The Cartan Classification

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May 1, 2020

This set of notes serves to supplement the material taught by N. Dorey in the Part III *Symmetries, Fields and Particles* course. Specifically, we classify all finite dimensional complex Lie algebras (Cartan, 1894).

Throughout these notes, $\mathfrak{g}$ is assumed to be a simple complex Lie algebra.

1 Preliminary Definitions

Recall that for each $X \in \mathfrak{g}$, we may define the **adjoint map**:

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g} \quad , \quad Y \to \mathrm{ad}_X(Y) = [X, Y] \tag{1.1}$$

Observe that ad_X is a linear map between vector spaces of dimension $d = \dim(\mathfrak{g})$ and we can therefore think of it as a $d \times d$ matrix. This defines the **adjoint representation** of $\mathfrak{g}$ (one needs to prove this, but it is easy and not the focus of these notes).

The **Killing Form** of a Lie algebra is the inner product defined by:

$$K : \mathfrak{g} \times \mathfrak{g} \to F \quad , \quad K(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \tag{1.2}$$

Where $F = \mathbb{R}$ or $\mathbb{C}$ (in the case studied in these notes, it is the latter). The Killing form enjoys **invariance under adjoint action**, that is:

$$K([Z, X], Y) + K(X, [Z, Y]) = 0 \qquad \forall X, Y, Z \in \mathfrak{g} \tag{1.3}$$

Theorem 1.1. (**Cartan's Theorem**) $\mathfrak{g}$ is semi-simple if and only if the Killing Form is non-degenerate. (A semi-simple Lie algebra is one with no non-trivial abelian ideals. Thus all simple Lie algebras are also semi-simple.)

Proof. The "if" direction is omitted. For the "only if" (forward) direction (outline only), assume for contradiction that $\mathfrak{g}$ is not semi-simple, and so has a non-trivial abelian ideal $\mathfrak{j}$. Then show that $K(X, Y) = 0$ for all $X \in \mathfrak{j},\ Y \in \mathfrak{g}$, which is a contradiction. Details are in handwritten lecture notes. $\square$

2 The Cartan-Weyl basis

We say that $X \in \mathfrak{g}$ is **ad-diagonalisable** if the matrix corresponding to the map ad_X is diagonalisable.
A **Cartan subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra containing only ad-diagonalisable elements.

Theorem 2.1. We have:

(i) $H \in \mathfrak{h} \implies H$ is ad-diagonalisable.

(ii) $H, H' \in \mathfrak{h} \implies [H, H'] = 0$.

(iii) If $X \in \mathfrak{g}$ and $[X, H] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.

Proof. The first two are trivial and follow immediately from the definition. The third is omitted (for now). $\square$

All Cartan subalgebras of $\mathfrak{g}$ have the same dimension, and their dimension is always at least one (proof omitted). Thus we may define the **rank** of the Cartan subalgebra to be $r = \dim(\mathfrak{h}) \in \mathbb{N}$.

We may choose a basis $\{H^i \mid i = 1, \ldots, r\}$ of $\mathfrak{h}$. Since each H^i provides a map $\mathrm{ad}_{H^i} : \mathfrak{g} \to \mathfrak{g}$ and they commute with one another, we see that they are simultaneously diagonalisable and that $\mathfrak{g}$ **is spanned by the simultaneous eigenvectors of** $\boldsymbol{\mathrm{ad}_{H^i}}$**,** $\boldsymbol{i = 1, ..., r}$**.** We see that eigenvectors have:

1. **Zero eigenvalues:** Observe that H^j is an eigenvector with eigenvalue zero of H^i: $\mathrm{ad}_{H^i}(H^j) = [H^i, H^j] = 0$ for all $i, j \in \{1, ..., r\}$.
2. **Non-zero eigenvalues:** Since $r \leq d$, we will in the general case require eigenvectors not in $\mathfrak{h}$. We write these as E^α, with $\alpha \in \mathbb{R}^r$ indexing the eigenvector, and $\mathrm{ad}_{H^i}(E^\alpha) = \alpha^i E^\alpha$. Observe that α^i are not all zero, since otherwise we would have $E^\alpha \in \mathfrak{h}$ by (iii) of the above theorem.

Claim 2.2. The roots α lie in the dual vector space of $\mathfrak{h}$, $\mathfrak{h}^*$.

