Cpt 3. Freeness over subalgebras

0. abstract

(1) main theorem (Nichols & Zoeller): f.d. Hopf alg. is free over any subHopf algebra

Schneider: a very nice basis exists for H over K

(2) Adjoint actions and normal sub-Hopf-algebras.

(3) In the infinite-dimensional case, H need not be free over K.

A more general question of the faithful flatness of H over K.

Cpt 3.1 The Nichols-Zoeller Theorem.

let $A = P_1^{n_1} \oplus \cdots \oplus P_t^{n_t}$ be the decomposition of A into indecomposables. P_i are called principal indecomposable modules of A. (write PIM for short)

1. Lemmas

① Let A be a Frobenius algebra, and let W be a finitely-generated right A-module, Then

(1) W is faithful $\Leftrightarrow$ each P_i appears as a summand of W.

(2) $\exists\ r \in \mathbb{Z}^+$ s.t. $W^{(r)} \cong F \oplus E$ as A modules, where F is free and E is not faithful.

pf: (1) ref: [CuR, 59.3]

(2) (Radford) If W is not faithful, then $W^{(1)} = 0 \oplus W$ is as required.

Suppose that W is faithful, then $W \cong P_1^{m_1} \oplus \cdots \oplus P_t^{m_t}$. We wish to find $r, t \geq 1$ s.t. $W^{(r)} = A^{(s)} \oplus E$, where E is not faithful.

This boils down to finding $r, s \geq 1$ s.t. $r m_i \geq s \cdot n_i$, for all $1 \leq i \leq t$ with equality holding in at least one instance. Let $r = n_{i_0}$ and $t = m_{i_0}$, where $\frac{n_{i_0}}{m_{i_0}}$ is the maximum of $\{\frac{n_1}{m_1}, \cdots, \frac{n_t}{m_t}\}$ #

② Krull-schmidt theorem.

Suppose that M is a right A-module and that

$$M_1 \oplus \cdots \oplus M_k = M = N_1 \oplus \cdots \oplus N_l$$

are two decompositions of M into a direct sum of indecomposable A-mods. Then $k=l$ and there exists a relabelling of N_i with $N_1 \cong M_1, \cdots, N_k \cong M_k$.

Note: This is an analogy of unique decomposition thm for numbers.

Corollary: 1. let M, N be A-modules, then $M^{(r)} \cong N^{(r)}$ for some $r \geq 1$ implies $M \cong N$

2. let $M^{(r)} = M_1^{(r_1)} \oplus \cdots \oplus M_n^{(r_n)}$ be the decomposition of $M^{(r)}$ into indecomposables, then $r_1, \cdots, r_n$ are divisible by r.

3. let $M = \bigoplus_{i=1}^{m} M_i^{r_i}$ and $N = \bigoplus_{i=1}^{n} N_i^{t_i}$ be decompositions of indecomposables. If $M^{(r)} \cong N^{(t)}$, then $m=n$ and there exists a relabelling of N_i with $N_i \cong M_i$, $r_i \cdot r = t_i \cdot t$ for any $1 \leq i \leq m$.

③. let A be a f.d. algebra, then M is finitely generated iff M is finite-dimensional.

pf: It follows from the fact that cyclic A-modules are finite-dimensional. That is,

$\dim Ax = \dim A/\mathrm{ann}\, x \leq \dim A < \infty$.

2. augmented

①. a $\mathbb{k}$-algebra A is called augmented if it has a non-zero algebra map $\epsilon: A \to \mathbb{k}$. In particular, hopf algebras are augmented.

Note: 1. $\mathbb{k}$ is an left A-module via ϵ, i.e. $a \cdot k = \epsilon(a) \cdot k$, $\forall a \in A, k \in \mathbb{k}$.

2. In general, $f: A \to B$ is an algebraic map, then B is a left (resp. right) module via $a \cdot b = f(a) \cdot b$ (resp. $b \cdot f(a)$) $\forall a \in A, b \in B$. Particularly, if B is commutative, then B is an A-bimodule.

