Today's Topics

9. The Discrete Wavelet Transform

Topic 7:

Discrete Wavelet Transform

- Wavelet transform vs. Laplacian pyramid
- Basic intuition: a simple wavelet-like 2D transform
- The 1D Haar wavelet transform
- 1D Haar wavelet transform as a matrix product
- Reconstructing a 1D image from its wavelet coefs
- Wavelet-based image compression
- The 2D Haar wavelet transform

Wavelet-Based Image Representations

The Wavelet-based representation of images touches upon several concepts covered so far:

- Image

$\Updownarrow$

Laplacian Pyramid

defined by a single filter

image represented as a pyramid of "detail" images

The Laplacian Pyramid Representation

- How many pixels does the Laplacian pyramid have?

$$(2^N+1)+(2^{N-1}+1)+\cdots(2^0+1)$$

L_0 $\qquad\qquad$ g_N

$$1 + 1/4 + 1/16 + \cdots \simeq 4/3$$

- $\Rightarrow$ Representation is <u>over complete</u> (more pixels in rep than pixels in image)

Wavelet-Based Image Representations

The Wavelet-based representation of images touches upon several concepts covered so far:

- Image $\Longleftrightarrow$ vector in a high dimensional space

$\Updownarrow$

Laplacian Pyramid

defined by a single filter

image represented as a pyramid of "detail" images

Wavelet-Based Image Representations

The Wavelet-based representation of images touches upon several concepts covered so far:

- Image $\Longleftrightarrow$ vector in a high dimensional space

$\Updownarrow$ $\Updownarrow$

Laplacian Pyramid

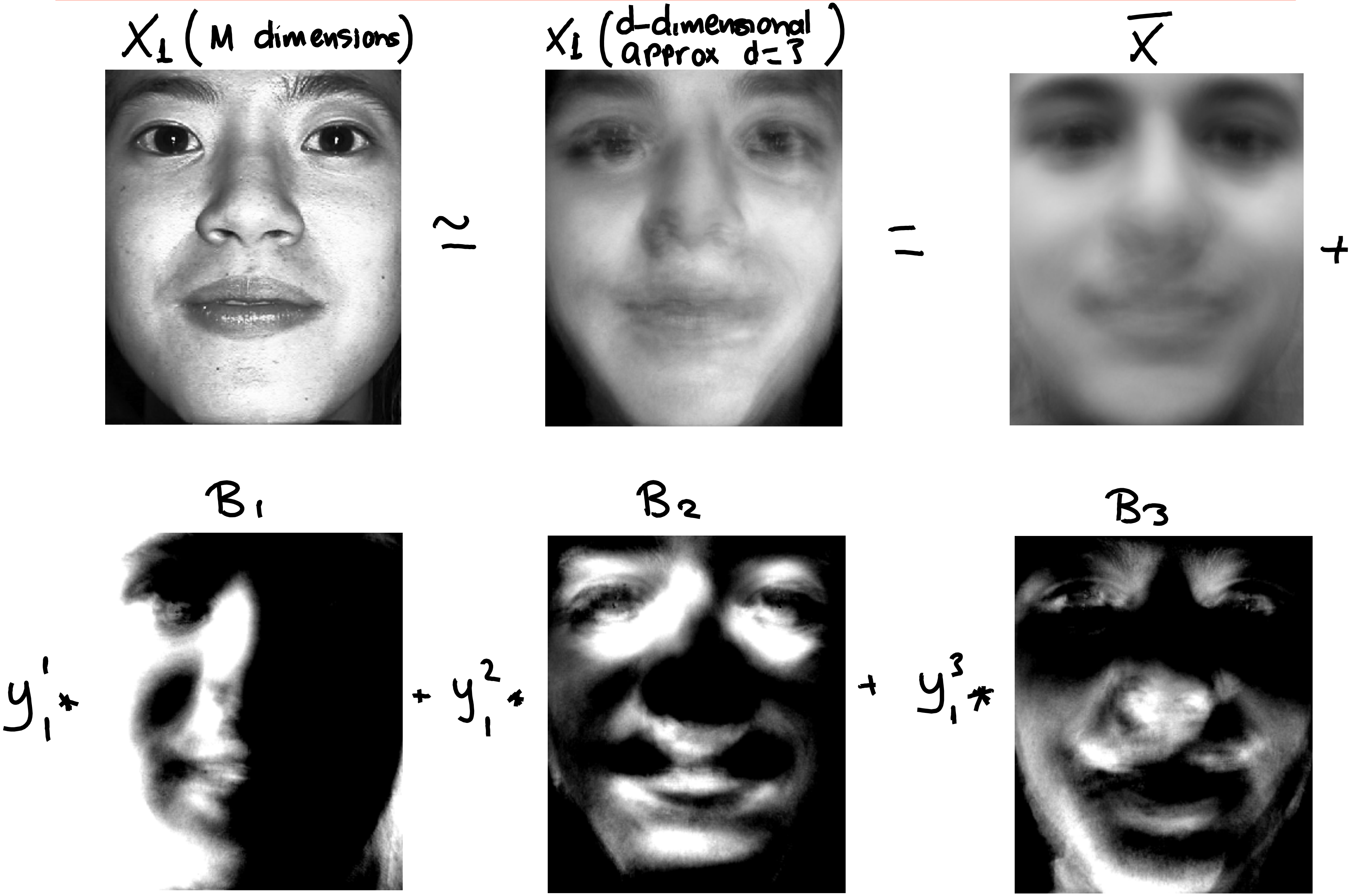
defined by a single filter

image represented as a pyramid of "detail" images

PCA basis (& eigenfaces) are an efficiently-computable & compact representation of images (from a known image class)

an image "coordinate" in this basis computed by a dot product

Reminder: The Eigenface/PCA Image Basis



Representing Images by their PCA Basis

all pixels of one image

j-th pixel of all images

$$\begin{bmatrix} z_1^1 & z_2^1 & \cdots & z_N^1 \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^M & z_2^M & \cdots & z_N^M \end{bmatrix} =$$

$$\begin{bmatrix} B_1 & B_2 & \cdots & B_M \end{bmatrix} \cdot$$

$$\begin{array}{l} \text{large} \left\{ \begin{array}{l} \\ \\ \end{array} \right. \\ \text{near zero} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. \end{array} \begin{bmatrix} y_1^1 & y_2^1 & \cdots & y_N^1 \\ y_1^d & y_2^d & \cdots & y_N^d \\ y_1^{d+1} & y_2^{d+1} & \cdots & y_N^{d+1} \\ \vdots & \vdots & & \vdots \\ y_1^M & y_2^M & \cdots & y_N^M \end{bmatrix}$$

$$\Longleftrightarrow \quad Z = B \cdot Y$$

eigenfaces

mean-subtracted images

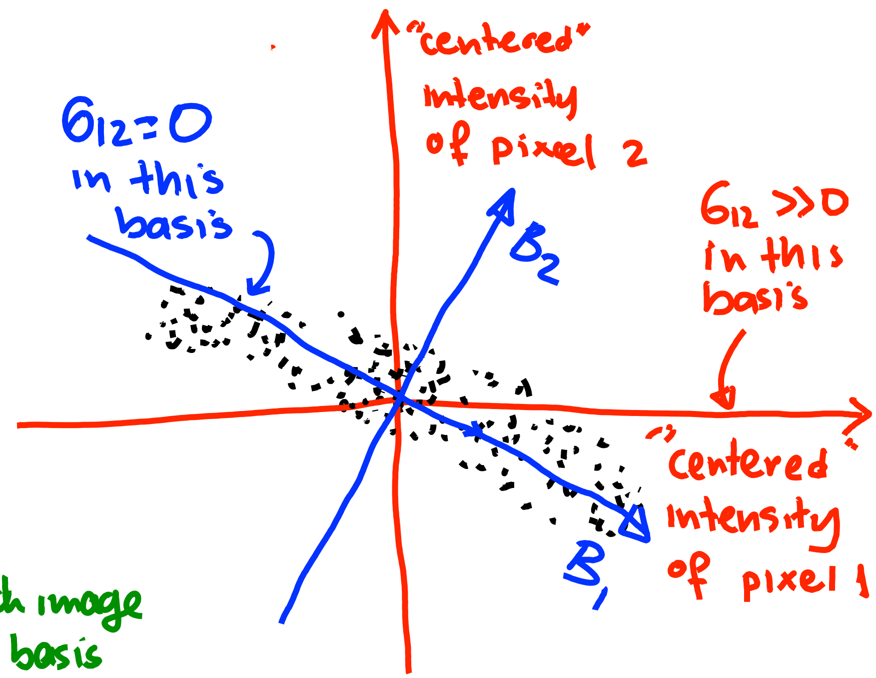
coordinates of each image in PCA/eigenface basis

- Image reconstruction:

$$\begin{bmatrix} x_i^1 \\ \vdots \\ x_i^M \end{bmatrix} = B \cdot \begin{bmatrix} y_i^1 \\ \vdots \\ y_i^M \end{bmatrix} + \overline{x}$$

- Image transform

$$[y_i] = B^T [x_i - \overline{x}]$$



The Discrete Wavelet Transform

all pixels of one image

j-th pixel of all images

$$\begin{bmatrix} x_1^1 & x_2^1 & \cdots & x_N^1 \\ x_1^2 & x_2^2 & \cdots & x_N^2 \\ \vdots & \vdots & & \vdots \\ x_1^M & x_2^M & \cdots & x_N^M \end{bmatrix} =$$

$$[B_1 \; B_2 \cdots B_M] \cdot \begin{array}{l} \\ \text{large}\left\{ \begin{array}{l} \\ \\ \end{array} \right. \\ \text{near zero}\left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. \end{array} \begin{bmatrix} y_1^1 & y_2^1 & \cdots & y_N^1 \\ y_1^d & y_2^d & \cdots & y_N^d \\ y_1^{d+1} & y_2^{d+1} & \cdots & y_N^{d+1} \\ \vdots & \vdots & & \vdots \\ y_1^M & y_2^M & \cdots & y_N^M \end{bmatrix}$$

$$\Longleftrightarrow$$

wavelet basis

$$X = B \cdot Y$$

wavelet coefficients

Image reconstruction:

(1) $$\begin{bmatrix} x_i^1 \\ \vdots \\ x_i^M \end{bmatrix} = B \cdot \begin{bmatrix} y_i^1 \\ \vdots \\ y_i^M \end{bmatrix} + \overline{x}$$

Image transform

(2) $$[y_i] = B^T [x_i - \overline{x}]$$

The (discrete) wavelet transform maps an image into yet another basis defined by a "special" matrix B:

- captures scale
- invertible, orthogonal, square
- image independent

The Discrete Wavelet Transform

all pixels of one image

j-th pixel

$$\begin{bmatrix} x_2^1 \\ x_2^2 \\ \vdots \\ x_2^M \end{bmatrix} = [B_1 \; B_2 \cdots B_M] \cdot \begin{bmatrix} y^1 \\ y^2 \\ y \\ \vdots \\ y^M \end{bmatrix}$$

coarse-scale details

fine-scale details

$$\Longleftrightarrow$$

wavelet basis

$$X = B \cdot Y$$

wavelet coefficients

Image reconstruction:

(1) $\begin{bmatrix} x_i^1 \\ \vdots \\ x_i^M \end{bmatrix} = B \cdot \begin{bmatrix} y_i^1 \\ \vdots \\ y_i^M \end{bmatrix} + \bar{x}$

Image transform

(2) $[y_i] = B^T [x_i - \bar{x}]$

The (discrete) wavelet transform maps an image into yet another basis defined by a "special" matrix B

- captures scale
- invertible, orthogonal, square
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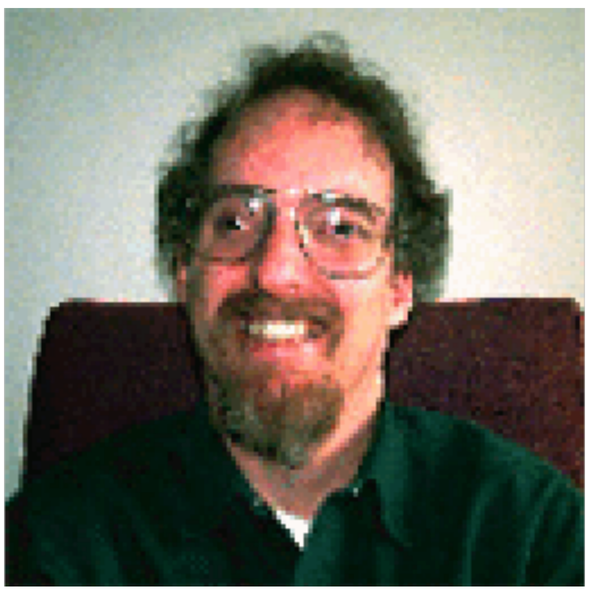
A Simple, Minimal 2-D Image Transform

Properties of transformation
- Minimal (no "wasted" pixels)
- Multiple scales represented simultaneously
- Invertible, linear

Input image ($2^N x 2^N$)



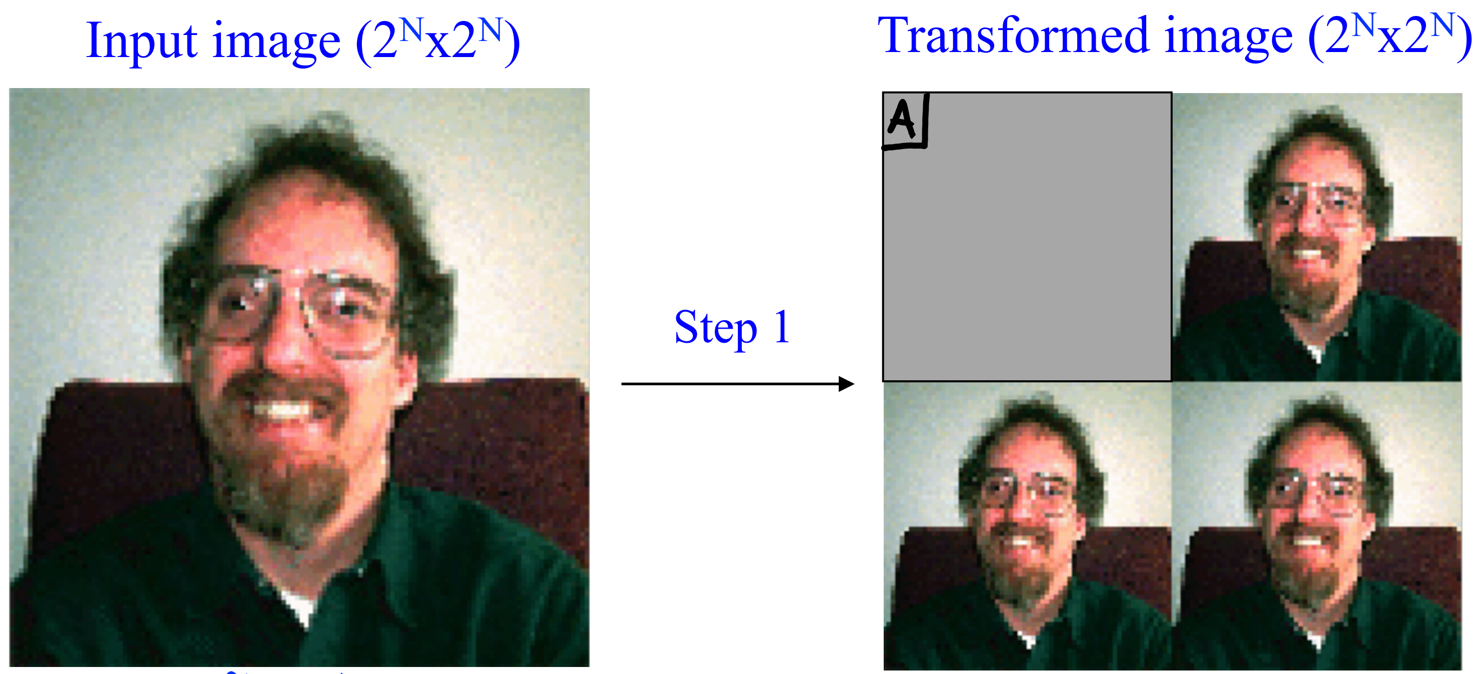
wavelet transform $\longrightarrow$

Transformed image ($2^N x 2^N$)

