Cpt 7. crossed product.

0. Abstract.

7.1 : defs & examples

7.2 : cleft extension

7.3 : equivalent ; inner action

7.4. Maschke-type problem.

7.5. twisted comod. alg.

Cpt 7.1. Defs & examples.

1. Group 2-cocycle :

① $A \hookrightarrow G \overset{\pi}{\twoheadrightarrow} H \Rightarrow H \cong G/A$

$\Rightarrow$ let $\gamma : H \to G$ s.t. $\pi \circ \gamma = 1_H$

i.e. $\gamma(h)$ is the representative of h in G.

$\Rightarrow G = A \cdot \gamma(H) = A \#_\sigma H$

② $a_1 \gamma(h_1) a_2 \gamma(h_2) = a_1 \gamma(h_1) a_2 \gamma(h_1)^{-1} \cdot \gamma(h_1)\gamma(h_2)$

$= a_1 a_2^{h_1} \cdot \gamma(h_1)\gamma(h_2)$

$= a_1 a_2^{h_1} \cdot \gamma(h_1)\gamma(h_2)\gamma(h_1h_2)^{-1}\gamma(h_1h_2)$

$= a_1 a_2^{h_1} \sigma(h_1,h_2)\gamma(h_1h_2)$

where $h\cdot : A \to A$, $\sigma : H \times H \to A$

$a \mapsto a^h = hah^{-1}$ $\qquad (h_1,h_2) \mapsto \gamma(h_1)\gamma(h_2)\gamma(h_1h_2)^{-1}$

hence $G = A\cdot\gamma(H) \cong A \#_\sigma H$ via γ, σ.

Remark : twisted action of $h\cdot$ on A

$h\cdot(k\cdot a) = \gamma(h)\gamma(k)\, a\, \gamma(k)^{-1}\gamma(h)^{-1}$

$= \underbrace{\gamma(h)\gamma(k)\gamma(hk)^{-1}}\gamma(hk)\, a\, \gamma(hk)^{-1} \underbrace{\gamma(hk)\gamma(k)^{-1}\gamma(h)^{-1}}$

$= \sigma(h,k)\,(hk\cdot a)\,\sigma(h,k)^{-1}$

2. crossed product.

let H be a Hopf algebra and A an algebra,

①. H measures A:

$$h\cdot(ab) = (h_1\cdot a)(h_2\cdot b)$$

$$h\cdot 1_A = \epsilon(h)\cdot 1_A$$

②. let σ be an invertible map in $Hom_{\mathbb{K}}(H\otimes H, A)$ and

let $A\#_\sigma H \cong A\otimes H$ as vector space, with multiplication

$$(a\#h)(b\#k) = a(h_1\cdot b)\sigma(h_2, k_1)\# h_3 k_2, \quad \forall h,k\in H, a,b\in A$$

Note: smash product $(a\#h)(b\#k) = a(h_1\cdot b)\# h_2 k$

3. lemma:

①. $1\#1$ is the identity of $A\#_\sigma H$ iff

$$\begin{cases} 1\cdot a = a \\ \sigma(h,1) = \sigma(1,h) = \epsilon(h)\cdot 1. \end{cases}$$ (normal condition)

Pf: $\Rightarrow$:

$(1\#1)(1\#h) = 1(1\cdot 1)\sigma(1,h_1)\# h_2 = \sigma(1,h_1)\# h_2 = 1\# h$

applying $1_A\otimes\epsilon$ on both sides, we get

$\epsilon(h) = \sigma(1,h)$, anagonsly, we have $\sigma(h,1) = \epsilon(h)$.

Now $(1\#1)(a\#1) = 1(1\cdot a)\sigma(1,1)\#1) = (1\cdot a)\#1 = a\#1$

Thus, we get $1\cdot a = a$.

$\Leftarrow$:

$(1\#1)(a\#h) = 1(1\cdot a)\sigma(1,h_1)\# h_2$

$= a\,\epsilon(h_1)\# h_2 = a\# h.$

Similarly, we have $(a\#h)(1\#1) = a\#h$.

②. let $1\#1$ be the unit of $A\#_\sigma H$, then

$A\#_\sigma H$ is associative iff

(1) $h\cdot(k\cdot a) = \sigma(h_1,k_1)(h_2k_2\cdot a)\sigma^{-1}(h_3,k_3)$

(2) $h_1\cdot\sigma(k_1,m_1)\sigma(h_2,k_2m_2) = \sigma(h_1,k_1)\sigma(h_2k_2,m)$

$\forall\ h,k,m\in H$

Pf: $\Rightarrow$:

(1) $[(1\#h)(1\#k)](a\#1) = (1\#h)[(1\#k)(a\#1)]$

left side $= (1(h_1\cdot 1)\sigma(h_2,k_1)\#h_3k_2)(a\#1)$

$= (\sigma(h_1,k_1)\#h_2k_2)(a\#1)$

$= \sigma(h_1,k_1)((h_2k_2)_1\cdot a)\sigma((h_2k_2)_2,1)\#(h_2k_2)_3 1$

$= \sigma(h_1,k_1)((h_2k_2)_1\cdot a)\#(h_2k_2)_2$

right side $= (1\#h)((k_1\cdot a)\sigma(k_2,1)\#k_3$

$=(1\#h)(k_1\cdot a)\#k_2$

$= h_1\cdot(k_1\cdot a)\sigma(h_2,k_2)\#h_3k_3$

applying $id\otimes\varepsilon$ on both sides, we get

$\sigma(h_1,k_1)((h_2k_2)_1\cdot a)\#(h_2k_2)_2$ $\quad$ $h_1\cdot(k_1\cdot a)\sigma(h_2,k_2)\#h_3k_3$

