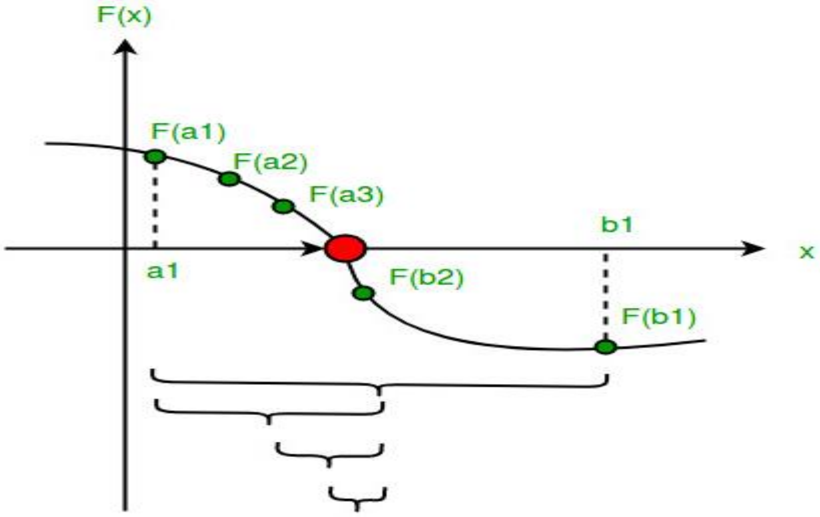
A Module on Non Linear Transcendental and Polynomial Function Techniques



Topics Covered:

1. Iterative bracketing method
   a. Graphical Method
   b. Incremental search method
   c. Bisection method
   d. False-position method
2. Iterative non-bracketing/open method
   a. Newton-raphson method
   b. Secant method
3. Iterative Polynomial Function Techniques
   a. Graeffe's root-squaring method
   b. Bairstow's method

General Objectives: After studying this topic you should:

Understand how to use numerical methods for solving algebraic and transcendental equations.

Specific Objectives: And you will be able to:

1. Understand the graphical interpretation of a root.
2. Know the graphical interpretation of the false-position method and why it is usually superior to the bisection method.
3. Understand the difference between bracketing and open methods for root location
4. Know why bracketing methods always converge, whereas open methods may sometimes diverge.
5. Know the fundamental difference between the false-position and secant methods and how it relates to convergence.
6. Understand the problems posed by multiple roots and the modifications available to mitigate them.

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Introduction:

In your Algebra subject, you learned to use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Eqn. 2.1}$$

To solve

$$f(x) = y = ax^2 + bc + c = 0 \qquad \text{Eqn. 2.2}$$

The values calculated with Eqn. 2.1 are called the roots of Eqn. 2.2 (quadratic equations). They represent the values of x that make a quadratic equation equal to zero. Thus, we can define the root of an equation as the value of x that makes f(x)=0.

This quadratic formula is very useful for solving quadratic equation, but there are many other functions for which the root cannot be determined easily. For these conditions, the numerical methods that will be described in this chapter provide efficient means to obtain the answer.

Mathematical Background:

By definition, a function given by y=f(x) is algebraic if it can be expressed in the form

$$f_n y^n + f_{n-1} y^{n-1} + \cdots + f_1 y + f_0 = 0$$

Where f_i=an ith-order polynomial in x. Polynomials are a simple class of algebraic functions that are represented generally by

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Where n=the order of the polynomial and the a's=constant.

Example:

1. $f_2(x) = 1 - 7.23x + 2.5x^2$
2. $f_5(x) = 3x + 5x^2 - x^3 + 7x^6$

A transcendental functions is one that is non-algebraic. These include trigonometric, exponential, logarithmic, and other, less familiar, functions.

