

LINEAR ALGEBRA - DS2103

CHAPTER 1: MATRIX ALGEBRA

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1.1 Definition :

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as A, B, C etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n}$$

A is a matrix of order $m \times n$. i^{th} row j^{th} column element of the matrix denoted by a_{ij} .

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

1.2 Special Types of Matrices:

1. Square matrix:

A matrix in which numbers of rows are equal to number of columns is called a square matrix.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 6 & 8 & 9 \end{pmatrix}$$

2. Diagonal matrix:

A square matrix $A = (a_{ij})_{n \times n}$ is called a diagonal matrix if each of its non-diagonal element is zero.

That is $a_{ij} = 0$ if $i \neq j$ and at least one element $a_{ii} \neq 0$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

3. Identity Matrix

A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n .

$$\text{That is } a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\text{Example: } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Upper Triangular matrix:

A square matrix said to be a Upper triangular matrix if $a_{ij} = 0$ if $i > j$.

$$\text{Example: } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 8 \\ 0 & -2 & 5 \\ 0 & 0 & 7 \end{pmatrix}$$

5. Lower Triangular Matrix:

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0$ if $i < j$.

Example: $A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 7 & 0 & 0 \\ 9 & 6 & 2 \end{pmatrix}$

6. Symmetric Matrix:

A square matrix $A = (a_{ij})_{n \times n}$ said to be a symmetric if $a_{ij} = a_{ji}$ for all i and j .

Example: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} 8 & -2 & 7 \\ -2 & -9 & 3 \\ 7 & 3 & 5 \end{pmatrix}$

7. Skew- Symmetric Matrix:

A square matrix $A = (a_{ij})_{n \times n}$ said to be a symmetric if $a_{ij} = -a_{ji}$ for all i and j .

Example: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} 8 & -2 & 7 \\ 2 & -9 & 3 \\ -7 & -3 & 5 \end{pmatrix}$

8. Zero Matrix:

A matrix whose all elements are zero is called as Zero Matrix and order $n \times m$ Zero matrix denoted by $0_{n \times m}$.

Example:

$$0_{3 \times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9. Row Vector

A matrix consist a single row is called as a row vector or row matrix.

Example:

$$A = (a_{11} \quad a_{12} \quad a_{13}) \qquad B = (7 \quad 4 \quad -3)$$

10. Column Vector

A matrix consist a single column is called a column vector or column matrix.

Example:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad B = \begin{pmatrix} 9 \\ -7 \\ 3 \end{pmatrix}$$

1.3 Algebra of Matrices:

1. Equality of two matrices:

Two matrices A and B are said to be equal if

- (i) They are of same order.
- (ii) Their corresponding elements are equal.

That is if $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ then $a_{ij} = b_{ij}$ for all i and j .

2. Scalar multiple of a matrix

Let k be a scalar then scalar product of matrix $A = (a_{ij})_{m \times n}$ given denoted by kA and

given by $kA = (ka_{ij})_{m \times n}$ or

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

3. Addition of two matrices:

Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ are two matrices with same order then sum of the two matrices given by

$$A + B = (a_{ij})_{m \times n} + (b_{ij})_{m \times n} = (a_{ij} + b_{ij})_{m \times n}$$

Example: let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & 8 \end{pmatrix}.$$

Find (i) $5B$ (ii) $A + B$ (iii) $4A - 2B$ (iv) $0 A$

4. Multiplication of two matrices:

Two matrices A and B are said to be confirmable for product AB if number of columns in A equals to the number of rows in matrix B . Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times r}$ be two matrices the product matrix $C = AB$, is matrix of order $m \times r$ where

$$c_{ik} = \sum_{j=1}^n a_{ij} b_{jk} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots \dots \dots + a_{in} b_{nj}$$

Example: Let $A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \\ -1 & -3 \end{pmatrix}$

Calculate (i) AB (ii) BA

(iii) is $AB = BA$?

Integer power of Matrices:

Let A be a square matrix of order n , and m be positive integer then we define

$$A^m = A \times A \times A \dots \dots \times A \quad (\text{m times multiplication})$$

1. 4. Properties of the Matrices

Let A , B and C are three matrices and λ and μ are scalars then

$$(i) \quad A + (B + C) = (A + B) + C \quad \text{Associative Law}$$

$$(ii) \quad \lambda (A + B) = \lambda A + \lambda B \quad \text{Distributive law}$$

$$(iii) \quad \lambda(\mu A) = (\lambda\mu)A \quad \text{Associative Law}$$

$$(iv) \quad (\lambda A)B = \lambda(AB) \quad \text{Associative Law}$$

$$(v) \quad A(BC) = (AB)C \quad \text{Associative Law}$$

$$(vi) \quad A(B + C) = AB + AC \quad \text{Distributive law}$$

1.5. Transpose:

The transpose of matrix $A = (a_{ij})_{m \times n}$, written A^t is the matrix obtained by writing the rows of A in order as columns.

