rush hour begins. A sample could be taken during the day and then the charts prepared for the expected onslaught.

The reader will undoubtably invent suitable examples for himself and some practical examples are to be found in [3]. The merit of this technique is that it is simple to understand and evidently tailor-made for a variety of practical situations.

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CAN REGULAR TETRAHEDRA BE GLUED TOGETHER FACE TO FACE TO FORM A RING?

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The Problem

Suppose someone gave you a large collection of identical regular tetrahedra. Would it be possible to glue them together face to face in such a way that you formed a nontrivial ring, or doughnut, in the sense that, during the glueing process, you could glue a tetrahedron both to your last tetrahedron and your first? Tetrahedra would be allowed to cut into each other if necessary.

The problem can arise in two contexts. When the concept of simplicial decomposition is introduced for the first time, it seems natural to ask how much can be accomplished using *regular* simplices instead of their image under a mapping, for instance, in decomposing a torus. The problem also arises as an offshoot of the problem of finding regular space-filling polyhedra, the tetrahedron being a natural generalization of an equilateral triangle, which fills the plane.

Solution

The answer to the problem is *negative*. It is not possible to form a ring from regular tetrahedra in the manner described. The following proof illustrates the use of *changing mathematical models* in order to make a problem tractable, and the use of *generalization*, in order to solve a specific problem.

Proof

Let A, B, C, D be the vertices of a regular tetrahedron. The effect of glueing a tetrahedron on the BCD face is the same as *reflecting* the *given* tetrahedron *in* that face. If O is a reference point in space then $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ are the position vectors of the vertices of the tetrahedron and the centroid M of BCD is represented by $\overrightarrow{OM} = \frac{1}{3}(\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD})$. The new position of A, namely A', is represented by

$$\overrightarrow{OA'} = 2\overrightarrow{OM} - \overrightarrow{OA} = -\overrightarrow{OA} + \tfrac{2}{3}OB + \tfrac{2}{3}OC + \tfrac{2}{3}\overrightarrow{OD}.$$

The reflection can then be represented by the matrix R_1 thought of as a right operator, where

$$R_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & 1 \end{pmatrix}$$

and $(\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD})R_1 = (\overrightarrow{OA'}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD})$.

Notice that $R_1{}^2 = I$; that is, if you reflect twice in the same face you are back to your first tetrahedron, forming a trivial 'ring'.

By symmetry it is easily seen that reflections in the other three faces are represented by

$$R_2 = \begin{pmatrix} 1 & \frac{2}{3} & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \end{pmatrix}$$

and

$$R_4 = \begin{pmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now the successive glueing together of tetrahedra can be thought of as a sequence of reflections in various faces, which corresponds to a sequence of matrices each member of which is one of R_1, R_2, R_3, R_4. The vertices of the last tetrahedron can be found in terms of the original $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ by calculating the product of the sequence of matrices. We can successfully form a ring of the tetrahedra if and only if the final vertices are some permutation of the original $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$; in other words, if and only if the product of the matrices is a permutation matrix.

There is one important point to consider here. We do not allow a 'trivial' sequence of the form R_1, R_1, or more generally of the form

$$R_{i_1}, R_{i_2}, \ldots, R_{i_{n-1}}, R_{i_n}, R_{i_n}, R_{i_{n-1}}, \ldots, R_{i_2}, R_{i_1},$$

since this sequence corresponds to 'going out and returning on the same path' rather than forming an honest ring. We can rule this case out by requiring that no matrix is ever repeated *consecutively* in what we will call a 'non-trivial' sequence.

Thus in order to show that no ring is possible, it is sufficient to show that no non-trivial sequence of the matrices R_1, R_2, R_3, R_4 is a permutation matrix. In group theoretical terms, which we will not pursue, we want to show that the group generated by R_1, R_2, R_3, R_4 is the free product of four copies of Z_2.

Having started with an amorphous problem of glueing tetrahedra, we have converted the problem into one involving direct calculations with matrices. It remains to find out what happens when you multiply sequences of the R_i.

If you start multiplying the matrices R_1, R_2, R_3, R_4 in various combinations you soon get the feeling that there is no hope of producing a permutation matrix in a non-trivial way. A proof can be obtained by considering a more general problem. The idea is to replace the entries $\frac{2}{3}$ by the letter a, so that the effect of the $\frac{2}{3}$ can be traced in the product matrices.

The resulting matrices, which we will call M_1, M_2, M_3, M_4 still satisfy the equations $M_1{}^2 = M_2{}^2 = M_3{}^2 = M_4{}^2 = I$, and of course $M_i = R_i$ when $a = \frac{2}{3}$. We now ask the more general question: for what values of a is there a non-trivial sequence of the M_i whose product is a permutation matrix? The question can be answered fairly easily, in the sense that we can rule out as possible solutions all values of a which are rational but not integral.

The method is to show that in successive non-trivial products of the M_i the entries are all polynomials in a with integral coefficients and leading coefficient ± 1; and further, that at least one entry has degree not less than n, where there are n matrices in the product. Consequently, in order that the non-trivial product of n of the matrices M_i be a permutation matrix, the values of a must make the polynomial entries zeros or ones. Since the entries have leading coefficients ± 1, rational but non-integral values of a will fail to produce a permutation matrix.

The problem, and the method, can of course be generalized to considering reflecting m-simplices in their k-faces.

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A NONRELATIVISTIC ANALOGY TO RELATIVISTIC TIME DILATATION

BY JOHN E. PRUSSING

In the theory of special relativity one encounters the notion of time dilatation. A commonly cited example is that "a clock moving relative to an observer is found to run more slowly than one at rest relative to him" [1]. This concept, although readily acceptable on a purely mathematical basis, often appears somewhat mystical to a student, when compared with his everyday experiences in the physical world. In order to bridge the gap between the mathematics and the physical world, a simple physical analogy (which could easily be constructed as a classroom demonstration) illustrates in a nonrelativistic manner this rather unintuitive notion of relativistic time dilatation.

The analogy is a one-dimensional linear oscillator of the form

$$\ddot{x} + \omega_0{}^2 x = 0,$$

which is rotated about an axis normal to the oscillation direction. A physical model would be a mass restrained by springs, oscillating in a slot in a horizontal disc which is rotated about a vertical axis through the centre of the disc. A second, nonrotating oscillator with identical characteristics could be used as a time reference.

In the analogy the variable corresponding to the velocity in special relativity is the rotational angular velocity of the disc about the vertical axis. The critical value of velocity, analogous to the speed of light, is the natural angular frequency of the oscillator, equal to the oscillation frequency of the nonrotating oscillator $\omega_0 = \sqrt{(k/m)}$, where m is the mass and k is the spring constant of the oscillator. In keeping with the spirit of special relativity one considers only uniform rotation of the disc (constant angular velocity). Thus the disc must be driven at a uniform rate.