$\downarrow id\otimes\varepsilon$ $\quad$ $\downarrow id\otimes\varepsilon$

$\sigma(h_1,k_1)(h_2k_2\cdot a) = h_1\cdot(k_1\cdot a)\sigma(h_2,k_2)$ $\quad \times\sigma^{-1}$

hence: $h\cdot(k\cdot a) = \sigma(h_1,k_1)(h_2k_2\cdot a)\sigma(h_3,k_3)^{-1}$

(2). $[(1\#h)(1\#k)](1\#m) = (1\#h)[(1\#k)(1\#m)]$

left side $= (1(h_1\cdot 1)\sigma(h_2,k_1)\#h_3k_2)(1\#m)$

$= (\sigma(h_1,k_1)\#h_2k_2)(1\#m)$

$= \sigma(h_1,k_1)((h_2k_2)_1\cdot 1)\sigma((h_2k_2)_2,m_1)\#(h_2k_2)_3m_2$

$= \sigma(h_1,k_1)\sigma((h_2k_2)_1,m_1)\#(h_2k_2)_2m_2$

right side $= (1\#h)(\sigma(k_1,m_1)\#k_2m_2)$

$= (h_1\cdot\sigma(k_1,m_1))\sigma(h_2,(k_2m_2)_1)\#h_3(k_2m_2)_2$

applying $id\otimes\varepsilon$ on both sides, we get

$$\sigma(h_1,k_1)\,\sigma((h_2k_2)_1, m_1) \# (h_2k_2)_2 m_2 \qquad (h_1\cdot\sigma(k_1,m_1))\,\sigma(h_2,(k_2m_2)_1) \# h_3 (k_2m_2)_2$$

$$\downarrow \mathrm{id}\otimes\varepsilon \qquad\qquad \downarrow \mathrm{id}\otimes\varepsilon$$

$$\sigma(h_1,k_1)\,\sigma(h_2k_2, m) = (h_1\cdot\sigma(k_1,m_1))\,\sigma(h_2, k_2m_2)$$

$\Leftarrow$: suppose (1) and (2) holds. let $a, b, c \in A$, $h, l, m \in H$. Then:

$$[(a\#h)(b\#l)](c\#m)$$

$$= (a(h_1\cdot b)\sigma(h_2,l_1) \# h_3 l_2)\, c\#m$$

$$= a(h_1\cdot b)\sigma(h_2,l_1)((h_3l_2)_1\cdot c)\,\sigma((h_3l_2)_2, m_1) \# (h_3l_2)_3 m_2$$

$$= a(h_1\cdot b)\,\sigma(h_2,l_1)(h_3 l_2\cdot c)\,\sigma(h_4 l_3, m_1) \# h_5 l_4 m_2$$

$\sigma(h_1,k_1)(h_2k_2\cdot a) = (h_1\cdot(k_1\cdot a))\sigma(h_2,k_2)$ (twisted module condiction)

$$= a(h_1\cdot b)\,(h_2\cdot(l_1\cdot c))\,\sigma(h_3,l_2)\,\sigma(h_4 l_3, m_1) \# h_5 l_4 m_2$$

$\sigma(h_1,k_1)\sigma(h_2k_2,m) = (h_1\cdot\sigma(k_1,m_1))\sigma(h_2,k_2m_2)$

$$= a(h_1\cdot b)\,(h_2\cdot(l_1\cdot c))\,(h_3\cdot\sigma(l_2,m_1))\,\sigma(h_4, l_3m_2) \# h_5 l_4 m_3$$

measure

$$= a\,(h_1\cdot[b(l_1\cdot c)\sigma(l_2,m_1)])\,\sigma(h_2, l_3m_2) \# h_3 l_4 m_3$$

$$= a\#h\,(b(l_1\cdot c)\sigma(l_2,m_1) \# l_3 m_2)$$

$$= a\#h((b\#l)(c\#m))$$

Corollary: $A\#_\sigma H$ is associative with unit $1\#1$ iff (1) A is a twisted H-module, i.e.

$$\begin{cases} 1\cdot a = a \\ h\cdot(k\cdot a) = \sigma(h_1,k_1)(h_2k_2\cdot a)\sigma^{-1}(h_3,k_3) \end{cases}$$

(2)

$$\sigma(h,1) = \sigma(1,h) = \varepsilon(h)\cdot 1$$

$$h_1\cdot\sigma(k_1,m_1)\,\sigma(h_2,k_2m_2) = \sigma(h_1,k_1)\,\sigma(h_2k_2,m)$$

4. examples

① let $H = \Bbbk G$ be a group algebra. Then the conditions for group crossed product become:

$$\begin{cases} 1\cdot a = a \\ g\cdot(h\cdot a) = \sigma(g,h)(gh\cdot a)\sigma^{-1}(g,h) \\ \sigma(h,1) = \sigma(1,h) = 1 \\ (g\cdot\sigma(h,k))\sigma(g,hk) = \sigma(g,h)\sigma(gh,k) \end{cases}$$

② In particular.

