This tutorial reflects the state of the most recent development version of agop available on *GitHub*.
If you use the “official” CRAN release, some of the features may be unavailable.

A Guide to the agop 0.1-0 Package for R

Aggregation Operators in R

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The package, as well as this tutorial, is still in its early days – any suggestions are welcome!

Contents

1 Getting started

> *“The process of combining several numerical values into a single representative one is called* ***aggregation****, and the numerical function performing this process is called* ***aggregation function****. This simple definition demonstrates the size of the field of application of aggregation: applied mathematics (e.g. probability, statistics, decision theory), computer science (e.g. artificial intelligence, operation research), as well as*

> *many applied fields (economics and finance, pattern recognition and image processing, data fusion, multicriteria decision making, automated reasoning etc.). Although history of aggregation is probably as old as mathematics (think of the arithmetic mean), its existence has reminded underground till only recent (...).” [15, p. xiii]*

R [21] is a free, open source software environment for statistical computing and graphics, which includes an implementation of a very powerful and quite popular high-level language called S. It runs on all major operating systems, i.e. Windows, Linux, and MacOS X. To install R and/or find some information on the S language please visit R Project’s Homepage at www.R-project.org. Perhaps you may also wish to install RStudio, a convenient development environment for R. It is available at www.rsudio.org.

agop is an open source (licensed under GNU LGPL 3) package for R $\geq$ 2.12 to which anyone can contribute. It started as a fork of the CITAN (*Citation Analysis Toolpack*, [8]) package.

To install latest “official” release of the package available on *CRAN* we type[1]:

```
install.packages('agop')
```

Alternatively, we may fetch its current development snapshot from *GitHub*:

```
install.packages('devtools')
library('devtools')
install_github('agop', 'Rexamine')
```

Note that in this case you will need a working C/C++ compiler.

Each session with agop should be preceded by a call to:

```
library('agop') # Load the package
```

To view the main page of the manual we type:

```
library(help='agop')
```

For more information please visit the package’s homepage [11]. In case of any problems, comments, or suggestions feel free to contact the authors. Good luck!

2 Theoretical Background

Let us establish some basic notation convention used throughout this tutorial. From now on let $\mathbb{I} = [a, b]$, possibly with $a = -\infty$ or $b = \infty$. Note that in many practical situations we choose $\mathbb{I} = [0, 1]$ or $\mathbb{I} = [0, \infty]$. A set of all vectors of arbitrary length with elements in $\mathbb{I}$ is denoted by $\mathbb{I}^{1,2,\dots} = \bigcup_{n=1}^{\infty} \mathbb{I}^n$.

For two equal-length vectors $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ we write $\mathbf{x} \leq \mathbf{y}$ if and only if for all $i = 1, \dots, n$ it holds $x_i \leq y_i$. Moreover, all binary arithmetic operations on vectors $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ will be performed element-wise, e.g. $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n) \in \mathbb{I}^n$. Similar behavior is assumed for $-$, $\cdot$, $/$, $\wedge$ (min), $\vee$ (max), etc. Additionally, each function of one variable $\mathsf{f} : \mathbb{I} \to \mathbb{I}$ can be extended to the vector space: we write $\mathsf{f}(\mathbf{x})$ to denote $(\mathsf{f}(x_1), \dots, \mathsf{f}(x_n))$.

[1] You are viewing the **development** version of the tutorial. Some of the features presented in this document may be missing in the CRAN release. Please, upgrade to the **latest** development version from *GitHub* if you need the new functionality. Note that you will need a working C/C++ compiler.

Let $x_{(i)}$ denote the ith order statistic, i.e. the ith smallest value in $\mathbf{x}$. Moreover, for convenience, let $x_{\{i\}} = x_{|\mathbf{x}|-i+1}$ denote the ith greatest value in $\mathbf{x}$.

For any $n \in \mathbb{N}$ and $c \in \mathbb{I}$, we set $(n * c) = (c, \ldots, c) \in \mathbb{I}^n$.

2.1 A Note on Representing Numeric Data and Applying Operations in R

Recall how we create numeric vectors in R:

```
(x1 <- c(5, 2, 3, 1, 0, 0))

## [1] 5 2 3 1 0 0

class(x1)

## [1] "numeric"

(x2 <- 10:1) # the same as seq(10, 1)

##  [1] 10  9  8  7  6  5  4  3  2  1

(x3 <- seq(1, 5, length.out=6))

## [1] 1.0 1.8 2.6 3.4 4.2 5.0

(x4 <- seq(1, 5, by=1.25))

## [1] 1.00 2.25 3.50 4.75
```

To obtain $(n * c)$, e.g. for $n = 10$ and $c = 3$, we call:

```
rep(10, 3)

## [1] 10 10 10
```

Note that in R all the arithmetic operations on vectors are performed element-wise, i.e. in a manner indicated above. This is called ***vectorization***. The same holds for mathematical functions: they are extended to the vector space.

```
x <- c(1, 3,  3, 2)
y <- c(2, 3, -1, 0)
x+y

## [1] 3 6 2 2

x*y

## [1]  2  9 -3  0

pmin(x,y)

## [1]  1  3 -1  0

pmax(x,y)

## [1] 2 3 3 2

abs(y)

## [1] 2 3 1 0
```

Thus, we calculated $\mathbf{x} + \mathbf{y}$, $\mathbf{x} \cdot \mathbf{y}$, $\mathbf{x} \wedge \mathbf{y}$, $\mathbf{x} \vee \mathbf{y}$, and $|\mathbf{x}|$ (try to determine yourself what happens if we deal with vectors of unequal length is R).

Moreover, for the $\leq$ relation we write:

```
all(x <= y)

## [1] FALSE
```

To get $x_{\{i\}}$ we have to sort the given vector nonincreasingly.

```
(xs <- sort(x, decreasing=TRUE)) # `decresing' may be misleading

## [1] 3 3 2 1

xs[3] # the third greatest value in x

## [1] 2
```

2.2 Aggregation Operators and Their Basic Properties

Dealing with huge amounts of data faces us with the problem of constructing their synthetic descriptions. The aggregation theory, a relatively new research domain at the border of mathematics and computer science, is interested in the analysis of functions that may be used in this task. Thus, we should start with the formal definition of objects of our interest. Here is the most general setting:

Definition 1. A function $\mathsf{F} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}$ is called an ***(extended) aggregation operator*** if it is at least ***nondecreasing*** in each variable, i.e. for all n and $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ if $\mathbf{x} \le \mathbf{y}$, then $\mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y})$.

Note that each aggregation operator is a mapping into $\mathbb{I}$, thus for all n we have $\inf_{\mathbf{x}\in\mathbb{I}^n} \mathsf{F}(\mathbf{x}) \ge a$ and $\sup_{\mathbf{x}\in\mathbb{I}^n} \mathsf{F}(\mathbf{x}) \le b$. By nondecreasingness, however, these conditions reduce to $\mathsf{F}(n * a) \ge a$ and $\mathsf{F}(n * b) \le b$.

