Algorithms & Data Structures I

Lesson 1: proof by induction

Marc Gaetano

Edition 2015-2016

Background on Induction

- Type of mathematical proof
- Typically used to establish a given statement for all natural numbers (e.g. integers > 0)
- Proof is a sequence of deductive steps
  1. Show the statement is true for the first number.
  2. Show that if the statement is true for any one number, this implies the statement is true for the next number.
  3. If so, we can infer that the statement is true for all numbers.

Think about climbing a ladder

1. Show you can get to the first rung (base case)

2. Show you can get between rungs (inductive step)

3. Now you can climb forever.

Why you should care

- Induction turns out to be a useful technique
  - AVL trees
  - Heaps
  - Graph algorithms
  - Can also prove things like $3^n > n^3$ *for* $n \geq 4$
- Exposure to rigorous thinking

Example problem

- Find the sum of the integers from 1 to *n*
- 1 + 2 + 3 + 4 + … + (n-1) + n

$$\sum_{i=1}^{n} i$$

- For any $n \geq 1$
- Could use brute force, but would be slow
- There's probably a clever shortcut

Finding the formula

- Shortcut will be some formula involving *n*
- Compare examples and look for pattern
- Start with n = 10:
  1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10
  - Large enough to be a pain to add up
  - Worthwhile to find shortcut

Finding the formula

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

$$= 5 \times 11$$

Finding the formula

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$$

$$= 4 \times 10 + 5$$

Finding the formula

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$$

$$= 4 \times 9$$

Finding the formula

$$1 + 2 + 3 + 4 + 5 + 6 + 7$$

$$= 3 \times 8 + 4$$

Finding the formula

n=7	$3 \times 8 + 4$
n=8	4×9
n=9	$4 \times 10 + 5$
n=10	5×11

Finding the formula

n=7	3×8 + 4	n is odd
n=8	4×9	n is even
n=9	4×10 + 5	n is odd
n=10	5×11	n is even

Finding the formula

When n is even

$$1 + 2 + 3 + 4 + \ldots + (n-3) + (n-2) + (n-1) + n$$

$$= (n/2) \times (n+1)$$

Finding the formula

3×8 + 4	
4×9	n(n+1)/2
4×10 + 5	
5×11	n(n+1)/2

Finding the formula

When n is odd

$$1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$$

$$= ((n-1)/2)\times(n+1) + (n+1)/2$$

$$= (n/2) \times (n+1)$$

Finding the formula

3×8 + 4	n(n+1)/2
4×9	n(n+1)/2
4×10 + 5	n(n+1)/2
5×11	n(n+1)/2

Are we done?

- The pattern seems pretty clear
  - Is there any reason to think it changes?
- But we want something for any $n \geq 1$
- A mathematical approach is skeptical

$$\frac{n(n+1)}{2}$$

Are we done?

- The pattern seems pretty clear
  - Is there any reason to think it changes?
- But we want something for *any* $n \geq 1$
- A mathematical approach is *skeptical*
- All we know is $n(n+1)/2$ works for 7 to 10
- We must *prove* the formula works in all cases
  - A *rigorous* proof

Proof by Induction

- Prove the formula works for all cases.
- Induction proofs have four components:

1. The thing you want to prove, e.g., *sum of integers from 1 to n = n(n+1)/2*
2. The base case (usually "let n = 1"),
3. The assumption step (“assume true for n = k")
4. The induction step (“now let n = k + 1").

n and *k* are just *variables*!

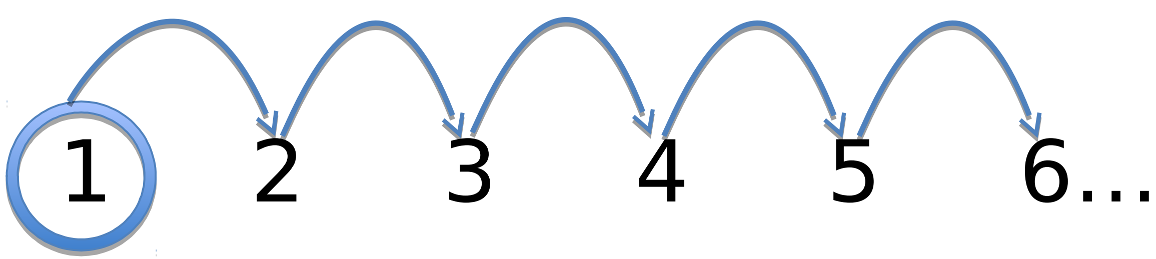
Proof by induction

- P(n) = sum of integers from 1 to n
- We need to do
  - Base case *prove for P(1)*
  - Assumption *assume for P(k)*
  - Induction step *show for P(k+1)*

- *n* and *k* are just *variables*!