(1) For any $N \triangleleft G$, consider $\pi: G \to G/N = \overline{G}$

let $\gamma: \overline{G} \to G$ s.t. $\pi\circ\gamma = Id_{\overline{G}}$.

i.e. $\gamma(\overline{G})$ is a section of $\overline{G}$ in G

(2) Since $G = N\cdot\gamma(\overline{G})$, we may multiply in $\mathbb{K}G$ by

$$n\gamma(\bar{x})m\gamma(\bar{y}) = n\underline{\gamma(\bar{x})m\gamma(\bar{x})^{-1}}\,\underline{\gamma(\bar{x})\gamma(\bar{y})\gamma(\bar{x}\bar{y})^{-1}}\,\gamma(\bar{x}\bar{y})$$

$$= n(\bar{x}\cdot m)\,\sigma(\bar{x},\bar{y})\,\gamma(\bar{x}\bar{y})$$

$\forall\, n, m \in N$, $\bar{x},\bar{y} \in \overline{G}$, where $\bar{x}\cdot m = \gamma(\bar{x})m\gamma(\bar{x})^{-1}$

$\sigma(\bar{x},\bar{y}) = \gamma(\bar{x})\gamma(\bar{y})\gamma(\bar{x}\bar{y})^{-1}$

Since $\mathbb{K}G = \mathbb{K}N \#_\sigma \mathbb{K}[G/N]$ is associative with unit $1\#1$, the conditions of crossed product hold.

③ transitive of crossed product.

Given a crossed product $A\#_\sigma\mathbb{K}G$ and $N \triangleleft G$, we have

$$(A\#_\sigma\mathbb{K}N)\#_\tau\mathbb{K}[G/N] \cong A\#_\sigma(\mathbb{K}N\#_\tau\mathbb{K}[G/N]) = A\#_\sigma\mathbb{K}G.$$

pf: $A\#_\sigma N$ is a crossed product via restriction (σ, measure) and a subalgebra of $A\#_\sigma G$

(1) let $\gamma: \mathbb{K}[G/N] \to A\#_\sigma G$, $\forall\, \bar{x} \in G/N$

$$\bar{x} \mapsto 1\#x$$

(2) Setting $\bar{x}\cdot(a\#n) = \gamma(x)(a\#n)\gamma(x)^{-1}$

$$\tau(\bar{x},\bar{y}) = \gamma(\bar{x})\gamma(\bar{y})\gamma(\bar{x}\bar{y})^{-1}$$

It's easy to verify that $(A \#_\sigma \mathbb{K}N) \#_\tau \mathbb{K}(G/N)$ is an algebra.

Note: 1. We may use this property for induction.

2. $f \in \mathrm{Hom}(C, A)$ convolution invertible, $c \in G(C) \Rightarrow f^{-1} * f(c) = f^{-1}(c) f(c) = 1$

hence $f(c)$ is invertible in A with inverse $f^{-1}(c)$. In particular

σ is invertible, $(1 \# x^{-1})(1 \# x) = \sigma(x^{-1}, x) \# 1 \Rightarrow (1 \# x)^{-1} = \sigma^{-1}(x^{-1}, x) \# x^{-1}$

$(1 \# x)(1 \# x^{-1}) = \sigma(x, x^{-1}) \# 1 \Rightarrow (1 \# x)^{-1} = (x^{-1} \cdot \sigma^{-1}(x, x^{-1})) \# x^{-1}$

Thus, $\sigma^{-1}(x^{-1}, x) = x^{-1} \cdot \sigma^{-1}(x, x^{-1})$, or equivalently,

④. let $H = U(\mathfrak{g})$, and $A \hookrightarrow B$ be $\mathbb{K}$-algebras s.t. $B \cong A \otimes U(\mathfrak{g})$ as vector spaces. let $\gamma: \mathfrak{g} \to B$ s.t. $\gamma(x) = \bar{x}$

$B = A \#_\tau U(\mathfrak{g})$ iff the following conditions hold:

(1) $x \cdot a = \bar{x} a - a \bar{x} = \delta_x(a) \in A$, $\delta_x \in \mathrm{Der}_{\mathbb{K}}(A)$

(2) $\tau(x, y) = [\bar{x}\, \bar{y}] - \overline{[xy]} \in A$

(3) for a given basis $\{x_\alpha\}$ of $\mathfrak{g}$, B is a free left and right A-module with the standard monomials in $\{\bar{x}_\alpha\}$ as a basis.

⑤. let H be cocomm. and A a twisted H-module which is measures by H. Then

A is an H-module iff $\sigma(H \otimes H) \subseteq Z(A)$

pf: $\Rightarrow$: omit (Cp † 5.6)

$\Leftarrow$: $h \cdot (k \cdot a) = \sigma(h_1, k_1)(h_2 k_2 \cdot a)\sigma^{-1}(h_3, k_3)$ cocomm. + Center

$= \sigma(h_1, k_1)\sigma^{-1}(h_2, k_2)(h_3 k_3 \cdot a)$

$= \sigma * \sigma^{-1}(h_1, k_1)(h_2 k_2 \cdot a)$

$= \epsilon(h_1) \cdot \epsilon(k_1)(h_2 k_2 \cdot a)$

$= hk \cdot a$.

In particular, if A is comm. then A is always an H-module

Cpt 7.2. cleft extension.

1. comodules.

① . comod alg.

(1) A is a right H-comodule via $\rho: A \to A \otimes H$

(2) m and u are H-comodule morphisms.

i.e. $(ab)_0 \otimes (ab)_1 = a_0 b_0 \otimes a_1 b_1$, $\rho(1) = 1 \otimes 1$, $\forall a, b \in A$.

② coinvariant :

$M^{coH} = \{ m \in M \mid \rho(m) = m \otimes 1 \}$

③ $A \xrightarrow{f} B$ algebraic $\Rightarrow$ B-modules are naturally A-modules.

$C \xrightarrow{g} D$ coalgebraic $\Rightarrow$ C-comodules are naturally D-comodules.

i.e.
$$\begin{tikzcd} C \arrow[r, "\rho'"] \arrow[dr, "\rho"'] & C \otimes D \\ & C \otimes C \arrow[u, "I_C \otimes g"'] \end{tikzcd}$$
, $\rho' = (I_C \otimes g) \circ \rho$

④ . let $A \#_\sigma H$ be an algebra with unit $1 \# 1$,

then $A \#_\sigma H$ is a free A-module and right H-comodule.

