G.C.E. (Advanced Level) Examination - August 2000
10 - Combined Mathematics - I
Three hours

- Answer six questions only.

(01) (i) α and β are roots of the equation $x^2 - px + q = 0$. Find the equation whose roots are $\alpha(\alpha+\beta)$ and $\beta(\alpha+\beta)$.

(ii) Find the values of λ for which the expression $f(x, y) = 2x^2 + \lambda xy + 3y^2 - 5y - 2$ can be written as the product of two linear factors.

(iii) Express $\dfrac{2x^2 - x + 3}{x(x-1)^2}$ in partial fractions.

(02) (a) Let $u_n = 1.n + 2.(n-1) + \ldots\ldots + (n-1).2 + n.1$ for any positive integer n.

Prove, by the Principle of Mathematical Induction, that $u_n = \dfrac{1}{6}n(n+1)(n+2)$.

Find v_n such that $\dfrac{1}{u_n} = v_n - v_{n+1}$ for any positive integer n.

Hence or otherwise show that

$$\sum_{r=1}^{n} \frac{1}{u_r} = \frac{3}{2} - \frac{3}{(n+1)(n+2)}$$

Deduce the value of $\sum_{n=1}^{\infty} \dfrac{1}{u_n}$.

(b) Let

$(1+kx)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{10}x^{10}$, $x \in \mathbb{R}$

Where $a_2 = \dfrac{20}{9}$ and k is a positive constant. Find the value of k.

Show that $a_1 + a_3 + a_5 + a_7 + a_9 = \dfrac{11^{10} - 7^{10}}{2.9^{10}}$

Deduce the value of

$a_0 + a_2 + a_4 + a_6 + a_8 + a_{10}$

(03) (a) Find algebraically the modulus and the argument of the complex number $\dfrac{(-1+i)^3}{(1+i)^4}$

(ii) The points P_1 and P_2 represent the complex numbers z_1 and z_2 respectively in the Argand diagram. Provide a geometrical construction to obtain the position of the point which represents the complex number $z_1 + z_2$ in the Argand diagram.

Plot the complex numbers $z_1 = \dfrac{1+i}{1-i}$ and $z_2 = \dfrac{\sqrt{2}}{1-i}$ in the Argand diagram. Using the above result find the Position of $z_1 + z_2$.

Deduce that $\tan\dfrac{\pi}{8} = \sqrt{2} - 1$.

(04) (a) Evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos^2(2\sin x)}{1-\cos 2x}$

(b) If $y = e^{k\sin^{-1}x}$, where k is a constant, show that

$\dfrac{dy}{dx}\sqrt{1-x^2} = ky$

Find $\dfrac{dy}{dx}$ When $x = \dfrac{1}{2}$

(c) Three towns A, B and C are connected by two straight roads AB and BC such that $A\hat{B}C = \dfrac{\pi}{2}$, AB = 15 km and BC = 50 km. There is a proposed project to build another straight road connecting the town A to a place D on the road BC. Maximum speeds allowed for a car on the section DC is 50 km h^{-1} and on the proposed road AD is 40 km h^{-1}.

If D is located x km from the town A, find $T(x)$ the total time taken in hours by a car to travel from A to C via D, assuming that it travels at the maximum permitted speeds.

Examine the sign of $\frac{dT}{dx}$ as x increases from 0 to 50km.

Find the most suitable place for D that will allow a car to complete the journey from A to C in the shortest time.

(05) (a) Using a suitable substitution evaluate

$$\int_1^8 \frac{1}{\left(x^{\frac{4}{3}} + x^{\frac{2}{3}}\right)} dx .$$

(b) Let $I = \int_0^{\pi} e^{-2x} \cos x \, dx$ and $J = \int_0^{\pi} e^{-2x} \sin x \, dx$.

By using the method of integration by parts. show that $I = 2J$ and $J = 1 + e^{-2\pi} - 2I$ Hence obtain the values of I and J.

(c) Find $\int \frac{x^2 - 5x}{(x-1)(x+1)^2} dx$.

(06) Obtain the equation of the straight line that makes intercepts a and b on the x and y axes respectively.

The fixed straight line l given by $\frac{x}{h} + \frac{y}{k} = 1$ meets the x and y axes at the points A and B respectively. A straight line l' perpendicular to l meets the x and y axes at the points P and Q respectively. Show that the point of intersection of the straight lines AQ and BP lies on the circle $x^2 + y^2 - hx - ky = 0$, with the point (h, k) deleted.

(07) Show that if the two circles given by $x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$ and $x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0$ intersect orthogonally then $2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$.

A circles S with centre on the x-axis intersects orthogonally the circle S' given by $x^2 + y^2 - 8x - 6y + 21 = 0$ and touches the circle S'' given by $x^2 + y^2 + 4x + 6y + 9 = 0$.

Show that there are two such circles of S, one touching the circle S'' externally and the other touching the circle S'' internally.

Find the equations of these two circles.

(08) (a) Show that $\frac{1 + \cos\theta + \sin\theta}{1 - \cos\theta + \sin\theta} = \frac{1 + \cos\theta}{\sin\theta}$ for $\theta \neq n\pi$ or $2n\pi - \frac{\pi}{2}$, $n \in \mathbb{Z}$.

(b) Show that $8(\cos^6 x + \sin^6 x) = 5 + 3\cos 4x$ for all real x.

Hence or otherwise sketch the graph of $y = \cos^6 x + \sin^6 x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Deduce the value or the rance of values of k for the equation $\cos^6 x + \sin^6 x = k$ to have

(i) no solution
(ii) only two solutions
(iii) only three solutions
(iv) only four solutions

in $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(09) (a) Solve the equation $4\sin^2 x + 12 \sin x \cos x - \cos^2 x + 5 = 0$ for $0 \le x \le 2\pi$.

(b) State the Sine Rule and the Cosine Rule for a triangle.

It is given that $\frac{b+c}{2k-1} = \frac{c+a}{2k} = \frac{a+b}{2k+1}$,

where k is a given integer greater than 2 but not equal to 4 and a, b, c are the sides of a triangle ABC \in the usual notation.

Show that $\frac{\sin A}{k+1} = \frac{\sin B}{k} = \frac{\sin C}{k-1}$.

Also obtain cos A in term of k and show that

$$\frac{\cos A}{(k-4)(k+1)} = \frac{\cos B}{k^2+2} = \frac{\cos C}{(k+4)(k-1)}$$

where A, B, C have the usual meanings.