A Simple, Minimal 2-D Image Transform

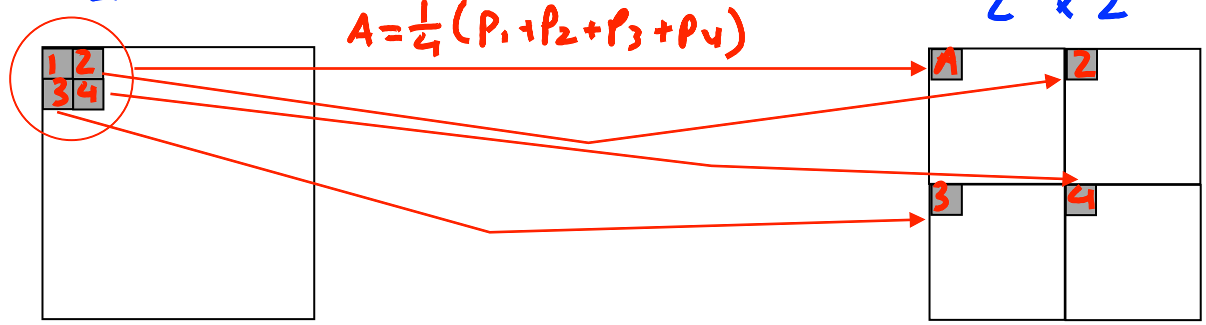
Step 1: Create 4 new images of size $2^{N-1} x 2^{N-1}$ as shown in figure

W_0:



$2^N \times 2^N$ $\qquad$ $2^N \times 2^N$

$A = \frac{1}{4}(p_1 + p_2 + p_3 + p_4)$

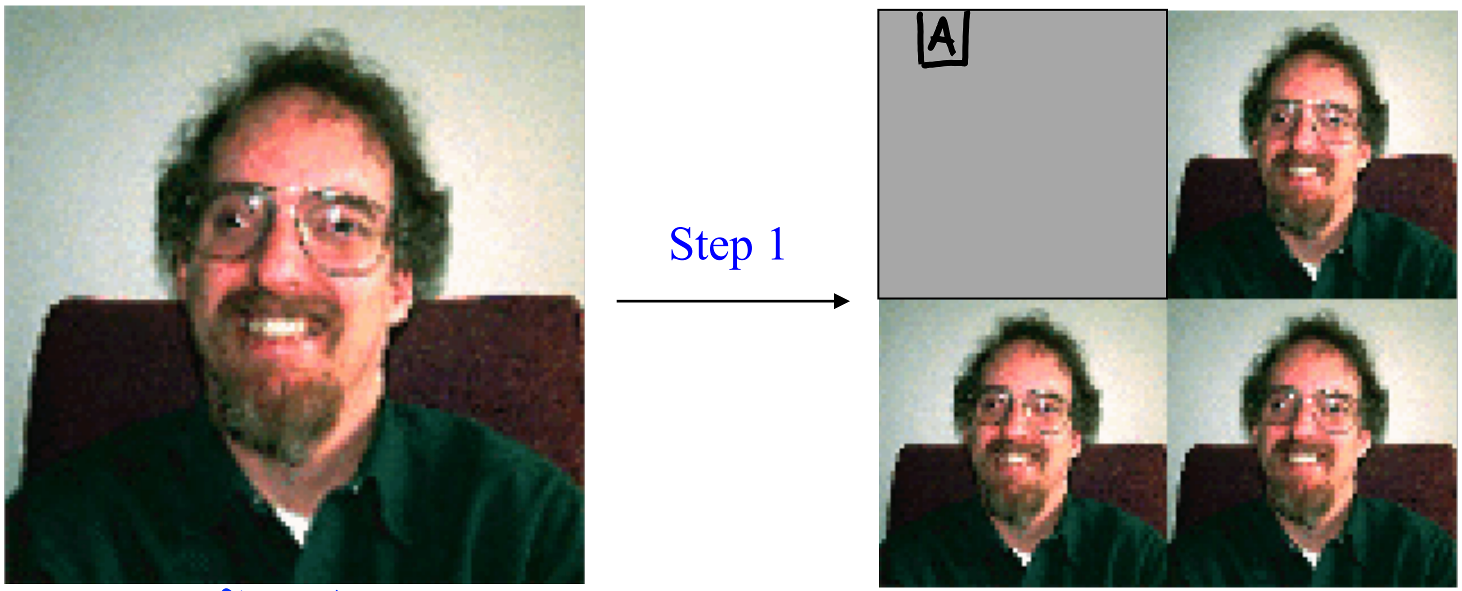


A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1}x2^{N-1}$ as shown in figure

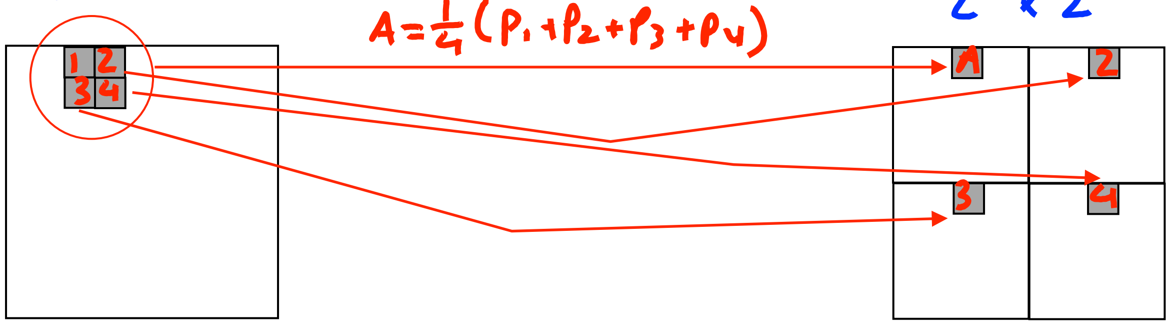
Input image ($2^N x 2^N$) | Transformed image ($2^N x 2^N$)

W_0:



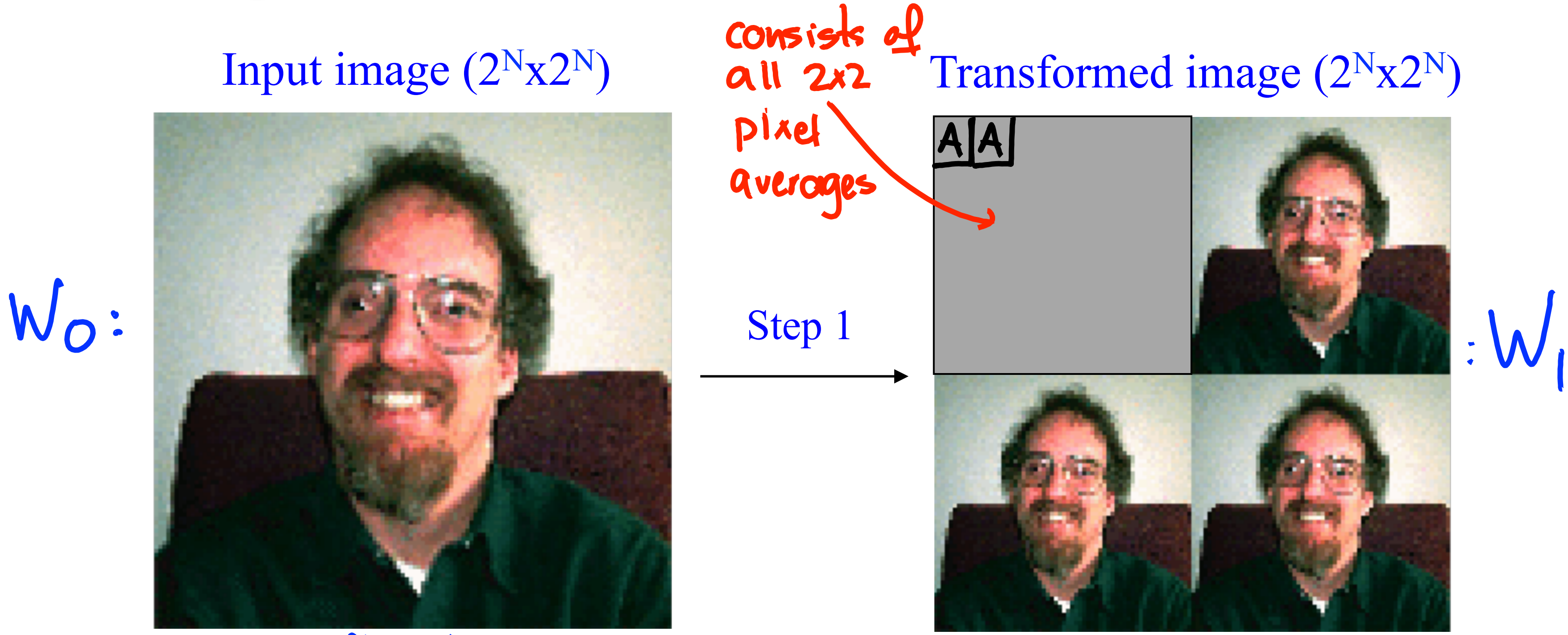
$2^N \times 2^N$ | $2^N \times 2^N$

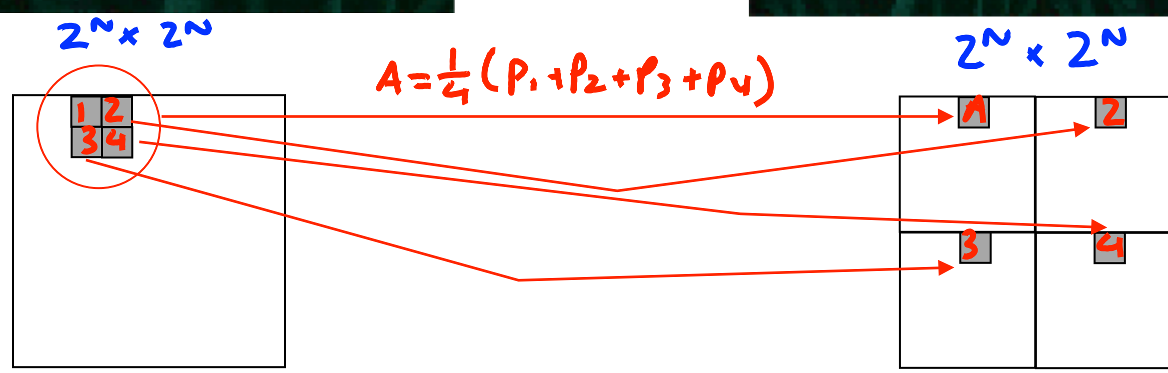
$A = \frac{1}{4}(p_1 + p_2 + p_3 + p_4)$



A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1}x2^{N-1}$ as shown in figure

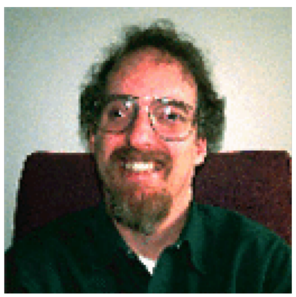




A Simple, Minimal 2-D Image Transform

Step 2: Recursively perform Step 1 for top-left quadrant of result

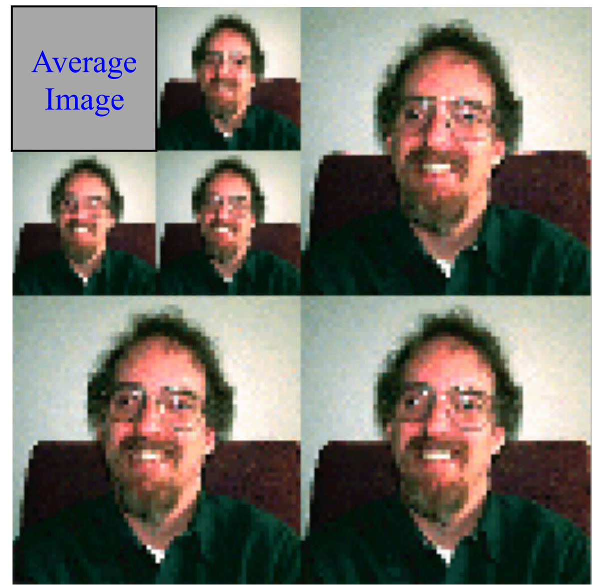
Result of Step 1 (2^{N-1}x2^{N-1})



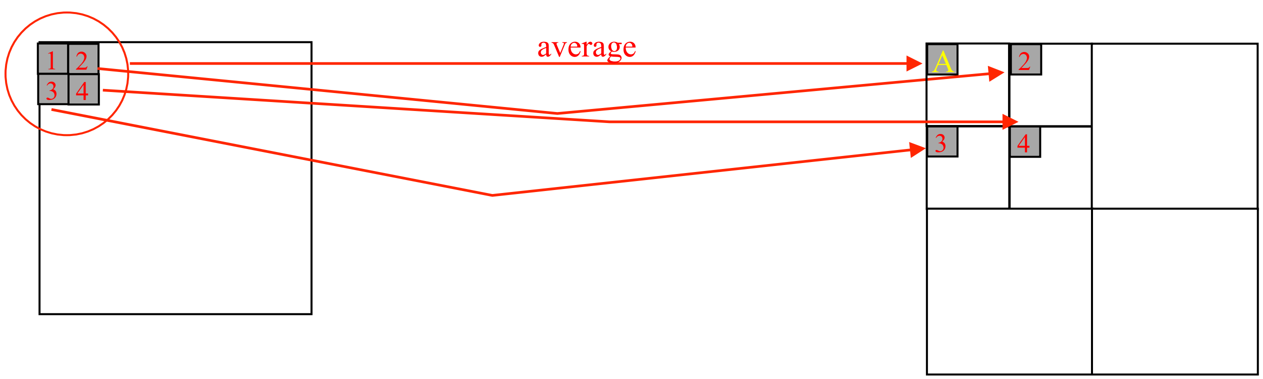
$2^{N-1} \times 2^{N-1}$

Step 2

Transformed image (2^Nx2^N)



$: W_2$



A Simple, Minimal 2-D Image Transform

Step 3: Recursion stops when average image is 1 pixel

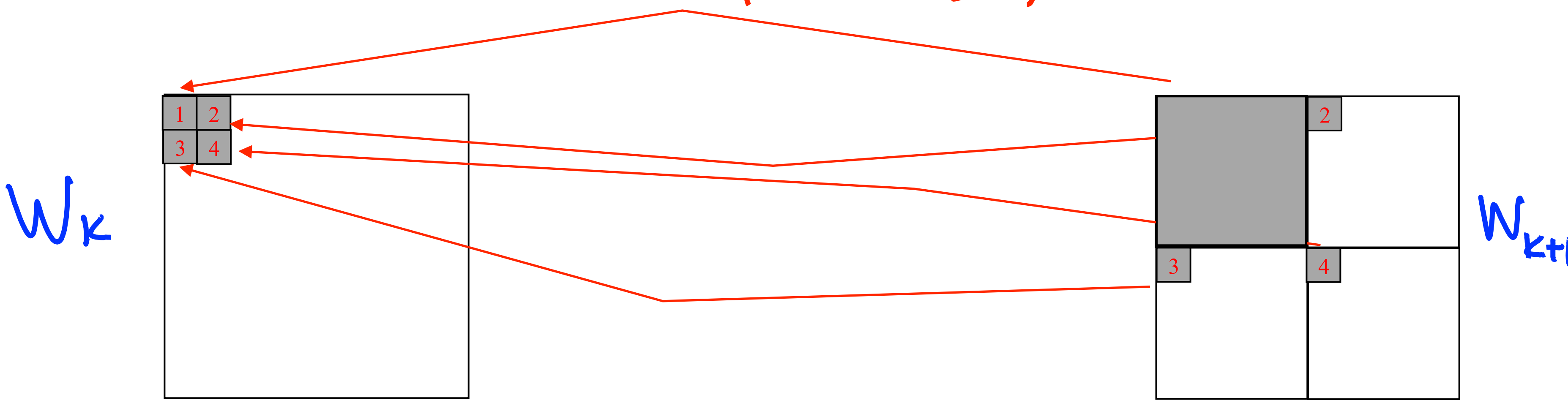
Transformed image (2^Nx2^N)

