

The Cartan Classification

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This set of notes serves to supplement the material taught by N. Dorey in the Part III *Symmetries, Fields and Particles* course. Specifically, we classify all finite dimensional complex Lie algebras (Cartan, 1984).

Throughout these notes, \mathfrak{g} is assumed to be a simple complex Lie algebra.

1 Preliminary Definitions

Recall that for each $X \in \mathfrak{g}$, we may define the **adjoint map**:

$$\mathrm{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g} \quad , \quad Y \mapsto \mathrm{ad}_X(Y) = [X, Y] \quad (1.1)$$

Observe that ad_X is a linear map between vector spaces of dimension $d = \dim(\mathfrak{g})$ and we can therefore think of it as a $d \times d$ matrix. This defines the **adjoint representation** of \mathfrak{g} (one needs to prove this, but it is easy and not the focus of these notes).

The **Killing Form** of a Lie algebra is the inner product defined by:

$$K : \mathfrak{g} \times \mathfrak{g} \rightarrow F \quad , \quad K(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \quad (1.2)$$

Where $F = \mathbb{R}$ or \mathbb{C} (in the case studied in these notes, it is the latter). The Killing form enjoys **invariance under adjoint action**, that is:

$$K([Z, X], Y) + K(X, [Z, Y]) = 0 \quad \forall X, Y, Z \in \mathfrak{g} \quad (1.3)$$

Theorem 1.1. (Cartan's Theorem) \mathfrak{g} is semi-simple if and only if the Killing Form is non-degenerate. (A semi-simple Lie algebra is one with no non-trivial abelian ideals. Thus all simple Lie algebras are also semi-simple.)

Proof. The "if" direction is omitted. For the "only if" (forward) direction (outline only), assume for contradiction that \mathfrak{g} is not semi-simple, and so has a non-trivial abelian ideal \mathfrak{j} . Then show that $K(X, Y) = 0$ for all $X \in \mathfrak{j}$, $Y \in \mathfrak{g}$, which is a contradiction. Details are in handwritten lecture notes. \square

2 The Cartan-Weyl basis

We say that $X \in \mathfrak{g}$ is **ad-diagonalisable** if the matrix corresponding to the map ad_X is diagonalisable. A **Cartan subalgebra** \mathfrak{h} of \mathfrak{g} is a maximal abelian subalgebra containing only ad-diagonalisable elements.

Theorem 2.1. We have:

- (i) $H \in \mathfrak{h} \implies H$ is ad-diagonalisable.
- (ii) $H, H' \in \mathfrak{h} \implies [H, H'] = 0$.
- (iii) If $X \in \mathfrak{g}$ and $[X, H] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.

Proof. The first two are trivial and follow immediately from the definition. The third is omitted (for now). \square

All Cartan subalgebras of \mathfrak{g} have the same dimension, and their dimension is always at least one (proof omitted). Thus we may define the **rank** of the Cartan subalgebra to be $r = \dim(\mathfrak{h}) \in \mathbb{N}$.

We may choose a basis $\{H^i \mid i = 1, \dots, r\}$ of \mathfrak{h} . Since each H^i provides a map $\text{ad}_{H^i} : \mathfrak{g} \rightarrow \mathfrak{g}$ and they commute with one another, we see that they are simultaneously diagonalisable and that \mathfrak{g} is **spanned by the simultaneous eigenvectors of ad_{H^i} , $i = 1, \dots, r$** . We see that eigenvectors have:

1. **Zero eigenvalues:** Observe that H^j is an eigenvector with eigenvalue zero of H^i : $\text{ad}_{H^i}(H^j) = [H^i, H^j] = 0$ for all $i, j \in \{1, \dots, r\}$.
2. **Non-zero eigenvalues:** Since $r \leq d$, we will in the general case require eigenvectors not in \mathfrak{h} . We write these as E^α , with $\alpha \in \mathbb{R}^r$ indexing the eigenvector, and $\text{ad}_{H^i}(E^\alpha) = \alpha^i E^\alpha$. Observe that α^i are not all zero, since otherwise we would have $E^\alpha \in \mathfrak{h}$ by (iii) of the above theorem.

Claim 2.2. The roots α lie in the dual vector space of \mathfrak{h} , \mathfrak{h}^* .

Proof. Write $H \in \mathfrak{h}$ as $H = \rho_i H^i$ (sum over index). Then $[H, E^\alpha] = \alpha(H)E^\alpha$, with $\alpha(H) = \rho_i \alpha^i$. Because of the linearity of the Lie bracket, we see that α is a linear map from $\mathfrak{h} \rightarrow \mathbb{C}$ and hence defines a linear map in \mathfrak{h}^* . \square

Claim 2.3. The roots α are non-degenerate.