Also keep in mind that some authors assume (cf. [15]) that aggregation operators must fulfill the two following ***strong boundary conditions***: for all n we have $\inf_{\mathbf{x}\in\mathbb{I}^n} \mathsf{F}(\mathbf{x}) = a$ and $\sup_{\mathbf{x}\in\mathbb{I}^n} \mathsf{F}(\mathbf{x}) = b$. In our case, this does not necessarily hold – we want to be more general.

Here are some interesting properties of aggregation operators. Later on we will characterize the classes of aggregation operators that fulfill them.

Definition 2. We call $\mathsf{F} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}$ ***symmetric*** if:

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \cong \mathbf{y} \Longrightarrow \mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{y}),$$

where $\mathbf{x} \cong \mathbf{y}$ if and only if there exists a permutation σ of $[n] := \{1, 2, \dots, n\}$ such that $\mathbf{x} = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$.

It may be shown, see [15, Thm. 2.34], that $\mathsf{F} : \mathbb{I}^n \to \mathbb{I}$ is symmetric if and only if there exists a function $\mathsf{G} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}$ such that $\mathsf{F}(x_1, \dots, x_n) = \mathsf{G}(x_{\{1\}}, \dots, x_{\{n\}})$, i.e. it may be defined solely using order statistics. Intuitively, F is symmetric if its value is independent of the aggregated vector's elements presentation.

Idempotence is well-known from algebra, where we say that element x is idempotent with respect to binary operator $*$ if we have $x * x = x$. The following definition extends this property to n-ary aggregation functions, cf. [15].

Definition 3. We call $\mathsf{F} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}$ ***idempotent*** if:

$$(\forall n \in \mathbb{N})\ (\forall x \in \mathbb{I})\ \mathsf{F}(n * x) = x.$$

Idempotent aggregation operators fulfilling the strong boundary conditions (see p. 4) are sometimes called ***averaging functions***, cf. [**?**]. An example of such object is the arithmetic mean or median.

Definition 4. We call $\mathsf{F} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}$ ***additive*** if:

$$\mathsf{F}(\mathbf{x}+\mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y}),$$

for all $n \in \mathbb{N}, \mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ such that $\mathbf{x}+\mathbf{y} \in \mathbb{I}^n$.

Definition 5. We call F ***minitive*** if:

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathsf{F}(\mathbf{x} \wedge \mathbf{y}) = \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y}).$$

Definition 6. We call F ***maxitive*** if:

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathsf{F}(\mathbf{x} \vee \mathbf{y}) = \mathsf{F}(\mathbf{x}) \vee \mathsf{F}(\mathbf{y}).$$

Definition 7. We call F ***modular*** if:

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathsf{F}(\mathbf{x} \vee \mathbf{y}) + \mathsf{F}(\mathbf{x} \wedge \mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y}).$$

It may easily be seen that each additive operator fulfiling the lower strong boundary condition is also modular (i.e. modularity is more general than additivity),

Apart from the "ordinary" minitivity, maxitivity, and modularity we may introduce their symmetrized versions, using $\mathbf{x} \overset{S}{+} \mathbf{y} = (x_{\{1\}}+y_{\{1\}}, \dots, x_{\{n\}}+y_{\{n\}})$, $\mathbf{x} \overset{S}{\vee} \mathbf{y} = (x_{\{1\}} \vee y_{\{1\}}, \dots, x_{\{n\}} \vee y_{\{n\}})$ and $\mathbf{x} \overset{S}{\wedge} \mathbf{y} = (x_{\{1\}} \wedge y_{\{1\}}, \dots, x_{\{n\}} \wedge y_{\{n\}})$.

2.3 Impact Functions and The Producers Assessment Problem

We already noticed the important class of aggregation operators: the averaging functions. They may be used to represent the most "typical" value of a numeric vector. Here is another interesting class that represent solutions to some very interesting practical issues.

The ***Producers Assessment Problem*** (PAP, [14]) concerns evaluation of a set of ***producers*** (e.g. scientists, artists, writers, craftsman) according to the ***products*** (e.g. scientific articles, works, books, artifacts) each entity has produced. It is assumed that for each of the products some quality or popularity ***rating*** (e.g. the number of articles' citations, the number of sold book copies etc.) is given.

Tab. 1. The Producer Assessment Problem – typical instances

	Producer	Products	Rating method	Discipline
A	Scientist	Scientific articles	Number of citations	Scientometrics
B	Scientific institute	Scientists	The h-index	Scientometrics
C	Web server	Web pages	Number of in-links	Webometrics
D	Artist	Paintings	Auction price	Auctions
E	Billboard company	Advertisements	Sale results	Marketing
F	R package author	Packages	PageRank values on the citation graph	Software Engineering

PAP instances may be found in many real-life situations, like those encountered for example scientometrics, webometrics, marketing, manufacturing, or quality engineering, see Table 1 and e.g. [5]. Our main interest here is focused on constructing and analyzing aggregation operators which may be used in the producers' rating task. Such functions should take into a count the two following aspects of a producer's quality:

- his/her ability to output highly-rated products,
- his/her overall productivity.

For the sake of illustration, we will consider PAP in the scientometric context, where scientists "produce" papers, which in turn themselves receive citations.

Let $\mathbb{I} = [0, \infty]$ represent the set of values that some a priori chosen paper quality measure may take. These may of course be non-integers, for example when we consider citations normalized with respect to the number of papers' authors.

It is widely accepted, see e.g. [29, 28, 27, 22, 19, 20, 14, 7, 6], that each aggregation operator $\mathsf{F} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}$ to be applied in PAP should at least be:

(a) nondecreasing in each variable (additional citations received by a paper or an improvement of its quality measure does not result in a decrease of the authors' overall evaluation),

(b) arity-monotonic (by publishing a new paper we never decrease the overall valuation of the entity),

(c) symmetric (independent of the order of elements' presentation, i.e. we may always assume that we aggregate vectors that are already sorted).

More formally, axiom (b) is fulfilled iff for any $\mathbf{x} \in \mathbb{I}^{1,2,\cdots}$ and $y \in \mathbb{I}$ it holds $\mathsf{F}(\mathbf{x}) \le \mathsf{F}(x_1, \dots, x_n, y)$. It may be seen that this property is ***arity-dependent***, i.e. it takes into account the number of elements to be aggregated.

Moreover, (a) and (c) were defined in the previous section.

3 Predefined Classes of Aggregation Operators in agop

3.1 A Note on Storing Multiple Numeric Vectors in R

Vectors of the same length can be conveniently store in a matrix. Please note that dimnames attribute for the matrix giving the row and column names. It value can be set as NULL (no names for columns and rows are given) or a list of length 2 (rows and columns names respectively). Another simply way to set columns and row names is using `colnames()` and `rownames()` functions.

