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BS PHYSICS 3-1

MATPHY 3 PS-3

(1) Find the general power series solution about $z=0$ of the equation

$$z\frac{d^2y}{dz^2}+(2z-3)\frac{dy}{dz}+\frac{4}{z}y=0. \quad —(1)$$

So, assume the solution of eq (1) to be

$$y=\sum_{n=0}^{\infty} a_n z^{(m+n)} \Rightarrow y=z^m\sum_{n=0}^{\infty} a_n z^n$$

$$y=z^m(a_0+a_1z+a_2z^2+a_3z^3+\dots) \quad —(2)$$

differentiate the equation 2 times.

$$\frac{dy}{dz}=z^m(0+a_1+2a_2z+3a_3z^2+\dots)+(a_0+a_1z+a_2z^2+a_3z^3+\dots)(mz^{m-1})$$

or eq (2) can be written as

$$y=(a_0z^m+a_1z^{m+1}+a_2z^{m+2}+a_3z^{m+3}+\dots a_nz^{m+n})$$

now,

$$\frac{dy}{dz}=a_0mz^{m-1}+a_1(m+1)z^m+(m+2)a_2z^{m+1}+a_3(m+3)z^{m+2}+\dots(m+n)a_nz^{(m+n-1)}$$

$$\frac{d^2y}{dz^2}=m(m-1)a_0z^{m-2}+a_1(m+1)mz^{m-1}+a_2(m+2)(m+1)z^m+a_3(m+3)(m+2)z^{m+1}+\dots a_n(m+n)(m+n-1)z^{(m+n-2)}$$

now put the values of y, $\frac{dy}{dz}$ & $\frac{d^2y}{dz^2}$ in eq (1)

$$z\left[a_0m(m-1)z^{m-2}+a_1m(m+1)z^{m-1}+a_2(m+1)(m+2)z^m\dots\right]$$
$$+(2z-3)\left[a_0mz^{m-1}+a_1(m+1)z^m+a_2(m+2)z^{m+1}+\dots\right]$$
$$+\frac{4}{z}\left[a_0z^m+a_1z^{m+1}+a_2z^{m+2}+\dots\right]=0$$

or

$$(a_0m(m-1)z^{m-1}+a_1m(m+1)z^m+a_2(m+1)(m+2)z^{m+1}+\dots)$$
$$+(2a_0mz^m+2a_1(m+1)z^{m+1}+2a_2(m+2)z^{m+2}+\dots)$$
$$-(3a_0mz^{m-1}+3a_1(m+1)z^m+3a_2(m+2)z^{m+1}+\dots)$$
$$+(4a_0z^{m-1}+4a_1(z^m)+4a_2z^{m+1}+\dots)=0$$

the lowest power of z is $(m-1)$ so, equating the coefficient of z^{m-1} equal to zero,

$$a_0 m(m-1) - 3a_0 m + 4a_0 = 0$$
$$a_0 [m^2 - m - 3m + 4] = 0 \quad \text{or} \quad \underline{m^2 - 4m + 4 = 0}$$

this differ is an integer, so,

$$\underline{m = 1 \,\&\, 2}$$

now, equate the zero of the coefficient of z^m

$$a_1 m(m+1) + 2a_0 m - 3a_1(m+1) + 4a_1 = 0$$
$$a_1(m^2 + m - 3m - 3 + 4) + 2a_0 m = 0$$
$$a_1 = \left(\frac{-2m}{m^2 - 2m + 1}\right) a_0.$$

now, the coefficient of z^{m+1} equate to zero.

$$a_2(m+1)(m+2) + 2a_1(m+1) - 3a_2(m+2) + 4a_2 = 0$$
$$a_2[(m+1)(m+2) - 3(m+2) + 4] + 2a_1(m+1) = 0$$
$$a_2[(m^2 + 2m + m + 2) - 3m - 6 + 4] = -2(m+1)a_1$$
$$a_2(m^2 - 0) = -2(m+1)a_1$$
$$a_2 = -2(m+1)a_1$$
$$a_2 = -\frac{2(m+1)}{m^2} a_1$$
$$= -\frac{2(m+1)}{m^{\cancel{2}}} \times \frac{(-2\cancel{m})}{(m^2 - 2m + 1)} a_0$$
$$a_2 = \frac{4(m+1)}{m(m^2 - 2m + 1)} a_0$$

