times 200 \times 50 - \frac{\pi \times 50^2}{2}}$$

$$\bar{y} = \frac{10^6 + 583333.33 - 83333.33}{20000 + 5000 - 3927}$$

$$\bar{y} = 71.18 \text{ mm}$$

(ii) To find I_{xOx}

$$I_{xOx} = \frac{200 \times 100^3}{3} + \left(\frac{200 \times 50^3}{36} + \frac{1}{2} \times 200 \times 50 \times 116.67^2\right) - \frac{\pi \times 50^4}{8}$$

$$I_{xOx} = 66666666.67 + 68753888.94 - 2454369.26$$

$$I_{xOx} = 132966186.4 \text{ mm}^4$$

(iii) To find I_{xGx}

By parallel axis theorem

$$I_{xOx} = I_{xGx} + A(\bar{y})^2$$

$$I_{xGx} = I_{xOx} - A(\bar{y})^2$$

$$I_{xGx} = 132966186.4 - 21073 \times (71.18)^2$$

$$I_{xGx} = 26197884.76 \text{ mm}^4$$

6. **(b)** Find the coordinate of the centroid and the areal moment of inertia about the centroidal axes parallel to the base for the shaded part of the Fig. 6(b)-i.

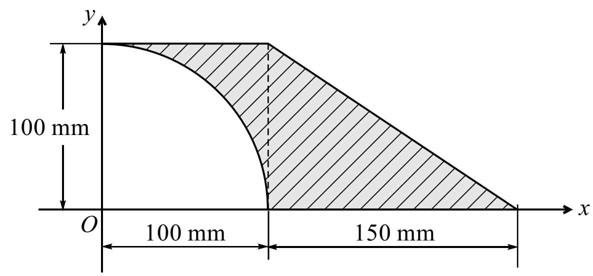


Fig. 6(b)-i

Solution :

(i) To find $\bar{y}$

$$\bar{y} = \frac{100 \times 100 \times 50 + \frac{1}{2} \times 150 \times 100 \times \left(\frac{100}{3}\right) - \frac{\pi \times 100^2}{4} \times \frac{4 \times 100}{3\pi}}{100 \times 100 + \frac{1}{2} \times 150 \times 100 - \frac{\pi \times 100^2}{4}}$$

$$\bar{y} = \frac{500000 + 250000 - 333333.33}{10000 + 7500 - 7853.98}$$

$$\bar{y} = 43.2 \text{ mm}$$

(ii) To find I_{xOx}

$$I_{xOx} = \frac{100 \times 100^3}{3} + \frac{150 \times 100^3}{12} - \frac{\pi \times 100^4}{16}$$

$$I_{xOx} = 26198379.25 \text{ mm}^4$$

(iii) To find I_{xGx}

By parallel axis theorem

$$I_{xOx} = I_{xGx} + A(\bar{y})^2$$

$$I_{xGx} = I_{xOx} - A(\bar{y})^2$$

$$I_{xGx} = 26198379.25 - 9646.02 \times (43.2)^2$$

$$I_{xGx} = 8196590.88 \text{ mm}^4$$

7. **(a)** Determine the Y coordinate of the centroid and find areal moment of inertia of the shaded area about the centroidal X axis for the composite figure shown in Fig. 7(a)-i.

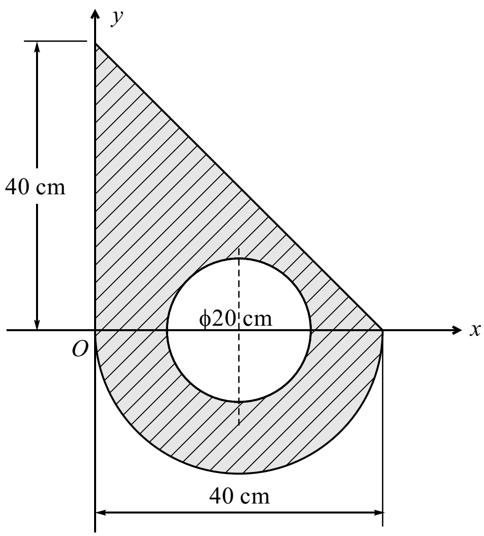


Fig. 7(a)-i

Solution :