Example:

1. $f(x) = \ln x^2 - 1$
2. $f(x) = e^{-0.2x} \sin(3x - 0.5)$
3. $f(x) = e^{-x} - x$

Roots of equations may be either real or complex. Although there are cases where complex roots of nonpolynomials are of interest, such situations are less common than for polynomials. As a consequence, the standard methods for locating roots typically fall into two somewhat related but primarily distinct problem areas:

1. The determination of the real roots of algebraic and transcendental equations. These techniques are usually designed to determine the value of a single real root on the basis of foreknowledge of its approximate location.
2. The determination of all real and complex roots of polynomials. These methods are specifically designed for polynomials. They systematically determine all the roots of the polynomial rather than determining a single real root given an approximate location.

Bracketing Methods:

This topic on roots of equations deals with methods that exploit the fact that a function typically changes sign in the vicinity of a root. These techniques are called bracketing methods because two initial guesses for the root are required. As the name implies, these guesses must “bracket,” or be on either side of, the root. The particular methods described herein employ different strategies to systematically reduce the width of the bracket and, hence, reduce the iterations to come up with the solution.

As a prerequisite to these methods, we will briefly discuss graphical methods to describe functions and their roots. Graphical techniques are very useful for visualizing the properties of the functions and the behavior of the various numerical methods.

1. Graphical Methods:

A simple method for obtaining an estimate of the root of the equation f(x)=0 is to make a plot of the function and observe where it crosses the *x* axis. This point, which represents the *x* value for which f(x) = 0, provides a rough approximation of the root.

Illustrative Example:

Use the graphical approach to determine the root of the equation

$$f(x) = e^{-x} - x.$$

Solution:

Various values of x can be substituted into the right side of this equation ($f(x) = e^{-x} - x$) to compute

x	f(x)=(e^-x)-x
0	1
0.2	0.61873075
0.4	0.27032005
0.6	-0.05118836
0.8	-0.35067104

These points are plotted as shown in the figure 2.1. The resulting curve crosses the x axis between 0.4 and 0.6. Visual inspection of the plotted curve provides a rough estimate of the root of 0.56.

