Homework #2

Due 10:10, Oct. 20, 2021

1. Consider a zero-mean stationary sequence $x(n)$ with a correlation sequence $r(i)$. Show that, if $r(i)$ is a real and even function of *i*, then its Fourier transform $R(e^{j\omega})$ is a real, even, and non-negative function of ω.

2. Find the autocorrelation function associated with the AR(2) process given by

$$x(n) = x(n-1) - 0.25x(n-2) + w(n)$$

where $w(n)$ is a zero-mean white noise sequence with variance σ_w^2.

3. For each of the following cases, show that $\left|H(e^{j\omega})\right|^2 = \text{constant}$.

(1) $H(z) = \dfrac{(1 - z_1 z^{-1})}{(z^{-1} - z_1)}$, where z_1 is a real.

(2) $H(z) = \dfrac{(1 - z_a z^{-1})(1 - z_a^* z^{-1})}{(z^{-1} - z_a)(z^{-1} - z_a^*)}$, where * denotes complex conjugate.

Also, use these results to infer that a general $H(z)$ expressible in the form of

$$H_{\text{ap}}(z) = \frac{(1 - z_{N_1+1} z^{-1}) \cdots (1 - z_N z^{-1})}{(z^{-1} - z_{N_1+1}) \cdots (z^{-1} - z_N)}$$

is all-pass.

4. A deterministic signal is estimated by averaging *M* noise corrupted measurements

$$y_j(n) = x(n) + v_j(n), \quad j = 0, 1, 2, \ldots, M-1$$

where $v_j(n)$ is a zero mean iid sequence and $E\{v_j(n) v_i(n)\} = \sigma^2 \delta(i-j)$. Find the variance of $\hat{x}(n) = (1/M)\sum_{j=1}^{M} y_j(n)$.

5. For each of the following signals, show that the sample mean $\hat{\mu} = (1/M)\sum_{i=0}^{M-1} x(i)$ is unbiased and consistent.

(1) $x(n)$ is an iid sequence with mean value μ and variance σ^2.

(2) $x(n) = w(n) + aw(n-1)$, where $w(n)$ is an iid sequence with zero mean and unit variance.

6. Consider the following two estimators for the correlation of a random signal:

$$\hat{r}(m)=\frac{1}{M-|m|}\sum_{n=0}^{M-|m|-1}x(n)x(n+|m|);\ \ r'(m)=\frac{1}{M}\sum_{n=0}^{M-|m|-1}x(n)x(n+|m|).$$

(1) Generate a stationary Gaussian random signal with samples $x(n)$ for $n = 0, 1, 2, \ldots, M-1$ using MATLAB, where M=100. Then estimate the correlation of the random signal based on each of the two estimators, i.e., compute $\hat{r}(m)$ and $r'(m)$ for $m = -M+1, \ldots, M-1$. From the simulation results, show that $\hat{r}(m)$ has high variability for $m > M/4$.

(2) Let **R** be a correlation matrix of the random signal $x(n)$. It is known that **R** is a positive semi-definite matrix if the true correlation values are used. How is the positive semi-definite property of **R** if the true correlation values are replaced with the estimates $\hat{r}(m)$ or $r'(m)$ for $m = 0, 1, 2, \ldots, 24$ and $m = 0, 1, 2, \ldots, 99$? Justify your answers.