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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2017 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2017

සංයුක්ත ගණිතය I / இணைந்த கணிதம் I / Combined Mathematics I

10 E I

පැය තුනයි / மூன்று மணித்தியாலம் / *Three hours*

Index Number							

Instructions:

* *This question paper consists of two parts;* **Part A** (Questions 1 - 10) and **Part B** (Questions 11 - 17).
* **Part A:** *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
* **Part B:** *Answer **five** questions only. Write your answers on the sheets provided.*
* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
* *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

For Examiners' Use only

(10) Combined Mathematics I		
Part	**Question No.**	**Marks**
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
	15	
	16	
	17	
	Total	
	Percentage	

Paper I	
Paper II	
Total	
Final Marks	

Final Marks

In Numbers	
In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

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Part A

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$ for all $n \in \mathbb{Z}^+$.

2. Find all real values of x satisfying the inequality $x^2 - 1 \geq |x+1|$.

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3. Sketch, in an Argand diagram, the locus l of the points that represent complex numbers z satisfying $\operatorname{Arg}(z - 2i) = \frac{\pi}{3}$.

Let P and Q be the points in the above Argand diagram that represent the complex numbers $2i$ and $\sqrt{3} + 5i$, respectively. Find the distance PQ and show that the point Q lies on l.

4. In how many different ways can the eight letters of the word INFINITY be arranged in a row? How many of these arrangements have

(i) all three I letters close to one another?

(ii) exactly one I letter and both N letters as the first three letters?

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5. Let $0 < \alpha < \frac{\pi}{2}$. Show that $\lim\limits_{x \to \alpha} \dfrac{x^3 - \alpha^3}{\tan x - \tan \alpha} = 3\alpha^2 \cos^2 \alpha$.

6. Let $0<a<b$. Show that $\dfrac{d}{dx}\sin^{-1}\left(\sqrt{\dfrac{b-a}{b}}\cos x\right) = -\dfrac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2 x + b\sin^2 x}}$.

Hence, find $\displaystyle\int \frac{\sin x}{\sqrt{a\cos^2 x + b\sin^2 x}}\,dx$.

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7. A curve C is given parametrically by $x=3\cos\theta-\cos^3\theta$, $y=3\sin\theta-\sin^3\theta$ for $0<\theta<\frac{\pi}{2}$.
Show that $\frac{dy}{dx}=-\cot^3\theta$.
Find the coordinates of the point P on the curve C at which the gradient of the tangent line is -1.

8. Let l_1 and l_2 be the straight lines given by $3x-4y=2$ and $4x-3y=1$ respectively.
(i) Write down the equations of the bisectors of the angles between l_1 and l_2.
(ii) Find the equation of the bisector of the acute angle between l_1 and l_2.

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9. Let S be the circle given by $x^2 + y^2 - 4 = 0$ and let l be the straight line given by $y = x + 1$. Find the equation of the circle which passes through the points of intersection of S and l, and also intersects the circle S orthogonally.

10. Show that $\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2 = 1 + \sin\theta$ for $-\pi < \theta \le \pi$. Hence, show that $\cos\frac{\pi}{12} + \sin\frac{\pi}{12} = \sqrt{\frac{3}{2}}$ and also find the value of $\cos\frac{\pi}{12} - \sin\frac{\pi}{12}$. Deduce that $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Department of Examinations, Sri Lanka

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Part B

* Answer five questions only.

11. (a) Let $f(x) = 3x^2 + 2ax + b$, where $a, b \in \mathbb{R}$.

It is given that the equation $f(x) = 0$ has two real distinct roots. Show that $a^2 > 3b$.

Let α and β be the roots of $f(x)=0$. Write down $\alpha+\beta$ in terms of a and $\alpha\beta$ in terms of b.

Show that $|\alpha - \beta| = \frac{2}{3}\sqrt{a^2 - 3b}$.

Show further that the quadratic equation with $|\alpha+\beta|$ and $|\alpha-\beta|$ as its roots is given by $9x^2 - 6\left(|a| + \sqrt{a^2 - 3b}\right)x + 4\sqrt{a^4 - 3a^2b} = 0$.

