Closing a Platonic Gap

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This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on.

Contributions are most welcome.

➢ Submissions should be uploaded to http://tmin.edmgr.com or sent directly to **Sergei Tabachnikov,** tabachni@math.psu.edu

For four of the five Platonic solids, there is a toroidal ring using copies of the solid that meet face-to-face. This is trivial for cubes and not too hard for octahedra, icosahedra, and dodecahedra (Fig. 1). Here we close a gap in this area by presenting such rings for tetrahedra. The problem is that, in 1958, S. Świerczkowski [SS, SS1] proved that no such ring exists!

Background

In 1957 Hugo Steinhaus [HS] asked whether there was a perfectly closed loop of congruent regular tetrahedra. Precisely: Is there a finite sequence $T_1, \ldots, T_n$ of regular tetrahedra that meet face-to-face and do not double back (such is called a *Steinhaus chain*), and are such that a face of T_n coincides with a face of T_1? Świerczkowski showed that such a closed chain cannot exist. Other approaches were found by Dekker [D] and Mason [M]. General chains are allowed to intersect, so let us call a Steinhaus chain *embedded* if each two tetrahedra in the chain have disjoint interiors.

We wondered how close one could get to a true *tetratorus* (an embedded tetrahedral torus made from regular tetrahedra). Initial explorations led to a loop of 48 tetrahedra (assume centimeter-long edges throughout) having a gap of 2 mm. It seemed that the overall error could be reduced by using perturbations to spread the gap around the tetrahedra; this did indeed work to yield the fake tetratorus in Figure 2. The loop closes up perfectly and has perfect 8-fold rotational symmetry, but 32 of the 48 tetrahedra are about 0.0014 cm away from being regular. Such an error is less than the tolerance of a 3-dimensional printer, the output of which is shown in Figure 2. The method of distributing the gap was to uniformly elongate 32 of the 144 edges. It came as a surprise that this could be done with an elongation factor that was expressible by radicals, but this approach became irrelevant with our later discoveries, which yielded chains of regular tetrahedra with vastly smaller gaps, less than a thousandth of the radius of a proton.

A loop of length 36 with very small error (0.0005 cm) is shown in Figure 3. See the third section for the details of how the sequence of reflections in such loops is found. A proof that a true tetratorus cannot exist is in the second section.

One surprising regular tetrahedral structure is the spiral chain (Fig. 4) known as the *tetrahelix*, or *Boerdijk-Coxeter helix* [BC], or *Bernal spiral*. It arises by repeating the sequence of reflections in the face opposite vertex 1, the face opposite vertex 2, the face opposite vertex 3, and the face opposite vertex 4. The largest example is an art museum in Mito City, Japan: a length-28 tetrahelix about 100 meters high (Fig. 4).

The Tetratorus Does Not Exist

This section contains Mason's approach to the impossibility proof; more details are in [W]. The essence of the

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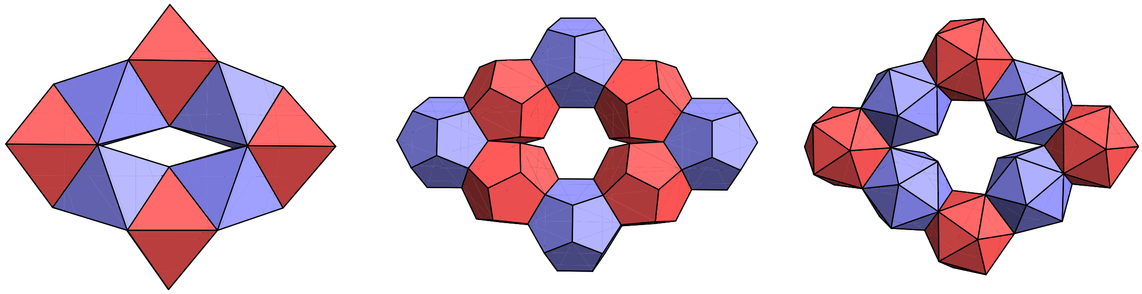

Figure 1. Octahedral, dodecahedral, and icosahedral tori.

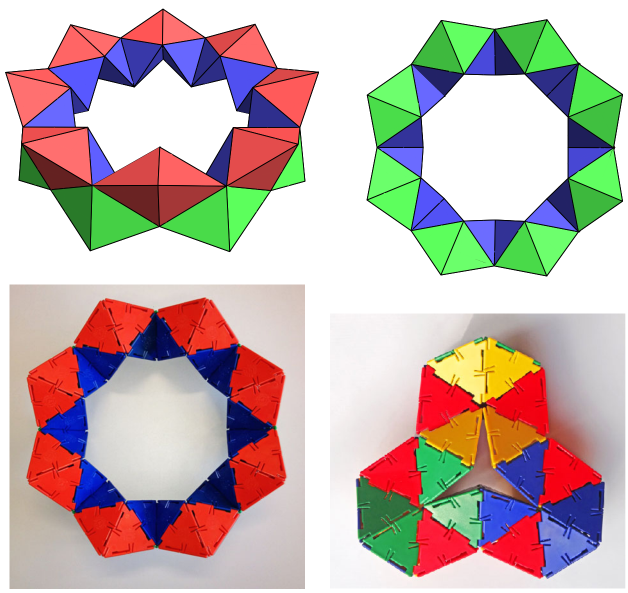

Figure 2. A fake tetratorus of length 48. The model at lower left is from a 3-dimensional printer via shapeways.com (available at [W1]), whereas the model at lower right was made by John Sullivan, using Polydron™ pieces.

theorem is that the group generated by the reflections in the four faces is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.