: W_N

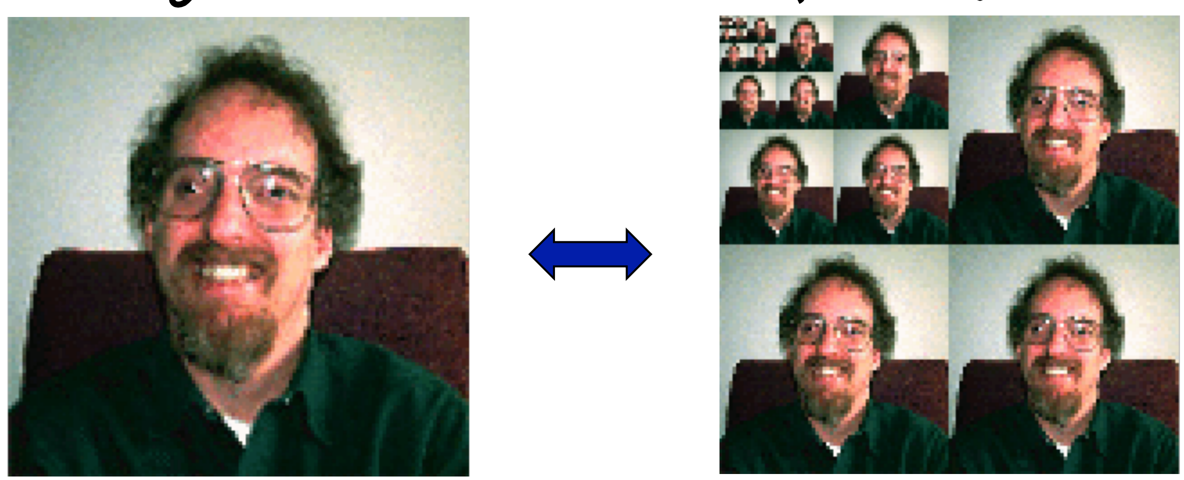
Invertibility of the Transformation

Property: W_k can be reconstructed from W_{k+1}

$P_1 = 4P_A - P_2 - P_3 - P_4$



$\Longleftrightarrow$ W_0 reconstructible from W_N



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1D Haar Wavelet Transform: Recursive Definition

Input image I^3

pixel #	0	1	2	...				2^3-1
	9	7	3	5	6	10	2	6

Average I^2

(analogous to level g_0 of Gaussian pyramid)

8	4	8	4

average of first two pixels

Detail coefs D^2

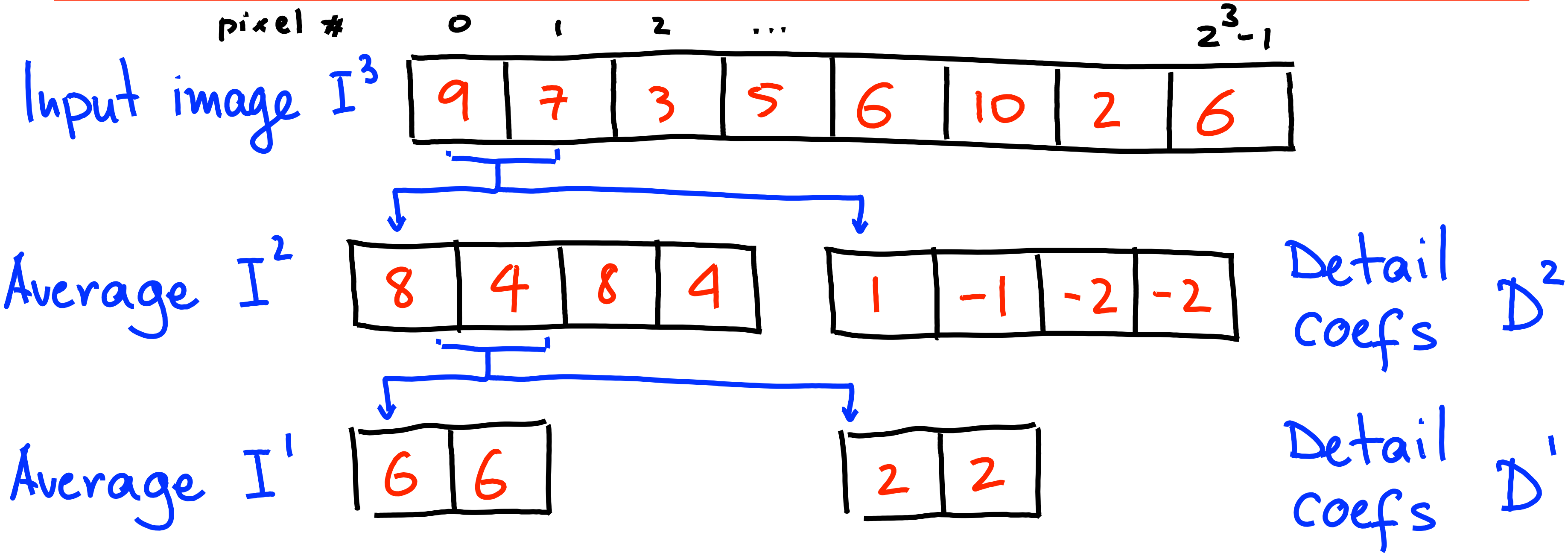
(analogous to level L_0 of Laplacian pyramid)

1	-1	-2	-2

difference of first pixel from the average

(note: we don't need to store difference of 2nd pixel from average $\Rightarrow$ D^0 has ½ the size of the corresponding Laplacian level L_0!)

1D Haar Wavelet Transform: Recursive Definition



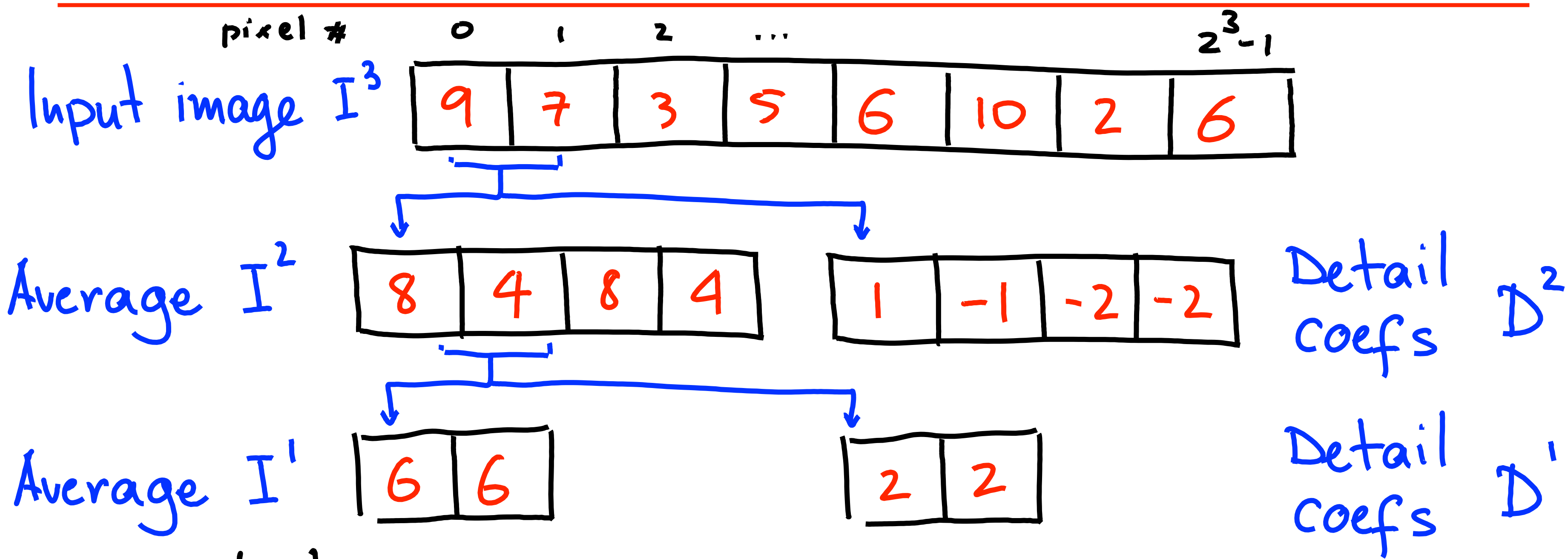
$$I_i^j = \frac{1}{2}\left(I_{2i}^{j+1} + I_{2i+1}^{j+1} \right)$$

j-th level of "pyramid" contains 2^j pixels

$$D_i^j = I_{2i}^{j+1} - \frac{1}{2}\left(I_{2i}^{j+1} + I_{2i+1}^{j+1} \right)$$

$$= \frac{1}{2}\left(I_{2i}^{j+1} - I_{2i+1}^{j+1} \right)$$

1D Haar Wavelet Transform: Recursive Definition



equivalent convolution mask ϕ: | 1/2 | 1/2 |

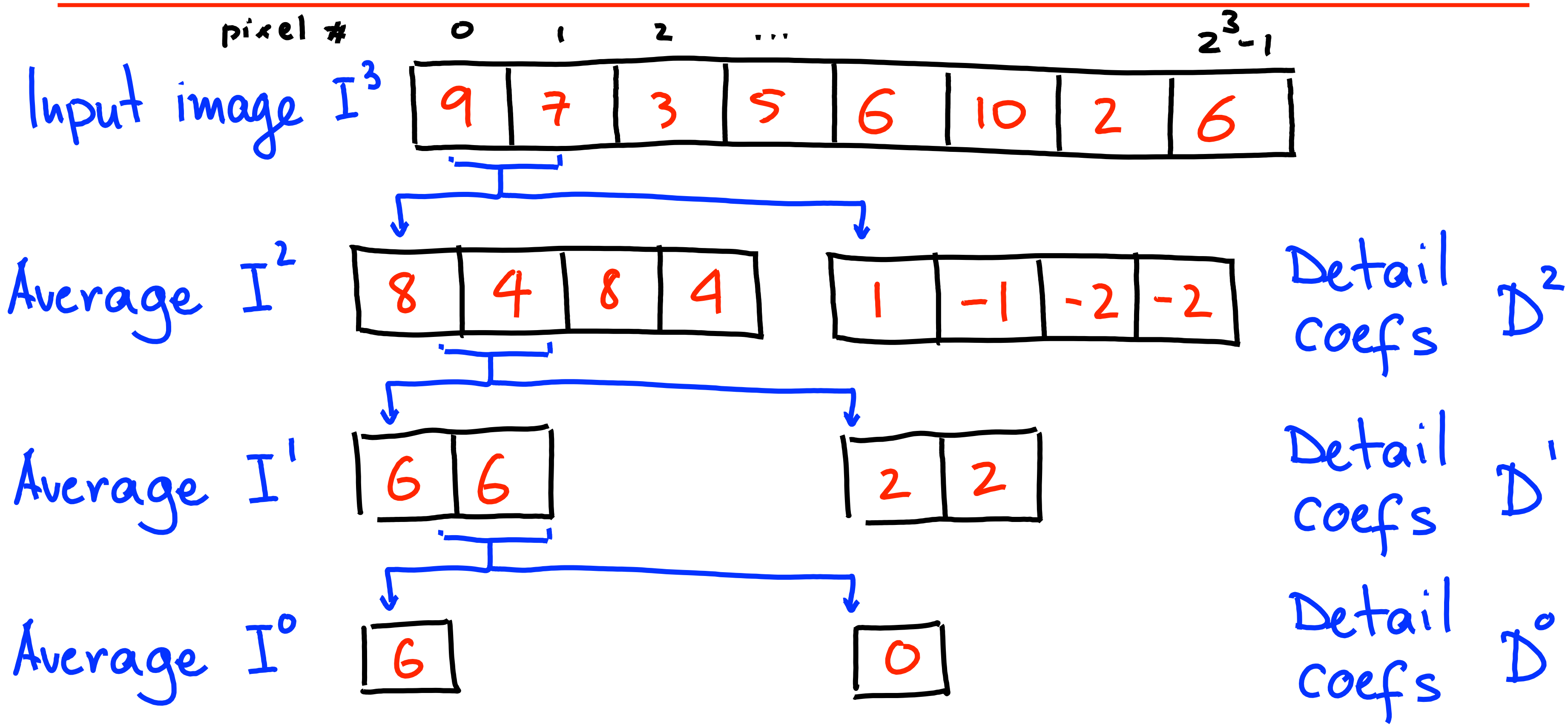
$$\boxed{I^j_i = \frac{1}{2}\left(I^{j+1}_{2i} + I^{j+1}_{2i+1}\right)}$$

equivalent convolution mask ψ: | -1/2 | 1/2 |

$$\boxed{D^j = \frac{1}{2}\left(I^{j+1}_{2i} - I^{j+1}_{2i+1}\right)}$$

j-th level of "pyramid" contains 2^j pixels

1D Haar Wavelet Transform: Recursive Definition

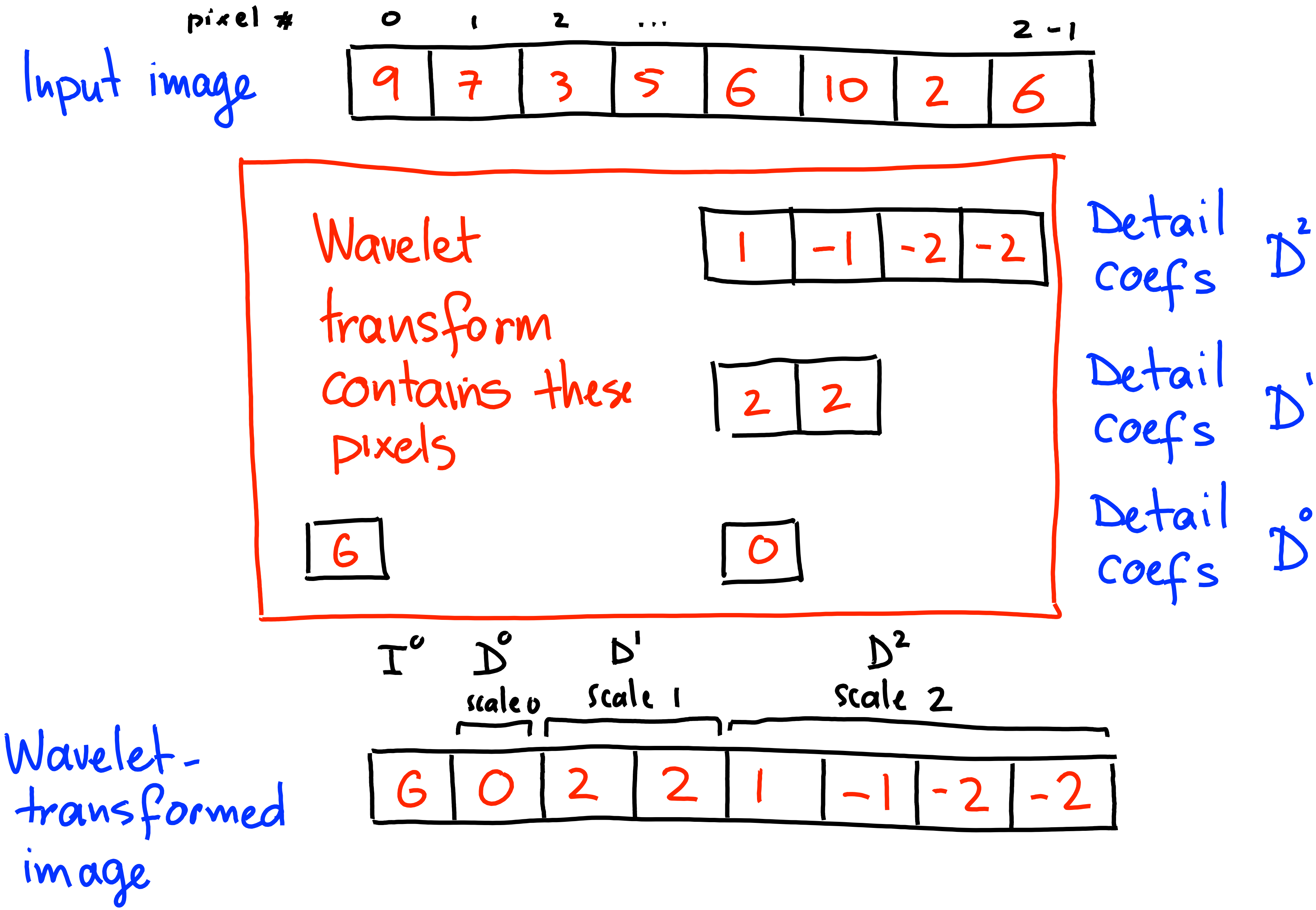


$$I_i^j = \frac{1}{2}\left(I_{2i}^{j+1} + I_{2i+1}^{j+1} \right)$$

$$D^j = \frac{1}{2}\left(I_{2i}^{j+1} - I_{2i+1}^{j+1} \right)$$

j-th level of "pyramid" contains 2^j pixels

1D Haar Wavelet Transform: Recursive Definition



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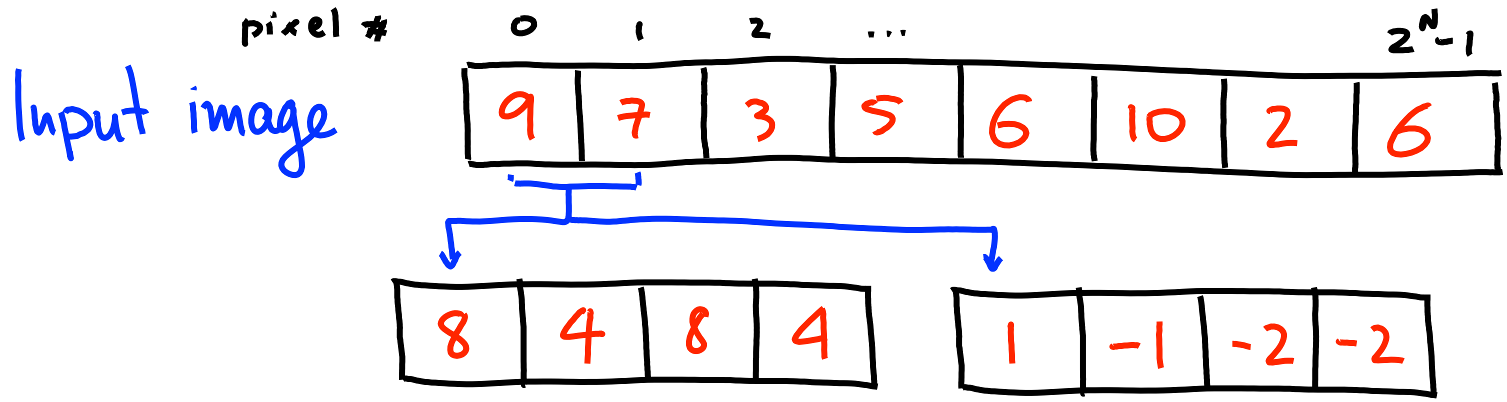
1D Haar Wavelet Transform as a Matrix Product

