

### 3.4. Additional Convexification Techniques

#### 3.4.1 Epigraph based SOCP relaxation

The cost function in SDP approach is

$$\min \sum \frac{1}{\sigma^2 d_{ij}^2} (\|x_i - x_j\| - d_{ij})^2$$

the cost func is non linear so we can get a simpler convex relaxation using epigraph formulation

Add a convex constraint  $t_{ij} \geq (\|x_i - x_j\| - d_{ij})^2 \rightarrow$  rotated soc form.

On top of this, introduce  $s_{ij}$  as upper bound on the distance  $s_{ij} \geq \|x_i - x_j\|$  b/w agents.

So the residual constraint becomes  $t_{ij} \geq (s_{ij} - d_{ij})^2$

The optimization problem becomes

$$\min_{x_i, s_{ij}, t_{ij}} \sum_{ij \in E} \frac{1}{\sigma^2 d_{ij}^2} t_{ij}$$

$$\text{st. } s_{ij} \geq \|x_i - x_j\|_2 \quad (\text{SOC})$$

$$t_{ij} \geq (s_{ij} - d_{ij})^2 \quad (\text{rotated SOC})$$

$$x_i \in \mathbb{R}^d, s_{ij} \geq 0, t_{ij} \geq 0$$

The constraints can further be simplified by converting to LMI

$$t_{ij} \geq (s_{ij} - d_{ij})^2 \iff \begin{bmatrix} t_{ij} & s_{ij} - d_{ij} \\ (s_{ij} - d_{ij})^T & 1 \end{bmatrix} \succeq 0 \quad \text{--- (2)}$$

for the SOC distance constraint,

$$s_{ij}^2 - \|x_i - x_j\|^2 \geq 0$$

$$\Rightarrow s_{ij} - s_{ij}^{-1} \|x_i - x_j\|^2 \geq 0 \quad s_{ij} \neq 0$$

$$\Rightarrow \begin{bmatrix} s_{ij} & (x_i - x_j) \\ (x_i - x_j)^T & s_{ij} \end{bmatrix} \succeq 0 \quad s_{ij} \neq 0 \quad \text{--- (2)}$$

Based on ①, ① and ② the final optimization problem is

$$\begin{aligned}
 \min_{x_i, s_{ij}, t_{ij}} \quad & \sum_{i,j \in \mathcal{E}} \frac{1}{\sigma^2 d_{ij}^2} t_{ij} \\
 \text{s.t.} \quad & \begin{bmatrix} s_{ij} & (x_i - x_j) \\ (x_i - x_j)^T & s_{ij} \end{bmatrix} \succeq 0 \quad (\text{SOC}) \\
 & \begin{bmatrix} t_{ij} & s_{ij} - d_{ij} \\ (s_{ij} - d_{ij})^T & 1 \end{bmatrix} \succeq 0 \quad (\text{rotated SOC}) \\
 & x_i \in \mathbb{R}^d, \quad s_{ij} > 0, \quad t_{ij} \geq 0
 \end{aligned}$$

### 3.4.2. Robust Huber Loss via Epigraph

The current loss uses squared residual of the form  $(\|x_i - x_j\| - d_{ij})^2$ . Squared loss is sensitive to outliers so in section 3.3 (MILP) binary trust variables are added. There is (I think) a simpler way to do that by replacing square loss with Huber loss. It is quadratic for small residuals and linear for large residuals.

For residual  $r$ , the Huber loss is

$$L_\delta(r) = \begin{cases} \frac{1}{2} r^2 & , |r| \leq \delta \\ \delta |r| - \frac{1}{2} \delta^2 & , |r| > \delta \end{cases}$$

Note:  $\delta$  is a user specified hyperparam.

i) When error is small, squared loss  $\rightarrow$  high accuracy

ii) where error is high, absolute loss  $\rightarrow$  suppress outlier.

Convex, smooth and does not need integer variables.

Epigraph formulation makes optimization problem clean despite the piecewise function.

the residual is  $r_{ij} = \|x_i - x_j\| - d_{ij} \quad r_{ij} \in \mathbb{R}$

$|r_{ij}|$  can be written as LP  $s_{ij} \geq r_{ij} \quad s_{ij} \geq -r_{ij}$

This gives  $s_{ij} = |r_{ij}|$  at optimality.

and introduce epigraph variable  $t_{ij}$

$$t_{ij} \geq L_s(r)$$

$$L_s(r) = \max \left( \frac{1}{2} r^2, \delta |r| - \frac{1}{2} s^2 \right)$$

This epigraph helps replace piecewise loss func with set of convex constraints given by

$$t_{ij} \geq \frac{1}{2} r_{ij}^2 \Rightarrow \|r_{ij}\|^2 \leq 2t_{ij} \quad \text{or } (r_{ij}, \sqrt{2t_{ij}}) \in \text{SOC}$$

$$t_{ij} \geq \delta |r_{ij}| - \frac{1}{2} s^2$$

Final optimization problem for this case is

$$\min_{x_i, s_{ij}, t_{ij}} \sum_{ij \in E} \frac{1}{\sigma^2 d_{ij}^2} t_{ij}$$

$$\text{st. } r_{ij} = \|x_i - x_j\| - d_{ij}$$

$$s_{ij} \geq r_{ij}$$

$$s_{ij} \geq -r_{ij}$$

$$t_{ij} \geq \delta s_{ij} - \frac{1}{2} s^2$$

$$r_{ij}^2 \leq 2t_{ij}$$

$$s_{ij} \geq 0, t_{ij} \geq 0$$