(b) Let $g(x) = x^3 + px^2 + qx + 1$, where $p, q \in \mathbb{R}$. When $g(x)$ is divided by $(x-1)(x+2)$, the remainder is $3x+2$. Show that the remainder when $g(x)$ is divided by $(x - 1)$ is 5, and that the remainder when $g(x)$ is divided by $(x+2)$ is -4.

Find the values of p and q, and show that $(x + 1)$ is a factor of $g(x)$.

12. (a) Write down the binomial expansion of $(5 + 2x)^{14}$ in ascending powers of x.

Let T_r be the term containing x^r in the above expansion for $r = 0, 1, 2, \ldots, 14$.

Show that $\frac{T_{r+1}}{T_r} = \frac{2(14-r)}{5(r+1)}x$ for $x \neq 0$.

Hence, find the value of r which gives the largest term of the above expansion, when $x = \frac{4}{3}$.

(b) Let $c \geq 0$. Show that $\frac{2}{(r+c)(r+c+2)} = \frac{1}{(r+c)} - \frac{1}{(r+c+2)}$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{n} \frac{2}{(r+c)(r+c+2)} = \frac{(3+2c)}{(1+c)(2+c)} - \frac{1}{(n+c+1)} - \frac{1}{(n+c+2)}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty} \frac{2}{(r+c)(r+c+2)}$ converges and find its sum.

By using this sum with suitable values for c, show that $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$.

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13.(*a*) Let $\mathbf{A} = \begin{pmatrix} 2 & a & 3 \\ -1 & b & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & a \\ 1 & b & 0 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$, where $a, b \in \mathbb{R}$.

It is given that $\mathbf{AB}^{\mathrm{T}} = \mathbf{P}$, where $\mathbf{B}^{\mathrm{T}}$ denotes the transpose of the matrix $\mathbf{B}$. Show that $a = 1$ and $b = -1$, and with these values for a and b, find $\mathbf{B}^{\mathrm{T}}\mathbf{A}$.

Write down $\mathbf{P}^{-1}$, and using it, find the matrix $\mathbf{Q}$ such that $\mathbf{PQ} = \mathbf{P}^2 + 2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Sketch in an Argand diagram, the locus C of the points representing complex numbers z satisfying $|z| = 1$.

Let $z_0 = a(\cos\theta + i\sin\theta)$, where $a > 0$ and $0 < \theta < \frac{\pi}{2}$. Find the modulus in terms of a and the principal argument, in terms of θ, of each of the complex numbers $\frac{1}{z_0}$ and z_0^2.

Let P, Q, R and S be the points in the above Argand diagram representing the complex numbers z_0, $\frac{1}{z_0}$, $z_0 + \frac{1}{z_0}$ and z_0^2, respectively.

Show that when the point P lies on C above,

(i) the points Q and S also lie on C, and

(ii) the point R lies on the real axis between 0 and 2.

14.(*a*) Let $f(x) = \dfrac{x^2}{(x-1)(x-2)}$ for $x \neq 1, 2$.

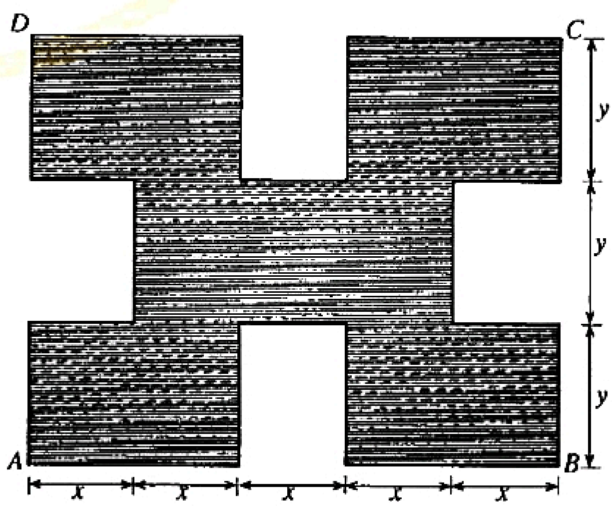
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{x(4-3x)}{(x-1)^2(x-2)^2}$ for $x \neq 1, 2$.

