4.4 $\alpha = 1 \times \det(A_{31}) - 4 \times \det(A_{32}) + 2 \times \det(A_{33}) - 2 \times \det(A_{34}) - 3 \times \det(A_{35})$

$\beta = 3 \times \det(B_{31}) - 12 \times \det(B_{32}) + 6 \times \det(B_{33}) - 6 \times \det(B_{34}) - 9 \times \det(B_{35})$

$$A = \begin{bmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 4 & 2 & 2 & -3 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{bmatrix}$$

$$B = \begin{bmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 3(1) & 3(4) & 3(2) & 3(2) & 3(-3) \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{bmatrix}$$

as we can see from B, the third row of A times 3 is the third row. By using row scalar property, we can $\alpha = 3\beta$.

4.6 (a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $\det(A) = ad - bc$.

Since a, b, c, d are integers, $ad - bc$ is also an integer as integers are closed under addition and multiplication.

(b) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ where $a \ldots i$ are integers

$$\det(A) = a\underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{X} - b\underbrace{\begin{vmatrix} d & f \\ g & i \end{vmatrix}}_{Y} + c\underbrace{\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{Z}$$

we know from (a) that they are integers.

and so, since addition and multiplication are closed for integers.

$aX + cZ - bY$ is also an integer.

(c) Let $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$

$$\det(A) = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} \cdots$$

Similar to (b), we can see that

$$\det(A) = aX - bY + cZ + dZ$$

We know that X, Y, Z and Z are integers. So, using same properties of (a) and (b), $\det(A)$ will also be an integer.

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4.9) $d_1 = a_2b_3 - b_2a_3$

$d_2 = b_1a_3 - a_1b_3$

$d_3 = a_1b_2 - b_1a_2$

$$z^t(x \times y) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{pmatrix} a_2b_3 - b_2a_3 \\ b_1a_3 - a_1b_3 \\ a_1a_2 - b_1a_2 \end{pmatrix}$$

$$= c_1(a_2b_3 - b_2a_3) + c_2(b_1a_3 - a_1b_3) + c_3(a_1a_2 - b_1a_2)$$

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = c_1(a_2b_3 - b_2a_3) - c_2(a_1b_3 - b_1a_3) + c_3(a_1b_2 - b_1a_2)$$

$$= c_1 \cancel{b_2 a_3} (a_2b_3 - b_2a_3) + c_2(a_1b_3 - b_1a_3) + c_3(a_1b_2 - b_1a_2)$$

Thus, proven.

4.12 (a)

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \xrightarrow{R_3 += R_1} \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 2 \end{vmatrix} \xrightarrow{R_3 \to R_2}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$ det = 0 (equal rows)

(b)

$$\begin{vmatrix} -3 & 2 & 2 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix}$$ $R_3 = R_3 - 7R_2$, $R_1 = R_1 + 3R_2$

$$\begin{vmatrix} 0 & 14 & 5 \\ 1 & 4 & 1 \\ 0 & -22 & -9 \end{vmatrix}$$ det = C_2

$= -1(-126 + 110)$

$= \underline{16}$

(c)

$$\begin{vmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 6 & 5 \end{vmatrix}$$ Expand along column 2.

$$\det = 1 \begin{vmatrix} 2 & 2 & 0 \\ 0 & 3 & 2 \\ 1 & 0 & 5 \end{vmatrix}$$

$= 1(2(15) + 1(4)) = \underline{34}$

4.17 $A = \begin{bmatrix} A_1 \\ A_2 \\ 2A_1 + A_2 \end{bmatrix}$

~~det (A) so~~ $A = \begin{bmatrix} A_1 \\ A_2 \\ 2A_1 \end{bmatrix} + \begin{bmatrix} A_1 \\ A_2 \\ A_2 \end{bmatrix}$

$$\det(A) = 2\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{11} & A_{12} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{21} & A_{22} & A_{23} \end{vmatrix}$$

Using row equal property.

$\det(A) = 0$.

4.16) To prove : $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$

$$\xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 - R_3} \begin{vmatrix} 0 & x-z & x^2-z^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z \end{vmatrix}$$

$$\det = 1\left((x-z)(y+z)(y-z) - (y-z)(x+z)(x-z)\right)$$

$$= (y-z)\left((x-z)(y+z) - (x+z)(x-z)\right)$$

$$= (x-z)(y-z)\left(y+z-(x+z)\right)$$

$$= \overset{\times(-)}{(x-z)}\overset{\times(-)}{(y-z)}\overset{\times(+)}{(y-x)}$$

$= \cancel{(x-z)}\ (y-x)(z-x)(z-y)$. Proved

For $\begin{bmatrix} 1, & x, & x^2 \end{bmatrix}$
$\begin{bmatrix} 1, & y, & y^2 \end{bmatrix}$ to be linearly dependat.
$\begin{bmatrix} 1, & z, & z^2 \end{bmatrix}$

the rows have to be a linear combination of each other, and because of column 1 we see that it must be $x = y = z$.

1. $$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$ $R_4 = R_4 - 4R_1$, $R_3 = R_3 - 3R_1$, $R_2 = R_2 - 2R_1$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{vmatrix} = \begin{vmatrix} -1 & -2 & -7 \\ -2 & -8 & -10 \\ -7 & -10 & -13 \end{vmatrix}$$

$$= -1(104 - 100) + 2(26 - 70) - 7(20 - 56)$$

$$= -4 - 88 - 7(-36)$$

$$= 252 - 92$$

$$= \underline{160}$$