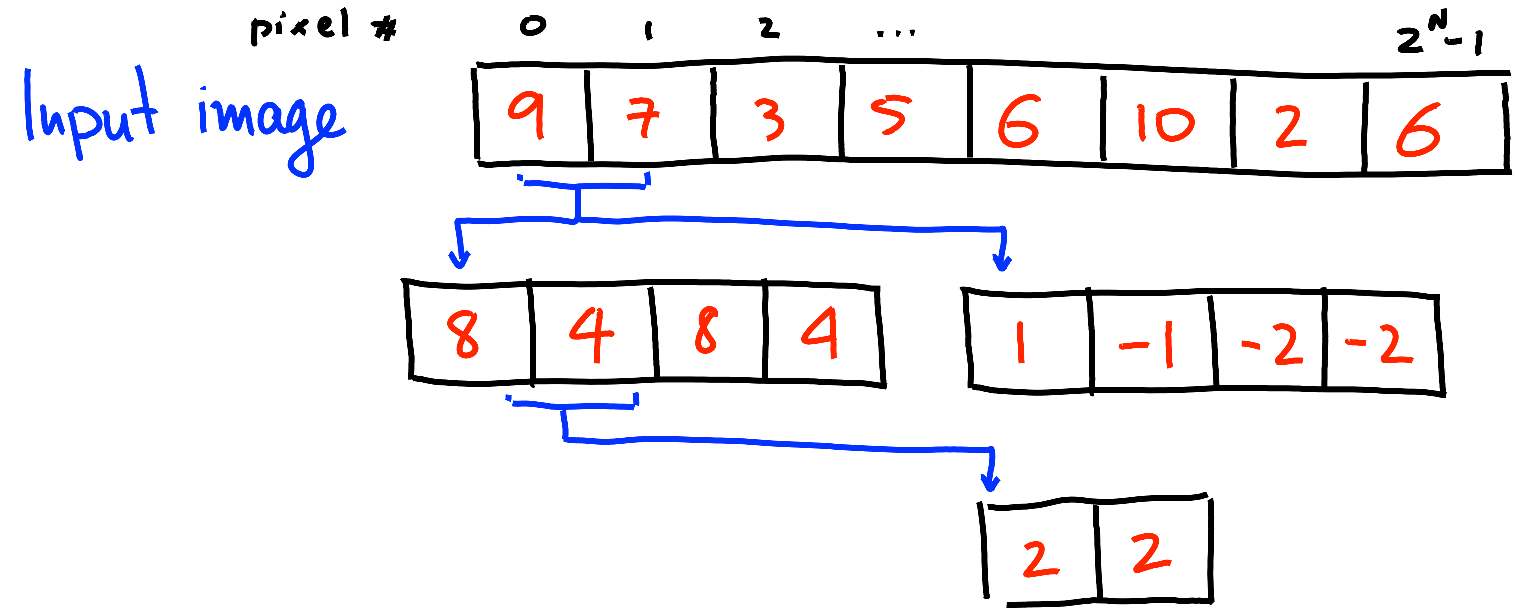


Wavelet transformed image

$$
\begin{array}{c}
I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\
\end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}
$$

Original image

1D Haar Wavelet Transform as a Matrix Product

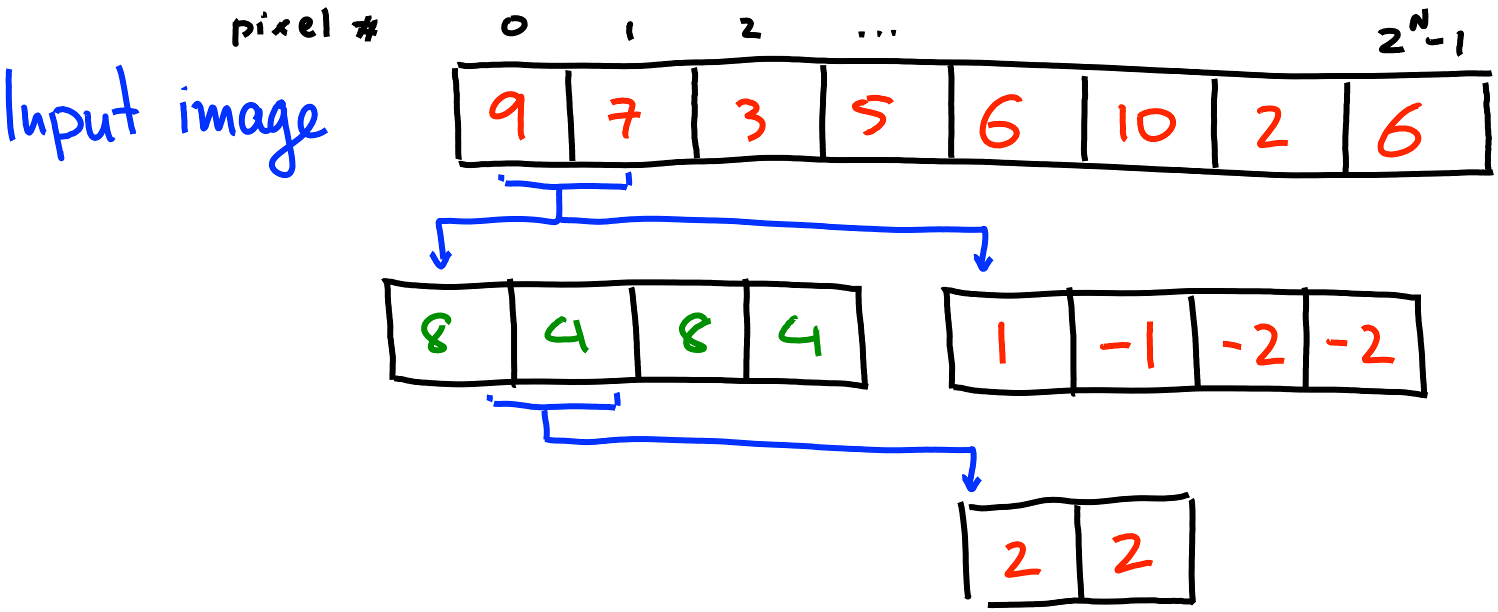


Wavelet transformed image

$$\begin{array}{cc} \begin{array}{c} I^0\\ D^0\\ D^1\\ \\ \\ D^2\\ \\ \\ \end{array} & \begin{bmatrix}6\\0\\2\\2\\1\\-1\\-2\\-2\end{bmatrix}\end{array} = \frac{1}{2}\begin{bmatrix} \multicolumn{8}{c}{?} \\ \\ \\ \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix}9\\7\\3\\5\\6\\10\\2\\6\end{bmatrix}$$

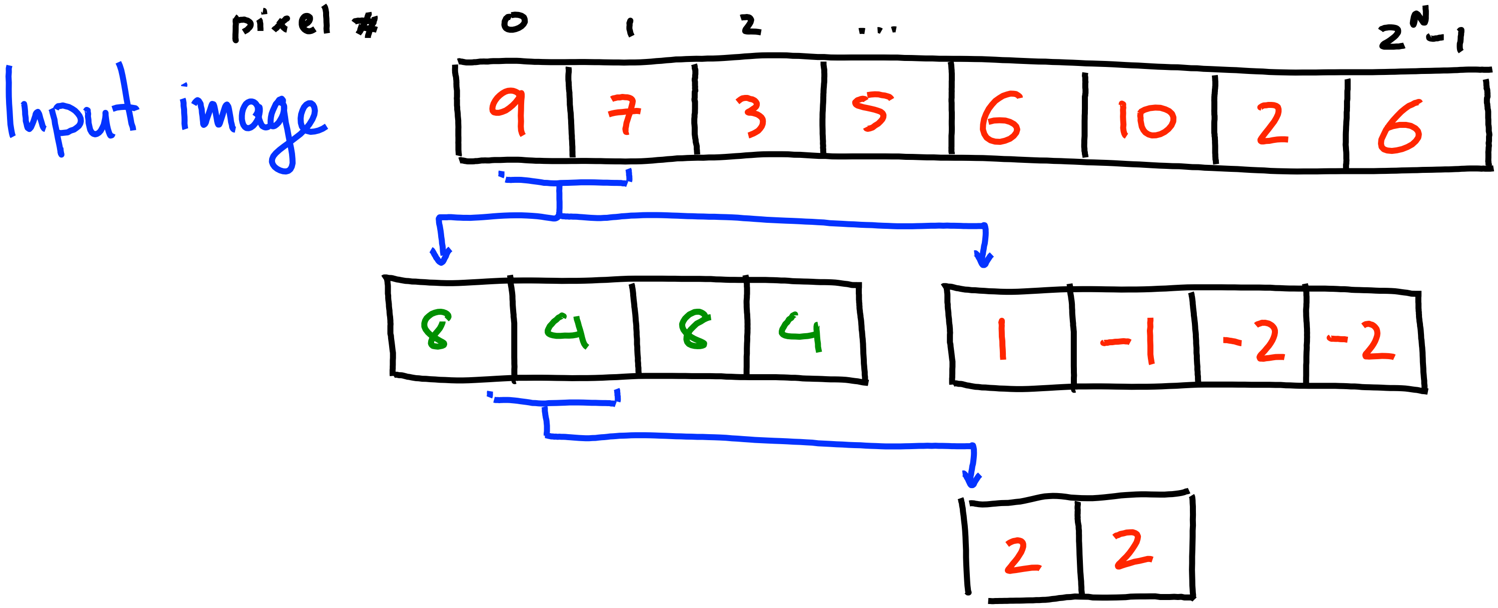
Original image

1D Haar Wavelet Transform as a Matrix Product



$$
\begin{array}{c}
I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\
\end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
$$

1D Haar Wavelet Transform as a Matrix Product



$$
\begin{array}{c}
I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\
\end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\cdot
\frac{1}{2} \cdot
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}
$$

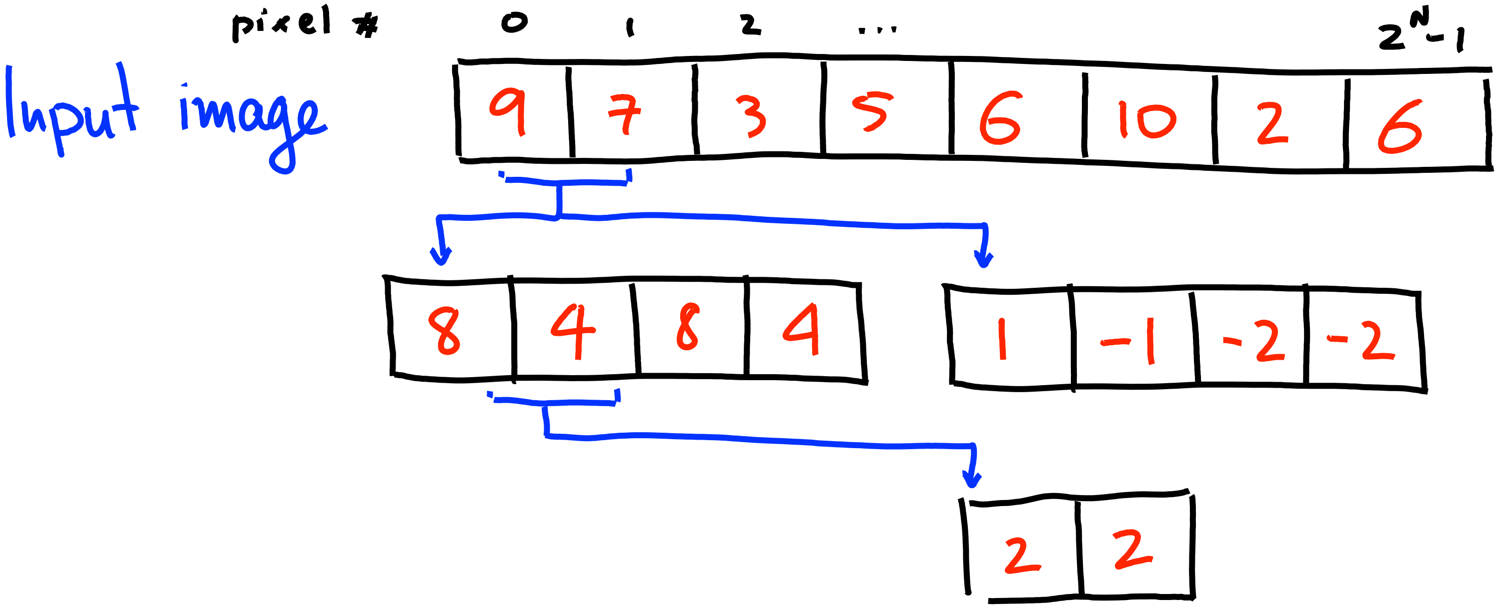
1D Haar Wavelet Transform as a Matrix Product

3rd & 4th rows of product

$$
\left[\begin{array}{cccccccc}
 & & & & & & & \\
\tfrac{1}{4} & \tfrac{1}{4} & -\tfrac{1}{4} & -\tfrac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tfrac{1}{4} & \tfrac{1}{4} & -\tfrac{1}{4} & -\tfrac{1}{4} \\
 & & & & & & & \\
\end{array}\right]
$$

$$
\begin{array}{c}
I^0 \\ D^0 \\ \\ D^1 \\ \\ \\ D^2 \\ \\
\end{array}
\left[\begin{array}{c}
6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2
\end{array}\right]
=
\left[\begin{array}{cccccccc}
 & & & & & & & \\
 & & & & & & & \\
\tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\cdot
\tfrac{1}{2} \cdot
\left[\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}\right]
\left[\begin{array}{c}
9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6
\end{array}\right]
$$