Function `apply()` evaluate the value of the given function on each row or column (row- or column-wise) of the given matrix (parameter MARGIN equal to 1 and 2 respectively).

```
expertopinions <- matrix(c(
      6,7,2,3,1, # this will be the first COLUMN
      8,3,2,1,9, # 2nd
      4,2,4,1,6  # 3rd
   ),
   ncol=3,
   dimnames=list(NULL, c("A", "B", "C")) # only column names set
)
class(expertopinions)

## [1] "matrix"

print(expertopinions)   # or print(authors)
```

```
##      A B C
## [1,] 6 8 4
## [2,] 7 3 2
## [3,] 2 2 4
## [4,] 3 1 1
## [5,] 1 9 6
```

```
apply(expertopinions, 2, mean) # on each COLUMN apply the mean() function
```

```
##   A   B   C
## 3.8 4.6 3.4
```

Vectors that are not of the same length can be store in the list, with possibly named elements. In that case functionality of `apply()` functions is given by `lapply()` and `sapply()` functions.

```
authors <- list(
   "John S." = c(7,6,2,1,0),
   "Kate F." = c(9,8,7,6,4,1,1,0)
)
class(authors)
```

```
## [1] "list"
```

```
str(authors)   # or print(authors)
```

```
## List of 2
##  $ John S.: num [1:5] 7 6 2 1 0
##  $ Kate F.: num [1:8] 9 8 7 6 4 1 1 0
```

```
index_h(authors[[1]]) # the h-index (see below) for 1st author
```

```
## [1] 2
```

```
sapply(authors, index_h) # calculate the h-index for all vectors in a list
```

```
## John S. Kate F.
##       2       4
```

```
index_h(authors) # index_h() expects an numeric vector on input
```

```
## Error:  argument ‘x‘ should be a numeric vector (or an object coercible to)
```

3.2 Examples of basic aggregation operators

Definition 8. Let $\mathbf{w} = c(w_1, \ldots, w_n) \in [0,1]^n$ be such that $\sum_{i=1}^n w_i = 1$. Then, for any $\mathbf{x} \in \mathbb{I}^n$:

1. The ***weighted arithmetic mean*** $\mathsf{WAM} : \mathbb{I}^n \to \mathbb{I}$ associated with the weight vector $\mathbf{w}$ is defined as
$$\mathsf{WAM}(\mathbf{x}) = \sum_{i=1}^n w_i x_i.$$
2. The ***ordered weighted averaging function*** $\mathsf{OWA} : \mathbb{I}^n \to \mathbb{I}$ associated with the weight vector $\mathbf{w}$ is defined as
$$\mathsf{OWA}(\mathbf{x}) = \sum_{i=1}^n w_i x_{(i)}.$$

Definition 9. Let $\mathbf{w} = c(w_1, \ldots, w_n) \in [0,1]^n$ be such that $\bigvee_{i=1}^n w_i = 1$, and assume $\mathbb{I} = [0,1]^n$. Then, for any $\mathbf{x} \in \mathbb{I}^n$:

1. The ***weighted maximum*** $\mathsf{WMax} : \mathbb{I}^n \to \mathbb{I}$ associated with the weight vector $\mathbf{w}$ is defined as
$$\mathsf{WMax}(\mathbf{x}) = \bigvee_{i=1}^{n} (w_i \wedge x_i).$$
2. The ***ordered weighted maximum*** $\mathsf{OWMax} : \mathbb{I}^n \to \mathbb{I}$ associated with the weight vector $\mathbf{w}$ is defined as
$$\mathsf{OWMax}(\mathbf{x}) = \bigvee_{i=1}^{n} (w_i \wedge x_{(i)}),$$
with $x_{(1)} \leq \cdots \leq x_{(n)}$ and $w_1 \geq w_2 \geq \cdots \geq w_n$.

Definition 10. Let $\mathbf{w} = c(w_1, \ldots, w_n) \in [0,1]^n$ be such that $\bigvee_{i=1}^n w_i = 1$, and assume $\mathbb{I} = [0,1]^n$. Then, for any $\mathbf{x} \in \mathbb{I}^n$:

1. The ***weighted minimum*** $\mathsf{WMin} : \mathbb{I}^n \to \mathbb{I}$ associated with the weight vector $\mathbf{w}$ is defined as
$$\mathsf{WMin}(\mathbf{x}) = \bigwedge_{i=1}^{n} ((1 - w_i) \vee x_i).$$
2. The ***ordered weighted minimum*** $\mathsf{OWMin} : \mathbb{I}^n \to \mathbb{I}$ associated with the weight vector $\mathbf{w}$ is defined as
$$\mathsf{OWMin}(\mathbf{x}) = \bigwedge_{i=1}^{n} ((1 - w_i) \vee x_{(i)}),$$
with $x_{(1)} \leq \cdots \leq x_{(n)}$ and $w_1 \leq w_2 \leq \cdots \leq w_n$.

Definition 11. The ***triangle of coefficients*** is a sequence $\triangle = (c_{i,n} \in \mathbb{I} : i \in [n], n \in \mathbb{N})$.

Definition 12. The ***L-statistic*** for a given triangle of coefficients $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}$ is a function $\mathsf{L}_\triangle : \mathbb{I}^n \to \mathbb{I}$ such that
$$\mathsf{L}_\triangle(\mathbf{x}) = \sum_{i=1}^{n} c_{i,n} x_{(n-i+1)},$$
for any $\mathbf{x} \in \mathbb{I}^n$.

Definition 13. The ***S-statistic*** for a given triangle of coefficients $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}$ is a function $\mathsf{S}_\triangle : \mathbb{I}^n \to \mathbb{I}$ such that
$$\mathsf{S}_\triangle(\mathbf{x}) = \bigvee_{i=1}^{n} c_{i,n} \wedge x_{(n-i+1)},$$
for any $\mathbf{x} \in \mathbb{I}^n$.

Definition 14. The ***I-statistic*** for a given triangle of coefficients $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}$ is a function $\mathsf{I}_\triangle : \mathbb{I}^n \to \mathbb{I}$ such that
$$\mathsf{I}_\triangle(\mathbf{x}) = \bigwedge_{i=1}^{n} c_{i,n} \vee x_{(n-i+1)},$$
for any $\mathbf{x} \in \mathbb{I}^n$.

Please note that the ***OWA*** functions are a special case of ***L-statistics***. Similarly, ***OWMax*** and ***OWMin*** operators are a special cases of ***S-statistics*** and ***I-statistics***, respectively.

Definition 15. The ***triangle of functions*** is a sequence $\triangle = (f_{i,n} \in \mathbb{I}^{\mathbb{I}} : i \in [n], n \in \mathbb{N})$.

