POLYTECH
NICE-SOPHIA

# Algorithms & Data Structures I

## Lesson 9: Binary Heaps

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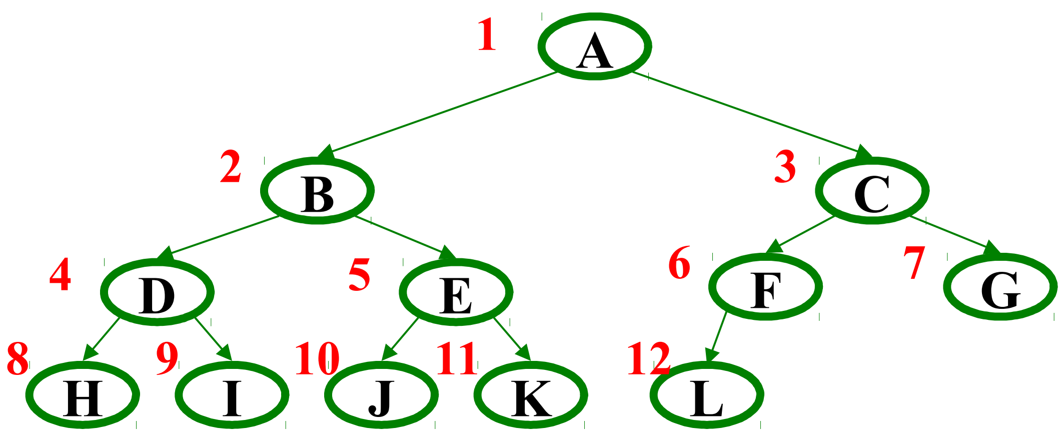
Edition 2015-2016

# *Array Representation of Binary Trees*




From node `i`:

left child: `i*2`
right child: `i*2+1`
parent: `i/2`

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **I** | **J** | **K** | **L** | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

# *Judging the array implementation*

Plusses:

- Non-data space: just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so *n*-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete
- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index **`size`**

Minuses:

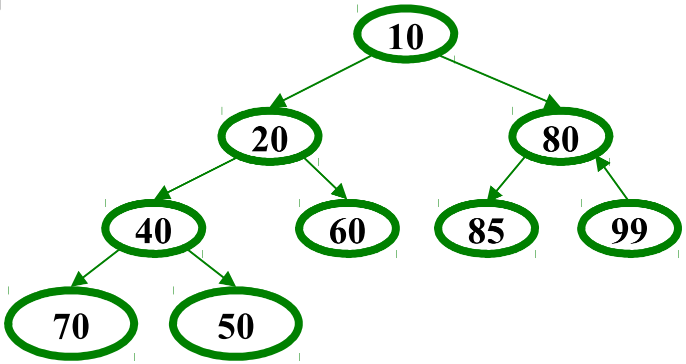
- Same might-be-empty or might-get-full problems we saw with stacks and queues (resize by doubling as necessary)

Plusses outweigh minuses: “this is how people do it”

# Pseudocode: insert into binary heap

```
void insert(int val) {
   if(size==arr.length-1)
     resize();
   size++;
   i=percolateUp(size,val);
   arr[i] = val;
}
```

```
int percolateUp(int hole,
                int val) {
   while(hole > 1 &&
         val < arr[hole/2])
     arr[hole] = arr[hole/2];
     hole = hole / 2;
   }
   return hole;
}
```

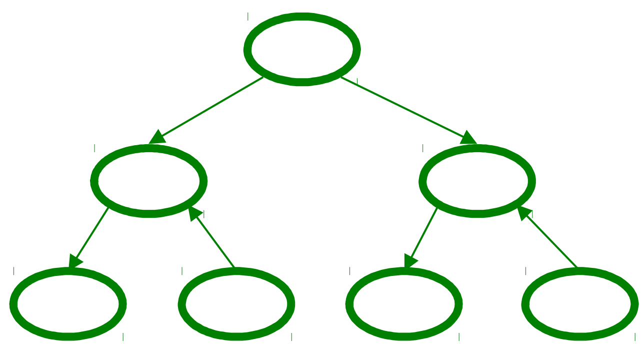



| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 10 | 20 | 80 | 40 | 60 | 85 | 99 | 70 | 50 | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

