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BS Physics 3-1
MathPhy 3
PS3

Problem 1:

1. $(x^2-4)y'' + xy' - y = 0$

$$p(x) = \frac{x}{(x^2-4)}, \quad q(x) = \frac{-1}{(x^2-4)}$$

$x = 0$, $x = \pm 2$

- At $x=0$, $P(x)=0$ & $q(x) = \frac{1}{4}$, so that $\boxed{\therefore x = 0 \text{ is an ordinary point}}$
- At $x = \pm 2$, both $p(x)$ & $q(x)$ are infinite, such that $p(x) = q(x) = \infty$, therefore, both $x = \pm 2$ are singular.
- To check the type of singularity, solve for $\lim_{x\to a} (x-a)\,p(x)$ & $\lim_{x\to a} (x-a)^2 f(x)$, such that

At $x = +2$

$$\star \lim_{x\to a} (x-a)\,p(x) = \lim_{x\to 2} (x-2)\frac{x}{(x^2-4)} = \lim_{x\to 2} \frac{x}{x+2} = \frac{1}{2}$$

$$\star \lim_{x\to a} (x-a)^2 q(x) = \lim_{x\to 2} \frac{-(x-2)^2}{(x^2-4)} = \lim_{x\to 2} \frac{-(x-2)(x-2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{-(x-2)}{(x+2)} = 0$$

$\boxed{\therefore x = +2 \text{ is a regular singular point because both limits remain finite.}}$

- When repeating this step to find the type of singularity for $x = -2$, we'll realize that $x = -2$ is also ~~an~~ a regular point.

Problem 2: Part 1

(2) $(1-z^2)y'' - 3z\,y' + \lambda y = 0.$ at $z=0$

let $z = x$ (for convenience).

$(1-x^2)y'' - 3xy' + \lambda = 0.$ at $x=0$

Solution:

$P_0(x) = 1-x^2$, $P_1(x) = -3x$, $P_2(x) = \lambda$

At point $x=0$:- $P_0(0) = 1-(0)^2 = 1 \neq 0$

$\therefore x=0$ is an ordinary point.

Now by power series method :-

let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots\ldots$ $\Rightarrow y$

$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n (n x^{n-1}) = \sum_{n=1} n a_n x^{n-1}$ $\Rightarrow y'$

$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} (n a_n)(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ $\Rightarrow y''$

Now put y, y' & y'' in given D.E.

$\Rightarrow (1-x^2)\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) - 3x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \lambda\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2+2} - 3\sum_{n=1}^{\infty} n a_n x^{n-1+1} + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 3\sum_{n=1}^{\infty} n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 3\sum_{n=1}^{\infty} n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow \left[(2)(1)a_2 x^0 + (3)(2)a_3 x^1 + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n\right]$

$- \sum_{n=2}^{\infty} n(n-1)a_n x^n - 3\left[(1)a_1 x^1 + \sum_{n=2}^{\infty} n a_n x^n\right] + \lambda\left[a_0 x^0 + a_1 x^1 + \sum_{n=2}^{\infty} a_n x^n\right] = 0$

$\Rightarrow 2a_2 + 6a_3 x^1 - 3a_1 x^1 + \lambda a_0 x^0 + \lambda a_1 x^1 + \sum_{n=2}^{\infty}\left[(n+1)(n+1)a_{n+2} - n(n-1)a_n\right.$

$\left. - 3n a_n + \lambda a_n\right] x^n = 0$

$\Rightarrow (\lambda a_0 + 2a_2) + (6a_3 + (\lambda-3)a_1)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (-n^2 + n - 3n + \lambda)a_n\right]x^n = 0$

$\Rightarrow (\lambda a_0 + 2a_2) + (6a_3 + (\lambda-3)a_1)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (-n^2 - 2n + \lambda)a_n\right]x^n = 0$

Problem 2: Part 2 (with final answer)

By comparing coefficient of power of x to zero.