Proof. Write $H \in \mathfrak{h}$ as $H = \rho_i H^i$ (sum over index). Then $[H, E^\alpha] = \alpha(H) E^\alpha$, with $\alpha(H) = \rho_i \alpha^i$. Because of the linearity of the Lie bracket, we see that α is a linear map from $\mathfrak{h} \to \mathbb{C}$ and hence defines a linear map in $\mathfrak{h}^*$. □

Claim 2.3. The roots α are non-degenerate.

Proof. Omitted (for now). □

The two above claims show that the set of roots Φ consists of $d - r$ distinct elements of $\mathfrak{h}^*$. Hence the E^αs are all linearly independent and there are $d - r$ of them. So we have found a basis of our Lie algebra: The **Cartan-Weyl basis**:

$$\boxed{\mathcal{B} = \{H^i \mid i = 1, 2, ..., r\} \cup \{E^\alpha \mid \alpha \in \Phi\}} \tag{2.1}$$

Because of Cartan's theorem, we know that our Killing form is non-degenerate. This fact is used extensively in proofs throughout the remainder of these notes.

Theorem 2.4. In the Cartan-Weyl basis, we have:

(i) For all $H \in \mathfrak{h}$ and $\alpha \in \Phi$, $K(H, \alpha) = 0$

(ii) For all $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$, $K(E^\alpha, E^\beta) = 0$

(iii) For all $H \in \mathfrak{h}$, there exists $H' \in \mathfrak{h}$ such that $K(H, H') \neq 0$

(iv) $\alpha \in \Phi \implies -\alpha \in \Phi$, and $K(E^\alpha, E^{-\alpha}) \neq 0$.

Proof. We have:

(i) For all $H' \in \mathfrak{h}$, observe that by (1.3) we may write: $\alpha(H')K(H, E^\alpha) = K(H, [H', E^\alpha]) = -K([H', H], E^\alpha) = 0$. Because $\alpha \neq 0$, we conclude that $K(H, E^\alpha) = 0$.

(ii) Again by (1.3), for all $H' \in \mathfrak{h}$ we have $\alpha(H') + \beta(H')K(E^\alpha, E^\beta) = 0$, so provided $\alpha + \beta \neq 0$, we have $K(E^\alpha, E^\beta) = 0$.

(iii) Assume no such H' exists. Then $K(H, H') = 0 \ \forall H' \in \mathfrak{h}$, but also by (i) $K(H, E^\alpha) = 0 \ \forall \alpha \in \Phi$. Since the H^is and E^αs form a basis, we conclude that $K(H, X) = 0 \ \forall X \in \mathfrak{g}$, which is a contradiction.

(iv) Assume not, so for $\alpha \in \Phi$ we have either $-\alpha \notin \Phi$ or $K(E^\alpha, E^{-\alpha}) = 0$. Then by (i) and (ii) we see that $K(E^\alpha, X) = 0$ for all $X \in \mathfrak{g}$, which is a contradiction.

□

From (iii) above, we see that K starts life as a non-degenerate Killing form on $\mathfrak{g}$, but yields a non-degenerate inner product on $\mathfrak{h}$ also. If we write $H = \rho_i H^i$ and $H' = \rho'_i H^i$ in components, then $K(H, H') = \rho_i \rho'_j K^{ij}$, where we write $K^{ij} = K(H^i, H^j)$. Because K is non degenerate, K^{ij} is an invertible matrix with inverse K^{-1}_{ij}, which may be used to define a non-degenerate inner product on $\mathfrak{h}^*$:

$$\boxed{(\alpha, \beta) := (K^{-1})_{ij} \alpha^i \beta^j} \tag{2.2}$$

3 Algebra in the Cartan-Weyl basis

So far, $[E^\alpha, E^\beta]$ $(\alpha, \beta \in \Phi)$ is undetermined. By use of the Jacobi identity, one has:

$$[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^j)[E^\alpha, E^\beta] \quad \text{for } \alpha, \beta \in \Phi \text{ and all } i \in \{1, ..., r\} \tag{3.1}$$

Then there are two cases to consider:

1. $\boldsymbol{\alpha + \beta \neq 0}$:
   (i) First, if $\alpha + \beta \notin \Phi$, we conclude that $[E^\alpha, E^\beta] = 0$.
   (ii) On the other hand, if $\alpha + \beta \in \Phi$, we have $[E^\alpha, E^\beta] = \mathcal{N}_{\alpha,\beta} E^{\alpha+\beta}$ for some $\mathcal{N}_{\alpha,\beta} \in \mathbb{C}$ (no sum - by non-degeneracy there is exactly one possible eigenvector with eigenvalue $\alpha + \beta$).
2. $\boldsymbol{\alpha + \beta = 0}$.

