Rabi $\longleftrightarrow$ Fermi's golden rule

is a bit complicated. I am giving you a simplified version (although it is a pretty standard one - see Schiff, Sec. 35, for example). Wikipedia ("Fermi's Golden Rule," not "Golden Rule") has an alternative set of initial conditions. Finally, here are some slightly cryptic and highbrow notes (by Jinx Cooper, JILA theorist 1970's-2000's) which might satisfy a need for a very complete story!

8) Two level atom in a coherent field

Thus far we have been dealing with perturbation theory, now we go beyond and consider semiclassically the interaction between a two-level atom and a coherent field.

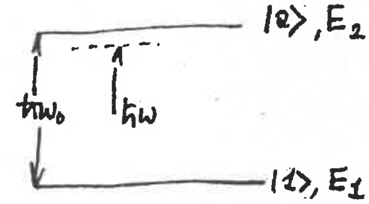


Fig 11.

$$i\hbar \frac{d\Psi}{\delta t} = [H_a - \vec{\mu}\cdot\vec{E}(t)]\Psi \qquad (8.1)$$

$$H'(t) = -\mu_z \mathcal{E}(t) = -\frac{1}{2}\mu_z(\mathcal{E}e^{i\omega t} + \mathcal{E}^* e^{-i\omega t}) \qquad (8.2)$$

Try

$$\Psi(\vec{r},t) = a_1(t)e^{-iE_1t/\hbar}\varphi_1(\vec{r}) + a_2(t)e^{-iE_2t/\hbar} \qquad (8.3)$$

$$H_0\varphi_n = E_n\varphi_n \qquad \hbar\omega_0 = E_2 - E_1 \qquad (8.4)$$

$$i\hbar\frac{da_1(t)}{dt} = \langle\varphi_1|H'|\varphi_2\rangle e^{-i\omega_0 t}a_2(t) \qquad (8.5)$$

$$i\hbar\frac{da_2(t)}{dt} = \langle\varphi_2|H'|\varphi_1\rangle e^{i\omega_0 t}a_1(t) \qquad (8.6)$$

or

$$\frac{da_1}{dt} = \frac{i}{2\hbar}\mu_{12}(\mathcal{E}e^{i(\omega-\omega_0)t} + \mathcal{E}^*e^{-i(\omega_0+\omega)t})a_2(t) \qquad (8.7)$$

$$\frac{da_2}{dt} = \frac{i}{2\hbar}\mu_{21}(\mathcal{E}e^{i(\omega+\omega_0)t} + \mathcal{E}^*e^{-i(\omega-\omega_0)t})a_1(t)$$

Make ROTATING WAVE APPROXIMATION – since terms $e^{\pm i(\omega+\omega_0)t}$ on RHS oscillate rapidly, they may be taken as zero when averaged over a time scale larger than $\sim 1/\omega$.

Then, with $\Omega_0 = \mu_{12}\mathcal{E}/\hbar$ (the Rabi frequency) (8.8)

$$\frac{da_1}{dt} = \frac{i\Omega_0}{2} e^{i\Delta\omega t} a_2 \qquad (8.9)$$

$$\frac{da_2}{dt} = \frac{i\Omega_0^*}{2} e^{-i\Delta\omega t} a_1 \qquad (8.10)$$

with $\Delta\omega = (\omega - \omega_0)$

$$\therefore \quad e^{-i\Delta\omega t}\frac{da_1}{dt} = \frac{i\Omega_0 a_2}{2}$$

$$-i\Delta\omega e^{-i\Delta\omega t}\frac{da_1}{dt} + e^{-i\Delta\omega t}\frac{d^2a_1}{dt^2} = \frac{i\Omega_0}{2}\frac{da_2}{dt} = -\frac{|\Omega_0|^2}{4}e^{-i\Delta\omega t}a_1$$

$$\frac{d^2a_1}{dt^2} - i\Delta\omega\frac{da_1}{dt} + \frac{|\Omega_0|^2}{4}a_1 = 0 \qquad (8.11)$$

Try $a_1 = Ae^{i\lambda t}$, then $\lambda^2 - \Delta\omega\lambda - |\Omega_0|^2/4$ (8.12)

$$\lambda = \frac{+\Delta\omega \pm \sqrt{\Delta\omega^2 + |\Omega_0|^2}}{2} = \frac{\Delta\omega}{2} \pm \frac{\Omega}{2} \qquad (8.13)$$

$$\text{with} \quad \Omega = \sqrt{\Delta\omega^2 + |\Omega_0|^2} \qquad (8.14)$$

$$a_1(t) = \left[A_1 e^{i\Omega t/2} + A_2 e^{-i\Omega t/2}\right] e^{i\Delta\omega t/2} \qquad (8.15)$$

$$\frac{da_1}{dt} = \frac{i}{2}\left[(\Omega + \Delta\omega)e^{i\Omega t/2}A_1 - (\Omega - \Delta\omega)A_2 e^{-i\Omega t/2}\right]e^{i\Delta\omega t/2} = \frac{i\Omega_0}{2}e^{i\Delta\omega t}a_2 \qquad (8.16)$$