1D Haar Wavelet Transform as a Matrix Product

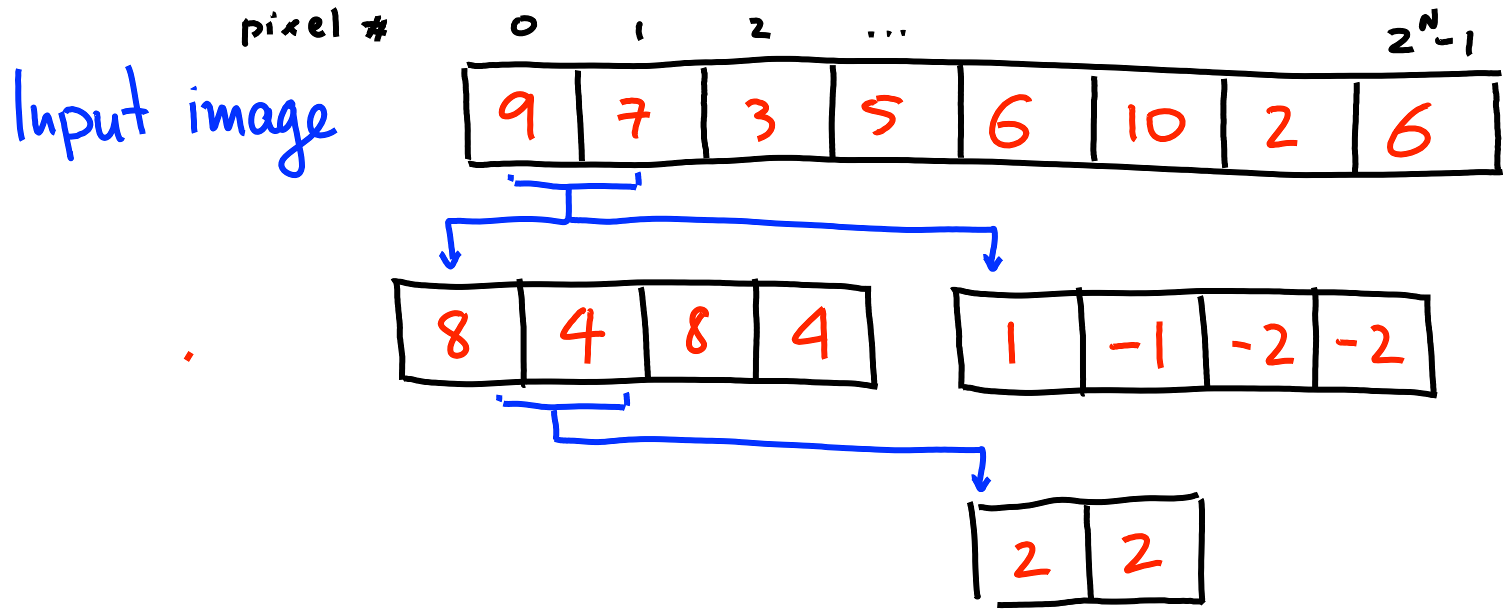


Wavelet transformed image

$$\begin{matrix} I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\ \end{matrix}\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{matrix} \\ \\ \frac{1}{4} \\ \\ \\ \frac{1}{2} \\ \\ \\ \end{matrix}\begin{bmatrix} & & & & & & & \\ & & & & & & & \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

1D Haar Wavelet Transform as a Matrix Product



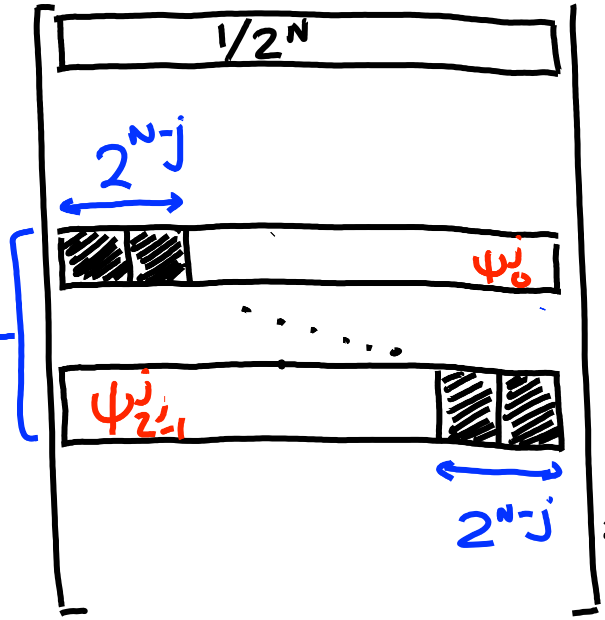
Wavelet transformed image

$$
\begin{array}{c}
I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\
\end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{array}{c}
1/8 \\ 1/8 \\ 1/4 \\ \\ 1/2 \\ \\ \\ \\
\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}
$$

Original image

The 1D Haar Wavelet Transform Matrix W

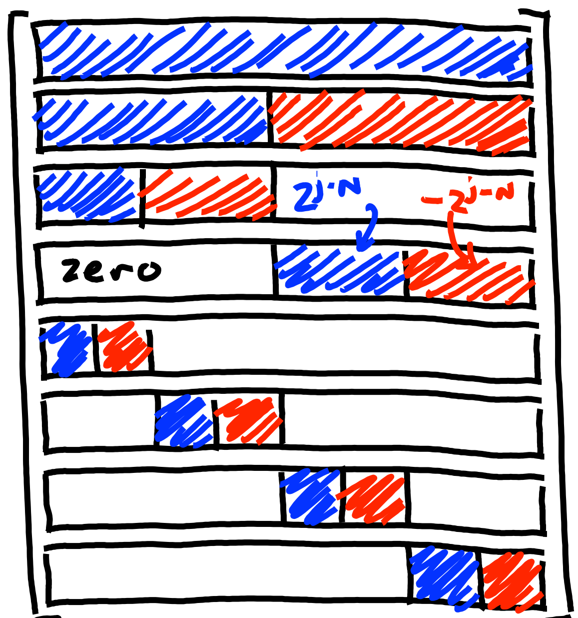
- Matrix contains N-1 scales
- Scale j represented by 2^j rows
  $\psi_0^j, \ldots, \psi_{2^j-1}^j$



- Row ψ_j^i has $\frac{2^N}{2^j} = 2^{N-j}$ non-zero pixels
- They are pixels $x = i\,2^{N-j}, \ldots, (i+1)\,2^{N-j}-1$ with $|\psi_j^i(x)| = \frac{1}{2^{N-j}}$

Wavelet transformed image

$$\begin{array}{c} I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\ \end{array}\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} =$$



$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

Topic 7:

Discrete Wavelet Transform

- Wavelet transform vs. Laplacian pyramid
- Basic intuition: a simple wavelet-like 2D transform
- The 1D Haar wavelet transform
- 1D Haar wavelet transform as a matrix product
- Reconstructing a 1D image from its wavelet coefs
- Wavelet-based image compression
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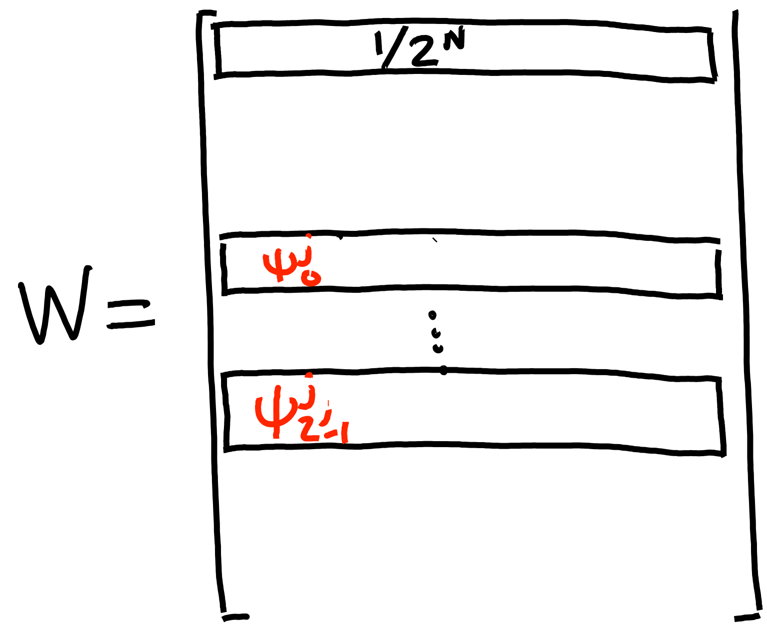
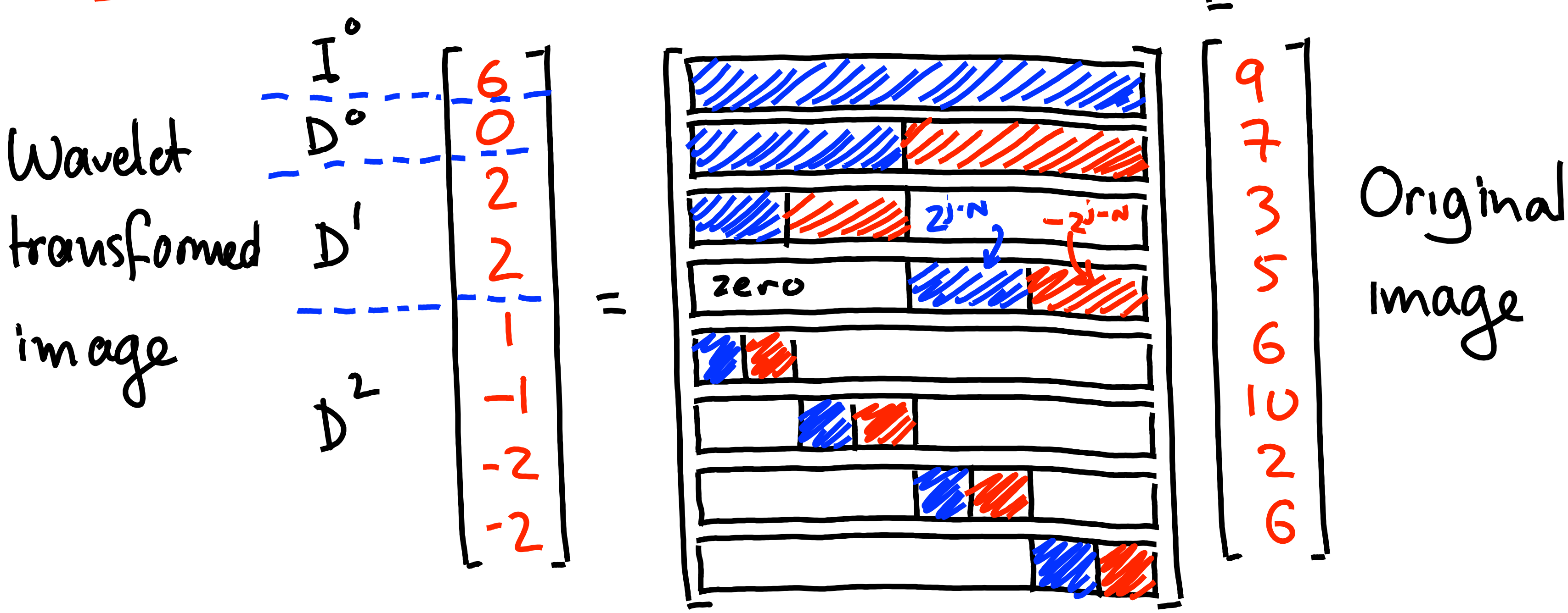
Reconstructing an Image from its Wavelet Coefs

Question: What is the dot product

$$\psi_i^j \cdot \psi_{i'}^{j'}$$

of two distinct rows of W?





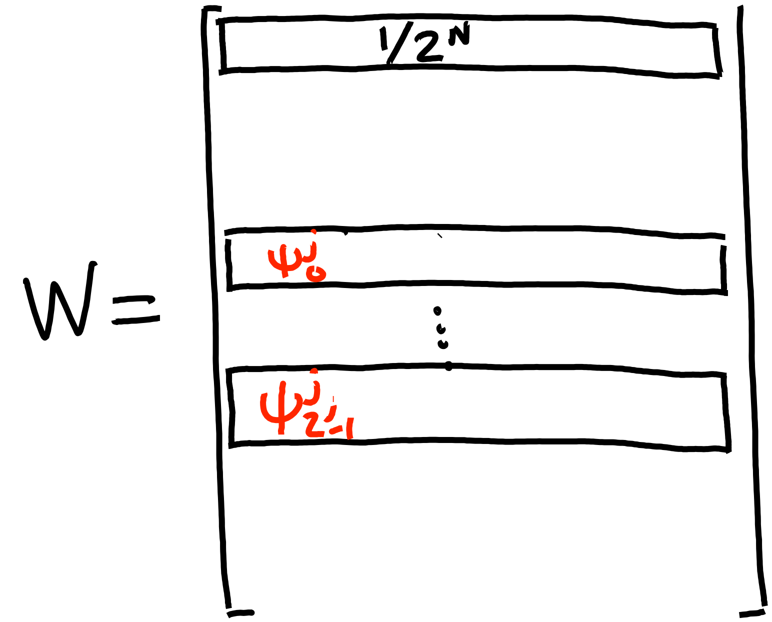
Reconstructing an Image from its Wavelet Coefs

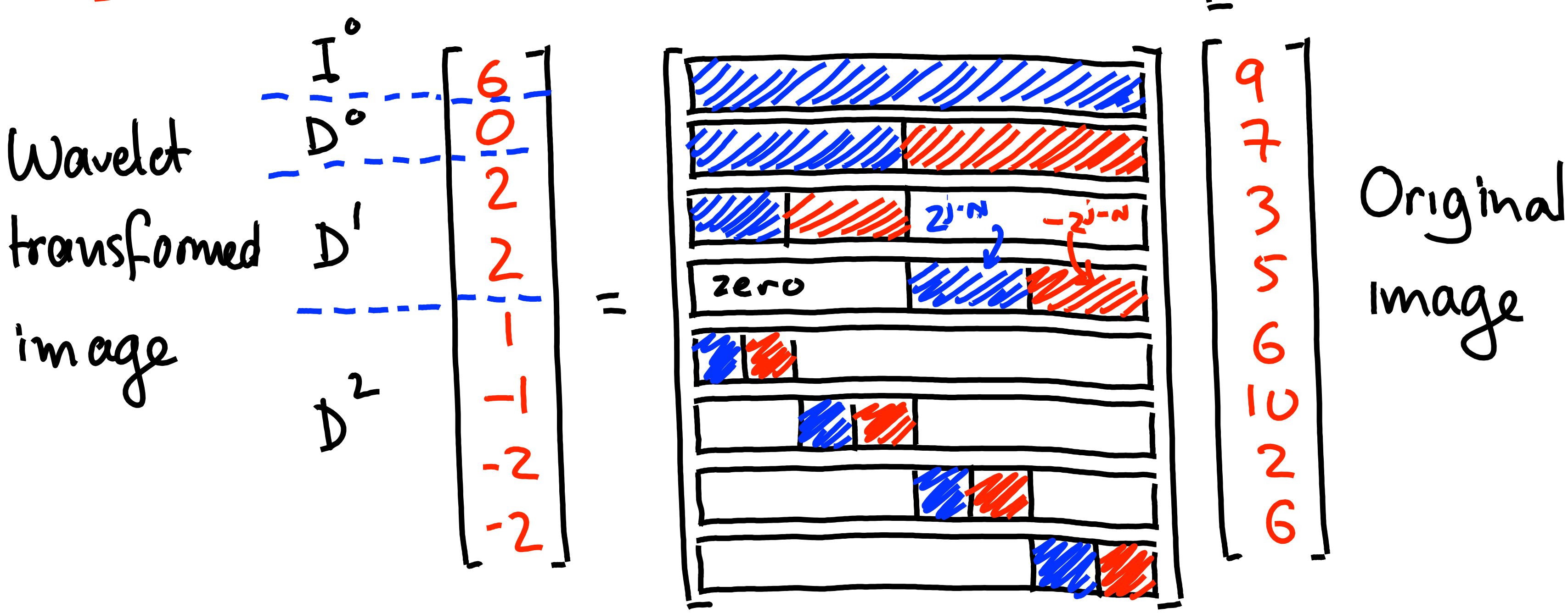
Answer:

$$\psi_i^j \cdot \psi_{i'}^{j'} = 0$$

for two distinct rows of W $\Longleftrightarrow$

$$WW^T = \text{diagonal}$$





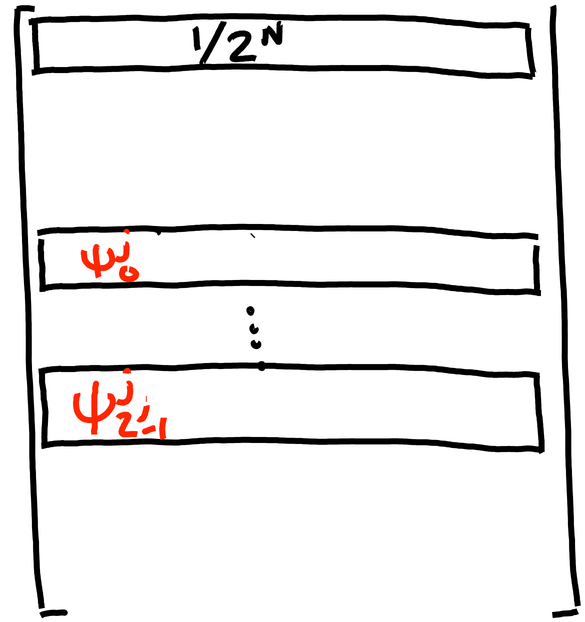
Reconstructing an Image from its Wavelet Coefs

$$WW^T = \text{diagonal}$$

$$= \begin{bmatrix} \ddots & & \\ & \frac{1}{2^{N-j}} & \\ & & \ddots \end{bmatrix}$$