For the second case, we begin by making use of the identity (1.3):

$$K([E^\alpha, E^{-\alpha}], H) = \alpha(H) K(E^\alpha, E^{-\alpha}) \tag{3.2}$$

This motivates us to define:

$$H^\alpha := \frac{[E^\alpha, E^{-\alpha}]}{K(E^\alpha, E^{-\alpha})} \tag{3.3}$$

(The denominator is non-zero by (iv) of the previous theorem). Then $K(H^\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. Because K^{ij} is invertible, as was discussed at the end of the preceding section, we may in-fact solve this equation to find the unique solution:

$$H^\alpha = (K^{-1})_{ij} \alpha^j H^i \tag{3.4}$$

This may then be plugged into the definition (3.3) and rearranged to give us $[E^\alpha, E^{-\alpha}] = K(E^\alpha, E^{-\alpha}) H^\alpha$, with H^α given as in (3.4). One can show that $[H^\alpha, E^\beta] = (\alpha, \beta) E^\beta$. This motivates the renormalizeation:

$$\begin{aligned} e^\alpha &= \frac{\sqrt{2}}{\sqrt{(\alpha,\alpha) K(E^\alpha, E^{-\alpha})}} E^\alpha \\ h^\alpha &= \frac{2}{(\alpha,\alpha)} H^\alpha \end{aligned} \tag{3.5}$$

(Strictly, one must show $(\alpha, \alpha) \neq 0$.) Then one has the algebra in its (almost) final form:

$$\begin{aligned} [h^\alpha, h^\beta] &= 0 \\ [h^\alpha, e^\beta] &= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} e^\beta \\ [e^\alpha, e^\beta] &= \begin{cases} n_{\alpha,\beta} e^{\alpha+\beta} & \alpha + \beta \in \Phi \\ h^\alpha & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.6}$$

Remark: We say 'almost' final because it is not clear at this point that we can replace the H^i's with h^α's - indeed, are the h^α's linearly independent and do they span $\mathfrak{h}$? The answers are no and yes respectively, which means we'll have to trim down these h's to get a basis. We will get to this shortly.

4 $\mathcal{L}_{\mathbb{C}}(SU(2))$ Subalgebra and Root Strings

Claim 4.1. Let $\alpha \in \Phi$. The only other multiples $k\alpha$, for $k \in \mathbb{C}$, in Φ are $k = -1$.

Proof. Omitted (for now). □

Observe that by the above and (iv) of Theorem 2.4, for each pair $\pm\alpha \in \Phi$ we have an $\mathcal{L}_{\mathbb{C}}(SU(2))$ subalgebra of $\mathfrak{g}$, the basis being $\{h^\alpha, e^\alpha, e^{-\alpha}\}$. We call this subalgebra $sl(2)_\alpha$.

4.1 Root Strings

For $\alpha,\ \beta \in \Phi$, we define the "α string through β" to be the set of roots of the form $\beta + n\alpha$ for $n \in \mathbb{Z}$, that is:

$$S_{\alpha,\beta} = \{\gamma \in \Phi \mid \gamma = \alpha + n\beta \text{ for } n \in \mathbb{Z}\} \tag{4.1}$$

This has the corresponding vector subspace:

$$V_{\alpha,\beta} = \text{span}_{\mathbb{C}}\{e^{\gamma} \mid \gamma \in S_{\alpha,\beta}\} \tag{4.2}$$

Now consider the action of $sl(2)_\alpha$ on $V_{\alpha,\beta}$. We have:

$$\begin{aligned} [h^{\alpha}, e^{\beta+n\alpha}] &= \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n\right)e^{\beta+n\alpha} \\ [e^{\pm\alpha}, e^{\beta+n\alpha}] &\propto \begin{cases} e^{\beta+(n\pm1)\alpha} & \text{if } \beta+(n\pm1)\alpha \in \Phi \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.3}$$

So $\boldsymbol{V_{\alpha,\beta}}$ is invariant under the action of $\boldsymbol{sl(2)_\alpha}$. Therefore it is isomorphic to the representation space of some representation $\mathcal{R}$ of $sl(2)_\alpha$ (convince yourself this is true), with weight set:

$$S_{\mathcal{R}} = \{\frac{2(\alpha,\beta)}{\alpha,\alpha} + 2n \mid \beta + n\alpha \in \Phi\} \tag{4.4}$$