(i) To find $\bar{y}$

$$\bar{y} = \frac{\frac{1}{2} \times 40 \times 40 \times \frac{40}{3} - \frac{\pi \times 20^2}{2} \times \frac{4 \times 20}{3\pi} - \pi \times 10^2 \times 0}{\frac{1}{2} \times 40 \times 40 + \frac{\pi \times 20^2}{2} - \pi \times 10^2}$$

$$\bar{y} = \frac{10666.67 - 5333.33 - 0}{800 + 628.32 - 314.16}$$

$$\bar{y} = 4.787 \text{ cm}$$

(ii) Shaded area $A = 800 + 628.32 - 314.16$

$$A = 1114.16 \text{ cm}^2$$

(iii) $$I_{xox} = \frac{40 \times 40^3}{12} + \frac{\pi \times 20^4}{8} - \frac{\pi \times 10^4}{4}$$

$$I_{xox} = 213333.33 + 62831.85 - 7853.98$$

$$I_{xox} = 268311.2 \text{ cm}^4$$

(iv) To find I_{xGx}

By parallel axis theorem

$$I_{xOx} = I_{xGx} + A(\bar{y})^2$$

$$I_{xGx} = I_{xOx} - A(\bar{y})^2$$

$$I_{xGx} = 268311.2 - 1114.16 \times (4.787)^2$$

$$\boxed{I_{xGx} = 242779.81 \text{ cm}^4}$$

7. (b) Locate the Y coordinate of centroid and find areal moment of inertia about OY axis of the composite area shown in Fig. 7(b)-i.

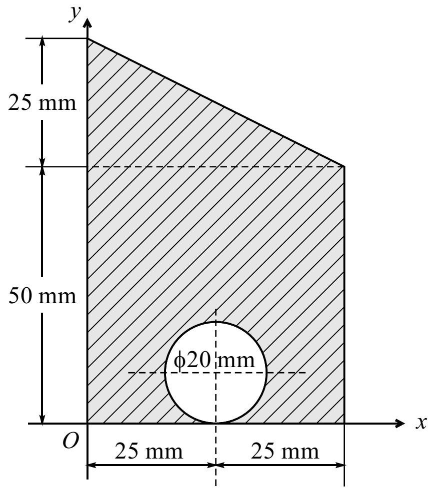


Fig. 7(b)-i

Solution :

(i) To find $\bar{y}$

$$\bar{y} = \frac{50 \times 50 \times 25 + \frac{1}{2} \times 50 \times 25 \times \left(50 + \frac{25}{3}\right) - \pi \times 10^2 \times 10}{50 \times 50 + \frac{1}{2} \times 50 \times 25 - \pi \times 10^2}$$

$$\bar{y} = \frac{62500 + 36458.33 - 3141.59}{2500 + 625 - 314.16}$$

$$\boxed{\bar{y} = 34.09 \text{ mm}}$$

(ii) $$I_{yoy} = \frac{50 \times 50^3}{3} + \frac{25 \times 50^3}{12} - \left(\frac{\pi \times 10^4}{4} + \pi \times 10^2 \times 25^2\right)$$

$$I_{yoy} = 2083333.33 + 260416.67 - 204203.52$$

$$\boxed{I_{yoy} = 2139546.48 \text{ mm}^4}$$

8. (a) A particle moves along a straight line with an acceleration prescribed, by the relation $a = (4t^2 - 3t + 2)$. Where 'a' is in m/s^2 and t is in seconds. The particle has a velocity of 10 m/s at origin at $t = 2$ seconds. Determine the position and velocity of the particle after 5 seconds.