Now coeff of $x^0 \Rightarrow \lambda a_0 + 2a_2 = 0 \Rightarrow \lambda a_0 = -2a_2 \Rightarrow \boxed{a_2 = -\lambda a_0/2}$

coeff of $x^1 \Rightarrow 6a_3 + (\lambda-3)a_1 = 0 \Rightarrow 6a_3 = -(\lambda-3)a_1 \Rightarrow \boxed{a_3 = -(\lambda-3)a_1/6}$

coeff of $x^n \Rightarrow (n+2)(n+1)a_{n+2} + (-n^2-2n+\lambda)a_n$

for $n \geq 2$.

$$\Rightarrow (n+2)(n+1)a_{n+2} = -(-n^2-2n+\lambda)a_n$$

$$\Rightarrow a_{n+2} = \frac{-(-n^2-2n+\lambda)a_n}{(n+2)(n+1)}$$

$$\Rightarrow a_{n+2} = \frac{(n^2+2n-\lambda)a_n}{(n+2)(n+1)} \Rightarrow (1)$$

Put $n=2$ in eq. (1) :- $a_4 = \frac{(8-\lambda)a_2}{12} \Rightarrow a_4 = \frac{-(8-\lambda)\lambda a_0}{(12)(2)}$

Put $n=3$ in eq (1) :- $a_5 = \frac{(15-\lambda)a_3}{20} \Rightarrow a_5 = \frac{-(15-\lambda)a_1(\lambda-3)}{(20)(6)}$

Put $n=4$ in eq (1) :- $a_6 = \frac{(24-\lambda)a_4}{30} \Rightarrow a_6 = \frac{-(24-\lambda)(8-\lambda)\lambda a_0}{(30)(24)}$

Now, put the values of a_2, a_3, a_4, a_5 & a_6 in Solution:-

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots$$

$$= a_0 + a_1 x + \left(\frac{-\lambda a_0}{2}\right)x^2 + \left(\frac{-(\lambda-3)a_1}{6}\right)x^3 + \left[\frac{-(8-\lambda)\lambda a_0}{24}\right]x^4 + \left[\frac{(\lambda-15)(\lambda-3)a_1}{120}\right]x^5$$
$$+ \left[\frac{(\lambda-24)(8-\lambda)\lambda a_0}{(30)(24)}\right]x^6$$

$$= a_0 + a_1 x + \left(\frac{-\lambda a_0}{2}\right)x^2 + \left(\frac{(3-\lambda)a_1}{6}\right)x^3 + \left[\frac{(\lambda-8)\lambda a_0}{24}\right]x^4 + \left[\frac{(\lambda-15)(\lambda-3)a_1}{120}\right]x^5$$
$$+ \left[\frac{(\lambda-24)(8-\lambda)\lambda a_0}{288}\right]x^6$$

$$= a_0 + a_1 x + \left(\frac{-\lambda a_0}{2}\right)x^2 + \frac{(3-\lambda)x^3 a_1}{6} + \left[\frac{(\lambda-8)\lambda x^4}{24}\right]a_0 + \left[\frac{(\lambda-15)(\lambda-3)x^5}{120}\right]a_1$$
$$+ \left[\frac{(\lambda-24)(8-\lambda)\lambda x^6}{288}\right]a_0$$

$$\boxed{\begin{aligned} &= a_0\left[1 - \frac{\lambda x^2}{2} + \frac{(\lambda-8)\lambda x^4}{24} + \frac{(\lambda-24)(8-\lambda)\lambda x^6}{720} + \ldots\right] \\ &+ a_1\left[x + \frac{(3-\lambda)x^3}{6} + \frac{(\lambda-15)(\lambda-3)x^5}{120} + \ldots\right] \end{aligned}}$$

Problem 3: Part 1

3. $4zy'' + 2(1-z)y' - y = 0$ at $z=0$

let $y = a_0 z^n + a_1 z^{n+1} + a_2 z^{n+2} + a_3 z^{n+3} + \dots$

$y' = n a_0 z^{n-1} + a_1(n+1) z^n + a_2(n+2) z^{n+1} + (n+3) a_3 z^{n+2} + \dots$

$y'' = a_0(n)(n-1) z^{n-2} + a_1(n+1)(n) z^{n-1} + a_2(n+2)(n+1) z^n$
$+ a_3(n+3)(n+2) z^{n+1} + \dots$

by substituting the y, y', y'' in given D.E.