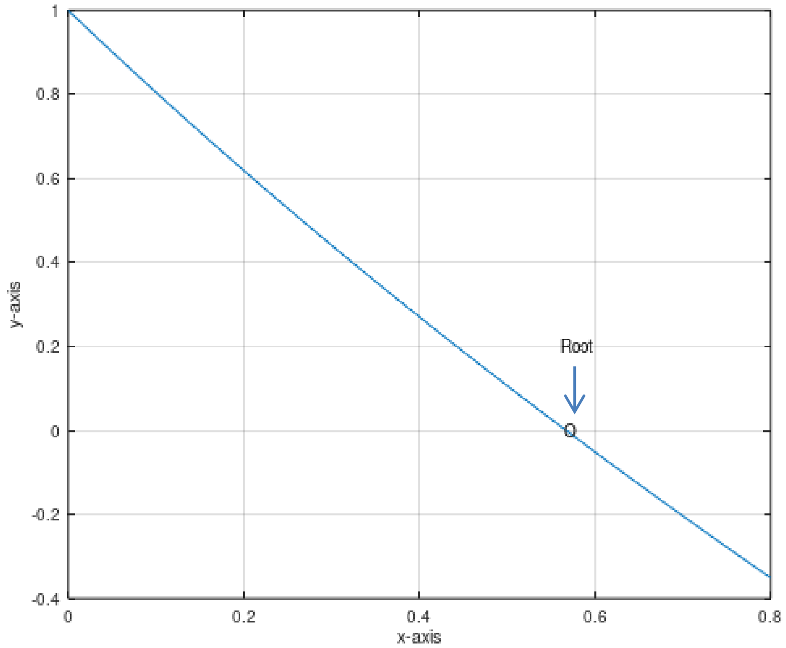


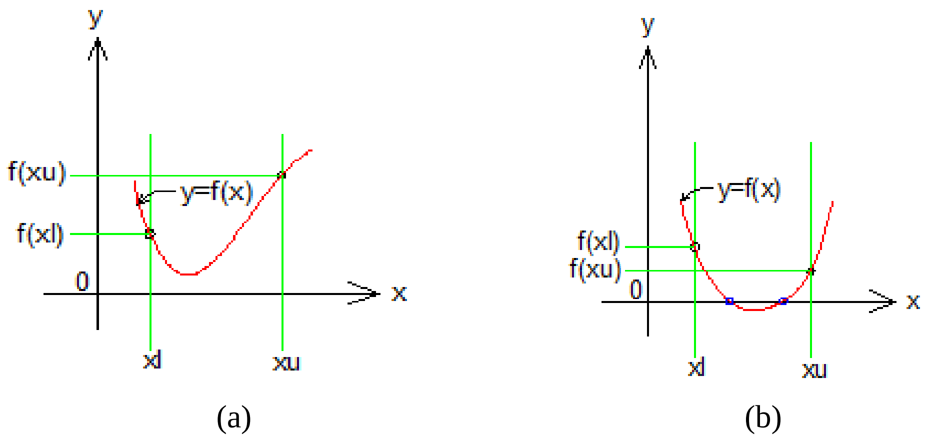
Figure 2.1 – The graphical approach for determining the root of an equation.

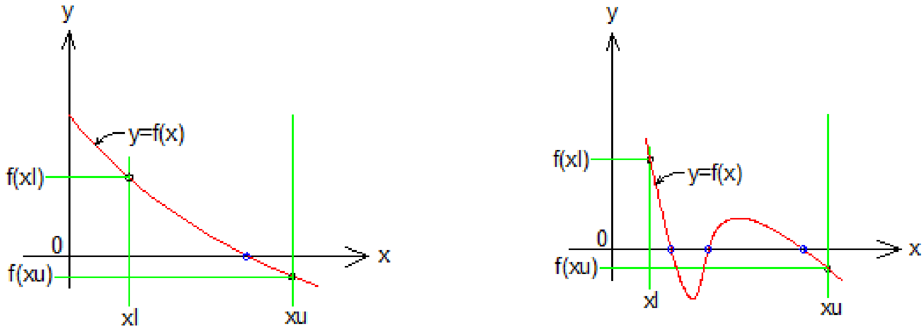
The graphical estimate can be checked by substituting it into the equation ($f(x) = e^{-x} - x$) to yield

$$f(0.56) = e^{0.56} - 0.56 = 0.0112$$

which is close to zero.

Graphical techniques are of limited practical value because they are not precise. However, graphical methods can be utilized to obtain rough estimates of roots. These estimates can be employed as starting guesses for numerical methods discussed in this chapter. Aside from providing rough estimates of the root, graphical interpretations are important tools for understanding the properties of the functions and anticipating the hidden problems of the numerical methods. For example, figures 2.2 illustrates a number of general ways that a root may occur in an interval prescribed by a lower bound x_l and an upper bound x_u.



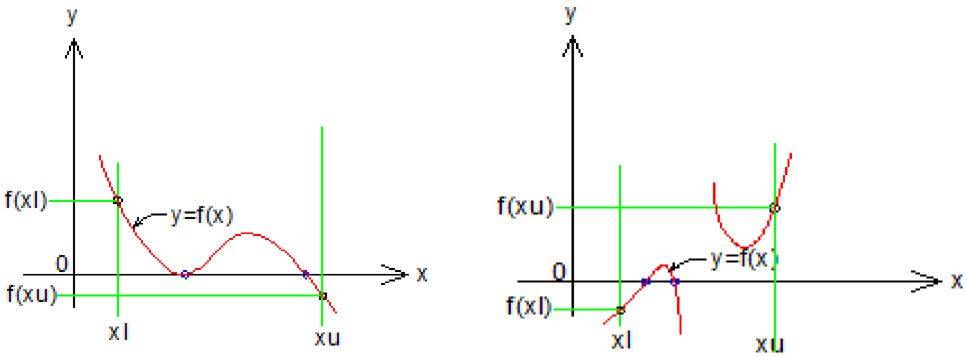


(c) (d)

Figure 2.2 – Parts (a) and (b) indicate that if both $f(x_l)$ and $f(x_u)$ have the same sign, either there will be no roots or there will an even number of roots within the interval. Parts (c) and (d) indicate that if the function has different signs at the end points, there will be an odd number of roots in the interval.