Suppose at $t = 0$ atom is initially in <u>upper</u> state ; $a_2 = 1$ $a_1 = 0$

$$\therefore \quad A_1 + A_2 = 0 \qquad A_1(\Omega + \Delta\omega) - A_2(\Omega - \Delta\omega) = \Omega_0$$

$$\therefore \quad A_1 = -A_2 = \Omega_0/2\Omega \qquad (8.17)$$

$$a_1(t) = i\frac{\Omega_0}{\Omega} e^{i\Delta\omega t/2} \sin\left(\frac{\Omega t}{2}\right) \qquad (8.18)$$

$$a_2(t) = e^{-i\Delta\omega t/2}\left(\cos(\Omega t/2) + i\left(\frac{\Delta\omega}{\Omega}\right)\sin\left(\frac{\Omega t}{2}\right)\right). \qquad (8.19)$$

Note $|a_1(t)|^2 + |a_2(t)|^2 = 1$

Ω_0 is called the Rabi frequency.

$$|a_1(t)|^2 = \frac{\Omega_0^2}{\Omega^2} \sin^2\left(\frac{\Omega t}{2}\right) \qquad (8.20)$$

Coherent oscillation

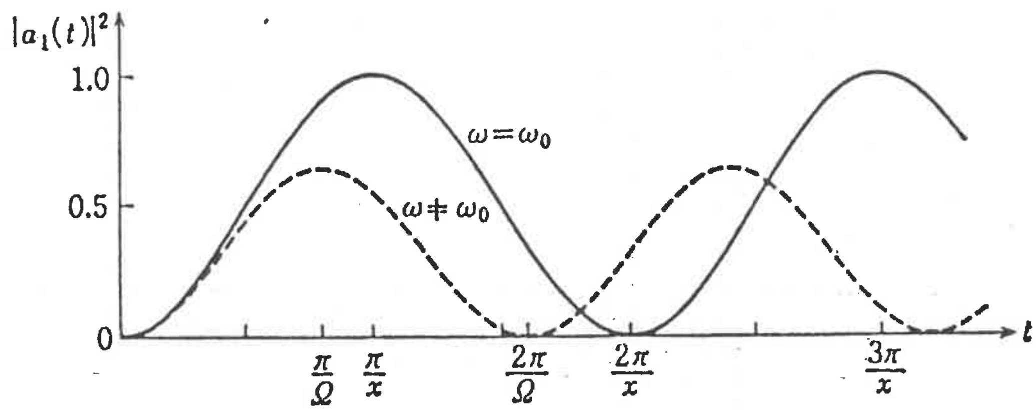


Fig 12. Time evolution of the transition probability for a two-level atom undergo interaction

Compare with perturbation theory result, which goes like $\sin^2\left(\frac{\Delta\omega t}{2}\right)/\left(\frac{\Delta\omega}{2}\right)^2$

Reaches first zero, $t = 2\pi/\Omega$ faster, hence spectrum (via $\Delta\omega.\Delta t \sim 1$) is broadened. Power broadened by Ω_0.

8.1) Induced dipole moment.

A non-degenerate atom in a stationary state does not have any dipole moment, but when subject to a coherent interaction, an oscillatory dipole moment $\mu(t)$ is induced. It expectation is

$$\vec{\mu}(t) = \int \Psi^*(\vec{r},t)\, \mu\, \Psi(\vec{r},t)\, dV = \langle \Psi | \mu | \Psi \rangle$$

$$= a_2^* a_1\, \mu_{21}\, e^{i\omega_0 t} + a_1^* a_2\, \mu_{12}\, e^{-i\omega_0 t} \qquad (8.21)$$

$$= \frac{i\Omega_0}{2\Omega}\, \mu_{21} \left[\sin(\Omega t) - i\frac{\Delta\omega}{\Omega}(1 - \cos(\Omega t)) \right] e^{i\omega t} + c.c \qquad (8.22)$$

NOTE: In spite of $e^{i\omega_0 t}$ factor in eq (8.21), $\vec{\mu}(t)$ actually oscillates as $e^{i\omega t}$ (i.e. at ω and not ω_0).

The power given to the field is $\int \vec{J}.\vec{E}\, dV = \vec{E}(t).\int \vec{J}\, dv = \frac{\partial \vec{\mu}}{\partial t}.\vec{E}$ (8.23)

i.e. Just the rate of doing work on the dipole by the external field $\vec{E}$.

∴ For $\omega = \omega_0$, averaging over a cycle of ω, and assuming $\omega >> \Omega$

$$P = \overline{\frac{\partial \vec{\mu}}{\partial t}.\vec{E}} = \hbar\omega \frac{\Omega_0^2}{2\Omega} \sin \Omega t. \qquad (8.24)$$

$$= \hbar\omega \frac{d}{dt} |a_1(t)|^2 \qquad (8.25)$$

Equation (8.25) holds for $\omega \neq \omega_0$.

The energy lost by a two level atom is, in general, equal to the work done by the external optical field on the induced dipole moment.

8.2) Incoherent excitation – low intensities

Based on the above considerations, let us now think about the induced absorption and emission in the atom caused by incoherent light.

