The Cartan Classification

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This set of notes serves to supplement the material taught by N. Dorey in the Part III *Symmetries, Fields and Particles* course. Specifically, we classify all finite dimensional complex Lie algebras (Cartan, 1984).

Throughout these notes, $\mathfrak{g}$ is assumed to be a simple complex Lie algebra.

1 Preliminary Definitions

Recall that for each $X \in \mathfrak{g}$, we may define the **adjoint map**:

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g} \quad , \quad Y \to \mathrm{ad}_X(Y) = [X, Y] \tag{1.1}$$

Observe that ad_X is a linear map between vector spaces of dimension $d = \dim(\mathfrak{g})$ and we can therefore think of it as a $d \times d$ matrix. This defines the **adjoint representation** of $\mathfrak{g}$ (one needs to prove this, but it is easy and not the focus of these notes).

The **Killing Form** of a Lie algebra is the inner product defined by:

$$K : \mathfrak{g} \times \mathfrak{g} \to F \quad , \quad K(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \tag{1.2}$$

Where $F = \mathbb{R}$ or $\mathbb{C}$ (in the case studied in these notes, it is the latter). The Killing form enjoys **invariance under adjoint action**, that is:

$$K([Z, X], Y) + K(X, [Z, Y]) = 0 \qquad \forall X, Y, Z \in \mathfrak{g} \tag{1.3}$$

Theorem 1.1. (**Cartan's Theorem**) $\mathfrak{g}$ is semi-simple if and only if the Killing Form is non-degenerate. (A semi-simple Lie algebra is one with no non-trivial abelian ideals. Thus all simple Lie algebras are also semi-simple.)

Proof. The "if" direction is omitted. For the "only if" (forward) direction (outline only), assume for contradiction that $\mathfrak{g}$ is not semi-simple, and so has a non-trivial abelian ideal $\mathfrak{j}$. Then show that $K(X, Y) = 0$ for all $X \in \mathfrak{j},\ Y \in \mathfrak{g}$, which is a contradiction. Details are in handwritten lecture notes. □

2 The Cartan-Weyl basis

We say that $X \in \mathfrak{g}$ is **ad-diagonalisable** if the matrix corresponding to the map ad_X is diagonalisable.
A **Cartan subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra containing only ad-diagonalisable elements.

Theorem 2.1. We have:

(i) $H \in \mathfrak{h} \implies H$ is ad-diagonalisable.

(ii) $H, H' \in \mathfrak{h} \implies [H, H'] = 0$.

(iii) If $X \in \mathfrak{g}$ and $[X, H] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.

Proof. The first two are trivial and follow immediately from the definition. The third is omitted (for now). □

All Cartan subalgebras of $\mathfrak{g}$ have the same dimension, and their dimension is always at least one (proof omitted). Thus we may define the **rank** of the Cartan subalgebra to be $r = \dim(\mathfrak{h}) \in \mathbb{N}$.

We may choose a basis $\{H^i \mid i = 1, \ldots, r\}$ of $\mathfrak{h}$. Since each H^i provides a map $\mathrm{ad}_{H^i} : \mathfrak{g} \to \mathfrak{g}$ and they commute with one another, we see that they are simultaneously diagonalisable and that **$\mathfrak{g}$ is spanned by the simultaneous eigenvectors of $\mathbf{ad}_{H^i}$, $i = 1, ..., r$.** We see that eigenvectors have:

1. **Zero eigenvalues:** Observe that H^j is an eigenvector with eigenvalue zero of H^i: $\mathrm{ad}_{H^i}(H^j) = [H^i, H^j] = 0$ for all $i, j \in \{1, ..., r\}$.
2. **Non-zero eigenvalues:** Since $r \leq d$, we will in the general case require eigenvectors not in $\mathfrak{h}$. We write these as E^α, with $\alpha \in \mathbb{R}^r$ indexing the eigenvector, and $\mathrm{ad}_{H^i}(E^\alpha) = \alpha^i E^\alpha$. Observe that α^i are not all zero, since otherwise we would have $E^\alpha \in \mathfrak{h}$ by (iii) of the above theorem.

