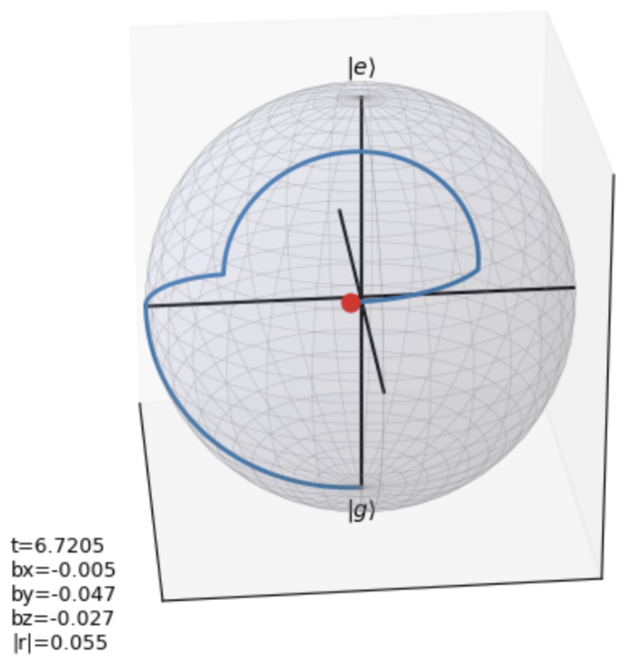
Dynamical Decoupling

Project Report

Aritrika
Duha Alsurdi
Sayan Biswas
Shreya JVS



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Contents

Project II: Dynamical Decoupling:

Problem

Consider two two-level systems (TLSs) that interact via a potential:

$$\hat{V} = V_0 \, |e_1 e_2\rangle\langle e_1 e_2|.$$

In addition, the excited states experience different detunings ("inhomogeneous broadening"), Δ_1 and Δ_2, and there is one laser that drives both TLSs.

1. Explain what inhomogeneous broadening is, and how it differs from homogeneous broadening. Give examples.
2. Formulate the Hamiltonian of the system and derive the equations of motion H.
3. If $\delta_1 = \delta_2$, show how the interaction can realize a CZ operation on the atoms, leaving them in an entangled state.
4. Compute the evolution of the system in the presence of finite detunings, and illustrate how the inhomogeneous broadening complicates the dynamics. Can you find a decoupling sequence that eliminates the influence of the inhomogeneous broadening?

Abstract

In realistic quantum systems, spatial and environmental inhomogeneities lead to qubit dependent frequency shifts that cause dephasing and limit the fidelity of entangling operations. In this project, we study a two qubit model consisting of two driven two–level systems interacting through a conditional interaction:

$$\hat{V} = V_0 \left| ee \right\rangle \left\langle ee \right|,$$

while experiencing unequal detunings that model inhomogeneous broadening. We show that, in the absence of detuning mismatch, this interaction naturally implements a Controlled Z (CZ) gate and generates entanglement from initially separable states using simple pulse sequences. When the detunings differ, unwanted relative phases accumulate and degrade the entangling dynamics. We analyze this effect explicitly and demonstrate how a two qubit spin echo (dynamical decoupling) sequence cancels single qubit phase errors while preserving the entangling interaction. The protocol restores high fidelity entanglement even in the presence of inhomogeneous broadening.

1 Definitions and Examples

Homogeneous broadening:

Spectral line is broadened by effects experienced equally by all atoms. As an example, the spontaneous emission case we took in class, we saw how the spectral line of emission showed a Lorentzian shape at resonance, and broadening by γ_r, the radiative decay rate. The broadening here is natural, meaning it's just a result of Heisenberg uncertainty principle which tells us that the inherent width of spectral lines is caused by the finite lifetime of an atom's excited state.

Inhomogeneous broadening:

An increase in the width of a spectral line of absorption or emission that occurs because different atoms in a medium have slightly different transition frequencies, that may happen due to differences in the local environment every atom lives in, that may cause energy shifts. The overall observed spectrum is an average of these individual, slightly shifted, spectral lines, resulting in a broader Gaussian peak.

Example of Inhomogeneous broadening:

Doppler broadening is a prominent example on inhomogeneous broadening. It is considered inhomogeneous because atoms in a Maxwell–Boltzmann thermal distribution move at different velocities, so each atom experiences a different Doppler shifted frequency of the incoming light.

2 Hamiltonian and Equations of Motion

We consider two driven TLSs, each described in the way we learned in class for a single TLS, j:

$$\hat{H}^{(j)} = -\frac{\hbar\delta_j}{2}\,\hat{\sigma}_z^{(j)} - \frac{\hbar\Omega}{2}\,\hat{\sigma}_x^{(j)}, \qquad j = 1, 2, \tag{1}$$

where δ_j is the detuning of TLS j ($\delta_j = \omega_L - \omega_{o,j}$), Ω is the Rabi frequency (driving field identical for both TLSs), and $\hat{\sigma}_{x,z}^{(j)}$ are Pauli matrices acting on TLS j only.

The two TLSs interact via:

$$\hat{V} = V_0\,|e_1 e_2\rangle\langle e_1 e_2|. \tag{2}$$

And as evident by the form of interaction, it's only "on" when both the electronic populations of the TLSs are in the excited states.

2.1 Full Hamiltonian

The total Hamiltonian is

$$\boxed{\hat{H} = -\frac{\hbar\delta_1}{2}\,\sigma_z^{(1)} - \frac{\hbar\delta_2}{2}\,\sigma_z^{(2)} - \frac{\hbar\Omega}{2}\big(\sigma_x^{(1)} + \sigma_x^{(2)}\big) + V_0\,|e_1e_2\rangle\langle e_1e_2|} \tag{3}$$

2.2 Equations of Motion

We use the basis

$$|g_1g_2\rangle, \quad |e_1g_2\rangle, \quad |g_1e_2\rangle, \quad |e_1e_2\rangle,$$

and expand the wavefunction as:

$$|\psi(t)\rangle = c_{gg}(t)|g_1g_2\rangle + c_{eg}(t)|e_1g_2\rangle + c_{ge}(t)|g_1e_2\rangle + c_{ee}(t)|e_1e_2\rangle. \tag{4}$$