Although these generalizations are usually true, there are cases where they do not hold. For example, functions that are tangential to the x axis (Figure 2.3 a) and discontinuous functions (Figure 2.3 b) can violate these principles. An example of a function that is tangential to the axis is the cubic equation $f(x) = (x = 2)(x = 2)(x = 4)$. Notice that $x = 2$ makes two terms in this polynomial equal to zero. Mathematically, $x = 2$ is called a multiple root. At the end of this chapter, we will present techniques that are expressly designed to locate multiple roots.

The existence of cases of the type depicted in Figure 2.3 makes it difficult to develop general computer algorithms guaranteed to locate all the roots in an interval. However, when used in conjunction with graphical approaches, the methods described in the following sections are extremely useful for solving many roots of equations problems confronted routinely by engineers and applied mathematicians.



(a) (b)

Figure 2.3 – Illustration of some exceptions to the general cases depicted in figure 2.2. (a) Multiple root that occurs when the function is tangential to the *x* axis. For this case, although the end points are of opposite signs, there are an even number of axis intersections for the interval. (b) Discontinuous function where end points of opposite sign bracket an even number of roots.

2. The Incremental-Search Method:

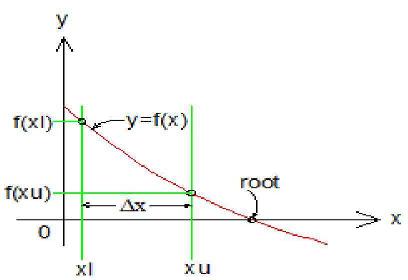
In this approach we determine values of f(x) for successive values of x in some interval to be search until a sign change occurs for f(x). A sign change occurs between x_l and x_u if $f(x_i).f(x_u)<0$. The sign change generally indicates that a roo has been passed (it could also indicate a discontinuity in the function as shown in figure 2.3 b). A closer approximation to the value of the root may then be obtained by reverting to the last x value preceding the sign change and, beginning with this x value, again determining values of f(x) for successive values of x, using a smaller increment than was used initially, until the sign change of f(x) changes again. This procedure is repeated with progressive smaller increments of x until a sufficient accurate value of the root is obtained. If additional roots are desired, the incrementation of x can be continued until the next root is approximately located by another sign change of f(x), and so on.

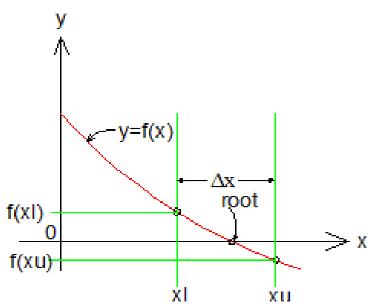
Care must be observed in selecting the initial value by which x is to be incremented, so that roots are not by-passed in an instance when two roots are close together in value. This is usually not a problem, if fairly small increments are used in the initial sequence. Additional insight to determine the location of the roots is by plotting and in understanding the physical problem.

Algorithm of Incremental-Search Method:

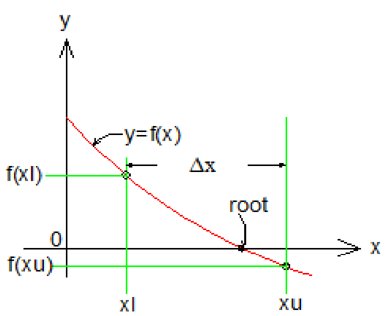
The steps to apply the incremental-search method to find the root of the equation f(x)=0 are:

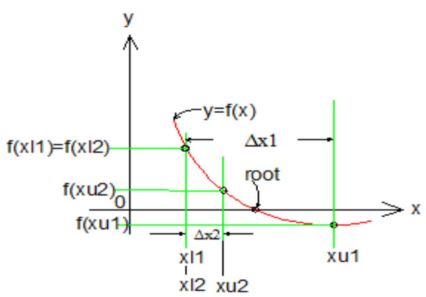
1. Select the initial value of x_l and Δx.
2. Determine the values of $f(x_l)$ and $f(x_u)$; $x_u = x_l + \Delta x$.
3. If $f(x_l).\ f(x_u) < 0$, indicates that the root has been passed. If not continue Iterations.





