Jackson's discussion of transformations is really specific to vector quantities. It's possible to do this in greater generality. To be fair, all we need ~~is the~~ to ~~study~~ do classical E&M is the story for vectors, and we basically have that with the Lorentz transformations, but lets be more general. We will learn some interesting things about relativistic field theory

Begin with our transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = \left[\delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu}\right] x^{\nu}$$

$$= x^{\mu} + \delta x^{\mu} \qquad \delta x^{\mu} = \epsilon^{\mu}{}_{\nu} x^{\nu} = \epsilon^{\mu\lambda} x_{\lambda}$$

Recall, $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ to preserve $x^{\mu} x_{\mu}$.

Write $\delta x^{\mu} = \epsilon^{\mu}{}_{\nu} x^{\nu}$

$$= \frac{i}{2} \epsilon^{\rho\sigma} L_{\rho\sigma} x^{\mu}$$

where $L_{\mu\nu} = i\left[x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right]$ is a sort of generalized angular momentum. $\partial_{\mu} = \left[\frac{\partial}{\partial t}, \vec{\nabla}\right]$

(note $L_{\nu\mu} = -L_{\nu\mu}$ - there are six L's)

Recall $\epsilon^{\rho\sigma}$ is a parameter. This is like with

$$\delta\vec{x} = i(\vec{\theta}\cdot\vec{L})\cdot\vec{x}$$

for a rotation.

Check this improbable result

$$\delta x^{\mu} = \frac{i}{2}\epsilon^{\rho\sigma} L_{\rho\sigma} x^{\mu} = \frac{i}{2}(i)\epsilon^{\rho\sigma}\left[x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}\right]x^{\mu}$$

$$\partial_{\sigma} x^{\mu} = \delta_{\sigma}^{\mu}, \quad \partial_{\rho} x^{\mu} = \delta^{\mu}{}_{\rho}$$

$$\delta x^{\mu} = -\frac{1}{2}\epsilon^{\rho\sigma}\left[x_{\rho}\delta_{\sigma}^{\mu} - x_{\sigma}\delta^{\mu}{}_{\rho}\right]$$

$$= -\frac{1}{2}\left[\epsilon^{\rho\mu} x_{\rho} - \epsilon^{\mu\sigma} x_{\sigma}\right] = \epsilon^{\mu\nu} x_{\nu}$$ — flips indices on first term.

Now the point is, ϵ is a number, $L_{\mu\nu}$ is an operator. What is its algebra? It is easy though tedious to grind out the commutator - it is

$$[L_{\mu\nu}, L_{\rho\sigma}] = i g_{\nu\rho} L_{\mu\sigma} - i g_{\mu\rho} L_{\nu\sigma}$$
$$- i g_{\nu\sigma} L_{\mu\rho} + i g_{\mu\sigma} L_{\nu\rho} \quad (*)$$

The L's are the generators of a Lie algebra. Actually, some of the entries are familiar. Look at μ, ν spacelike, define $L_i = \frac{1}{2} \epsilon_{ijk} L_{jk}$

(ex $L_1 = \frac{1}{2}(L_{23} - L_{32}) = L_{23}$ - L is antisymmetric

$$[L_1, L_2] = [L_{23}, L_{31}] = i g_{33} L_{21} = -i L_{21} = i L_{12} = i L_3$$

$(\mu, \nu, \rho, \sigma = 2331)$

which is the usual angular momentum commutator.

L doesnt know about spin, but (in complete analogy to orbital vs spin angular momentum) we can imagine that we have states which are characterized by a set of internal labels, ~~Spin~~ affected by operators with the same ~~quantum numbers~~ commutation relations as $*$, call them $S_{\mu\nu}$, and

$$[L_{\mu\nu}, S_{\mu\nu}] = 0$$

Then the most general representation of the generator is

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$$

and we have a generalized infinitesimal rotation matrix $\quad D(\epsilon) = 1 + \frac{i}{2} \epsilon^{\mu\nu} M_{\mu\nu}$

Now for the miracle. Define

$$J_i \equiv \frac{1}{2}\epsilon_{ijk} M_{jk} \quad 3\ J\text{'s}$$

$$K_i = M_{0i} \quad 3\ K\text{'s}$$

and two linear combinations

$$A_i = \frac{1}{2}[J_i + iK_i]$$

$$B_i = \frac{1}{2}[J_i - iK_i]$$

discover

$[A_i, B_j] = 0$

$[A_i, A_j] = i\epsilon_{ijk} A_k$

$[B_i, B_j] = i\epsilon_{ijk} B_k$

A & B obey the algebra of SU(2).