# *Example of insert*

insert: 16, 32, 4, 67, 105, 43, 2

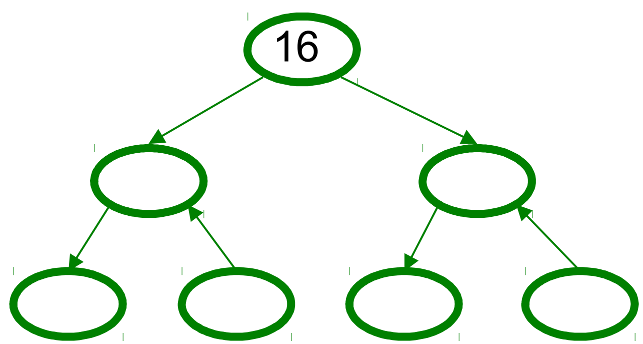
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



# *Example of insert*

insert: **16**, 32, 4, 67, 105, 43, 2

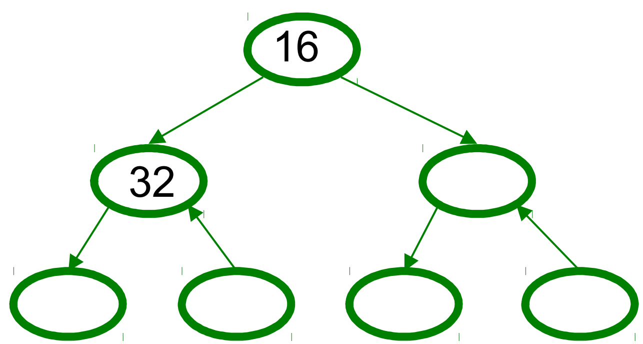
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **16** | | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



# *Example of insert*

insert: **<u>16</u>**, **<u>32</u>**, 4, 67, 105, 43, 2

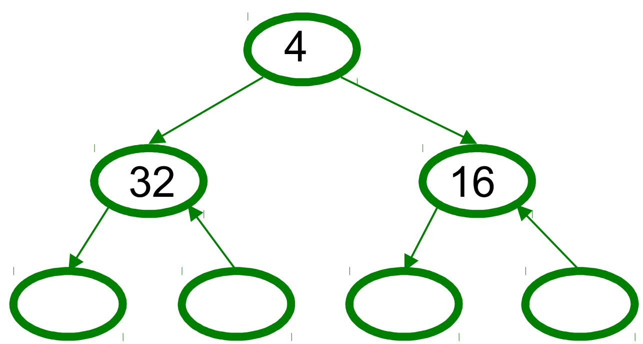
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **16** | **32** | | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



# *Example of insert*

insert: **<u>16</u>**, **<u>32</u>**, **<u>4</u>**, 67, 105, 43, 2

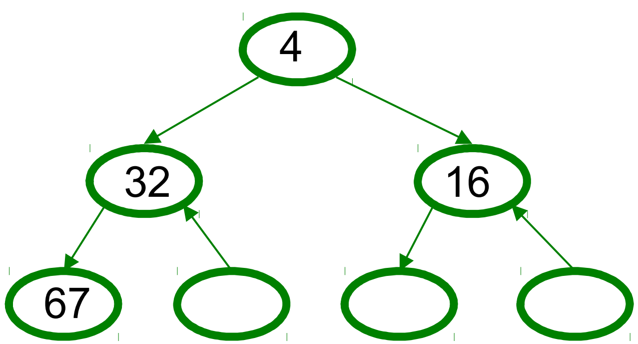
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **4** | **32** | **16** | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



# *Example of insert*

insert: **<u>16</u>**, **<u>32</u>**, **<u>4</u>**, **<u>67</u>**, 105, 43, 2

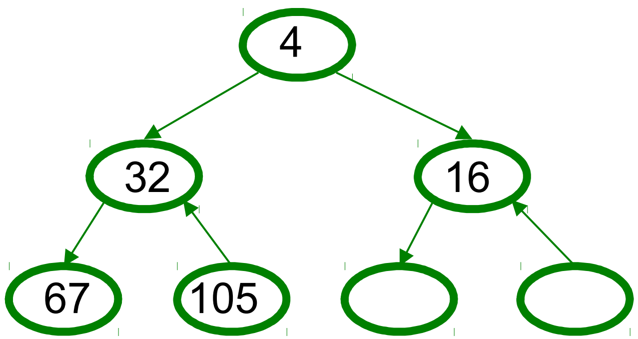
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **4** | **32** | **16** | **67** | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



