

### 3.4. Additional Convexification Techniques

#### 3.4.1 Epigraph based SOCP relaxation

The cost function in SDP approach is

$$\min \sum_{ij} \frac{1}{\sigma^2 d_{ij}^2} (\|x_i - x_j\| - d_{ij})^2$$

the cost func is non linear so we can get a simpler convex relaxation using epigraph formulation

Add a convex constraint  $t_{ij} \geq (\|x_i - x_j\| - d_{ij})^2 \rightarrow$  rotated soc form.

On top of this, introduce  $s_{ij}$  as upper bound on the distance  $s_{ij} \geq \|x_i - x_j\|$  b/w agents.

So the residual constraint becomes  $t_{ij} \geq (s_{ij} - d_{ij})^2$

The optimization problem becomes

$$\begin{aligned} & \min_{x_i, s_{ij}, t_{ij}} \sum_{ij \in \Sigma} \frac{1}{\sigma^2 d_{ij}^2} t_{ij} \\ & \text{s.t. } s_{ij} \geq \|x_i - x_j\|_2 \quad (\text{SOC}) \\ & \quad t_{ij} \geq (s_{ij} - d_{ij})^2 \quad (\text{rotated SOC}) \end{aligned} \quad \longrightarrow \textcircled{1}$$

$$x_i \in \mathbb{R}^d, s_{ij} \geq 0, t_{ij} \geq 0$$

The constraints can further be simplified by converting to LMI

$$t_{ij} \geq (s_{ij} - d_{ij})^2 \iff \begin{bmatrix} t_{ij} & s_{ij} - d_{ij} \\ (s_{ij} - d_{ij})^\top & 1 \end{bmatrix} \succeq 0 \quad \longrightarrow \textcircled{2}$$

for the SOC distance constraint,

$$\begin{aligned} & s_{ij}^2 - \|x_i - x_j\|^2 \geq 0 \\ & \Rightarrow s_{ij} - s_{ij}^{-1} \|x_i - x_j\|^2 \geq 0 \quad s_{ij} \neq 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} s_{ij} & (x_i - x_j) \\ (x_i - x_j)^\top & s_{ij} \end{bmatrix} \succeq 0 \quad s_{ij} \neq 0 \quad \longrightarrow \textcircled{2}$$

Based on ①, ② and ③ the final optimization problem is

$$\begin{aligned}
 & \min_{x_i, s_{ij}, t_{ij}} \sum_{ij \in \Sigma} \frac{1}{\sigma^2 d_{ij}^2} t_{ij} \\
 \text{s.t. } & \begin{bmatrix} s_{ij} & (x_i - x_j) \\ (x_i - x_j)^T & s_{ij} \end{bmatrix} \succeq 0 \quad (\text{SOC}) \\
 & \begin{bmatrix} t_{ij} & s_{ij} - d_{ij} \\ (s_{ij} - d_{ij})^T & 1 \end{bmatrix} \succeq 0 \quad (\text{rotated SOC}) \\
 & x_i \in \mathbb{R}^n, s_{ij} \geq 0, t_{ij} \geq 0
 \end{aligned}$$

### 3.4.2. Robust Huber Loss via Epigraph

The current loss uses squared residual of the form  $(\|x_i - x_j\| - d_{ij})^2$ . Squared loss is sensitive to outliers so in section 3.3 (MQIP) binary trust variables are added. There is (I think) a simpler way to do that by replacing square loss with Huber loss. It is quadratic for small residuals and linear for large residuals.

For residual  $r$ , the Huber loss is

$$L_\delta(r) = \begin{cases} \frac{1}{2}r^2, & |r| \leq \delta \\ \delta|r| - \frac{1}{2}\delta^2, & |r| > \delta \end{cases}$$

Note:  $\delta$  is a user specified hyper param.

- i) When error is small, squared loss  $\rightarrow$  high accuracy
- ii) where error is high, absolute loss  $\rightarrow$  suppress outlier.

Convex, smooth and does not need integer variables.

Epigraph formulation makes optimization problem clean despite the piecewise function.

the residual is  $r_{ij} = \|x_i - x_j\| - d_{ij} \quad r_{ij} \in \mathbb{R}$

$|r_{ij}|$  can be written as LP  $s_{ij} \geq r_{ij} \quad s_{ij} \geq -r_{ij}$

This gives  $s_{ij} = |r_{ij}|$  at optimality.

and introduce epigraph variable  $t_{ij}$

$$t_{ij} \geq L_\delta(r)$$

$$L_\delta(r) = \max\left(\frac{1}{2}r^2, \delta|r| - \frac{1}{2}\delta^2\right)$$

This epigraph helps replace piecewise loss func with set of convex constraints

given by

$$t_{ij} \geq \frac{1}{2}r_{ij}^2 \Rightarrow \|r_{ij}\|^2 \leq 2t_{ij} \text{ or } (r_{ij}, \sqrt{2}t_{ij}) \in \text{SOC}$$

$$t_{ij} \geq \delta|r_{ij}| - \frac{1}{2}\delta^2$$

Final optimization problem for this case is

$$\min_{x_i, s_{ij}, t_{ij}} \sum_{ij \in E} \frac{1}{\sigma^2 d_{ij}^2} t_{ij}$$

$$\text{s.t. } r_{ij} = \|x_i - x_j\| - d_{ij}$$

$$s_{ij} \geq r_{ij}$$

$$s_{ij} \geq -r_{ij}$$

$$t_{ij} \geq \delta s_{ij} - \frac{1}{2}\delta^2$$

$$r_{ij}^2 \leq 2t_{ij}$$

$$s_{ij} \geq 0, t_{ij} \geq 0$$