THEOREM (ŚWIERCZKOWSKI) The last tetrahedron in a Steinhaus chain cannot coincide with the first.

PROOF Let ϕ_1, ϕ_2, ϕ_3, ϕ_4 be the four reflections in the faces of a regular tetrahedron T in $\mathbb{R}^3$. Any point in $\mathbb{R}^3$ may be represented uniquely as $x_1V_1 + x_2V_2 + x_3V_3 + x_4V_4$, where the V_i are T's vertices and $\sum x_i = 1$; these are barycentric coordinates with respect to T. Each ϕ_i may be represented by a 4×4 matrix acting on barycentric coordinates, where the columns are the vectors $\phi_i(V_i)$. Composition corresponds to matrix multiplication.

Because the reflection in the face $x_i = 0$ sends V_i to $C + (C - V_i) = 2C - V_i = 2\left(\frac{1}{3}\Sigma_{j\neq i} V_j\right) - V_i$, where C is the centroid of the face opposite V_i (Fig. 5), the barycentric matrices for the ϕ_i are

$$M_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & \frac{2}{3} & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \end{pmatrix} \quad M_4 = \begin{pmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now if the last tetrahedron in a chain agrees with the first, there is a sequence $i_1, \ldots, i_s$ with no consecutive pair the same so that $M_{i_1}M_{i_2} \cdots M_{i_s}$ is a permutation matrix. We can and will assume $i_1 = 1$. The next claim gives the structure of the matrix product, which will imply that it cannot be a permutation matrix.

CLAIM Consider the product $M_1M_{i_2} \cdots M_{i_s}$ with 2/3 replaced by x. The polynomials in the second row have

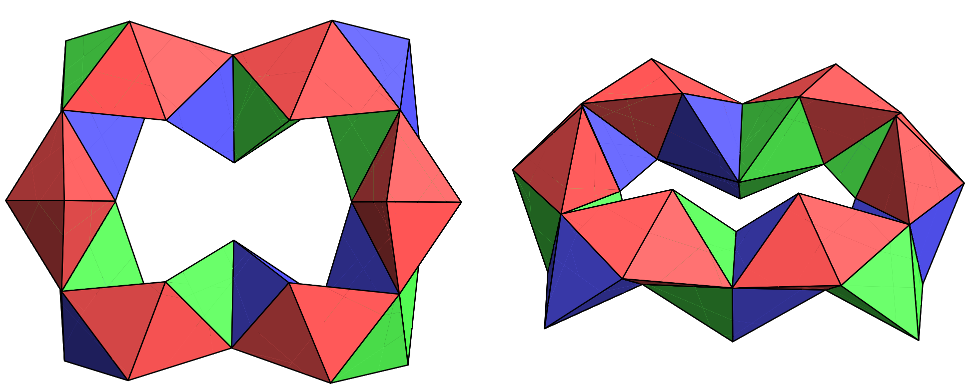

Figure 3. A length-36 fake tetratorus with a final gap of about 0.0005 cm.

Figure 4. The Mito Art Tower, Mito City, Japan: a 100-meter tall titanium tetrahelix.

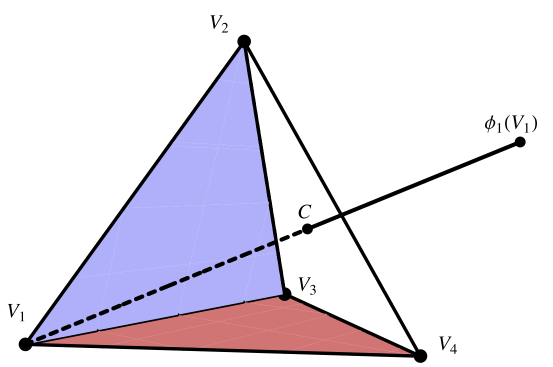

Figure 5. Reflection in a side of a regular tetrahedron.

x-degree less than s except for the one in the i_sth column, which has degree s. And they all have leading coefficient $+1$.

Proof of Claim By induction; it is clear for $s = 1$. Consider what happens when the matrix of a word that ends in M_j, assumed to have the claimed form, is multiplied on the right by M_n, with $n \neq j$. The multiplications by x preserve the claimed property, as the degree becomes $s + 1$ in the nth position of row 2, but does not rise at all elsewhere in the row. And the leading coefficient's sign is affected only by the x multipliers.