When the atom is perturbed incoherently, the wavefunction can also be expressed as a linear superposition of $\Psi_1(t)$ and $\Psi_2(t)$. However, since $\mathcal{E}(t)$ fluctuates the probability amplitudes $a_1(t)$ and $a_2(t)$ fluctuate in an indeterministic manner. Even though $\mu(t)$ has an instantaneous value, the ensemble average (on depending $a_1 a_2^*$ etc) for a large number of atoms or time average $\langle \mu(t) \rangle = 0$.

Although $\langle \mathcal{E}(t) \rangle = 0$, $\langle |\mathcal{E}(t)|^2 \rangle$ is not equal to zero, so that the probability of the induced process is not zero, even for incoherent light.

In fact for low intensities we have just the same B coefficient – thus

$$|a_1(t)|^2 = \frac{\Omega_0^2}{\Omega^2} \sin^2\left(\frac{\Omega t}{2}\right) \rightarrow \frac{\langle \Omega_0^2(\omega) \rangle}{4} \cdot \frac{\sin^2 \frac{\Delta\omega t}{2}}{\left(\frac{\Delta\omega}{2}\right)^2} = \frac{\langle \Omega_0^2(\omega) \rangle 2\pi t\, \delta(\Delta\omega)}{4} \qquad (8.26)$$

$$\langle \Omega_0^2(\omega) \rangle = \frac{\mu_z^2}{\hbar^2} \langle |\mathcal{E}(\omega)| \rangle^2 \quad \& \quad \rho(\omega) = \tfrac{1}{2}\epsilon_0 \langle |\mathcal{E}(\omega)|^2 \rangle \qquad \langle \mu_z^2 \rangle = \langle \mu^2 \rangle / 3 \qquad (8.27)$$

Basically the same argument as previously

We shall see later (in § 10) how the B coefficient may be obtained for intense broadband radiation.

9. Density matrix equations

We are normally interested in populations, such as $|a_1|^2$ or if we require the dipole moment, $a_1^* a_2$ etc. (called coherences).

We organize these quantities to form a density matrix – firstly, it leads to simpler mathematics and secondly, when only probabilities are known, it is useful for statistical ensemble averages.

Consider for simplicity a two level atom

$$i\frac{d\Psi}{dt} = [H_a + H']\Psi \quad \ldots (9.1)$$

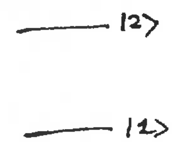


Fig 13

If $\Psi(t) = c_1(t)\varphi_1(\vec{r}) + c_2(t)\varphi_2(\vec{r})$

$$\frac{dc_1}{dt} = -i\omega_1 c_1 - \frac{i}{\hbar} H'_{12} c_2 \quad (9.2)$$

$$\frac{dc_2}{dt} = -i\omega_2 c_2 - \frac{i}{\hbar} H'_{21} c_1 \quad (9.3)$$

$\langle 1|H'|2\rangle = H'_{12}$ etc

$\rho_{11} = c_1 c_1^*$ is the probability of being in the lower level, and similarly for ρ_{22}.

$\rho_{12} = c_1 c_2^*$ & $\rho_{21} = \rho_{12}^* = c_2 c_1^*$ determines the dipole moment.

The expectation value for an operater O is $\langle\Psi|O|\Psi\rangle$

Thus,

$$\langle\Psi|O|\Psi\rangle = (\rho_{11}O_{11} + \rho_{12}O_{21}) + (\rho_{21}O_{12} + \rho_{22}O_{22}) \quad (9.4)$$

$$= \sum_i \sum_j \rho_{ij} O_{ji} = \mathrm{Tr}[\rho O] \quad (9.5)$$

(9.6)

In fact $\langle\mu\rangle = \mu_{12}\rho_{21} + \mu_{21}\rho_{12}$

In this case the density operator is derived from the pure state $\Psi(t)$ (of eq 9.1) which can be written in terms of the eigenstates φ_1 & φ_2.

$$\text{Hence} \quad \rho = |\Psi\rangle\langle\Psi| \quad (9.7)$$

(eg: $\langle\varphi_1|\rho|\varphi_2\rangle = \rho_{12} = c_1 c_2^*$)

Suppose we do not know the state vector precisely but only the probability P_Ψ of being in eg. state Ψ.

Can still define

$$\rho = \sum{}' P_\psi |\Psi\rangle\langle\Psi| . \tag{9.8}$$

For example, we could consider an ensemble of two level atoms each being driven by a chaotic field ; so that H' for the j^{th} atom has a random phase ϕ_j associated with the field. One could imagine a model in which all state vectors are identical except for a distribution of phases, i.e.

$$|\Psi_j\rangle = c_1|1\rangle + c_2 e^{i\phi_j}|2\rangle \tag{9.9}$$

Then $$\rho_{12} = \sum{}' P_j c_1 (c_{2j})^* = \sum{}' P_j c_1 c_2^* e^{-i\phi_j} = c_1 c_2^* \sum{}' P_j e^{-i\phi_j} = 0 \tag{9.10}$$

for random phase.

Thus, ρ is a convenient way to perform the ensemble average.