Claim 2.2. The roots α lie in the dual vector space of $\mathfrak{h}$, $\mathfrak{h}^*$.

Proof. Write $H \in \mathfrak{h}$ as $H = \rho_i H^i$ (sum over index). Then $[H, E^\alpha] = \alpha(H) E^\alpha$, with $\alpha(H) = \rho_i \alpha^i$. Because of the linearity of the Lie bracket, we see that α is a linear map from $\mathfrak{h} \to \mathbb{C}$ and hence defines a linear map in $\mathfrak{h}^*$. □

Claim 2.3. The roots α are non-degenerate.

Proof. Omitted (for now). □

The two above claims show that the set of roots Φ consists of $d - r$ distinct elements of $\mathfrak{h}^*$. Hence the E^αs are all linearly independent and there are $d - r$ of them. So we have found a basis of our Lie algebra: The **Cartan-Weyl basis**:

$$\boxed{\mathcal{B} = \{H^i \mid i = 1, 2, ..., r\} \cup \{E^\alpha \mid \alpha \in \Phi\}} \tag{2.1}$$

Because of Cartan's theorem, we know that our Killing form is non-degenerate. This fact is used extensively in proofs throughout the remainder of these notes.

Theorem 2.4. In the Cartan-Weyl basis, we have:

(i) For all $H \in \mathfrak{h}$ and $\alpha \in \Phi$, $K(H, \alpha) = 0$

(ii) For all $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$, $K(E^\alpha, E^\beta) = 0$

(iii) For all $H \in \mathfrak{h}$, there exists $H' \in \mathfrak{h}$ such that $K(H, H') \neq 0$

(iv) $\alpha \in \Phi \implies -\alpha \in \Phi$, and $K(E^\alpha, E^{-\alpha}) \neq 0$.

Proof. We have:

(i) For all $H' \in \mathfrak{h}$, observe that by (1.3) we may write: $\alpha(H')K(H, E^\alpha) = K(H, [H', E^\alpha]) = -K([H', H], E^\alpha) = 0$. Because $\alpha \neq 0$, we conclude that $K(H, E^\alpha) = 0$.

(ii) Again by (1.3), for all $H' \in \mathfrak{h}$ we have $\alpha(H') + \beta(H')K(E^\alpha, E^\beta) = 0$, so provided $\alpha + \beta \neq 0$, we have $K(E^\alpha, E^\beta) = 0$.

(iii) Assume no such H' exists. Then $K(H, H') = 0 \ \forall H' \in \mathfrak{h}$, but also by (i) $K(H, E^\alpha) = 0 \ \forall \alpha \in \Phi$. Since the H^is and E^αs form a basis, we conclude that $K(H, X) = 0 \ \forall X \in \mathfrak{g}$, which is a contradiction.

(iv) Assume not, so for $\alpha \in \Phi$ we have either $-\alpha \notin \Phi$ or $K(E^\alpha, E^{-\alpha}) = 0$. Then by (i) and (ii) we see that $K(E^\alpha, X) = 0$ for all $X \in \mathfrak{g}$, which is a contradiction.

□

From (iii) above, we see that K starts life as a non-degenerate Killing form on $\mathfrak{g}$, but yields a non-degenerate inner product on $\mathfrak{h}$ also. If we write $H = \rho_i H^i$ and $H' = \rho'_i H^i$ in components, then $K(H, H') = \rho_i \rho'_j K^{ij}$, where we write $K^{ij} = K(H^i, H^j)$. Because K is non degenerate, K^{ij} is an invertible matrix with inverse K^{-1}_{ij}, which may be used to define a non-degenerate inner product on $\mathfrak{h}^*$:

$$\boxed{(\alpha, \beta) := (K^{-1})_{ij} \alpha^i \beta^j} \tag{2.2}$$

3 Algebra in the Cartan-Weyl basis

So far, $[E^\alpha, E^\beta]$ $(\alpha, \beta \in \Phi)$ is undetermined. By use of the Jacobi identity, one has:

$$[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^j)[E^\alpha, E^\beta] \quad \text{for } \alpha, \beta \in \Phi \text{ and all } i \in \{1, ..., r\} \tag{3.1}$$

Then there are two cases to consider:

1. $\boldsymbol{\alpha + \beta \neq 0}$:
   - (i) First, if $\alpha + \beta \notin \Phi$, we conclude that $[E^\alpha, E^\beta] = 0$.
   - (ii) On the other hand, if $\alpha + \beta \in \Phi$, we have $[E^\alpha, E^\beta] = \mathcal{N}_{\alpha,\beta} E^{\alpha+\beta}$ for some $\mathcal{N}_{\alpha,\beta} \in \mathbb{C}$ (no sum - by non-degeneracy there is exactly one possible eigenvector with eigenvalue $\alpha + \beta$).
2. $\boldsymbol{\alpha + \beta = 0}$.

For the second case, we begin by making use of the identity (1.3):

$$K([E^\alpha, E^{-\alpha}], H) = \alpha(H) K(E^\alpha, E^{-\alpha}) \tag{3.2}$$

This motivates us to define:

$$H^\alpha := \frac{[E^\alpha, E^{-\alpha}]}{K(E^\alpha, E^{-\alpha})} \tag{3.3}$$

(The denominator is non-zero by (iv) of the previous theorem). Then $K(H^\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. Because K^{ij} is invertible, as was discussed at the end of the preceding section, we may in-fact solve this equation to find the unique solution:

$$H^\alpha = (K^{-1})_{ij} \alpha^j H^i \tag{3.4}$$

This may then be plugged into the definition (3.3) and rearranged to give us $[E^\alpha, E^{-\alpha}] = K(E^\alpha, E^{-\alpha}) H^\alpha$, with H^α given as in (3.4). One can show that $[H^\alpha, E^\beta] = (\alpha, \beta) E^\beta$. This motivates the renormalizeation:

$$\begin{aligned} e^\alpha &= \frac{\sqrt{2}}{\sqrt{(\alpha,\alpha) K(E^\alpha, E^{-\alpha})}} E^\alpha \\ h^\alpha &= \frac{2}{(\alpha,\alpha)} H^\alpha \end{aligned} \tag{3.5}$$

(Strictly, one must show $(\alpha, \alpha) \neq 0$.) Then one has the algebra in its (almost) final form:

$$\begin{aligned} [h^\alpha, h^\beta] &= 0 \\ [h^\alpha, e^\beta] &= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} e^\beta \\ [e^\alpha, e^\beta] &= \begin{cases} n_{\alpha,\beta} e^{\alpha+\beta} & \alpha + \beta \in \Phi \\ h^\alpha & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.6}$$

Remark: We say 'almost' final because it is not clear at this point that we can replace the H^i's with h^α's - indeed, are the h^α's linearly independent and do they span $\mathfrak{h}$? The answers are no and yes respectively, which means we'll have to trim down these h's to get a basis. We will get to this shortly.

4 $\mathcal{L}_\mathbb{C}(SU(2))$ Subalgebra and Root Strings

Observe that by (iv) of (2.4), for each pair $\pm\alpha \in \Phi$ we have an $\mathcal{L}_\mathbb{C}(SU(2))$ subalgebra of $\mathfrak{g}$, the basis being $\{h^\alpha, e^\alpha, e^{-\alpha}\}$. We call this subalgebra $sl(2)_\alpha$.

4.1 Root Strings

For $\alpha,\ \beta \in \Phi$, we define the "α string through β" to be the set of roots of the form $\beta + n\alpha$ for $n \in \mathbb{Z}$, that is:

$$S_{\alpha,\beta} = \{\gamma \in \Phi \mid \gamma = \alpha + n\beta \text{ for } n \in \mathbb{Z}\} \tag{4.1}$$

This has the corresponding vector subspace:

$$V_{\alpha,\beta} = \text{span}_{\mathbb{C}}\{e^{\gamma} \mid \gamma \in S_{\alpha,\beta}\} \tag{4.2}$$

Now consider the action of $sl(2)_\alpha$ on $V_{\alpha,\beta}$. We have:

$$\begin{aligned} [h^\alpha, e^{\beta+n\alpha}] &= \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n\right) e^{\beta+n\alpha} \\ [e^{\pm\alpha}, e^{\beta+n\alpha}] &\propto \begin{cases} e^{\beta+(n\pm1)\alpha} & \text{if } \beta + (n \pm 1)\alpha \in \Phi \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.3}$$