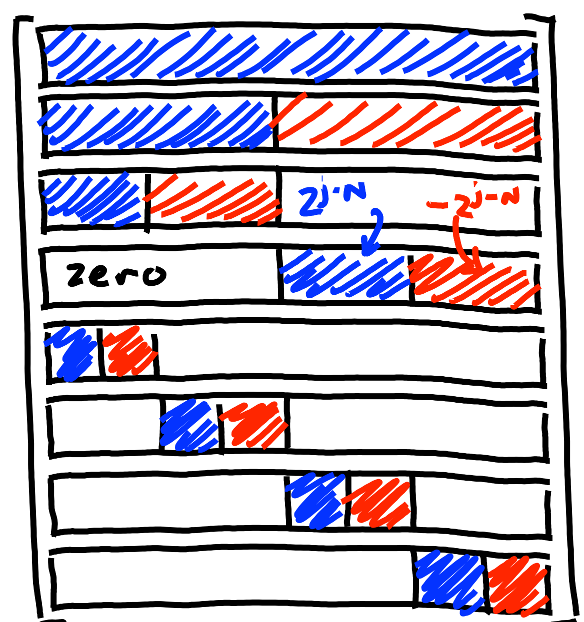
$$(\psi_i^j)\cdot(\psi_i^j)^T \longrightarrow \frac{1}{2^{N-j}}$$

W =



Wavelet transformed image

$$\begin{array}{c} I^0 \\ D^0 \\ D^1 \\ \\ \\ D^2 \\ \\ \\ \end{array}\begin{bmatrix}6\\0\\2\\2\\1\\-1\\-2\\-2\end{bmatrix} =$$



$$\begin{bmatrix}9\\7\\3\\5\\6\\10\\2\\6\end{bmatrix}$$

Original image

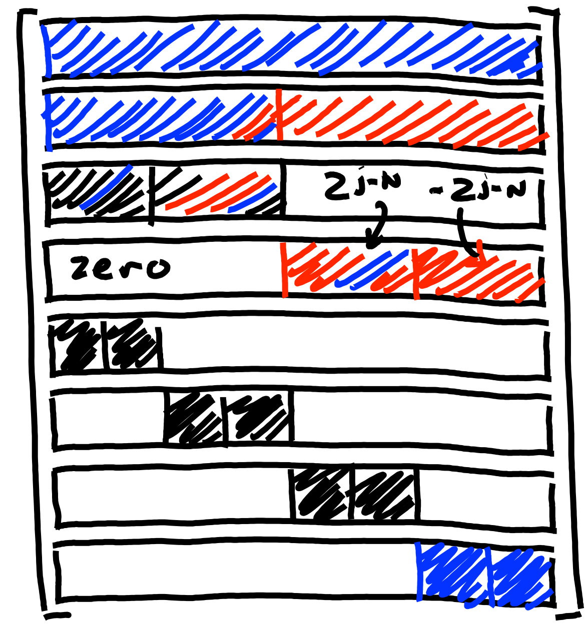
Reconstructing an Image from its Wavelet Coefs

Define $\Lambda = WW^T$

with

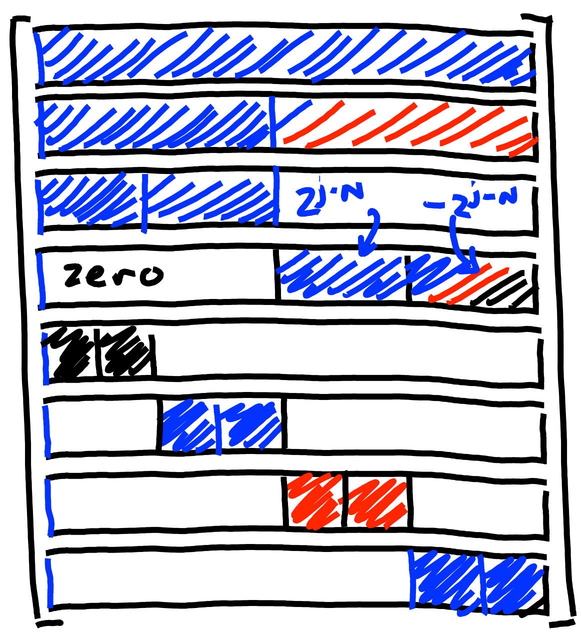
$$\Lambda = \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2^N-1} \end{bmatrix}$$

of the form $\frac{1}{2^{N-j}}$

$W =$ 

Multiply W^T on both sides:

$$W^T \cdot \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = W^T \cdot$$



$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

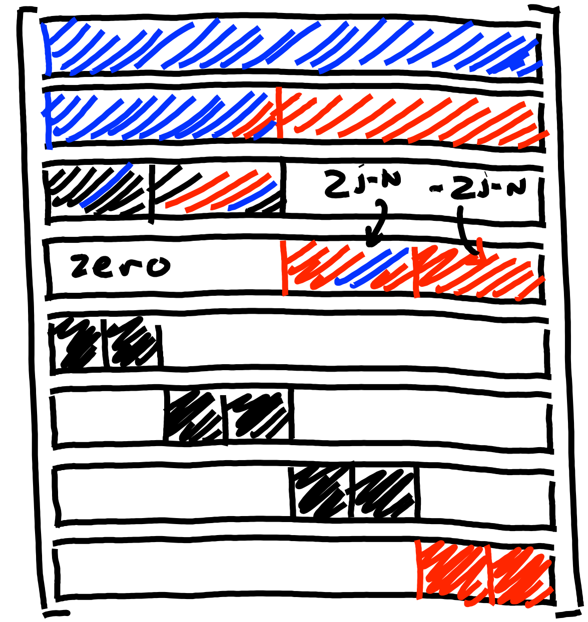
Reconstructing an Image from its Wavelet Coefs

Define $\Lambda = WW^T$

with

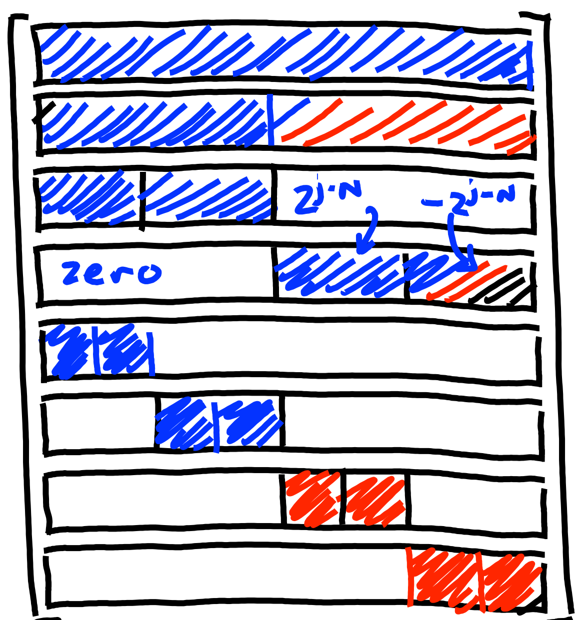
$$\Lambda = \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2^N-1} \end{bmatrix}$$

of the form $\frac{1}{2^{N-j}}$

$W =$ 

Multiply $\Lambda^{-1} W^T$ on both sides:

$$\Lambda^{-1} W^T \cdot \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \Lambda^{-1} W^T \cdot$$



$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

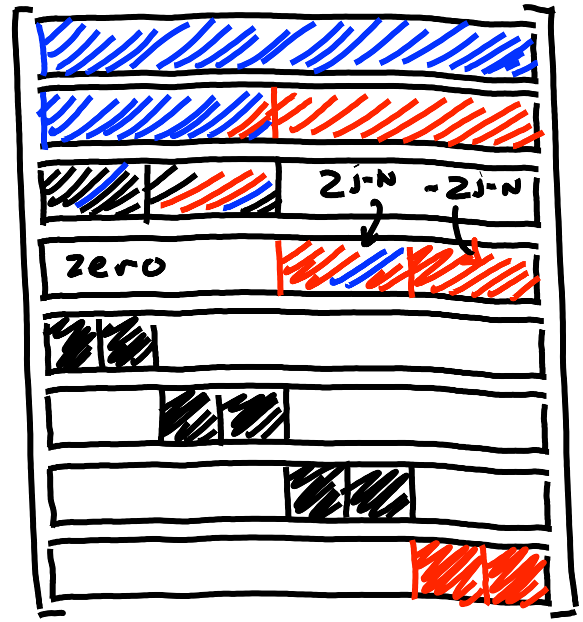
Reconstructing an Image from its Wavelet Coefs

Observation:

$\Lambda^{-1/2} \cdot W$ is orthogonal because

$$(\Lambda^{-1/2} \cdot W)(\Lambda^{-1/2} \cdot W)^T =$$
$$\Lambda^{-1/2} W \cdot W^T \cdot \Lambda^{-1/2} =$$
$$\Lambda^{-1/2} \cdot \Lambda \cdot \Lambda^{-1/2} = I$$

$W =$



Therefore

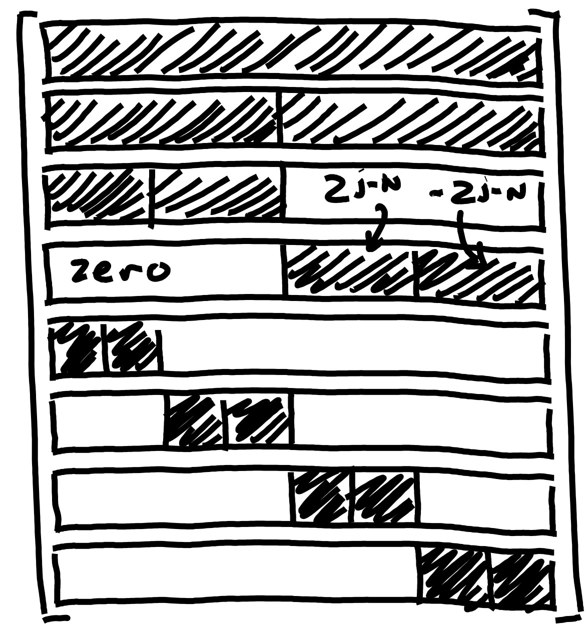
$$\Lambda^{-1} W^T \cdot \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

Reconstructing an Image from its Wavelet Coefs

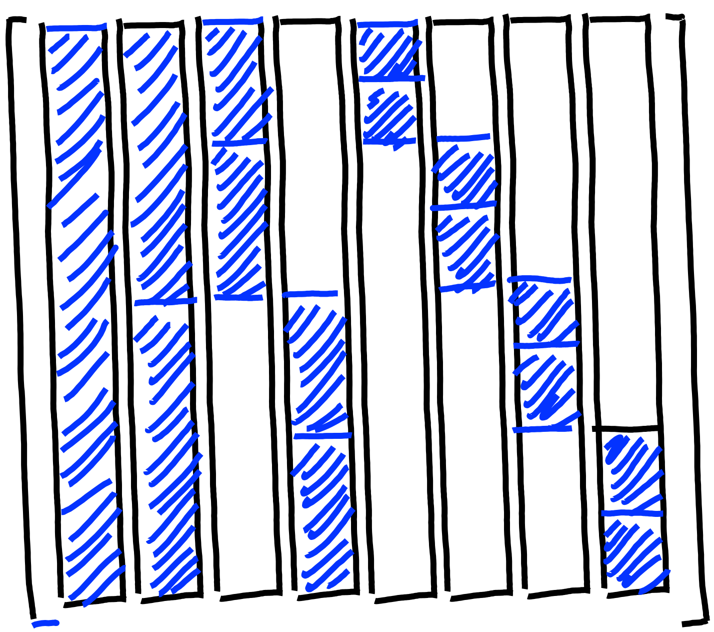
$$\Lambda^{-1} = \begin{bmatrix} \lambda_0^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2^N-1}^{-1} \end{bmatrix}$$