We still have $$\langle O\rangle = \mathrm{Tr}[\rho O] \tag{9.11}$$

since
$$\begin{aligned}\langle O\rangle &= \sum_\Psi{}' P_\Psi \langle\Psi|O|\Psi\rangle \\ &= \sum_{\Psi,k}{}' P_\Psi \langle\Psi|O|k\rangle\langle k|\Psi\rangle \\ &= \sum_{\Psi,k} \langle k|\Psi\rangle P_\Psi \langle\Psi|O|k\rangle \\ &= \sum_k{}' \langle k|\rho O|k\rangle \\ &= \mathrm{Tr}[\rho O]\end{aligned}$$

Average

$i\hbar|\dot\psi\rangle = H|\psi\rangle$

Also
$$\begin{aligned}\frac{d\rho}{dt} &= \sum{}' P_\psi [|\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi|] \\ &= -i/\hbar \left[\sum{}' P_\psi [H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| H]\right. \\ &= -i/\hbar\, [H,\rho]\end{aligned} \tag{9.12}$$

with $[H,\rho] = \{H\rho - \rho H\}$ (9.13)

Note : $\mathrm{Tr}[\rho] = 1$ (9.14)

Writing out elements, $\omega_{21}=\omega_0$

$$\dot{\rho}_{22} = -\tfrac{i}{\hbar} H'_{21}\rho_{12} + \tfrac{i}{\hbar} H'_{12}\rho_{21} \tag{9.15}$$

$$\dot{\rho}_{11} = -\tfrac{i}{\hbar} H'_{12}\rho_{21} + \tfrac{i}{\hbar} H'_{21}\rho_{12} \tag{9.16}$$

$$\dot{\rho}_{12} = -i\omega_{12}\rho_{12} + \tfrac{i}{\hbar} H'_{12}(\rho_{11}-\rho_{22}) \tag{9.17}$$

$$\rho_{12} = \rho_{21}^* \tag{9.18}$$

(9.19)

Note $\dot{\rho}_{11}+\dot{\rho}_{22}=0$ (follows also from $\rho_{11}+\rho_{22}=1$)

$$\text{Now} \quad H'_{12} = -\frac{\mu_{12}}{2}\left(\mathcal{E} e^{i\omega t} + \mathcal{E}^* e^{-i\omega t}\right) \tag{9.20}$$

Make a RWA by writing

$$\rho_{12} = \tilde{\rho}_{12} e^{i\omega t} \tag{9.21}$$

$$\rho_{21} = \tilde{\rho}_{21} e^{-i\omega t} \tag{9.22}$$

$\omega_0=\omega_{21}$

$$\dot{\rho}_{22} = +\tfrac{i}{\hbar}\frac{\mu_{21}}{2}\left(\mathcal{E}e^{i\omega t}+\mathcal{E}^*e^{-i\omega t}\right)\tilde{\rho}_{12}e^{i\omega t} + \text{c.c.} \tag{9.23}$$

$$\dot{\tilde{\rho}}_{12} = -i(\omega-\omega_0)\tilde{\rho}_{12} - \tfrac{i}{\hbar} e^{-i\omega t}\mu_{12}\left(\mathcal{E}e^{i\omega t}+\mathcal{E}^*e^{-i\omega t}\right)(\rho_{11}-\rho_{22}) \tag{9.24}$$

The sign of the exponential in (9.21) was chosen to given $\omega-\omega_0$ in (9.24) (rather than $\omega+\omega_0$)

Ignoring $e^{\pm 2i\omega t}$ terms

$$\dot{\rho}_{22} = i\frac{\Omega_0^*}{2}\tilde{\rho}_{12} - i\frac{\Omega_0}{2}\tilde{\rho}_{21} \tag{9.25}$$

$$\dot{\tilde{\rho}}_{12} = -i(\omega-\omega_0)\tilde{\rho}_{12} - i\frac{\Omega_0}{2}(\rho_{11}-\rho_{22}) \tag{9.26}$$

$$\Omega_0 = \frac{\mu_{12}\mathcal{E}}{\hbar} \tag{9.27}$$

9.1. Decay phenomena

The states which we deal with are not eigenstates — they can decay by spontaneous emission or perturbed by collisions

For example, for spontaneous emission (with $A=\Gamma$) the rate of decay of excited state is $d\rho_{22}/dt = -\Gamma\rho_{22}$ (9.28)

This can be included phenomenologically by $c_2(t) = e^{-\Gamma t/2}$ (note: factor of 2 for $|c_2|^2$ for ρ_{22}) — Wigner-Weisskopf theory gives the same result.

This indicates decay of coherence as $\frac{d\rho_{21}}{dt} = -\Gamma/2\,\rho_{21}$ (9.29)

In general, ρ_{11} & ρ_{22} do not depend on phase of the wavefunction, whereas ρ_{12} does depend on relative phase (see eq (9.10)). Collisions can interrupt the phase of the dipole moment without causing inelastic transitions.
[The phase of ρ_{12} determines the phase of (the expectation value of) the dipole].

Thus, in general damping effects on diagonal elements (populations) are different from off-diagonal elements (coherences).

For populations $k = \frac{1}{T_1}$ (Longitudinal relaxation) (9.30)

coherences $k' = \frac{1}{T_2}$ (Transverse relaxation) (9.31)

For a two level atom, often a good approximation

$$k = \Gamma \qquad (9.32)$$

$$k' = \Gamma/2 + \gamma_{ph} \qquad (9.33)$$

where γ_{ph} is due to phase interrupting (line broadening) collisions — having made what is known as the IMPACT approximation.