Solution : *Given* : $a = 4t^2 - 3t + 2$; at $t = 2$ s ; $v = 10$ m/s

Find : $s = ?$; $v = ?$ at $t = 5$ s

Integrating the given expression of a w.r.t. time t,

$$v = \frac{4t^3}{3} - \frac{3t^2}{3} + 2t + c_1$$

At $t = 2$ s, $v = 10$ m/s

$$10 = \frac{4 \times 2^3}{3} - \frac{3 \times 2^2}{2} + 2 \times 2 + c_1$$

$$c_1 = 10 - \frac{32}{3} + \frac{12}{2} - 4 \qquad \therefore\ c_1 = 1.33$$

$$v = \frac{4t^3}{3} - \frac{3t^2}{2} + 2t + 1.33 \qquad \text{..... (I)}$$

Further integrating

$$s = \frac{4t^4}{3 \times 4} - \frac{3t^3}{2 \times 3} + \frac{2t^2}{2} + 1.33t + c_2$$

At $t = 2$ s, $s = 0$

$$0 = \frac{4 \times 2^4}{3 \times 4} - \frac{3 \times 2^3}{2 \times 3} + \frac{2 \times 2^2}{2} + 1.33 \times 2 + c_2$$

$$c_2 = -5.33 + 4 - 4 - 2.66 \qquad \therefore\ c_2 = -7.99$$

$$s = \frac{t^4}{3} - \frac{t^3}{2} + t^2 + 1.33t - 7.99 \qquad \text{..... (II)}$$

Put $t = 5$ s in equation (I) and (II), we get

$$v = \frac{4 \times 5^3}{3} - \frac{3 \times 5^2}{2} + 2 \times 5 + 1.33$$

$$v = 166.67 - 37.5 + 10 + 1.33$$

$$\boxed{v = 140.5 \text{ m/s}}$$

$$s = \frac{5^4}{3} - \frac{5^3}{2} + 5^2 + 1.33 \times 5 - 7.99$$

$$s = 208.33 - 62.5 + 25 + 6.65$$

$$\boxed{s = 177.48 \text{ m}}$$

8. **(b)** An automobile enters a curved road at: 30 km/hr and then leaves at 48 km/hr. The curved road is in the form of quarter of a circle and has a length 400 m. If the car travels at constant acceleration along the curve, calculate the resultant acceleration and its direction at both ends of the curve.

Solution : *Given* : $s = 400$ m,

$u = 30$ km/hr $= 8.33$ m/s,

$v = 48$ km/hr $= 13.33$ m/s

Find : Acceleration at point A and B.

Refer to Fig. 8(b)-i.

$$s = \frac{2\pi\rho}{4}$$

$$400 = \frac{2\pi\rho}{4}$$

$$\rho = 254.65 \text{ m (Radius of curvature)}$$

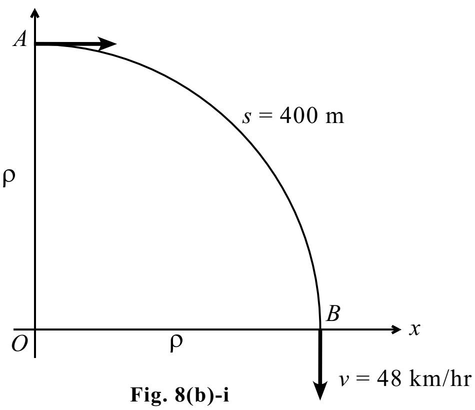
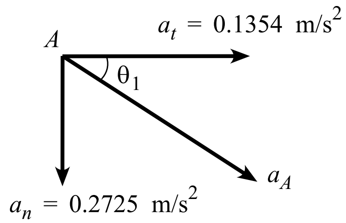


Fig. 8(b)-i

For constant acceleration, we have



$$v^2 = u^2 + 2a_t s$$

$$13.33^2 = 8.33^2 + 2 \times a_t \times 400$$

$$a_t = 0.1354\ \text{m/s}^2$$

At the starting point A

$$a_n = \frac{u^2}{\rho} = \frac{8.33^2}{254.65}$$

$$a_n = 0.2725\ \text{m/s}^2$$

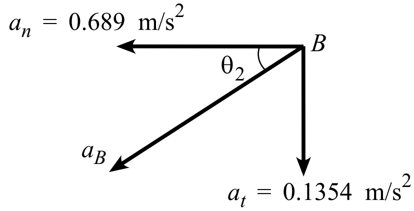
$$\tan\theta_1 = \frac{a_n}{a_t} = \frac{0.1354}{0.2725}$$