Proof. Omitted (for now). \square

The two above claims show that the set of roots Φ consists of $d - r$ distinct elements of \mathfrak{h}^* . Hence the E^α s are all linearly independent and there are $d - r$ of them. So we have found a basis of our Lie algebra: The **Cartan-Weyl basis**:

$$\mathcal{B} = \{H^i \mid i = 1, 2, \dots, r\} \cup \{E^\alpha \mid \alpha \in \Phi\} \quad (2.1)$$

Because of Cartan's theorem, we know that our Killing form is non-degenerate. This fact is used extensively in proofs throughout the remainder of these notes.

Theorem 2.4. In the Cartan-Weyl basis, we have:

- (i) For all $H \in \mathfrak{h}$ and $\alpha \in \Phi$, $K(H, \alpha) = 0$
- (ii) For all $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$, $K(E^\alpha, E^\beta) = 0$
- (iii) For all $H \in \mathfrak{h}$, there exists $H' \in \mathfrak{h}$ such that $K(H, H') \neq 0$
- (iv) $\alpha \in \Phi \implies -\alpha \in \Phi$, and $K(E^\alpha, E^{-\alpha}) \neq 0$.

Proof. We have:

- (i) For all $H' \in \mathfrak{h}$, observe that by (1.3) we may write: $\alpha(H')K(H, E^\alpha) = K(H, [H', E^\alpha]) = -K([H', H], E^\alpha) = 0$. Because $\alpha \neq 0$, we conclude that $K(H, E^\alpha) = 0$.
- (ii) Again by (1.3), for all $H' \in \mathfrak{h}$ we have $\alpha(H') + \beta(H')K(E^\alpha, E^\beta) = 0$, so provided $\alpha + \beta \neq 0$, we have $K(E^\alpha, E^\beta) = 0$.
- (iii) Assume no such H' exists. Then $K(H, H') = 0 \forall H' \in \mathfrak{h}$, but also by (i) $K(H, E^\alpha) = 0 \forall \alpha \in \Phi$. Since the H^i s and E^α s form a basis, we conclude that $K(H, X) = 0 \forall X \in \mathfrak{g}$, which is a contradiction.
- (iv) Assume not, so for $\alpha \in \Phi$ we have either $-\alpha \notin \Phi$ or $K(E^\alpha, E^{-\alpha}) = 0$. Then by (i) and (ii) we see that $K(E^\alpha, X) = 0$ for all $X \in \mathfrak{g}$, which is a contradiction. \square

From (iii) above, we see that K starts life as a non-degenerate Killing form on \mathfrak{g} , but yields a non-degenerate inner product on \mathfrak{h} also. If we write $H = \rho_i H^i$ and $H' = \rho'_i H^i$ in components, then $K(H, H') = \rho_i \rho'_j K^{ij}$, where we write $K^{ij} = K(H^i, H^j)$. Because K is non degenerate, K^{ij} is an invertible matrix with inverse K_{ij}^{-1} , which may be used to define a non-degenerate inner product on \mathfrak{h}^* :

$$(\alpha, \beta) := (K^{-1})_{ij} \alpha^i \beta^j \quad (2.2)$$

3 Algebra in the Cartan-Weyl basis

So far, $[E^\alpha, E^\beta]$ ($\alpha, \beta \in \Phi$) is undetermined. By use of the Jacobi identity, one has:

$$[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^j)[E^\alpha, E^\beta] \quad \text{for } \alpha, \beta \in \Phi \text{ and all } i \in \{1, \dots, r\} \quad (3.1)$$

Then there are two cases to consider:

1. $\alpha + \beta \neq 0$:

- (i) First, if $\alpha + \beta \notin \Phi$, we conclude that $[E^\alpha, E^\beta] = 0$.
- (ii) On the other hand, if $\alpha + \beta \in \Phi$, we have $[E^\alpha, E^\beta] = \mathcal{N}_{\alpha, \beta} E^{\alpha+\beta}$ for some $\mathcal{N}_{\alpha, \beta} \in \mathbb{C}$ (no sum - by non-degeneracy there is exactly one possible eigenvector with eigenvalue $\alpha + \beta$).

2. $\alpha + \beta = 0$.