Corresponding to the different possibilities os states that can be occupied.
Applying Schrodinger's equation $i\hbar\dot{\psi} = H\psi$, we obtain, for the RHS:

$$H|\psi\rangle = c_{gg}H|gg\rangle + c_{eg}H|eg\rangle + c_{ge}H|ge\rangle + c_{ee}H|ee\rangle. \tag{5}$$

Insert each expression:

$$\begin{aligned} H|\psi\rangle = c_{gg}&\left[\frac{\hbar}{2}(\delta_1+\delta_2)|gg\rangle - \frac{\hbar\Omega}{2}\big(|eg\rangle + |ge\rangle\big)\right] \\ &+ c_{eg}\left[-\frac{\hbar}{2}(\delta_1-\delta_2)|eg\rangle - \frac{\hbar\Omega}{2}|gg\rangle - \frac{\hbar\Omega}{2}|ee\rangle\right] \\ &+ c_{ge}\left[\frac{\hbar}{2}(\delta_1-\delta_2)|ge\rangle - \frac{\hbar\Omega}{2}|gg\rangle - \frac{\hbar\Omega}{2}|ee\rangle\right] \\ &+ c_{ee}\left[-\frac{\hbar}{2}(\delta_1+\delta_2)|ee\rangle - \frac{\hbar\Omega}{2}\big(|ge\rangle + |eg\rangle\big) + V_0|ee\rangle\right]. \end{aligned} \tag{6}$$

Grouping terms with the same states/kets, we obtain

$$\begin{aligned} H|\psi\rangle = &\left[\tfrac{\hbar}{2}(\delta_1+\delta_2)c_{gg} - \tfrac{\hbar\Omega}{2}(c_{eg}+c_{ge})\right]|gg\rangle \\ &+ \left[-\tfrac{\hbar\Omega}{2}c_{gg} - \tfrac{\hbar}{2}(\delta_1-\delta_2)c_{eg} - \tfrac{\hbar\Omega}{2}c_{ee}\right]|eg\rangle \\ &+ \left[-\tfrac{\hbar\Omega}{2}c_{gg} + \tfrac{\hbar}{2}(\delta_1-\delta_2)c_{ge} - \tfrac{\hbar\Omega}{2}c_{ee}\right]|ge\rangle \\ &+ \left[-\tfrac{\hbar\Omega}{2}(c_{eg}+c_{ge}) + \big(-\tfrac{\hbar}{2}(\delta_1+\delta_2)+V_0\big)c_{ee}\right]|ee\rangle. \end{aligned} \tag{7}$$

On the RHS, we have:

$$i\hbar\frac{d}{dt}|\psi\rangle = i\hbar\dot{c}_{gg}|gg\rangle + i\hbar\dot{c}_{eg}|eg\rangle + i\hbar\dot{c}_{ge}|ge\rangle + i\hbar\dot{c}_{ee}|ee\rangle. \tag{8}$$

Equating both sides and reading off the equations of motion, we obtain

$$i\hbar\dot{c}_{gg} = \tfrac{\hbar}{2}(\delta_1+\delta_2)c_{gg} - \tfrac{\hbar\Omega}{2}(c_{eg}+c_{ge}) \tag{9}$$

$$i\hbar\dot{c}_{eg} = -\tfrac{\hbar\Omega}{2}c_{gg} - \tfrac{\hbar}{2}(\delta_1-\delta_2)c_{eg} - \tfrac{\hbar\Omega}{2}c_{ee} \tag{10}$$

$$i\hbar\dot{c}_{ge} = -\tfrac{\hbar\Omega}{2}c_{gg} + \tfrac{\hbar}{2}(\delta_1-\delta_2)c_{ge} - \tfrac{\hbar\Omega}{2}c_{ee} \tag{11}$$

$$i\hbar\dot{c}_{ee} = -\tfrac{\hbar\Omega}{2}(c_{eg}+c_{ge}) + \big(-\tfrac{\hbar}{2}(\delta_1+\delta_2)+V_0\big)c_{ee}. \tag{12}$$

Canceling the $\hbar$s:

$$i\dot{c}_{gg} = \frac{1}{2}(\delta_1 + \delta_2)c_{gg} - \frac{\Omega}{2}(c_{eg} + c_{ge}), \tag{13}$$

$$i\dot{c}_{eg} = -\frac{\Omega}{2}c_{gg} - \frac{1}{2}(\delta_1 - \delta_2)c_{eg} - \frac{\Omega}{2}c_{ee}, \tag{14}$$

$$i\dot{c}_{ge} = -\frac{\Omega}{2}c_{gg} + \frac{1}{2}(\delta_1 - \delta_2)c_{ge} - \frac{\Omega}{2}c_{ee}, \tag{15}$$

$$i\dot{c}_{ee} = -\frac{\Omega}{2}(c_{eg} + c_{ge}) + \left(-\frac{1}{2}(\delta_1 + \delta_2) + \frac{V_0}{\hbar}\right)c_{ee}. \tag{16}$$

Or, more compactly,

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle, \tag{13}$$

which in components reads

$$i\hbar\begin{pmatrix}\dot{c}_{gg}\\ \dot{c}_{eg}\\ \dot{c}_{ge}\\ \dot{c}_{ee}\end{pmatrix} = \hat{H}\begin{pmatrix}c_{gg}\\ c_{eg}\\ c_{ge}\\ c_{ee}\end{pmatrix}, \tag{14}$$

with the Hamiltonian matrix in the ordered basis $\{|gg\rangle, |eg\rangle, |ge\rangle, |ee\rangle\}$ given by

$$\hat{H} = \begin{pmatrix} \frac{\delta_1+\delta_2}{2} & -\frac{\Omega}{2} & -\frac{\Omega}{2} & 0 \\ -\frac{\Omega}{2} & -\frac{\delta_1-\delta_2}{2} & 0 & -\frac{\Omega}{2} \\ -\frac{\Omega}{2} & 0 & \frac{\delta_1-\delta_2}{2} & -\frac{\Omega}{2} \\ 0 & -\frac{\Omega}{2} & -\frac{\Omega}{2} & -\frac{\delta_1+\delta_2}{2} + \frac{V_0}{\hbar} \end{pmatrix}. \tag{15}$$