So $V_{\alpha,\beta}$ is invariant under the action of $sl(2)_\alpha$. Therefore it is isomorphic to the representation space of some representation $\mathcal{R}$ of $sl(2)_\alpha$ (convince yourself this is true), with weight set:

$$S_{\mathcal{R}} = \{\frac{2(\alpha,\beta)}{\alpha,\alpha} + 2n \mid \beta + n\alpha \in \Phi\} \tag{4.4}$$

(This follows from the first part of (4.3)). We can go further - because all the roots of $\mathfrak{g}$ are finite in number and non-degenerate, **$\mathcal{R}$ is finite and irreducible**.[1] Hence $\mathcal{R} = \mathcal{R}_\Lambda$ for some $\Lambda \in \mathbb{Z}_{\geq 0}$, so that:

$$S_{\mathcal{R}} = \{-\Lambda, -\Lambda + 2,\ \dots\ ,\ \Lambda - 2, \Lambda\} \tag{4.5}$$

Now we can compare our two expressions for $S_{\mathcal{R}}$, to conclude:

$$\exists n_\pm \in \mathbb{Z} \text{ such that } \begin{cases} -\Lambda &= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_- \\ \Lambda &= \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_+ \end{cases} \tag{4.6}$$

If we add the expressions for $n_\pm$, we conclude:

$$\boxed{\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}} \tag{4.7}$$

The set of roots are said to form an **unbroken string** of width $n_+ + n_- + 1$, so $S_{\alpha,\beta} = \{\beta + n\alpha \mid n_- \leq n \leq n_+\}$.

We now wish to show that $(\alpha,\beta) \in \mathbb{R}$ also. To do this, we will begin by finding an expression for the Killing form in terms of our roots:

$$K^{ij} = K(H^i, H^j) = \frac{1}{\mathcal{N}} \text{Tr}(\text{ad}_{H^i} \circ \text{ad}_{H^j}) = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} \delta^i \delta^j \tag{4.8}$$

For some (arbitrary) real and non-zero normalisation $\mathcal{N}$. The second equality follows from the fact that the trace is the sum of the eigenvalues. Defining $\alpha_i = (K^{-1})_{ij}\alpha^j$ for ease, we thus have:

$$(\alpha,\beta) = \alpha_i \beta_j K^{ij} = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\alpha,\delta)(\beta,\delta) \tag{4.9}$$

If we multiply this equation through by $4/[(\alpha,\alpha)(\beta,\beta)]$, we get:

$$\frac{2}{(\beta,\beta)} \underbrace{\left(\frac{2(\alpha,\beta)}{\alpha,\alpha}\right)}_{real} = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} \underbrace{\left(\frac{2(\alpha,\delta)}{(\alpha,\alpha)}\right)}_{real} \underbrace{\left(\frac{2(\beta,\delta)}{(\beta,\beta)}\right)}_{real} \tag{4.10}$$

We see that by using our result (4.7) and the fact $\mathcal{N}$ is real, (β,β) must also be real for all $\beta \in \Phi$, because β was arbitrary. Using (4.7) once more, we conclude:

$$\boxed{(\alpha,\beta) \in \mathbb{R} \quad \forall \alpha,\beta \in \Phi} \tag{4.11}$$

[1] For irreducibility, observe that the weights are non-degenerate and spaced by 2.

5 Real Geometry of Roots

We showed that $\alpha \in \Phi$ are elements of the dual space $\mathfrak{h}^*$. We will will investigate this line of thinking further. The first thing one would like to establish is exactly how much of $\mathfrak{h}^*$ we can get to with these roots.

Theorem 5.1. The roots span $\mathfrak{h}^*$.