9.2 Bloch equations

In RWA

BLOCH equations

$$\frac{d\rho_{22}}{dt} = -K\rho_{22} - \frac{i}{2}\left[\Omega_0\tilde{\rho}_{21} - \Omega_0^*\tilde{\rho}_{12}\right] \qquad (9.34)$$

$$\frac{d\rho_{11}}{dt} = K\rho_{22} + i/2\left[\Omega_0\tilde{\rho}_{21} - \Omega_0^*\tilde{\rho}_{12}\right] \qquad (9.35)$$

$$\frac{d\tilde{\rho}_{12}}{dt} = \left[-i(\omega-\omega_0)-K'\right]\tilde{\rho}_{12} - i\frac{\Omega_0}{2}(\rho_{11}-\rho_{22}) \qquad (9.36)$$

$$\frac{d\tilde{\rho}_{21}}{dt} = \left[\ i(\omega-\omega_0)-K'\right]\tilde{\rho}_{21} + i\frac{\Omega_0^*}{2}(\rho_{11}-\rho_{22}) \qquad (9.37)$$

Note: $\frac{d}{dt}(\rho_{11}+\rho_{22}) = 0$ (9.38)

For monochromatic fields, $\Omega_0 =$ constant, these equations are best solved by Laplace transform methods.

In steady state $d/dt = 0$ $\quad \rho_{11}+\rho_{22} = 1 \quad \Delta\omega = \omega-\omega_0$

from (9.37)

$$\Omega_0\tilde{\rho}_{21} = \frac{i|\Omega_0|^2}{2}\frac{(\rho_{11}-\rho_{22})}{[K'-i\Delta\omega]} \qquad (9.39)$$

& subst. in (9.34)

$$K\rho_{22} = \frac{|\Omega_0|^2}{4}\frac{2K'(\rho_{11}-\rho_{22})}{[K'^2+\Delta\omega^2]}$$

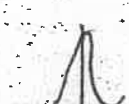
$$= \frac{|\Omega_0|^2}{2}\frac{K'}{[\Delta\omega^2+K'^2]}(1-2\rho_{22}) \qquad (9.40)$$

Hence,

$$\rho_{22} = \frac{\frac{1}{2}\frac{K'|\Omega_0|^2}{(\Delta\omega^2+K'^2)}}{K + \frac{|\Omega_0|^2K'}{(\Delta\omega^2+K'^2)}} \rightarrow 1/2 \text{ as } |\Omega_0|^2 \to \infty \qquad (9.41)$$

also

$$(\rho_{11}-\rho_{22}) = \frac{(\Delta\omega^2+K'^2)}{\Delta\omega^2+K'^2+\frac{K'}{K}|\Omega_0|^2} \qquad (9.42)$$



Note: (1) Atomic response via $K'/(\Delta\omega^2+K'^2)$

(2) $\sqrt{K'^2+K'|\Omega_0|^2/K}$ sometimes referred to as power broadened rate (see §10)

10) Maxwell-Bloch equations

Here we consider the effect of a two level medium on the propagation of laser light. In an actual medium, there will be many transitions other than the two levels in resonance with the incident light, which can be characterized by a linear susceptibility χ_L. We will denote by $\vec{P}$ the polarization caused by the two levels. Then total $\vec{P}$, $\vec{P}_{total} = \chi_L \vec{E} + \vec{P}$ (10.1)

$$\therefore \quad \vec{D} = \epsilon_0 \vec{E} + \vec{P}_{total} = \epsilon \vec{E} + \vec{P} \quad \text{with } \epsilon = \epsilon_0 + \chi_L = n^2 \tag{10.2}$$

$$\nabla \times (\nabla \times \vec{E}) + \epsilon \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \vec{P}}{\partial t^2} \tag{10.3}$$

In general, $\vec{P}$ is in response to $\vec{E}$ and although saturation (dependence on $|\vec{E}|^2$) occurs, it is expected to be in the direction of $\vec{E}$ i.e. transverse

$$\therefore \quad \operatorname{div} \vec{P} = -i\vec{k}.\vec{P} = 0 \quad \& \operatorname{div} \vec{D} = \epsilon \operatorname{div} \vec{E} = 0 \tag{10.4}$$

Take z direction in direction of propagation, and $v^2 = 1/\epsilon\mu_0 = c^2/n^2$

$$\frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2} \tag{10.5}$$

and express $\vec{E}$ and $\vec{P}$ for plane waves (with $\hat{\epsilon}.\hat{k} = 0$) as

$$\vec{E}(z,t) = \frac{1}{2} E(z,t) e^{i(\omega t - kz)} \hat{\epsilon} + c.c \tag{10.6}$$

$$\vec{P}(z,t) = \frac{1}{2} P(z,t) e^{i(\omega t - kz)} \hat{\epsilon} + c.c \tag{10.7}$$

$$\therefore \quad \frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{2}\left(\frac{\partial^2 E}{\partial z^2} - 2ik\frac{\partial E}{\partial z} - k^2 E\right) e^{i(\omega t - kz)} \hat{\epsilon} + c.c. \tag{10.8}$$

If spatial variation of E is gradual compared to k, then $\left|\frac{\partial E}{\partial z}\right| \ll k|E|$ and $\frac{\partial^2 E}{\partial z^2}$ can be neglected. Similarly, $\left|\frac{\partial E}{\partial t}\right| \ll \omega |E|$ and $\frac{\partial^2 E}{\partial t^2}$ can be neglected, for temporal variation of E slow compared to ω.