For the second case, we begin by making use of the identity (1.3):

$$K([E^\alpha, E^{-\alpha}], H) = \alpha(H)K(E^\alpha, E^{-\alpha}) \quad (3.2)$$

This motivates us to define:

$$H^\alpha := \frac{[E^\alpha, E^{-\alpha}]}{K(E^\alpha, E^{-\alpha})} \quad (3.3)$$

(The denominator is non-zero by (iv) of the previous theorem). Then $K(H^\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. Because K^{ij} is invertible, as was discussed at the end of the preceding section, we may in-fact solve this equation to find the unique solution:

$$H^\alpha = (K^{-1})_{ij} \alpha^j H^i \quad (3.4)$$

This may then be plugged into the definition (3.3) and rearranged to give us $[E^\alpha, E^{-\alpha}] = K(E^\alpha, E^{-\alpha})H^\alpha$, with H^α given as in (3.4). One can show that $[H^\alpha, E^\beta] = (\alpha, \beta)E^\beta$. This motivates the renormalization:

$$\begin{aligned} e^\alpha &= \frac{\sqrt{2}}{\sqrt{(\alpha, \alpha)K(E^\alpha, E^{-\alpha})}} E^\alpha \\ h^\alpha &= \frac{2}{(\alpha, \alpha)} H^\alpha \end{aligned} \quad (3.5)$$

(Strictly, one must show $(\alpha, \alpha) \neq 0$.) Then one has the algebra in its (almost) final form:

$$\begin{aligned} [h^\alpha, h^\beta] &= 0 \\ [h^\alpha, e^\beta] &= \frac{2(\alpha, \beta)}{(\alpha, \alpha)} e^\beta \\ [e^\alpha, e^\beta] &= \begin{cases} n_{\alpha, \beta} e^{\alpha+\beta} & \alpha + \beta \in \Phi \\ h^\alpha & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (3.6)$$

Remark: We say 'almost' final because it is not clear at this point that we can replace the H^i 's with h^α 's - indeed, are the h^α 's linearly independent and do they span \mathfrak{h} ? The answers are no and yes respectively, which means we'll have to trim down these h 's to get a basis. We will get to this shortly.

4 $\mathcal{L}_{\mathbb{C}}(SU(2))$ Subalgebra and Root Strings

Observe that by (iv) of (2.4), for each pair $\pm\alpha \in \Phi$ we have an $\mathcal{L}_{\mathbb{C}}(SU(2))$ subalgebra of \mathfrak{g} , the basis being $\{h^\alpha, e^\alpha, e^{-\alpha}\}$. We call this subalgebra $sl(2)_\alpha$.

4.1 Root Strings

For $\alpha, \beta \in \Phi$, we define the " α string through β " to be the set of roots of the form $\beta + n\alpha$ for $n \in \mathbb{Z}$, that is:

$$S_{\alpha,\beta} = \{\gamma \in \Phi \mid \gamma = \alpha + n\beta \text{ for } n \in \mathbb{Z}\} \quad (4.1)$$

This has the corresponding vector subspace:

$$V_{\alpha,\beta} = \text{span}_{\mathbb{C}}\{e^\gamma \mid \gamma \in S_{\alpha,\beta}\} \quad (4.2)$$

Now consider the action of $sl(2)_\alpha$ on $V_{\alpha,\beta}$. We have:

$$\begin{aligned} [h^\alpha, e^{\beta+n\alpha}] &= \left(\frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2n \right) e^{\beta+n\alpha} \\ [e^{\pm\alpha}, e^{\beta+n\alpha}] &\propto \begin{cases} e^{\beta+(n\pm 1)\alpha} & \text{if } \beta + (n \pm 1)\alpha \in \Phi \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (4.3)$$

So $V_{\alpha,\beta}$ is invariant under the action of $sl(2)_\alpha$. Therefore it is isomorphic to the representation space of some representation \mathcal{R} of $sl(2)_\alpha$ (convince yourself this is true), with weight set:

$$S_{\mathcal{R}} = \left\{ \frac{2(\alpha, \beta)}{\alpha, \alpha} + 2n \mid \beta + n\alpha \in \Phi \right\} \quad (4.4)$$

(This follows from the first part of (4.3)). We can go further - because all the roots of \mathfrak{g} are finite in number and non-degenerate, **\mathcal{R} is finite and irreducible.**¹ Hence $\mathcal{R} = \mathcal{R}_\Lambda$ for some $\Lambda \in \mathbb{Z}_{\geq 0}$, so that:

$$S_{\mathcal{R}} = \{-\Lambda, -\Lambda + 2, \dots, \Lambda - 2, \Lambda\} \quad (4.5)$$

Now we can compare our two expressions for $S_{\mathcal{R}}$, to conclude:

$$\exists n_{\pm} \in \mathbb{Z} \text{ such that } \begin{cases} -\Lambda &= \frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2n_- \\ \Lambda &= \frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2n_+ \end{cases} \quad (4.6)$$

If we add the expressions for n_{\pm} , we conclude:

$$\boxed{\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}} \quad (4.7)$$

The set of roots are said to form an **unbroken string** of width $n_+ + n_- + 1$, so $S_{\alpha,\beta} = \{\beta + n\alpha \mid n_- \leq n \leq n_+\}$.