Definition 16. The ***quasi-L-statistic*** for a given triangle of functions $\triangle = (f_{i,n})_{i\in[n],n\in\mathbb{N}}$ is a function $\mathsf{qL}_\triangle : \mathbb{I}^n \to \mathbb{I}$ such that

$$\mathsf{qL}_\triangle(\mathbf{x}) = \sum_{i=1}^{n} f_{i,n}(x_{(n-i+1)}),$$

for any $\mathbf{x} \in \mathbb{I}^n$.

Definition 17. The ***quasi-S-statistic*** for a given triangle of functions $\triangle = (f_{i,n})_{i\in[n],n\in\mathbb{N}}$ is a function $\mathsf{qS}_\triangle : \mathbb{I}^n \to \mathbb{I}$ such that

$$\mathsf{qS}_\triangle(\mathbf{x}) = \bigvee_{i=1}^{n} f_{i,n}(x_{(n-i+1)}),$$

for any $\mathbf{x} \in \mathbb{I}^n$.

Definition 18. The ***quasi-I-statistic*** for a given triangle of functions $\triangle = (f_{i,n})_{i\in[n],n\in\mathbb{N}}$ is a function $\mathsf{qI}_\triangle : \mathbb{I}^n \to \mathbb{I}$ such that

$$\mathsf{qI}_\triangle(\mathbf{x}) = \bigwedge_{i=1}^{n} f_{i,n}(x_{(n-i+1)}),$$

for any $\mathbf{x} \in \mathbb{I}^n$.

It is easily seen that ***quasi-L-statistics*** generalize ***L-statistics***. We obtain an ***L-statistic*** by taking $f_{i,n}(x) = c_{i,n}x$. Similarly, by setting $f_{i,n}(x) = x \wedge c_{i,n}$ we obtain ***S-statistic*** from ***quasi-S-statistics***, and by setting $f_{i,n}(x) = x \vee c_{i,n}$ we obtain ***I-statistic*** from ***quasi-I-statistics***.

Let now us introduce the class of symmetric maxitive, minitive and modular aggregation operators introduced in [2].

Definition 19. A sequence of nondecreasing functions $\mathbf{w} = (\mathsf{w}_1, \mathsf{w}_2, \dots)$, $\mathsf{w}_i : \mathbb{I} \to \mathbb{I}$, and a triangle of coefficients $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}$, $c_{i,n} \in \mathbb{I}$ such that $(\forall n)\ c_{1,n} \le c_{2,n} \le \cdots \le c_{n,n}$, $0 \le \mathsf{w}_n(0) \le c_{1,n}$, and $\mathsf{w}_n(b) = c_{n,n}$, generates a nondecreasing ***OM3 operator*** $\mathsf{M}_{\triangle,\mathbf{w}} : \mathbb{I}^n \to \mathbb{I}$ such that for $\mathbf{x} \in \mathbb{I}^n$ we have:

$$\begin{aligned}\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) &= \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n} = \bigwedge_{i=1}^{n} (\mathsf{w}_n(x_{(n-i+1)}) \vee c_{i-1,n}) \wedge c_{n,n} \\ &= \sum_{i=1}^{n} \left(\left(\mathsf{w}_n(x_{(n-i+1)}) \vee c_{i-1,n} \right) \wedge c_{i,n} - c_{i-1,n} \right).\end{aligned}$$

We see that the above contains i.a. all order statistics (whenever $\mathsf{w}_n(x) = x$, and $c_{i,n} = 0$, $c_{j,n} = b$ for $i < k$, $j \ge k$, and some k), OWMax operators (for $\mathsf{w}_n(x) = x$), and the famous Hirsch h-index ($\mathsf{w}_n(x) = \lfloor x \rfloor$, $c_{i,n} = i$).

3.3 Bibliometric Impact Indices

Below we assume that $\mathbb{I} = [0, \infty]$.

The h-index. Given a sequence $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{I}^{1,2,\dots}$, the ***Hirsch index*** [16] of $\mathbf{x}$ is defined as $\mathsf{H}(x) = \max\{i = 1, \dots, n : x_{\{i\}} \ge i\}$ if $n \ge 1$ and $x_{\{1\}} \ge 1$, or $\mathsf{H}(x) = 0$ otherwise. It may be shown that the h-index is an zero-insensitive OM3 aggeration operator, see [9], with:

$$\mathsf{H}(\mathbf{x}) = \bigvee_{i=1,\dots,n}^{n} i \wedge \lfloor x_{\{i\}} \rfloor.$$

Interpretation: "an author has h-index of H if H of his/her n most cited papers have at least H citations each, and the other $n-H$ papers are cited no more that H times each". The h-index may also be expressed as a Sugeno integral [24] w.r.t. to a counting measure, cf. [25]. agop implementation: `index_h()`.

```
index_h(c(6,5,4,2,1,0,0,0,0,0,0))

## [1] 3

index_h(c(-1,3,4,2)) # only for x>=0

## Error:  all elements in ‘x‘ should be in [0,Inf]
```

Hirsch index is a zero-insensitive impact function, denoted $\mathsf{H} \in \mathcal{P}_{(\mathrm{a0})}$, which means that for each $\mathbf{x} \in \mathbb{I}^{1,2,\dots}$ it holds $\mathsf{H}(\mathbf{x}, 0) = \mathsf{H}(\mathbf{x})$.

We have $\mathsf{H}(\mathbf{x}) \le \min\{n, x_1\}$.

The g-index. Egghe's g-index [4]: $\mathsf{G}(\mathbf{x}) = \max\{g = 1, \dots, n : \sum_{i=1}^{g} x_{\{g\}} \ge g^2\}$, available in agop as `index_g()`. We have $\mathsf{G}(\mathbf{x}) \ge \mathsf{H}(\mathbf{x})$ with $\mathsf{G}(n * n) = \mathsf{H}(n * n) = n$

Note that this aggregation operator is not zero-insensitive, for example $\mathsf{G}(9,0) = 2$ and $\mathsf{G}(9,0,0) = 3$. Thus, we also provide the `index_g_zi()` function, which treats $\mathbf{x}$ as it would be padded with 0s.

```
index_g(9)

## [1] 1

index_g(c(9,0,0))

## [1] 3

index_g_zi(9)

## [1] 3
```

The index is interesting from the computational point of view – it may be calculated on the nondecreasing vector of cumulative sums, `cumsum(sort(x, decreasing=TRUE))`, however, it cannot be expressed as a symmetric maxitive aggregation operator.

Interestingly, it might be shown that if $\mathbf{x}$ is sorted nondecreasingly, then:

$$\mathsf{G}(\mathbf{x}) = \mathsf{H}(\mathbf{x})(0 \vee \mathtt{cummin}(\mathtt{cumsum}(x) - (1:n)^2 + (1:n))),$$

where $1:n = (1,2,3,\dots,n)$.

The w-index. The w-index (cf. [29]) defined as

$$\mathsf{W}(\mathbf{x}) = \max\left\{w = 0,1,2,\dots : x_{\{i\}} \ge w - i + 1, i = 1,\dots,w\right\} \tag{1}$$

is available in agop as `index_w()`.

