5260 MIDTERM

Begin each problem on a separate piece of paper. Take your time and think before you write. Show all your work. When an explanation is required, write complete sentences in grammatical English.

1) [30 points] Make a variational estimate of the ground energy of a particle of mass m confined in a three dimensional linear potential $V(r) = \sigma r$, assuming a hydrogen-like trial wave function $u(r) = rR(r) = r\exp(-\lambda r)$. A useful integral, maybe:

$$\int_0^\infty x^n \exp(-ax)dx = n!(\frac{1}{a})^{n+1} \tag{1}$$

Solution:

$$E(\lambda) = \left\langle \lambda | -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \sigma r | \lambda \right\rangle = T + V. \tag{2}$$

The normalization factor is

$$\frac{1}{N^2} = \int_0^\infty drr^2 \exp(-2\lambda r) = \frac{2!}{(2\lambda)^3} = \frac{1}{4\lambda^3}. \tag{3}$$

Then

$$\begin{aligned} \langle V \rangle &= 4\lambda^3\sigma \int_0^\infty drr^3 \exp(-2\lambda r) = 4\lambda^3\sigma\frac{3!}{(2\lambda)^4} \\ &= \frac{4!\sigma}{16\lambda} \\ &= \frac{3\sigma}{2\lambda}. \end{aligned} \tag{4}$$

Next,

$$\begin{aligned} \frac{\partial^2}{\partial r^2} r\exp(-\lambda r) &= \frac{\partial}{\partial r}[\exp(-\lambda r) - \lambda r\exp(-\lambda r)] \\ &= (-\lambda - \lambda + \lambda^2 r)\exp(-\lambda r) \end{aligned} \tag{5}$$

so

$$\begin{aligned} T &= \frac{\hbar^2}{2m}4\lambda^3\sigma \int_0^\infty drr\exp(-\lambda r)(2\lambda - \lambda^2 r)\exp(-2\lambda r) \\ &= \frac{\hbar^2}{2m}4\lambda^3\sigma[2\lambda\frac{1}{(2\lambda)^2} - \lambda^2\frac{2!}{(2\lambda)^3}] \end{aligned}$$

$$
\begin{aligned}
&= \frac{2\hbar^2\lambda^2}{m}[\frac{1}{2}-\frac{1}{4}] \\
&= \frac{\hbar^2\lambda^2}{2m}
\end{aligned}
\tag{6}
$$

so

$$E(\lambda) = \frac{\hbar^2\lambda^2}{2m} + \frac{3\sigma}{2\lambda}. \tag{7}$$

Now

$$\frac{\partial E(\lambda}{\partial\lambda} = 0 = \frac{\hbar^2\lambda}{m} - \frac{3\sigma}{2\lambda^2} \tag{8}$$

or

$$\lambda^3 = \frac{3\sigma m}{2\hbar^2}. \tag{9}$$

Plug back into E:

$$
\begin{aligned}
E &= \frac{\hbar^2}{2m}[\frac{3\sigma m}{2\hbar^2}^{2/3} + + \frac{3\sigma}{2}[2\frac{2\hbar^2}{3\sigma m}]^{1/3} \\
&= (\frac{\hbar^2\sigma^2}{m})^{1/3}[\frac{1}{2}(\frac{3}{2})^{2/3} + (\frac{3}{2})^{1/3}] \\
&= (\frac{\hbar^2\sigma^2}{m})^{1/3}(\frac{3}{2})^{5/3}.
\end{aligned}
\tag{10}
$$

2) [30 points] A nucleus N_1 decays to a nucleus N_2 by the emission of an alpha particle. The masses of the two nuclei are much greater than the mass of the alpha particle, so the situation is like the decay of an atom in that we only have to consider the phase space for the alpha particle. However, the alpha particle is nonrelativistic. Its energy is $E_\alpha = m_\alpha c^2 + p_\alpha^2/(2m_\alpha)$. Parameterizing the matrix element for the decay $\langle f|H_I|i\rangle = \lambda/\sqrt{V}$ where we are working in a box of volume V, compute the Golden Rule lifetime for the decay $N_1 \to N_2 + \alpha$. A useful variable might be $\Delta E = M(N_1)c^2 - M(N_2)c^2 - m_\alpha c^2$. ($\lambda$ is the quantity we computed last semester a la WKB-Gamow-Condon-Gurney.)

Solution: The Golden Rule says

$$d\Gamma = \frac{2\pi}{\hbar}|\langle f|H_I|i\rangle|^2\delta(M_1c^2 - M_2c^2 - E_\alpha) \times V\frac{d^3p_\alpha}{(2\pi\hbar)^3} \tag{11}$$

where $\langle f|H_i|i\rangle$ is parameterized as λ/V and $E_\alpha = m_\alpha c^2 + K_\alpha$ and

$$K_\alpha = \frac{p_\alpha^2}{2m_\alpha}. \tag{12}$$

Then

$$d\Gamma = \frac{2\pi}{\hbar}|\lambda|^2\delta(K_\alpha - \Delta E)\frac{d^3p_\alpha}{(2\pi\hbar)^3}. \tag{13}$$