<u>now</u>

when $m = 1$ because $m = 1 \,\&\, 2$

$$a_1 = \infty$$
$$a_2 = \frac{4(1+1)}{1(1-2+1)} = \infty$$

when $m = 2$,

$$a_1 = \frac{-2\times 2}{4-4+1}\, a_0 \Rightarrow \underline{a_1 = -4a_0}$$

& $$a_2 = \frac{4(2+1)}{2(4\cancel{t}-4+1)}\, a_0 \Rightarrow \underline{a_2 = 6a_0}$$

y at $m=2$ from eq (2)

$$y_2 = z^2(a_0 + (-4a_0 z) + 6a_0 z^2 + \cdots)$$
$$y_2 = a_0 z^2(1 - 4z + 6z^2 + \cdots) \quad \text{—— (3)}$$

and y at $m=1$ does not exist

so,

complete Sol'n

$$y = C_1(y)_{m=1} + C_2(y_{m=2})$$

So, $$\boxed{y = C_2 a_0 z^2 (1-4z+6z^2+\cdots)}$$ } general power series.

(I) a.) Equating to zero, the general term's coefficient

$$(m+r+2)(m+r-1)a_{r+2} + a_r = 0$$

$$\Rightarrow a_{r+2} = \frac{-a_r}{(m+r+2)(m+r-1)} \quad r = 0, 1, 2, 3, \ldots$$

putting $r = 0, 1, 2, 3, 4, \ldots$

$$a_2 = \frac{-a_0}{(m+2)(m-1)}$$

$$a_3 = \frac{-a_1}{(m+3)m} = 0$$

$$a_4 = \frac{-a_2}{(m+4)(m+1)} = \frac{a_0}{(m+4)(m+1)(m+2)(m-1)}$$

$$a_5 = 0$$

$$a_6 = \frac{-a_4}{(m+6)(m+3)} = \frac{-a_0}{(m+4)(m+1)(m+2)(m-1)(m+6)(m+3)}$$

$$a_7 = 0$$

$$a_8 = \frac{-a_6}{(m+8)(m+5)} = \frac{a_0}{(m+8)(m+5)(m+4)(m+1)(m+2)(m+1)(m+6)}.$$

So, $y = z^m \Big[a_0 - \frac{a_0}{(m+2)(m-1)} z^2 + \frac{a_0 \, z^4}{(m+4)(m+1)(m+2} -$

$$\frac{a_0 z^6}{(m+4)(m+1)(m+2)(m-1)(m+6)(m+3)} + \ldots$$

when $m=0$

$$y = a_0\left[1 - \frac{z^2}{2(-1)} + \frac{z^4}{(4)(1)(2)(-1)} - \frac{z^6}{(4)(2)(1)(-1)(6)(3)} + \cdots\right]$$

$$= a_0\left[1 + \frac{z^2}{2} - \frac{z^4}{2(4)} + \frac{z^6}{1\cdot 2\cdot 4\cdot 6} + \cdots\right]$$

$$= a_0\left[1 + \frac{z^2}{2} - \frac{3z^4}{4!} + \frac{5z^6}{6!} + \cdots\right]$$

$$= a_0\left[1 + \frac{x^2}{2!} - \frac{3z^4}{4!} + \frac{5z^6}{6!} + \cdots\right]$$

$$\boxed{y_2(z) = a_0 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2n-1)}{(2n)!} z^{2n}}$$

when $m=3$

$$y_1(x) = z^3\left[a_0 - \frac{a_0 z^2}{5\cdot 2} + \frac{a_0 z^4}{7\cdot 4\cdot 5\cdot 2} - \frac{a_0 z^6}{7\cdot 4\cdot 5\cdot 2\cdot 9\cdot 6} + \cdots\right]$$

$$= a_0\left[a_0 z^3 - \frac{a_0 z^5}{2\cdot 5} + \frac{a_0 z^7}{7\cdot 5\cdot 6\cdot 2} - \frac{a_0 z^9}{9\cdot 7\cdot 5\cdot 5\cdot 4\cdot 2}\right]$$

$$= a_0\left[\frac{2\cdot 3\cdot z^3}{3!} - \frac{3\cdot 4\cdot z^5}{5!} + \frac{3\cdot 6\cdot z^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} - \frac{3\cdot 8\cdot z^9}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \cdots\right]$$

$$= 3a_0\left[\frac{2z^3}{3!} - \frac{4z^5}{5!} + \frac{6z^7}{7!} - \frac{8z^9}{9!} + \cdots\right]$$

$$\boxed{y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)\, z^{2n+1}}{(2n+1)!}}$$

b.) Using the Maclaurin series of $\sin z$ { $\cos z$ } substituting it into the expression $3a_0(\sin z - z\cos z)$ we have,