Now look at the polynomial in the $(2, i_s)$ position: $x^s + a_1x^{s-1} + \ldots + a_s$, where $a_i \in \mathbb{Z}$. Setting $x = 2/3$ and taking a common denominator yields $(2^s + 3a_1 2^{s-1} + \ldots + 3^{s-1}a_s)/3^s$, the numerator of which is not divisible by 3; the fraction is therefore not 0 or 1, as required. $\square$

The Great Tetratorus Hunt

The hunt should not be abandoned simply because the quarry does not exist! Here we will present several fake tetratori that will satisfy the most demanding practical geometer. Whether there are examples that satisfy *any* positive tolerance is an open question (see the fourth section); evidence suggests that the answer is yes.

The key to finding loops with small error is exploring many strings using the barycentric representations M_i of the four reflections of the preceding section. We use 1, 2, 3, and 4 to encode the matrices; so the example in Figure 2 comes from the length-48 string $(121434)^8$, whereas the length-36 example of Figure 3 corresponds to $(123141321241232142)^2$. And the tetrahelix is simply a power of 1234.

The search space is gigantic—3^{n-2} strings of length n, because we can assume a start of 12—so some shortcuts are necessary and one uses symmetry and intuition to focus on likely candidates. A *Mathematica* program was constructed that allowed easy exploration, and then periodicity could be used in a number of ways. This allowed the examination of long chains and led to some remarkable examples, including γ_{164}, with error at the end a minutely small 10^{-14} cm.

Suppose K is a product of r reflection matrices M_i. A bit of linear algebra shows that the eigenvalues of K are the eigenvalues of $\begin{pmatrix} & & & t_1 \\ & (-1)^r R & & t_2 \\ & & & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where R is a rotation fixing the origin and (t_i) represents the translational part of the isometry determined by K, which is

$$X \to (-1)^r R \cdot X + (t_1, t_2, t_3).$$

It follows that K's eigenvalues are $\{1, (-1)^r, e^{i\theta}, e^{-i\theta}\}$, for some θ. Our main search mechanism was a generalization of the 48-loop that has been very fruitful. One obtains K from a special sequence, one of the form $Sj\bar{S}(Sj\bar{S})_p$, where S starts with 1, $j \in \{1, 2, 3, 4\}$ and differs from the last integer in S, the subscript p refers to a permutation that does not fix 1, and the bar denotes string reversal. Note that this is the concatenation of two palindromes. The search looks for such examples for which a power of K is close to the identity. If K's θ-value is near a rational multiple of π, then some power of K will almost equal the identity. It turns out that the special form has a very strong preference for θ-values near π or $2\pi/3$ (see Fig. 8).

The current record holder for the special form is γ_{164} (Table 1; Fig 7); the discrepancy from perfect closure (the *gap error*: the maximum distance between corresponding vertices) is $1.1 \cdot 10^{-14}$ cm, smaller than the radius of a proton. But a much smaller error was obtained by considering the new form $Sj\bar{S}(Sj\bar{S})_p\, Sj\bar{S}(Sj\bar{S})_{p^{-1}}$. We found an example with error under $6 \cdot 10^{-18}$ cm: $\gamma_{540} = (Z\, Z_{(1234)}\, Z\, Z_{(1432)})^3$, where $Z = S4\bar{S}$ and S = 1234123434132341213412. It can be important to consider possible shifts of the string, as that can reduce the error; when we present errors, we will always use the minimal error for a shift that leads to an embedded chain. Such a shift is equivalent to conjugating the final matrix.

A noteworthy chain is the 174-loop defined by $\gamma_{174} = \left(S4\bar{S}(S4\bar{S})_{(134)}\right)^3$, where $S = 12342342321423$; the

