Lower bound amortized analysis

The complexity of problems

Problem P, Algorithm A

$$\text{Inputs: } \mathcal{X}_n \text{ of size } n$$
$$W_A(n) = \max_{x\in\mathcal{X}_n} T_A(x)$$
$$B_A(n) = \min_{x\in\mathcal{X}_n} T_A(x)$$
$$A_A(n) = \sum_{x\in\mathcal{X}_n} T_A(x)\cdot P(x) = \mathbb{E}\left[T_A\right] = \sum_{t\in T_A(\mathcal{X}_n)} t\cdot P(T=t)$$
$$T_P(n) = \min_{A \text{ solves } P} W_A(n) = \min_{A \text{ solves } P}\max_{x\in\mathcal{X}_n} T_A(x)$$

解决方法：

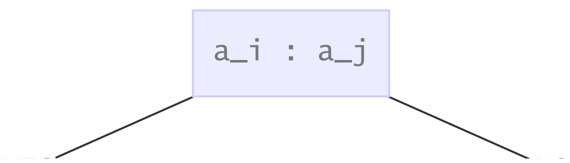
- Decision Tree
- Adversary Argument

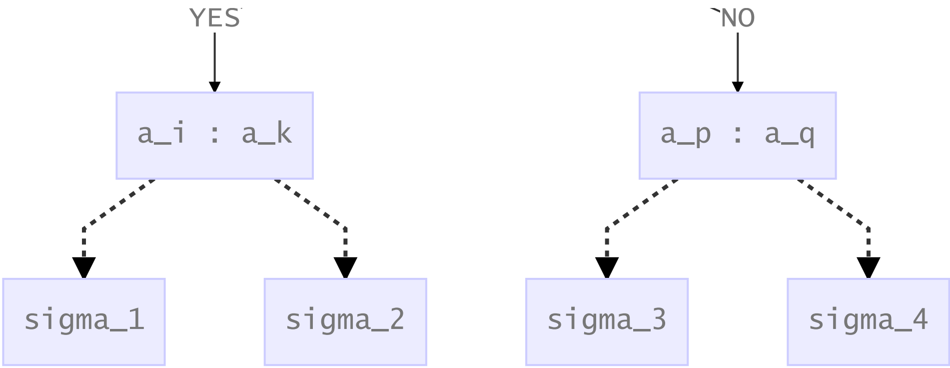
Decision tree

e.g., Lower bound for **Comparison-based** Sorting

非基于比较的排序：如煎饼排序

> Decision tree model
>
> - Nodes: comparisons $a_i : a_j$
> - Edges: two-way decisions
> - Leaves: possible permutations





Assumption: All the elements are distinct

- 不影响原命题
- 简化证明

任意 Comparison-based Sorting 算法都可由 decision tree 描述

算法 A 的 worst-case 即是决策树的高度

问题的下界是所有决策树高度的最小值

作为排序算法

$$L = \# \text{ of leaves } \geqslant n!$$

作为二叉树

$$L = \# \text{ of leaves } \leqslant 2^h$$

故

$$\begin{aligned} n! \leqslant L = \# \text{ of leaves } \leqslant 2^h \\ h \geqslant \log n! = \Omega(n \log n) \end{aligned}$$

K-sorted Array

QuickSort (with median as pivot) stops after the $\log k$ recursions.

复杂度：$\Theta(n \log k)$

Decision Tree 叶结点个数

$$L \geqslant \binom{n}{n/k}\binom{n-n/k}{n/k}\cdots\binom{n/k}{n/k} = \binom{n}{n/k, \ldots, n/k} = \frac{n!}{\left(\left(\frac{n}{k}\right)!\right)^k}$$

高度下界

$$H \geqslant \log\left(\frac{n!}{\left(\left(\frac{n}{k}\right)!\right)^k}\right) = \Omega(n \log k)$$

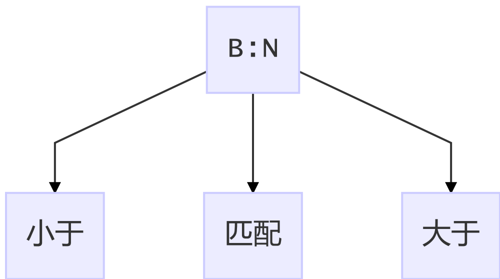
依据：

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \Longrightarrow \log n! \sim n \log n$$

Bolts and Nuts

变相的 QuickSort

$$A(n) = O(n \log n)$$



$$3^H \geqslant L \geqslant n! \Longrightarrow H \geqslant \log n! \Longrightarrow H = \Omega(n \log n)$$

Repeated Elements

$\Omega(n \log k)$

Adversary argument

Searching in matrix

一个 m 行 n 列的矩阵，行从左至右递增，列从上至下递增，$x \in M$?

思路：从左下角开始，每次均可减少一行或一列

复杂度 $m + n - 1$

Assume: $M: n \times n$

已知

$$W(n) \leqslant 2n-1$$

求证

$$\color{red}{W(n) \geqslant 2n-1}$$

Adversary strategy

对角线没有准确的大小关系，至少要比较 $2n-1$ 次

$$\begin{aligned} i+j \leqslant n-1 &\Longrightarrow x > M_{ij} \\ i+j > n-1 &\Longrightarrow x < M_{ij} \end{aligned}$$

Amortized analysis

> Amortized analysis is an algorithm analysis technique for analyzing a sequence of operations irrespective of the input to show that the average cost per operation is small, even though a single operation within the sequence might be expensive.

- Summation method
- Accounting method
- Potential method

Array merging dictionary

$$\sum_{i=1}^{n} c_i = \sum_{j=0}^{\lfloor \log n \rfloor} \left\lfloor \frac{n}{2^j} \right\rfloor 2^j \leq n(\lfloor \log n \rfloor + 1)$$
$$\forall i, \hat{c}_i = 1 + \lfloor \log n \rfloor$$