Slowly varying envelope approximation — SVEA

The equation for $\frac{\partial^2 \vec{P}}{\partial t^2}$ is similar, but $\vec{P}$ is a perturbation to the wave equation and so can neglect both $\partial P/\partial t$ and $\partial^2 P/\partial t^2$.

With $k = \omega n/c$

$$\frac{\partial E}{\partial z} + \frac{n}{c}\frac{\partial E}{\partial t} = -\frac{j k P}{2\epsilon} \tag{10.9}$$

If $\vec{P} = 0$, $\frac{dE}{d\tau} = 0$ with $\tau = t - \frac{n}{c}z$, so $E(\tau) = \text{constant}$, thus the waveform propagates without dispersion (ie. without change of shape). If $P \neq 0$, the waveform changes.

Now, for two level atom, we have $\vec{\mu} = \mu\hat{e}$ since $\vec{\mu}$ is in the direction of field $\vec{E}$

$$\langle \mu \rangle = Tr[\rho\mu] = \rho_{12}\mu_{21} + \rho_{21}\mu_{12}$$

$$= \tilde{\rho}_{12}\mu_{21}e^{j\omega t} + c.c. \tag{10.10}$$

$$= -i\frac{|\mu_{21}|^2}{2\hbar}\frac{E(\rho_{11}-\rho_{22})}{i\Delta\omega + K'}e^{j\omega t} + c.c. \tag{10.11}$$

In steady state

(for atom not at origin must include $e^{\pm ikz}$ factors)

$$\vec{P} = N\langle\vec{\mu}\rangle \quad \text{where } N \text{ is the number of dipoles / unit volume} \tag{10.12}$$

$$\therefore P(z,t) = (\chi' - i\chi'')E(z,t) = \chi E(z,t) \tag{10.13}$$

$$\text{with } \chi' = -N\frac{|\mu_{21}|^2}{\hbar}\frac{\Delta\omega(\rho_{11}-\rho_{22})}{\Delta\omega^2 + K^2} = N\frac{|\mu_{01}|^2}{\hbar}\frac{\Delta\omega}{\Delta\omega^2 + K'^2 + \frac{K'}{K}|\Omega_0|^2} \tag{10.14}$$

(using eq. 9.42)

$$\chi'' = N\frac{|\mu_{21}|^2}{\hbar}\frac{K'}{\Delta\omega^2 + K'^2 + \frac{K'}{K}|\Omega_0|^2} = N\frac{|\mu_{21}|^2}{\hbar}\frac{K'(\rho_{11}-\rho_{22})}{(\Delta\omega^2 + K'^2)} \tag{10.15}$$

Note (1) with $E(z,t) = E_0 e^{-i(k'-k)z}$, eq (10.9) implies $k' = k(1 + \chi/2\epsilon) = \frac{n\omega}{c}(1 + \chi/2\epsilon) \sim \frac{\omega}{c}\sqrt{\epsilon_0(1 + \chi_i + \chi)}$

This is as expected. However, χ is a function of $|E|^2$ and is not constant

(2) χ' & $\chi'' \to 0$ as $|\Omega_0|^2 \to \infty$. Saturation.

(3) As a function of $\Delta\omega$, the width of the response χ'' is not K' but $\sqrt{K'^2 + \frac{K'}{K}|\Omega_0|^2}$ ie. power broadening [If $K' = \Gamma'/2$ & $K = \Gamma'$, $W \approx \Gamma'/2 + \frac{|\Omega_0|^2}{2\Gamma'}$ for $|\Omega_0|^2 \ll 1$]

Suppose the laser is propagating in some solid angle $\Delta\Omega$ ($\sim(\lambda/w)^2$ where w is the beam waist)

Then we can define an intensity (cycle averaged) via

$$I_L = I\,\Delta\Omega = c\,\frac{\epsilon}{2}EE^* \tag{10.16}$$

(This is consistent with $I = c\rho(\omega)/4\pi$ for isotropic light & $\int I\,d\Omega = c\rho(\omega)$ in general)

Hence, in steady state $\partial/\partial t = 0$ and using eq (10.9)

$$\Delta\Omega.\frac{dI}{dz} = \frac{c\epsilon}{2}\left(E\frac{dE^*}{dz} + E^*\frac{dE}{dz}\right) = \frac{c\epsilon}{2}.2\,\mathrm{Im}\left[\frac{kPE^*}{2\epsilon}\right] = -2\alpha(\omega)\,I\,\Delta\Omega \tag{10.17}$$

$$\text{or}\quad \frac{dI}{dz} = -2\alpha(\omega)I \tag{10.18}$$

Thus $\alpha(\omega) = \dfrac{k\chi''(\omega)}{2\epsilon}$ is the amplitude absorption coefficient.