$[4a_0 n(n-1) z^{n-1} + 4a_1(n+1)(n) z^n + 4a_2(n+2)(n+1) z^{n+1} + 4a_3(n+3)(n+2) z^{n+2} + \dots$
$+ [2a_0 n z^{n-1} + 2a_1(n+1) z^{n} + 2a_2(n+2) z^{n+1} + 2a_3(n+3) z^{n+2} + \dots]$
$- [2a_0 n z^n + 2a_1(n+1) z^{n+1} + 2a_2(n+2) z^{n+2} + 2a_3(n+3) z^{n+3} + \dots]$
$- [a_0 z^n + a_1 z^{n+1} + a_2 z^{n+2} + a_3 z^{n+3} + \dots] = 0.$

look for the coefficients of z^n on both a_0 & a_1:

~~Wearing the least power of z in z^{n-1} for a_0 & the least power of z for a_1 in z^n.~~

$4a_1(n+1)(n) + 2a_1(n+1) - 2a_0 n - a_0 = 0$

$a_1[4n(n+1) + 2(n+1)] = a_0[2n+1]$

$a_1[(n+1)(4n+2)] = a_0[2n+1]$

$a_1[2(n+1)\cancel{(2n+1)}] = a_0[\cancel{2n+1}]$

$$\boxed{a_1 = \frac{a_0}{2(n+1)}}$$

Now look for the coefficients of z^{n+1} on both a_1 & a_2:

$a_2[4(n+2)(n+1) + 2(n+2)] = a_1[2(n+1)+1]$

$a_2[2(n+2)\cancel{[2(n+1)+1]}] = a_1\cancel{[2(n+1)+1]}$

$$\boxed{a_2 = \frac{a_1}{2(n+2)} = \frac{a_0}{4(n+1)(n+2)}}$$

Now look for the coefficients of z^{n+2} on both a_2 & a_3:

$a_3[2(n+3)[2(n+2)+1]] = a_2[2(n+2)+1]$

$$\boxed{a_3 = \frac{a_2}{2(n+3)} = \frac{a_0}{8(n+1)(n+2)(n+3)}}$$

Problem 3: Part 2 (with final answer)

$$\therefore\ y = a_0 z^n + \frac{a_0}{2(n+1)} z^{n+1} + \frac{a_0}{4(n+1)(n+2)} z^{n+2} + \frac{a_0}{8(n+1)(n+2)(n+3)} z^{n+3} + \dots$$

the solution can be written as

$$y = a_0 z^n \left[1 + \frac{1}{2(n+1)} z + \frac{1}{4(n+1)(n+2)} z^2 + \frac{1}{8(n+1)(n+2)(n+3)} z^3 + \dots \right]$$

for $n=0$, we have

$$y_1 = a_0 \left[1 + \frac{1}{2} z + \frac{1}{8} z^2 + \frac{1}{48} z^3 + \dots \right]$$

$$\boxed{y_1 = a_0 + \frac{a_0}{2} z + \frac{a_0}{8} z^2 + \frac{a_0}{48} z^3 + \dots}$$

One of the solutions of the given D.E.

Verifying the founded solution.

$$y_1 = 1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48} + \dots$$

$$y_1' = \frac{1}{2} + \frac{z}{4} + \frac{z^2}{16} + \dots$$

$$y_1'' = \frac{1}{4} + \frac{z}{8} + \dots$$

by substitution in the given D.E.

$$\Rightarrow 4z \left[\frac{1}{4} + \frac{z}{8} + \dots \right] + 2\left[\frac{1}{2} + \frac{z}{4} + \frac{z^2}{16} \right] - 2z\left[\frac{1}{2} + \frac{z}{4} + \frac{z^2}{16} + \dots \right] - \left(1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48}\right)$$

$$\Rightarrow \left(z + \frac{z^2}{2} + \dots\right) + \left(1 + \frac{z}{2} + \frac{z^2}{8} + \dots\right) - \left(z + \frac{z^2}{2} + \frac{z^3}{8} + \dots\right) - \left(1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48} + \dots\right)$$

$$\Rightarrow \cancel{z} + \cancel{\frac{z^2}{2}} + \cancel{1} + \cancel{\frac{z}{2}} + \cancel{\frac{z^2}{8}} - \cancel{z} - \cancel{\frac{z^2}{2}} - \cancel{\frac{z^2}{8}} \cancel{1} - \cancel{\frac{z}{2}} - \cancel{\frac{z^2}{8}} - \cancel{\frac{z^3}{48}} + \dots$$

$$= \boxed{0}$$

it's clear that the given differential equation is satisfied by $y = 1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48} + \dots$