# *Example of insert*

insert: **16**, **32**, **4**, **67**, **105**, 43, 2

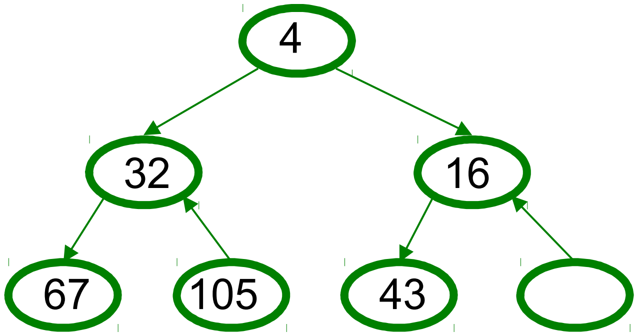
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **4** | **32** | **16** | **67** | **105** | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



# *Example of insert*

insert: **<u>16</u>**, **<u>32</u>**, **<u>4</u>**, **<u>67</u>**, **<u>105</u>**, **<u>43</u>**, 2

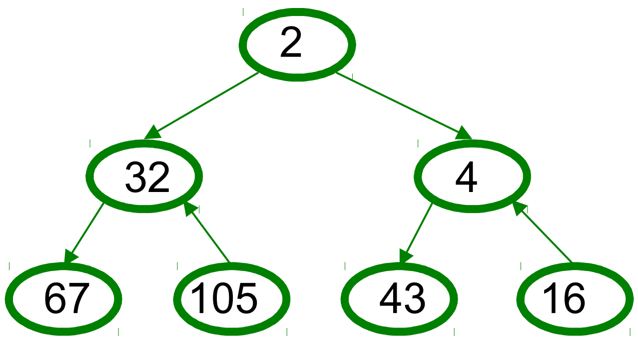
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **4** | **32** | **16** | **67** | **105** | **43** | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



# *Example of insert*

insert: **<u>16</u>**, **<u>32</u>**, **<u>4</u>**, **<u>67</u>**, **<u>105</u>**, **<u>43</u>**, **<u>2</u>**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **2** | **32** | **4** | **67** | **105** | **43** | **16** |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



# *Other operations*

- **`decreaseKey`**: given pointer to object in priority queue (e.g., its array index), lower its priority value by *p*
  - Change priority and percolate up

- **`increaseKey`**: given pointer to object in priority queue (e.g., its array index), raise its priority value by *p*
  - Change priority and percolate down

- **`remove`**: given pointer to object in priority queue (e.g., its array index), remove it from the queue
  - **`decreaseKey`** with $p = \infty$, then **`deleteMin`**

Running time for all these operations?

# *Build Heap*

- Suppose you have *n* items to put in a new (empty) priority queue
  - Call this operation **`buildHeap`**

- *n* **`insert`**s works
  - Only choice if ADT doesn't provide **`buildHeap`** explicitly
  - $O(n$ **`log`** $n)$

- Why would an ADT provide this unnecessary operation?
  - Convenience
  - Efficiency: an $O(n)$ algorithm called Floyd's Method
  - Common issue in ADT design: how many specialized operations

# *Floyd's Method*

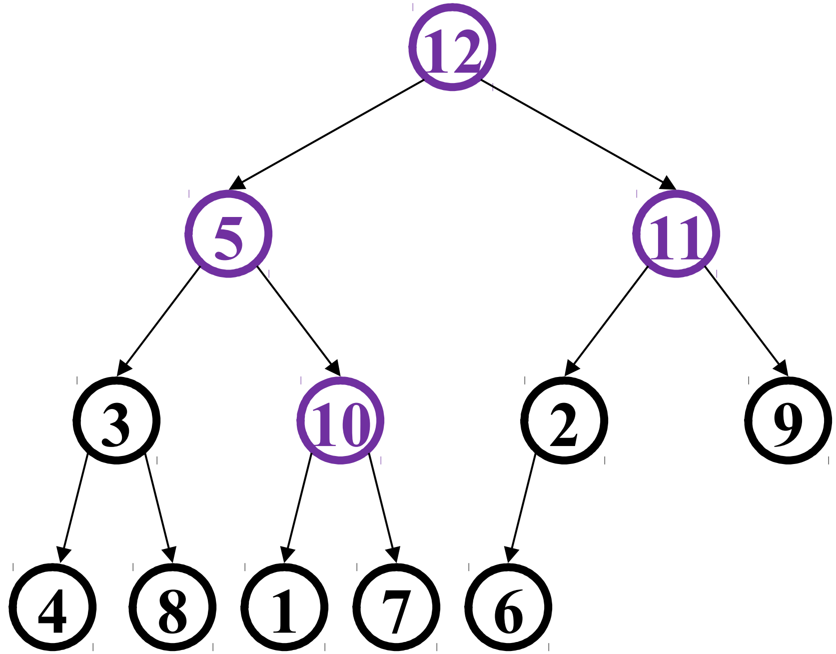
1. Use *n* items to make any complete tree you want
   - That is, put them in array indices 1,…,*n*