(This follows from the first part of (4.3)). We can go further - because all the roots of $\mathfrak{g}$ are finite in number and non-degenerate, **$\boldsymbol{\mathcal{R}}$ is finite and irreducible**.[1] Hence $\mathcal{R} = \mathcal{R}_\Lambda$ for some $\Lambda \in \mathbb{Z}_{\geq 0}$, so that:

$$S_{\mathcal{R}} = \{-\Lambda, -\Lambda+2,\ \dots\ ,\ \Lambda-2, \Lambda\} \tag{4.5}$$

Now we can compare our two expressions for $S_{\mathcal{R}}$, to conclude:

$$\exists n_{\pm} \in \mathbb{Z} \text{ such that } \begin{cases} -\Lambda &= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_- \\ \Lambda &= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_+ \end{cases} \tag{4.6}$$

If we add the expressions for $n_\pm$, we conclude:

$$\boxed{\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}} \tag{4.7}$$

The set of roots are said to form an **unbroken string** of width $n_+ + n_- + 1$, so $S_{\alpha,\beta} = \{\beta + n\alpha \mid n_- \leq n \leq n_+\}$.

We now wish to show that $(\alpha, \beta) \in \mathbb{R}$ also. To do this, we will begin by finding an expression for the Killing form in terms of our roots:

$$K^{ij} = K(H^i, H^j) = \frac{1}{\mathcal{N}}\text{Tr}(\text{ad}_{H^i} \circ \text{ad}_{H^j}) = \frac{1}{\mathcal{N}}\sum_{\delta\in\Phi}\delta^i\delta^j \tag{4.8}$$

For some (arbitrary) real and non-zero normalisation $\mathcal{N}$. The second equality follows from the fact that the trace is the sum of the eigenvalues. Defining $\alpha_i = (K^{-1})_{ij}\alpha^j$ for ease, we thus have:

$$(\alpha,\beta) = \alpha_i\beta_j K^{ij} = \frac{1}{\mathcal{N}}\sum_{\delta\in\Phi}(\alpha,\delta)(\beta,\delta) \tag{4.9}$$

If we multiply this equation through by $4/[(\alpha,\alpha)(\beta,\beta)]$, we get:

$$\frac{2}{(\beta,\beta)}\underbrace{\left(\frac{2(\alpha,\beta)}{\alpha,\alpha}\right)}_{real} = \frac{1}{\mathcal{N}}\sum_{\delta\in\Phi}\underbrace{\left(\frac{2(\alpha,\delta)}{(\alpha,\alpha)}\right)}_{real}\underbrace{\left(\frac{2(\beta,\delta)}{(\beta,\beta)}\right)}_{real} \tag{4.10}$$

We see that by using our result (4.7) and the fact $\mathcal{N}$ is real, (β,β) must also be real for all $\beta \in \Phi$, because β was arbitrary. Using (4.7) once more, we conclude:

$$\boxed{(\alpha,\beta) \in \mathbb{R} \quad \forall \alpha,\beta \in \Phi} \tag{4.11}$$

[1]For irreducibility, observe that the weights are non-degenerate and spaced by 2.

5 Real Geometry of Roots

We showed that $\alpha \in \Phi$ are elements of the dual space $\mathfrak{h}^*$. We will will investigate this line of thinking further. The first thing one would like to establish is exactly how much of $\mathfrak{h}^*$ we can get to with these roots.

Theorem 5.1. The roots span $\mathfrak{h}^*$.

Proof. Assume the roots do not span $\mathfrak{h}^*$. Then there exists $\lambda \neq 0$ in $\mathfrak{h}^*$ such that $(\lambda, \alpha) = 0$ for all $\alpha \in \Phi$. Then $h_\lambda := \lambda_i H^i$ has the property:

$$[h_\lambda, h] = 0 \ \forall h \in \mathfrak{h} \text{ and } [h_\lambda, e^\alpha] = 0 \ \forall \alpha \in \Phi \implies [h_\lambda, X] = 0 \quad \forall X \in \mathfrak{g} \tag{5.1}$$

So $\mathfrak{g}$ has a non-trivial abelian ideal $\implies$ it is not semi-simple, and the result follows by the contrapositive. □

Hence we can find a basis of r roots for $\mathfrak{h}^*$, which we shall write as $\{\alpha_{(i)} \mid i = 1, ..., r\}$.