3. If A is commutative, then any A-module can be treat as A-bimodule.

② Let A be a f.d. augmented algebra, and let $A = P_1^{r_1} \oplus \cdots \oplus P_n^{r_n}$ be the decomposition of A into right PIMs. Then $(r_1, \cdots, r_n) = 1$

pf: Suppose that $r_0 = (r_1, \cdots, r_n) \neq 1$. Let $t_i = \frac{r_i}{r_0} \in \mathbb{Z}^+$ for $1 \leq i \leq n$, and let $M = P_1^{t_1}\oplus\cdots\oplus P_n^{t_n}$ be a proper submodule of A. Then

$r_0\cdot \dim_{\mathbb{K}}(M\otimes_A \mathbb{K}) = \dim_{\mathbb{K}}(M^{(r_0)}\otimes_A \mathbb{K}) = \dim_{\mathbb{K}}(A\otimes_A \mathbb{K}) = 1$, contradicting the

fact that $r_0 \neq 1$.

Note: $(\sum_{i\in I} M_i)\otimes_A(\sum_{j\in J} N_j) = \sum_{i,j} M_i\otimes_A N_j$, in particular, $(M\otimes_A N)^{(2)} = M^{(2)}\otimes_A N = M\otimes_A N^{(2)}$

③ (1) A is an augmented algebra via ϵ iff $A = \ker\epsilon \oplus \mathbb{K}.1_A$, where $\ker\epsilon$ is a

maximal ideal of A. (2) Let M be any submodule of A, then $M\otimes_A \mathbb{K} = 0$ iff $M\subseteq \ker\epsilon$

pf: (2) It follows directly from the definition: $a\otimes_A k = a\cdot 1_A\otimes_A k = 1_A\otimes_A \epsilon(a)\cdot k \mapsto \epsilon(a)\cdot k$

Note: Let $A = \mathbb{K}\oplus\mathbb{K}$, viewing A as right a module, the two $\mathbb{K}$s are not isomorphic.

④ Prop: Let A be a f.d. augmented algebra and W a finitely-generated

right A-mod. If $\exists\, r\in \mathbb{Z}^+$ s.t. $W^{(r)}$ is free over A, then W is free.

pf: By assumption $W^{(r)} \cong A^{(n)}$ for some $n\in\mathbb{Z}^+$. Since A is augmented,

via ϵ, $\mathbb{K}$ becomes a left A-module via $a\cdot\alpha = \epsilon(a)\cdot\alpha$, $\forall a\in A, \alpha\in k$

It follows that $r\cdot\dim_{\mathbb{K}} W\otimes_A \mathbb{K} = \dim_{\mathbb{K}} W^{(r)}\otimes_A \mathbb{K} = \dim_{\mathbb{K}} A^{(n)}\otimes_A \mathbb{K} = n$

Thus, $r\mid n$, $W^{(r)} \cong (A^{(\frac{n}{r})})^{(r)}$, and hence $W\cong A^{(\frac{n}{r})}$ by 1.② corollary.

Note: By 1.② corollary 3, $W = \bigoplus_{i=1}^{t} P_i^{m_i}$ s.t. $m_i\cdot r = n_i\cdot n$, $\forall\, 1\le i\le t$.

Because $(n_1,\cdots,n_t) = 1$, we must have $r\mid n$.

3. review some results about modules

Let A be an algebra, B a bialgebra and H a Hopf algebra.

① trivial modules and comodules

(1) M is a trivial (right) B-module: $m\cdot b = \epsilon(b)\cdot m$, $\forall m\in M, b\in B$

M is a trivial (right) B-comodule: $\rho(m) = m\otimes 1$, $\forall m\in M$.

