Theorem

Adjacent transposition graph of a poset is connected.

Proof

Let $L \neq L'$ extend P. The number of inversions of elements between L and L', $Inv(L, L')$, is the minimum number of incomparable pairs in P.

Since $L \neq L'$, there exists at least a pair x, y such that $x||y$ for P. Among all such pairs, there exists at least one pair a, b such that $a \prec b \in L$, because, however you compose the elements, we can choose a, b to be the pair of the first neighboring pair with each element from the two upward chain.

Because $a||b$ and are adjacent, we can safely swap a, b in L without violating P. Starting from L, we can swap such a, b to obtain a new L while reducing $Inv(L, L')$ by one, and we would eventually arrive at L'.

From this we see that there is a sequence of adjacent transposition on L such that the linear order obtained at every step of it also extends P. Thus, there exists a path between L and L'.

> I believe the part missing in Ruskey's original paper is the justification that there exists at least one adjacent pair that is incomparable which we can swap.
>
> My justification is: For all $x||y$, let $PREx = \{z|z < x\}$. Let $MINx \in PREx$ be such that $\nexists e, e < MINx$. We can always find at least one adjacent pair a, b with a being some $MINx$.
>
> I am absolutely unsure if this checks out, but this is how I think of it: since getting a linearization is just a topological sort on the hasse diagram of poset, and that incomparable pairs are just elements on different branches, the traversal algorithm must at some point decides to take the branch if it exists, and thus the first node it take on that branch would be not only incomparable but also adjacent to our last taken node.

Theorem

For any pair x, y such that $x||y$ for poset P, there exist linear orders L, L' such that $P \subseteq L, L'$ with $x \prec y \in L$, $y \prec x \in L'$ and $L \leftrightarrow L'$.

Proof

Let $PRE = \{z|z < x \in P\}$ and $POST = \{z|x < z \in P\}$. Since $x||y$, $x \nless y$ and $y \nless x$ in P. Furthermore, for all $z \in PRE$, $y \nless z$ in P, and, for all $z \in POST$, $z \nless y$ in P. Thus, we can construct Q as the transitive closure of $P \cup \{x < y\}$ and Q' the transitive closure of $P \cup \{y < x\}$. Since $x \prec y \in Q$ and $y \prec x \in Q'$, there exist linear orders L, L' such that $Q \subseteq L$, $Q' \subseteq L'$, and $L \leftrightarrow L'$.

> This was proved by H&N but I thought I'd give it a go at simplifying things a bit.