Pf: (1) $(a \# 1)(b \# h) = a(1 \cdot b)\sigma(1, h_1) \# h_2 = a \# h$

$\Rightarrow$ Any $\mathbb{K}$-vector space basis of H is a free A-module basis of A.

(2) $A \#_\sigma H$ is an H-comodule via $\rho = I_A \otimes \Delta$. In fact,

$a \# h \in (A \#_\sigma H)^{coH} \Rightarrow \rho(a \# h) = a \# h_1 \otimes h_2 = (a \# h) \otimes 1$

$\xRightarrow{I_A \otimes \varepsilon \otimes I_H}$ $a \# 1 \otimes h = a \# 1 \otimes 1$

$\Rightarrow h = 1$

hence $(A \#_\sigma H)^{coH} = A$.

2. cleft extension

let $A \subseteq B$ be $\mathbb{K}$-algebras, and H a Hopf algebra.

① $A \subseteq B$ is a right H-extension if B is a H-comodule alg. with $B^{\mathrm{coH}} = A$

② The H-extension $A \subseteq B$ is H-cleft if there exists a (convolution) invertible right H-comodule map $\gamma : H \to B$

Note: 1. We may assume that $\gamma(1) = 1$, for if not, we may replace γ by $\gamma' = \gamma(1)^{-1}\gamma$ (pf: see below)

2. $\forall a \in B^{\mathrm{coH}}$, $l(a) : B \to B$, $b \mapsto ab$ is an H-comodule map.

Pf:
$$\begin{array}{ccc} B & \to & B \otimes H \\ \downarrow & & \downarrow \\ B & \to & B \otimes H \end{array} \quad : \quad \begin{array}{ccccc} b & \mapsto & b_0 \otimes b_1 & \to & ab_0 \otimes b_1 \\ \downarrow & & & & \parallel \\ ab & \mapsto & (ab)_0 \otimes (ab)_1 & = & ab_0 \otimes b_1 \end{array}$$

3. If $f : M \to N$ is an H-comodule map then $f(M^{\mathrm{coH}}) \subseteq N^{\mathrm{coH}}$

In particular, $1 \in H^{\mathrm{coH}} \Rightarrow \gamma(1) \in B^{\mathrm{coH}}$.

4. γ is convolution invertible, then $\gamma(g)^{-1} = \gamma^{-1}(g)$, $\forall g \in G(H)$

3. Main Theorem.

①. An H-extension $A \subseteq B$ is H-cleft $\Leftrightarrow B \cong A \#_\sigma H$

②. $A \subseteq B$ be an H-cleft extension via $\gamma \in \mathrm{Hom}(H, B)$ s.t. $\gamma(1) = 1$. Then $A \#_\sigma H$ is a crossed product via

(1). $h \cdot a = \gamma(h_1)\, a\, \gamma^{-1}(h_2)$, $\forall a \in A, h \in H$

(2) $\sigma(h, k) = \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2 k_2)$, $\forall h, k \in H$

Moreover, $\Phi : A \#_\sigma H \to B$, $a \# h \mapsto a\gamma(h)$ is an algebra isomorphism

It is both a left A-module and a right H-comodule map.

Main step: (1) $h \cdot a$, $\sigma(h, k)$ is well-defined

(2) find Ψ s.t. $\Phi \circ \Psi = I_B$, $\Psi \circ \Phi = I_{A \#_\sigma H}$ and that Φ is an algebra map.

③ lemma:

let $A \subseteq B$ be H-cleft via $\gamma \in \mathrm{Hom}(H,B)$ s.t. $\gamma(1)=1$

then: (1) $\rho \circ \gamma^{-1} = (\gamma^{-1} \otimes S) \circ \tau \circ \Delta$

(2) $b_0 \gamma^{-1}(b_1) \in B^{\omega H}$

Note: 1. γ is an H-comod. map $\Rightarrow \rho^n \circ \gamma(h) = \gamma(h_1) \otimes h_2 \otimes \cdots \otimes h_{n+1}$

2. $(\rho \circ \gamma^{-1}) * (\rho \circ \gamma) = u_{B \otimes H} \epsilon_H$ since $\gamma^{-1} * \gamma = u_B \epsilon_H$

pf: $(\rho \circ \gamma^{-1}) * (\rho \circ \gamma)(h) = \rho(\gamma^{-1}(h_1))\, \rho(\gamma(h_2))$ ∂ ρ alg map.

$= \rho(\gamma^{-1}(h_1)\gamma(h_2))$

$= \rho(\epsilon(h) \cdot 1) = \epsilon(h)$

pf: (1). let $\theta = (\gamma^{-1} \otimes S) \circ \tau \circ \Delta$, then

$(\rho \circ \gamma) * \theta(h) = \left((\gamma \otimes I_H) \circ \Delta(h_1)\right)\left((\gamma^{-1} \otimes S) \circ \tau \circ \Delta(h_2)\right)$

$= (\gamma(h_1) \otimes h_2)(\gamma^{-1}(h_4) \otimes S(h_3))$

$= \gamma(h_1)\gamma^{-1}(h_4) \otimes h_2 S h_3$

$= \gamma(h_1)\gamma^{-1}(h_3) \otimes \epsilon(h_2)$

$= \gamma(h_1)\gamma^{-1}(h_2) \otimes 1 = \epsilon(h) 1_B \otimes 1_H$

$\rho \circ \gamma^{-1} = \theta$ since $(\rho \circ \gamma) * (\rho \circ \gamma^{-1}) = (\rho \circ \gamma^{-1}) * (\rho \circ \gamma) = u_B \epsilon_H$