(2) Any $\mathbb{K}$-subspace of M is a trivial submodule (resp. sub-comodule)

(3) Any $\mathbb{K}$-vector space V can be treat as trivial B-module via

$v\cdot b = \epsilon(b)\cdot v$ and as trivial B-comodule via $\rho(v) = v\otimes 1$

② tensor product

(1) M, N are B-module, then $M\otimes N$ is a B-module via

$(m\otimes n)\cdot b = mb_1\otimes nb_2$, for all $m\in M, n\in N, b\in B$

(2) M, N are B-comodule, then $M\otimes N$ is a B-comodule via

$\rho(m\otimes n) = m_0\otimes n_0\otimes m_1n_1$, for all $m\in M, n\in N$

③. bialgebra module

(1) (M, ρ, γ) is a B-bialgebra module if the following holds

I. (M, γ) is a B-module

II. (M, ρ) is a B-comodule

III. ρ is a B-module map

Note: This is exact the definition of Hopf modules when B is a Hopf algebra.

(2) M is a B-module, then $M\otimes B$ is a B-bialgebra module via

$(m\otimes x)b = mb_1\otimes xb_2$, $\rho(m\otimes x) = m\otimes\Delta(x)$, for all $m\otimes x\in M\otimes B$, $b\in B$.

(3) M is a B-comodule, then $M\otimes B$ is a B-bialgebra module via

$(m\otimes x)b = m\otimes xb$, $\rho(m\otimes x) = m_0\otimes x_1\otimes m_1x_2$ for all $m\otimes x\in M\otimes B$.

(4) In particular, when M is a trivial module (resp. comodule), $M\otimes B$ is a trivial module via (1) (resp. (2)). That is to say,

$(m\otimes x)\cdot b = m\otimes x\cdot b$, $\rho(m\otimes x) = m\otimes\Delta(x)$ for all $m\otimes x\in M\otimes B$, $b\in B$.

④ direct sum

(1) $\mathbb{K}$ is a B-module via $k\cdot b = \varepsilon(b)\cdot k$, $\forall\, b\in B$, $k\in\mathbb{K}$; hence $\mathbb{K}$ is a trivial B-module. In fact, any trivial B-module is direct sum of $\mathbb{K}$s as B-modules.

(2) let M be an A-module, V be any $\mathbb{K}$-vector space. Then $M\otimes_{\mathbb{K}} V$ is an A-module via $(m\otimes v)\cdot a = ma\otimes v$, $\forall\, a\in A$, $m\otimes v\in M\otimes V$.

Let $\{v_i \mid i\in I\}$ be a basis of V, then $M\otimes V \cong \bigoplus_{i\in I} M$

(3) Let M be a B-module, and V be defined as above. Viewing V as a trivial B-module, the B-module structure of $M\otimes V$ via tensor

product is the same as defined in (2). Thus $M \otimes V = M^{(\dim V)}$

⑤ Hopf modules

let M be an H-module, then $M \otimes H \cong H^{(\dim_k M)}$. Also, if F is a free H-module, then $V \otimes F \cong F^{(\dim V)}$

Note: $F = \bigoplus_{i \in I} H \Rightarrow V \otimes F = V \otimes (\bigoplus_{i \in I} H) = \bigoplus_{i \in I} V \otimes H = \bigoplus_{i \in I} H^{(\dim V)} = F^{(\dim V)}$

4. Main results.

① Let K be a f.d. Hopf algebra and W a f.g. right K-module. Suppose that there exists a f.g. faithful right K-module V s.t. $W \otimes V \cong W^{\dim V}$ as right K-modules. Then W is free over K.

Pf: (1) Since K is a Frobenius algebra, $\exists\, r, s > 0$ s.t. $W^{(r)} = F \oplus E$, $V^{(s)} = F' \oplus E'$, F and F' free, E and E' not faithful, and $F' \neq 0$.