3 CZ gate ideation

In this part, we assume that the two atoms have identical detunings $\delta_1 = \delta_2$, (to be understood why later), and we want to show two things:

1. The **interaction** behaves like a CZ gate. A CZ gate is a two-qubit gate that acts as follows:

$$\begin{aligned} \mathrm{CZ}\,|gg\rangle &= |gg\rangle, \\ \mathrm{CZ}\,|ge\rangle &= |ge\rangle, \\ \mathrm{CZ}\,|eg\rangle &= |eg\rangle, \\ \mathrm{CZ}\,|ee\rangle &= -\,|ee\rangle. \end{aligned}$$

   i.e;

$$\mathrm{CZ} = \mathrm{diag}(1, 1, 1, -1).$$

   That is, the CZ gate leaves $|gg\rangle, |ge\rangle, |eg\rangle$ unchanged, but multiplies $|ee\rangle$ by a minus sign. Which fits the interaction Hamiltonian, since it's only "on" when it acts on $|ee\rangle$, but we're left with preparing the correct state to be acted on.

2. Acting with this CZ gate creates entanglement from a non-entangled state. If we start in a suitable superposition (not entangled), after the CZ gate the system becomes entangled. The given interaction Hamiltonian,

$$\hat{H}_{\mathrm{int}} = V_0\,|ee\rangle\langle ee|,$$

   acts only when both atoms are excited. We want to show that if we prepare a suitable superposition containing $|ee\rangle$, then letting the system evolve under $\hat{H}_{\mathrm{int}}$ for the right amount of time produces exactly the CZ phase on $|ee\rangle$.

So the overall logic is:

$$|gg\rangle \xrightarrow{\pi/2\ \text{pulses}} \text{product superposition} \xrightarrow{\hat{H}_{\text{int}}} \text{entangled state.}$$

The story start when both atoms are in the ground states in the two emitters, now, we want to generate the right state, whatever that state is, it should contain the ($|ee\rangle$), so that we can act on it by the interaction Hamiltonian.

Hamiltonian without interaction

At the beginning, both atoms are in the ground state ($|gg\rangle$). We can consider each atom as an independent two-level system driven by the laser:

$$\hat{H}^{(j)}_{\text{single}} = -\frac{\hbar\delta_j}{2}\,\sigma_z^{(j)} - \frac{\hbar\Omega}{2}\,\sigma_x^{(j)}, \qquad j = 1, 2, \tag{16}$$

If we ignore the interaction Hamiltonian in the question at this stage, the total Hamiltonian is just the sum of the two single-atom Hamiltonians:

$$\hat{H}_{\text{no int}} = \hat{H}^{(1)}_{\text{single}} \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes \hat{H}^{(2)}_{\text{single}}, \tag{17}$$

where ($\mathbb{I}_1, \mathbb{I}_2$) are identity operators on atom 1 and 2, respectively.

Later, when we turn off the driving (set $\Omega = 0$) and consider only the interaction, the relevant Hamiltonian will simply be

$$\hat{H}_{\text{int}} = V_0\,|ee\rangle\langle ee|. \tag{18}$$

This term does nothing unless both atoms are in $|e\rangle$.

Single-atom Bloch vector and $\pi/2$ pulse

The general pure state of a single driven two-level atom can be written on the Bloch sphere as

$$|\psi(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right)|g\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|e\rangle, \tag{19}$$

where

- $\theta \in [0, \pi]$ is the polar angle,
- $\phi \in [0, 2\pi)$ is the azimuthal (phase) angle.

If we start in $|g\rangle$, in the south pole of the Bloch sphere, a pulse of area $\theta = \Omega * t$ rotates the Bloch vector by an angle θ and produces

$$|g\rangle \xrightarrow{\text{pulse with angle } \theta} |\psi(\theta, 0)\rangle = \cos\left(\frac{\theta}{2}\right)|g\rangle + \sin\left(\frac{\theta}{2}\right)|e\rangle, \tag{20}$$

For a $\pi/2$ pulse the final state becomes:

$$|\psi_{\pi/2}\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle). \tag{21}$$

Thus, a $\pi/2$ pulse acting on an atom initially in $|g\rangle$ creates a superposition of $|g\rangle$ and $|e\rangle$ with equal amplitudes (up to an overall phase). On the other hand, a π pulse acting on an atom initially in $|g\rangle$ creates an atom that's in $|e\rangle$, and tensor product this $|e\rangle$ with the second $|e\rangle$coming from the second $|g\rangle$ will result in non entangeled state that can't be entangled by applying the interaction Hamiltonian, and that's why we need the $\pi/2$ pulse.

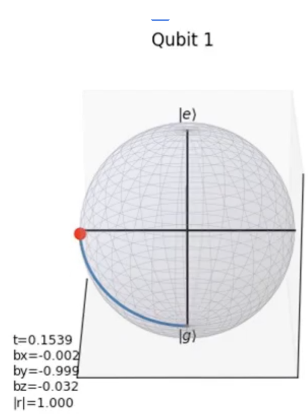


Figure 1: $pi/2$ pulse

3.1 Two-atom state after $\pi/2$ pulses

We start with both atoms in the ground state:

$$|\psi_{\text{initial}}\rangle = |g_1 g_2\rangle = |gg\rangle. \tag{22}$$

We apply the same $\pi/2$ pulse (with appropriate phase) to each atom:

$$|g_j\rangle \xrightarrow{\pi/2} \frac{|g_j\rangle + |e_j\rangle}{\sqrt{2}}, \qquad j = 1, 2. \tag{23}$$

Therefore the joint state becomes the tensor product of two single-atom superpositions:

$$|\psi_{\text{after pulses}}\rangle = \left(\frac{|g_1\rangle + |e_1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|g_2\rangle + |e_2\rangle}{\sqrt{2}}\right). \tag{24}$$

Expanding this product,

$$|\psi_{\text{after pulses}}\rangle = \frac{1}{2}\Big(|g_1 g_2\rangle + |g_1 e_2\rangle + |e_1 g_2\rangle + |e_1 e_2\rangle\Big) = \frac{1}{2}\Big(|gg\rangle + |ge\rangle + |eg\rangle + |ee\rangle\Big). \tag{25}$$

This state is still a *product* of two single-atom superpositions (no entanglement), but importantly it has non-zero amplitudes for all four basis states. In particular, the $|ee\rangle$ component is present, so the interaction Hamiltonian $\hat{H}_{\text{int}} = V_0 |ee\rangle\langle ee|$ will now have something to act on.

