Homework #4

Due 10:10, Dec. 8, 2021

1. Assume that a least mean squares (LMS) adaptive filter has the following input:

$$x(n)=\begin{cases}1, & n=0,(N-1),(2N-1),...\\0, & \text{otherwise.}\end{cases}$$

(1) Find an equivalent transfer function $H(z)$ relating the error signal $e(n)$ and the desired signal $d(n)$.

(2) Determine the poles and zeros of the transfer function.

(3) Determine a stability limit for the adaptation constant α.

2. Consider an LMS adaptive filter with input $x(n)=w(n)*h(n)$, where $w(n)$ is a zero-mean independent, identically distributed (iid) sequence. Let $S_{\max}$ and $S_{\min}$ be the largest and smallest components of the power spectral density of $x(n)$, respectively, and $\lambda_{\max}$ and $\lambda_{\min}$ be the largest and smallest eigenvalues of the autocorrelation matrix $\mathbf{R}$ of $x(n)$, respectively. Use the relation $\lambda_{\max}/\lambda_{\min}\le S_{\max}/S_{\min}$ to bound the eigenvalue disparity for each of the following cases:

(1) $h(n)=\delta(n)+c\delta(n-1)$, c is a constant.

(2) $h(n)=a^n u(n)$, where $u(n)$ is the unit step sequence and $|a|<1$.

3. Consider an LMS adaptive filter with input vector $\mathbf{x}_n$, coefficient vector $\mathbf{f}_n$, output signal $y(n)$, desired signal $d(n)$, error signal $e(n)=d(n)-y(n)$.

(1) Assume that the input signal is zero-mean stationary and

$$E\{e(n)\mathbf{x}_i^T\mathbf{x}_n\}=E\{e(n)\}E\{\mathbf{x}_i^T\mathbf{x}_n\}.$$

Show that, for the LMS update $\mathbf{f}_{n+1}=\mathbf{f}_n+\alpha e(n)\mathbf{x}_n$, an equivalent transfer function relating the mean output signal $y(n)$ and the mean desired signal $d(n)$ can be expressed as

$$\bar{H}(z)=\frac{\bar{Y}(z)}{\bar{D}(z)}=\left[\frac{\alpha LR(z)}{1+\alpha LR(z)}\right]$$

where $\bar{Y}(z)=\sum_{n=0}^{\infty}E\{y(n)\}z^{-n}$ and $\bar{D}(z)=\sum_{n=0}^{\infty}E\{d(n)\}z^{-n}$.

(2) Use the result in (1) to find $\bar{y}(\infty)$ if

$$d(n)=v(n)+1$$
$$x(n)=w(n)+aw(n-1)$$

where $w(n)$ and $v(n)$ are zero-mean white Gaussian sequences, and a is a constant.

4. Consider an LMS adaptive filter with input data vector $\mathbf{x}_n$ and coefficient vector $\mathbf{f}_n$ at the nth iteration. Let $\mathbf{f}^*$ be the optimal filter (coefficient vector) and $\mathbf{v}_n = \mathbf{f}_n - \mathbf{f}^*$ be a coefficient error vector. Show that the LMS update equation may be written as

$$\mathbf{v}_{n+1} = \left(\mathbf{I} - \alpha \mathbf{x}_n \mathbf{x}_n^T\right)\mathbf{v}_n + e_{\mathrm{o}}(n)\mathbf{x}_n$$

where α is an adaptation constant and $e_{\mathrm{o}}(n)$ is the error associated with $\mathbf{f}^*$.

5. Consider a two-tap LMS adaptive filter with an input sequence $x(n)$ given by

$$x(n) = ax(n-1) + \sqrt{1-a^2}\,w(n)$$

where $0 < a < 1$ and $w(n)$ is a zero-mean white Gaussian noise with unity variance.

(1) Compute the eigenvectors and eigenvalues of the autocorrelation matrix $\mathbf{R}$ of $x(n)$.

(2) Express autocorrelation matrix as $\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$, where $\mathbf{Q}$ is an orthonormal matrix.

(3) Explain how to use the orthonormal matrix $\mathbf{Q}$ to show convergence of the corresponding modified LMS update equation $\mathbf{u}_n = (\mathbf{I} - \alpha\mathbf{R})\mathbf{u}_n$, where α is an adaptation constant, $\mathbf{u}_n = E\{\mathbf{f}_n\} - \mathbf{f}^*$ with $\mathbf{f}_n$ denoting a filter coefficient vector at the nth iteration and $\mathbf{f}^*$ denoting the optimal solution.

(4) Determine a range of α that will guarantee stability/convergence.

(5) Estimate the time constant for convergence as a function of a and α.