4. If there is a sign changed, revert back to x_l and reduce Δx to smaller increment.





5. Determine $f(x_l)$ and $f(x_u)$ progressively until another sign change.
6. Repeat steps until $f(x_u) \approx 0$ or $|\epsilon_a|$ is achieved.

Illustrative Example:

Use the incremental-search method to determine the root of the equation $f(x) = e^{-x} - x$. (Consider n=2)

Solution:

The search will start at $x_l=0$ and an increment(Δx)=0.5, that makes $x_u=0+0.5=0.5$ for the first iteration. The next iteration is shown in the table below. For n=2, $\epsilon_s = 0.5x10^{2-n}\%$=0.5%. The iteration should stop at $|\epsilon_a| \leq \epsilon_s$.

Iteration	x_l	Δx	x_u	$f(x_l)$	$f(x_u)$	$f(x_l).f(x_u)$	Remark
1	0	0.5	0.5	1	0.10653	> 0	Go to next interval
2	0.5	0.5	1	0.106531	-0.63212	< 0	Revert back to xl & consider smaller interval
3	0.5	0.1	0.6	0.106531	-0.05119	< 0	Revert back to xl & consider smaller interval
4	0.5	0.05	0.55	0.106531	0.02695	> 0	Go to next interval
5	0.55	0.05	0.6	0.02695	-0.05119	< 0	Revert back to xl & consider smaller interval
6	0.55	0.01	0.56	0.02695	0.01121	> 0	Go to next interval
7	0.56	0.01	0.57	0.011209	-0.00447	< 0	Revert back to xl & consider smaller interval
8	0.56	0.001	0.561	0.011209	0.00964	> 0	Go to next interval
9	0.561	0.001	0.562	0.009638	0.00807	> 0	Go to next interval
10	0.562	0.001	0.563	0.008068	0.0065	> 0	Go to next interval
11	0.563	0.001	0.564	0.006498	0.00493	> 0	Go to next interval
12	0.564	0.001	0.565	0.004929	0.00336	> 0	Go to next interval
13	0.565	0.001	0.566	0.00336	0.00179	> 0	Go to next interval
14	0.566	0.001	0.567	0.001792	0.00022	> 0	Go to next interval

The value of the percent relative approximate error at iteration no. 9 (with an approximate root of 0.562) is already 0.178%. Therefore, the iteration will stop at that iteration because, $|\epsilon_a|$ *is less than* ϵ_s, and also conclude that the root is 0.562. The other iterations will only show more computations, and you will notice in the table that the value of f(0.567)= 0.00022 which is the lowest value approaching zero.

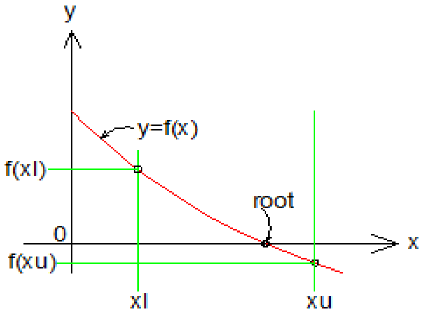
3. The Bisection Method:

The bisection method, which is alternatively called binary chopping, interval halving, or Bolzano's method, is one type of incremental search method in which the interval is always divided in half. If a function changes sign over an interval, the function value at the midpoint is evaluated. The location of the root is then determined as lying at the midpoint of the subinterval within which the sign change occurs. The process is repeated to obtain refined estimates.