(2) $\rho(b_0 \gamma^{-1}(b_1)) \overset{\text{alg. map}}{=} \rho(b_0)\, \rho(\gamma^{-1}(b_1))$

$= (b_0 \otimes b_1)\, (\gamma^{-1} \otimes S) \circ \tau \circ \Delta(b_2)$

$= (b_0 \otimes b_1)(\gamma^{-1}(b_3) \otimes S(b_2))$

$= b_0 \gamma^{-1}(b_3) \otimes b_1 S(b_2)$

$= b_0 \gamma^{-1}(b_1) \otimes 1$

proof of (2):

pf:

(1) $(h \cdot a) \in A$ and H measures A

$\rho(h \cdot a) = \rho(\gamma(h_1) a \gamma^{-1}(h_2))$

$\overset{\text{alg. map}}{=} \rho(\gamma(h_1))\, \rho(a)\, \rho(\gamma^{-1}(h_2))$ $\rho \circ \gamma^{-1} = (\gamma^{-1} \otimes S) \circ \tau \circ \Delta$

$$= (\gamma(h_1)\otimes h_2)(a\otimes 1)(\gamma^{-1}(h_4)\otimes S(h_3))$$

$$= \gamma(h_1)\, a\, \gamma^{-1}(h_4)\otimes h_2 S(h_3)$$

$$= \gamma(h_1)\, a\, \gamma^{-1}(h_2)\otimes 1$$

$$= (h\cdot a)\otimes 1$$

H measures A:

$$h\cdot 1 = \gamma(h)\gamma^{-1}(h) = \epsilon(h)\cdot 1$$

$$h\cdot(ab) = \gamma(h_1)(ab)\gamma^{-1}(h_2)$$

$$= \gamma(h_1)\, a\, \gamma^{-1}(h_2)\ \gamma(h_3)\, b\, \gamma^{-1}(h_4)$$

$$= (h_1\cdot a)(h_2\cdot b)$$

(2). Similarly. $\sigma(h,k)\in A$. i.e.

$$\rho(\sigma(h,k)) = \rho(\gamma(h_1)\gamma(k_1)\ \gamma^{-1}(h_2k_2))\in A$$

$\rho\circ\gamma = (\gamma\otimes I_H)\circ\Delta$

$\rho\circ\gamma^{-1} = (\gamma^{-1}\otimes S)\circ\tau\circ\Delta$

$$= \rho\circ\gamma(h_1)\ \rho\circ\gamma(k_1)\ \rho\circ\gamma^{-1}(h_2k_2)$$

$$= (\gamma(h_1)\otimes h_2)(\gamma(k_1)\otimes k_2)\ \gamma^{-1}((h_2k_2)_2)\otimes S(h_2k_2)_1$$

$$= (\gamma(h_1)\otimes h_2)(\gamma(k_1)\otimes k_2)(\gamma^{-1}(h_4k_4)\otimes S(h_3k_3))$$

$$= \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_4k_4)\otimes h_2k_2S(h_3k_3)$$

$$= \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2)\otimes 1$$

$$= \sigma(h,k)\otimes 1$$

recall: $\Phi : A\#_\sigma H \to B$, let $\Psi : B\to A\#_\sigma H$

$a\#h \mapsto a\gamma(h)$ $\qquad$ $b\mapsto \underbrace{b_0\gamma^{-1}(b_1)}_{\text{well-defined.}}\# b_2$

(3). Φ and Ψ are mutual inverses:

$$\Phi\circ\Psi(b) = \Phi(b_0\gamma^{-1}(b_1)\# b_2)$$

$$= b_0\gamma^{-1}(b_1)\gamma(b_2) = b$$

$$\Psi\circ\Phi(a\#h) = \Psi(a\gamma(h))$$ def

$$= (a\gamma(h))_0\ \gamma^{-1}((a\gamma(h))_1)\#(a\gamma(h))_2$$ $\rho(a) = a\otimes 1$

$= a\,\gamma(h)_0\,\gamma^{-1}(\gamma(h)_1)\ \#\ \gamma(h)_2$ $\quad \rho\circ\gamma = (\gamma\otimes I_H)\circ\Delta$

$\gamma(h)_0\otimes\gamma(h)_1\otimes\gamma(h)_2 = \rho^2\circ\gamma(h) = \gamma(h_1)\otimes h_2\otimes h_3$

$= a\,\gamma(h_1)\,\gamma^{-1}(h_2)\ \#\ h_3$

$= a\ \#\ h$

(4) Φ is an algebra map:

$\Phi(a\#h)\,\Phi(b\#k) = a\gamma(h)\,b\gamma(k)$

$= a\,\gamma(h_1)\,b\,\gamma^{-1}(h_2)\gamma(h_3)\,\gamma(k_1)\,\gamma^{-1}(h_4k_2)\,\gamma(h_5k_3)$

$= a\,(h_1\cdot b)\,\sigma(h_2,k_1)\,\gamma(h_3k_2)$

$= \Phi(\,a(h_1\cdot b)\,\sigma(h_2,k_1)\ \#\ h_3k_2)$

$= \Phi((a\#h)(b\#k))$

Thus $B\cong A\#_\sigma H$, the conditions of crossed product follow from 7.1.2.

(5) Φ is left A-mod. map

right H-comod map.

Pf: $\forall\, b\in A,\ a\#h\in A\#_\sigma H$.

