This tutorial reflects the state of the most recent development version of FuzzyNumbers available on *GitHub*.
If you use the "official" CRAN release, some of the features may be unavailable.

A Guide to the FuzzyNumbers 0.3-0 Package for R

Marek Gagolewski[1,2]

[1] Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01-447 Warsaw, Poland

[2] Rexamine, Email: `gagolews@rexamine.com`
www.rexamine.com/resources/fuzzynumbers/

June 21, 2013

The package, as well as this tutorial, is still in its early days – any suggestions and contributions are welcome!

Contents

1 Getting Started

Fuzzy set theory lets us quite intuitively represent imprecise or vague information. Fuzzy numbers (FNs), introduced by Dubois and Prade in [7], form a particular subclass of fuzzy sets of the real line. Formally, a fuzzy set A with membership function $\mu_A : \mathbb{R} \to [0, 1]$ is a fuzzy number, if it possess at least the three following properties:

(i) it is a normalized fuzzy set, i.e. $\mu_A(x_0) = 1$ for some $x_0 \in \mathbb{R}$,

(ii) it is fuzzy convex, i.e. for any $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$ it holds $\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$,

(iii) the support of A is bounded, where $\operatorname{supp}(A) = \operatorname{cl}(\{x \in \mathbb{R} : \mu_A(x) > 0\})$.

Fuzzy numbers play a significant role in many practical applications (cf. [13]) since we often describe our knowledge about objects through numbers, e.g. "I'm about 180 cm tall" or "The rocket was launched between 2 and 3 p.m.".

FuzzyNumbers is an Open Source (licensed under GNU LGPL 3) package for R – a free software environment for statistical computing and graphics, which runs on all major operating systems, i.e. Windows, Linux, and MacOS X[1].

FuzzyNumbers has been created in order to deal with fuzzy numbers conveniently and effectively. To install latest "official" release of the package available on *CRAN* we type[2]:

```
install.packages('FuzzyNumbers')
```

Alternatively, we may fetch its current development snapshot from *GitHub*:

```
install.packages('devtools')
library('devtools')
install_github('FuzzyNumbers', 'Rexamine')
```

Each session with FuzzyNumbers should be preceded by a call to:

```
library('FuzzyNumbers') # Load the package
```

To view the main page of the manual we type:

```
library(help='FuzzyNumbers')
```

For more information please visit the package's homepage [9]. In case of any problems, comments, or suggestions feel free to contact the author. Good luck!

[1]Please visit R Project's homepage at www.R-project.org for more details. Perhaps you may also wish to install RStudio, a convenient development environment for R. It is available at www.rsudio.com/ide.

[2]You are viewing the **development** version of the tutorial. Some of the features presented in this document may be missing in the current *CRAN* release. Please, upgrade to the **latest** development version from *GitHub* if you need the new functionality.

2 How to Create Instances of Fuzzy Numbers

2.1 Arbitrary Fuzzy Numbers

A fuzzy number A may be defined by specifying its core, support, and either its left/right side functions or lower/upper α-cut bounds. Please note that many algorithms that deal with FNs assume we provide at least the latter, i.e. α-cuts.

2.1.1 Definition by Side Functions

A fuzzy number A specified by side functions[3] has membership function of the form:

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < \mathtt{a1}, \\ \mathtt{left}\left(\frac{x-\mathtt{a1}}{\mathtt{a2}-\mathtt{a1}}\right) & \text{if } \mathtt{a1} \le x < \mathtt{a2}, \\ 1 & \text{if } \mathtt{a2} \le x \le \mathtt{a3}, \\ \mathtt{right}\left(\frac{x-\mathtt{a3}}{\mathtt{a4}-\mathtt{a3}}\right) & \text{if } \mathtt{a3} < x \le \mathtt{a4}, \\ 0 & \text{if } \mathtt{a4} < x, \end{cases} \tag{1}$$

where $\mathtt{a1}, \mathtt{a2}, \mathtt{a3}, \mathtt{a4} \in \mathbb{R}$, $\mathtt{a1} \le \mathtt{a2} \le \mathtt{a3} \le \mathtt{a4}$, $\mathtt{left} : [0,1] \to [0,1]$ is a nondecreasing function (called the *left side generator of* A), and $\mathtt{right} : [0,1] \to [0,1]$ is a nonincreasing function (*right side generator of* A). In our package, it is assumed that these functions fulfill the conditions $\mathtt{left}(0) \ge 0$, $\mathtt{left}(1) \le 1$, $\mathtt{right}(0) \le 1$, and $\mathtt{right}(1) \ge 0$.

An example: a fuzzy number A_1 with linear sides (a trapezoidal fuzzy number, see also Sec. 2.3).

```
A1 <- FuzzyNumber(1, 2, 4, 7,
    left=function(x) x,
   right=function(x) 1-x
)
```

This object is an instance of the following R class:

```
class(A1)

## [1] "FuzzyNumber"
## attr(,"package")
## [1] "FuzzyNumbers"
```

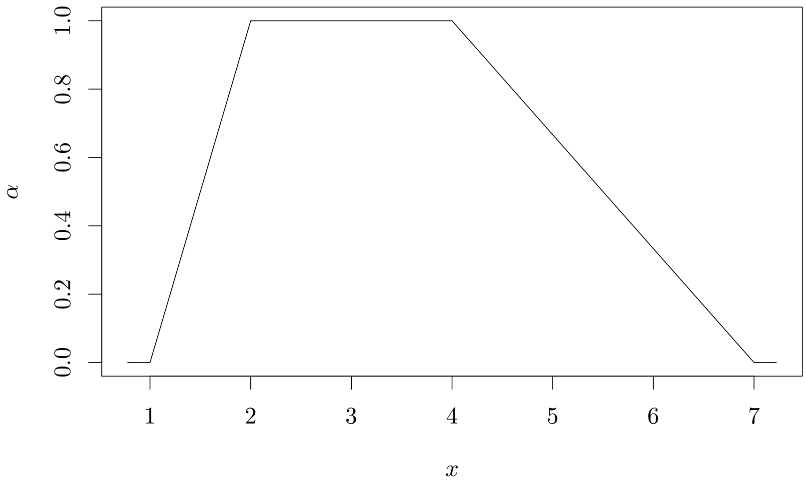
We may print some basic information on A_1 by calling `print(A1)` or simply by typing:

```
A1

## Fuzzy number with:
##    support=[1,7],
##       core=[2,4].
```

To depict A_1 we call:

```
plot(A1)
```

[3] Side functions are sometimes called branches or shape functions in the literature.



Remark. Please note that by using side generating functions defined on $[0,1]$ we really make (in the author's humble opinion) the process of generating examples for our publications much easier. A similar concept was used e.g. in [14] (LR-fuzzy numbers).

Assume, however, that we are given two fancy side functions $f : [a_1, a_2] = [-4, -2] \to [0, 1]$, and $g : [a_3, a_4] = [-1, 10] \to [1, 0]$, for example:

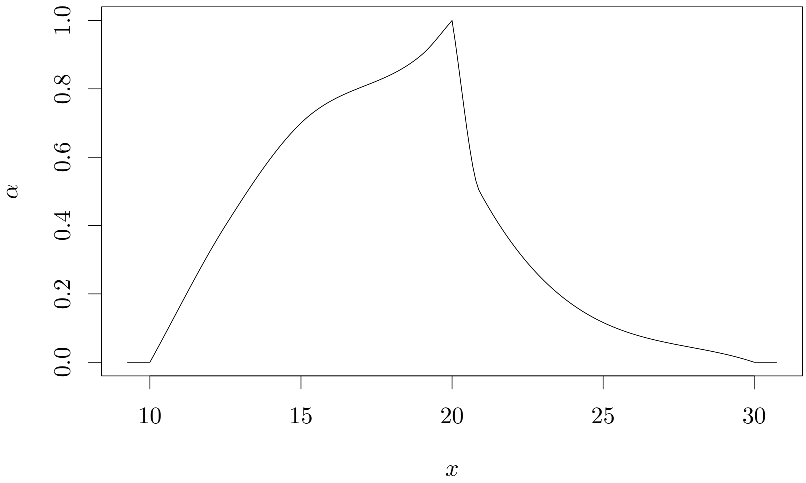
```
f <- splinefun(c(-4,-3.5,-3,-2.2,-2), c(0,0.4,0.7,0.9,1), method='monoH.FC')
g <- splinefun(c(-1,0,10), c(1,0.5,0), method='monoH.FC')
```

We should convert them to side *generating* functions, which shall be defined on the interval $[0, 1]$. This may easily be done with the `convertSide()` function. It returns a new function that calls the original one with linearly transformed input.

```
convertSide(f, -4, -2)(c(0,1))
## [1] 0 1
convertSide(g, -1, 10)(c(0,1))
## [1] 1 0
convertSide(g, 10, -1)(c(0,1)) # interesting!
## [1] 0 1
```

These functions may be used to define a fuzzy number, now with arbitrary support and core.

```
B <- FuzzyNumber(10,20,20,30,
    left=convertSide(f, -4, -2),
   right=convertSide(g, -1, 10)
)
plot(B, xlab='$x$', ylab='$\\alpha$')
```



2.1.2 Definition by α-cut Bounds

Alternatively, a fuzzy number A may be defined by specifying its α-cuts. We have (for $\alpha \in (0,1)$ and $\mathtt{a1} \le \mathtt{a2} \le \mathtt{a3} \le \mathtt{a4}$):

$$A_\alpha := [A_L(\alpha), A_U(\alpha)] \tag{2}$$
$$= \big[\mathtt{a1} + (\mathtt{a2} - \mathtt{a1}) \cdot \mathtt{lower}(\alpha), \mathtt{a3} + (\mathtt{a4} - \mathtt{a3}) \cdot \mathtt{upper}(\alpha)\big], \tag{3}$$

where $\mathtt{lower} : [0,1] \to [0,1]$ is a nondecreasing function (called *lower α-cut bound generator of A*), and $\mathtt{upper} : [0,1] \to [0,1]$ is a nonincreasing function (*upper bound generator*). In our package, we assume that $\mathtt{lower}(0) = 0$, $\mathtt{lower}(1) = 1$, $\mathtt{upper}(0) = 1$, and $\mathtt{upper}(1) = 0$.