Algorithm of Bisection Method:

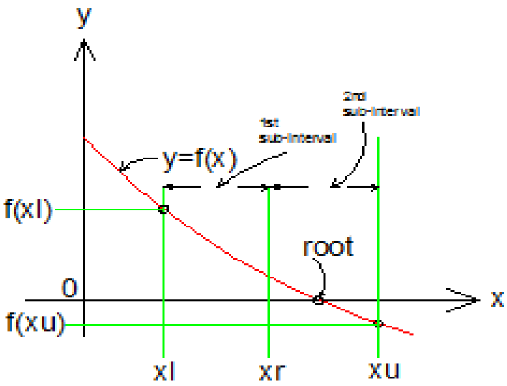
The steps to apply the bisection method to find the root of the equation f(x)=0 are:

1. Choose x_l and x_u as two guesses for the root such that $f(x_l).f(x_u)<0$, or in other words, f(x) changes sign between x_l and x_u.



2. Estimate the root, x_r of the equation f(x)=0 as the midpoint between x_l and x_u as

$$x_r = \frac{\mathrm{xl} + \mathrm{xu}}{2}$$



3. Make the following evaluation to determine in which subinterval the root lies:
   a. If $f(x_l)$. $f(x_r)$ < 0, then the root lies between x_l and x_r (1^{st} subinterval); then set $x_l = x_l$; $x_u = x_r$.
   b. If $f(x_l)$. $f(x_r)$ > 0, then the root lies between x_r and x_u (2^{nd} subinterval); then set $x_l = x_r$; $x_u = x_u$.
   c. If $f(x_l)$. $f(x_r)$ =0, then the root is x_r. Stop the algorithm if this is true.

4. Find the new estimate of the root

$$x_r = \frac{\mathrm{xl} + \mathrm{xu}}{2}$$

Find the absolute relative approximate error as

$$|\epsilon_a| = \left|\frac{x_r^{new} - x_r^{old}}{x_r^{new}}\right| x100$$

where:

x_r^{new}=estimated root from present iteration

x_r^{old}=estimated root from previous iteration

5. Compare the absolute relative approximate error $|\epsilon_a|$ with the pre-specified relative error tolerance ϵ_s. If $|\epsilon_a|>\epsilon_s$, then go to step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Illustrative Example:

Use the bisection method to determine the root of the equation $f(x) = e^{-x} - x$. Use n=2 as stopping criterion.

Solution:

$\epsilon_s = 0.5x10^{n-2} = 0.5x10^{2-2} = 0.5\%$

From the example in graphical approach, consider lower x_l=0.4 and upper x_u=0.6. The function changes sign between values of 0.4 and 0.6 (see figure 2.1).

Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_r)$	$\|\epsilon_a\|$, %	$f(x_l).f(x_r)$	Remark
1	0.4	0.5	0.6	0.27032	0.1065307		> 0	2nd subinterval
2	0.5	0.55	0.6	0.106531	0.0269498	9.090909091	> 0	2nd subinterval
3	0.55	0.575	0.6	0.02695	-0.012295	4.347826087	< 0	1st subinterval
4	0.55	0.5625	0.575	0.02695	0.0072828	2.222222222	> 0	2nd subinterval
5	0.5625	0.56875	0.575	0.007283	-0.002517	1.098901099	< 0	1st subinterval
6	0.5625	0.565625	0.56875	0.007283	0.00238	0.552486188	> 0	2nd subinterval
7	0.565625	0.567188	0.56875	0.00238	-6.93E-05	0.275482094	< 0	1st subinterval

In iteration 7, the percent absolute relative approximate error is 0.27548, which is less than $\epsilon_s = 0.5\%$. Therefore, the root is equal to 0.567188.

Problems:

1. Determine the first positive root of the equation $y = f(x) = 1 + 5.25x - sec\sqrt{0.68x}$ by the graphical and incremental-search methods. Use a stopping criterion below $\epsilon_s = 0.5\%$ for the bisection method.

2. Solve the first positive root of $f(x) = x^2|\sin x| - 4$ using bisection method, where x is in radian. Use a stopping criterion below $\epsilon_s = 0.1\%$.