Proof. Assume the roots do not span $\mathfrak{h}^*$. Then there exists $\lambda \neq 0$ in $\mathfrak{h}^*$ such that $(\lambda, \alpha) = 0$ for all $\alpha \in \Phi$. Then $h_\lambda := \lambda_i H^i$ has the property:

$$[h_\lambda, h] = 0 \ \forall h \in \mathfrak{h} \text{ and } [h_\lambda, e^\alpha] = 0 \ \forall \alpha \in \Phi \implies [h_\lambda, X] = 0 \quad \forall X \in \mathfrak{g} \tag{5.1}$$

So $\mathfrak{g}$ has a non-trivial abelian ideal $\implies$ it is not semi-simple, and the result follows by the contrapositive. □

Hence we can find a basis of r roots for $\mathfrak{h}^*$, which we shall write as $\{\alpha_{(i)} \mid i = 1, ..., r\}$.

So far, we have been working with a complex vector space. We'll define a real subspace $\mathfrak{h}^*_\mathbb{R} \subset \mathfrak{h}$ by:

$$\mathfrak{h}^*_\mathbb{R} = \text{span}_\mathbb{R}\{\alpha_{(i)} \mid i = 1, ..., r\} \tag{5.2}$$

Since the $\alpha_{(i)}$ form a basis, for any root $\beta \in \Phi$ we have:

$$\beta = \sum_{i=1}^r \beta^i \alpha_{(i)} \implies \underbrace{(\beta, \alpha_{(j)})}_{real} = \sum_{i=1}^r \beta^i \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{real} \qquad j = 1, ..., r \tag{5.3}$$

For some complex coefficients β^i. We can view the equation after the implication arrow as a matrix equation, the RHS involving the invertible[2] matrix with entries $(\alpha_{(i)}, \alpha_{(j)})$. Then we multiply through by the inverse to get the solution, and deduce that in fact $\beta^i \in \mathbb{R}$, and hence:

$$\boxed{\beta \in \mathfrak{h}^*_\mathbb{R} \quad \forall \beta \in \Phi} \tag{5.4}$$

This tells us that our basis vectors $\alpha_{(i)}$ span $\mathfrak{h}^*_\mathbb{R}$. Further, we now have enough to show that our inner-product is positive definite:

$$\begin{aligned}
\lambda = \sum_{i=1}^r \lambda^i \alpha_{(i)} \quad , \quad \mu = \sum_{i=1}^r \mu^i \alpha_{(i)} \qquad \lambda^i, \mu^i \in \mathbb{R} \\
\implies (\lambda, \mu) = \sum_{i,j=1}^r \lambda^i \mu^j \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{real} \in \mathbb{R} \\
\text{and } (\lambda, \lambda) = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\lambda, \delta)^2 \geq 0
\end{aligned} \tag{5.5}$$

Because of the non-degeneracy of our inner product, the last line has equality iff $\lambda = 0$.

In summary, the roots $\alpha \in \Phi$ live in a **real vector space** $\mathfrak{h}^*_\mathbb{R} \equiv \mathbb{R}^r$, equipped with a **Euclidean inner-product**:

$$\boxed{\begin{aligned}
&\forall \lambda, \mu \in \mathfrak{h}^*_\mathbb{R} : \\
&\quad (i) \quad (\lambda, \mu) \in \mathbb{R} && (5.6) \\
&\quad (ii) \quad (\lambda, \lambda) \geq 0 && (5.7) \\
&\quad (iii) \quad (\lambda, \lambda) = 0 \iff \lambda = 0 && (5.8)
\end{aligned}}$$

[2]Suppose the columns of the matrix with entries $A_{ij} = (\alpha_{(i)}, \alpha_{(j)})$ are not linearly independent. Then there exists coefficients γ^i not all zero such that $(\gamma^i \alpha_{(i)}, \alpha_{(j)}) = 0$ for all $j = 1, ..., r$. Then by the non-degeneracy of the inner-product, deduce that $\gamma^i \alpha_{(i)} = 0$. But this is a contradiction because the $\alpha_{(i)}$s are linearly independent, since they form a basis. Therefore the matrix A_{ij} has linearly independent columns, and hence it is invertible.