W =



So we have

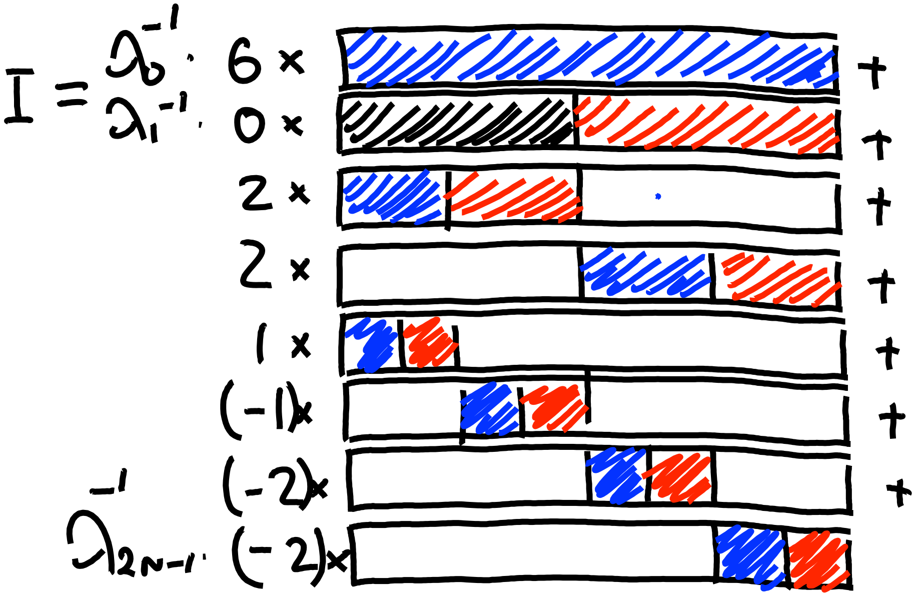
Λ^{-1}

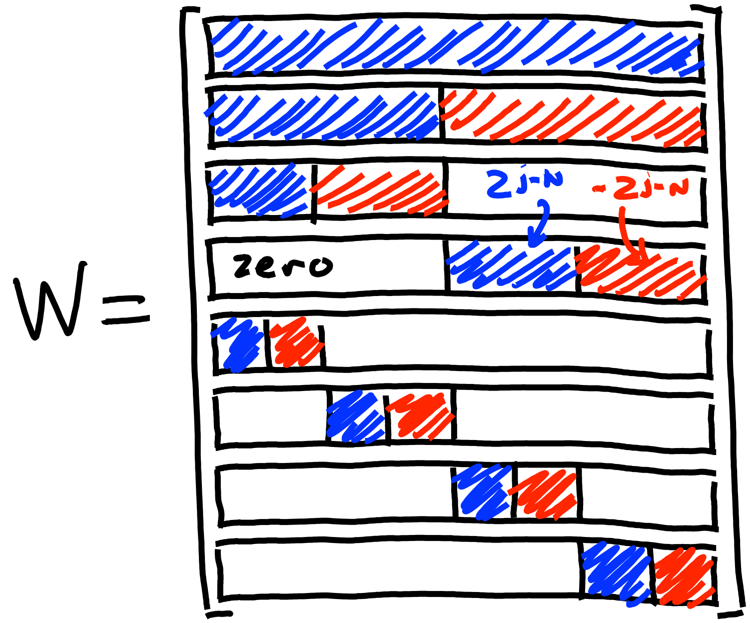


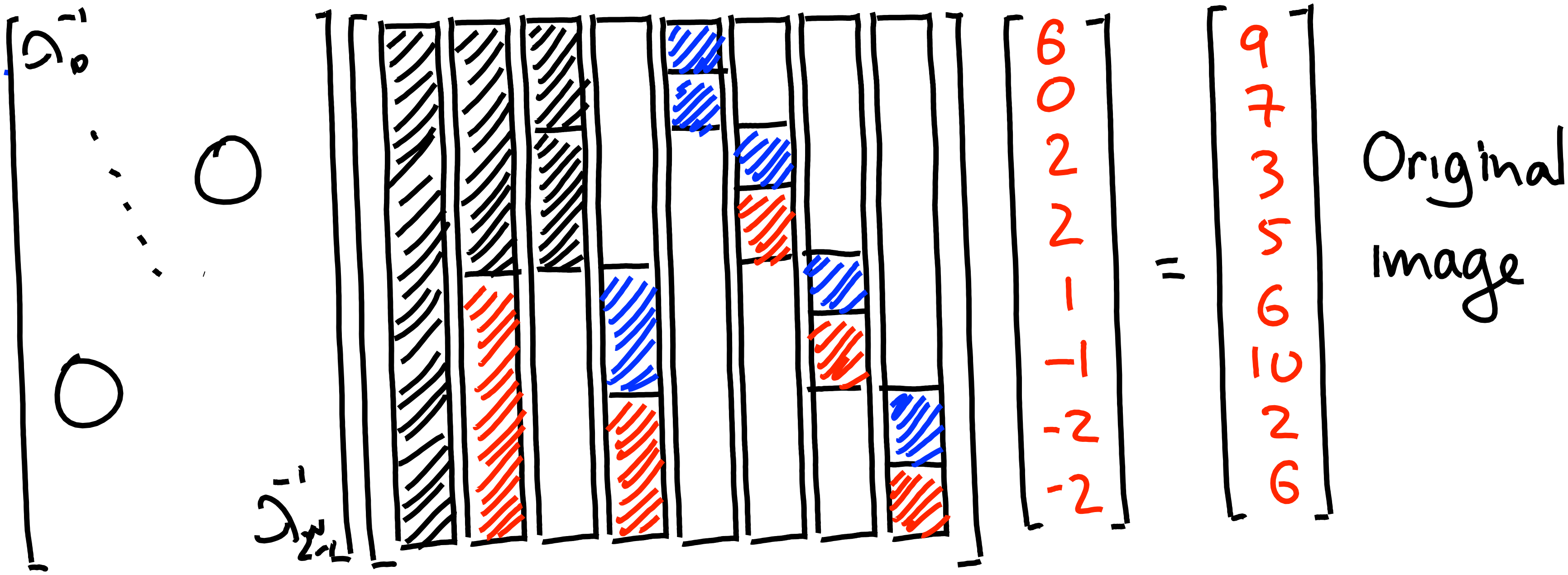
$$\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

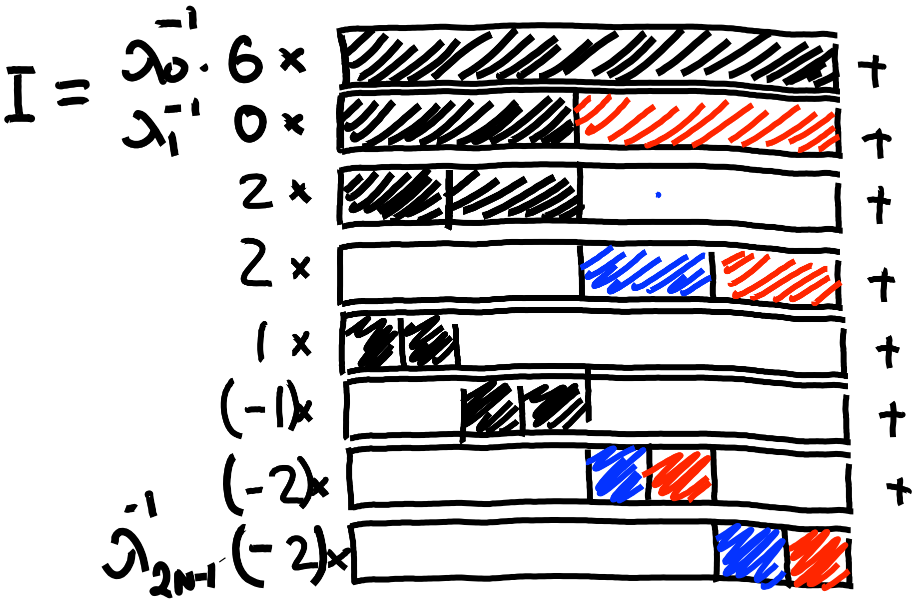
Interpreting the Wavelet Coefficients

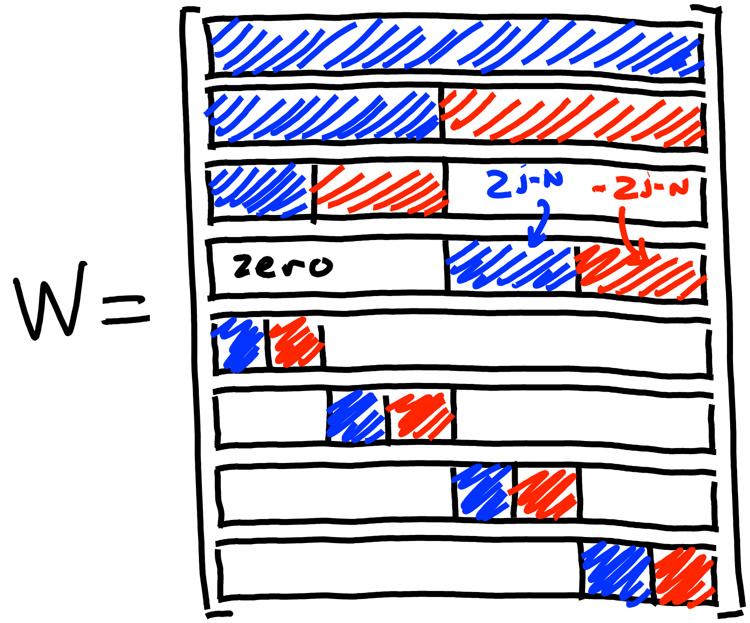






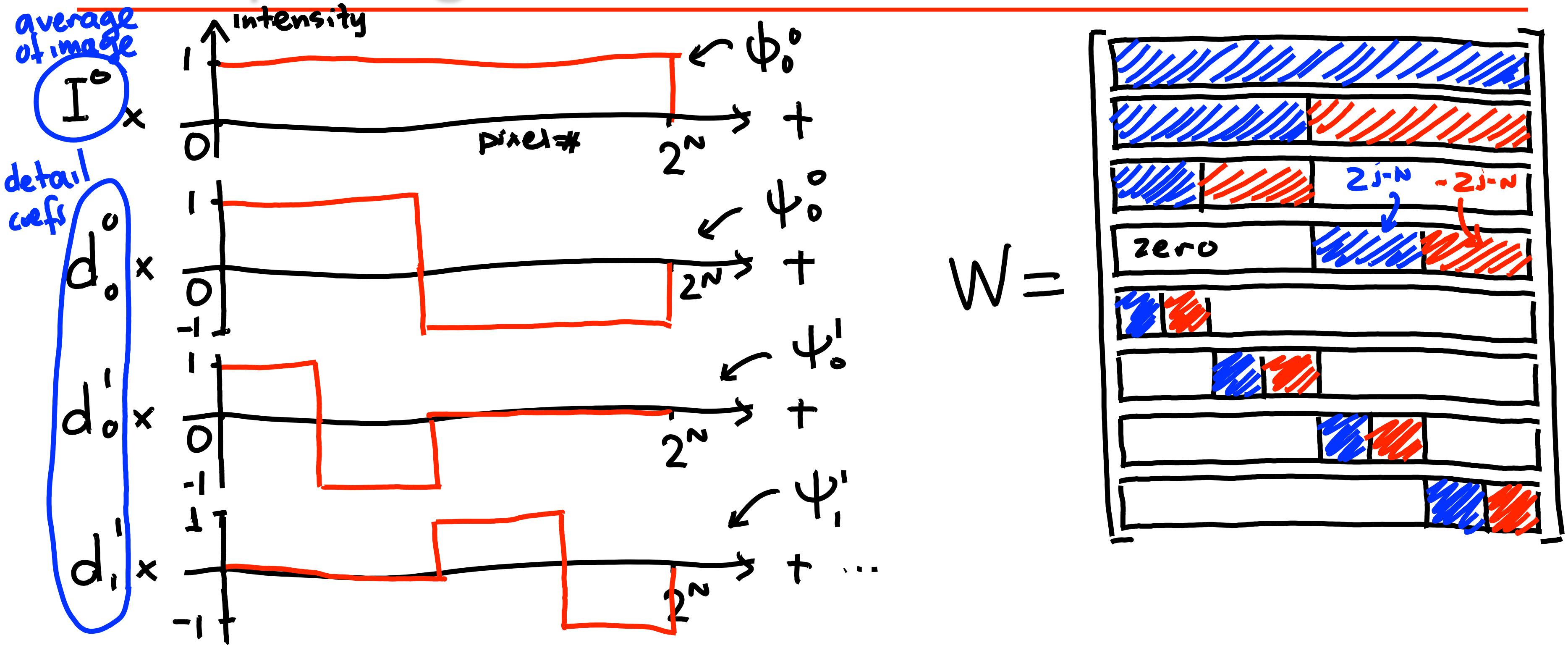
Interpreting the Wavelet Coefficients





⇒ By multiplying I with W we obtain a decomposition of the image into a sequence of basis images $\psi_0^0, \psi_0^1, \dots \psi_j^i, \dots$ that form an orthogonal basis of $\mathbb{R}^{2^N}$

Interpreting the Wavelet Coefficients



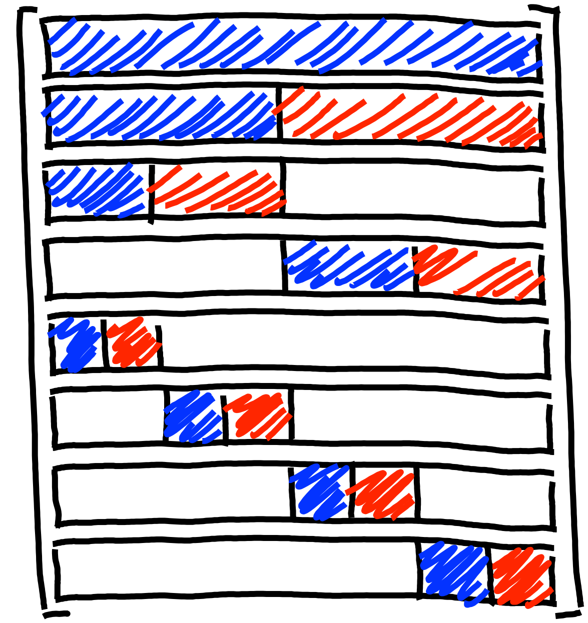
$\Rightarrow$ The wavelet coefficients are the coordinates of the image, considered as a vector in $\mathbb{R}^{2^N}$, in the basis defined by images $\phi_0^0, \psi_0^0, \psi_0^1, \ldots$

The Normalized Haar Wavelet Matrix

We can normalize the wavelet transform matrix by multiplying

$$\tilde{W} = \begin{bmatrix} \sqrt{\lambda_1^{-1}} & & \\ & \ddots & \\ & & \sqrt{\lambda_{2^N-1}^{-1}} \end{bmatrix} \cdot W$$

$\tilde{W} =$

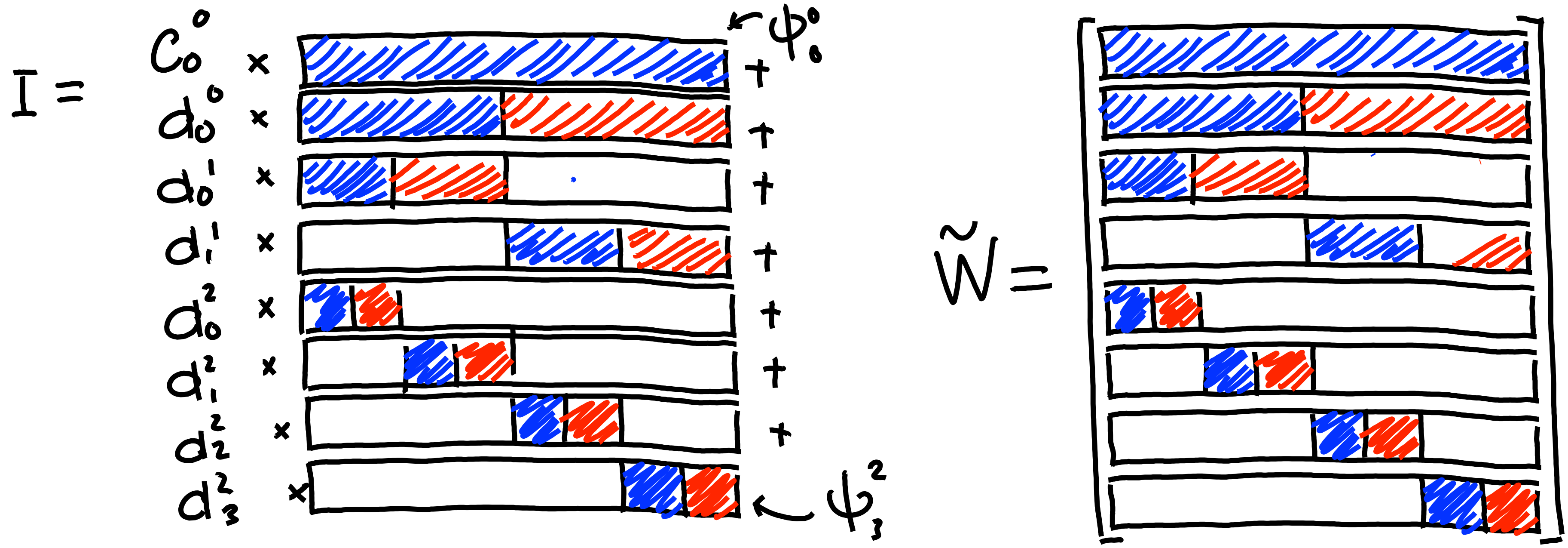


normalized wavelet coefficients

$$\begin{bmatrix} c_0^0 \\ d_0^1 \\ d_1^1 \\ \vdots \\ \\ \\ \\ \end{bmatrix} = \tilde{W} \cdot \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

The Normalized Haar Wavelet Coefficients



$\Rightarrow$ By multiplying I with $\tilde{W}$ we obtain a set of wavelet coefficients $c_0^0, d_0^0, \dots$ that express I as a linear combination of the basis images $\phi_0^0, \psi_0^0, \psi_0^1, \psi_1^1, \dots$

Topic 7:

Discrete Wavelet Transform

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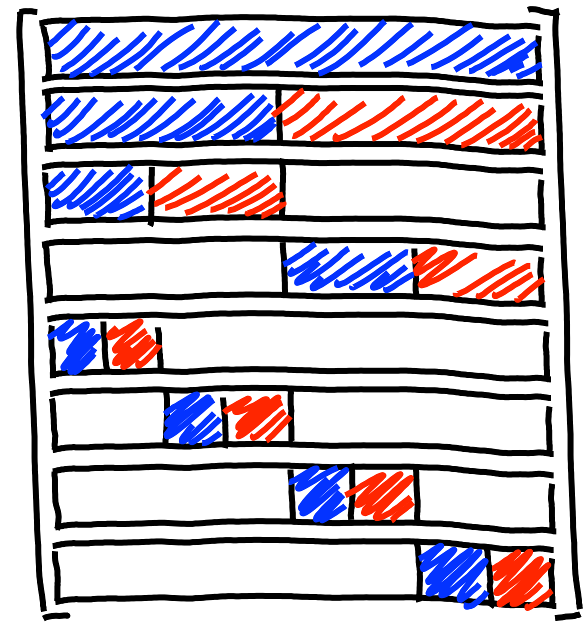
Wavelet Compression Algorithm #1

Input: 1D image I, desired compression k
Output: $k \cdot 2^N$ coefficients

1. Compute $\tilde{W} I$
2. Sort the coefficients $c_0^0, d_0^1, d_1^1, \ldots$ In order of decreasing absolute value
3. keep the top $k \cdot 2^N$ coefs