Interaction, CZ, and entanglement

After the $\pi/2$ pulses, the system is prepared in a product superposition. In this section we compute the full time evolution under the Hamiltonian (with the laser turned off), first for the general case $\delta_1 \neq \delta_2$, and then show how the special case $\delta_1 = \delta_2$ produces a clean CZ operation and generates entanglement.

Hamiltonian with the laser off

With the driving turned off ($\Omega = 0$), the effective Hamiltonian is

$$\hat{H} = -\frac{\hbar\delta_1}{2}\sigma_z^{(1)} - \frac{\hbar\delta_2}{2}\sigma_z^{(2)} + V_0|ee\rangle\langle ee| \equiv \hat{H}_Z + \hat{H}_{\text{int}}. \tag{26}$$

Since

$$[\hat{H}_Z, \hat{H}_{\text{int}}] = 0,$$

the evolution operator factorizes:

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\hat{H}_Z t/\hbar}\, e^{-i\hat{H}_{\text{int}} t/\hbar} \equiv \hat{U}_Z(t)\, \hat{U}_{\text{int}}(t). \tag{27}$$

Action of $\hat{U}_Z(t)$

The energies of the computational basis states under $\hat{H}_Z$ are:

$$E_{gg} = \frac{\hbar}{2}(\delta_1 + \delta_2), \tag{28}$$

$$E_{ge} = \frac{\hbar}{2}(\delta_1 - \delta_2), \tag{29}$$

$$E_{eg} = \frac{\hbar}{2}(\delta_2 - \delta_1), \tag{30}$$

$$E_{ee} = -\frac{\hbar}{2}(\delta_1 + \delta_2). \tag{31}$$

Thus the phase evolution is:

$$\hat{U}_Z(t)|gg\rangle = e^{-i(\delta_1+\delta_2)t/2}|gg\rangle, \tag{32}$$

$$\hat{U}_Z(t)|ge\rangle = e^{-i(\delta_1-\delta_2)t/2}|ge\rangle, \tag{33}$$

$$\hat{U}_Z(t)|eg\rangle = e^{-i(\delta_2-\delta_1)t/2}|eg\rangle, \tag{34}$$

$$\hat{U}_Z(t)|ee\rangle = e^{+i(\delta_1+\delta_2)t/2}|ee\rangle. \tag{35}$$

3.2 Action of the interaction evolution $\hat{U}_{\text{int}}(t)$

The interaction Hamiltonian is a projector:

$$\hat{H}_{\text{int}} = V_0 |ee\rangle\langle ee|. \tag{36}$$

Expanding the exponential,

$$e^A = \mathbb{I} + A + \frac{A^2}{2!} + \cdots, \tag{37}$$

with

$$A = -i\frac{V_0 t}{\hbar}|ee\rangle\langle ee|, \tag{38}$$

gives

$$A^n = \left(-i\frac{V_0 t}{\hbar}\right)^n |ee\rangle\langle ee|. \tag{39}$$

Factoring out the projector,

$$\hat{U}_{\text{int}}(t) = \mathbb{I} + \left(e^{-iV_0 t/\hbar} - 1\right)|ee\rangle\langle ee|. \tag{40}$$

Thus,

$$\hat{U}_{\text{int}}(t)|gg\rangle = |gg\rangle, \tag{41}$$
$$\hat{U}_{\text{int}}(t)|ge\rangle = |ge\rangle, \tag{42}$$
$$\hat{U}_{\text{int}}(t)|eg\rangle = |eg\rangle, \tag{43}$$
$$\hat{U}_{\text{int}}(t)|ee\rangle = e^{-iV_0 t/\hbar}|ee\rangle. \tag{44}$$

3.3 Full evolution for $\delta_1 \neq \delta_2$

Acting with $\hat{U}(t)$ on the initial state in Eq. 28,

$$\boxed{|\psi(t)\rangle = \frac{1}{2}\left[e^{-i(\delta_1+\delta_2)t/2}|gg\rangle + e^{-i(\delta_1-\delta_2)t/2}|ge\rangle + e^{-i(\delta_2-\delta_1)t/2}|eg\rangle + e^{+i(\delta_1+\delta_2)t/2}e^{-iV_0 t/\hbar}|ee\rangle\right].} \tag{45}$$

If we let the system evolve for the right time, i.e, until $|ee\rangle$ gets a minus sign; and others stay the same, we get:

$$e^{-iV_0 t/\hbar} = -1 \quad \Longrightarrow \quad t = \frac{\pi\hbar}{V_0}, \tag{46}$$

we obtain

$$|\psi(t)\rangle = \frac{1}{2}\left[e^{-i(\delta_1+\delta_2)t/2}|gg\rangle + e^{-i(\delta_1-\delta_2)t/2}|ge\rangle + e^{-i(\delta_2-\delta_1)t/2}|eg\rangle - e^{+i(\delta_1+\delta_2)t/2}|ee\rangle\right]. \tag{47}$$

This shows explicitly that unequal detunings $\delta_1 \neq \delta_2$ introduce different phases on each component, spoiling the clean CZ structure.

3.4 Special case $\delta_1 = \delta_2$

Setting $\delta_1 = \delta_2 = \delta = 0$ (so that what's left is the pure interaction evolution, which is what the question asks us to prove), Eq. 48 reduces to

$$|\psi(t)\rangle = \frac{1}{2}\big(|gg\rangle + |ge\rangle + |eg\rangle + e^{-iV_0 t/\hbar}|ee\rangle\big), \tag{48}$$

At $t = \pi\hbar/V_0$,

$$|\psi_{\text{final}}\rangle = \frac{1}{2}\big(|gg\rangle + |ge\rangle + |eg\rangle - |ee\rangle\big). \tag{49}$$

The corresponding unitary is

$$\hat{U}_{int}\left(\tfrac{\pi\hbar}{V_0}\right) = \text{diag}(1, 1, 1, -1), \tag{50}$$

which is exactly the controlled-Z gate.

