G.C.E. (Advanced Level) Examination - April 2003
10 - Combined Mathematics - I
Three hours

- Answer six questions only.

(01) (a) $\lambda \in \mathbb{R}$ and $p(x) = (\lambda - 2)x^2 - 3(\lambda + 2)x + 6\lambda$.

(i) Find the least integral value of λ for which $p(x)$ is positive for all $x \in \mathbb{R}$

(ii) For what values of λ does the equation $p(x) = 0$ have two distinct real roots?

(iii) If the roots of $p(x) = 0$ are real and if the difference of the roots is equal to 3, find λ.

(02) (a) There are 8 students in a certain class. The class teacher wants to divide those students into four teams to compete in a contest. The sizes of the teams need not all be equal and a team may consist of even one person.

Show that the required four teams can be formed in 1701 ways.

(b) Show that, in the usual notation

${}^{n}C_{r+1} + {}^{n}C_{r} = {}^{n+1}C_{r+1}$ for $0 \le r \le n-1$

Deduce that

${}^{2003}C_r + {}^{2004}C_r + \ldots\ldots + {}^{2013}C_r = {}^{2014}C_{r+1} - {}^{2003}C_{r+1}$

for $0 \le r \le 2002$

(03) (a) Prove by using the Principle of Mathematical Induction that $8(n+1)! > 2^{n+3}(n+2)$ for every positive interger n.

Deduce that $\displaystyle\sum_{k=1}^{n} \frac{k!}{2^k} > \frac{1}{16}\left(n^2 + 3n + 4\right)$

Hence, show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{2^k}$ is not convergent.

(b) Find the set of all real values of x satisfying the inequality $|x+2| + |x-1| > 5$.

(04) Express the complex number $\omega = \sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and θ is in radians with $0 \le \theta < 2\pi$.

Obtain ω^2, ω^3, ω^4 and ω^5 in the above form.

Shade the region R consisting of the points represention the complex numbers z in the Argand diagram, for which

$6 < |z| < 30$ and $\frac{\pi}{6} < \arg z < \frac{5\pi}{6}$

Determine which of the points represnting complex number ω^n ($n = 1, 2, \ldots\ldots, 5$) lie in the region R.

(05) (a) If $y = e^{\cos x}$

find $\left(\frac{dy}{dx}\right)_{x=0}$, $\left(\frac{d^2y}{dx^2}\right)_{x=0}$, $\left(\frac{d^3y}{dx^3}\right)_{x=0}$, $\left(\frac{d^4y}{dx^4}\right)_{x=0}$ and $\left(\frac{d^5y}{dx^5}\right)_{x=0}$

(b) Given that $y = \frac{2x}{1+x^2}$, find the values of x for which $\frac{dy}{dx} = 0$. Considering only the behaviour of the first derivative, investigate the muture of those stationary values of y.

Sketch the curve $y = \frac{2x}{1+x^2}$.

(06) (a) By making a suitable substitution, evaluate the intergral $\int_1^8 \frac{dx}{1+\sqrt[3]{x}}$.

(b) By using intergration by parts, evaluate the intergral $\int_0^1 x^3 e^{x^2+1}\, dx$

(c) Find $\int \frac{dx}{x\left(x^2+3\right)}$

(07) Two sides of a parallelogram are given by the equations $y = x - 2$ and $4y = x+4$. The diagonals of the parallelogram intersect at the origin. Obtain

(i) the equations of the remaining sides of the parallelogram.

and (ii) the equations of its diagonals.

Also, find the area of the parallelogram.

(08) If the two circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ intersect orthogonally, show that $2gg' + 2ff' = c + c'$.

Let P and Q be the points on the circle $S \equiv x^2 + y^2 - a^2 = 0$ with the coordinates $(-a, 0)$ and $(a \cos\theta, a \sin\theta)$ respectively. The chord PQ is extended to a point R so that PQ = QR. Find the coordinates of R and show that, as θ varies, R lies on a circle S'. Obtain the equation of S'.

A third circle S'', which touches the y axis, intersects both circles S and S' orthogonally. Show that there are two such circles S'' and obtain their equations.

(09) (a) If $x = \sin\theta - \cos\theta$ and $y = \tan\theta + \cot\theta$, where θ is a real number not equal to a multiple of $\frac{\pi}{2}$, obatain $\sin 2\theta$

(i) in terms of x only.

(ii) in terms of y only.

Hence obtain a relationship between x and y.

(b) Show that
$\sin 2x + \sin 4x + \sin 6x = (1 + 2\cos 2x)\sin 4x$

Hence show that

$\sin x\,(\sin 2x + \sin 4x + \sin 6x) = \sin 3x \sin 4x$

Deduce that $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

(c) State the Sine Rule for a triangle.
In a triangle ABC, in the usual notation,
$a = b + \lambda c$, where $\lambda \in \mathbb{R}$.

Show that $\lambda \cos\frac{C}{2} = \cos\left(B + \frac{C}{2}\right)$