Lambda Algorithm

For f

$$
\begin{aligned}
f &= \bigvee_{n=1}^{s} \bigwedge_{m}^{t_n} \ell_m^n = \bigvee_{n=1}^{s} T_n \\
f(x+a_k) &= \bigvee_{n=1}^{s} ((\bigwedge_{\substack{x_i \in P\ell^n \\ a_k[i]=0}} x_i) \wedge (\bigwedge_{\substack{x_i \in P\ell^n \\ a_k[i]=1}} \neg x_i) \wedge (\bigwedge_{\substack{x_j \in N\ell^n \\ a_k[j]=0}} \neg x_j) \wedge (\bigwedge_{\substack{x_j \in N\ell^n \\ a_k[j]=1}} x_j)) \\
&= \bigvee_{n=1}^{s} T_n^{a_k} \\
\mathscr{M}(f(x+a_k)) &= \bigvee_{n=1}^{s} ((\bigwedge_{\substack{x_i \in P\ell^n \\ a_k[i]=0}} x_i) \wedge (\bigwedge_{\substack{x_j \in N\ell^n \\ a_k[j]=1}} x_j)) \\
&= \bigvee_{n=1}^{s} \{ \bigwedge_{(T_n^{a_k}+a_k)[i]=1} x_i \mid x \in \ell^n \} \\
&= \bigvee_{n=1}^{s} \mathscr{T}_k = \bigvee_{n=1}^{s} \mathscr{M}(T_n^{a_k}) \\
\mathscr{M}_{a_k}(f) &= \mathscr{M}(f(x+a_k))(x+a_k) = (\bigvee_{n=1}^{s} \mathscr{T}_k)(x+a_k)
\end{aligned}
$$

For $S_i = \{\widehat{v_n} + a_i \mid n\}$

$$
\begin{aligned}
\mathrm{M}_{\mathrm{DNF}}(S_i) &= \{\mathrm{M}_{\mathrm{TERM}}(\widehat{v_n} + a_i)) \mid (\widehat{v_n} + a_i) \in S_i\} \\
&= \{ \bigwedge_{(\widehat{v_n}+a_i)[j]=1} x_j \mid (\widehat{v_n} + a_i) \in S_i\} \\
\bigvee_{(\widehat{v_n}+a_i)\in S_i} \mathrm{M}_{\mathrm{DNF}}(S_i) &= \bigvee_{(\widehat{v_n}+a_i)\in S_i} \bigwedge_{(\widehat{v_n}+a_i)[j]=1} x_j \\
H_i &= (\bigvee_{(\widehat{v_n}+a_i)\in S_i} \mathrm{M}_{\mathrm{DNF}}(S_i))(x+a_i) \\
&= \bigvee_{(\widehat{v_n}+a_i)\in S_i} ((\bigwedge_{\substack{\widehat{v_n}[j]=1 \\ a_i[j]=0}} x_j) \wedge (\bigwedge_{\substack{\widehat{v_n}[j]=0 \\ a_i[j]=1}} \neg x_j))
\end{aligned}
$$

The idea is to fit $\mathrm{M}_{\mathrm{DNF}}(S_i)$ to $\mathscr{T}_i$ in each a_i dimension s.t. H_i fits to $\mathscr{M}_{a_k}(f)$.

Proposition A

Let f be a boolean function. If a is an assignment such that $f(a) = 1$ and for all i where $a[i] = 1$ we have $f(a + \neg x_i) = 0$, then for any DNF $\bigvee_{i=1}^{s} T_i$ of f there is a term T_a such that $\mathscr{M}(T_a) = \mathrm{M}_{\mathrm{TERM}}(a)$.

Proof Assume

1. v^δ is a positive counterexample.
2. I^δ is non-empty.
3. $\mathrm{M}_{\mathrm{TERM}}(\widehat{v}^\delta + a_i) \in \mathscr{T}_i - \mathrm{M}_{\mathrm{DNF}}(S_i^{\delta-1})$ for $i \in I^\delta$.
4. $\mathrm{M}_{\mathrm{DNF}}(S_i^\delta) \subseteq \mathscr{T}_i$ for $i = 1..t$.
5. $H_i^\delta \to \mathscr{M}_{a_i}(f)$ for $i = 1..t$.