$[J_i, J_j] = i\epsilon_{ijk} J_k$

$[J_i, K_j] = i\epsilon_{ijk} K_k$

$[K_i, K_j] = -i\epsilon_{ijk} J_k$

J's generate rotations

K's generate boosts

Recall how eigenstates of angular momentum behave - states are labeled by a J which tell us its transformation properties under rotations - in fact, there are 2J+1 states, labelled by m, $-J < m < J$, which mix under rotations.

For the Lorentz group there are 2 such indices. States are labelled by pairs of integers a & b

$$A^2 |\psi_{AB}\rangle = a(a+1)|\psi_{AB}\rangle$$

$$B^2 |\psi_{AB}\rangle = b(b+1)|\psi_{AB}\rangle$$

$a, b = 0, \frac{1}{2}, 1 \ldots$

Interesting - and useful - this is where "intrinsic spin of particle" comes from.

Note under parity – J is an axial vector, K is a vector

$J_i \to J_i$

$K_i \to -K_i$

this means – reflection is equivalent to exchange $A \leftrightarrow B$

○ Irreducible representations of Lorentz group are <u>not</u> necessarily parity eigenstates

Example – left handed neutrino – $A = 1/2 \quad B = 0$

Since $J = A + B$, usual spin of reps is $J = a + b$

States conveniently labelled

$$\begin{pmatrix} - \\ - \\ - \\ \hline - \\ - \\ - \end{pmatrix} \begin{matrix} \leftarrow 2a+1 \text{ entries for } A \text{ quantum \#'s} \\ \leftarrow 2b+1 \text{ entries for } B \end{matrix}$$

Spin-$\frac{1}{2}$ $\quad a = 1/2 \quad b = 0 \quad$ or $\quad a = 0 \quad b = \frac{1}{2}$ $\quad \left(\begin{matrix}2\\ \text{comps}\end{matrix}\right)$

<u>not</u> parity eigenstate. Dirac particle <u>is</u> parity eigenstate $\quad (a, b) = (\frac{1}{2}, 0) + (0, \frac{1}{2})$ direct product

$$\begin{pmatrix} 1 \\ 2 \\ \hline 3 \\ 4 \end{pmatrix} \quad \text{4-component spinor}$$

Spin-0 $\quad (0,0)$ is overkill – just a one-component state.

How do states transform?

$$D(\epsilon) = 1 + \frac{1}{2}\epsilon^{\mu\nu} M_{\mu\nu} = 1 + i(\vec{\theta}_A \cdot \vec{A}) + i(\vec{\theta}_B \cdot \vec{B})$$

$$\longrightarrow \left[\begin{array}{c|c} e^{iA\cdot\theta_A} & 0 \\ \hline 0 & e^{iB\cdot\theta_B} \end{array}\right]$$

easy to transform in (A,B) basis.

Also $\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K})$ $\quad B = \frac{1}{2}(\vec{J} - i\vec{K})$

$$D = 1 + \frac{i}{2}\vec{J}\cdot(\vec{\theta}_A + \vec{\theta}_B) - \frac{\vec{K}}{2}\cdot(\vec{\theta}_A - \vec{\theta}_B)$$

$$\equiv 1 + i\,J\cdot\vec{\omega} - \vec{K}\cdot\vec{\zeta}$$

3 ω's, 3 ζ's

and we are back to Jackson

<u>Vector</u> <u>fields</u> are usually treated more simply.
The analogy is like cartesian vectors $\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

vs spherical vectors $\begin{pmatrix} A_+ = \frac{A_x + iA_y}{\sqrt{2}} \\ A_0 = A_z \\ A_- = \frac{A_x - iA_y}{\sqrt{2}} \end{pmatrix}$

Case of pure boost, $\vec{\omega} = 0$, $\vec{\zeta} \neq 0$ is interesting.