$\Phi((b\#1)\cdot(a\#h)) = \Phi(ba\#h) = ba\gamma(h) = b\,\Phi(a\#h)$

$\rho(\Phi(a\#h)) = \rho(a\gamma(h))$

$= \rho(a)\,\rho\circ\gamma(h)$

$= (a\otimes 1)\,\gamma(h_1)\otimes h_2$

$= a\gamma(h_1)\otimes h_2$

$= \Phi(a\#h_1)\otimes h_2$

$$\begin{array}{ccc} A\#_\sigma H & \xrightarrow{\Phi} & B \\ \downarrow{\scriptstyle I_A\otimes\Delta} & & \downarrow{\scriptstyle\rho} \\ (A\#_\sigma H)\otimes H & \xrightarrow{\Phi\otimes I_H} & B\otimes H \end{array}$$

So far: $A\subseteq B$ H-cleft via $\gamma: H\to B$

$\Rightarrow B\xrightarrow{\Psi} A\#H \xrightarrow{\Phi} B$ left A-mod, right H-comod

$$\begin{cases} h\cdot a = \gamma(h_1)\,a\,\gamma^{-1}(h_2) \\ \sigma(h,k) = \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2) \end{cases}$$

(6). Let $A\#_\sigma H$ be a crossed product satisfies the cocycle condiction and twisted module condition.

Let $\gamma: H \to A\#_\sigma H$, then γ is an H-comod. map
$h \mapsto 1\#h$

with convolution inverse:

$$\gamma^{-1}: H \to A\#_\sigma H$$
$$h \mapsto \sigma^{-1}(Sh_2, h_3)\# Sh_1$$

In particular, $A \hookrightarrow A\#_\sigma H$ is H-cleft

pf: It's trivial that γ is an H-comod. map and that $A=(A\#_\sigma H)^{coH}$

(1) $\gamma^{-1}*\gamma = u_{A\#_\sigma H}\,\epsilon_H$

Set $\mu(h) = \sigma^{-1}(Sh_2, h_3)\# Sh_1$, then

$$\mu * \gamma(h) \overset{\text{convolution}}{=} (\sigma^{-1}(Sh_2, h_3)\# Sh_1)(1\# h_4)$$
$$= (\sigma^{-1}(Sh_2, h_3)((Sh_1)_1\cdot 1)\,\sigma((Sh_1)_2, h_5)\#(Sh_1)_3 h_5$$
展开 S
$$= \sigma^{-1}(\underline{Sh_3, h_4)\,\sigma(Sh_2, h_5)}\# Sh_1h_6$$
$$= \sigma^{-1}*\sigma(Sh_2, h_3)\# Sh_1h_4$$
$$= 1\#1$$

(2) $\gamma*\gamma^{-1} = u_{A\#_\sigma H}\,\epsilon_H$

By a similar computation, we have

$$\gamma*\mu(h) = (1\#h_1)(\sigma^{-1}(Sh_3, h_4)\# Sh_2)$$ def of $(a\#h)\cdot(b\#k)$
$$= ((h_1)_1\cdot\sigma^{-1}(Sh_3, h_4))\,\sigma((h_1)_2, (Sh_2)_1)\#(h_1)_3(Sh_2)_2$$
$$= (h_1\cdot\sigma^{-1}(Sh_6, h_7))\,\sigma(h_2, Sh_5)\# h_3Sh_4$$
$$= (h_1\cdot\sigma^{-1}(Sh_4, h_5))\,\sigma(h_2, Sh_3)\# 1$$

recall: $\sigma(h_1, k_1)\sigma(h_2k_2, m) = (h_1\cdot\sigma(k_1, m_1))\,\sigma(h_2, k_2m_2)$ $*\sigma^{-1}$

$$\Rightarrow \sigma(h_1, k_1)\sigma(h_2k_2, m_1)\sigma^{-1}(h_3, k_3m_2) = h\cdot\sigma(k, m)$$

$$\Rightarrow h\cdot\sigma^{-1}(k, m) = (h\cdot\sigma(k, m))^{-1}$$
$$= (\sigma(h_1, k_1)\sigma(h_2k_2, m_1)\sigma^{-1}(h_3, k_3m_2))^{-1}$$

$$= \sigma(h_1, k_1 m_1)\,\sigma(h_2 k_2, m_2)\,\sigma^{-1}(h_3, k_3)$$

hence $(h_1 \cdot \sigma^{-1}(Sh_4, h_5))\,\sigma(h_2, Sh_3) \overset{\text{def}}{=}$

$$= \sigma((h_1)_1, (Sh_4)_1 (h_5)_1)\,\sigma^{-1}((h_1)_2 (Sh_4)_2, (h_5)_2)\,\sigma^{-1}((h_1)_3, (Sh_4)_3)\,\underline{\sigma(h_2, Sh_3)}$$

$$= \underline{\sigma(h_1, Sh_8 h_9)}^{\sigma\sigma}\,\sigma^{-1}(h_2 Sh_7, h_{10})\,\sigma^{-1}(h_3, Sh_6)\,\sigma(h_4, Sh_5)$$

$$= \sigma^{-1}(h_1 Sh_6, h_7)\,\underline{\sigma^{-1}(h_2, Sh_5)\,\sigma(h_3, Sh_4)}^{\sigma\sigma}$$

$$= \sigma^{-1}(h_1 Sh_2, h_3)$$

$$= \epsilon(h) \cdot 1$$

Note: 1. for smash product, $\gamma(h) = 1 \# Sh$.