Since the final state (Eq. (49)) cannot be written as a tensor product of single atom states, the interaction has generated entanglement!

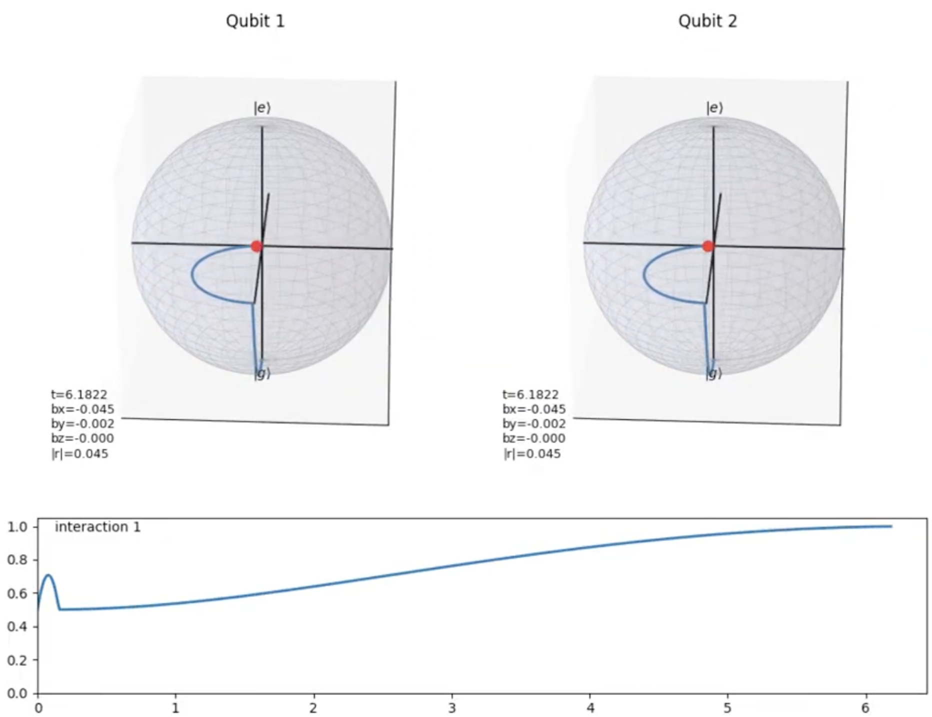


Figure 2: $\pi/2$ pulse followed by interaction for equal detunings. $\Omega = 10$, $V_0 = 0.5$, $\delta_1 = \delta_2 = 0$

3.5 Realizing a CNOT gate for $\Delta_1 = \Delta_2$ and generating entangled states

When the drive is switched off ($\Omega = 0$) and both atoms have the same detuning $\Delta_1 = \Delta_2 \equiv \Delta$, the rotating–frame Hamiltonian in the computational basis $\{|gg\rangle, |ge\rangle, |eg\rangle, |ee\rangle\}$ is

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \hbar\Delta & 0 & 0 \\ 0 & 0 & \hbar\Delta & 0 \\ 0 & 0 & 0 & 2\hbar\Delta + V_0 \end{pmatrix}. \tag{51}$$

It is convenient to split this into a local (single–qubit) contribution and a purely two–qubit interaction,

$$H = H_{\text{loc}} + H_{\text{int}}, \tag{52}$$

with

$$H_{\text{loc}} = \hbar\Delta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \qquad H_{\text{int}} = V_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{53}$$

In operator form,

$$H_{\text{loc}} = \hbar\Delta \left(|e_1\rangle \langle e_1| + |e_2\rangle \langle e_2| \right), \tag{54}$$

$$H_{\text{int}} = V_0 \, |e_1 e_2\rangle \langle e_1 e_2| \,. \tag{55}$$

Both H_{loc} and H_{int} are diagonal in the same basis, so they commute,

$$[H_{\text{loc}}, H_{\text{int}}] = 0. \tag{56}$$

CZ gate in the interaction picture

We now move to the interaction picture with respect to H_{loc}. The interaction–picture evolution operator is

$$U_I(t) = e^{-iH_{\mathrm{int}}t/\hbar} = \exp\!\left(-i\frac{V_0 t}{\hbar}\,|ee\rangle\langle ee|\right) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & e^{-iV_0 t/\hbar}\end{pmatrix}. \tag{57}$$

Thus, the states $|gg\rangle$, $|ge\rangle$ and $|eg\rangle$ are unchanged, while $|ee\rangle$ acquires an additional phase. Choosing the interaction time

$$t_{\mathrm{CZ}} = \frac{\pi\hbar}{V_0} \tag{58}$$

gives

$$U_I(t_{\mathrm{CZ}}) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} = \mathrm{CZ}. \tag{59}$$

In other words, for equal detuning the interaction produces a clean controlled–phase gate that flips the sign of $|ee\rangle$ and leaves the other basis states unchanged. Any additional phases from H_{loc} are local Z rotations on the individual qubits and can be corrected by single–qubit pulses.

CNOT from CZ

A CNOT gate with qubit 1 as control and qubit 2 as target is related to CZ by

$$\mathrm{CNOT}_{1\to 2} = (\mathbb{I}\otimes H)\,\mathrm{CZ}\,(\mathbb{I}\otimes H), \tag{60}$$

where

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{61}$$

is the single–qubit Hadamard on the *target* qubit. Experimentally, this sequence corresponds to

1. Apply a $\pi/2$ pulse (Hadamard) to qubit 2;
2. Turn off the laser and let the system evolve freely for $t_{\mathrm{CZ}} = \pi\hbar/V_0$ (CZ gate);
3. Apply a second $\pi/2$ pulse to qubit 2.