$\delta x^\mu = \epsilon^{\mu\nu} x_\nu$ is easiest starting point

$\delta x^0 = \epsilon^{01} x_1 = -\epsilon^{10} x_1 = +\epsilon^{10} x^1$

$\delta x^1 = \epsilon^{10} x^0$

i.e. $\delta\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \epsilon^{10}\begin{pmatrix} x^1 \\ x^0 \end{pmatrix} = \epsilon^{10}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$

call $\epsilon^{10} \equiv \delta\zeta$)

infinitesimal $D(\delta\zeta) = 1 + \zeta\sigma_x \to \exp(\zeta\sigma_x)$

$$= \begin{bmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{bmatrix}$$

or $$V'= \begin{bmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} V$$

for boost along x

Finally: $[J_i, J_j] = i\epsilon_{ijk} J_k$ – rotations don't commute

$[J_i, K_j] = i\epsilon_{ijk} K_k$
rotation + boost don't commute

$[K_i, K_j] = -i\epsilon_{ijk} J_k$
non collinear boosts don't commute – there is a rotation

Thomas Precession

Recall electron ~~spin~~ magnetic moment — a la Goudsmit & Uhlenbeck 1926

$$\vec{\mu} = \frac{ge}{2mc}\vec{s}$$

$$g = 2$$

$$s_z = \pm \hbar/2$$

$g = 2$ explains anomalous Zeeman effect

$$H = \frac{e}{2mc}(\vec{L} + \underbrace{2\vec{S}}_{g=2}) \cdot B$$

but Fine structure too ~~[illegible]~~ big ~~[illegible]~~ by $\times 2$

Twenty years later, Einstein heard something about the Lorentz group that greatly surprised him. It happened while he was in Leiden. In October 1925 George Eugene Uhlenbeck and Samuel Goudsmit had discovered the spin of the electron [U1] and thereby explained the occurrence of the alkali doublets, but for a brief period it appeared that the magnitude of the doublet splitting did not come out correctly. Then Llewellyn Thomas supplied the missing factor, 2, now known as the Thomas factor [T1]. Uhlenbeck told me that he did not understand a word of Thomas's work when it first came out. 'I remember that, when I first heard about it, it seemed unbelievable that a relativistic effect could give a factor of 2 instead of something of order v/c. . . . Even the cognoscenti of the relativity theory (Einstein included!) were quite surprised' [U2]. At the heart of the Thomas precession lies the fact that a Lorentz transformation with velocity $\vec{v}_1$ followed by a second one with a velocity $\vec{v}_2$ in a different direction does not lead to the same inertial frame as one single Lorentz transformation with the velocity $\vec{v}_1 + \vec{v}_2$ [K1]. (It took Pauli a few weeks before he grasped Thomas's point.*)

Pais, "Subtle is the Lord"

Recall derivation of fine structure: electron moves with velocity $\vec{v}$ in external fields $\vec{E}$ & $\vec{B}$

$$\frac{d\vec{s}}{dt} = \vec{\mu} \times \vec{B}$$

In NR limit $B' \sim \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right)$ — we'll derive this later

$$\mu = \frac{ge}{2mc}\vec{S}$$

$$\left.\frac{d\vec{S}}{dt}\right)_{\text{rest frame}} = \vec{\mu}\times\vec{B}'$$ eqn of motion of electron's spin

or $$U' = -\vec{\mu}\cdot\vec{B}'$$ energy of interaction

$$\vec{B}' = B - \frac{v}{c}\times\vec{E}$$

$$e\vec{E} = \frac{\vec{r}}{r}\frac{\partial V}{\partial r} \Rightarrow B' = B - \frac{\vec{v}\times\vec{r}}{cr}\frac{\partial V}{\partial r}$$

$$L = \vec{r}\times m\vec{v} \Rightarrow U' = -\mu\cdot\left[B + \frac{L}{mcr}\frac{\partial V}{\partial r}\right]$$

$$U' = \cancel{-}\frac{ge}{2mc}\vec{S}\cdot\vec{B} + \frac{ge}{2m^2c^2}L\cdot S\,\frac{1}{r}\frac{\partial V}{\partial r}$$

This isn't quite right, because as it happens, the electron's rest frame rotates with time w/ frequency $\vec{\omega}_T$