That is $A^t = (a_{ji})_{n \times m}$.

Properties of Transpose:

$$(i) \quad (A + B)^t = (A^t + B^t)$$

$$(ii) \quad (A^t)^t = A$$

$$(iii) \quad (kA)^t = k A^t \text{ for scalar } k.$$

$$(iv) \quad (AB)^t = B^t A^t$$

Example: Using the following matrices A and B, Verify the transpose properties

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 6 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

Remakes:

(i) A square matrix A is said to be symmetric if $A = A^t$.

Example:

$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$, A is symmetric by the definition of symmetric matrix.

Then

$$A^t = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$

That is $A = A^t$

(ii). A square matrix A is said to be skew-symmetric if $A = -A^t$

Example:
$$A = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -5 & 8 \\ 1 & 8 & 9 \end{pmatrix}$$

The matrix A is skew-symmetric since $A = -A^t$.

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix}$$

- (i) AA^t is a symmetric matrix.
- (ii) $A + A^t$ is a symmetric matrix.
- (iii) $A - A^t$ is a skew-symmetric matrix.

1.6. Determinant:

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n , then the number $|A|$ called determinant of the matrix A .

(i) Determinant of 2×2 matrix

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ then } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

(ii) Determinant of 3×3 matrix

$$\text{Let } B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\text{Then } |B| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|B| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

(iii). Determinant of a $n \times n$ matrix by expanding cofactor

- Let $A = (a_{ij})_{n \times n}$ is a square matrix. Then M_{ij} denote a sub matrix of A with order $(n-1) \times (n-1)$ obtained by deleting its i^{th} row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A .

The cofactor of a_{ij} denoted by A_{ij} and is equal to $(-1)^{i+j} |M_{ij}|$.

$$\det(A) = |A| = \sum_{j=1}^n a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$$

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Ex: Determinant of 3×3 matrix

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

If A is a square matrix (of order 2 or greater), then the determinant of A is the sum of the entries in the first row of A multiplied by their cofactors. That is,

$$\det(A) = |A| = \sum_{j=1}^n a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}.$$

$$\text{Then } |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 4: Calculate the determinants of the following matrices

$$(i) A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{pmatrix} \quad (ii) B = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$$

$$\begin{bmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

n × n matrix

Properties of the Determinant:

a. The determinant of a matrix A and its transpose A^t are equal.

$$|A| = |A^t|$$

b. Let A be a square matrix

(i) If A has (column) of zeros then $|A| = 0$.

(ii) If A has two identical rows (or columns) then $|A| = 0$.

(i) If A is triangular matrix then $|A|$ is product of the diagonal elements.

1.7. Singular Matrix

If A is square matrix of order n , the A is called singular matrix when $|A| = 0$ and non-singular otherwise.

1.8. Minor and Cofactors:

Let $A = (a_{ij})_{n \times n}$ is a square matrix. Then M_{ij} denote a sub matrix of A with order $(n-1) \times (n-1)$ obtained by deleting its i^{th} row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A .

The cofactor of a_{ij} denoted by A_{ij} and is equal to $(-1)^{i+j} |M_{ij}|$.

Example 4: Let $A = \begin{pmatrix} 5 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & -2 & -1 \end{pmatrix}$

- (i) Compute determinant of A.
- (ii) Find the cofactor matrix.

3. 9. Adjoin Matrix:

The transpose of the matrix of cofactors of the element a_{ij} of A denoted by $adj A$ is called adjoin of matrix A.

Example 5: Find the adjoint matrix of example 4.

Theorem 1:

For any square matrix A ,

$A (adj A) = (adj A) A = |A| I$ where I is the identity matrix of same order.

$$\text{If } |A| \neq 0 \quad A^{-1} = \frac{1}{|A|} (adj A)$$

Example 6: Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ find A^{-1} .

Theorem 2: (Existence of the Inverse)

The necessary and sufficient condition for a square matrix A to have an inverse is that $|A| \neq 0$ (That is A is non singular).

Theorem 3: (Uniqueness of the Inverse)

Inverse of a matrix if it exists is unique.