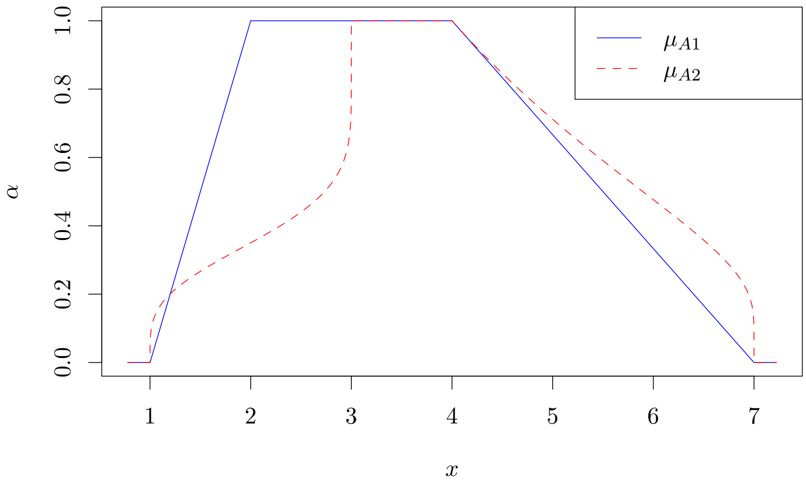
It is easily seen that for $\alpha \in (0,1)$ we have the following relationship between generating functions:

$$\mathtt{lower}(\alpha) = \inf\{x : \mathtt{left}(x) \ge \alpha\}, \tag{4}$$
$$\mathtt{upper}(\alpha) = \sup\{x : \mathtt{right}(x) \ge \alpha\}. \tag{5}$$

Moreover, if side generating functions are continuous and strictly monotonic, then α-cut bound generators are their inverses.

An example:

```
A1 <- FuzzyNumber(1, 2, 4, 7,
    left=function(x) x,
   right=function(x) 1-x
)
A2 <- FuzzyNumber(1, 3, 4, 7,
   lower=function(alpha) pbeta(alpha, 5, 9), # CDF of a beta distr.
   upper=function(alpha) pexp(1/alpha-1) # transformed CDF of an exp. distr.
)
plot(A1, col='blue')
plot(A2, col='red', lty=2, add=TRUE)
legend('topright', c(expression(mu[A1]), expression(mu[A2])),
   col=c('blue', 'red'), lty=c(1,2))
```



Remark. The `convertAlpha()` function works similarly to `convertSide()`. It scales the output values of a given function, thus it may be used to create an α-cut generator conveniently.

2.1.3 Definition with Generating Functions Omitted: Shadowed Sets

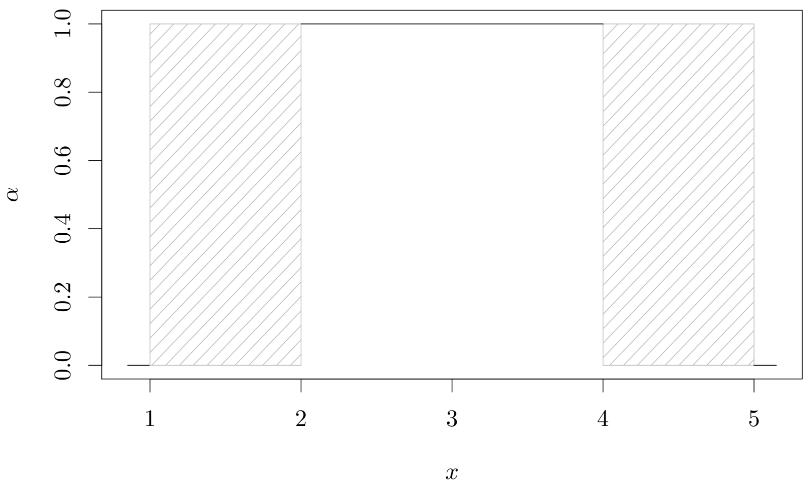
In the above examples either side generating functions or α-cut generators were passed to the `FuzzyNumber()` function. Let us note what will happen if we omit both of them.

```
A3 <- FuzzyNumber(1, 2, 4, 5)
A3
```

```
## Fuzzy number with:
##    support=[1,5],
##       core=[2,4].
```

The object seems to be defined correctly: R does not make any complaints. However…

```
plot(A3)
```



It turns out that we have obtained a *shadowed set*! Indeed, this behavior is quite reasonable: we have provided no information on the "partial knowledge" part of our fuzzy number. In fact, the object has been initialized with generating functions always returning `NA` (*Not-Available* or *any* value). Does it mean that when we define a FN solely by side generators, we cannot compute its α-cuts? Indeed!

```
alphacut(A2, 0.5) # A2 has alpha-cut generators defined
##            L        U
## 0.5 2.733154 5.896362
alphacut(A1, 0.5) # A1 hasn't got them
##      L  U
## 0.5 NA NA
```

Another example: evaluation of the membership function.

```
evaluate(A1, 6.5) # A1 has side generators defined
##       6.5
## 0.1666667
evaluate(A2, 6.5) # A2 hasn't got them
## 6.5
##  NA
```

2.2 Using Numeric Approximations of α-cut or Side Generators

The reason for setting `NA`s[4] as return values of omitted generators is simple. Finding a function inverse numerically requires lengthy computations and is always done locally (for a given point, not for "whole" the function at once). R is not a symbolic mathematical solver. If we had defined such procedures (it is really easy to do by using the `uniroot()` function), then an inexperienced user would have used it in his/her algorithms and wondered why everything runs so slow. To get more insight, let us look at the internals of `A2`:

```
A2['lower']
## function(alpha) pbeta(alpha, 5, 9)
A2['upper']
## function(alpha) pexp(1/alpha-1)
A2['left']
## function (x)
## rep(NA_real_, length(x))
## <environment: 0x329a828>
A2['right']
## function (x)
## rep(NA_real_, length(x))
## <environment: 0x329a828>
```

Note that all generators are properly vectorized (for input vectors of length n they always give output of the same length). Thus, general rules are as follows. If you want α-cuts (e.g. for finding trapezoidal approximations of FNs), specify them. If you would like to calculate the membership function (by the way, the `plot()` function automatically detects what kind of knowledge we have), assure the side generators are provided.

[4] To be precise, it's `NA_real_`.

However, we also provide a convenient short-cut method to *interpolate* generating functions of one type to get some crude numeric approximations of their inverses: the `approxInvert()` function[5], which may of course be applied on results returned by `convertAlpha()` and `convertSide()`. This is a simple wrapper to R's `approxfun()` (piecewise linear interpolation, the `'linear' method`) and `splinefun()` (monotonic splines: `methods` `'hyman'` and `'monoH.FC'`; the latter is default and recommended).

```
l <- function(x) pbeta(x, 1, 2)
r <- function(x) 1-pbeta(x, 1, 0.1)
A4 <- FuzzyNumber(-2, 0, 0, 2,
   left  = l,
   right = r,
   lower = approxInvert(l),
   upper = approxInvert(r)
)

x <- seq(0,1,length.out=1e5)
max(abs(qbeta(x, 1, 2) - A4['lower'](x)))     # sup-error estimator
## [1] 0.0001389811
max(abs(qbeta(1-x, 1, 0.1) - A4['upper'](x))) # sup-error estimator
## [1] 0.0008607773
```

2.3 Trapezoidal Fuzzy Numbers

A trapezoidal fuzzy number (TFN) is a FN which has linear side generators and linear α-cut bound generators. To create a trapezoidal fuzzy number T_1 with, for example, $\mathrm{core}(T_1) = [1.5, 4]$ and $\mathrm{supp}(T_1) = [1, 7]$ we call:

```
T1 <- TrapezoidalFuzzyNumber(1, 1.5, 4, 7)
```

Thus, we have:

$$\mu_{T_1}(x) = \begin{cases} 0 & \text{for} \quad x \in (-\infty, 1), \\ (x-1)/0.5 & \text{for} \quad x \in [1, 1.5), \\ 1 & \text{for} \quad x \in [1.5, 4], \\ (7-x)/3 & \text{for} \quad x \in (4, 7], \\ 0 & \text{for} \quad x \in (7, +\infty). \end{cases}$$

$$T_{1\alpha} = [1 + 0.5\,\alpha, 7 - 3\,\alpha].$$

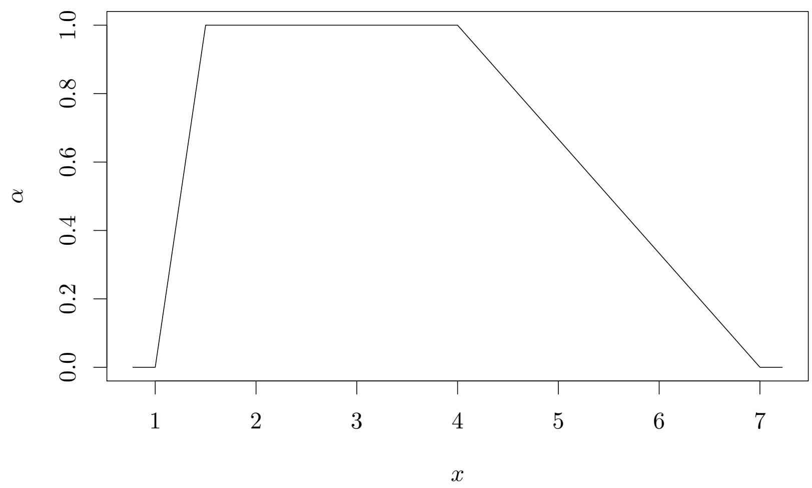
Note that the above equations have been atomatically generated by knitr and LaTeX by calling `cat(as.character(T1, toLaTeX=TRUE, varnameLaTeX='T_1'))`, see Sec. 3.
The `T1` object is an instance of the following R class:

```
class(T1)
## [1] "TrapezoidalFuzzyNumber"
## attr(,"package")
## [1] "FuzzyNumbers"
```

[5] The `n` argument, which sets the number of interpolation points, controls the trade-off between accuracy and computation speed. Well, world's not ideal, remember that "some" is better than "nothing" sometimes.