So far, we have been working with a complex vector space. We'll define a real subspace $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}$ by:

$$\mathfrak{h}^*_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{\alpha_{(i)} \mid i = 1, ..., r\} \tag{5.2}$$

Since the $\alpha_{(i)}$ form a basis, for any root $\beta \in \Phi$ we have:

$$\beta = \sum_{i=1}^{r} \beta^i \alpha_{(i)} \implies \underbrace{(\beta, \alpha_{(j)})}_{real} = \sum_{i=1}^{r} \beta^i \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{real} \qquad j = 1, ..., r \tag{5.3}$$

For some complex coefficients β^i. We can view the equation after the implication arrow as a matrix equation, the RHS involving the invertible[2] matrix with entries $(\alpha_{(i)}, \alpha_{(j)})$. Then we multiply through by the inverse to get the solution, and deduce that in fact $\beta^i \in \mathbb{R}$, and hence:

$$\boxed{\beta \in \mathfrak{h}^*_{\mathbb{R}} \quad \forall \beta \in \Phi} \tag{5.4}$$

This tells us that our basis vectors $\alpha_{(i)}$ span $\mathfrak{h}^*_{\mathbb{R}}$. Further, we now have enough to show that our inner-product is positive definite:

$$\begin{aligned}
\lambda = \sum_{i=1}^{r} \lambda^i \alpha_{(i)} \quad , \quad \mu = \sum_{i=1}^{r} \mu^i \alpha_{(i)} \qquad \lambda^i, \mu^i \in \mathbb{R} \\
\implies (\lambda, \mu) = \sum_{i,j=1}^{r} \lambda^i \mu^j \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{real} \in \mathbb{R} \\
\text{and } (\lambda, \lambda) = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\lambda, \delta)^2 \geq 0
\end{aligned} \tag{5.5}$$

Because of the non-degeneracy of our inner product, the last line has equality iff $\lambda = 0$.

In summary, the roots $\alpha \in \Phi$ live in a **real vector space** $\mathfrak{h}^*_{\mathbb{R}} \equiv \mathbb{R}^r$, equipped with a **Euclidean inner-product**:

$$\forall \lambda, \mu \in \mathfrak{h}^*_{\mathbb{R}} :$$

$$(i) \quad (\lambda, \mu) \in \mathbb{R} \tag{5.6}$$

$$(ii) \quad (\lambda, \lambda) \geq 0 \tag{5.7}$$

$$(iii) \quad (\lambda, \lambda) = 0 \iff \lambda = 0 \tag{5.8}$$

[2] Suppose the columns of the matrix with entries $A_{ij} = (\alpha_{(i)}, \alpha_{(j)})$ are not linearly independent. Then there exists coefficients γ^i not all zero such that $(\gamma^i \alpha_{(i)}, \alpha_{(j)}) = 0$ for all $j = 1, ..., r$. Then by the non-degeneracy of the inner-product, deduce that $\gamma^i \alpha_{(i)} = 0$. But this is a contradiction because the $\alpha_{(i)}$s are linearly independent, since they form a basis. Therefore the matrix A_{ij} has linearly independent columns, and hence it is invertible.

This allows us to define **lengths** and **angles**:

$$|\alpha| = (\alpha,\alpha)^{\frac{1}{2}} \qquad \text{(length of } \alpha.\text{)} \tag{5.9}$$

$$\cos\phi = \frac{(\alpha,\beta)}{|\alpha||\beta|} \qquad \text{(angle between } \alpha \text{ and } \beta.\text{)} \tag{5.10}$$

By use of (4.7), we see that $2|\beta||\alpha|^{-1}\cos\phi \in \mathbb{Z}$. By exchanging the roles of α and β, we conclude that $4\cos^2\phi \in \mathbb{Z}$. Hence:

$$\begin{aligned} \cos\phi &= \pm\frac{\sqrt{n}}{2} \quad \text{for } n \in \{0,1,2,3,4\} \\ \implies \phi &\in \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi\right\} \end{aligned} \tag{5.11}$$

6 Simple Roots

There are finitely many roots $\alpha \in \Phi$, so we may divide them into positive and negative by drawing a hyperplane in $\mathfrak{h}^*_{\mathbb{R}}$ of our choosing.