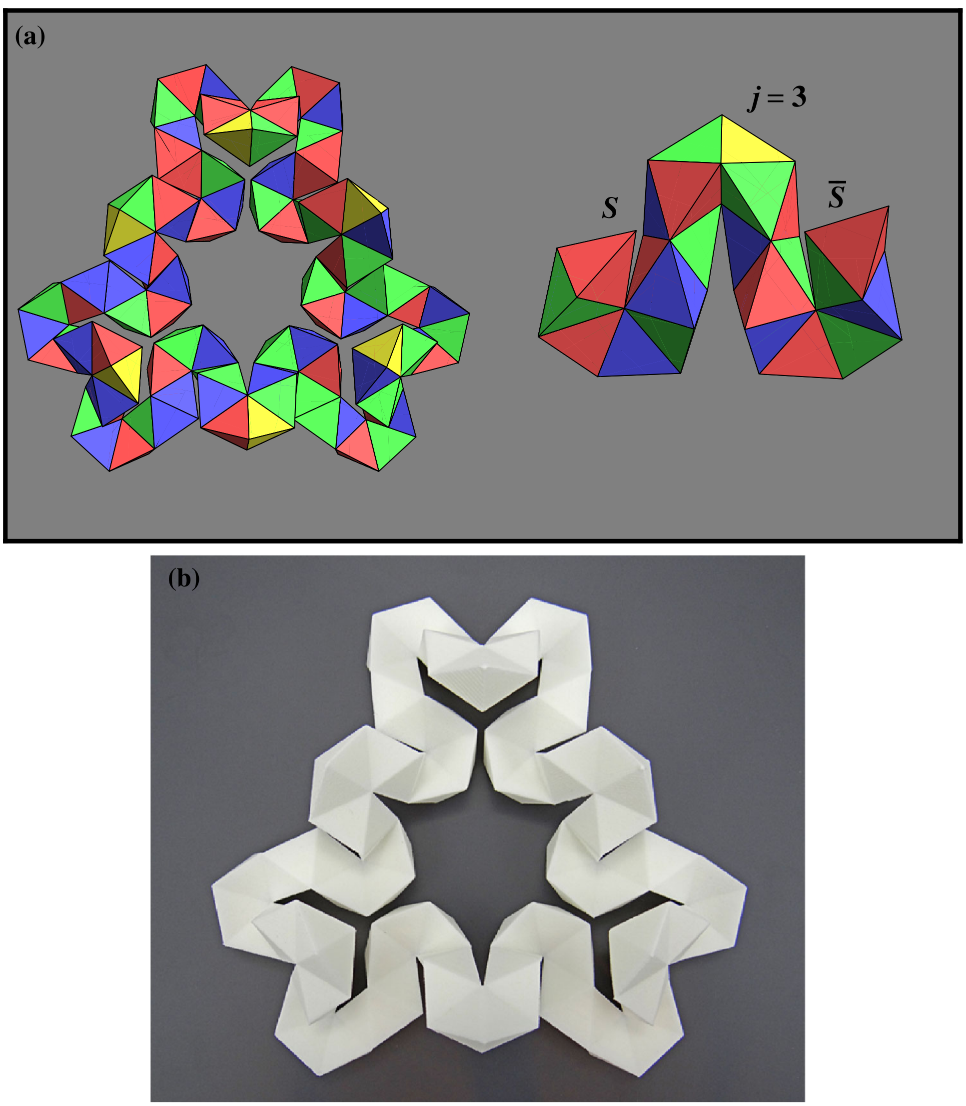

Figure 6. (a) A fake tetratorus using 174 tetrahedra: the terminal gap has size $1.4 \cdot 10^{-13}$ cm. It is made from six copies of the 29-chain at right, which is labeled to show how the 14-string S, the pivot 3, and $\bar{S}$ combine. Yellow indicates the location of the pivot. (b) A 3-dimensional printer model with 1-cm edge length; the final gap has size about the diameter of a proton.

gap error in γ_{174} is just over 10^{-13} cm (see Fig. 6). The chain γ_{174} as given actually has a small collision at the end, as opposed to a small gap, and so is not embedded. But if one applies λ, a leftward shift of 17 characters, then the collision becomes a gap, and the difference between the identity matrix and the product of the 174 matrices M_i corresponding to $\lambda(\gamma_{174})$ is

$$\begin{pmatrix} 3.2\times10^{-14} & 3.\times10^{-14} & -3.\times10^{-16} & 6.6\times10^{-14} \\ -1.3\times10^{-14} & -1.7\times10^{-14} & 5.2\times10^{-15} & -3.7\times10^{-14} \\ 8.7\times10^{-14} & 3.\times10^{-14} & 5.3\times10^{-14} & 4.1\times10^{-14} \\ -1.3\times10^{-13} & -6.8\times10^{-14} & -8.1\times10^{-14} & -9.2\times10^{-14} \end{pmatrix}$$

The error in the matrix difference is of the same order of magnitude as the gap error. One can express the final matrix K^3 in rationals as a guard against roundoff error. For γ_{174}, the upper left entry is exactly

$$\frac{1290070078170121 < 50 > 0046975423053339}{1290070078170102 < 50 > 4597492642263849}$$

A 3D-printed version of γ_{174} using edges that are one centimeter long (Fig. 6(b)) is available at [W2] (at that site one can rotate the image).

Our general approach in finding near-tetratori was first to isolate strings that came close to closure and then to discard the intersecting ones. This last step was eventually automated using the separating plane part of the algorithm in [GPR].

Searching through increasing lengths of the special form was quite fruitful; Table 1 lists 16 embedded chains having small error, as well as an elegant loop (length 30) with a collision instead of a gap. Several are pictured in Figure 7. The demonstration at [EW] allows the user to view and rotate many of these loops, as well as experiment with other sequences.

The Ultimate Tetratorus

The various small-error specimens lead naturally to the following conjecture.

CONJECTURE For any $\epsilon > 0$, there is an embedded Steinhaus chain that closes up with terminal gap of size less than ϵ.