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$3a_0(\sin z - z\cos z)$$

$$3a_0\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} - z\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}\right)$$

$$3a_0\sum_{n=0}^{\infty} (-1)^n z^{2n+1}\left(\frac{1}{(2n+1)!} - \frac{1}{(2n)!}\right)$$

$$3a_0\sum_{n=0}^{\infty} (-1)^n z^{2n+1}\left[\frac{(2n)! - (2n+1)!}{(2n)!(2n+1)!}\right]$$

$$3a_0\sum_{n=0}^{\infty} (-1)^n z^{2n+1}\left[\frac{1-2n-1}{(2n+1)!}\right]$$

$$3a_0\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n)z^{2n+1}}{(2n+1)!}$$

Adjust the index into $n=1$.

$$3a_0\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)z^{2n+1}}{(2n+1)!}$$

$$y_1(z) = 3a_0\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)z^{2n+1}}{(2n+1)!} = 3a_0(\sin z - z\cos z)$$

Using the wronskian method to solve the second solution, we have,

$$y_2(z) = y_1(z)\int^{z} \frac{1}{y_1^2(u)} \exp\left\{-\int^{u} P(v)dv\right\} du$$

by substituting $P(z)$ and $y_1(z)$

$$y_2(z) = \frac{y_1(z)}{3a_0}\int^{z} \frac{1}{(\sin u - u\cos u)^2} \exp\left\{-\int^{u} \frac{-2}{v} dv\right\} dv$$

by integrating the term inside e we have,

$$2\int^{u} \frac{1}{v} dv \Rightarrow 2\ln u$$

Hence

$$\exp\{2\ln u\} = u^2$$

We have,

$$y_2(z) = \frac{y_1(z)}{3a_0}\int^{z} \frac{u^2}{(\sin u - u\cos u^2)^2} du$$

let $u^2 = \frac{u}{\sin u}(u\sin u)$, then

$$y_2(z) = \frac{y_1(z)}{3a_0}\int^{z} \frac{u}{\sin u} \frac{u\sin u}{(\sin u - u\cos u)^2} du$$

let

$x = \frac{u}{\sin u}$ $\quad dx = \frac{\sin u - u\cos u}{\sin^2 u} du$

let $w = \sin u - u\cos u$ $\quad dw = u\sin u\, du$

let $w = \sin u - u\cos u$ $\quad dw = u\sin u\, du$

Then

$$y = \int \frac{1}{w^2} dw \Rightarrow y = \frac{-1}{w} = \frac{-1}{\sin u - u\cos u}$$

by Integration by Parts

$$y_2(z) = \frac{y_1(z)}{3a_0}\left(\left[\frac{u}{\sin u}\frac{-1}{\sin u - u\cos u}\right] + \int^z \frac{1}{\sin u - u\cos u}\frac{\sin u - u\cos u}{\sin^2 u}du\right)$$

$$y_2(z) = \frac{y_1(z)}{3a_0}\left(\frac{z}{\sin z}\frac{-1}{\sin z - z\cos z} - \cot z\right)$$

$$= \frac{3a_0(\sin z - z\cos z)}{3a_0}\left(\frac{z}{\sin z}\frac{-1}{\sin z - z\cos z} - \frac{\cos z}{\sin z}\right)$$

$$= \frac{-z}{\sin z} - \cos z + \frac{z\cos^2 z}{\sin z}$$

$$= \frac{-z}{\sin z} + \frac{z}{\sin z} - \cos z - z\sin z$$

$$y_2(z) = -(\cos z + z\sin z)$$

When $n=0$

Series | Sinusoidal form
1 | $-1 = -(\cos z + z\sin z)$

Hence

$$\boxed{y_2(z) = \cos z + z\sin z}$$

c.) let $y_1 = \sin z - z\cos z$

$y_1' = z\sin z$

$y_2 = \cos z + z\sin z$

$y_2' = z\cos z$

$W(z) = y_1 y_2' - y_2 y_1'$

$\cancel{= (\sin z - z\cos z - z\sin z\cos}$

$= (\sin z - z\cos z)z\cos z - (\cos z + z\sin z)z\sin z$

$= z\sin z\cos z - z\sin z\cos z - z^2(\cos^2 z + \sin^2 z)$

$= \boxed{-z^2}$