Entanglement from CNOT: one Bell state

To show that this interaction can generate entanglement, we start from the separable ground state

$$|\psi_0\rangle = |gg\rangle\,. \tag{62}$$

First, we create a superposition on the *control* qubit by applying a Hadamard to qubit 1 (a $\pi/2$ pulse with appropriate phase):

$$|\psi_1\rangle = (H\otimes\mathbb{I})\,|gg\rangle = \frac{1}{\sqrt{2}}\big(|gg\rangle + |eg\rangle\big). \tag{63}$$

Next, we apply $\mathrm{CNOT}_{1\to 2}$:

$$\mathrm{CNOT}_{1\to 2}\,|gg\rangle = |gg\rangle\,, \tag{64}$$
$$\mathrm{CNOT}_{1\to 2}\,|eg\rangle = |ee\rangle\,, \tag{65}$$

so that

$$|\psi_2\rangle = \mathrm{CNOT}_{1\to 2}\,|\psi_1\rangle = \frac{1}{\sqrt{2}}\big(|gg\rangle + |ee\rangle\big) \equiv |\Phi^+\rangle\,, \tag{66}$$

which is one of the four maximally entangled Bell states. The reduced density matrix of either qubit is maximally mixed,

$$\rho_1 = \mathrm{Tr}_2\big(|\Phi^+\rangle\langle\Phi^+|\big) = \frac{1}{2}\left(|g\rangle\langle g| + |e\rangle\langle e|\right), \tag{67}$$

confirming that $|\Phi^+\rangle$ is entangled.

Full sequence and generation of all four Bell states

The total unitary corresponding to the complete pulse sequence discussed above is

$$U_{\text{seq}} = (\mathbb{I} \otimes H)\, \text{CZ}\, (\mathbb{I} \otimes H)\, (H \otimes \mathbb{I}) = \text{CNOT}_{1\to 2}\, (H \otimes \mathbb{I}), \tag{68}$$

i.e. a Hadamard on the control followed by CNOT. Acting on the four computational basis states, we obtain

$$U_{\text{seq}} |gg\rangle = \frac{1}{\sqrt{2}}\big(|gg\rangle + |ee\rangle \big) = |\Phi^+\rangle\,, \tag{69}$$

$$U_{\text{seq}} |ge\rangle = \frac{1}{\sqrt{2}}\big(|ge\rangle + |eg\rangle \big) = |\Psi^+\rangle\,, \tag{70}$$

$$U_{\text{seq}} |eg\rangle = \frac{1}{\sqrt{2}}\big(|gg\rangle - |ee\rangle \big) = |\Phi^-\rangle\,, \tag{71}$$

$$U_{\text{seq}} |ee\rangle = \frac{1}{\sqrt{2}}\big(|ge\rangle - |eg\rangle \big) = |\Psi^-\rangle\,. \tag{72}$$

Thus, by starting from any of the four computational basis states and applying the same sequence

(i) H on qubit 1 $\to$ (ii) H on qubit 2 $\to$ (iii) CZ interaction $\to$ (iv) H on qubit 2,

we obtain *all four* Bell states $\{|\Phi^\pm\rangle, |\Psi^\pm\rangle\}$. This makes explicit how, for equal detuning $\Delta_1 = \Delta_2$, the interaction $V_0\, |ee\rangle\langle ee|$ together with single–qubit rotations realizes a CNOT gate and allows the preparation of maximally entangled two–qubit states.

3.6 Connection to Ramsey Interferometry

The sequence that we described above is exactly what is done in Ramsey Interferometry, a technique used to precisely measure transition frequencies of particles. In this technique, two short pi/2 pulses, separated by a much longer time period, are applied to the atoms. The population of the excited state is then measured, which will depend on the detuning of the atom. A single qubit in

$$\frac{|g\rangle + |e\rangle}{\sqrt{2}}$$

accumulates a phase $e^{-i\Delta t}$, giving the Ramsey fringe

$$P_e(t) = \frac{1}{2}\big(1 + \cos(\Delta t)\big) = \cos^2(\frac{\Delta t}{2}) \tag{73}$$

If Δ fluctuates across the ensemble, the contrast decays. The same dephasing mechanism occurs for the $|gg\rangle \leftrightarrow |ee\rangle$ coherence.

In fact, in the system we have considered, we can actually extract the detunings of the TLSs from the frequency at which the fidelity oscillates. To show this, we first derive what the fundamental frequency modes of the fidelity should be.

Consider the most general case. Let the basis sates be $|k\rangle$

$$H_0|k\rangle = E_k|k\rangle$$

Let $\quad |\psi(0)\rangle = \sum_k c_k|k\rangle \quad (c_k = \langle k \mid \psi(0)\rangle)$ Now, evolve the state by time t.

$$\Rightarrow \quad |\psi(t)\rangle = \sum_k c_k e^{-iE_k t}|k\rangle$$

Overlap is given by $b_k = \langle k \mid \psi_{\text{target}}\rangle$

$$\Rightarrow \quad F(t) = |\langle \psi_{\text{target}} \mid \psi(t)\rangle|^2 = \left| \sum_k b_k^* c_k e^{-iE_k t} \right|^2$$

$$= \left(\sum_k b_k^* c_k e^{-iE_k t} \right) \left(\sum_k b_c c_c^* e^{+iE_c t} \right)$$

$$= \sum_{k,l} (b_k^* c_k)\,(b_c c_c^*)\, e^{-i(E_k - E_c)t}$$

This implies that fidelity oscillates at frequencies

$$\omega_{kl} = E_k - E_l \tag{74}$$

We previously found that the eigenenergies for our system are

$$\begin{array}{ll} E_{00} = \frac{\delta_1+\delta_2}{2} & E_{10} = \frac{\delta_2-\delta_1}{2} \\ E_{01} = \frac{\delta_1-\delta_2}{2} & E_{11} = \frac{-\delta_1-\delta_2}{2} + v_0 \end{array}$$

This shows that when $\delta_1 = \delta_2, \quad \omega_F = v_0 - (\delta_1 + \delta_2)$ and when $\delta_1 \neq \delta_2, \omega_F = |\delta_1 - \delta_2|$.

