Pigeonhole Principle and Extended Pigeonhole Principle

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1 Pigeonhole Principle

The **Pigeonhole Principle** is a fundamental concept in discrete math that states that if n items are put into m containers, with $n > m$, then at least one container must have more than one item. In mathematical terms, let's consider a mapping $f : X \to Y$, where X represents the set of pigeons and Y represents the set of pigeonholes. If the cardinality of set X is greater than the cardinality of set Y, denoted as $|X| > |Y|$, then according to the Pigeonhole Principle, there must exist at least one pair of distinct elements x_1 and x_2 in set X such that $x_1 \neq x_2$ and yet their mappings are the same, i.e., $f(x_1) = f(x_2)$.

$$f : X \to Y,\ |X| > |Y| \Rightarrow \exists\, x_1, x_2 \in X :\ x_1 \neq x_2 \wedge f(x_1) = f(x_2)$$

1.1 Example

If there is a group of 25 people, prove that at least 3 of them must have a birthday in the same month.

1.2 Proof

Assuming we have the worst-case scenario where the birth months are distributed as evenly as possible. This means that every month must have at least 2 people who have their birthdays in that month, with a total of 24 people. No matter where the last person's birth month is, it will always result in 1 month having at least 3 people with the same birth month.

2 Extended Pigeonhole Principle

The **Extended Pigeonhole Principle** extends the classic Pigeonhole Principle that takes into account scenarios where distribution is not uniform in size. Given a set of pigeons X and a set of pigeonholes Y, if the cardinality of X is greater than the k times the cardinality of Y, where $|X| > k \cdot |Y|$, the Extended Pigeonhole Principle guarantees that there must be a pigeonhole that contains at least $k + 1$ pigeons.

$$\left\lceil \frac{|X|}{|Y|} \right\rceil \quad \text{OR} \quad \left\lfloor \frac{|X| - 1}{|Y|} \right\rfloor + 1$$

2.1 Example

If there are 30 dictionaries in a library that contain a total of 61327 pages, prove that one of the dictionaries must have at least 2045 pages.

2.2 Proof

Assuming the worst-case scenario again, if we were to set the 61327 pages to X and the 30 dictionaries to Y, using one of the equation above would result in 2045. This means that a dictionary in the set of pages to have at least 2045 pages.

3 Typography

- Set: $A = \{a_1, a_2, \ldots, a_n\}$
- Member of: $x \in A$
- Cardinality: $|A|$ denotes the number of elements in set A
- Mapping: $f : A \to B$
- Equal: $a = b$
- Not equal: $a \neq b$
- Floor: $\lfloor x \rfloor$
- Ceiling: $\lceil x \rceil$
- Fraction: $\frac{a}{b}$
- Sequence: $(a_1, a_2, \ldots, a_n)$