Interestingly, it might be shown that if $\mathbf{x}$ is sorted nondecreasingly, then:

$$\mathsf{W}(\mathbf{x}) = \mathsf{H}(\mathbf{x})(\mathtt{cummin}(\mathbf{x} + (1:n) - 1)).$$

Thus, it is easily seen that this is a zero-insensitive impact function. What is more we have $\mathsf{H}(\mathbf{x}) \le \mathsf{W}(\mathbf{x}) \le 2\mathsf{H}(\mathbf{x})$ and $\mathsf{W}(\mathbf{x}) \le \min\{n, x_1\}$.

The r_p-indices. The r_p-index, for $p \geqslant 1$ is expressed as

$$\mathsf{r}_p(\mathbf{x}) = \sup\{r > 0 : \mathsf{s}^{p,r} \trianglelefteq \mathbf{x}\},$$

where $\mathbf{s}^{p,r} = \left(\sqrt[p]{r^p - 0^p}, \sqrt[p]{r^p - 1^p}, \ldots, \sqrt[p]{r^p - \lfloor r \rfloor^p}\right)$. For more details see [7, 12].

Please note that for integer vectors we have $r_1 = \mathsf{W}$ and $r_\infty = \mathsf{H}$ (cf. [12]). Hence it easily seen that, this is a zero-insensitive impact function.

The l_p-indices. The l_p-index (cf. [7, 12]) for $p \in [1, \infty)$, $u > 0$ and $v > 0$ is a function $\mathsf{l}_p : \mathbb{I}^{1,2,\ldots} \to \mathbb{I}^2$ given by the equation

$$\mathsf{l}_p(\mathbf{x}) = \mathop{\mathsf{arg\,sup}}_{(u,v)} \{uv : \mathsf{e}^{p,u,v} \trianglelefteq \mathbf{x}\},$$

where $\mathsf{e}^{p,u,v} = \left(\sqrt[p]{v^p - (\frac{v}{u}0)^p}, \sqrt[p]{v^p - (\frac{v}{u}1)^p}, \ldots, \sqrt[p]{v^p - (\frac{v}{u}\lfloor u \rfloor)^p}\right)$.

The MAXPROD**-index.** The MaxProd-index (cf. [18]) given by the equation

$$\mathsf{MP}(\mathbf{x}) = \max\left\{i \cdot x_{\{i\}} : i = 1, 2, \ldots\right\} \tag{2}$$

is another example of zero-insensitive impact function. Interestingly, this index is a particular case of a projected l_∞-index, see [12], and can be also expressed in terms of Shilkret integral [23].

In agop MaxProd-index is implemented as `index_maxprod()`.

Simple transformations of the h-index. For example, The $h(2)$-index [17]:

$$\mathsf{H2}(\mathbf{x}) = \max\left\{h = 0, 1, 2, \ldots : x_h \geq h^2\right\}. \tag{3}$$

Note that the $h(2)$-index is one of the many examples of very simple, direct modifications of the h-index. Many authors considered other settings than "h^2" on the right side of (3), e.g. "$2h$", "αh" for some $\alpha > 0$, or "h^β", $\beta \geq 1$, cf. [1].

It may easily be shown that these reduce to the h-index for properly transformed input vectors.....

4 Visualization

4.1 Depicting producers

The `plot_producer()` function may be used to draw a graphical representation of a given numeric vector, i.e. what is sometimes called a citation function in scientometrics.

A given vector $\mathbf{x} = (x_1, \ldots, x_n)$ can be represented by a step function defined for $0 \leq y < n$ and given by:

$$\pi(y) = x_{(n - \lfloor y+1 \rfloor + 1)}.$$

This function may be obtained by setting `type == 'right.continuous'` argument in `plot_producer()`. Recall that $x_{(i)}$ denotes i-th smallest value in $\mathbf{x}$.

On the other hand, for `type == 'left.continuous'` (the default), we get

$$\pi(y) = x_{(n - \lfloor y \rfloor + 1)}$$

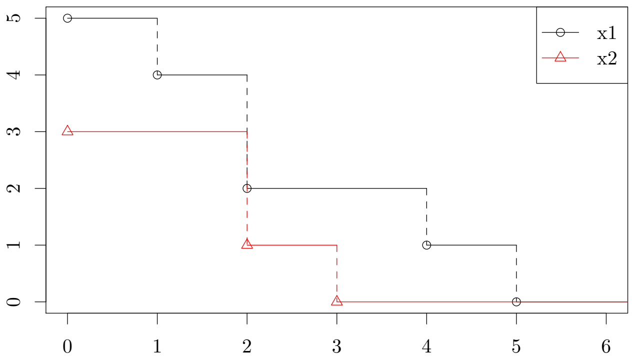
for $0 < y \le n$.

Moreover, this function may depict the curve joining the sequence of points $(0, x_{(n)}), (1, x_{(n)}), (1, x_{(n-1)}), (2, x_{(n-1)}), \dots, (n, x_{(1)})$.

The **plot_producer()** function behaves much like the well-known R’s **plot.default()** and allows for passing all its graphical parameters.

For example, let us depict the state of two given producers, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

```
x1 <- c(5, 4, 2, 2, 1)
x2 <- c(3, 3, 1, 0, 0, 0, 0)
plot_producer(x1, extend=TRUE)
plot_producer(x2, add=TRUE, col=2, pch=2, extend=TRUE)
legend('topright', c('x1', 'x2'), col=c(1, 2), lty=1, pch=c(1, 2))
```



5 Pre-orders

....

Let us consider the following relation on $\mathbb{I}^{1,2,\dots}$. For any $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$ we write $\mathbf{x} \trianglelefteq \mathbf{y}$ if and only if $n \le m$ and $x_{\{i\}} \le y_{\{i\}}$ for all $i \in \min\{n, m\}$. Of course, $\trianglelefteq$ is a pre-order – it would have been a partial order, if we had defined it on the set of *sorted* vectors.

In other words, we say that an author X is (weakly) dominated by an author Y, if X has no more papers than Y and each the ith most cited paper of X has no more citations than the ith most cited paper of Y. Not that the $m - n$ least cited Y’s papers are not taken into account here. Most importantly, however, there exist pairs of vectors that are *incomparable* with respect to $\trianglelefteq$ (see the illustration below).

This pre-order in agop as **pord_weakdom()**.

```
c(pord_weakdom(5:1, 10:1), pord_weakdom(10:1, 5:1)) # 5:1 <= 10:1

## [1]  TRUE FALSE

c(pord_weakdom(3:1, 5:4),  pord_weakdom(5:4, 3:1))  # 3:1 ?? 5:4

## [1] FALSE FALSE
```

We have the following result (Gagolewski, Grzegorzewski, [14]). Let $\mathsf{F} \in \mathcal{E}(\mathbb{I})$. Then F is symmetric, nondecreasing in each variable and arity-monotonic if and only if for any $\mathbf{x}, \mathbf{y}$ if

$\mathbf{x} \trianglelefteq \mathbf{y}$, then $\mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y})$. Therefore, the class of impact functions may be equivalently defined as all the aggregation operators that are nondecreasing with respect to this preorder.