This allows us to define **lengths** and **angles**:

$$|\alpha| = (\alpha,\alpha)^{\frac{1}{2}} \qquad \text{(length of } \alpha.) \tag{5.9}$$

$$\cos\phi = \frac{(\alpha,\beta)}{|\alpha||\beta|} \qquad \text{(angle between } \alpha \text{ and } \beta.) \tag{5.10}$$

By use of (4.7), we see that $2|\beta||\alpha|^{-1}\cos\phi \in \mathbb{Z}$. By exchanging the roles of α and β, we conclude that $4\cos^2\phi \in \mathbb{Z}$. Hence:

$$\begin{aligned} \cos\phi &= \pm\frac{\sqrt{n}}{2} \quad \text{for } n \in \{0,1,2,3,4\} \\ \implies \phi &\in \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi\right\} \end{aligned} \tag{5.11}$$

6 Simple Roots

There are finitely many roots $\alpha \in \Phi$, so we may divide them into positive and negative by drawing a hyperplane in $\mathfrak{h}^*_{\mathbb{R}}$ of our choosing.

This decomposes Φ into Φ_+ and Φ_-, the sets of positive and negative roots, $\Phi = \Phi_+ + \Phi_-$. Observe also that $\alpha \in \Phi_\pm \iff -\alpha \in \Phi_\mp$, and $\alpha, \beta \in \Phi_\pm \implies \alpha + \beta \in \Phi_\pm$.

A root is called **simple** if it is a positive root that cannot be written as the sum of two positive roots. We will denote the set of simple roots $\Phi_s = \{\delta \in \Phi_+ \mid \delta \neq \alpha + \beta \quad \forall \alpha, \beta \in \Phi_+\}$.

Theorem 6.1. (Properties of simple roots).

(i) If α and β are simple roots, then $\alpha - \beta$ is not a root.

(ii) If α and β are simple roots, then the α string through β has length:

$$l_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \tag{6.1}$$

(iii) $(\alpha,\beta) \leq 0$ for all $\alpha, \beta \in \Phi_s$ with $\alpha \neq \beta$.

(iv) Any positive root can be written as a linear combination of simple roots with positive integer coefficients:

$$\beta \in \Phi_+ \implies \beta = \sum_i c^i \alpha_{(i)} \quad , \quad \text{with } \alpha_{(i)} \in \Phi_s \text{ and } c^i \in \mathbb{N} \tag{6.2}$$

(v) Simple roots are linearly independent.

Proof. We have:

(i) Let α and β be simple roots. Suppose $\alpha - \beta$ is a root. Then either $\alpha - \beta \in \Phi_+$ or $\beta - \alpha \in \Phi_+$. In the first case, $\alpha = \beta + (\alpha - \beta)$ is not simple, and in the second $\beta = (\beta - \alpha) + \alpha$ is not simple, both cases being a contradiction. So $\alpha - \beta$ is not a root.

(ii) If $\alpha, \beta \in \Phi_s$, then by the above $\beta - \alpha$ is not a root. Then recalling that $S_{\alpha,\beta} = \{\beta + n\alpha \in \Phi \mid n \in \mathbb{Z},\ n_- \leq n \leq n_+\}$, we see that $n_- = 0$. Then using (4.6), we have:

$$n_+ + n_- = \frac{-2(\alpha,\beta)}{(\alpha,\alpha)} \implies n_+ = \frac{-2(\alpha,\beta)}{(\alpha,\alpha)} \tag{6.3}$$

Since the length of the root string is $l_{\alpha,\beta} = n_- + n_+ + 1$, the result follows.

(iii) This is a consequence of $0 = n_- \leq n_+$ in the above, and the non-negativity of norms (5.7).

(iv) Let $\beta \in \Phi_+$. If $\beta \in \Phi_s$, we are done. Else, $\beta = \beta_1 + \beta_2$ for $\beta_1, \beta_2 \in \Phi_+$. Continue this process iteratively. It will always terminate, since $|\beta| < |\beta_1| + |\beta_2|$ and there are finitely many roots (rigorously, must also show β_1, β_2 are not multiples of another - this is hard).