The examples in the third section show that the conjecture is likely true even when restricted to the special form.

CHALLENGE Find an embedded Steinhaus chain that closes up with error below our best, $5.6 \cdot 10^{-18}$.

We can use the pigeonhole principle (suggested by John Sullivan) to show that the minimum closure error for all chains (ignoring the intersection issue) decreases exponentially. First, a tetrahelix fact:

LEMMA The tetrahelix with n tetrahedra has the largest diameter—asymptotically $n/\sqrt{10}$—of any chain of the same length.

PROOF One easily checks that for $n \leq 4$, a length-n tetrahelix has a larger diameter than any other chain of length n. Now if a chain of length n had a larger diameter than the corresponding tetrahelix 1234123…, then, by the triangle inequality, some consecutive 4-tuple (or shorter) would have a longer span than the corresponding tetrahelix, which cannot be. The tetrahedral vertices in a vertical tetrahelix are $(3\sqrt{3}\cos(n\theta), 3\sqrt{3}\sin(n\theta), n)/\sqrt{10}$, and the asymptotic result follows.

Table 1. Seventeen Steinhaus chains with small error; one (30) is not embedded, but it is very elegant (see Fig. 7). All but one (60) have the special form $(Sj\bar{S})(Sj\bar{S})_p$, where the bar is string reversal; S is shown in red, whereas j and p (in cycle notation) are in the last two columns. The error is always the least error obtained by any rotation of the final string that leads to an embedded chain; such a rotation will not have the special form, but will yield essentially the same physical chain. The boldface ones are shown in Figure 2, 3, 6, or 7

Length	Base sequence	Periods	Error	j	Permutation
48	**1 2 1 4 3 4**	**8**	**0.186**	**2**	**(14)(23)**
60	12343	12	0.165	—	—
30	**12 3 21 32 4 23 (not embedded)**	**3**	**0.015**	**3**	**(134)**
36	**1231 4 1321 2412 3 2142**	**2**	**0.00027**	**4**	**(1243)**
68	12313143 1 34131321 23424214 2 41242432	2	0.00021	1	(1234)
60	**1234212 3 2124321 2341323 4 3231432**	**2**	**$1.09\cdot10^{-5}$**	**3**	**(1234)**
84	**1234231243 2 3421324321 4312314321 3 1234132134**	**2**	**$1.31\cdot10^{-6}$**	**2**	**(1423)**
92	**12423142131 4 13124132421 23134213242 1 24231243132**	2	$1.02\cdot10^{-6}$	4	(1234)
174	**12342321342132 4 23124312324321 23143132143213 4 31234123134132**	3	$6.22\cdot10^{-7}$	4	(123)
116	**12323132432314 3 41323423132321 23434243143421 4 12434134243432**	**2**	**$5.09\cdot10^{-11}$**	**3**	**(1234)**
124	1234231412321343 2 3431232141324321 3421423134243212 4 2123424313241243	2	$2.43\cdot10^{-12}$	4	(1324)
132	1234231412321343 2 3431232141324321 3421423134243212 4 2123424313241243	2	$1.23\cdot10^{-12}$	2	(1324)
140	1232412342431434 1 2 14341342432142321 34241342141231213 4 31213214124314243	2	$8.6\cdot10^{-13}$	2	(1324)
108	1234243412421 3 1242143424321 2413431324342 1 2434231343142	2	$8.16\cdot10^{-13}$	3	(1243)
174	**12342342321423 4 32412324324321 24314314342143 1 34124341341342**	**3**	**$1.23\cdot10^{-13}$**	**4**	**(124)**
148	**12342313141243 2123 4 321234214131324321 314214343231241314 2 413142132343412413**	**2**	**$7.4\cdot10^{-14}$**	**4**	**(1342)**
164	**123214123234143 24124 3 421423414323214123 21 241423241413231432 43 1 342341323141423241 42**	**2**	**$5.05\cdot10^{-14}$**	**3**	**(1243)**