2. Treat it as a heap and fix the heap-order property
   - Bottom-up: leaves are already in heap order, work up toward the root one level at a time

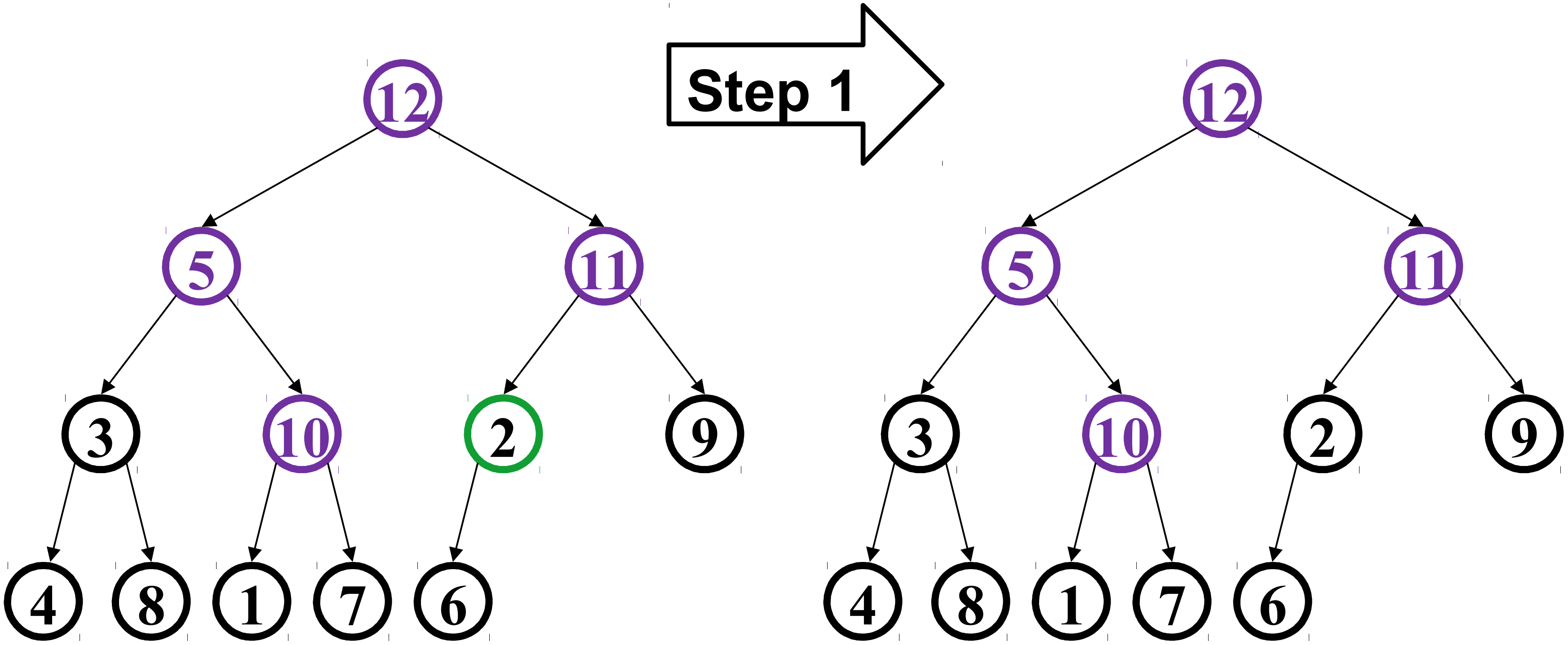
```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

# *Example*

- In tree form for readability
  - Purple for node not less than descendants
    - heap-order problem
  - Notice no leaves are purple
  - Check/fix each non-leaf bottom-up (6 steps here)

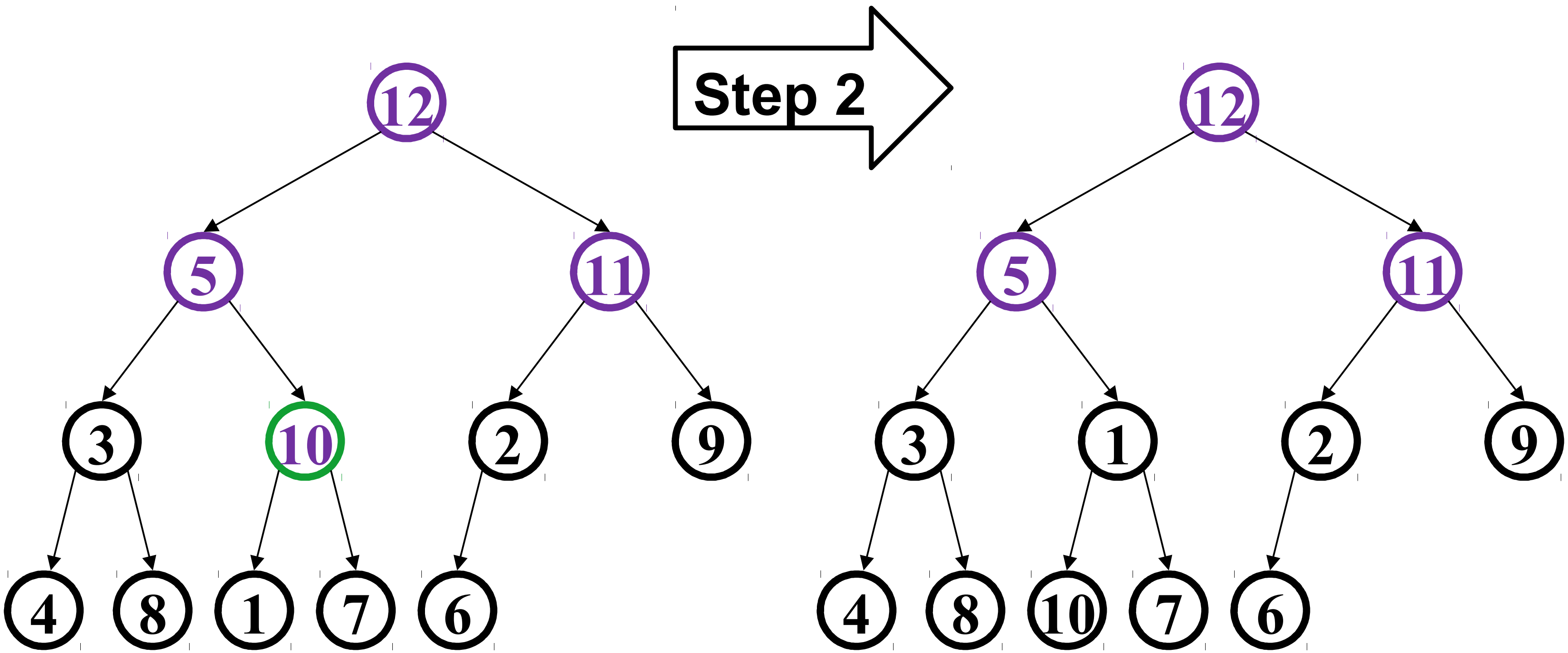


# Example



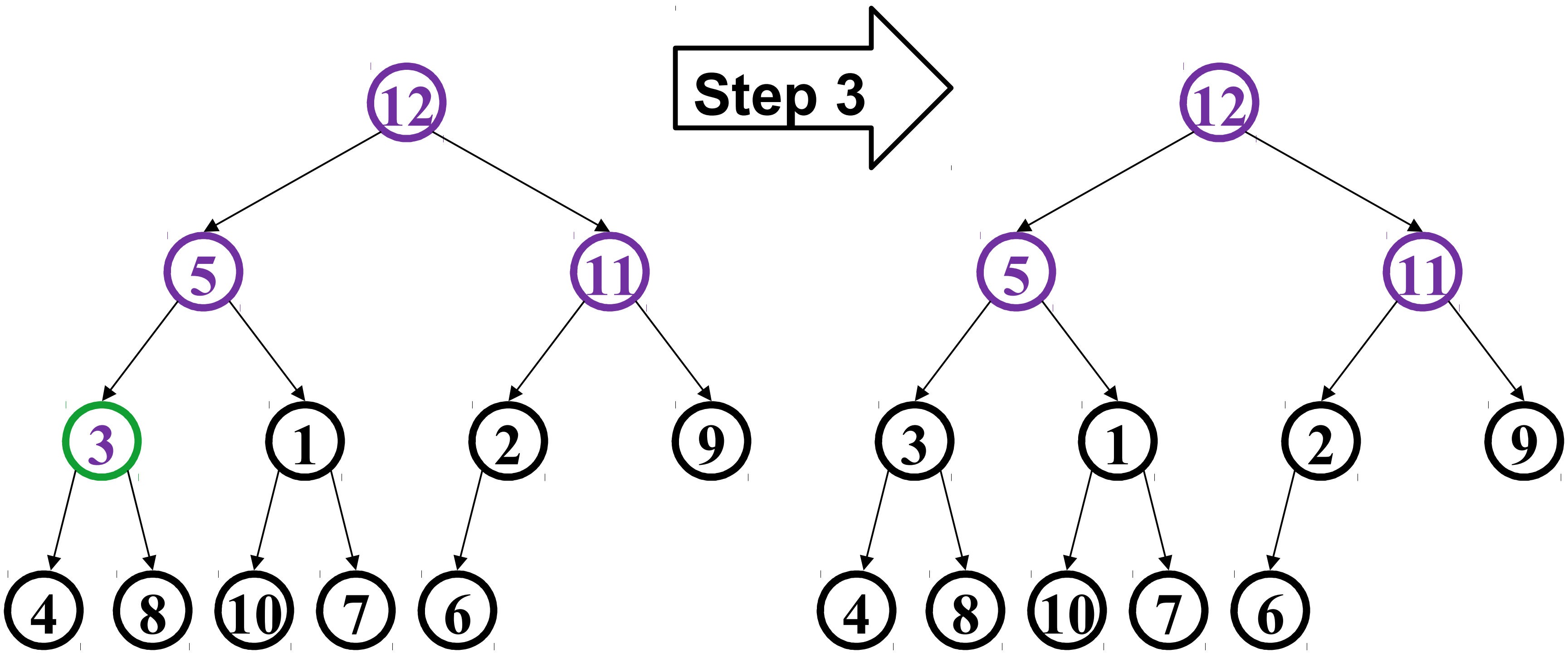

