LINEAR ALGEBRA - DS2103

Elementary Transformations & Applications

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Elementary Transformations:

Some operations on matrices called as elementary transformations. There are six types of elementary transformations, three of then are row transformations and other three of them are column transformations. There are as follows

(i) Interchange of any two rows (or columns).

(ii) Multiplication of the elements of any row (or column) by a nonzero number k.

(iii) Multiplication to elements of any row (or column) by a scalar k and addition of it to the corresponding elements of any other row (or column).

- We adopt the following notations for above transformations

(i) Interchange of ith row and jth row is denoted by $R_i \leftrightarrow R_j$.

(ii) Multiplication by k to all elements in the ith row $R_i \rightarrow kR_i$.

(iii) Multiplication to elements of jth row by k and adding them to the corresponding elements of ith row is denoted by $R_i \rightarrow R_i + kR_j$.

Equivalent Matrix:

A matrix B is said to be equivalent to a matrix A if B can be obtained from A, by for forming finitely many successive elementary transformations on a matrix A.

Denoted by A~ B.

Rank of a Matrix:

Definition:

A positive integer 'r' is said to be the rank of a non- zero matrix A if

(i) There exists at least one non-zero minor of order r of A and

(ii) Every minor of order greater than r of A is zero.

The rank of a matrix A is denoted by $\rho(A)$.

❖ Elementary matrices

An elementary matrix is a square matrix which is obtained from an identity matrix by the application of a single elementary row operation.

Examples of elementary matrices.

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

- If an elementary row operation is performed on an $\mathrm{m} \times n$ matrix A, the resulting matrix can be written as EA, where the $\mathrm{m} \times m$ matrix E is created by performing the same row operation on I_m.

- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Examples:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Check the effects of pre-multiplying A by E_1, E_2 and E_3 above.

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}.$$

(The first two rows are interchanged.)

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 35 & 40 & 45 \end{bmatrix}$$

(The third row is multiplied by 5.)

Echelon Matrices:

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

❖ The following matrices are in echelon form. The leading entries (▪) may have any nonzero value; the starred entries (*) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

❖ The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below and above each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- Any nonzero matrix may be row reduced(that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

- A pivot position in a matrix Ai s a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Example:

Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \longrightarrow$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

$$\downarrow$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Next pivot column

$$\longrightarrow \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

General form: $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Pivot positions

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot columns

Example 7:

Reduce following matrices to row reduce echelon form

(i) $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

(ii) $B = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix}$

Gauss-Jordan Method of finding the inverse

Those elementary row transformations which reduce a given square matrix A to the unit matrix, when applied the unit matrix I give the inverse of A.

$$R_i R_{i-1} \dots \dots R_2 R_1 A = I$$

$$R_i R_{i-1} \dots \dots R_2 R_1 A A^{-1} = I A^{-1}$$

$$R_i R_{i-1} \dots \dots R_2 R_1 I = A^{-1}$$

Example :

Using the Gauss-Jordan method, find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$$