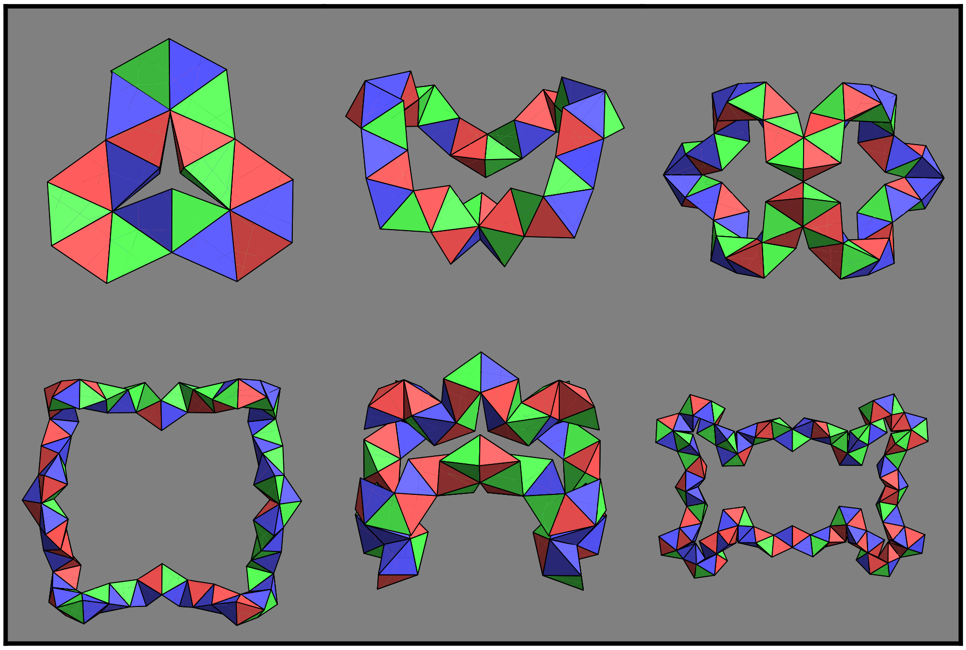

Figure 7. Six specimens from the tetratorus zoo. Clockwise from top left, the lengths are 30, 60, 84, 164, 116, and 148. These are all embedded period-2 chains, except the first, which is not embedded and has period 3. The 84-loop has a small space between the central parts that seem to touch. The gap errors are in Table 1; γ_{164} at lower right, is the current record holder for the special form, with error under $1.1 \cdot 10^{-14}$.

Proposition For sufficiently large n, there is a Steinhaus chain of length between 2 and n that closes up to within error $0.86\sqrt{n}3^{-n/12}$.

Proof Because of the asymptotic nature of the assertion, we can ignore the parity issue for n; take it to be odd. Consider all $4 \cdot 3^{(n+1)/2-2}$ chains of length $(n+1)/2$ that start from a tetrahedron anchored at the origin. By the lemma, and working asymptotically, the farthest any such chain can extend is $\rho = \frac{n+1}{2}\frac{1}{\sqrt{10}}$. The pigeonhole principle will tell us that, for an appropriate ε, there must be two such chains whose last tetrahedra coincide to within ε. Start at the end of one of them, backtrack to the anchor, and then continue out to the other end; this yields an n-chain that, within error ε, is a closed loop. However, this chain might well have some cancellation at the center; e.g., if the two $(n+1)/2$-chains were 1232124 and 1232142, they would merge to 42123211232142, which reduces to 4242. The chains are not identical, so they cannot fully cancel.

Given an $(n+1)/2$-chain, consider the centroid V of the tetrahedron farthest from the origin; it lies in the ball of radius ρ. After V is specified, the last tetrahedron is completely determined by three angles: two (longitude and latitude) determine a unit vector in a hemisphere that defines a line from V to the farthest vertex, and one provides a rotation about that line to determine the whole tetrahedron. The first two of these angles are in $[0, 2\pi)$ and $\left[0, \tan^{-1}\sqrt{23}\right]$, respectively, whereas the last one can be taken in $[0, 2\pi/3)$.

To conclude, we need to divide the 6-dimensional state space for all such terminal tetrahedra into pigeonholes so that two states in the same pigeonhole lead to tetrahedra that coincide to within ε. The aim is to find ε so that the number of pigeonholes is just less than the number of chains. So we divide the ball's volume into small boxes of side ε and the three angular intervals into subintervals of length ε. This leads to the following inequality, involving ε^6, to guarantee that there are more pigeons than holes.

$$\frac{2\pi}{\epsilon}\,\frac{\tan^{-1}\sqrt{23}}{\epsilon}\,\frac{2\pi}{3\epsilon}\,\frac{1}{\epsilon^3}\,\frac{4}{3}\pi\rho^3 < 4 \cdot 3^{(n-3)/2}.$$

This can be solved, yielding: $\epsilon > 0.853\sqrt{n}\,3^{-n/12}$. The use of 0.86 in the statement takes care of small issues such as the parity of n. □

The proposition ignores intersections, so it gives no information about embedded chains. But several examples in Table 1 have error much smaller than the preceding bound: e.g., the bound for $n = 164$ is about 10^{-6}, whereas the gap error for γ_{164} is a million times smaller. Putting error aside and looking only at geometry, we looked at the number of embedded chains of length n. Experiments up to $n = 15$ suggest that, among the 3^{n-2} chains of length n beginning with the reflections M_1M_2, about $1.4 \cdot 2.8^{n-2}$ of them are embedded. The fit to the data is quite good; perhaps there is a heuristic geometric argument for the asymptotic number of embedded chains. Experiments show that for our special form, the probability of an embedded chain is vastly greater than for the general case.