2. $(h \cdot \sigma(k \otimes m))^{-1} = (- \cdot \sigma(- \otimes -))^{-1}(h \otimes k \otimes m) = f(h \otimes k \otimes m)$

$h \cdot \sigma^{-1}(k \otimes m) = (- \cdot \sigma^{-1}(- \otimes -))(h \otimes k \otimes m) = g(h \otimes k \otimes m)$

$\Rightarrow f * g(h \otimes k \otimes m) = [h_1 \cdot \sigma(k_1 \otimes m_1)][h_2 \cdot \sigma^{-1}(k_2 \otimes m_2)]$

$$= h \cdot \sigma(k_1 \otimes m_1)\,\sigma^{-1}(k_2 \otimes m_2)$$

$$= h \cdot \sigma * \sigma^{-1}(k \otimes m)$$

$$= \epsilon(k)\epsilon(m)\epsilon(h) \cdot 1_A$$

example:

$A \#_\sigma \mathbb{K}G$, $N \lhd G \Rightarrow (A \#_\sigma \mathbb{K}N) \#_\tau \mathbb{K}[G/N] \cong A \#_\sigma \mathbb{K}G$, via

$$\gamma: \mathbb{K}[G/N] \to A \#_\sigma G$$
$$\bar{x} \mapsto 1 \# x$$

$A \#_\sigma \mathbb{K}G$ is a $\mathbb{K}[G/N]$ comodule algebra via

$$\begin{tikzcd}
\rho: A \#_\sigma \mathbb{K}G \arrow[rr] \arrow[dr, "I_A \otimes \Delta"'] & & A \#_\sigma \mathbb{K}G \otimes \mathbb{K}[G/N] \\
& A \#_\sigma \mathbb{K}G \otimes \mathbb{K}G \arrow[ur, "I_{A \# \mathbb{K}G} \otimes \pi"'] &
\end{tikzcd}$$

where $\pi: \mathbb{K}G \to \mathbb{K}[G/N]$,

$$\begin{tikzcd}
a \# g \arrow[rr] \arrow[dr] & & a \# g \otimes \bar{g} \\
& a \# g \otimes g \arrow[ur] &
\end{tikzcd}$$

We show that: $(A \#_\sigma \mathbb{K}G)^{\mathrm{co}\,\mathbb{K}[G/N]} \overset{(1)}{=} A \#_\sigma \mathbb{K}N$

$$\overset{(2)}{\rho \circ \gamma(\bar{x})} = \gamma(\bar{x}) \otimes \bar{x}$$

(3) $\gamma^{-1}(\bar{x}) = 1\# x^{-1}$

Pf: (2) $\rho(\gamma(\bar{x})) = \rho(1\# x) = 1\# x\otimes \bar{x} = \gamma(x)\otimes\bar{x}$

(4) $\rho(a\# n) = a\# n\otimes\bar{n} = a\# n\otimes\bar{1} \Rightarrow A\#_{\sigma}N \subseteq (A\#_{\sigma}\mathbb{K}G)^{\mathrm{co}\,\mathbb{K}[G/N]}$

$\rho(a\# x) = a\# x_1\otimes\bar{x}_2 = a\# x\otimes\bar{1}$

$\overset{I_A\otimes\epsilon\otimes I_{\mathbb{K}G/N}}{\Longrightarrow}$ $a\# 1\otimes\bar{x} = \epsilon(x)\cdot a\# 1\otimes\bar{1}$

$\Rightarrow$ $\bar{x} = \epsilon(x)\cdot\bar{1}$ i.e. $x\in N$

$\Rightarrow$ $a\# x\in A\#_{\sigma}N$.

7.2.9. $\pi: H\to\bar{H}$ hopf algebra surjection $\not\Rightarrow$ $H = A\#_{\sigma}H$

Counter example: (3.5.2)

$G = \{\sigma, 1\}$, $H = (\mathbb{Z}_2)^G$, $K = (\mathbb{Z}(n\mathbb{Z}))^G \subseteq H$, n is a even number

K is normal $\Rightarrow$ $I = HK^{+} = K^{+}H$ is a hopf ideal

$\Rightarrow$ $\bar{H} = H/I$ s.t. $H^{\mathrm{co}\bar{H}} = K$

$H = K\#_{\sigma}\bar{H}$ contradicting the fact that H is not free over K

Known results:

1. H is pointed cocomm.
2. H is f.d.

7.2.11 Let $A\#_{\sigma}H$ be a crossed product. Then $A\#_{\sigma}H \cong H\otimes A$ as right A-modules, provided the antipode S of H is bijective.

Pf:

Let $\bar{S}$ denote the composition inverse of S and

set $\mu = \gamma^{-1}\circ\bar{S}$, $\hat{\mu} = \gamma\circ\bar{S}$: $H^{op}\to A$

then $\mu * \hat{\mu}(h) = \gamma^{-1}(\bar{S}(h_1))\gamma(\bar{S}(h_2))$

$= \gamma^{-1}(\bar{S}(h)_2)\gamma(\bar{S}(h)_1) = \epsilon(\bar{S}(h)) = \epsilon(h)$

Let $B = A\#_{\sigma}H$, B is a right H-comod alg

$\Rightarrow B^{op}$ is a right H^{op}-comod alg

B is H-cleft via $\gamma \Rightarrow B^{op}$ is H-cleft via μ

$\Rightarrow B^{op} \cong A^{op} \otimes H^{op}$ as left A-mod.

$\Rightarrow B \cong H \otimes A$ as right A-mod.