To depict T_1 we call:

```
plot(T1)
```



Trapezoidal fuzzy numbers are among the simplest FNs. Despite their simplicity, however, they include triangular FNs, "crisp" real intervals, and "crisp" reals. Please note that currently no separate classes for these particular TFNs types are implemented in the package.

```
TrapezoidalFuzzyNumber(1,2,2,3)  # triangular FN
## Trapezoidal fuzzy number with:
##    support=[1,3],
##       core=[2,2].
TrapezoidalFuzzyNumber(2,2,3,3)  # `crisp' interval
## Trapezoidal fuzzy number with:
##    support=[2,3],
##       core=[2,3].
TrapezoidalFuzzyNumber(5,5,5,5)  # `crisp' real
## Trapezoidal fuzzy number with:
##    support=[5,5],
##       core=[5,5].
```

T_1 is (roughly) equivalent to the trapezoidal fuzzy number A_1 defined in the previous subsection. The `TrapezoidalFuzzyNumber` class inherits all the goodies from the `FuzzyNumber` class, but is more specific (guarantees faster computations, contains more detailed information, etc.). Of course, in this case the generating functions are known *a priori* (A_1 had no α-cut generators) so there is no need to provide them manually (what is more, this has been disallowed for safety reasons). Thus, is we wanted to define a trapezoidal FN next time, we would do it not like with A_1 but rather as with T_1.

```
T1['lower']
## function (alpha)
## alpha
## <bytecode: 0x2761548>
```

```
## <environment: namespace:FuzzyNumbers>
T1['upper']
## function (alpha)
## 1 - alpha
## <bytecode: 0x2761bf0>
## <environment: namespace:FuzzyNumbers>
T1['left']
## function (x)
## x
## <bytecode: 0x275f760>
## <environment: namespace:FuzzyNumbers>
T1['right']
## function (x)
## 1 - x
## <bytecode: 0x275fe08>
## <environment: namespace:FuzzyNumbers>
```

2.4 Piecewise Linear Fuzzy Numbers

Trapezoidal fuzzy numbers are generalized by piecewise linear FNs (PLFNs), i.e. fuzzy numbers which side generating functions and α-cut generators are piecewise linear functions. Each PLFN is given by:

- four coefficients $\mathtt{a1} \le \mathtt{a2} \le \mathtt{a3} \le \mathtt{a4}$ defining its support and core,
- the number of "knots", `knot.n`≥ 0,
- a vector of α-cut coordinates, `knot.alpha`, consisting of `knot.n` elements $\in [0, 1]$,
- a nondecreasingly sorted vector `knot.left` consisting of `knot.n` elements $\in [\mathtt{a1}, \mathtt{a2}]$, defining interpolation points for the left side function, and
- a nondecreasingly sorted vector `knot.right` consisting of `knot.n` elements $\in [\mathtt{a2}, \mathtt{a3}]$, defining interpolation points for the right side function.

If `knot.n`≥ 1, then the membership function of a piecewise linear fuzzy number P is defined as:

$$\mu_P(x) = \begin{cases} 0 & \text{if} \quad x < \mathtt{a1}, \\ \alpha_i + (\alpha_{i+1} - \alpha_i)\left(\frac{x - l_i}{l_{i+1} - l_i}\right) & \text{if} \ \ l_i \le x < l_{i+1} \\ & \ \text{for some } i \in \{1, \dots, n+1\}, \\ 1 & \text{if } \mathtt{a2} \le x \le \mathtt{a3}, \\ \alpha_{n-i+2} + (\alpha_{n-i+3} - \alpha_{n-i+2})\left(1 - \frac{x - r_i}{r_{i+1} - r_i}\right) & \text{if} \ \ r_i < x \le r_{i+1} \\ & \ \text{for some } i \in \{1, \dots, n+1\}, \\ 0 & \text{if } \mathtt{a4} < x, \end{cases} \tag{6}$$

and its α-cuts for $\alpha \in [\alpha_i, \alpha_{i+1}]$ (for some $i \in \{1, \dots, n+1\}$) are given by:

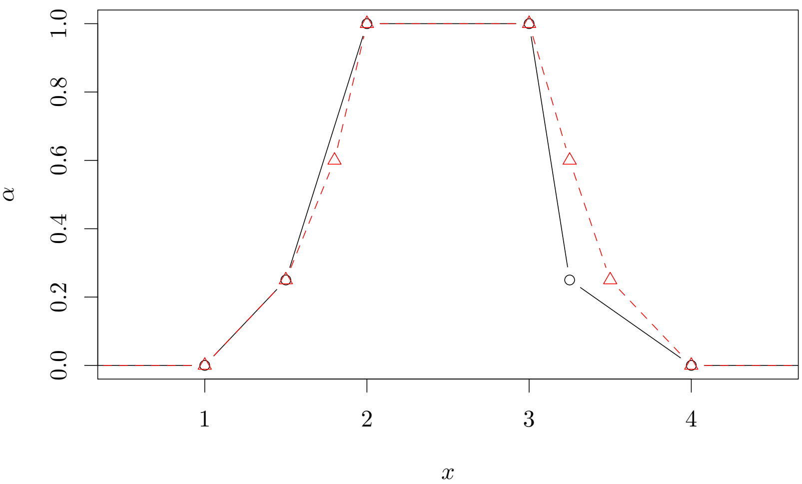
$$P_L(\alpha) = l_i + (l_{i+1} - l_i)\left(\frac{\alpha - \alpha_i}{\alpha_{i+1} - \alpha_i}\right), \tag{7}$$

$$P_U, (\alpha) = r_{n-i+2} + (r_{n-i+3} - r_{n-i+2})\left(1 - \frac{\alpha - \alpha_i}{\alpha_{i+1} - \alpha_i}\right), \tag{8}$$

where $n = \mathtt{knot.n}$, $(l_1, \dots, l_{n+2}) = (\mathtt{a1}, \mathtt{knot.left}, \mathtt{a2})$, $(r_1, \dots, r_{n+2}) = (\mathtt{a3}, \mathtt{knot.right}, \mathtt{a4})$, and $(\alpha_1, \dots, \alpha_{n+2}) = (0, \mathtt{knot.alpha}, 1)$.

PLFNs in our package are represented by the `PiecewiseLinearFuzzyNumber` class.

```
P1 <- PiecewiseLinearFuzzyNumber(1, 2, 3, 4,
   knot.n=1, knot.alpha=0.25, knot.left=1.5, knot.right=3.25)
class(P1)

## [1] "PiecewiseLinearFuzzyNumber"
## attr(,"package")
## [1] "FuzzyNumbers"

P1

## Piecewise linear fuzzy number with 1 knot(s),
##    support=[1,4],
##       core=[2,3].

P2 <- PiecewiseLinearFuzzyNumber(1, 2, 3, 4,
   knot.n=2, knot.alpha=c(0.25,0.6),
   knot.left=c(1.5,1.8), knot.right=c(3.25, 3.5))
P2

## Piecewise linear fuzzy number with 2 knot(s),
##    support=[1,4],
##       core=[2,3].

plot(P1, type='b', from=0, to=5, xlim=c(0.5,4.5))
plot(P2, type='b', col=2, lty=2, pch=2, add=TRUE, from=0, to=5)
```



The following operators return matrices with all knots of a PLFN. Each matrix has three columns: α-cuts, left side coordinates, and right side coordinates.

```
P1['knots']

##        alpha left right
## knot_1  0.25  1.5  3.25

P1['allknots'] # including a1,a2,a3,a4

##        alpha left right
## supp    0.00  1.0  4.00
## knot_1  0.25  1.5  3.25
## core    1.00  2.0  3.00
```

We have, for example:

$$\mu_{P_1}(x) = \begin{cases} 0 & \text{for} \quad x \in (-\infty, 1), \\ 0 + 0.25\,(x+1)/0.5 & \text{for} \quad x \in [1, 1.5), \\ 0.25 + 0.75\,(x+1.5)/0.5 & \text{for} \quad x \in [1.5, 2), \\ 1 & \text{for} \quad x \in [2, 3], \\ 0.25 + 0.75\,(3.25-x)/0.25 & \text{for} \quad x \in [3, 3.25), \\ 0 + 0.25\,(4-x)/0.75 & \text{for} \quad x \in [3.25, 4), \\ 0 & \text{for} \quad x \in (4, +\infty). \end{cases}$$

$$P_{1\alpha} = [P_{1L}(\alpha), P_{1U}(\alpha)],$$

where

$$P_{1L}(\alpha) = \begin{cases} 1 + 0.5\,(\alpha-0)/0.25 & \text{for} \quad \alpha \in [0, 0.25], \\ 1.5 + 0.5\,(\alpha-0.25)/0.75 & \text{for} \quad \alpha \in [0.25, 1], \end{cases}$$

$$P_{1U}(\alpha) = \begin{cases} 3.25 + 0.75\,(0.25-\alpha)/0.25 & \text{for} \quad \alpha \in [0, 0.25], \\ 3 + 0.25\,(1-\alpha)/0.75 & \text{for} \quad \alpha \in [0.25, 1]. \end{cases}$$

If you want to obtain a PLFN with equally distributed knots, then you may use the more convenient version of the `PiecewiseLinearFuzzyNumber()` function.

```
PiecewiseLinearFuzzyNumber(knot.left=c(0,0.5,0.7,1),
                           knot.right=c(2,2.2,2.4,3))['allknots']
```

```
##            alpha left right
## supp   0.0000000  0.0   3.0
## knot_1 0.3333333  0.5   2.4
## knot_2 0.6666667  0.7   2.2
## core   1.0000000  1.0   2.0
```

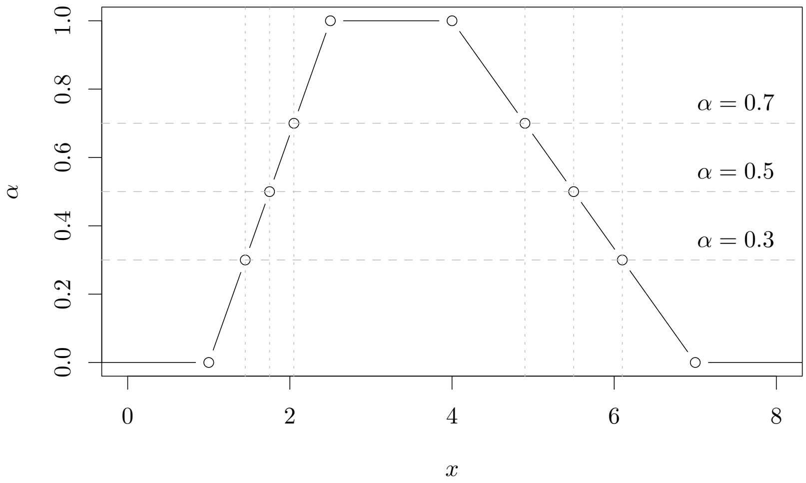
Note that if $a_1, \ldots, a_4$ are omitted, then they are taken from `knot.left` and `knot.right` (their lengths should then be equal to `knot.n`+2).