We conclude with some convincing evidence for the conjecture. When we examine all strings having the special form $Sj\bar{S}\,(Sj\bar{S})_p$ (where S starts with 12, which is no loss of generality), a wonderful pattern emerges. The

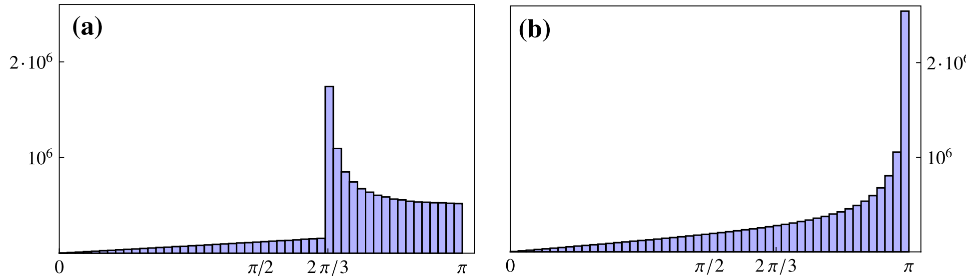

Figure 8. The distribution of θ for the special form, where the length of the initial string S is 15. Chart (a) uses only permutations of order 3; (b) uses order 4.

histograms in Figure 8 summarize over 28 million θ values arising from the special form, with S of length 15. One can divide the count according to the order of the permutation p. The order-2 case yields a flat histogram, the order-3 case yields a dramatic spike at $2\pi/3$, and the order-4 case yields an even sharper spike at π. These spikes show that the special form is inordinately likely to lead to 2- and 3-periodic chains with small error. The spike pattern is the same as saying that the trace of the matrix for the special string is near 0 (resp., 1) for permutations of order 4 (resp., 3). The existence of the spikes begs for an explanation.

Challenge Explain the spikes in the histograms of Figure 8.

There are some algebraic ideas surrounding the special form that might shed light on various issues. If K is the matrix for the special form $Sj\bar{S}\,(Sj\bar{S})_p$ and we use Z for the matrix of the first half, then $K = Z\,P^{-1}Z\,P$, where P is the permutation matrix of p. Experiments showed that for small-error period-3 loops, Z typically has the property that one row is very close to (0, 0, 0, 1), or a permutation thereof. And for period 2, the good Z usually has the property that each column has two numbers that sum to near 0. These properties were used in some of our searches and that is how γ_{148} was found. The two Z-matrices corresponding to the small-error loops γ_{164} and γ_{174} are as follows. The first has four pairs of nearly canceling numbers in the columns and the second has a row that is almost (0, 0, 1, 0).

$$\begin{pmatrix} 0.22 & \mathbf{-0.6524422} & \mathbf{-0.8741583} & -0.75 \\ \mathbf{6.9105604} & 6.81 & 7.79 & \mathbf{6.6888445} \\ \mathbf{-6.9105597} & -5.81 & -6.79 & \mathbf{-6.6888439} \\ 0.776 & \mathbf{0.6524416} & \mathbf{0.8741575} & 1.75 \end{pmatrix}$$

$$\begin{pmatrix} -0.68 & -0.88 & -1.44 & -1.77 \\ 3.96 & 3.07 & 3.4 & 4.16 \\ \mathbf{2.9\cdot 10^{-7}} & \mathbf{1.5\cdot 10^{-7}} & \mathbf{1+3\cdot 10^{-7}} & \mathbf{3.1\cdot 10^{-7}} \\ -2.28 & -1.19 & -1.95 & -1.39 \end{pmatrix}$$

At first glance the different forms seem unrelated, but they are manifestations of the same phenomenon. Let $\hat{K} = K - P^2$; then $\hat{K}$ in the two cases is:

$$\begin{pmatrix} -7.56300197 & -7.56300204 & -7.56300215 & -7.56300205 \\ 6.03640217 & 6.03640206 & 6.03640216 & 6.03640226 \\ -6.03640226 & -6.03640216 & -6.03640206 & -6.03640217 \\ 7.56300205 & 7.56300215 & 7.56300204 & 7.56300197 \end{pmatrix}$$

$$\begin{pmatrix} -4.83886832 & -4.83886826 & -4.83886801 & -4.83886822 \\ 5.34838502 & 5.34838508 & 5.34838487 & 5.34838512 \\ 9\cdot 10^{-7} & 7\cdot 10^{-7} & 8\cdot 10^{-7} & 6\cdot 10^{-7} \\ -0.50951762 & -0.50951756 & -0.50951755 & -0.50951752 \end{pmatrix}$$

All the columns of these matrices are nearly equal. If the columns were exactly equal, then $\hat{K}$ would be of rank 1. Because K's columns are barycentric coordinates, each column of $\hat{K}$ sums to exactly 0 and $\hat{K}$ has rank at most 3. We observed experimentally that, for our fake tetratori, $\hat{K}$ sometimes has exact rank 2, and is often very close to a matrix of rank 2. The third singular value of a matrix is an exact measure of how far it is from rank 2.