This decomposes Φ into Φ_+ and Φ_-, the sets of positive and negative roots, $\Phi = \Phi_+ + \Phi_-$. Observe also that $\alpha \in \Phi_\pm \iff -\alpha \in \Phi_\mp$, and $\alpha,\beta \in \Phi_\pm \implies \alpha+\beta \in \Phi_\pm$.

A root is called **simple** if it is a positive root that cannot be written as the sum of two positive roots. We will denote the set of simple roots $\Phi_s = \{\delta \in \Phi_+ \mid \delta \neq \alpha+\beta \quad \forall \alpha,\beta \in \Phi_+\}$.

Theorem 6.1. (Properties of simple roots).

(i) If α and β are simple roots, then $\alpha - \beta$ is not a root.

(ii) If α and β are simple roots, then the α string through β has length:

$$l_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \tag{6.1}$$

(iii) $(\alpha,\beta) \leq 0$ for all $\alpha,\beta \in \Phi_s$ with $\alpha \neq \beta$.

(iv) Any positive root can be written as a linear combination of simple roots with positive integer coefficients:

$$\beta \in \Phi_+ \implies \beta = \sum_i c^i \alpha_{(i)} \quad , \quad \text{with } \alpha_{(i)} \in \Phi_s \text{ and } c^i \in \mathbb{N} \tag{6.2}$$

(v) Simple roots are linearly independent.

(vi) Any finite dimensional simple complex Lie algebra (as we are considering!) has simple roots of at most two different lengths.

Proof. We have:

(i) Let α and β be simple roots. Suppose $\alpha - \beta$ is a root. Then either $\alpha - \beta \in \Phi_+$ or $\beta - \alpha \in \Phi_+$. In the first case, $\alpha = \beta + (\alpha - \beta)$ is not simple, and in the second $\beta = (\beta - \alpha) + \alpha$ is not simple, both cases being a contradiction. So $\alpha - \beta$ is not a root.

(ii) If $\alpha,\beta \in \Phi_s$, then by the above $\beta - \alpha$ is not a root. Then recalling that $S_{\alpha,\beta} = \{\beta + n\alpha \in \Phi \mid n \in \mathbb{Z},\ n_- \leq n \leq n_+\}$, we see that $n_- = 0$. Then using (4.6), we have:

$$n_+ + n_- = \frac{-2(\alpha,\beta)}{(\alpha,\alpha)} \implies n_+ = \frac{-2(\alpha,\beta)}{(\alpha,\alpha)} \tag{6.3}$$

Since the length of the root string is $l_{\alpha,\beta} = n_- + n_+ + 1$, the result follows.

(iii) This is a consequence of $0 = n_- \leq n_+$ in the above, and the non-negativity of norms (5.7).

(iv) Let $\beta \in \Phi_+$. If $\beta \in \Phi_s$, we are done. Else, $\beta = \beta_1 + \beta_2$ for $\beta_1, \beta_2 \in \Phi_+$. Continue this process iteratively. It will always terminate, since $|\beta| < |\beta_1| + |\beta_2|$ and there are finitely many roots (rigorously, must also show β_1, β_2 are not multiples of another - this is hard).

(v) Consider $\lambda = \sum_{i\in\mathcal{J}} c^i \alpha_{(i)}$, with $c^i \in \mathbb{R}$, $\alpha_{(i)} \in \Phi_s$ and $\mathcal{J}$ being an index set. We split $\lambda = \lambda_- \lambda_+$, with $\lambda_\pm = \sum_{i\in\mathcal{J}_\pm} c^i \alpha_{(i)}$, $\mathcal{J}_\pm = \{i \mid i \in \mathcal{J} \text{ and } c^i \lesseqgtr 0\}$. Then:

$$(\lambda, \lambda) = (\lambda_+, \lambda_+) + (\lambda_-, \lambda_-) + 2(\lambda_-, \lambda_+) \geq 2 \sum_{i\in\mathcal{J}_+} \sum_{j\in\mathcal{J}_-} \underbrace{c^i}_{+} \underbrace{c^j}_{-} \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{-} \geq 0 \tag{6.4}$$

With equality occurring if and only if $\lambda = 0$ (by non-degeneracy). So $\lambda = 0$ if and only if $c^i = 0$ for all $i \in \mathcal{J}$, as required.

(vi) Omitted (for now).

□

Corollary 6.1.1. We have shown that the roots span $\mathfrak{h}$ in Theorem 5.1, and we may (v) above to see that the simple roots must also span $\mathfrak{h}$.