Since we have considered only the z-component $B = \dfrac{\pi}{\epsilon_0\hbar^2}|\mu_{21}|^2 = \dfrac{c}{4\pi}B^I$

(remember $\langle\mu_{12}\rangle^2 = \langle\mu^2\rangle/3$)

B^I being defined in terms of intensity rather than $\rho(\omega)$

Then

$$2\alpha(\omega) = \frac{k\chi''(\omega)}{\epsilon_0} = \frac{\hbar\omega}{c}\,\frac{|\mu_{12}|^2\pi}{\hbar^2\epsilon_0}\,\frac{k'/\pi}{\Delta\omega^2+k'^2}\,(N\rho_{11}-N\rho_{22})$$

$$= \frac{\hbar\omega}{c}Bg(\omega)(N_{11}-N_{22}) \tag{10.19}$$

with $N\rho_{11} = N_{11}$ & $g(\omega) = \dfrac{k'/\pi}{\Delta\omega^2+k'^2}$

$$\text{or}\quad 2\alpha(\omega) = \frac{\hbar\omega}{4\pi}B^I g(\omega)(N_{11}-N_{22}) \tag{10.20}$$

$$\frac{dI}{dz} = -\frac{\hbar\omega}{4\pi}B^I g(\omega)(N_{11}-N_{22})\,I \tag{10.21}$$

or, multiplying by $\Delta\Omega$

$$\frac{dI_L}{dz} = -\frac{\hbar\omega}{4\pi}B^I g(\omega)(N_{11}-N_{22})\,I_L \tag{10.22}$$

10.1) Radiative transfer

The above equation (10.22) may be obtained by simple energy arguments.

The transfer equation is obtained by considering the change in energy as a beam of intensity $I(z)$ and solid angle $\Delta\Omega$, passes through a cylindrical volume of length Δz. (as shown in Fig 14)

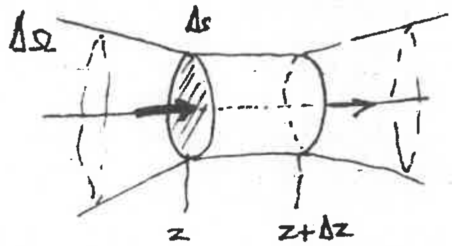


Fig 14

$I \Delta\Omega \Delta s \Delta t$ is the amount of energy entering the volume.

$\therefore \dfrac{dI}{dz} \cdot \Delta z \, \Delta\Omega \, \Delta s \, \Delta t$ is change in energy (10.23)

Energy absorbed from beam in volume in time dt is, taking difference between absorption and stimulated emission,

$$\hbar\omega \, (N_{11} - N_{22}) \, B \, \rho(\omega) g(\omega) \, \Delta t \, \Delta s \, \Delta z \qquad (10.24)$$

Note: each absorption takes up energy $\hbar\omega$, and $\rho(\omega) = \dfrac{I \Delta\Omega}{c}$ for beam (see eq 10.16)

Spontaneous emission at rate A_{21} with emission profile $g(\omega)$ can also add energy to the beam, i.e. $\hbar\omega \, N_{22} \, A_{21} g(\omega) \Delta t \, \Delta s \, \Delta z \, \dfrac{\Delta\Omega}{4\pi}$ (10.25)

The factor $\Delta\Omega/4\pi$ accounts for amount of isotropically radiated spontaneous emission which goes into solid angle

$$\therefore \frac{dI}{dz} = -\hbar\omega (N_{11} - N_{22}) B \frac{I}{c} g(\omega) + N_{22} A_{21} \frac{\hbar\omega}{4\pi} g(\omega) \qquad (10.26)$$

$$= -\frac{\hbar\omega}{4\pi} (N_{11} - N_{22}) B^{I} g(\omega) \, I + N_{22} A_{21} \frac{\hbar\omega}{4\pi} g(\omega) \qquad (10.27)$$

EQUATION OF TRANSFER

$$\frac{dI}{dz} = -K(\omega) I + j(\omega) \qquad (10.28)$$

$$K(\omega) = \frac{\hbar\omega}{4\pi} (N_{11} - N_{22}) B^{I} g(\omega) \qquad (10.29)$$

$$j(\omega) = N_{22} A_{21} \frac{\hbar\omega}{4\pi} \qquad (10.30)$$

Note: For a laser, which is coherent, the incoherent spontaneous emission (or possible scattering from other possible modes) is unimportant.

This is because $I(\omega)\Delta\Omega = I_L$ - a constant as $\Delta\Omega \to 0$, whereas

$A_{21}\Delta\Omega \to 0$.

10.2) Incoherent excitation

The equation of transfer (eq. (10.29)) holds for incoherent radiation being obtained by considering conservation of radiant energy. For laser the Maxwell-Bloch equation (10.9) (or its equivalent, eq. (10.22)) should be used.