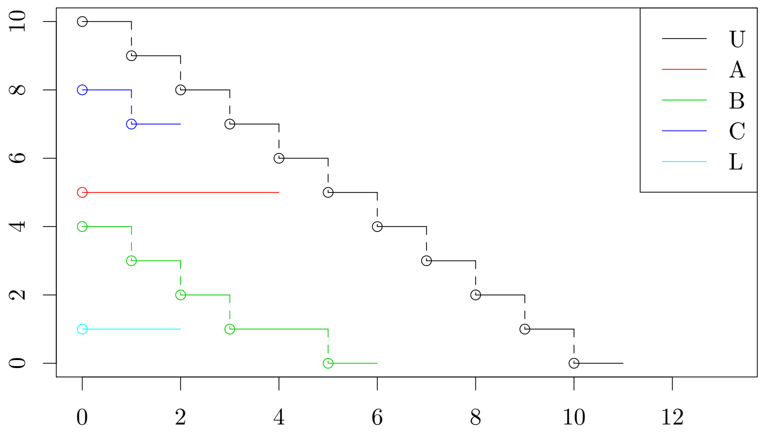
Additionally, we will write $\mathbf{x} \triangleleft \mathbf{y}$ if $\mathbf{x} \trianglelefteq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ (strict dominance).

Example. Let us consider the 5 following vectors.

```
ex1 <- list(
   U = 10:0,            # some upper bound
   A = c(5,5,5,5),      # moderate productivity & quality
   B = c(4,3,2,1,1,0), # high productivity
   C = c(8,7),          # high quality
   L = c(1,1)           # some lower bound
)
```

Plot of "citation" curves:

```
for (i in seq_along(ex1))
plot_producer(ex1[[i]], add=(i>1), col=i)
legend("topright", legend=names(ex1), col=1:length(ex1), lty=1)
```



get adjacency matrix for $(\{A, B, C, L, U\}, \trianglelefteq)$....

```
ord <- rel_graph(ex1, pord_weakdom)
print(ord)

## 5 x 5 sparse Matrix of class "dtCMatrix"
##   U A B C L
## U 1 . . . .
## A 1 1 . . .
## B 1 . 1 . .
## C 1 . . 1 .
## L 1 1 1 1 1

is_reflexive(ord)  # is reflexive

## [1] TRUE

is_transitive(ord) # is transitive

## [1] TRUE
```

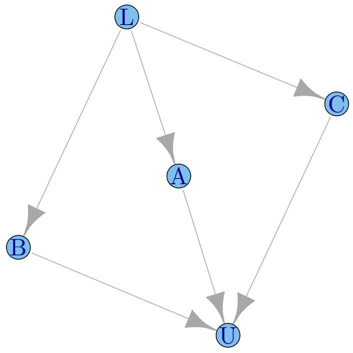
```
is_total(ord)      # not a total preorder...

## [1] FALSE
```

We see that we have $A??B$, $A??C$, $B??C$ (no pair from $\{A, B, C\}$ is comparable w.r.t. $\trianglelefteq$):

```
incomp <- get_incomparable_pairs(ord)
incomp <- incomp[incomp[,1]<incomp[,2],] # remove permutations: ((1,2), (2,1))->(1,2)
incomp[,] <- rownames(ord)[incomp]
print(incomp) # all incomparable pairs

##      [,1] [,2]
## [1,] "A"  "B"
## [2,] "A"  "C"
## [3,] "B"  "C"

# the other way: generate maximal independent sets
lapply(get_independent_sets(ord), function(set) rownames(ord)[set])

## [[1]]
## [1] "A" "B" "C"
```

To draw the Hasse diagram, it will be good to de-transitivize the graph (for æsthetic reasons)....

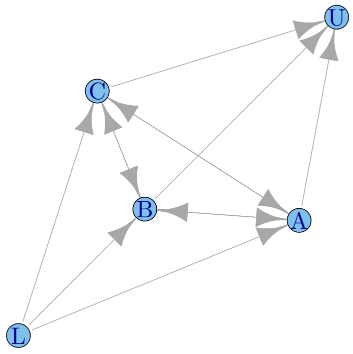
```
require(igraph)
hasse <- graph.adjacency(de_transitive(ord))
set.seed(1234567) # igraph's draving facilities are far from perfect
plot(hasse, layout=layout.fruchterman.reingold(hasse, dim=2))
```



$(\{A, B, C, L, U\}, \trianglelefteq)$ is not totally ordered, let's apply fair totalization (set $x \trianglelefteq'' y$ and $y \trianglelefteq'' x$ whenever $\neg(x \trianglelefteq y$ or $y \trianglelefteq x)$ + calculate transitive closure

```
ord_total <- closure_transitive(closure_total_fair(ord)) # a total preorder
print(ord_total)

## 5 x 5 sparse Matrix of class "dgCMatrix"
##   U A B C L
## U 1 . . . .
## A 1 1 1 1 .
## B 1 1 1 1 .
## C 1 1 1 1 .
## L 1 1 1 1 1
```

```
hasse <- graph.adjacency(de_transitive(ord_total))
set.seed(1234)
plot(hasse, layout=layout.fruchterman.reingold(hasse, dim=2))
```



...Note that each total preorder $\trianglelefteq''$ induces an equivalence relation ($x \simeq y$ iff $x \trianglelefteq'' y$ and $y \trianglelefteq'' x$; the equivalence classes may be ordered with $\trianglelefteq''$). These may be explored with the get_equivalence_classes() function....

```
sapply(get_equivalence_classes(ord_total), function(set) rownames(ord)[set])
```

```
## [[1]]
## [1] "L"
##
## [[2]]
## [1] "A" "B" "C"
##
## [[3]]
## [1] "U"
```

Thus, we've obtained $L \prec (A \simeq B \simeq C) \prec U$.

6 Aggregation Operators from the Probabilistic Perspective

Theory of aggregation looks at the aggregation operators mainly from the algebraic perspective. Of course, we may also be interested in their probabilistic properties, e.g. in i.i.d. RVs models (the simplest and the most "natural" ones in statistics), cf. [7] for discussion. In such case we assume that input data to be aggregated are in fact realizations of some random samples.

In probability, an aggregation operator is simply called a ***statistic*** (formalism......)

Let $(X_1, \ldots, X_n)$ i.i.d. F, where $\operatorname{supp} F = \mathbb{I}$.

....