Thus, we may now assume that our basis elements $\boldsymbol{\alpha_{(i)}}$ are simple.

The inner products between our simple basis roots are encoded in the **Cartan matrix**:

$$\boxed{A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})}} \tag{6.5}$$

By (4.7), we know that $A^{ij} \in \mathbb{Z}$.

Looking back at (3.6), we see that for each simple root $\alpha_{(i)}$, $i = 1, ..., r$, we have an associated $\mathcal{L}_{\mathbb{C}}(SU(2))$ subalgebra with generators:

$$\begin{aligned} &\{h^i := h^{\alpha_{(i)}} \ , \ e^i_\pm = e^{\pm\alpha_{(i)}}\} \\ &[h^i, e^i_\pm] = \pm 2 e^i_\pm \qquad i = 1, ..., r \\ &[e^i_+, e^i_-] = h^i \qquad\quad i = 1, ..., r \end{aligned} \tag{6.6}$$

These are the generators of the **Chevalley basis** for $\mathfrak{g}$. The algebra becomes:

$$\boxed{\begin{aligned} [h^i, h^j] &= 0 \\ [h^i, e^j_\pm] &= \pm A^{ji} e^j_\pm \\ [e^i_\pm, e^j_\pm] &= \delta^{ij} h^i \end{aligned}} \tag{6.7}$$

Observe that the e^is and h^is are not always enough to span $\mathfrak{g}$. The full algebra must then be generated by repeated brackets subject to the **Chevalley-Serre relations**:

$$\boxed{(\mathrm{ad}_{e^i_\pm})^{1-A^{ji}} e^j_\pm = 0 \qquad \forall i, j \in \{1, ..., r\}} \tag{6.8}$$

(The add map is just applying $[\cdot, \cdot]$ operator many times).

Motivation of the Chevalley-Serre relations: We know that $[e^i_+, e^j_+] \propto e^{\alpha_{(i)} + \alpha_{(j)}}$ if $\alpha_{(i)} + \alpha_{(j)} \in \Phi$, and is equal to zero otherwise. Continuing this train of thought, $[e^i_+[e^i_+, e^j_+]] \propto e^{2\alpha_{(i)} + \alpha_{(j)}}$ if $2\alpha_{(i)} + \alpha_{(j)} \in \Phi$, and is equal to zero otherwise, and so on. Indeed, if $\alpha_{(j)} + n\alpha_{(i)} \in \Phi$, must have:

$$n < l_{ij} = 1 - \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(i)}, \alpha_{(i)})} = 1 - A^{ji} \tag{6.9}$$

l_{ij} being the length of the $\alpha_{(i)}$ string through $\alpha_{(j)}$.

7 Constraints on the Cartan Matrix

Claim 7.1. We have:

(i) $A^{ii} = 2$ for $i = 1, ..., r$.

(ii) $A^{ij} = 0 \iff A^{ji} = 0$.

(iii) $A^{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$

(iv) $A^{ij}A^{ji} \leq 4$ for all $i, j \in \{1, ..., r\}$, with equality if and only if $i = j$.

(v) $\det A > 0$

(vi) (For simple Lie algebras) A is irreducible.

Proof. Observe that (i), (ii) and (iii) follow immediately from our previous definitions: (i) can be seen by setting $i = j$ in (6.5), (ii) follows from the symmetry of our inner product, and (iii) is a consequence of Theorem 6.1 part (iii).

For part (iv), we first note that $A^{ij}A^{ji} = 4\cos^2\varphi_{ij}$, with φ_{ij} being the angle between the simple roots $\alpha_{(i)}$ and $\alpha_{(j)}$, and hence the first part of the result. For the if and only if part, we use the result of Claim 4.1 to see that if $4\cos^2\varphi_{ij} = 4$, then the roots must be (anti-)parallel to each other, but this means they are equal and hence $i = j$ (we cant have α and $-\alpha$, say, because these can't both be simple).

For part (v), we first prove the following lemma:

Lemma 7.2. For the Killing form K, we have $\det(K^{-1}) > 0$.