(v) Consider $\lambda = \sum_{i \in \mathcal{J}} c^i \alpha_{(i)}$, with $c^i \in \mathbb{R}$, $\alpha_{(i)} \in \Phi_s$ and $\mathcal{J}$ being an index set. We split $\lambda = \lambda_- \lambda_+$, with $\lambda_\pm = \sum_{i \in \mathcal{J}_\pm} c^i \alpha_{(i)}$, $\mathcal{J}_\pm = \{i \mid i \in \mathcal{J} \text{ and } c^i \lesseqgtr 0\}$. Then:

$$(\lambda, \lambda) = (\lambda_+, \lambda_+) + (\lambda_-, \lambda_-) + 2(\lambda_-, \lambda_+) \geq 2 \sum_{i \in \mathcal{J}_+} \sum_{j \in \mathcal{J}_-} \underbrace{c^i}_{+} \underbrace{c^j}_{-} \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{-} \geq 0 \tag{6.4}$$

With equality occurring if and only if $\lambda = 0$ (by non-degeneracy). So $\lambda = 0$ if and only if $c^i = 0$ for all $i \in \mathcal{J}$, as required.

□

Corollary 6.1.1. We have shown that the roots span $\mathfrak{h}$ in Theorem 5.1, and we may (v) above to see that the simple roots must also span $\mathfrak{h}$.

Thus, we may now assume that our basis elements $\boldsymbol{\alpha_{(i)}}$ are simple.

The inner products between our simple basis roots are encoded in the **Cartan matrix**:

$$\boxed{A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})}} \tag{6.5}$$

By (4.7), we know that $A^{ij} \in \mathbb{Z}$.

Looking back at (3.6), we see that for each simple root $\alpha_{(i)}$, $i = 1, ..., r$, we have an associated $\mathcal{L}_\mathbb{C}(SU(2))$ subalgebra with generators:

$$\begin{aligned} &\{h^i := h^{\alpha_{(i)}} \ , \ e^i_\pm = e^{\pm\alpha_{(i)}}\} \\ &[h^i, e^i_\pm] = \pm 2 e^i_\pm \qquad i = 1, ..., r \\ &[e^i_+, e^i_-] = h^i \qquad\quad i = 1, ..., r \end{aligned} \tag{6.6}$$

These are the generators of the **Chevalley basis** for $\mathfrak{g}$. The algebra becomes:

$$\boxed{\begin{aligned} [h^i, h^j] &= 0 \\ [h^i, e^j_\pm] &= \pm A^{ji} e^j_\pm \\ [e^i_\pm, e^j_\pm] &= \delta^{ij} h^i \end{aligned}} \tag{6.7}$$

Observe that the e^is and h^is are not always enough to span $\mathfrak{g}$. The full algebra must then be generated by repeated brackets subject to the **Chevalley-Serre relations**:

$$\boxed{(\mathrm{ad}_{e^i_\pm})^{1-A^{ji}} e^j_\pm = 0 \qquad \forall i, j \in \{1, ..., r\}} \tag{6.8}$$

(The add map is just applying $[\cdot, \cdot]$ operator many times).

Motivation of the Chevalley-Serre relations: We know that $[e^i_+, e^j_+] \propto e^{\alpha_{(i)} + \alpha_{(j)}}$ if $\alpha_{(i)} + \alpha_{(j)} \in \Phi$, and is equal to zero otherwise. Continuing this train of thought, $[e^i_+[e^i_+, e^j_+]] \propto e^{2\alpha_{(i)} + \alpha_{(j)}}$ if $2\alpha_{(i)} + \alpha_{(j)} \in \Phi$, and is equal to zero otherwise, and so on. Indeed, if $\alpha_{(j)} + n\alpha_{(i)} \in \Phi$, must have:

$$n < l_{ij} = 1 - \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(i)}, \alpha_{(i)})} = 1 - A^{ji} \tag{6.9}$$

l_{ij} being the length of the $\alpha_{(i)}$ string through $\alpha_{(j)}$.

7 Constraints on the Cartan Matrix

Claim 7.1. We have:

(i) $A^{ii} = 2$ for $i = 1, ..., r$.

(ii) $A^{ij} = 0 \iff A^{ji} = 0$.

(iii) $A^{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$

(iv) $\det A > 0$

(v) (For simple Lie algebras) A is irreducible.

Proof. Observe that (i), (ii) and (iii) follow immediately from our previous definitions: (i) can be seen by setting $i = j$ in (6.5), (ii) follows from the symmetry of our inner product, and (iii) is a consequence of Theorem 6.1 part (iii). $\square$