Conjecture The third singular value σ_3 of $\hat{K}$ is bounded above by $\sqrt{2\|K^{2+\mathrm{tr}(P)} - \text{identity matrix}\|}$, where $\|\cdot\|$ is the largest-singular-value matrix norm, and the 2 is sharp.

Note that when the permutation p is of order 4 (resp., 3), then $K^{2+\mathrm{tr}(P)}$ is just K^2 (resp., K^3).The conjectured bound is very small for chains that have small gap error, and so this sheds conjectural light on why $\hat{K}$ is almost of rank 2 when powers of K are near the identity. Even more seems to be true, as a strong majority (90%) of the second singular values σ_2 of $\hat{K}$ seem to obey the same bound. For the 16 special-form loops of Table 1, all the period 3 ones have $\hat{K}$ with σ_3 equal to 0 exactly, whereas the period-2 loops have $\sigma_2 \approx \sigma_3$, and both are between 25% and 50% of the bound in the conjecture. For the last six period-2 loops in Table 1, $\sigma_3(\hat{K})$ is under $1.3\cdot 10^{-6}$.

One wonders next about how many of the special-form chains are both almost periodic and embedded. Figure 9 is a logarithmic view of an exhaustive count (rigorously, only a lower bound) of embedded, almost 2-periodic chains, where L, the length of S, runs from 3 to 16 (so n, the full length, runs from 28 to 132) and S is assumed to start with 12; "almost" means that the angular error $|\pi - \theta\,|$ is less than $(2/3)^L$. The number of such embedded almost-closing chains appears exponential in L, strong evidence for the truth of the ϵ-conjecture even when restricted to the special form.

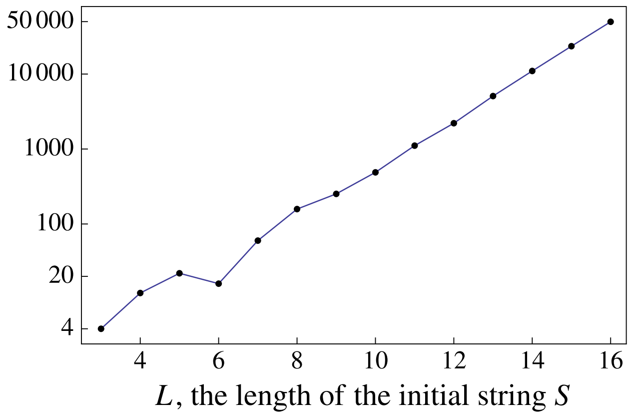

Figure 9. A logarithmic plot of the number of embedded period-2 chains arising from the special form and having gap error under $(2/3)^L$, where L is the length of S in $Sj\bar{S}(Sj\bar{S})_p$.

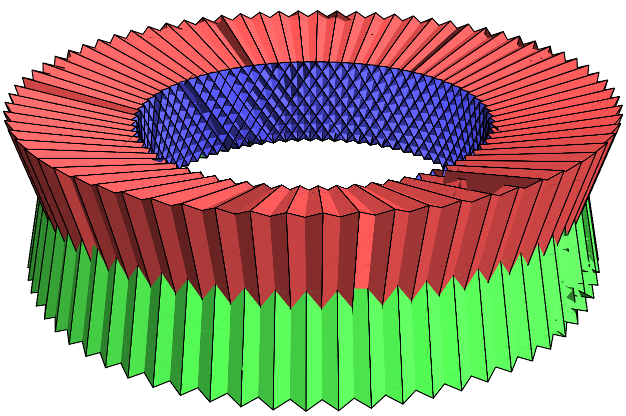

Figure 10. Continuing the 48-chain around 9 revolutions (using 73 copies of the chain) leads to very small error because $\cos^{-1}(25/27)/\pi$ is within about 10^{-6} of 9/73.

A final comment is that, allowing intersections, one can give a constructive proof that arbitrarily small error exists. One can use the 48-chain of the first section (before the elongation), where the basic repeating set of six tetrahedra (121434) defines an angle $\theta = 2\cos^{-1}(25/27) \sim 44.4°$. If $n\,\theta$ is close to $2\pi m$, then n repetitions of the set of six will yield a nearly perfect loop. The convergents to the continued fraction for $2\pi/\theta$ yield the desired approximations; the first ones are:

$$\frac{1}{8}, \frac{8}{65}, \frac{9}{73}, \frac{1736}{14081}, \frac{1745}{14154}, \ldots < 6 > \ldots, \frac{485749333}{3939997750}$$

Thus taking 3939997750 repetitions of the basic six-chain yields a long chain going round a circular loop, but closing up to within error about 10^{-8} of a degree after 485749333 full revolutions. Because π/θ is irrational, Kronecker's Theorem tells us that integer multiples of θ have values arbitrarily close to an integer multiple of 2π, yielding the result that the error can be made arbitrarily small. Figure 10 shows the result of using 73 copies of the basic set.

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