# G.C.E. (Advanced Level) Examination - August 2001
## 10 - Combined Mathematics - I
## Three hours

- Answer six questions only.

(01) (a) Let $\alpha$ and $\beta$ be the roots of theequation $x^2 + px + 1 = 0$ and let $\gamma$ and $\delta$ be the roots of the equation $x^2 + \frac{1}{p}x + 1 = 0$.

Show that

$$(\alpha-\gamma)(\beta-\gamma)(\alpha-\delta)(\beta-\delta) = (\gamma^2 + p\gamma + 1)(\delta^2 + p\delta + 1)$$

and deduce that

$$(\alpha-\gamma)(\beta-\gamma)(\alpha-\delta)(\beta-\delta) = \left(p - \frac{1}{p}\right)^2$$

(b) If a and b are positive real numbers, show that

$$\log_a b = \frac{1}{\log_b a}.$$

Show that

$$\frac{1}{\log_2 2001} + \frac{1}{\log_3 2001} + \frac{1}{\log_4 2001} + \dots\dots\dots + \frac{1}{\log_{100} 2001} = \frac{1}{\log_{100!} 2001}$$

(02) (a) Let $A_{n+1} = (1-\alpha)(1-A_n) + A_n$ for n = 1, 2, 3 ...... and $A_1 = \beta$. Where $\alpha$ and $\beta$ are real numbers. Prove, by the Principle of Mathemaical Induction, that for every positive integer n.

$$A_n = 1 - (1-\beta)\alpha^{n-1}$$

Find $\sum_{r=1}^{n} A_r$

(b) Show that for integers k and n such that $1 \le k \le n$,

$$K\,{}^nC_k = n\,{}^{n-1}C_{k-1}$$

Hence or otherwise prove that for any $x \in \mathbb{R}$ and $n \ge 0$,

$$\sum_{k=0}^{n} K\,{}^nC_k\, x^k (1-x)^{n-k} = nx.$$

(03) (a) In how many ways can 7 boys and 7 girl be lined up if a girl must be first in line and girls and boys alternate positions in line.

(b) Sketch the graphs of $y = 2|x+1| - 3$ and $y = x + 2|x-1|$ in the same diagram.

Hence find the set of $x$ values satisfying $x + 2|x-1| > 2|x+1| - 3$.

Solve the equation $x + 2|x-1| = 2|x+1| - 3$

(04) (a) If arg (z - a) = $\alpha$, where $a \in \mathbb{R}$ and $0 < \alpha < \pi$ describe the locus of z.

It is given that $\arg(z+1) = \frac{\pi}{6}$ and $\arg(z-1) = \frac{2\pi}{3}$

Using the first part find z.

(b) Show that the complex number $\frac{5-i}{2-3i}$ can be expressed in the form $\lambda(1+i)$, where $\lambda$ is real. State the value of $\lambda$.

Hence, show that $\left(\frac{5-i}{2-3i}\right)^4$ is imaginary and determine its value.

(05) (a) If $x = t - \sin t$ and $y = 1 - \cos t$, show that

$$y\left(\frac{d^3y}{dx^3}\right) + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0 \text{ for } t \ne 2n\pi,\ n \in \mathbb{Z}.$$

(b) Three towns A, B and C are located at the vertices of an isosceles triangle such that the distances from A to B and A to C are equal. The distance from B to C is 12 km and the altitude through A is 16km.

How far from A, along the altitude through A should a well be located so that it will use minimum pipe when supplying water to all three towns A, B and C.

(06) (*a*) By making a suitable substitution evaluate the

intergral $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$

(b) By using intergration by parts, show that the

intergral $\displaystyle\int x \ln x \, dx = a\ln b + c$ ,

where a, b and c are integers to be determined.

(c) Find $\displaystyle\int \frac{(7x - x^2)}{(2-x)(x^2+1)}\,dx$

(07) The st. ight line $y = mx + c$ intersects the two non-parallel straight lines $u_1 \equiv y - m_1x - c_1 = 0$ and $u_2 \equiv y - m_2x - c_2 = 0$ at A and B respectively. R is a point on AB such that $AR = kRB$. Show that the equation of the straight line joining R to the point of intersection of

$u_1 = 0$ and $u_2 = 0$ is $u_1 + \dfrac{k(m - m_1)}{m - m_2} u_2 = 0$ ,

The sides AB, BC, CA of a triangle ABC lie along the lines $3x + 2y - 6 = 0$ , $2x + y - 2 = 0$, $x + y - 3 = 0$ respectively. R is a point on AB and Q is a point on AC such that $2\,AR = RB$ and $3\,AQ = 2\,QC$.

(i) Find the coordinates of A.

(ii) Write the equations of the lines BQ and CR.

(iii) If BQ and CR meet at D and P is the point of intersection of AD and BC, find the ratio AP : PB.

(08) Show that the equation of the chord of contact of the tangents drawn from an external point $(x_0, y_0)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_0 + yy_0 + g(x + x_0) + f(y + y_0) + c = 0$.

$x^2 + y^2 + 2x + 6y + 1 = 0$ and $4x + 3y - 5 = 0$ are the equations of a given circle and a given straight line respectively. Show that the line does not cut the circle.

A Variable straight line intersects the given circle at two distinct points P and Q, and the tangents to the circle at P and Q meet on the given straight line. Show that this variable line passes through a fixed point, and find the coordinates of this point.

(09) (*a*) Show that for any real number $x$,

$$\sin^3 2x \cos 6x + \cos^3 2x \sin 6x = \frac{3}{4}\sin 8x$$

Deduce the values of a for which the equation $\sin^3 2x \cos 6x + \cos^3 2x \sin 6x = a$ is solvable.

(b) In a triangle, the largest angle is twice the size of the smallest angle and the longest side is $1\frac{1}{2}$ times the length of the shortest side. Show that smallest angle of the triangle is $\cos^{-1}\left(\frac{3}{4}\right)$.

Given that the length of the middle side is 10cm, find the lengths of the other two sides.