In the modeling of social phenomena, if F is continuous, we often assume that the underlying density f is decreasing and convex on $\mathbb{I}$, possibly with heavy-tails. E.g. in the bibliometric impact assessment problem, this assumption reflects the fact that higher paper valuations are more difficult to obtain than the lower ones, most of the papers have very small valuation (near 0), and the probability of attaining a high note decreases in at least linear pace.

6.1 Some Notable Probability Distributions

6.1.1 Pareto-Type II Distribution

Many generalizations of the Pareto distribution have been proposed (GPD, *Generalized Pareto Distributions*, cf. e.g. [26, 30]). Here we will introduce the so-called Pareto-Type II (Lomax) distribution, which has support $\mathbb{I} = [0, \infty]$.

Formally, X follows the Pareto-II distribution with shape parameter $k > 0$ and scale parameter $s > 0$, denoted $X \sim \mathrm{P2}(k, s)$, if its density is of the form

$$f(x) = \frac{ks^k}{(s+x)^{k+1}} \quad (x \geq 0). \tag{4}$$

The cumulative distribution function of X is then:

$$F(x) = 1 - \frac{s^k}{(s+x)^k} \quad (x \geq 0). \tag{5}$$

TO DO: agop: `dpareto2()` – (4), `ppareto2()` – (5), and `qpareto2()`... `rpareto2()`......

Properties. The expected value of $X \sim \mathrm{P2}(k, s)$ exists for $k > 1$ and is equal to

$$\mathbb{E}X = \frac{s}{k-1}.$$

Variance exists for $k > 2$ and is equal to

$$\operatorname{Var} X = \frac{ks^2}{(k-2)(k-1)^2}.$$

More generally, the i-th raw moment for $k > i$ is given by:

$$\mathbb{E}X^i = \frac{\Gamma(i+1)\Gamma(k-i)}{\Gamma(k+1)} ks^i.$$

For a fixed s, if $X \sim \mathrm{P2}(k_x, s)$ and $Y \sim \mathrm{P2}(k_y, s)$, $k_x < k_y$, then X stochastically dominates Y, denoted $X \succ Y$. On the other hand, for a fixed k, $X \sim \mathrm{P2}(k, s_x)$, $Y \sim \mathrm{P2}(k, s_y)$, $s_x > s_y$, implies $X \succ Y$.

Interestingly, if $X \sim \mathrm{P2}(k, s)$, then the conditional distribution of $X - t$ given $X > t$, is $\mathrm{P2}(k, s + t)$ $t \geq 0$.

Additionally, it might be shown that if $X \sim \mathrm{P2}(k, s)$, then $\ln(s + X)$ has c.d.f. $F(x) = 1 - s^k e^{-kx}$ and density $f(x) = ks^k e^{-kx}$ for $x \geq \ln s$, i.e. has the same distribution as $Z + \ln s$, where $Z \sim \mathrm{Exp}(k) \equiv \Gamma(1, 1/k)$.

Parameter estimation. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a realization of the Pareto-Type II i.i.d. sample with known $s > 0$. The unbiased (corrected) maximum likelihood estimator for k:

$$\widehat{k}(\mathbf{x}) = \frac{n-1}{\sum_{i=1}^{n} \ln\left(1 + \frac{1}{s} x_i\right)}.$$

It may be shown that for $n > 2$ it holds $\mathrm{Var}\,\widehat{k}(\mathbf{x}) = k^2 \frac{1}{n-2}$.

TO DO: agop: `pareto2.mlekestimate()`

For both unknown k and s we have:

$$\begin{cases} \widehat{k} = \frac{n}{\sum_{i=1}^{n} \ln\left(1+x_i/\widehat{s}\right)}, & \\ 1 + \frac{1}{n}\sum_{i=1}^{n} \ln\left(1 + x_i/\widehat{s}\right) - \frac{n}{\sum_{i=1}^{n} \left(1+x_i/\widehat{s}\right)^{-1}} & = \quad 0. \end{cases} \tag{6}$$

Unfortunately, the second equation must be solved numerically. The estimation procedure has been implemented in agop as TO DO: `pareto2.mleksestimate()`.... It is worth noting that the above system of equations may sometimes have no solution (as the local minimum of the likelihood function may not exist, see [3] for discussion).

In this case one of the estimators worth noting (and often better than MLE) was proposed in [31]. The Zhang-Stevens MMS (*minimum mean square error*) (Bayesian) estimator has relatively small bias (often positive) and mean squared error. In agop it is available as TO DO: `pareto2.zsestimate()`.

Goodness-of-fit tests. TO BE DONE....

Applications. TO DO

Two-sample F-test. The following simple test was introduced in [7]. Let $(X_1, X_2, \dots, X_{n_1})$ *i.i.d.* $\mathrm{P2}(k_1, s)$ and $(Y_1, Y_2, \dots, Y_{n_2})$ *i.i.d.* $\mathrm{P2}(k_2, s)$, where s is an a-priori known scale parameter. We are going to verify the null hypothesis $H_0 : k_1 = k_2$ against the two-sided alternative hypothesis $K : k_1 \neq k_2$.

It might be shown that $\sum_{i=1}^{n} \ln(s + X_i) - n \ln s \sim \Gamma(n, 1/k)$. This implies that under H_0, the following test statistic follows the Snedecor F distribution:

$$R(\mathbf{X}, \mathbf{Y}) = \frac{n_1}{n_2} \frac{\sum_{i=1}^{n_2} \ln\left(1 + \frac{Y_i}{s}\right)}{\sum_{i=1}^{n_1} \ln\left(1 + \frac{X_i}{s}\right)} \overset{H_0}{\sim} \mathrm{F}^{[2n_2, 2n_1]}. \tag{7}$$

The null hypothesis is accepted iff

$$R(\mathbf{x}, \mathbf{y}) \in \left[\texttt{qf}(\tfrac{\alpha}{2}, 2n_2, 2n_1),\ \texttt{qf}(1 - \tfrac{\alpha}{2}, 2n_2, 2n_1)\right],$$

where $\texttt{qf}(q, d_1, d_2)$ denotes the q-quantile of $\mathrm{F}^{[d_1, d_2]}$

The p-value may be determined as follows:

$$p = 2\left(\frac{1}{2} - \left|\texttt{pf}(R(\mathbf{x}, \mathbf{y}), 2n_2, 2n_1) - \frac{1}{2}\right|\right), \tag{8}$$

where $\texttt{pf}(x, d_1, d_2)$ is the c.d.f. of $\mathrm{F}^{[d_1, d_2]}$.

TO DO: `pareto2.ftest()`.

6.2 Stochastic Properties of Aggregation Operators

Let us assume that $(X_1, \ldots, X_n)$ denotes a sequence of independent, identically distributed random variables and let c.d.f F of X_i $i = 1, \ldots, n$ be continuous and strictly increasing in $[a, b]$, such that $a = \inf\{x : \mathsf{F}(x) > 0\}$ and $b = \sup\{x : \mathsf{F}(x) < 1\}$, $a < b$.