Usual classical mechanics - $$\left.\frac{d\vec{G}}{dt}\right|_{\text{nonrotating frame}} = \left.\frac{dG}{dt}\right)_{rot} + \vec{\omega}_T\times\vec{G}$$

$$\frac{d\vec{S}}{dt} = \vec{S}\times\left[\frac{geB'}{2mc} - \vec{\omega}_T\right]$$

$$U = U' + \vec{S}\cdot\vec{\omega}_T.$$

Where does the rotation come from? Suppose at time t, velocity of e^- rest frame wrt lab is

$$\vec{V}(t) = c\vec{\beta} \longleftrightarrow \underset{e \text{ rest frame}}{x'} = A_{boost}(\beta)\underset{\text{lab coord}}{x}$$

later time $$\vec{V}(t+\delta t) = c(\vec{\beta}+\delta\vec{\beta}) \longleftrightarrow x'' = A_{boost}(\beta+\delta\beta)x$$

These A's are pure boosts. But what is $x'' = A_T x'$

$$A_T = A_{boost}(\vec{\beta}+\vec{\delta}\beta)A^{-1}_{boost}(\beta) = A_{boost}(\vec{\beta}+\vec{\delta}\beta)A_{boost}(-\vec{\beta})$$

Particle (electron) moves along a trajectory under external forces

At time t_1, $v/c = \vec{\beta}$

At time t_2 $\frac{v}{c} = \vec{\beta} + \delta\vec{\beta}$

To L.T. from frame 1 to frame 2, go $1 \to lab \to 2$ by pure boosts

$$x_2 = \Lambda x_1$$

$$\Lambda = \Lambda(\vec{\beta} + \delta\vec{\beta})\, \Lambda(-\vec{\beta})$$

contrast this with a pure boost from 1 to 2

$$x_2' = \Lambda(\delta\vec{\beta})\, x_1$$

Is $x_2' = x_2$?!

call $\vec{\zeta}(\beta) \equiv \zeta'$ $(= \hat{\beta} \tanh \beta)$

$\vec{\zeta}(\beta + \delta\beta) \equiv \zeta$

$\Lambda = \exp -\zeta \cdot K$ K = 4x4 matrix (generator of LT

~~[illegible]~~

Now we will find that the product is a pure boost and a pure rotation, $A(\vec{\beta}+\delta\vec{\beta})A(-\vec{\beta}) \simeq R(\Delta\Omega)A(\Delta\beta)$

~~[illegible]~~ pure boost gives $B \to B - \frac{v}{c}\times E$.

But ~~[illegible]~~ the rotation contaminates the NR equation of motion. What we really want to think about is the ~~[illegible]~~ rest frame coordinates at time $t+\delta t$, found by a boost alone:

$$x''' = A_{boost}(\Delta\vec{\beta})\,x'$$

$$= R(-\Delta\Omega)A_b(\vec{\beta}+\delta\vec{\beta})\,x$$

We find $\Delta\Omega$

The proper time rate of precession $\frac{dG}{d\tau} = \omega_T \times G$

where $\omega_T = -\frac{\Delta\Omega}{\delta t}$; $\frac{dG}{dt} = \frac{1}{\gamma}\frac{dG}{d\tau}$;

and ~~[illegible]~~ $U' = -\vec{s}\cdot\left[\frac{g e B'}{2mc} - \frac{\vec{\omega}_T}{\gamma}\right]$

So if we can find $\Delta\Omega$ we can get ω_T.

~~A quick way to go~~

A tedious way to proceed is to follow Jackson, and explicitly multiply the matrices. A quicker way to go (though it only works cleanly in the NR limit) is to look at the product of two infinitesimal boosts

$$A\cdot A = \exp\left[-\vec{\zeta}(\beta+\delta\beta)\cdot\vec{K}\right]\exp\left[\vec{\zeta}(\beta)\cdot\vec{K}\right]$$

and compare it to the single boost without rotation.

$$A'' = \exp\left[\left[-\vec{\zeta}(\beta+\delta\beta)+\vec{\zeta}(\beta)\right]\cdot\vec{K}\right]$$