Thus $W^{(r)} \otimes V^{(s)} \cong W \otimes V^{(s \cdot r)} = W^{(s \cdot r \cdot \dim V)} = (W^{(r)})^{(\dim V^{(s)})}$, where $V^{(s)}$ is faithful, and $W^{(r)}$ is free iff W is free. Hence we might replace W by $W^{(r)}$, V by $V^{(s)}$ and assume that $W = F \oplus E$, $V = F' \oplus E'$.

(2) Setting $t = \dim V$, our hypothesis gives

$$F^{(t)} \oplus E^{(t)} = (F \oplus E)^{(t)} = W^{(t)} \cong W \otimes V \cong (F \oplus E) \otimes V = (F \otimes V) \oplus (E \otimes V)$$

By 3.①, the right side is isomorphic to $F^{(t)} \oplus (E \otimes V)$. Using Krull-Schmidt Thm, we obtain $E^{(t)} \cong E \otimes V = E \otimes (E' \oplus F') = (E \otimes E') \oplus (E \otimes F') \cong (E \otimes E') \oplus F'^{(\dim E)}$

Again by Krull-Schmidt Thm, E must be zero since it's not faithful. Thus, we obtain $W = F$ and complete the proof.

Steps: 1. WLOG, assume that $W = F \oplus E$, $V = F' \oplus E'$

2. $F^{(t)} \oplus E^{(t)} = F^{(t)} \oplus F'^{(\dim E)} \oplus (E \otimes E')$ and use K-S Thm to get $E = 0$.

② lemma:

let H be a Hopf algebra with bijective antipode, let K be a sub-Hopf-algebra, and let $M \in M_K^H$. Then $M \otimes H \cong M^{(\dim H)}$ as K-modules.

pf:

(1) Let H_0 denote H considered as a trivial K-module. Since $M\otimes H_0 \cong M^{(\dim H)}$, it will suffice to prove $M\otimes H \cong M\otimes H_0$.

(2) let $\varphi: M\otimes H_0 \to M\otimes H$ and $\psi: M\otimes H \to M\otimes H_0$

$m\otimes h \mapsto m_0\otimes hm_1$ $\qquad$ $m\otimes h \mapsto m_0\otimes h\bar{S}m_1$

where $\bar{S}$ is the inverse of S.

(3) Since $\varphi(m\otimes h\cdot x) = \varphi(mx\otimes h) = (mx)_0\otimes h(mx)_1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\varphi(m\otimes h)\cdot x = m_0\otimes hm_1\cdot x = m_0x_1\otimes hm_1x_2$

$\psi(m\otimes h\cdot x) = \psi(mx_1\otimes hx_2) = (mx_1)_0\otimes hx_2\bar{S}(mx_1)_1 = m_0x_1\otimes hx_3\bar{S}(m_1x_2)$

$= m_0x_1\otimes hx_3\bar{S}(x_2)\bar{S}(m_1) = m_0x_1\otimes h\,\epsilon(x_2)\cdot\bar{S}(m_1) = m_0x\otimes h\cdot\bar{S}m_1$

$\psi(m\otimes h)\cdot x = (m_0\otimes h\bar{S}m_1)\cdot x = m_0x\otimes h\bar{S}m_1$

φ,ψ are homomorphism of K-modules

(4). $\varphi\circ\psi(m\otimes h) = \varphi(m_0\otimes h\bar{S}m_1) = m_0\otimes h\bar{S}(m_2)\cdot m_1 = m_0\otimes h\,\epsilon(m_1) = m\otimes h$

$\psi\circ\varphi(m\otimes h) = \psi(m_0\otimes hm_1) = m_0\otimes hm_2\bar{S}m_1 = m_0\otimes h\cdot\epsilon(m_1) = m\otimes h$ #

Note: 1. Here H might be infinite-dimensional. In this case, $M^{(\dim H)}$ denotes the infinite direct sum of M.

2. It's easy to check that φ and ψ are as required. The harder part is "How to use antipodes to construct the desirable maps".