If `knot.n` is equal to 0 or all left and right knots lie on common lines, then a PLFN reduces to a TFN. Please note that, however, the `TrapezoidalFuzzyNumber` class does not inherit from `PiecewiseLinearFuzzyNumber` for efficiency reasons. If, however, we wanted to convert an object of the first mentioned class to the other, we would do that by calling:

```
alpha <- c(0.3, 0.5, 0.7)
P3 <- as.PiecewiseLinearFuzzyNumber(
   TrapezoidalFuzzyNumber(1,2.5,4,7),
         knot.n=3, knot.alpha=alpha
)
P3
```

```
## Piecewise linear fuzzy number with 3 knot(s),
##    support=[1,7],
##       core=[2.5,4].
```

```
plot(P3, type='b', from=-1, to=9, xlim=c(0,8))
abline(h=alpha, col='gray', lty=2)
abline(v=P3['knot.left'], col='gray', lty=3)
abline(v=P3['knot.right'], col='gray', lty=3)
text(7.5, alpha, sprintf('a=%g', alpha), pos=3)
```



More generally, each PLFN or TFN may be converted to a direct `FuzzyNumber` class instance if needed (hope we will never not).

```
(as.FuzzyNumber(P3))

## Fuzzy number with:
##    support=[1,7],
##       core=[2.5,4].
```

On the other hand, to "convert" (with possible information loss) more general FNs to TFNs or PLFNs, we may use the approximation procedures described in Sec. 6.

2.5 Fuzzy Numbers with Sides Given by Power Functions

Fuzzy numbers which sides are given by power functions are defined using four coefficients $\mathtt{a1} \le \mathtt{a2} \le \mathtt{a3} \le \mathtt{a4}$, and parameters $\mathtt{p.left}, \mathtt{p.right} > 0$ which determine exponets for the side functions:

$$\begin{aligned} \mathtt{left}(x) &= x^{\mathtt{p.left}}, && (9)\\ \mathtt{right}(x) &= (1-x)^{\mathtt{p.right}}. && (10) \end{aligned}$$

We also have:

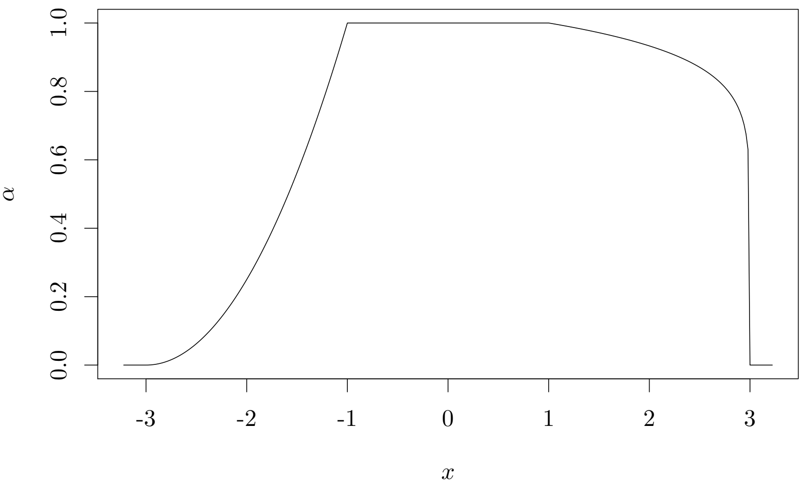
$$\begin{aligned} \mathtt{lower}(\alpha) &= \sqrt[\mathtt{p.left}]{\alpha}, && (11)\\ \mathtt{upper}(\alpha) &= 1 - \sqrt[\mathtt{p.right}]{\alpha}. && (12) \end{aligned}$$

These fuzzy numbers are another natural generalization of trapezoidal FNs.

An example:

```
X <- PowerFuzzyNumber(-3, -1, 1, 3, p.left=2, p.right=0.1)
class(X)

## [1] "PowerFuzzyNumber"
## attr(,"package")
## [1] "FuzzyNumbers"

X
```

```
## Fuzzy number given by power functions, and:
##     support=[-3,3],
##        core=[-1,1].
plot(X)
```



We have:

$$\mu_X(x) = \begin{cases} 0 & \text{for} \quad x \in (-\infty, -3), \\ ((x+3)/2)^2 & \text{for} \quad x \in [-3, -1), \\ 1 & \text{for} \quad x \in [-1, 1], \\ ((3-x)/2)^{0.1} & \text{for} \quad x \in (1, 3], \\ 0 & \text{for} \quad x \in (3, +\infty), \end{cases}$$

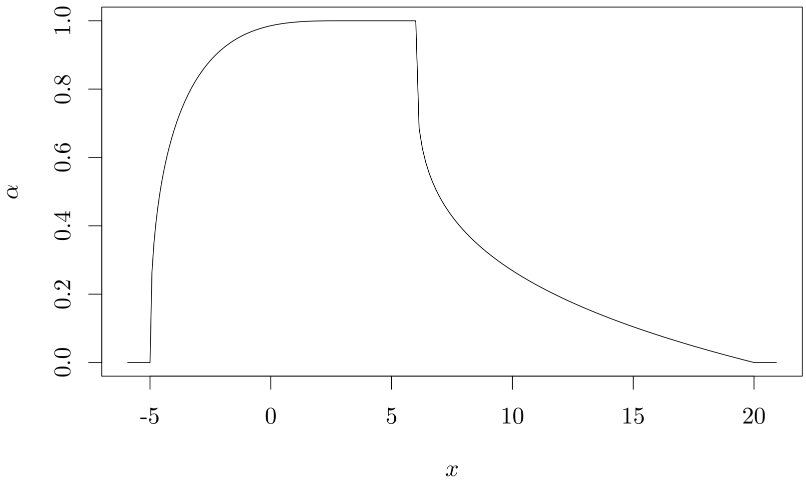
$$X_\alpha = [-3 + 2\,\alpha^{0.5}, 1 + 2\,(1 - \alpha^{10})].$$

3 Depicting Fuzzy Numbers

To draw FNs we call the `plot()` method, which uses similar parameters as the R-built-in `curve()` function / `plot.default()` method. If you are new to R, you may wish to read the manual on the most popular graphical routines by calling `?plot`, `?plot.default`, `?curve`, `?abline`, `?par`, `?lines`, `?points`, `?legend`, `?text` (some of these functions have already been called in this tutorial).

Let us consider the following FN:

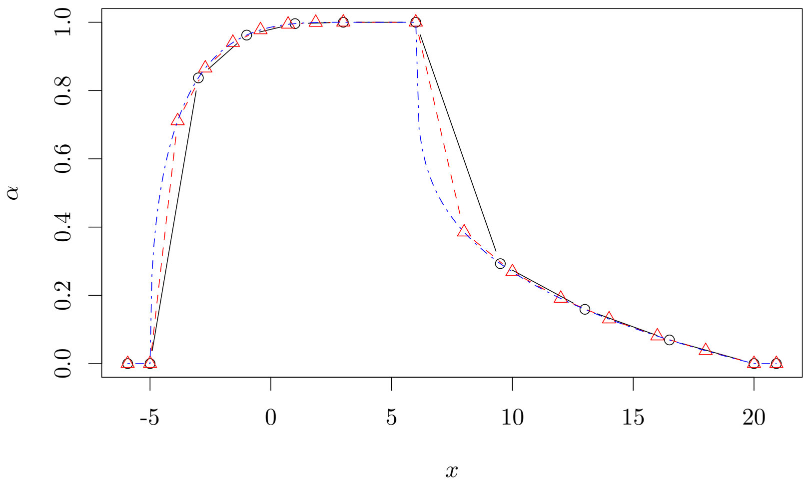
```
A <- FuzzyNumber(-5, 3, 6, 20,
    left=function(x) pbeta(x,0.4,3),
   right=function(x) 1-x^(1/4),
   lower=function(alpha) qbeta(alpha,0.4,3),
   upper=function(alpha) (1-alpha)^4
)
plot(A)
```



Plotting issues: discretization. Side functions or α-cut bounds of objects of the `FuzzyNumber` class (not including its derivatives) when plotted are naïvely approximated by piecewise linear functions with equidistant knots at one of the axes. Therefore, if we probe them at too few points, we may obtain very rough graphical representations. To control the number of points at which the interpolation takes place, we use the `n` argument (which defaults to `101`, i.e. "quite accurate").