Call $\vec{\zeta} = \zeta(\beta+\delta\beta)$, $\zeta' = \zeta(\beta)$

For small ζ, ζ' we expand

$$A \cdot A = \left[1 - \zeta\cdot K + \tfrac{1}{2}(\zeta\cdot K)^2 + \cdots\right]\left[1 + \zeta'\cdot K + \tfrac{1}{2}(\zeta'\cdot K)^2 + \cdots\right]$$

$$= 1 + (\zeta'-\zeta)\cdot K + K_i K_j \left(\tfrac{1}{2}\zeta_i\zeta_j - \zeta_i\zeta'_j + \tfrac{1}{2}\zeta'_i\zeta'_j\right)$$

while the pure boost is

$$A'' = 1 + (\zeta'-\zeta)\cdot K + \tfrac{1}{2} K_i K_j (\zeta'-\zeta)_i(\zeta'-\zeta)_j + \cdots$$

writing the square in components.

The difference between A'' and A is

$$\tfrac{1}{2} K_i K_j \left\{ \zeta'_i\zeta'_j - \zeta'_i\zeta_j - \zeta_i\zeta'_j + \zeta_i\zeta_j \right. \quad \leftarrow A''$$

$$\left. - \zeta_i\zeta_j + 2\zeta_i\zeta'_j - \zeta'_i\zeta'_j \right\} \quad \leftarrow A \cdot A'$$

$$= \tfrac{1}{2} K_i K_j \left[\zeta_i\zeta'_j - \zeta'_i\zeta_j\right]$$

$A'' = A - i$

$$= \tfrac{1}{2}\zeta_i\zeta'_j\left[K_iK_j - K_jK_i\right]$$ flipping a dummy index.

But $[K_i, K_j] = -i\,\epsilon_{ijk} S_k$ ~~[illegible]~~

$A'' - AA' = -\frac{i}{2} S\times(\zeta\times\zeta')$

$$A\cdot A' \cong A''\left(1 + \tfrac{1}{2} i\, S\cdot(\vec{\zeta}\times\vec{\zeta}')\right)$$

This is a rotation by $\Delta\vec{\Omega} = \frac{1}{2}\vec{\zeta}\times\vec{\zeta}'$

Now ~~for small~~ $\vec{\zeta} = \hat{\beta}\tanh^{-1}\beta \simeq \vec{\beta}$ for small β

$\zeta' \simeq \vec{\beta} + \delta\vec{\beta}$

$$\Delta\vec{\Omega} = \tfrac{1}{2}\beta\times\delta\vec{\beta}$$

The exact result is $\frac{\gamma^2}{\gamma+1} \simeq \frac{1}{2}$ in the NR limit.

So

$$\vec{\omega}_T = -\lim_{\delta t \to 0} \frac{1}{2}\frac{v}{c} \times \frac{\delta \vec{v}}{c\,\delta t} = \frac{1}{2}\frac{\vec{a} \times \vec{v}}{c^2}$$

$$\vec{a} = \text{acceleration} = \frac{\vec{F}}{m} = -\frac{\vec{r}}{mr}\frac{\partial V}{\partial r}$$

$$\vec{\omega}_T = -\frac{1}{2mc^2}\, \vec{r} \times \vec{v}\, \frac{1}{r}\frac{\partial V}{\partial r} = -\frac{1}{2m^2c^2}\, \vec{L}\, \frac{1}{r}\frac{\partial V}{\partial r}$$

and

$$U = -\frac{ge}{2mc}\, \vec{s}' \cdot B + \frac{(g-1)}{2m^2c^2}\, \vec{L} \cdot \vec{s}'\, \frac{1}{r}\frac{\partial V}{\partial r}$$

since $g = 2$ $g - 1 = 1 \Rightarrow$ factor of $1/2$.

Comments: 1) For sophisticated treatment, see "BMT" equation

$$\frac{d}{dt}(\hat{\beta} \cdot \vec{s}') \propto \frac{g-2}{2}\, \vec{s} \cdot (\vec{\beta} \times \vec{B})$$

precession of longitudinal polarization - very useful to measure g-2 - or calibrate ~~B~~ B field in beam

2) If potential is not like electromagnetism (Φ = 4th component of 4 vector) only Thomas term present.

$$U = \frac{-1}{2mc^2}\, L \cdot S\, \frac{1}{r}\frac{\partial V}{\partial r} \rightarrow \text{inverted multiplets.}$$

3) Very easy to pop Thomas - $\frac{1}{2}$ out of Dirac equation.