- Happens to already be less than children (er, child)

# *Example*



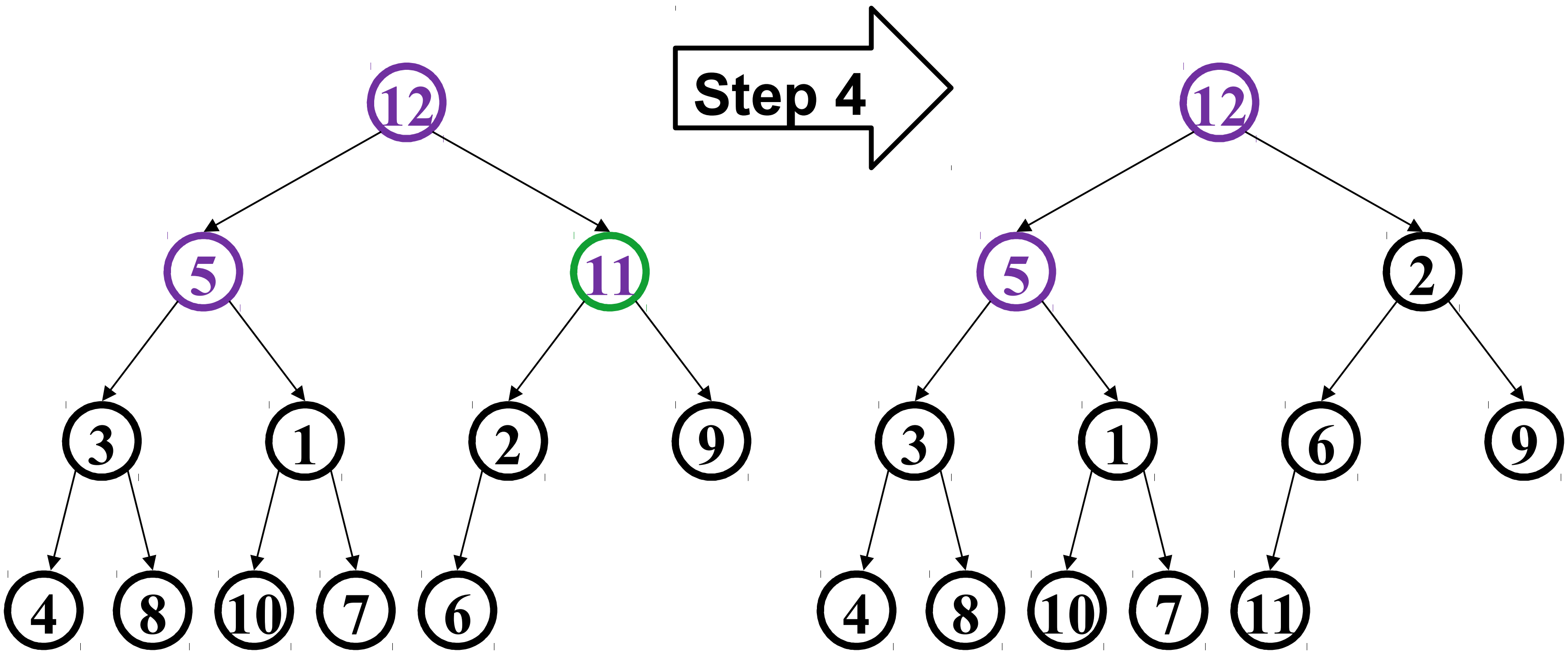

- Percolate down (notice that moves 1 up)

# *Example*



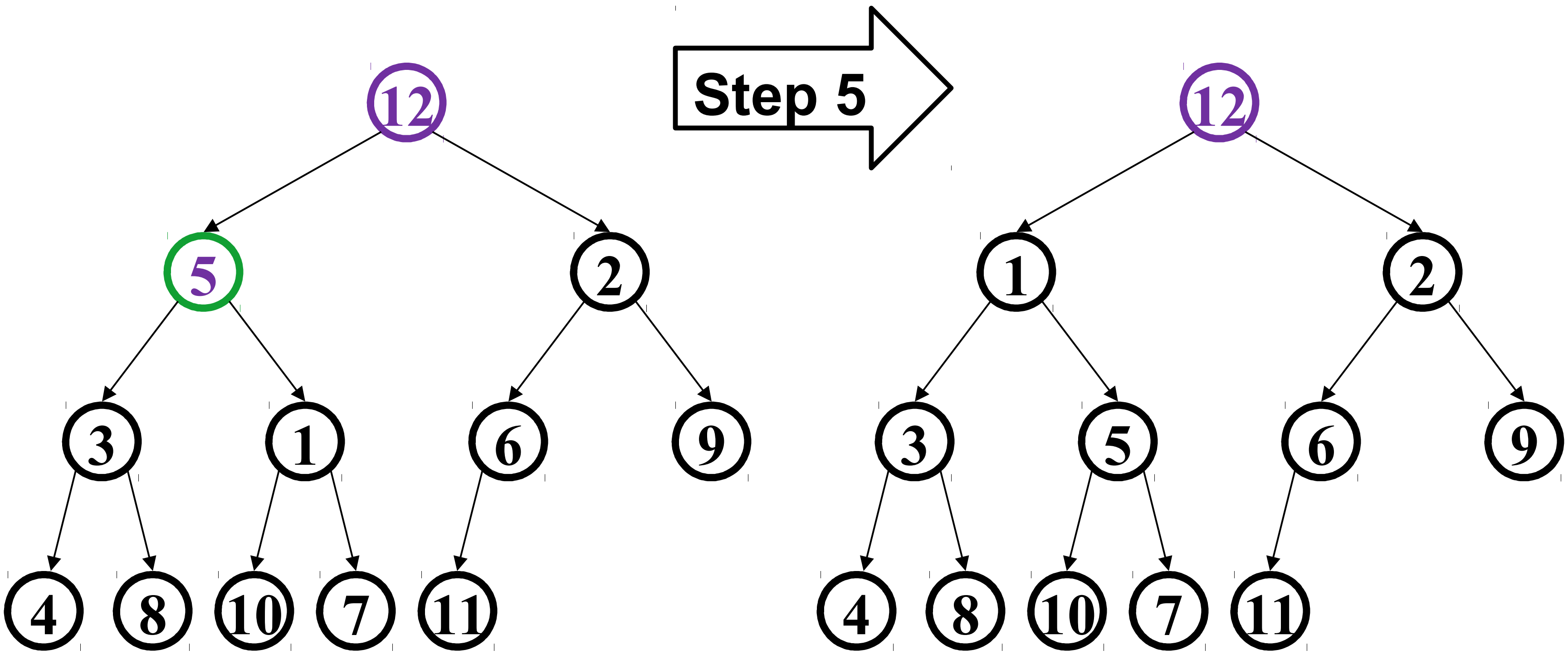

- Another nothing-to-do step

# Example

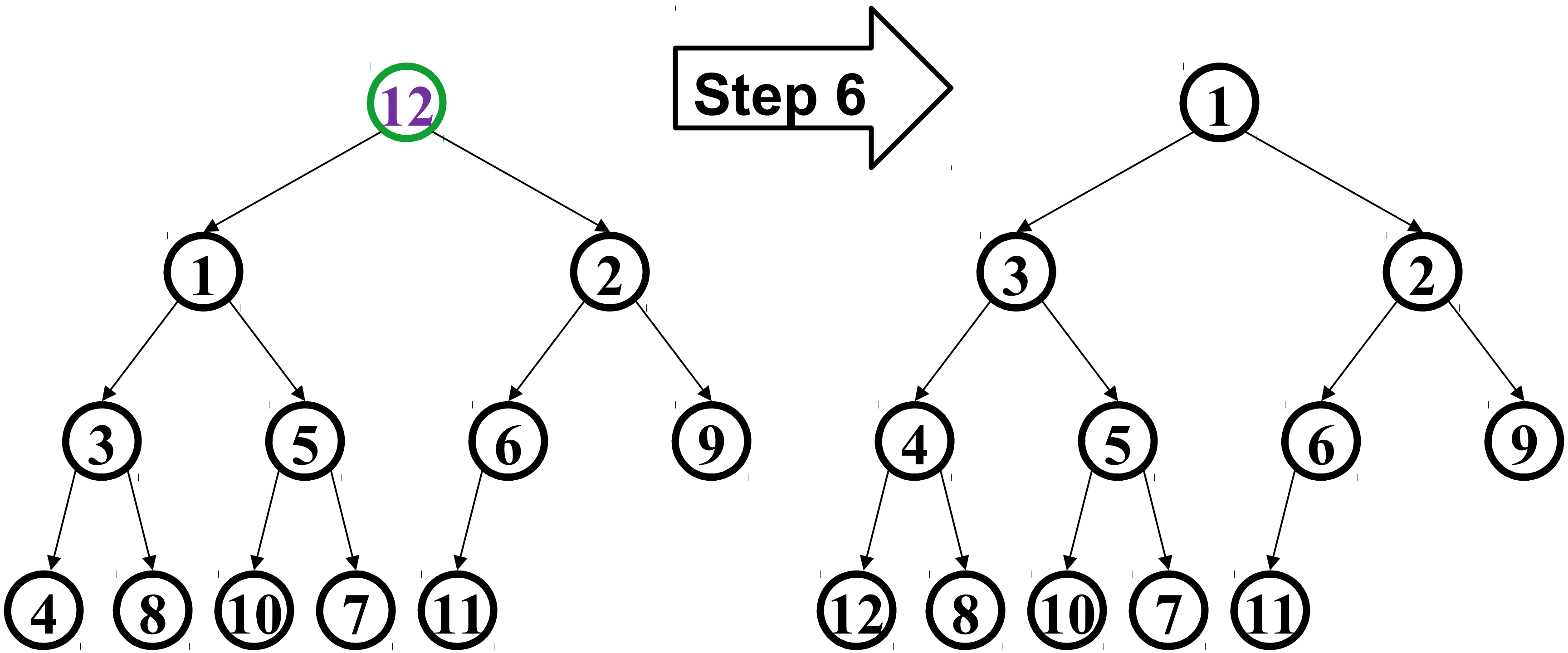



- Percolate down as necessary (steps 4a and 4b)

# Example



# Example



# *But is it right?*

- “Seems to work”
  - Let’s *prove* it restores the heap property (correctness)
  - Then let’s *prove* its running time (efficiency)

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

# Correctness

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

*Loop Invariant:* For all `j>i`, `arr[j]` is less than its children

- True initially: If `j > size/2`, then `j` is a leaf
  - Otherwise its left child would be at position > `size`
- True after one more iteration: loop body and `percolateDown` make `arr[i]` less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

# Efficiency

```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Easy argument: **`buildHeap`** is $O(n \textbf{ log } n)$ where $n$ is **`size`**

- **`size/2`** loop iterations
- Each iteration does one **`percolateDown`**, each is $O(\textbf{log } n)$

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

# Efficiency

```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Better argument: **`buildHeap`** is $O(n)$ where $n$ is **`size`**

- **`size/2`** total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps
- …
- ((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + …) < 2 (page 4 of Weiss)
  - So at most **`2(size/2)`** *total* percolate steps: $O(n)$

# *Lessons from* `buildHeap`

- Without **`buildHeap`**, our ADT already let clients implement their own in $O(n \texttt{ log } n)$ worst case
- By providing a specialized operation internal to the data structure (with access to the internal data), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness:
    - Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was $O(n \texttt{ log } n)$
    - Tighter analysis shows same algorithm is $O(n)$