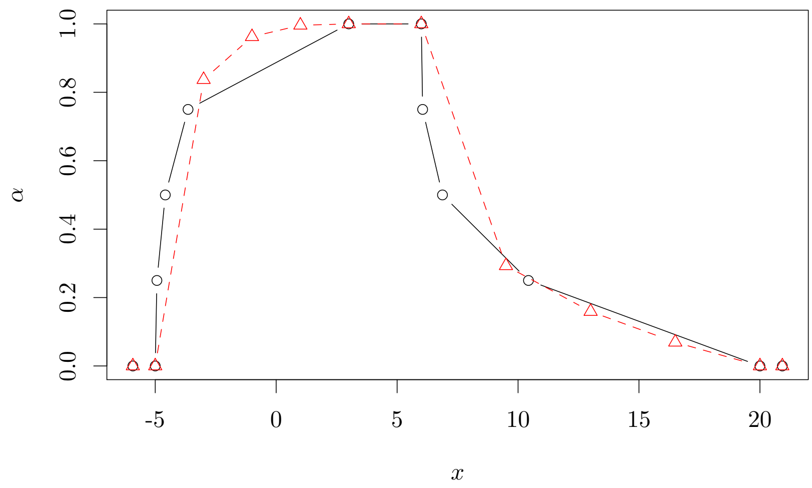
All three calls to the `plot()` method below depict the membership function of the same fuzzy number, but with different accuracy.

```
plot(A, n=3, type='b')
plot(A, n=6, add=TRUE,  lty=2, col=2, type='b', pch=2)
plot(A, n=101, add=TRUE, lty=4, col=4) # default n
```



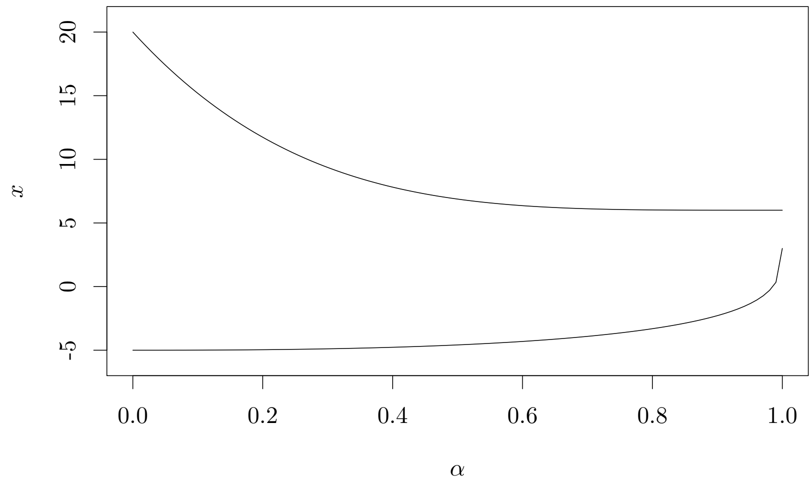
Making use of different generating functions' types. Please note (if you have not already) that to draw the membership function we do not need to provide necessarily the FN with side generators: the α-cuts will also suffice. The function is smart enough to detect the internal representation of the FN and use the kind representation it has. It both types of generators are given, then side functions are used. If we want, for some reasons, to use α-cuts, then we may do as follows:

```
plot(A, n=3, at.alpha=numeric(0), type='b') # use alpha-cuts
plot(A, n=3, type='b', col=2, lty=2, pch=2, add=TRUE) # use sides
```



We may also illustrate an α-cut representation of a fuzzy number:

```
plot(A, draw.alphacuts=TRUE)
```



Exporting figures. If we would like to generate figures for our publications, then we will surely be interested in storing them e.g. as PDF files. This may be done by calling:

```
pdf('figure1.pdf', width=8, height=5) # create file
plot(A)
```

```
dev.off() # close graphical device and save the file
```

Postscript (PS) files are generated by substituting the call to `pdf()` for the call to the `postcript()` function.

Conversion to LaTeX. Another way to depict a FN is to... give a mathematical expression which defines it.

```
cat(as.character(A, toLaTeX=TRUE, varnameLaTeX='A'))
```

This gives the following LaTeX code...

```
\[
\mu_{A}(x) = \left\{
\begin{array}{lll}
0       & \text{for} & x\in(-\infty,-5), \\
l_{A}(x) & \text{for} & x\in[-5,3), \\
1       & \text{for} & x\in[3,6], \\
r_{A}(x) & \text{for} & x\in(6,20], \\
0       & \text{for} & x\in(20,+\infty), \\
\end{array}
\right.
\]
where $l_{A}=\mathtt{left}_A((x+5)/8)$,
$r_{A}=\mathtt{right}_A((x-6)/14)$.

\[
{A}_\alpha = [{A}_L(\alpha), {A}_U(\alpha)],
\]
where ${A}_L(\alpha)=-5+8\,\mathtt{lower}_{A}(\alpha)$,
${A}_U(\alpha)=6+14\,\mathtt{upper}_{A}(\alpha)$.
```

...and, after compiling:

$$
\mu_{A}(x) = \left\{
\begin{array}{lll}
0 & \text{for} & x\in(-\infty,-5), \\
l_{A}(x) & \text{for} & x\in[-5,3), \\
1 & \text{for} & x\in[3,6], \\
r_{A}(x) & \text{for} & x\in(6,20], \\
0 & \text{for} & x\in(20,+\infty), \\
\end{array}
\right.
$$

where $l_{A}=\mathtt{left}_A((x+5)/8)$, $r_{A}=\mathtt{right}_A((x-6)/14)$.

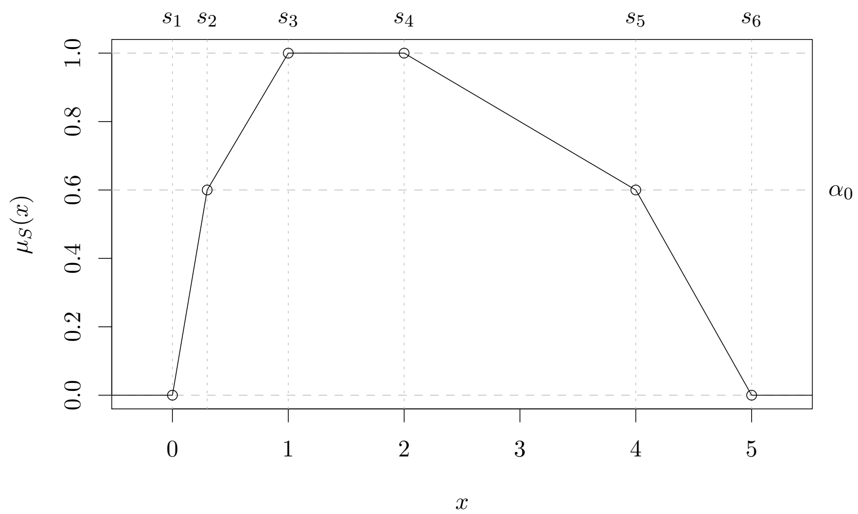
$$
{A}_\alpha = [{A}_L(\alpha), {A}_U(\alpha)],
$$

where ${A}_L(\alpha)=-5+8\,\mathtt{lower}_{A}(\alpha)$, ${A}_U(\alpha)=6+14\,\mathtt{upper}_{A}(\alpha)$.

The code may of course be modified manually to suit your needs.

Tuning your figures. Finally, we leave you with a quite complex example from one of our papers:

```
X <- PiecewiseLinearFuzzyNumber(0, 1, 2, 5, knot.n=1,
   knot.alpha=0.6, knot.left=0.3, knot.right=4)

plot.default(NA, xlab='$x$', ylab='$\\mu_S(x)$',
   xlim=c(-0.3,5.3), ylim=c(0,1)) # empty window

xpos <- c(X['a1'], X['knot.left'],  X['a2'],
          X['a3'], X['knot.right'], X['a4'])
xlab <- c('$s_1$', '$s_2$', '$s_3$',
          '$s_4$', '$s_5$', '$s_6$')
abline(v=xpos, col='gray', lty=3)
text(xpos, 1.05, xlab, pos=3, xpd=TRUE)

abline(h=c(0, X['knot.alpha'], 1), col='gray', lty=2)
text(5.55, X['knot.alpha'], sprintf('$\\alpha_0$'), pos=4, xpd=TRUE)

plot(X, add=TRUE, type='l', from=-1, to=6)
plot(X, add=TRUE, type='p', from=-1, to=6)
```



Please note that we use TEX commands in plot labels. They are interpreted by the tikzDevice package for R to generate beautiful figures, but setting this all up requires higher level of skills. . . and patience.

4 Basic Computations on and Characteristics of Fuzzy Numbers

In this section we consider the following FN:

```
A <- FuzzyNumber(-5, 3, 6, 20,
    left=function(x) pbeta(x,0.4,3),
   right=function(x) 1-x^(1/4),
   lower=function(alpha) qbeta(alpha,0.4,3),
   upper=function(alpha) (1-alpha)^4
)
```

4.1 Support and Core, and Other α-cuts

The support of A, i.e. $\mathrm{supp}(A) = [\mathtt{a1}, \mathtt{a4}]$, may be obtained by calling:

```
supp(A)

## [1] -5 20
```

We get the core of A, i.e. $\mathrm{core}(A) = [\mathtt{a2}, \mathtt{a3}]$, with:

```
core(A)

## [1] 3 6
```

To compute arbitrary α-cuts we use:

```
alphacut(A, 0) # same as supp(A) (if alpha-cut generators are defined)

##    L  U
## 0 -5 20

alphacut(A, 1) # same as core(A)

##   L U
## 1 3 6

(a <- alphacut(A, c(0, 0.5, 1)))

##             L      U
## 0.0 -5.000000 20.000
## 0.5 -4.583591  6.875
## 1.0  3.000000  6.000

a[1, ]

##  L  U
## -5 20

a[2, 2]

## [1] 6.875

a[, "L"]

##       0.0       0.5       1.0
## -5.000000 -4.583591  3.000000
```

Note that `alphacut()` always outputs a matrix with two columns. The matrix has named dimensions (names stand for only auxiliary information). The `alphacut()` method may only be used when α-cut generators are provided by the user during the declaration of A, even for $\alpha = 0$ or $\alpha = 1$.

4.2 Membership Function Evaluation

If side generators are defined, we may calculate the values of the membership function at different points by calling:

```
evaluate(A, 1)

##         1
## 0.9960291

evaluate(A, c(-3,0,3))

##        -3         0         3
## 0.8371139 0.9855322 1.0000000

evaluate(A, seq(-1, 2, by=0.5))
```

```
##      -1.0      -0.5       0.0       0.5       1.0       1.5       2.0
## 0.9624800 0.9760168 0.9855322 0.9919531 0.9960291 0.9983815 0.9995357
```

4.3 "Typical" Value

Let us first introduce the notion of the *expected interval* of A [8].

$$\begin{aligned} \mathrm{EI}(A) &:= [\mathrm{EI}_L(A), \mathrm{EI}_U(A)] && (13)\\ &= \left[\int_0^1 A_L(\alpha)\,d\alpha, \int_0^1 A_U(\alpha)\,d\alpha\right]. && (14)\end{aligned}$$

To compute the expected interval of A we call:

```
expectedInterval(A)
## [1] -4.058824  8.800000
```

In case of objects of the `FuzzyNumber` class, the expected interval is approximated by numerical integration. This method calls the `integrate()` function and its accuracy (quite fine by default) may be controlled by the `subdivisions`, `rel.tol`, and `abs.tol` parameters (call `?integrate` for more details). On the other hand, for e.g. TFNs and PLFs this method returns exact results.