Proof by induction

- P(n) = sum of integers from 1 to n
- We need to do
  - Base case *prove for P(1)*
  - Assumption *assume for P(k)*
  - Induction step *show for P(k+1)*



Proof by induction

- What we are trying to prove:

  $P(n) = n(n+1)/2$
- Base case
  - $P(1) = 1$
  - $1(1+1)/2 = 1(2)/2 = 1(1) = 1$

Proof by induction

- What we are trying to prove:
  $P(n) = n(n+1)/2$
- Assume true for k: $P(k) = k(k+1)/2$
- Induction step:
  - Now consider
    $P(k+1) = 1 + 2 + \ldots + k + (k+1)$

Proof by induction

- What we are trying to prove:

  $P(n) = n(n+1)/2$
- Assume true for k: $P(k) = k(k+1)/2$
- Induction step:
  - Now consider

    $P(k+1) = 1 + 2 + \ldots + k + (k+1)$

    $= k(k+1)/2 + (k+1)$

Proof by induction

- What we are trying to prove:

  $P(n) = n(n+1)/2$
- Assume true for k: $P(k) = k(k+1)/2$
- Induction step:
  - Now consider

    $P(k+1) = 1 + 2 + \ldots + k + (k+1)$

    $= k(k+1)/2 + (k+1)$

    $= k(k+1)/2 + 2(k+1)/2$

Proof by induction

- What we are trying to prove:

  $P(n) = n(n+1)/2$
- Assume true for k: $P(k) = k(k+1)/2$
- Induction step:
  - Now consider

    $P(k+1) = 1 + 2 + \ldots + k + (k+1)$

    $= k(k+1)/2 + (k+1)$

    $= k(k+1)/2 + 2(k+1)/2$

    $= (k(k+1) + 2(k+1))/2 = (k+1)((k+1)+1)/2$

We're done!

- P(n) = sum of integers from 1 to n
- We have shown
  - Base case *proved for P(1)*
  - Assumption *assumed for P(k)*
  - Induction step *proved for P(k+1)*

Success: we have proved that P(n) is true for any $n \geq 1$

Another one to try

- What is the sum of the first *n* powers of 2?
- $2^0 + 2^1 + 2^2 + ... + 2^{n-1}$

- k = 1: $2^0 = 1$
- k = 2: $2^0 + 2^1 = 1 + 2 = 3$
- k = 3: $2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$
- k = 4: $2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$

- For general n, the sum is $2^n - 1$

How to prove it

P(n) = “the sum of the first *n* powers of 2 is 2^n-1”

Theorem: *P(n)* holds for all $n \geq 1$

Proof: By induction on *n*

- Base case: *n*=1. Sum of first 1 power of 2 is 2^0 , which equals 1 = 2^1 - 1.
- Inductive case:
  - Assume the sum of the first *k* powers of 2 is 2^k-1
  - Show the sum of the first (*k*+1) powers of 2 is 2^{k+1}-1

How to prove it

- The sum of the first k+1 powers of 2 is

$$\underbrace{2^0 + 2^1 + 2^2 + \ldots + 2^{(k-1)}}_{\text{sum of the first k powers of 2}} + 2^k$$

sum of the first k powers of 2

by inductive hypothesis $= 2^k - 1 + 2^k$

$= 2(2^k) - 1 = 2^{k+1} - 1$

Conclusion

- Mathematical induction is a technique for proving something is true for all integers starting from a small one, usually 0 or 1.
- A proof consists of three parts:
  1. Prove it for the base case.
  2. Assume it for some integer k.
  3. With that assumption, show it holds for k+1
- It can be used for complexity and correctness analyses.