We now consider excitation due to an incoherent, multimode field.

$$E(t) = \sum_p \left(\frac{\mathcal{E}_p}{2} e^{i\omega_p t} + c.c\right) \qquad (10.31)$$

$$= \left[\frac{\mathcal{E}(t)}{2} e^{i\bar{\omega}t} + c.c\right] \qquad (10.32)$$

The second form of equation (10.32) writes the field with respect to the mean frequency $\bar{\omega}$. Now $\mathcal{E}(t) = \sum_p \mathcal{E}_p e^{i(\omega_p - \bar{\omega})t}$. For a monochromatic beam at frequency $\bar{\omega}$, $\mathcal{E}(t)$ is of course constant. Here $\langle \mathcal{E}(t)\rangle = 0$

For a broadband field it is useful to form the autocorrelation function $\langle \mathcal{E}(t)\mathcal{E}^*(t')\rangle$ where $\langle \ldots \rangle$ represents the ensemble average.

$$\langle \mathcal{E}(t)\mathcal{E}^*(t')\rangle = \sum_p |\mathcal{E}_p|^2 e^{i(\omega_p - \bar{\omega})(t-t')} \qquad (10.33)$$

To obtain this result we assumed that the modes were uncorrelated i.e. $\langle \mathcal{E}_p \mathcal{E}_\nu^* \rangle = |\mathcal{E}_p|^2 \delta_{p\nu}$ (10.34)

$|\mathcal{E}_p|^2$ is just proportional to $\rho(\omega_p)$, so replacing the sum by an integral

$$\langle \mathcal{E}(t)\mathcal{E}^*(t')\rangle = \frac{2}{\epsilon_0}\int d\omega' \rho(\omega') e^{i(\omega' - \bar{\omega})(t-t')} \qquad \rho(\omega') = \tfrac{1}{2}\epsilon_0 |E^2(\omega')| \qquad (10.34)$$

Performing the Fourier transform, for eg. a Lorentzian $\rho(\omega') = \dfrac{b/\pi}{(\omega' - \bar{\omega})^2 + b^2}$, show that

$$\langle \mathcal{E}(t)\mathcal{E}^*(t')\rangle \sim e^{-b|t-t'|} \qquad (10.35)$$

Effectively, the correlation time for the field is $1/b$ where b is the "bandwidth".

The Bloch equations (9.34) and (9.37) now become

$$\frac{d\rho_{22}}{dt} = -K\rho_{22} - \tfrac{i}{2}\left[\Omega(t)\tilde{\rho}_{21} - \Omega_0^*(t)\tilde{\rho}_{12}\right] \tag{10.36}$$

& $$\frac{d\tilde{\rho}_{21}}{dt} = \left[i(\bar{\omega}-\omega_0) - K'\right]\tilde{\rho}_{21} + i\frac{\Omega_0^*(t)}{2}\left[\rho_{11}-\rho_{22}\right] \tag{10.37}$$

with $\Omega(t) = \dfrac{\mu_{12}\mathcal{E}(t)}{\hbar}$ (10.38)

Integrate eq (10.38) $$\tilde{\rho}_{21}(t) = i\int_{-\infty}^{t} \frac{\Omega^*(t')}{2} e^{[i(\bar{\omega}-\omega_0)-K'](t-t')}\left[\rho_{11}(t')-\rho_{22}(t')\right]dt' \tag{10.39}$$

and substitute into (10.36)

$$\frac{d\rho_{22}}{dt} = -K\rho_{22} + \frac{1}{4}\int_{-\infty}^{t} \Omega(t)\Omega^*(t') e^{[i(\bar{\omega}-\omega_0)-K'](t-t')}\left[\rho_{11}(t')-\rho_{22}(t')\right] + c.c. \tag{10.34}$$

This must now be averaged over an ensemble of atoms. The populations $\rho_{22}(t')$ etc vary on a time scale of order $1/K$, whereas $\langle\Omega(t)\Omega^*(t')\rangle$ goes to zero in a time of order $1/b$. $\rho_{22}(t')$ is determined by fields occuring in the interval (of order $1/K$) up to t' and $\Omega(t)\Omega^*(t')$ depends on the fields in the interval $t-t'$ ($\sim 1/b$). When $b \gg K$ the correlation time is short, and we can make the DECORRELATION APPROXIMATION

ie. $$\langle\Omega(t)\Omega^*(t')\rho(t')\rangle \sim \langle\Omega(t)\Omega^*(t')\rangle\langle\rho(t')\rangle \sim \langle\Omega(t)\Omega^*(t')\rangle\langle\rho(t)\rangle \tag{10.35}$$

since fields in interval $t-t'$ are uncorrelated with those determining the overall behavior of $\rho(t')$.
The last term in eq (10.35) follows because $\langle\rho(t)\rangle$ is slowly varying over the correlation time.

$$\therefore \frac{d\rho_{22}}{dt} = -K\rho_{22} + \frac{|\mu_{21}|^2}{2\hbar^2\varepsilon_0}\left[\rho_{11}(t)-\rho_{22}(t)\right]\left[\int_{-\infty}^{t}dt'\int d\omega'\rho(\omega')e^{[i(\omega'-\omega_0)-K'](t-t')} + c.c.\right] \tag{10.36}$$

$$= -K\rho_{22}(t) + \left[\rho_{11}(t)-\rho_{22}(t)\right]B\int d\omega'\rho(\omega')g(\omega') \tag{10.37}$$

Note: $\bar{\omega}$ cancells; with $g(\omega') = \dfrac{K'/\pi}{(\omega'-\omega_0)^2 + K'^2}$ (10.38)

This is the usual Einstein type RATE EQUATION

– broadband field shows no Rabi oscillations.