The midpoint of the expected interval is called the *expected value* of a fuzzy number. It is given by:

$$\mathrm{EV}(A) := \frac{\mathrm{EI}_L(A) + \mathrm{EI}_U(A)}{2}. \tag{15}$$

Let us calculate $\mathrm{EV}(A)$.

```
expectedValue(A)
## [1] 2.370588
```

Note that this method uses a call to `expectedInterval(A)`, thus in case of `FuzzyNumber` class instances it also uses numerical approximation.

Sometimes a generalization of the expected value, called *weighted expected value*, is useful. For given $w \in [0, 1]$ it is defined as:

$$\mathrm{EV}_w(A) := (1 - w)\mathrm{EI}_L(A) + w\mathrm{EI}_U(A). \tag{16}$$

It is easily seen that $\mathrm{EV}_{0.5}(A) = \mathrm{EV}(A)$.
Some examples:

```
weightedExpectedValue(A, 0.5) # equivalent to expectedValue(A)
## [1] 2.370588
weightedExpectedValue(A, 0.25)
## [1] -0.8441176
```

The *value* of A [5] is defined by:

$$\mathrm{val}(A) := \int_0^1 \alpha\left(A_L(\alpha) + A_U(\alpha)\right)\,d\alpha. \tag{17}$$

It may be calculated by calling:

```
value(A)
## [1] 1.736177
```

Please note that the expected value or value may be used for example to "defuzzify" A.

4.4 Measures of "Nonspecificity"

The *width* of A [3] is defined as:

$$\mathrm{width}(A) := \mathrm{EI}_U(A) - \mathrm{EI}_L(A). \tag{18}$$

An example:

```
width(A)
## [1] 12.85882
```

The *ambiguity* of A [5] is defined as:

$$\mathrm{amb}(A) := \int_0^1 \alpha \left(A_U(\alpha) - A_L(\alpha)\right) \, d\alpha. \tag{19}$$

```
ambiguity(A)
## [1] 5.197157
```

Additionally, to express "nonspecificity" of a fuzzy number we may use e.g. the width of its support:

```
diff(supp(A))
## [1] 25
```

5 Operations on Fuzzy Numbers

5.1 Arithmetic Operations

The basic binary arithmetic operations for FNs are often defined by means of the so-called extension principle (see [13]) and interval arithmetic. For each $\alpha \in [0,1]$:

$$(A \circledast B)_\alpha = A_\alpha \circledast B_\alpha,$$

where $\circledast = +, -, *$ or $/$, and A, B are arbitrary FNs.

For example, we define the sum $A + B$ for every $\alpha \in [0,1]$ as:

$$(A+B)_\alpha = A_\alpha + B_\alpha = \left[A_L(\alpha) + B_L(\alpha), A_U(\alpha) + B_U(\alpha)\right],$$

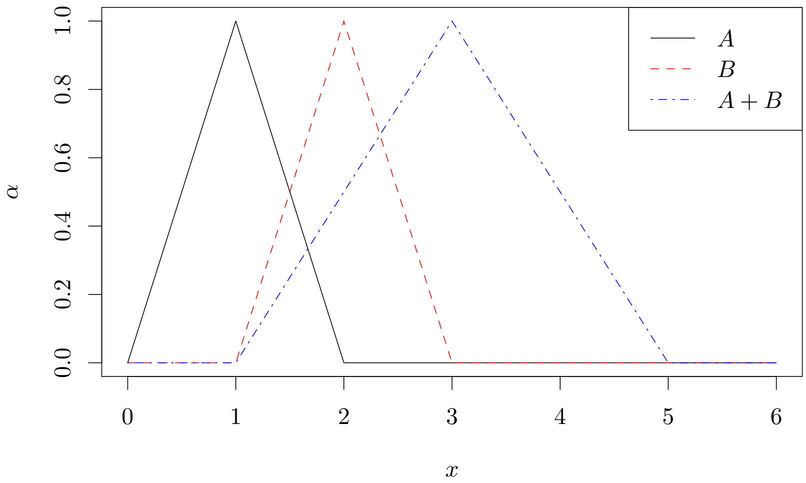
see [7, 6]. Moreover, for $\lambda \in \mathbb{R}$, the scalar multiplication is given by:

$$(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} \left[\lambda A_L(\alpha), \lambda A_U(\alpha)\right], & \text{if } \lambda \geq 0, \\ \left[\lambda A_U(\alpha), \lambda A_L(\alpha)\right], & \text{if } \lambda < 0, \end{cases}$$

for each $\alpha \in [0,1]$.

In the FuzzyNumbers package we have defined the `+`, `-`, `*` and `/` operators, which implements the basic arithmetic operations as defined in [13].

```
A <- TrapezoidalFuzzyNumber(0, 1, 1, 2)
B <- TrapezoidalFuzzyNumber(1, 2, 2, 3)
plot(A, xlim=c(0,6), xlab='$x$', ylab='$\\alpha$')
plot(B, add=TRUE, col=2, lty=2)
plot(A+B, add=TRUE, col=4, lty=4)
legend('topright', c('$A$', '$B$', '$A+B$'), lty=c(1,2,4), col=c(1,2,4))
```



Currently all the operations are available for piecewise linear FNs only, and addition and scalar multiplication is also implemented for trapezoidal FNs. Note that the computer arithmetic has anyway a discrete nature, and a PLFN with large number of knots often approximates (cf. Sec. 6) an arbitrary FN sufficiently well. The computations are always exact (well, up to the computer floating-point arithmetic errors) at knots.

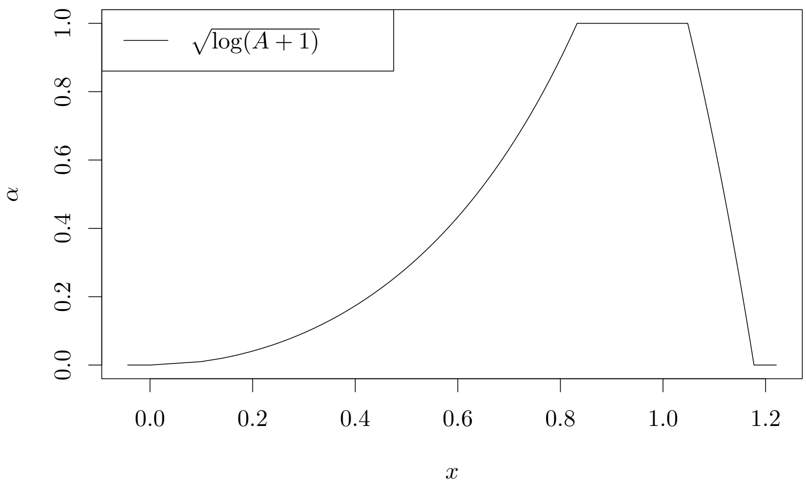
In theory the class of PLFNs is not closed under the operations `*` and `/`. However, if you operate on a large number of knots, the results should be satisfactory.

```
A <- piecewiseLinearApproximation(PowerFuzzyNumber(1,2,3,4,p.left=2,p.right=0.5),
   method="Naive", knot.n=20)
B <- piecewiseLinearApproximation(PowerFuzzyNumber(2,3,4,5,p.left=0.1,p.right=3),
   method="Naive", knot.n=40)
A+A # the same as 2*A

## Piecewise linear fuzzy number with 20 knot(s),
##     support=[2,8],
##        core=[4,6].

A+B # note the number of knots has increased

## Piecewise linear fuzzy number with 60 knot(s),
##     support=[3,9],
##        core=[5,7].
```

5.2 Applying Functions

To apply a monotonic transformation on a piecewise linear fuzzy number (using the extension principle) we call `fapply()`.

```
A <- as.PiecewiseLinearFuzzyNumber(TrapezoidalFuzzyNumber(0,1,2,3), knot.n=100)
plot(fapply(A, function(x) log(x+1)^0.5), xlab='$x$', ylab='$\\alpha$')
legend('topleft', '$\\sqrt{\\log(A+1)}$', lty=1)
```



The operation being applied should be a properly vectorized R function object.

6 Approximation of Fuzzy Numbers

6.1 Metrics in the Space of Fuzzy Numbers

It seems that the most suitable metric for approximation problems is an extension of the Euclidean (L_2) distance (cf. [10]), d, defined by the equation:

$$d^2(A,B) = \int_0^1 (A_L(\alpha) - B_L(\alpha))^2 \, d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 \, d\alpha. \tag{20}$$

The following metric `type`s are currently available in the `distance()` method: `"Euclidean"` (default), `"EuclideanSquared"`.

```
T1 <- TrapezoidalFuzzyNumber(-5, 3, 6, 20)
T2 <- TrapezoidalFuzzyNumber(-4, 4, 7, 21)
distance(T1, T2, type='Euclidean') # L2 distance /default/

## [1] 1.414214

distance(T1, T2, type='EuclideanSquared') # Squared L2 distance

## [1] 2
```

6.2 Approximation by Trapezoidal Fuzzy Numbers

TO BE DONE... Problem statement... Given a fuzzy number A we seek for a trapezoidal fuzzy number $\mathcal{T}(A)$....

In this subsection we will use the following fuzzy number A for the sake of illustration:

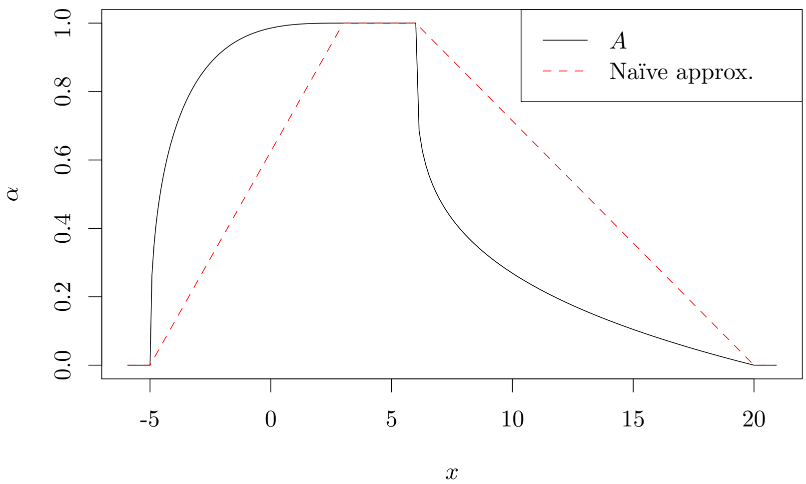
```
A <- FuzzyNumber(-5, 3, 6, 20,
   left=function(x) pbeta(x,0.4,3),
   right=function(x) 1-x^(1/4),
```

```
   lower=function(alpha) qbeta(alpha,0.4,3),
   upper=function(alpha) (1-alpha)^4
)
```

6.2.1 Naïve Approximation

The "Naive" method generates a trapezoidal FN with the same core and support as A.