In this section we are going to recall some basic statistical properties of ***L-*** and ***S-statistics***.

Definition 20. An ***S-statistic*** associated with function κ and a random sample $(X_1, \ldots, X_n)$ is a function

$$\mathsf{V}_{n,k}(X_1, \ldots, X_n) = \bigvee_{i=1}^{n} \kappa\Big(\frac{i}{n}\Big) \wedge X_{(n-i+1)},$$

where $\kappa : [0,1] \to [a,b]$, called ***control function***, is a strictly increasing continuous function such that $\kappa(0) = a$ and $\kappa(1) = b$.

It is easily seen that $\mathsf{V}_{n,k}$ is equivalent with ***S-statistic*** $\mathsf{S}_{\triangle}|_{\mathbb{I}^n}$ (cf. Definition 13) associated with triangle of coefficients $\triangle = (c_{i,n})_{n\in\mathbb{N}, i\in[n]}$ such that $c_{i,n} = \kappa\left(\frac{i}{n}\right)$.

Also please note that without loss of generality we may consider ***S-statistic*** form Definition 20 of a form

$$\mathsf{V}_n(Y_1, \ldots, Y_n) = \bigvee_{i=1}^{n} \frac{i}{n} \wedge Y_{(n-i+1)},$$

where $(Y_1, \ldots, Y_n) = (\kappa^{-1}(X_1), \ldots, \kappa^{-1}(X_n))$ is a sequence of i.i.d. random variables given by the continuous c.d.f. $\mathsf{G} := \mathsf{F} \circ \kappa$. For more details see [13].

The exact distribution is given by the following Theorem.

Theorem 21. *The c.d.f. of* $\mathsf{V}_n(Y_1, \ldots, Y_n)$ *is given by*

$$\begin{aligned} \mathsf{D}_n(x) &= 1 - \sum_{i=1}^{n} \binom{n}{i} [1 - \mathsf{G}(x)]^i [G(x)]^{n-1} \\ &= \mathcal{I}(\mathsf{G}(x); n - \lfloor xn \rfloor, \lfloor xn \rfloor + 1), \end{aligned}$$

for $x \in [0,1)$, *where* $\mathcal{I}(p; a, b)$ *is the regularized incomplete beta function.*

Let us recall that Hirsch’s h-index, for $x \in \mathbb{N}_0^n$ reduced itself to a ***OWMax*** operator, which is a special case of ***S-statistic***. Therefore, the exact distribution of H, given by the following theorem, comes straightforward from the previous results (for more details see. [10]).

Theorem 22. *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *be a sequence of i.i.d. random variables with continuous c.d.f.* F *defined on* $\mathbb{R}_{0+}$. *Then the c.d.f. of* $\mathsf{H}(\mathbf{X})$ *for* $x \in [0, n)$ *is given by*

$$\mathsf{D}_n(x) = \mathcal{I}(\mathsf{F}(\lfloor x + 1 \rfloor^{-0}); n - \lfloor x \rfloor, \lfloor x \rfloor + 1),$$

where $\mathcal{I}(p; a, b)$ *is the regularized incomplete beta function.*

Definition 23. A κ-index of a random variable given by a c.d.f. F with respect to control function κ is a number $\varrho_\kappa \in [0,1]$ such that

$$\varrho_\kappa = 1 - \mathsf{F}(\kappa(\varrho_\kappa)).$$

For survival function, $\mathsf{S}(x) = 1 - \mathsf{F}(x)$, ϱ_κ satisfies $\varrho_\kappa = \mathsf{S}(\kappa(\varrho_\kappa)) = \mathbb{P}(X > \kappa(\varrho_\kappa))$. It is clear to see that, intuitively, ϱ_κ is a number such that the probability of assuming a value greater than $\kappa(\varrho_\kappa)$ is equal to ϱ_κ. What is more, in [13] we have following results.

Theorem 24. *V_n is a strongly consistent estimator of $\varrho = \varrho_{id}$.*

The asymptotic distribution of ***S-statistic*** is given by the following theorem.

Theorem 25. *If G is a c.d.f. differentiable at ϱ, then*

$$\mathsf{V}_n \xrightarrow{D} \mathcal{N}\left(\varrho, \frac{1}{1+\mathsf{G}'(\varrho)}\sqrt{\frac{\varrho(1-\varrho)}{n}}\right).$$

Let us now consider asymptotic distribution of ***L-statistics***. Please note, that sometimes it is used to represent $\mathsf{L}_\triangle$ as

$$\mathsf{L}_n = \frac{1}{n}\sum_{i=1}^{n} \mathsf{J}\left(\frac{i}{n}\right) X_{(i)},$$

where function $\mathsf{J}(u)$ for $u \in [0,1]$ is such that $\mathsf{J}(i/n) = nc_{i,n}$, $i = 1, \dots, n$.

We present below the asymptotic normality result for normed L_n, cf **??**. To this end, first define

$$\mu(\mathsf{J},\mathsf{F}) = \int_0^1 \mathsf{J}(u)\mathsf{F}^{-1}(u)du$$

and

$$\sigma^2(\mathsf{J},\mathsf{F}) = 2\int\int_{0<u<v<1} \mathsf{J}(u)\mathsf{J}(v)u(1-v)d\mathsf{F}^{-1}(u)d\mathsf{F}^{-1}(v).$$

Theorem 26. *Assume that $\mathsf{var}(X_1) < \infty$ and that $\mathsf{J}(u)$ is bounded and continuous a.e. F^{-1}. Then*

$$\lim_{n\to\infty} n\mathbb{E}(\mathsf{L}_n) = \mu(\mathsf{J},\mathsf{F}),$$

$$\lim_{n\to\infty} n\mathsf{var}(\mathsf{L}_n) = \sigma^2(\mathsf{J},\mathsf{F}),$$

and if $\sigma^2(\mathsf{J},\mathsf{F}) > 0$,

$$\mathsf{L}_n^* = \frac{\mathsf{L}_n - \mathbb{E}(\mathsf{L}_n)}{\sqrt{\mathsf{var}(\mathsf{L}_n)}} \xrightarrow{D} \mathcal{N}(0,1).$$

What is more, it can be shown (cf. **??**) that if $\int_0^1 [u(1-u)]^{1/2} d\mathsf{F}^{-1}(u) < \infty$ and J is bounded and satisfies a Lipschitz condition of order $\alpha > \frac{1}{2}$ except perhaps at finite number of continuity points of F^{-1}, then we have

$$\lim_{n\to\infty} n^{1/2}[\mathbb{E}(\mathsf{L}_n) - \mu(\mathsf{J},\mathsf{F})] = 0.$$

7 NEWS/CHANGELOG

```
                          ** agop package NEWS **

***************************************************************************

0.1-0 /under development/

* initial release
  [the package started as a lightweight fork of the CITAN package]

***************************************************************************
```

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