$$\theta = 63.58°$$

$$a_A = \sqrt{a_t^2 + a_n^2} = \sqrt{0.2725^2 + 0.1354^2}$$

$$a_A = 0.304\ \text{m/s}^2$$

At the end point B



$$a_n = \frac{v^2}{\rho} = \frac{13.33^2}{254.65}$$

$$a_n = 0.689\ \text{m/s}^2$$

$$a_t = 0.1354\ \text{m/s}^2$$

$$\tan\theta_1 = \frac{a_t}{a_n} = \frac{0.1354}{0.689}$$

$$\theta = 10.98°$$

$$a_A = \sqrt{a_t^2 + a_n^2}$$

$$= \sqrt{0.689^2 + 0.1354^2}$$

$$a_B = 0.711\ \text{m/s}^2$$

9. **(a)** A block of mass 5 kg resting on a 30° inclined plane is released. The block after traveling a distance of 0.5 m along the inclined plane hits a spring of stiffness 15 N/cm, as shown in Fig. 9(a)-i. Find the maximum compression of spring. Assume the coefficient of friction between the block and the inclined plane as 0.2.

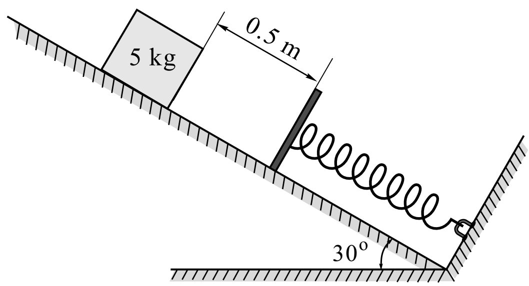


Fig. 9(a)-i

Solution :

By work energy principal, we have

Work done = Change in kinetic energy

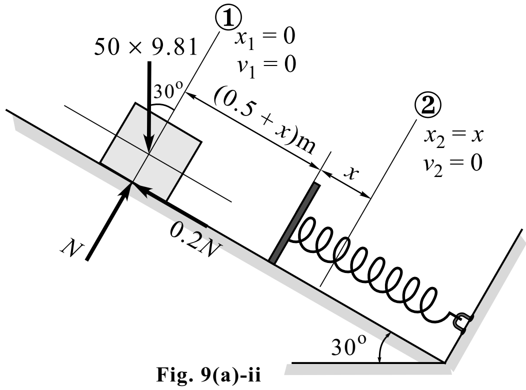


Fig. 9(a)-ii

$$\frac{1}{2} \times 15 \times 10^2 \times (0^2 - x^2) + 5 \times 9.81 \sin 30^\circ (0.5 + x) - 0.2 \times 5 \times 9.81 \cos 30^\circ = 0 - 0$$

$$-750\, x^2 + 12.26 + 24.53\, x - 8.496 = 0$$

$$750\, x^2 - 24.53\, x + 3.764 = 0$$

$$\therefore \; x = 0.089 \text{ m}$$

9. **(b)** A soldier fires a bullet with a velocity of 31.32 m/s at an angle α upwards from the horizontal from his position on a hill to strike a target which is 100 m away and 50 m below his position. Find the angle of projection α. Find the velocity with which the bullet strikes the object,

Solution :

(i) By general equation of projectile motion,

$$y = x \tan \alpha - \frac{gx^2}{2u^2} (1 + \tan^2 \alpha)$$

$$-50 = 100 \tan \alpha - \frac{9.81 \times 100^2}{2 \times 31.32^2} (1 + \tan^2 \alpha)$$