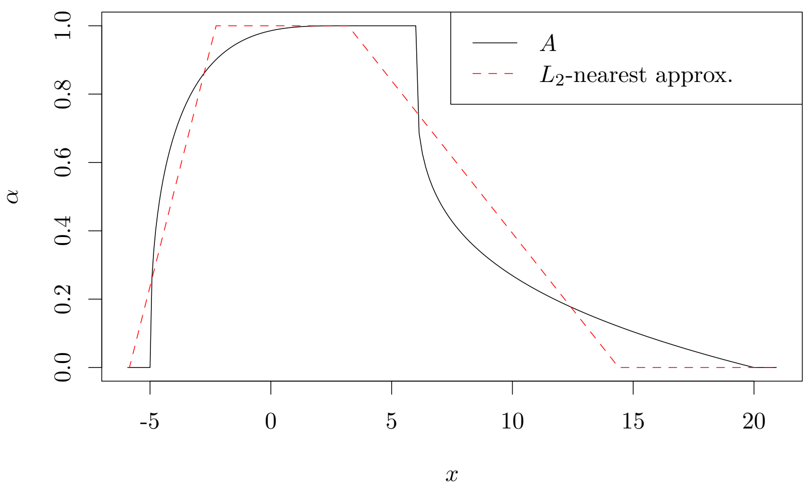
```
(T1 <- trapezoidalApproximation(A, method='Naive'))
## Trapezoidal fuzzy number with:
##    support=[-5,20],
##       core=[3,6].
distance(A, T1)
## [1] 5.761482
```



6.2.2 L_2-nearest Approximation

The "NearestEuclidean" method gives the nearest L_2-approximation of A [2, Corollary 8].

```
(T2 <- trapezoidalApproximation(A, method='NearestEuclidean'))
## Trapezoidal fuzzy number with:
##    support=[-5.85235,14.4],
##       core=[-2.26529,3.2].
distance(A, T2)
## [1] 1.98043
```



6.2.3 Expected Interval Preserving Approximation

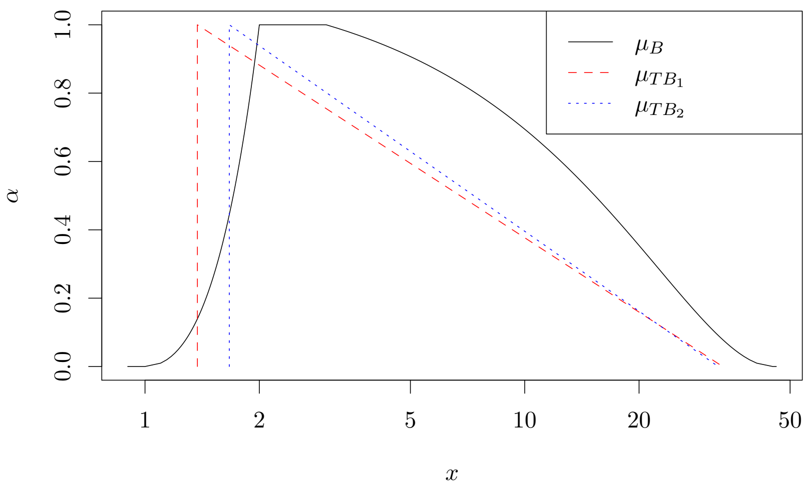
The "ExpectedIntervalPreserving" method gives the nearest L_2-approximation of A preserving the expected interval [1, 11, 15]. Note that if $\mathrm{amb}(A) \geq \mathrm{width}(A)/3$, then we get the same result as in the "NearestEuclidean" method.

```
(T3 <- trapezoidalApproximation(A, method='ExpectedIntervalPreserving'))
## Trapezoidal fuzzy number with:
##    support=[-5.85235,14.4],
##       core=[-2.26529,3.2].
distance(A, T3)
## [1] 1.98043
expectedInterval(A)
## [1] -4.058824  8.800000
expectedInterval(T3)
## [1] -4.058824  8.800000
```

Unfortunately, for highly skewed membership functions this method (as well as the previous one) reveals sometimes quite unfavorable behavior. E.g. if B is a FN such that $\mathrm{Val}(B) < \mathrm{EV}_{1/3}(B)$ or $\mathrm{Val}(B) > \mathrm{EV}_{2/3}(B)$, then it may happen that the core of the output and the core of the original fuzzy number B are disjoint, cf. [12].

```
(B  <- FuzzyNumber(1, 2, 3, 45,
   lower=function(x) sqrt(x),
   upper=function(x) 1-sqrt(x)))
## Fuzzy number with:
##    support=[1,45],
##       core=[2,3].
(TB1 <- trapezoidalApproximation(B, 'NearestEuclidean'))
## Trapezoidal fuzzy number with:
##    support=[1.37333,33.2133],
##       core=[1.37333,1.37333].
```

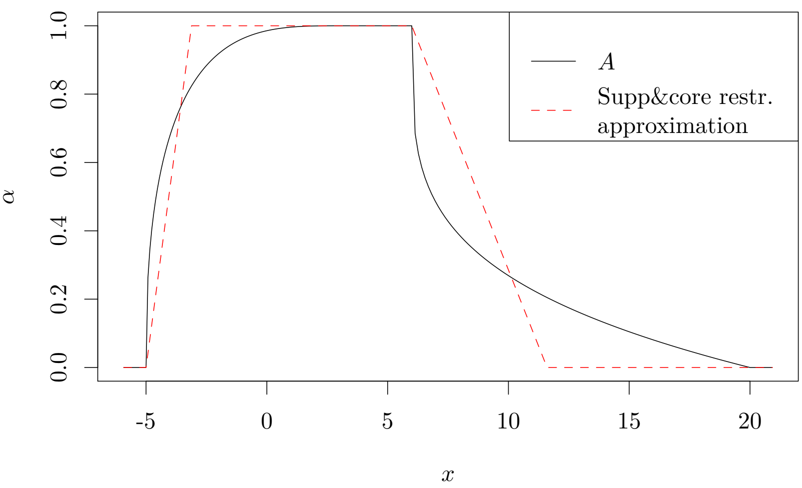
```
(TB2 <- trapezoidalApproximation(B, 'ExpectedIntervalPreserving'))

## Trapezoidal fuzzy number with:
##    support=[1.66667,32.3333],
##       core=[1.66667,1.66667].

distance(B, TB1)

## [1] 2.098994

distance(B, TB2)

## [1] 2.166239
```



6.2.4 Approximation with Restrictions on Support and Core

The "SupportCoreRestricted" method was proposed in [12]. It gives the L_2-nearest trapezoidal approximation with constraints $\mathrm{core}(A) \subseteq \mathrm{core}(\mathcal{T}(A))$ and $\mathrm{supp}(\mathcal{T}(A)) \subseteq \mathrm{supp}(A)$, i.e. for which each point that surely belongs to A also belongs to $\mathcal{T}(A)$, and each point that surely does not belong to A also does not belong to $\mathcal{T}(A)$.

```
(T4 <- trapezoidalApproximation(A, method='SupportCoreRestricted'))

## Trapezoidal fuzzy number with:
##    support=[-5,11.6],
##       core=[-3.11765,6].

distance(A, T4)

## [1] 2.603383
```



6.3 Approximation by Piecewise Linear Fuzzy Numbers

TO BE DONE... Problem statement... Given a fuzzy number A and fixed `knot.alpha` we seek for a piecewise linear fuzzy number $\mathcal{P}(A)$....

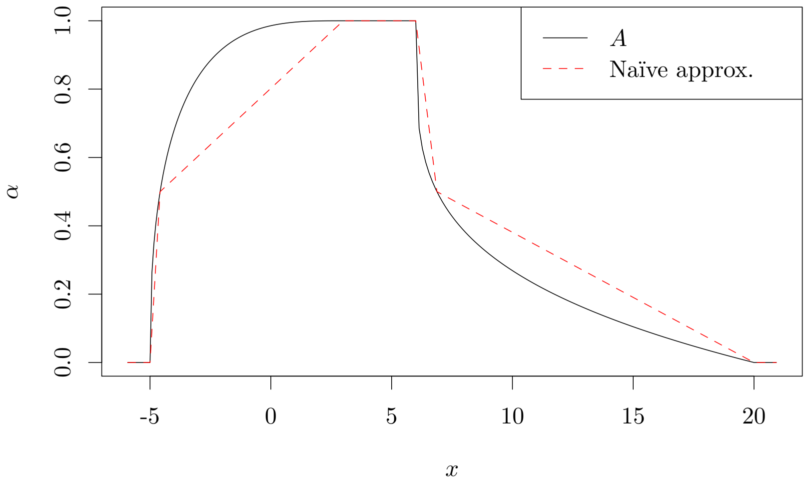
In this subsection we will use the following fuzzy number A for the sake of illustration:

```
A <- FuzzyNumber(-5, 3, 6, 20,
   left=function(x) pbeta(x,0.4,3),
   right=function(x) 1-x^(1/4),
   lower=function(alpha) qbeta(alpha,0.4,3),
   upper=function(alpha) (1-alpha)^4
)
```

6.3.1 Naïve Approximation

The `"Naive"` method generates a PLFN with the same core and support as A and with sides interpolating the membership function of A at given α-cuts.

```
P1 <- piecewiseLinearApproximation(A, method='Naive',
         knot.n=1, knot.alpha=0.5)
P1['allknots']

##         alpha      left  right
## supp      0.0 -5.000000 20.000
## knot_1    0.5 -4.583591  6.875
## core      1.0  3.000000  6.000

print(distance(A, P1), 8)

## [1] 2.4753305
```



The approximation error may be quite high. However, it may be shown that e.g. for equidistant knots if `knot.n` $\to \infty$, then it approaches 0.

6.3.2 L_2-nearest Approximation

Exact algorithm for fixed `knot.alpha`. TO BE DONE...

