Integer solutions to n dimensional Pythagoras' equations

Eddie Revell

July 11, 2021

Introduction

$3^2 + 4^2 = 5^2$ is a well known integer solution of Pythagoras' equation, $a^2 + b^2 = c^2$. Here we will study integer solutions to n dimensional Pythagoras' equation:

$$\sum_{i=1}^{n} a_i^2 = b^2 \qquad \text{for } n \in \{2, 3, 4, \dots\} \tag{0.1}$$

Method

Divide (0.1) by b^2, so that the problem is now a question of finding rational points on an n dimensional sphere:

$$\sum_{i=1}^{n} x_i^2 = 1 \tag{0.2}$$

Observe that (1,0,...,0) is a trivial rational solution to this equation. Suppose more generally that $(\alpha_1, \alpha_2, ..., \alpha_n)$ is a rational solution to this equation. Consider the straight line parameterised by the x_1 coordinate defined by:

$$x_i = \alpha_i + k_i(\alpha_1 - x_1) \qquad i \geq 2$$

Where k_i are rational numbers. This line intersects with the n sphere defined by (0.2) when:

$$x_1^2 + \sum_{i=2}^{n} (\alpha_i + k_i(\alpha_1 - x_1))^2 = 1 \tag{0.3}$$

This is quadratic in x_1. Because the solution $x_1 = \alpha_1$ is rational and all the coefficients of the equation are rational, it follows that the other solution $x_1 = \beta_1$ must be rational. Since the k_i's are rational, it follows also that the remaining coordinates $x_2 = \beta_2$,..., $x_n = \beta_n$ are all rational. Thus we have found another rational solution to the equation (0.2). Since any two rational solutions can be connected with such a line, it follows that all solutions can be found in this way.

By starting at the point (1,0,...,0), we may simplify the equation (0.3) to:

$$(x_1 - 1)((1 + \sum_{i=2}^{n} k_i^2 + 1 - \sum_{i=2}^{n} k_i^2)) = 0 \tag{0.4}$$

And hence:

$$\boxed{x_1 = \frac{\kappa - 1}{\kappa + 1} \quad , \quad x_i = \frac{2k_i}{\kappa + 1}} \tag{0.5}$$

Where $\kappa := \sum_{i=2}^{n} k_i^2$. For any choice of rational numbers $k_2, ..., k_n$, we may thus find a solution (0.2), and hence by multiplying through by the lowest common multiple of the denominators of $x_1, ..., x_n$, an integer solution to (0.1).

Example

By choosing $k_2 = 1/2$, $k_3 = 3/5$ and $k_4 = 7/11$, we have:

$$181^2 + 12100^2 + 14520^2 + 15400^2 = 23481^2 \tag{0.6}$$