Sketch the graph of $y = f(x)$ indicating the asymptotes and the turning points.

Using the graph, solve the inequality $\dfrac{x^2}{(x-1)(x-2)} \leq 0$.

(*b*) The shaded region shown in the adjoining figure is of area 385 m². This region is obtained by removing four identical rectangles each of length y metres and width x metres from a rectangle $ABCD$ of length $5x$ metres and width $3y$ metres. Show that $y = \frac{35}{x}$ and that the perimeter P of the shaded region, measured in metres, is given by $P = 14x + \frac{350}{x}$ for $x > 0$.

Find the value of x such that P is minimum.



15. (*a*) (i) Express $\dfrac{1}{x(x+1)^2}$ in partial fractions and **hence**, find $\displaystyle\int \frac{1}{x(x+1)^2}\,dx$.

(ii) Using integration by parts, find $\displaystyle\int xe^{-x}\,dx$ and **hence**, find the area of the region enclosed by the curve $y = xe^{-x}$ and the straight lines $x = 1$, $x = 2$ and $y = 0$.

(*b*) Let $c > 0$ and $I = \displaystyle\int_0^c \frac{\ln(c+x)}{c^2+x^2}\,dx$. Using the substitution $x = c\tan\theta$,

show that $I = \dfrac{\pi}{4c}\ln c + \dfrac{1}{c}J$, where $J = \displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$.

Using the formula $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where a is a constant, show that $J = \dfrac{\pi}{8}\ln 2$.

Deduce that $I = \dfrac{\pi}{8c}\ln(2c^2)$.

16. Let $m \in \mathbb{R}$. Show that the point $P \equiv (0,1)$ does not lie on the straight line l given by $y = mx$.

Show that the coordinates of any point on the straight line through P perpendicular to l can be written in the form $(-mt,\ t+1)$, where t is a parameter.

Hence, show that the coordinates of the point Q, the foot of the perpendicular drawn from P to l, are given by $\left(\dfrac{m}{1+m^2}, \dfrac{m^2}{1+m^2}\right)$.

Show that, as m varies, the point Q lies on the circle S given by $x^2 + y^2 - y = 0$, and sketch the locus of Q in the xy-plane.

Also, show that the point $R \equiv \left(\dfrac{\sqrt{3}}{4}, \dfrac{1}{4}\right)$ lies on S.

Find the equation of the circle S' whose centre lies on the x-axis, and touches S externally at the point R.

Write down the equation of the circle having the same centre as that of S' and touching S internally.

17. (*a*) (i) Show that $\dfrac{2\cos(60° - \theta) - \cos\theta}{\sin\theta} = \sqrt{3}$ for $0° < \theta < 90°$.

(ii) In the quadrilateral $ABCD$ shown in the figure, $AB = AD$, $A\hat{B}C = 80°$, $C\hat{A}D = 20°$ and $B\hat{A}C = 60°$.

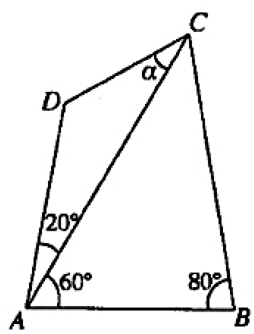
Let $A\hat{C}D = \alpha$. Using the Sine Rule for the triangle ABC, show that $\dfrac{AC}{AB} = 2\cos 40°$.

Next, using the Sine Rule for triangle ADC, show that $\dfrac{AC}{AD} = \dfrac{\sin(20° + \alpha)}{\sin\alpha}$.

Deduce that $\sin(20° + \alpha) = 2\cos 40° \sin\alpha$.

Hence, show that $\cot\alpha = \dfrac{2\cos 40° - \cos 20°}{\sin 20°}$.

Now, using the result in (i) above, show that $\alpha = 30°$.



(*b*) Solve the equation $\cos 4x + \sin 4x = \cos 2x + \sin 2x$.