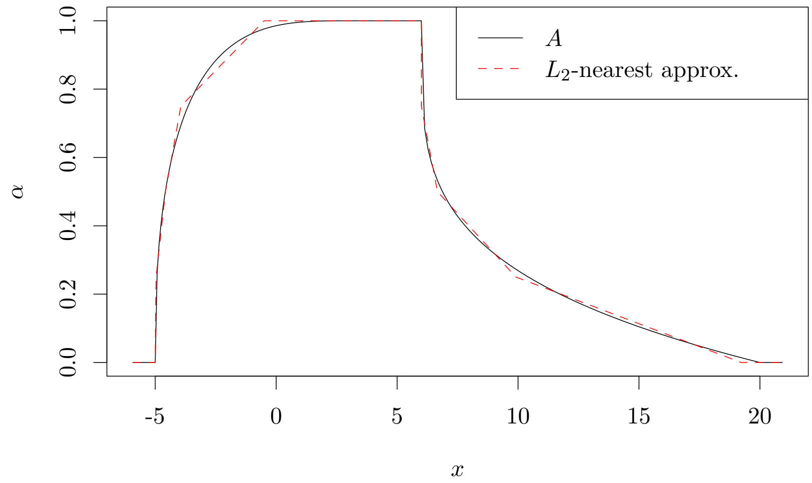
For `knot.n==1` the method proposed in [4] is used. [WORK ON `knot.n>1` ALMOST FINISHED]

```
P2 <- piecewiseLinearApproximation(A,
   method='NearestEuclidean', knot.n=3, knot.alpha=c(0.25,0.5,0.75))
print(P2['allknots'], 6)
```

```
##         alpha      left    right
## supp     0.00 -5.003841 19.22964
## knot_1   0.25 -4.966165  9.91416
## knot_2   0.50 -4.578596  6.66686
## knot_3   0.75 -3.941608  6.00278
## core     1.00 -0.494012  6.00278
```

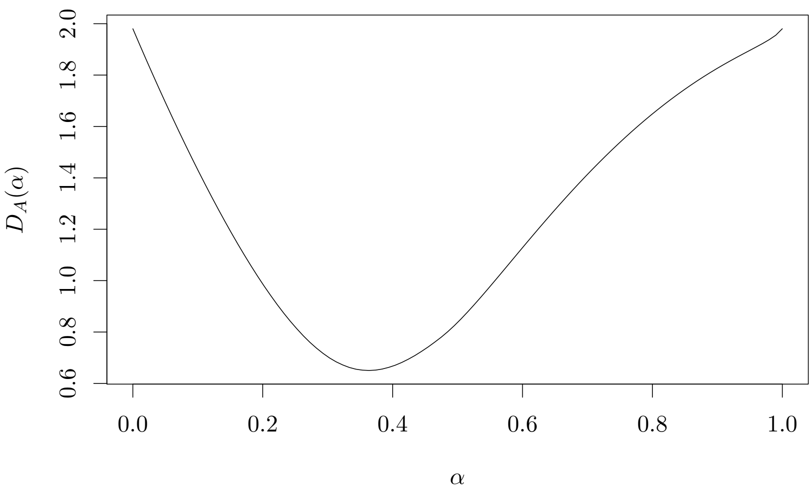
```
print(distance(A, P2), 12)
```

```
## [1] 0.288979920511
```



Finding best `knot.alpha` for `knot.n` = 1 numerically. Let us depict the "best" L_2 distance as a function of α, i.e. the $D_A(\alpha)$ function.

```
a <- seq(1e-9, 1-1e-9, length.out=100) # many alphas from (0,1)
d <- numeric(length(a)) # distances /to be calculated/
for (i in 1:length(a))
{
   P1 <- piecewiseLinearApproximation(A, method='NearestEuclidean',
            knot.n=1, knot.alpha=a[i])
   d[i] <- distance(A, P1)
}
plot(a, d, type='l', xlab=expression(alpha), ylab=expression(D[A](alpha)))
```



For `knot.n` $= 1$ we may find best `knot.alpha` using numerical optimization. It may be shown, see [4], that the distance function $D_A(\alpha)$ is continuous, but in general the minimum is not necessarily unique.

```
for (i in 1:5) # 5 iterations
{
   a0 <- runif(1,0,1) # random starting point
   optim(a0,
      function(a)
      {
         P1 <- piecewiseLinearApproximation(A, method='NearestEuclidean',
                                             knot.n=1, knot.alpha=a)

         distance(A1, P1)
      }, method='L-BFGS-B', lower=1e-9, upper=1-1e-9) -> res
   cat(sprintf('%.9f %6g *%.9f* %.9f\n', a0, res$counts[1], res$par, res$value))
}
## Error:  L-BFGS-B needs finite values of 'fn'
```

7 NEWS/CHANGELOG

```
                   ** FuzzyNumbers Package NEWS **

****************************************************************************

0.3-0 /under development/

* piecewiseLinearApproximation() - general case (any knot.n)
  for method="NearestEuclidean" now available.
  Thus, method="ApproximateNearestEuclidean" is now deprecated.

* New binary arithmetic operators, especially
   for PiecewiseLinearFuzzyNumbers: +, -, *, /

* New function: fapply() - applies a function on a PLFN
  using the extension principle

* New methods: as.character(); also used by show().
  This function also allows to generate LaTeX code defining the FN
  (toLaTeX arg thanks to Jan Caha).

* as.PiecewiseLinearFuzzyNumber() is now an S4 method,
  and can be called on objects of type numeric, as well as on
  various FNs

* piecewiseLinearApproximation() and as.PiecewiseLinearFuzzyNumber()
  argument `knot.alpha` now defaults to equally distributed knots
  (via given `knot.n`). If `knot.n` is missing, then it is guessed
  from `knot.alpha`.

* PiecewiseLinearFuzzyNumber() now accepts missing `a1`, `a2`, `a3`, `a4`,
  and `knot.left`, `knot.right` of length `knot.n`+2. Moreover, if `knot.n`
  is not given, then it is guessed from length(knot.left).
  If `knot.alpha` is missing, then the knots will be equally distributed
  on the interval [0,1].
```

```
* alphacut() now always returns a named two-column matrix.
  evaluate() returns a named vector.

* Function renamed: convert.side to convertSide, convert.alpha
  to convertAlpha, approx.invert to approxInvert

* Added a call to setGeneric("plot", function(x, y, ...) ...
  to avoid a warning on install

* The FuzzyNumbers Tutorial has been properly included
  as the package's vignette

* Man pages update & cleanup

***************************************************************************

0.2-1 /2012-12-27/

* approx.invert(): a new function to find the numerical
  inverse of a given side/alpha-cut generating function
  (by default via Hermite monotonic spline interpolation)

* convert.side(), convert.alpha():
  new functions to convert sides and alpha cuts
  to side generating funs and alpha cut generators

* FuzzyNumber class validity check for lower, upper, left, right:
   * checks whether each function is properly vectorized
     and gives numeric results
   * does not check for the number of formal arguments,
     but just uses the first from the list

* suggests `testthat`

* each object has been documented

* first CRAN release

***************************************************************************

0.1-1 /2012-07-01/

* initial release

***************************************************************************
```

Acknowledgments. This document has been generated with LaTeX, knitr and the tikzDevice package for R. Their authors' wonderful work is fully appreciated. Many thanks to Jan Caha for contributions to the package's source code, and also to Przemysław Grzegorzewski, Lucian Coroianu and Pablo Villacorta Iglesias for stimulating discussion.

The contribution of Marek Gagolewski was partially supported by the European Union from resources of the European Social Fund, Project PO KL "Information technologies: Research and their interdisciplinary applications", agreement UDA-POKL.04.01.01-00-051/10-00 (March-June 2013), and by FNP START Scholarship from the Foundation for Polish Science (2013).

Bibliography

[1] Ban, A. Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected interval. *Fuzzy Sets and Systems 159* (2008), 1327–1344.

[2] Ban, A. On the nearest parametric approximation of a fuzzy number – revisited. *Fuzzy Sets and Systems 160* (2009), 3027–3047.

[3] Chanas, S. On the interval approximation of a fuzzy number. *Fuzzy Sets and Systems 122* (2001), 353–356.

[4] Coroianu, L., Gagolewski, M., and Grzegorzewski, P. Nearest piecewise linear approximation of fuzzy numbers. *Fuzzy Sets and Systems* (2013). doi:10.1016/j.fss.2013.02.005.

[5] Delgado, M., Vila, M., and Voxman, W. On a canonical representation of a fuzzy number. *Fuzzy Sets and Systems 93* (1998), 125–135.

[6] Diamond, P., and Kloeden, P. *Metric Spaces of Fuzzy Sets. Theory and Applications.* World Scientific, Singapore, 1994.

[7] Dubois, D., and Prade, H. Operations on fuzzy numbers. *Int. J. Syst. Sci. 9* (1978), 613–626.

[8] Dubois, D., and Prade, H. The mean value of a fuzzy number. *Fuzzy Sets and Systems 24* (1987), 279–300.

[9] Gagolewski, M. *FuzzyNumbers: Tools to deal with fuzzy numbers in R*, 2013. www.rexamine.com/resources/fuzzynumbers/.

[10] Grzegorzewski, P. Metrics and orders in space of fuzzy numbers. *Fuzzy Sets and Systems 97* (1998), 83–94.

[11] Grzegorzewski, P. Algorithms for trapezoidal approximations of fuzzy numbers preserving the expected interval. In *Foundations of Reasoning Under Uncertainty* (2010), B.-M. B. et al, Ed., Springer, pp. 85–98.

[12] Grzegorzewski, P., and Pasternak-Winiarska, K. Trapezoidal approximations of fuzzy numbers with restrictions on the support and core. In *Proc. EUSFLAT/LFA 2011* (2011), Atlantic Press, pp. 749–756.

[13] Klir, G., and Yuan, B. *Fuzzy sets and fuzzy logic. Theory and applications.* Prentice Hall, New Jersey, 1995.

[14] Stefanini, L., and Sorini, L. Fuzzy arithmetic with parametric lr fuzzy numbers. In *Proc. IFSA/EUSFLAT 2009* (2009), pp. 600–605.

[15] Yeh, C.-T. Trapezoidal and triangular approximations preserving the expected interval. *Fuzzy Sets and Systems 159* (2008), 1345–1353.