$$-50 = 100 \tan \alpha - 50 - 50 \tan^2 \alpha$$

$$0 = 100 \tan \alpha - 50 \tan^2 \alpha$$

$$100 - 50 \tan \alpha = 0$$

$$\tan \alpha = \frac{100}{50}$$

$$\alpha = 63.44^\circ$$

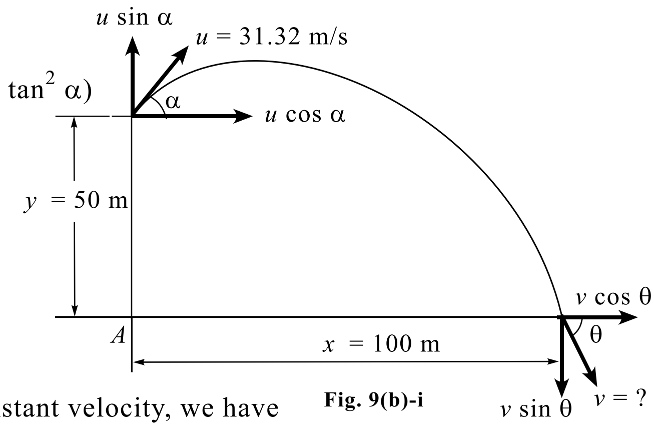


Fig. 9(b)-i

(ii) Consider horizontal motion with constant velocity, we have

$$v \cos \theta = u \cos \alpha = 31.32 \cos 63.44^\circ$$

$$v \cos \theta = 14 \text{ m/s} \qquad \text{... (I)}$$

(iii) Considering vertical motion under gravity, we have

$$(v \sin \theta)^2 = (u \sin \alpha)^2 + 2 \times 9.81 \times 50$$

$$(v \sin \theta)^2 = (31.32 \sin 63.44^\circ)^2 + 2 \times 9.81 \times 50$$

$$v \sin \theta = 31.52 \text{ m/s} \qquad \text{... (II)}$$

(iii) Dividing equation (II) by (I), we get

$$\frac{v \sin \theta}{v \cos \theta} = \frac{31.52}{14} \quad \text{i.e. } \tan \theta = 2.25 \qquad \therefore \; \theta = 66.05^\circ$$

From equation (I), we get

$$v \cos \theta = 14 \text{ m/s}$$

$$v = \frac{14}{\cos 66.05^\circ} \qquad v = 34.49 \text{ m/s} \; (\theta)$$

10. (a) A fly wheel 0.5 m in diameter accelerates uniformly from rest to 360 rpm in 12 seconds. Determine the velocity and acceleration of a point on the rim of the fly wheel 0.1 it has started from rest.

Solution : *Given* : $d = 0.5$ m, $\omega_0 = 0$, $t = 12$ seconds, and $N = 360$ rpm

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 360}{60}$$

$$\omega = 37.7 \text{ rad/s}$$

At $t = 12$ seconds

$$\omega = \omega_0 + \alpha t$$

$$37.7 = 0 + \alpha \times 12$$

$$\alpha = 3.14 \text{ rad/s}^2 \text{ (Angular acceleration)}$$

At $t = 0.1$ seconds

$$\omega = \omega_0 + \alpha \text{t}$$

$$\omega = 0 + 3.14 \times 0.1$$

$$\omega = 0.314 \text{ rad/s (Angular velocity)}$$

10. (b) A wheel rotating about a fixed axis at 20 revolutions per minute is uniformly accelerating for 70 seconds during which it makes 50 revolutions. Find the angular velocity at the end of this interval and time required for the velocity to reach 100 revolutions per minute.

Solution : *Given* : $N_0 = 20$ rpm, $t = 70$ seconds, $\theta = 2\pi \times 50$ rad

$$\omega_0 = \frac{2\pi N_0}{60} = \frac{2\pi \times 20}{60}$$

$$\omega_0 = 2.094 \text{ rad/s}$$

Now,

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$2\pi \times 50 = 2.094 \times 70 + \frac{1}{2}\alpha (70)^2$$

$$\alpha = 0.0684 \text{ rad/s}^2$$

$$\omega = \omega_0 + \alpha t = 2.094 + 0.0684 \times 70$$

$$\omega = 5 \text{ rad/s}$$

To find t_f

$$\omega_f = \frac{2\pi \times 100}{60} = 10.47 \text{ rad/s}$$

$$\omega_f = \omega_0 + \alpha t_f$$

$$10.47 = 2.094 + 0.0684 \times t_f$$

$$\therefore t_f = 122.46 \text{ seconds}$$

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