Then induction is as follows.

a. By hypothesis 5,

$$H_i^\delta \to \mathscr{M}_{a_i}(f)$$
$$\bigwedge_{i=1}^{t} H_i^\delta \to \bigwedge_{i=1}^{t} \mathscr{M}_{a_i}(f) = f$$

Therefore $v^{\delta+1}$ is a positive counterexample.

b. Since (a), $v^{\delta+1}$ is a positive counterexample, $(\bigwedge_{i=1}^{t} H_i^\delta)(v^{\delta+1}) = 0$ and so

$$I^{\delta+1} = \{i \mid H_i^\delta(v^{\delta+1}) = 0\}$$

is non-empty.

c. Let $i \in I^{\delta+1}$. Since (b), $H_i^\delta(v^{\delta+1}) = 0$. By definition,

$$H_i^\delta(v^{\delta+1}) = (\bigvee_{(\widehat{v_n}+a_i)\in S_i^\delta} \mathrm{M}_{\mathrm{DNF}}(S_i^\delta))(v^{\delta+1} + a_i) = 0$$

After walking, $v^{\delta+1}$ becomes $\widehat{v}^{\delta+1}$ and $\widehat{v}^{\delta+1} \leq_{a_i} v^{\delta+1}$ and

$$\widehat{v}^{\delta+1} + a_i \leq v^{\delta+1} + a_i$$

Since $(\bigvee_{(\widehat{v_n}+a_i)\in S_i^\delta} \mathrm{M}_{\mathrm{DNF}}(S_i^\delta))$ is monotone,

$$(\bigvee_{(\widehat{v_n}+a_i)\in S_i^\delta} \mathrm{M}_{\mathrm{DNF}}(S_i^\delta))(\widehat{v}^{\delta+1} + a_i) = 0$$

so

$$\mathrm{M}_{\mathrm{TERM}}(\widehat{v}^{\delta+1} + a_i) \notin \mathrm{M}_{\mathrm{DNF}}(S_i^{\delta})$$

Since $\widehat{v}^{\delta+1}$ is walked towards a_i,

$$f(\widehat{v}^{\delta+1}) = f((\widehat{v}^{\delta+1} + a_i) + a_i) = 1$$

and for all $(\widehat{v}^{\delta+1} + a_i)[j] = 1$

$$f(\widehat{v}^{\delta+1} + \neg x_j) = f(\widehat{v}^{\delta+1} + \neg x_j + a_i + a_i) = f((\widehat{v}^{\delta+1} + a_i) + \neg x_j + a_i) = 0$$

So by Proposition A, there is a term T in any DNF of $f(x+a_i)$ such that $\mathscr{M}(T) = \mathrm{M}_{\mathrm{TERM}}(\widehat{v}^{\delta+1} + a_i)$. Therefore

$$\mathrm{M}_{\mathrm{TERM}}(\widehat{v}^{\delta+1} + a_i) \in \mathscr{T}_i = \{\mathscr{M}(T) \mid T \in f(x + a_i)\}$$

This is the crux of the proof.

d. If $i \notin I^{\delta+1}$, then $S_i^{\delta+1} = S_i^{\delta}$. If $i \in I^{\delta+1}$, then

$$S_i^{\delta+1} = S_i^{\delta} \cup \{\widehat{v}^{\delta+1} + a_i\}.$$

Either way, by hypothesis 4 and (c), we have $\mathrm{M}_{\mathrm{DNF}}(S_i^{\delta+1}) \subseteq \mathscr{T}_i$.

e. By definition,

$$H_i^{\delta+1}(x) = (\bigvee_{(\widehat{v_n}+a_i)\in S_i^{\delta+1}} \mathrm{M}_{\mathrm{DNF}}(S_i^{\delta+1}))(x + a_i)$$

and

$$\mathscr{M}_{a_k}(f) = (\bigvee_{n=1}^{s} \mathscr{T}_k)(x + a_k)$$

By (d), $\mathrm{M}_{\mathrm{DNF}}(S_i^{\delta+1}) \subseteq \mathscr{T}_i$, so

$$\bigvee_{(\widehat{v_n}+a_i)\in S_i^{\delta+1}} \mathrm{M}_{\mathrm{DNF}}(S_i^{\delta+1}) \to \bigvee_{n=1}^{s} \mathscr{T}_k$$

Therefore

$$H_i^{\delta+1}(x) \to \mathscr{M}_{a_k}(f)$$