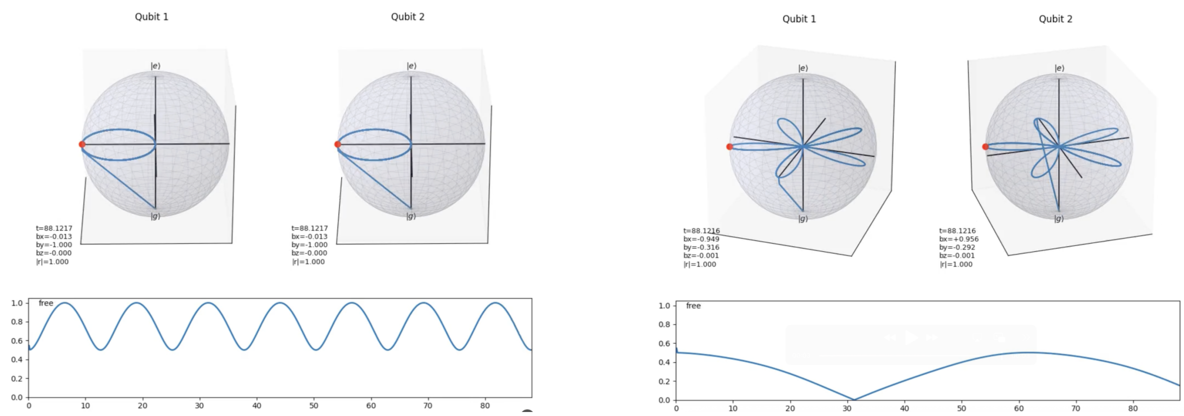


Figure 3: (a) Here, $\Omega = 10$, $V_0 = 0.5$, $\delta_1 = \delta_2 = 0$. We see that the fidelity has a frequency of $2\pi/T = 2\pi/13 = 0.5 = V_0$. (b) Here, $\Omega = 10$, $V_0 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.3$. We see that the fidelity has a frequency of $2\pi/T = 2\pi/62 = 0.1 = \delta_2 - \delta_1$

4 Effect of Inhomogeneous Broadening and Dynamical Decoupling

4.1 Inhomogeneous Broadening and Phase Errors

During the entangling period, when the laser drive is turned off, the two-qubit system evolves under the rotating-frame Hamiltonian

$$H = \Delta_1 Z_1 + \Delta_2 Z_2 + V_0 \left|ee\right\rangle \left\langle ee\right| . \tag{75}$$

Using the identity $|e\rangle\langle e| = (\mathbb{I} + Z)/2$, the interaction term becomes

$$\left|ee\right\rangle \left\langle ee\right| = \frac{1}{4}\left(\mathbb{I} + Z_1 + Z_2 + Z_1 Z_2\right), \tag{76}$$

showing that the Hamiltonian contains detuning terms and the entangling term $Z_1 Z_2$.

If $\Delta_1 = \Delta_2 = 0$, evolution for a time

$$T_{\mathrm{CZ}} = \frac{\pi\hbar}{V_0} \tag{77}$$

implements the ideal CZ gate,

$$\mathrm{CZ} = \mathrm{diag}(1, 1, 1, -1).$$

However, if $\Delta_1 \neq \Delta_2$, the state $|ee\rangle$ acquires an unwanted phase $e^{-i(\Delta_1+\Delta_2)t}$. Thus the prepared Bell state

$$|\Phi^+\rangle = \frac{|gg\rangle + |ee\rangle}{\sqrt{2}}$$

evolves into

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|gg\rangle + e^{-i(\Delta_1+\Delta_2)t}\,|ee\rangle\right). \tag{78}$$

Averaging over detuning noise yields the fidelity

$$F = \frac{1}{2}\left[1 + \left\langle e^{-i(\Delta_1+\Delta_2)t}\right\rangle\right]. \tag{79}$$

4.2 Spin Echo for Two Qubits

A global π pulse about x satisfies

$$XZX = -Z,$$

so $X \otimes X$ flips both Z_1 and Z_2, while

$$(X \otimes X)Z_1 Z_2 (X \otimes X) = Z_1 Z_2.$$

Thus the echo-protected entangling gate is

$$U_{\mathrm{CZ}}^{(\mathrm{echo})} = e^{-iHT/(2\hbar)}(X \otimes X)e^{-iHT/(2\hbar)}, \qquad T = \frac{\pi\hbar}{V_0}. \tag{80}$$

Detuning effects cancel; the entangling term accumulates.

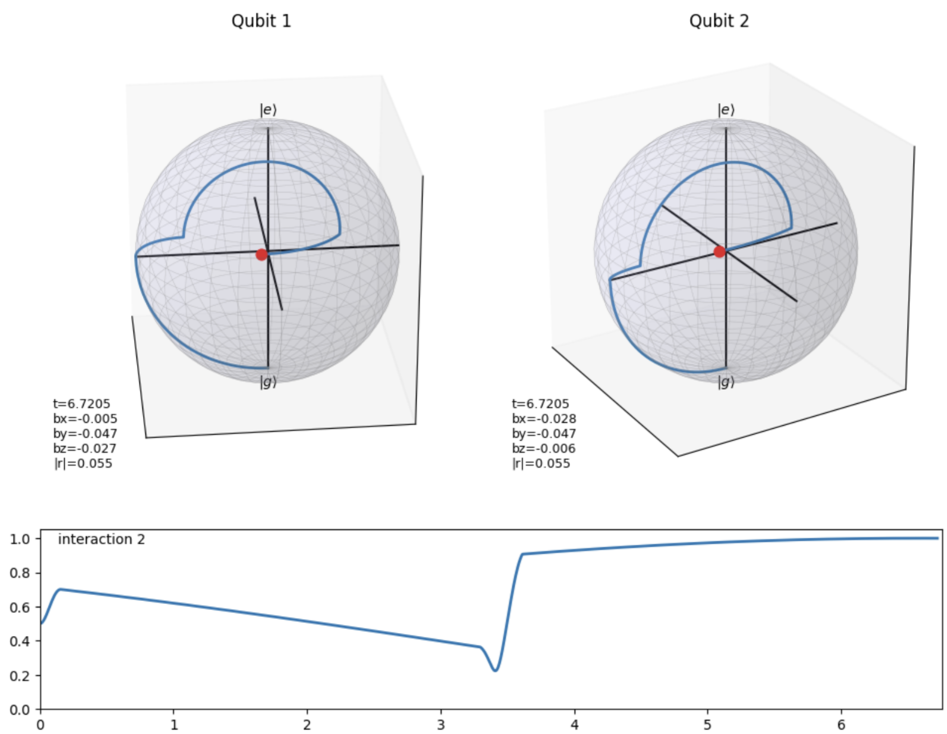


Figure 4: Spin echo protocol for unequal detunings. $\Omega = 10$, $V_0 = 0.5$, $\delta_1 = 0.1$, $\delta_2 = 0.2$

4.3 DD-Protected Entanglement Protocol Starting from $|gg\rangle$

We summarize the full CNOT-based entanglement protocol, showing where the DD-protected CZ block is inserted.