③. Let H be a f.d. Hopf algebra and let K be a sub-Hopf-algebra. Then every f.g. right (H,K)-Hopf module is free as a right K-module.

pf: Assume that $M\in M_K^H$ is finitely generated. By ②, $M\otimes H\cong M^{(\dim H)}$ as right K-modules. Since H is faithful, by ①, M is free.

④. Let H be a bialgebra and K a f.d. subbialgebra. If every f.d. $M\in M_K^H$ is free as a right K-module, then every $M\in M_K^H$ is free as a right K-module.

pf: [Radford]. Choose $0\neq M\in M_K^H$. We show that M is free.

(1) A set $L \subseteq M$ is called a partial basis if $LK \in \mathcal{M}_K^H$ and is a free K-modules with L as its basis.

(2) Let $\mathcal{L}$ denote the set of partial bases of M. $\mathcal{L} \neq \emptyset$ since $0 \in \mathcal{L}$. If $\mathcal{L}$ is ordered by inclusion, Zorn's lemma applies, and thus there exists a maximal partial basis L. We claim that $LK = M$.

(3) If $LK \neq M$, consider $\overline{M} = M/LK \neq 0$ and let $\pi : M \to \overline{M}$ be the natural projection. Since $\overline{M}$ is an H-comodule, it contains a non-zero f.d. subcomodule $\overline{M}'$. Then $M' = \pi^{-1}(\overline{M}')$ is a subcomodule of M.

(4) Let $M'K$ be the K-submodule of M generated by M', it is a H-subcomodule since $\rho(M'K) = \rho(M')K = (M'_0 \otimes M'_1)K = M'_0K \otimes M'_1K \subseteq M'K \otimes H$.
Thus, $M'K$, $\overline{M}'K \in \mathcal{M}_K^H$. $\overline{M}'K$ is f.d. since K and $\overline{M}'$ are f.d, hence $\overline{M}'K = \pi(M'K) \in \mathcal{M}_K^H$ and is free over K by hypotheses.

(5) Let $\overline{L'}$ be a basis of $\overline{M}'K$ as K-module, and let $L' \subseteq M$ s.t. $\pi(L') = \overline{L'}$. Then $M'K \in \mathcal{M}_K^H$ and is free as K-module with $L' \dot\cup L$ as its basis, contradicting the maximality of L.

Note: 1. π is a morphism of both K-modules and H-comodules.

2. If $M \in \mathcal{M}_K^H$ and N is both a K-submodule and a H-subcomodule of M, then $N \in \mathcal{M}_K^H$.

Corollary: If H is f.d. and K is any subHopfalgebra of H, then H is free as K-modules.

Remark: some results about comodule.

① (1) $f : M \to N$ is a homomorphism of A-modules if the following diagram commutes:

$$\begin{tikzcd} M \otimes A \arrow[r, "f \otimes I_A"] \arrow[d, "\gamma_M"'] & N \otimes A \arrow[d, "\gamma_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

(2) $g : M \to N$ is a homomorphism of C-comodules if the following

diagram commutes:

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\rho_M"'] & N \arrow[d, "\rho_N"] \\ M\otimes C \arrow[r, "g\otimes I_C"] & N\otimes C \end{tikzcd}$$

② (1) $\operatorname{Im} f$ (resp. $\ker f$) is A-submodule of N (resp. M)

(2) $\operatorname{Im} g$ (resp. $\ker g$) is C-subcomodule of N (resp. M)

pf: (1) $f(M)\cdot A = f(MA) \subseteq f(A)$,

$f(\ker f\cdot A) = f(\ker f)\cdot A = 0 \Rightarrow \ker f\cdot A \subseteq \ker f$

(2) $\rho_N(g(M)) = g\otimes I_C\circ \rho_M(M) = g(M_0)\otimes g(M_1) \subseteq g(M)\otimes C$

$g\otimes I_C\circ \rho_M(\ker g) = \rho_N(g(\ker g)) = 0 \Rightarrow \rho_M(\ker g) \subseteq \ker g\otimes C.$

③ (1) let N be a submodule of M, then M/N has a unique module structure making $\pi: M\to M/N$ an A-module morphism.