We now wish to show that $(\alpha, \beta) \in \mathbb{R}$ also. To do this, we will begin by finding an expression for the Killing form in terms of our roots:

$$K^{ij} = K(H^i, H^j) = \frac{1}{\mathcal{N}} \text{Tr}(\text{ad}_{H^i} \circ \text{ad}_{H^j}) = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} \delta^i \delta^j \quad (4.8)$$

For some (arbitrary) real and non-zero normalisation \mathcal{N} . The second equality follows from the fact that the trace is the sum of the eigenvalues. Defining $\alpha_i = (K^{-1})_{ij} \alpha^j$ for ease, we thus have:

$$(\alpha, \beta) = \alpha_i \beta_j K^{ij} = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\alpha, \delta) (\beta, \delta) \quad (4.9)$$

If we multiply this equation through by $4/[(\alpha, \alpha)(\beta, \beta)]$, we get:

$$\frac{2}{(\beta, \beta)} \underbrace{\left(\frac{2(\alpha, \beta)}{\alpha, \alpha} \right)}_{\text{real}} = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} \underbrace{\left(\frac{2(\alpha, \delta)}{(\alpha, \alpha)} \right)}_{\text{real}} \underbrace{\left(\frac{2(\beta, \delta)}{(\beta, \beta)} \right)}_{\text{real}} \quad (4.10)$$

We see that by using our result (4.7) and the fact \mathcal{N} is real, (β, β) must also be real for all $\beta \in \Phi$, because β was arbitrary. Using (4.7) once more, we conclude:

$$\boxed{(\alpha, \beta) \in \mathbb{R} \quad \forall \alpha, \beta \in \Phi} \quad (4.11)$$

¹For irreducibility, observe that the weights are non-degenerate and spaced by 2.

5 Real Geometry of Roots

We showed that $\alpha \in \Phi$ are elements of the dual space \mathfrak{h}^* . We will investigate this line of thinking further. The first thing one would like to establish is exactly how much of \mathfrak{h}^* we can get to with these roots.

Theorem 5.1. The roots span \mathfrak{h}^* .

Proof. Assume the roots do not span \mathfrak{h}^* . Then there exists $\lambda \neq 0$ in \mathfrak{h}^* such that $(\lambda, \alpha) = 0$ for all $\alpha \in \Phi$. Then $h_\lambda := \lambda_i H^i$ has the property:

$$[h_\lambda, h] = 0 \quad \forall h \in \mathfrak{h} \quad \text{and} \quad [h_\lambda, e^\alpha] = 0 \quad \forall \alpha \in \Phi \implies [h_\lambda, X] = 0 \quad \forall X \in \mathfrak{g} \quad (5.1)$$

So \mathfrak{g} has a non-trivial abelian ideal \implies it is not semi-simple, and the result follows by the contrapositive. \square

Hence we can find a basis of r roots for \mathfrak{h}^* , which we shall write as $\{\alpha_{(i)} \mid i = 1, \dots, r\}$.

So far, we have been working with a complex vector space. We'll define a real subspace $\mathfrak{h}_{\mathbb{R}}^* \subset \mathfrak{h}^*$ by:

$$\mathfrak{h}_{\mathbb{R}}^* = \text{span}_{\mathbb{R}}\{\alpha_{(i)} \mid i = 1, \dots, r\} \quad (5.2)$$

Since the $\alpha_{(i)}$ form a basis, for any root $\beta \in \Phi$ we have:

$$\beta = \sum_{i=1}^r \beta^i \alpha_{(i)} \implies \underbrace{(\beta, \alpha_{(j)})}_{\text{real}} = \sum_{i=1}^r \beta^i \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{\text{real}} \quad j = 1, \dots, r \quad (5.3)$$

For some complex coefficients β^i . We can view the equation after the implication arrow as a matrix equation, the RHS involving the invertible² matrix with entries $(\alpha_{(i)}, \alpha_{(j)})$. Then we multiply through by the inverse to get the solution, and deduce that in fact $\beta^i \in \mathbb{R}$, and hence:

$$\boxed{\beta \in \mathfrak{h}_{\mathbb{R}}^* \quad \forall \beta \in \Phi} \quad (5.4)$$