Main idea: $B = A \#_\sigma H$: free left A-mod 余模部分不变

$\Rightarrow B^{op} \cong A^{op} \#_{\sigma^r} H^{op}$: free left A-mod

$\Rightarrow B \cong H \otimes A$ as free left A-mod

e.g. $(a \# 1)(b \# k) = a(1 \cdot b)\sigma(1, k_1) \# k_2 = ab \# k$, $\not\Rightarrow$ free left A-mod.

twisted product $\Rightarrow (a \otimes h)(b \otimes k) = ab\,\sigma(h_1, k_1) \otimes h_2 k_2$

$\Rightarrow (a \otimes h)(b \otimes 1) = ab \otimes h \not\Rightarrow$ free right A-mod.

Question: $A \# H \overset{?}{=} (1 \# H)(A \# 1)$

weaker fact:

S bijective $\Rightarrow \forall$ H-stable ideal I of A,

$I \#_\sigma H = (1 \#_\sigma H)(I \# 1)$

小结:

1. crossed product.

① H measures A: $\begin{cases} h \cdot ab = (h_1 \cdot a)(h_2 \cdot b) \\ h \cdot 1_A = \epsilon(h) 1_A \end{cases}$

② $\sigma: H \otimes H \to A$, convolution invertible.

$A \#_\sigma H$ is an algebra (not necessarily asso. nor with unit).

(1) $1 \# 1$ is the identity $\Leftrightarrow$ $1 \cdot a = a$ (normal condition)

$\sigma(h, 1) = \sigma(1, h) = \epsilon(h) \cdot 1$.

under (1),

(2) $A \#_\sigma H$ is associative

$$\Longleftrightarrow \begin{cases} h\cdot(k\cdot a) = \sigma(h_1,k_1)(h_2k_2\cdot a)\,\sigma^{-1}(h_3,k_3) & \text{(twisted module condition)} \\ h_1\cdot\sigma(k_1,m_1)\,\sigma(h_2,k_2m_2) = \sigma(h_1,k_1)\,\sigma(h_2k_2,m) & \text{(cocycle condition)} \end{cases}$$

Note: P102, In the rest ····, crossed product 视语境定义.

2. Group

$N \lhd G \Rightarrow G \xrightarrow{\pi} G/N = \bar{G} \xrightarrow{\gamma} G$, γ a section of G

$\Rightarrow G = N\gamma(\bar{G})$

$$\Rightarrow n\gamma(\bar{x})\, m\gamma(\bar{y}) = n\underbrace{\gamma(\bar{x})\, m\gamma(\bar{x})^{-1}}\gamma(\bar{x})\gamma(\bar{y})$$

$$= n(\bar{x}\cdot m)\,\underbrace{\gamma(\bar{x})\gamma(\bar{y})\gamma(\bar{x}\cdot\bar{y})^{-1}}\cdot\gamma(\bar{x}\cdot\bar{y})$$

$$= n(\bar{x}\cdot m)\,\gamma(\bar{x})\gamma(\bar{y})\gamma(\bar{x}\cdot\bar{y})^{-1}\cdot\gamma(\bar{x}\cdot\bar{y})$$

$$= n(\bar{x}\cdot m)\,\sigma(\bar{x},\bar{y})\,\gamma(\bar{x}\cdot\bar{y})$$

hence $\bar{x}\cdot m = \gamma(\bar{x})\, m\, \gamma(\bar{x})^{-1}$

$\sigma(\bar{x},\bar{y}) = \gamma(\bar{x})\gamma(\bar{y})\gamma(\bar{x}\cdot\bar{y})^{-1}$

例: $A\#_\sigma H$ with unit $1\#1$, $x\in G(x) \Rightarrow 1\# x$ is invertible

Pf: $(1\# x^{-1})(1\# x) = \sigma(x^{-1},x)\# 1 \Rightarrow (1\# x)^{-1} = (\sigma(x^{-1},x)\# 1)^{-1}(1\# x^{-1})$

(left) $= (\sigma^{-1}(x^{-1},x)\# 1)(1\# x^{-1})$

$= \sigma^{-1}(x^{-1},x)\# x^{-1}$

(right)

$(1\# x)(1\# x^{-1}) = \sigma(x,x^{-1})\# 1 \Rightarrow (1\# x)^{-1} = (1\# x^{-1})(\sigma(x,x^{-1})\# 1)^{-1}$

$= (1\# x^{-1})(\sigma^{-1}(x,x^{-1})\# 1)$

$= x^{-1}\cdot\sigma^{-1}(x,x^{-1})\# x^{-1}$

hence $(1\# x)^{-1} = \sigma^{-1}(x^{-1},x)\# x^{-1} = x^{-1}\cdot\sigma^{-1}(x,x^{-1})\# x^{-1}$

3. $A\#_\sigma H$, H cocomm. A a twisted H-module:

A is an H-module $\Leftrightarrow \sigma(H\otimes H)\subseteq Z(A)$

In particular, A comm $\Rightarrow A$ is an H-module

Cqt 7.2.

H-cleft: 1. $H \overset{\text{comod alg}}{\curvearrowright} B$,

2. $A = B^{\mathrm{coH}} \hookrightarrow B$ subalg.

3. $\gamma : H \to B$ comod map. convolution invertible.

} $\to$ H-extension

$\Rightarrow \begin{cases} h \cdot a = \gamma(h_1)\, a\, \gamma^{-1}(h_2) \\ \sigma(h, k) = \gamma(h_1)\gamma(k_1)\, \gamma^{-1}(h_2 k_2) \end{cases}$

$\Phi : A \#_\sigma H \to B$ algebra map. Left A-mod, right H-comod.

$a \# h \mapsto a\gamma(h)$

Conversely, $A \#_\sigma H$ is asso. with $1\#1$ as its unit

$\Rightarrow A \hookrightarrow A\#_\sigma H$ is H-cleft via $\gamma : H \to A\#_\sigma H$

$h \mapsto 1 \# h$.