(2) let N be a subcomodule of M, then M/N has a unique comodule structure making $\pi: M\to M/N$ a C-comodule morphism.

pf: Denote $\overline{m} = m+N \in M/N$ for all $m\in M$.

(1) $\overline{m}\cdot a = \pi(m)\cdot a = \pi(ma) = \overline{ma}$;

(2) $\overline{\rho}(\overline{m}) = \overline{\rho}(\pi(m)) = (\pi\otimes I_C)(\rho(m)) = \overline{m_0}\otimes m_1.$

Hence the structure of M/N is unique #

④ let M be a C-comodule. For any element $m\in M$, m lies in a f.d. subcomodule of M.

pf: Let $\rho(m) = m_0\otimes m_1$, $N = \operatorname{span} m_0$. we claim that N is such a subcomodule of M as required.

(1) $m\in N$ since $m = (I_M\otimes \varepsilon)\circ\rho(m) = I_M(m_0)\cdot\varepsilon(m_1) \in \operatorname{span} m_0$.

(2) let $\rho(m) \overset{F}{=} \sum_i m_{0i}\otimes m_{1i}$, where $\{m_{0i}\}$ and $\{m_{1i}\}$ are linearly independent sets.

Let $\{m_{1i}^*\}$ be a dual set of $\{m_{1i}\}$, then

$\sum_i \rho(m_{0i})\otimes m_{1i} = (\rho\otimes I_C)\circ\rho(m)$

$= (I_C\otimes\Delta)\circ\rho(m) = \sum_i m_{0i}\otimes \Delta(m_{1i})$

Applying $I_M \otimes I_C \otimes m_{ij}^*$ to both sides, we deduce

$$\rho(m_{0j}) = \sum_i m_{0i} \otimes ((m_{1i})_1 \cdot m_{ij}^*(m_{1i})) \in N \otimes C$$

Hence $\rho(N) \subseteq N \otimes C$ and N is a subcomodule of M.

Corollary: $\rho(m_0) \subseteq \operatorname{span} m_0 \otimes C$, $\Delta(m_1) \subseteq C \otimes \operatorname{span} m_1$

Cpt 3.2. Applications.

1. If H is a f.d. Hopf algebra and K a sub-Hopf-algebra, then $\dim K$ divides $\dim H$. In particular if $\dim H = p$, a prime, then H has no proper sub Hopf algebra.

2. Conjecture.

   Let $\mathbb{K}$ be algebraically closed of char 0, and let H be a Hopf algebra of dimension p, a prime. Then $H \cong \mathbb{K}.\mathbb{Z}_p$, the group algebra over $\mathbb{Z}_p$.

(incomplete)

pf: It's equivalent to show that H (or H^*) contains a non-trivial group-like element g; for then $K = \mathbb{K}\langle g\rangle$ is a sub-Hopf-algebra of H and hence $K = H$.

If no such elements exists, then $\alpha = \epsilon$, where α is the distinguished grouplike element. Thus one may assume that both H and H^* are unimodular.

Note: Let G be the grouplike elements of H, and G' a subgroup of G, then $\mathbb{K}.G'$ is a sub-Hopf-algebra of H.

Corollary: $\# G(H) \mid \dim H$, if H is f.d.

3. If H is a f.d. s.s. Hopf algebra, then every sub-Hopf-algebra is also s.s.

pf: 2.2.2 and 3.1.5.

Remark: 1. Recall that if H is s.s., then H is unimodular.

2. However, sub-Hopf-algebra of unimodular Hopf algebra might not be unimodular.

3. Cpt 10: Any f.d. Hopf algebra may be imbedded in a unimodular one.