$\tilde{W} =$



normalized wavelet coefficients

$$\begin{bmatrix} c_0^0 \\ d_0^1 \\ d_1^1 \\ \vdots \end{bmatrix} = \tilde{W} \cdot \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

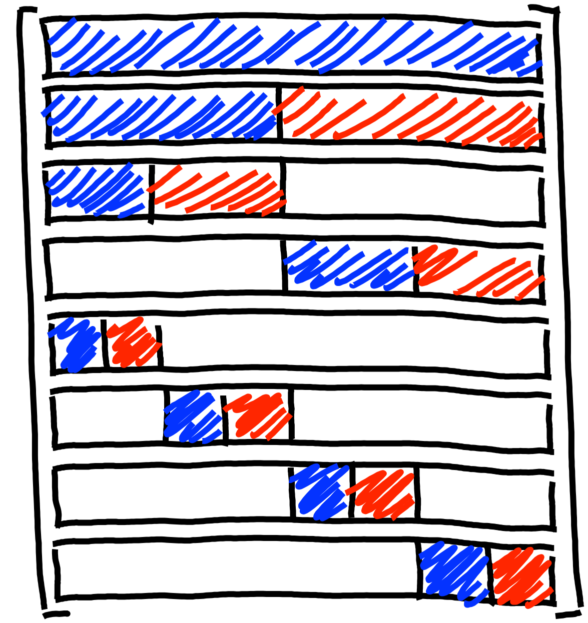
Original image I

* Readings show that the algorithm gives the best least-squares approx of the image for the given compression level

Wavelet Compression Algorithm #2

Input: 1D image I, max error ϵ
Output: $K \cdot 2^N$ coefficients

1. Compute $\tilde{W} I$
2. Sort the coefficients $c_0^0, d_0^1, d_1^1, \ldots$ In order of decreasing absolute value
3. Keep the top $K \cdot 2^N$ coefs with K such that $|\tilde{I} - I| < \epsilon$, where $\tilde{I}$ is the image reconstructed from the top $K 2^N$ coefs

$\tilde{W} =$ 

normalized wavelet coefficients

$$\begin{bmatrix} c_0^0 \\ d_0^1 \\ d_1^1 \\ \vdots \end{bmatrix} = \tilde{W} \cdot \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image I

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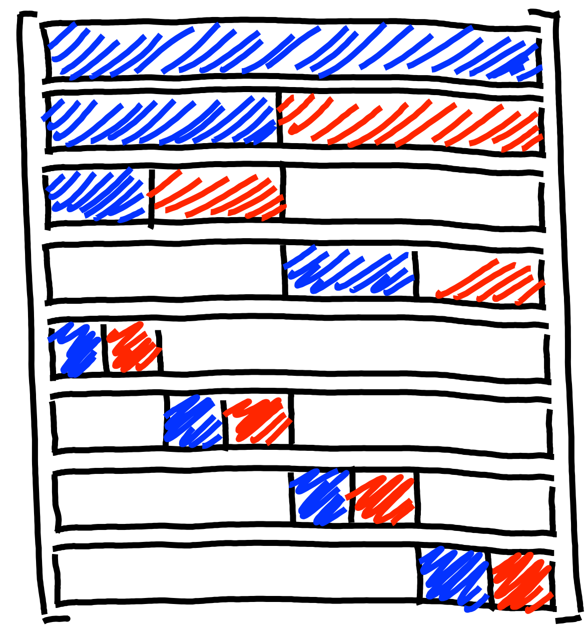
The 2D Haar Wavelet Transform

$$I = \begin{bmatrix} I_0 & I_1 & I_2 & \cdots & \end{bmatrix}$$

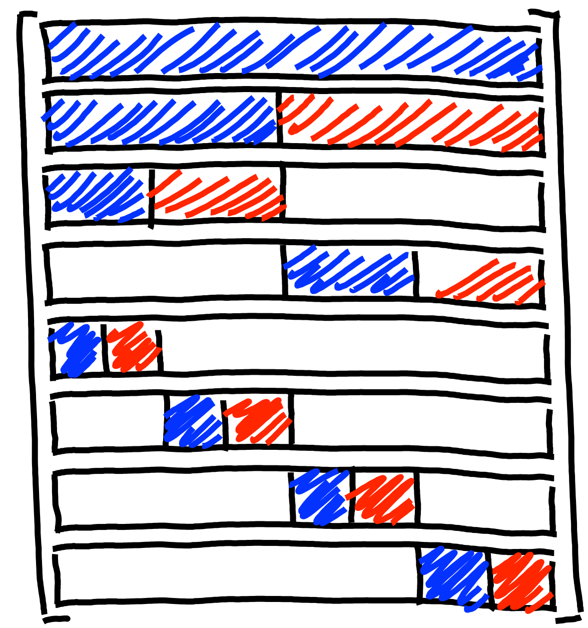
$\tilde{W} =$



- To compute the wavelet transform of a 2D image.
  1. Compute the 1D transform for each column and place the vectors $\tilde{W} I_i$ in a new image I'
  2. Compute the 1D transform of each row of I'

The 2D Haar Wavelet Transform

Exercise: Show that every 2D wavelet coefficient can be expressed as the result of a dot product of the image I and an image defined by $(\varphi_i^j)^T \cdot (\varphi_{i'}^{j'})$ where φ_i^j are 1D Haar basis images

$\tilde{W} =$ 

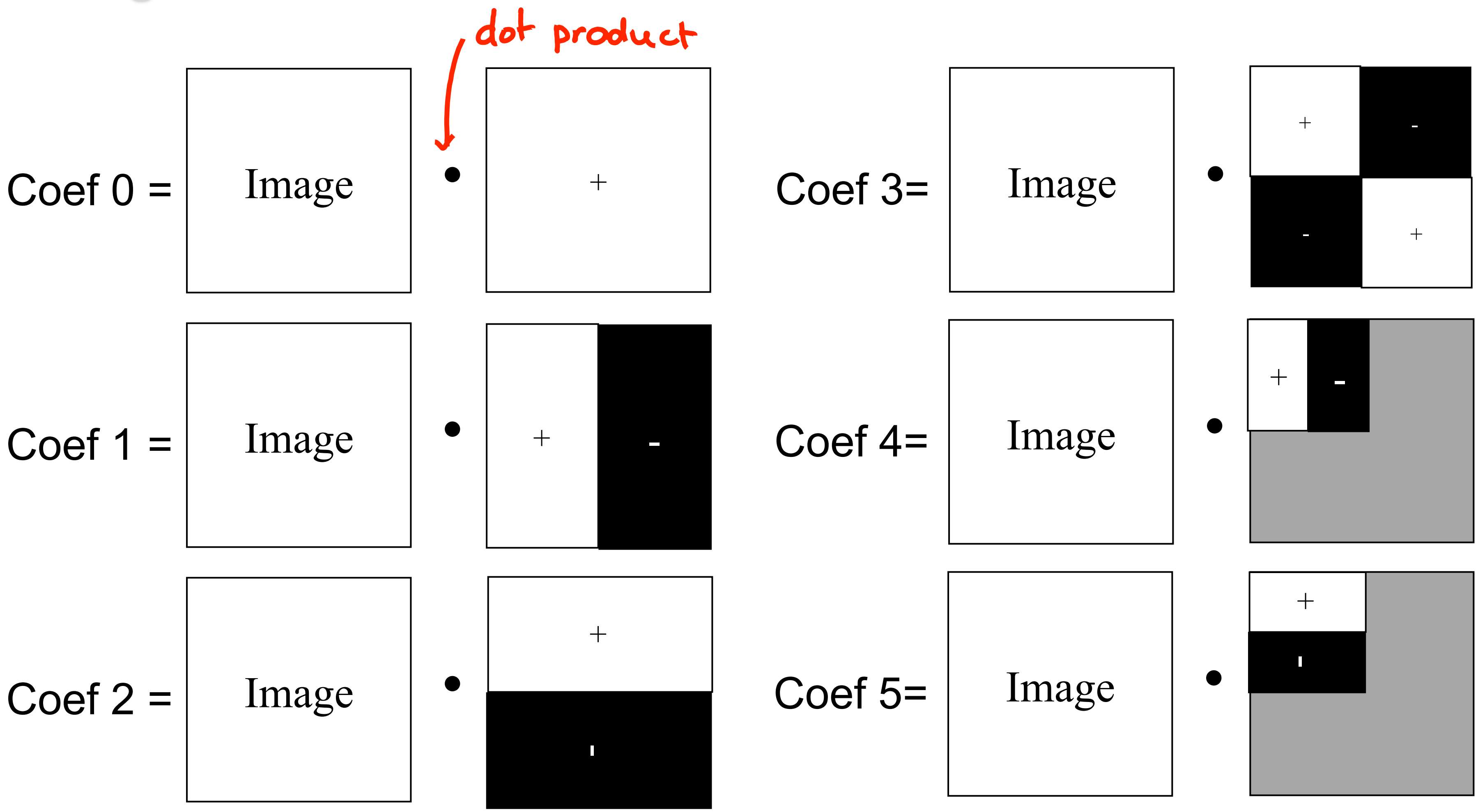
- To compute the wavelet transform of a 2D image:

① Compute the 1D transform for each column and place the vectors $\tilde{W} I_i$ in a new image I'

② Compute the 1D transform of each row of I'

The 2-D Haar Wavelet Basis

Definition of the first few (coarsest scale) wavelet coefficients of an image of dimensions of $2^N \times 2^N$

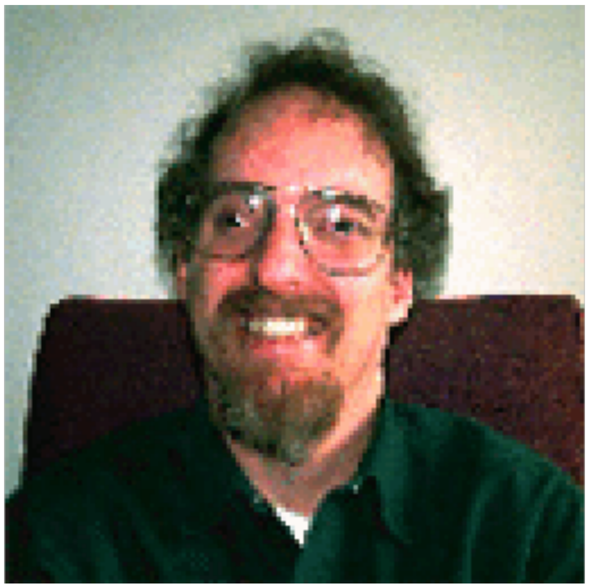


A Simple, Minimal 2-D Image Transform

The Haar 2-D Wavelet Transform

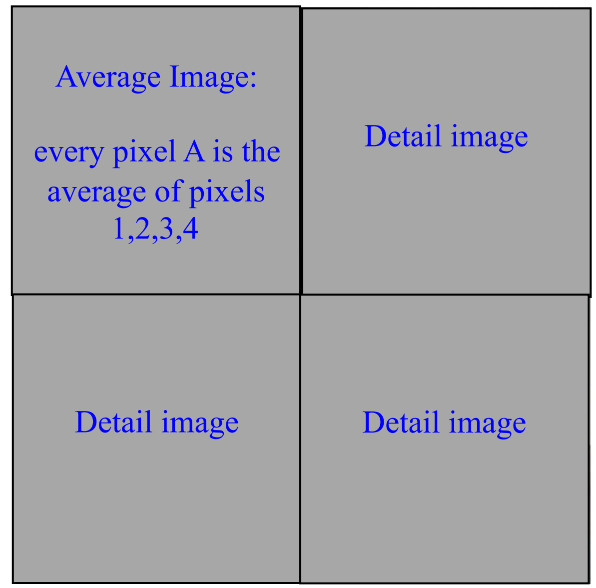
The 2-D Haar Wavelet Transform corresponds to a modification of this minimal recursive transform

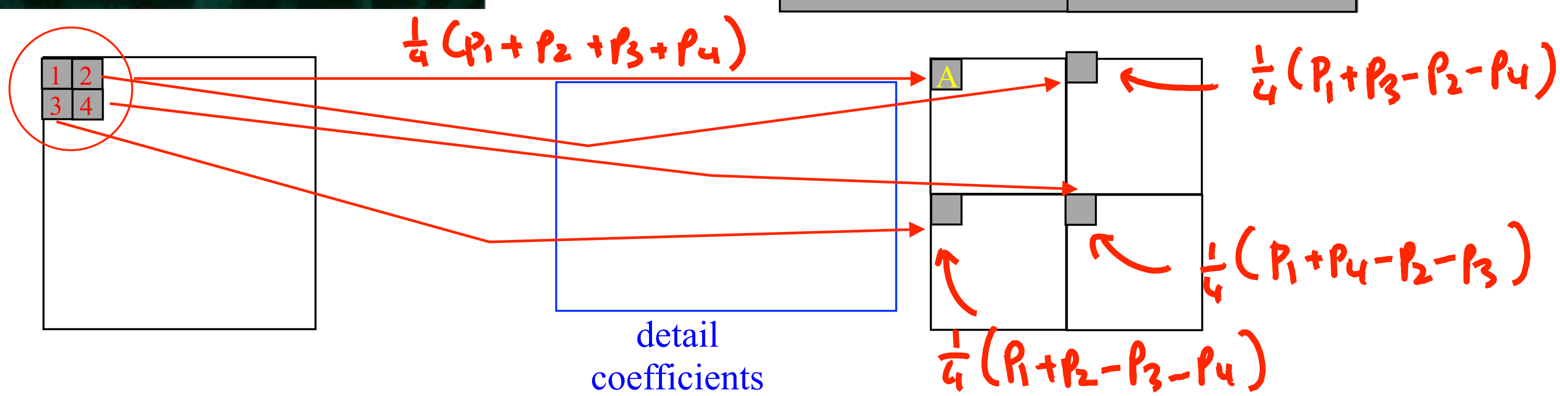
Input image (2^Nx2^N)



Step 1

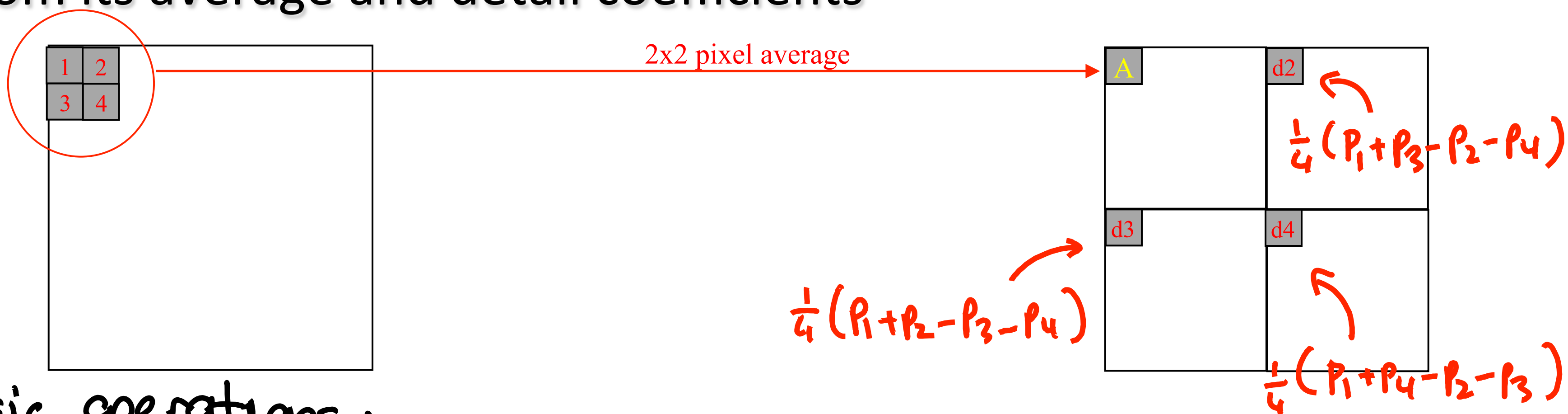
Wavelet image representation (2^Nx2^N)





Invertibility of the 2D Haar Transform

We can recursively reconstruct the intensities of every 2x2 window from its average and detail coefficients



2 basic operations:

- sum of 4 pixels
- difference of pairwise sums of pixels