This tells us that our basis vectors $\alpha_{(i)}$ span $\mathfrak{h}_{\mathbb{R}}^*$. Further, we now have enough to show that our inner-product is positive definite:

$$\begin{aligned} \lambda &= \sum_{i=1}^r \lambda^i \alpha_{(i)} \quad , \quad \mu = \sum_{i=1}^r \mu^i \alpha_{(i)} \quad \lambda^i, \mu^i \in \mathbb{R} \\ \implies (\lambda, \mu) &= \sum_{i,j=1}^r \lambda^i \mu^j \underbrace{(\alpha_{(i)}, \alpha_{(j)})}_{\text{real}} \in \mathbb{R} \\ \text{and } (\lambda, \lambda) &= \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\lambda, \delta)^2 \geq 0 \end{aligned} \quad (5.5)$$

Because of the non-degeneracy of our inner product, the last line has equality iff $\lambda = 0$.

In summary, the roots $\alpha \in \Phi$ live in a **real vector space** $\mathfrak{h}_{\mathbb{R}}^* \equiv \mathbb{R}^r$, equipped with a **Euclidean inner-product**:

$$\boxed{\forall \lambda, \mu \in \mathfrak{h}_{\mathbb{R}}^* :} \quad (5.6)$$

$$(i) \quad (\lambda, \mu) \in \mathbb{R} \quad (5.7)$$

$$(ii) \quad (\lambda, \lambda) \geq 0 \quad (5.8)$$

$$(iii) \quad (\lambda, \lambda) = 0 \iff \lambda = 0 \quad (5.9)$$

²Suppose the columns of the matrix with entries $A_{ij} = (\alpha_{(i)}, \alpha_{(j)})$ are not linearly independent. Then there exists coefficients γ^i not all zero such that $(\gamma^i \alpha_{(i)}, \alpha_{(j)}) = 0$ for all $j = 1, \dots, r$. Then by the non-degeneracy of the inner-product, deduce that $\gamma^i \alpha_{(i)} = 0$. But this is a contradiction because the $\alpha_{(i)}$ s are linearly independent, since they form a basis. Therefore the matrix A_{ij} has linearly independent columns, and hence it is invertible.

This allows us to define **lengths** and **angles**:

$$|\alpha| = (\alpha, \alpha)^{\frac{1}{2}} \quad (\text{length of } \alpha.) \quad (5.10)$$

$$\cos \phi = \frac{(\alpha, \beta)}{|\alpha||\beta|} \quad (\text{angle between } \alpha \text{ and } \beta.) \quad (5.11)$$

By use of (4.7), we see that $2|\beta||\alpha|^{-1} \cos \phi \in \mathbb{Z}$. By exchanging the roles of α and β , we conclude that $4 \cos^2 \phi \in \mathbb{Z}$. Hence:

$$\begin{aligned} \cos \phi &= \pm \frac{\sqrt{n}}{2} \quad \text{for } n \in \{0, 1, 2, 3, 4\} \\ \implies \phi &\in \left\{ 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi \right\} \end{aligned} \quad (5.12)$$

6 Simple Roots

There are finitely many roots $\alpha \in \Phi$, so we may divide them into positive and negative by drawing a hyperplane in $\mathfrak{h}_{\mathbb{R}}^*$ of our choosing.

This decomposes Φ into Φ_+ and Φ_- , the sets of positive and negative roots, $\Phi = \Phi_+ + \Phi_-$. Observe also that $\alpha \in \Phi_{\pm} \iff -\alpha \in \Phi_{\mp}$, and $\alpha, \beta \in \Phi_{\pm} \implies \alpha + \beta \in \Phi_{\pm}$.

A root is called **simple** if it is a positive root that cannot be written as the sum of two positive roots. We will denote the set of simple roots $\Phi_s = \{\delta \in \Phi_+ \mid \delta \neq \alpha + \beta \quad \forall \alpha, \beta \in \Phi_+\}$.

Theorem 6.1 (Properties of simple roots.). (i) If α and β are simple roots, then $\alpha - \beta$ is not a root.

(ii) If α and β are simple roots, then the α string through β has length:

$$l_{\alpha, \beta} = 1 - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \quad (6.1)$$

(iii) $(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in \Phi_s$ with $\alpha \neq \beta$.

(iv) Any positive root can be written as a linear combination of simple roots with positive integer coefficients:

$$\beta \in \Phi_+ \implies \beta = \sum_i c^i \alpha_{(i)} \quad , \quad \text{with } \alpha_{(i)} \in \Phi_s \text{ and } c^i \in \mathbb{N} \quad (6.2)$$

(v) Simple roots are linearly independent.