1. Prepare a superposition on qubit 1:

$$|gg\rangle \xrightarrow{H_1} |\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|gg\rangle + |eg\rangle\right).$$

2. Begin constructing CNOT: apply Hadamard on qubit 2

$$|\psi_1\rangle \xrightarrow{H_2} |\psi_{\mathrm{in}}\rangle = \frac{1}{2}\left(|gg\rangle + |ge\rangle + |eg\rangle + |ee\rangle\right).$$

3. Apply CZ. *(a) Ideal case, no detuning)*:

$$|\psi_{\rm in}\rangle \xrightarrow{\rm CZ} \cdots$$

(b) With dynamical decoupling):

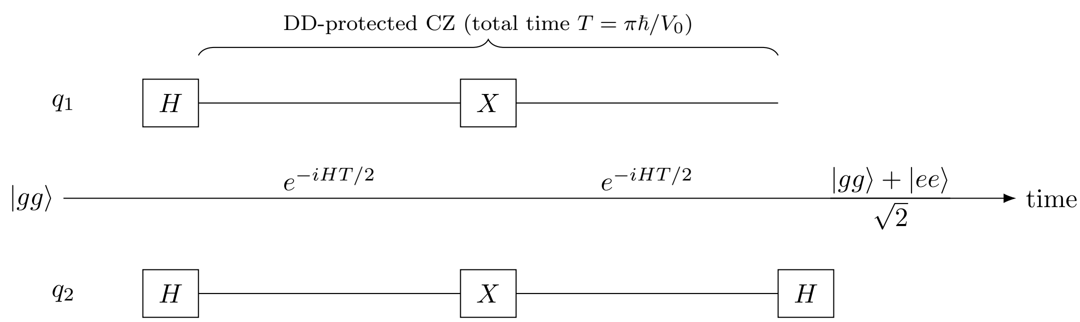
$$|\psi_{\rm in}\rangle \xrightarrow{U_{\rm CZ}^{\rm (DD)}} e^{-iHT/(2\hbar)}\,(X\otimes X)\,e^{-iHT/(2\hbar)}\,|\psi_{\rm in}\rangle\,.$$

4. Finish CNOT: apply Hadamard on qubit 2 again

$$\cdots \xrightarrow{H_2} \frac{|gg\rangle + |ee\rangle}{\sqrt{2}},$$

which is the desired Bell state.

Timeline Diagram of the DD-Protected CZ Sequence



Explanation. The first Hadamard prepares a superposition on qubit q_1, and the Hadamard on q_2 implements the standard CNOT construction. During the evolution segments of duration $T/2$, the laser is turned off, so the system evolves only under detuning and the interaction term $V_0\,|ee\rangle\langle ee|$. A global π pulse reverses the sign of the detuning contributions while preserving the entangling term, thereby cancelling dephasing from inhomogeneous broadening.

4.4 Spin Echo Collapse and Revival

There are a few cases when the spin echo protocol does not work. When the detunings are significantly smaller than the Rabi driving, and the sum of the detunings of the two TLSs is equal to the interaction strength V_0, spin echo fails.

This can be explained by focusing on the implementation of the π pulse in the protocol. The π pulse essentially exchanges the coefficients of the $|gg\rangle$, $|ee\rangle$ and the coefficients of the $|eg\rangle$, $|ge\rangle$ states. Although if the coefficients of these states are equal initially, the *pi* pulse would have no effect and spin echo would fail, i.e., we see a "collapse". Looking at the final state we derived in Eq. 45, we find

$$\frac{-(\delta_1+\delta_2)t}{2} = \frac{(\delta_1+\delta_2)t}{2} - \frac{V_0}{2}$$
$$\delta_1 + \delta_2 = V_0$$

And if the detunings are small, then the coefficients of $|eg\rangle$, $|ge\rangle$ states can be approximated to 1. This is the behaviour we observe in Fig. This can also be understood from the perspective of the frequency modes of the fidelity becoming 0, which means the fidelity can never reach 1.

In the case of an ensemble of two-level systems, the interactions are a lot more complicated than the system we have and in those cases, it is possible for all of the phases to coherently add up in such a way that there is a spin echo

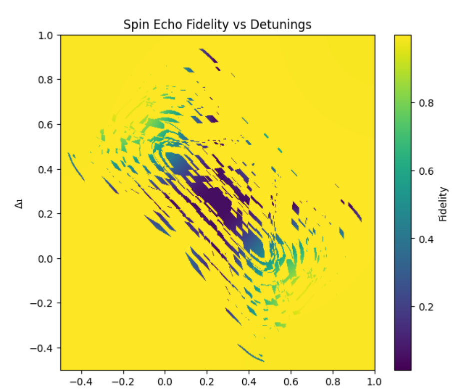


Figure 5: Fidelity map vs. detunings

collapse at a certain time. Consequently, a *revival* of the echo is also observed (Note that revival of the spin echo is not possible with the Hamiltonian we are considering).

Summary

Inhomogeneous broadening produces random phases that degrade the Bell-state coherence. Ramsey interferometry illustrates this dephasing, and two-qubit spin echo (or more general dynamical decoupling) removes unwanted single-qubit phases while preserving the entangling interaction. This principle has a plethora of applications across a wide range of fields, including the working principle behind MRI machines, precision NMR spectroscopy, solid-state qubits, and ultracold atomic systems. In these platforms, spin echo protocols are essential for counteracting dephasing caused by inhomogeneous broadening and interaction-induced phase buildup. In this project, using a simple model Hamiltonian, we illustrated how using a controlled pulse sequence we can effectively reverse unwanted phase evolution, improving the measurement fidelity.

Simulation Access

The simulations of this project can be accessed in the GitHub repository linked below.
https://github.com/ReyshaJV/Dynamical-Decoupling

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