Proof. We may write arbitrary $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{R}}$ as:

$$\lambda = \sum_{i=1}^{r} \lambda^i \alpha_{(i)} \quad , \quad \mu = \sum_{i=1}^{r} \mu^i \alpha_{(i)} \qquad \lambda^i, \mu^i \in \mathbb{R}\ \forall i$$

Then$(\lambda, \mu) = (K^{-1})_{ij}\lambda^i\mu^j$, with (as usual) $(K^{-1})_{ij} = (\alpha_{(i)}, \alpha_{(j)})$. Since K^{-1} is real and symmetric, we may diagonalise it:

$$\exists \text{ orthogonal } Q \text{ such that } \mathrm{diag}(\rho_1, ..., \rho_r) = Q^T K^{-1} Q$$

With $\rho_1, ..., \rho_r$ being the real eigenvectors of K^{-1}. In basis, these can be written as:

$$\boldsymbol{v}_{\rho_k} = \sum_{i=1}^{r} v^i_{\rho_k} \alpha_{(i)} \qquad v^i_{\rho_k} \in \mathbb{R}\ \forall i, k \in \{1, ..., r\}$$

$$\text{so that} \qquad \sum_{j=1}^{r} (K^{-1})_{ij} v^j_{\rho_k} = \sum_{j=1}^{r} \rho^k \delta_{ij} v^j_{\rho_k} \qquad \text{(by the eigenvalue property.)}$$

$$\implies (\boldsymbol{v}_{\rho_k}, \boldsymbol{v}_{\rho_k}) = \sum_{i,j} \rho_k \delta_{ij} v^j_{\rho_k} v^i_{\rho_k}$$

$$\implies |\boldsymbol{v}_{\rho_k}|^2 = \rho_k \sum_{i=1}^{r} (v^i_{\rho_k})^2$$

$$\implies \rho_k > 0 \qquad \forall k$$

(Note that the norm squared being greater than zero in the penultimate line is valid since we showed $(\lambda, \lambda) \geq 0\ \forall \lambda \in \mathfrak{h}^*_{\mathbb{R}}$ with equality iff $\lambda = 0$ in (5.7) and (5.8).) □

Now, we may write the Cartan matrix as $A^{ij} = (K^{-1})^{ik} D^j_k$ with $D^i_k = 2\delta^i_k[(\alpha_{(l)}, \alpha_{(l)}]^{-1}$. Then $\det(A) = \det(K^{-1}) \det D > 0$ and we are done.

(vi) is omitted (for now). □

8 Dynkin Diagrams and the Classification

8.1 Dynkin Diagrams

The information in the Cartan matrix can be represented in a diagrammatic form:

- Draw a node $\circ$ for each simple root $\alpha_{(i)} \in \Phi_s$
- Join the nodes corresponding to simple roots $\alpha_{(i)}$ and $\alpha_{(j)}$ by:

$$\max\left(|A^{ij}|, |A^{ji}|\right) \tag{8.1}$$

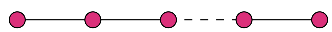
  Geometrically, this is the ratio of the lengths of the two roots (largest to smallest).

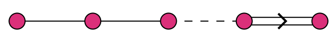
- If roots have different lengths, draw an arrow pointing from the node corresponding to the larger root towards the shorter one.

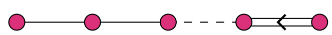
Examples of these diagrams can be found in the next subsection.

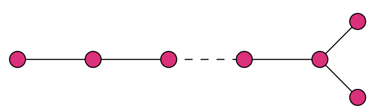
8.2 The Cartan Classification

Here it is (finally). The classification of Cartan matrices/Dynkin diagrams satisfying the constraints given in Claim 7.1[3] is a purely combinatoric problem, solved by Cartan in 1894. There are **four infinite families**, labelled by $n \in \mathbb{N}$ ($n = \text{rank}(\mathfrak{g})$) given by:

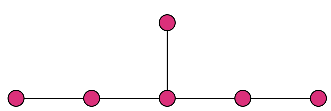
A_n : 

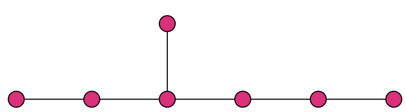
B_n : 

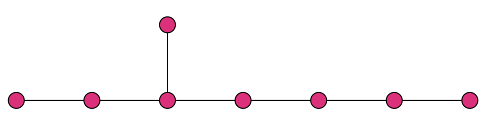
C_n : 

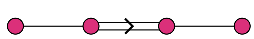
D_n : 

With **five exceptional cases:**

E_6 : 

E_7 : 

E_8 : 

F_4 : 

G_2 : 

[3](Minus constraint (iv), if you like, since this is implied by the others)