There is no angular dependence so $d^3p_\alpha \to 4\pi p_\alpha^2 dp_\alpha = 4\pi p_\alpha(p_\alpha dp_\alpha)$. From $p_\alpha^2 = 2m_\alpha K_\alpha$, we have $p_\alpha dp_\alpha = m_\alpha dK_\alpha$ and the Golden Rule shrinks still more to

$$d\Gamma = \frac{8\pi^2}{(2\pi)^3\hbar^4}|\lambda|^2\delta(K_\alpha - \Delta E)p_\alpha m_\alpha dK_\alpha. \tag{14}$$

Then doing the last integral gives

$$\Gamma = \frac{|\lambda|^2}{\pi\hbar^4}m_\alpha p_\alpha \tag{15}$$

where now $p_\alpha = (2m_\alpha\Delta E)^{1/2}$.

3) [25 points] All these QI talks reminded me of the Ising model in a transverse field, a one-dimensional array of spins (sorry, qubits, have to be contemporary) with a Hamiltonian

$$H = \frac{\epsilon}{2}\sum_{j=1}^{N}\sigma_z(j) - \Delta_0\sum_{j=1}^{N-1}\sigma_x(j)\sigma_x(j+1). \tag{16}$$

Suppose $N = 2$. In this case the system is exactly solvable, but never mind: do a perturbative calculation assuming $\epsilon \gg \Delta_0$ and find the energy spectrum (and corresponding eigenfunctions) to lowest nontrivial order in Δ_0/ϵ.

Solution: In strong coupling the ground state is $|\downarrow\downarrow\rangle$ where $|\uparrow\rangle$ and $|\downarrow\rangle$ are (of course) eigenstates of σ_z. The whole spectrum is

$$\begin{array}{ll} |\uparrow\uparrow\rangle, & E = \epsilon \\ |\uparrow\downarrow\rangle, & E = 0 \\ |\downarrow\uparrow\rangle, & E = 0 \\ |\downarrow\downarrow\rangle, & E = -\epsilon \end{array} \tag{17}$$

Of course, any linear combination of the $|\downarrow\uparrow\rangle$ and $|\uparrow\downarrow\rangle$ states also has zero energy. Then we have to use $\sigma_x|\uparrow\rangle = |\downarrow\rangle$ and $\sigma_x|\downarrow\rangle = |\uparrow\rangle$. The complete Hamiltonian, in a basis ($|\uparrow\uparrow\rangle$ $|\downarrow\downarrow\rangle$ $|\uparrow\downarrow\rangle$ $|\downarrow\uparrow\rangle$), is

$$H = \begin{pmatrix} -\epsilon & -\Delta_0 & 0 & 0 \\ -\Delta_0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & -\Delta_0 \\ 0 & 0 & -\Delta_0 & 0 \end{pmatrix} \tag{18}$$

Ordinary perturbation for the $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ gives two states,

$$\begin{aligned}
|\psi_1\rangle &= |\uparrow\uparrow\rangle - \frac{\Delta_0}{2\epsilon}|\downarrow\downarrow\rangle \\
E_1 &= -\epsilon - \frac{\Delta_0^2}{2\epsilon} \\
|\psi_2\rangle &= |\downarrow\downarrow\rangle + \frac{\Delta_0}{2\epsilon}|\uparrow\uparrow\rangle \\
E_2 &= \epsilon + \frac{\Delta_0^2}{2\epsilon}.
\end{aligned} \tag{19}$$

The other two states are degenerate and we must take account of that by exactly diagonalizing the degenerate subspace. That is trivial and we have

$$\begin{aligned}
|\psi_3\rangle &= (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \\
E_3 &= -\Delta_0 \\
|\psi_4\rangle &= (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} \\
E_4 &= \Delta_0.
\end{aligned} \tag{20}$$

4) [15 points] Tell me about selection rules for electric dipole transitions in atoms: what they are, where they come from, and give examples, in a page or less.

Solution: the transition probability per unit time for an electric dipole transition from an initial state $|i\rangle$ to a final state $|f\rangle$ is proportional to the absolute square of the matrix element of the electric dipole moment operator $\langle f|e\vec{r}|i\rangle$. A selection rule is a statement about when this matrix element vanishes, so the transition from $|i\rangle$ to $|f\rangle$ does not occur. Typically, selection rules involve the angular momentum quantum numbers of the state. For states $|l, m_l, s, m_s\rangle$ the selection rule is $\Delta m_s = 0$ because the perturbation $e\vec{r}$ does not involve spin. The Wigner-Eckart theorem tells us that $\Delta m_l = \pm 1, 0$ because $\vec{r}$ is a vector operator (a rank-1 spherical tensor) and $\Delta l = \pm 1$ from the law for addition of angular momentum ($l_i + 1 = l_i + 1, l_i, l_i - 1$) and $l_f = l_i$ transitions forbidden by parity. For eigenstates labelled by total angular momentum (its square and one component, $|jm\rangle$) the selection rules are $\Delta m = \pm 1, 0$ and $\Delta j = \pm 1, 0$ with $j = 0 \rightarrow j = 0$ transitions forbidden by angular momentum conservation.

5260 spring 2023 midterm

average = 71

σ = 15