G.C.E. (Advanced Level) Examination - August 2008
10 - Combined Mathematics - I
Three hours

- Answer six questions only

(01) (a) α and β are the roots of the equation $x^2 + bx + c = 0$, where $c \neq 0$. Find the quadratic equation, in terms of b and c, whose roots are α^4 and β^4.

Hence, find the quadratic equation in terms of b and c, whose roots are $\frac{\alpha^4}{\beta^4} + 1$ and $\frac{\beta^4}{\alpha^4} + 1$.

(b) Show that the remainder when the polynomial $f(x)$ is divided by $(x - \alpha)$ is $f(\alpha)$.

When the polynomial $f(x)$ is divided by $(x - \alpha)(x - \beta)$, where $\alpha \neq \beta$, the remainder takes the form $Ax + B$. Express the constants A and B in terms of α, β, $f(\alpha)$ andy $f(\beta)$.

Hence, find the value of the constant k for which the remainder when $x^3 + kx^2 + k$ is divided by $(x - 1)(x + 2)$ contains **no** constant term.

(02) (a) It is required to select 5 students in order to form a debating team from among 7 girls and 8 boys. Find the number of terms that can be selected if

(i) the teams must consist of two girls and three boys,

(ii) the teams must consist of at most three boys,

(iii) a certain boy and a certain girl cannot be selected to the same team.

(b) Three consecutive coefficients in the expassion of $(1 + x)^n$, where n is a positive interger, are 45, 120 and 210.

Find the value of n.

(c) Is it possible for three consecutive coefficients in the expansion of $(1 + x)^n$, where n is a positive integer, to be in geometric progression? Justify your answer.

(03) (a) By using the principle of Marthematical Induction, prove that $5^{n+1} - 2^{n+1} - 3^{n+1}$ is divisible by 6, for positive integral n.

(b) (i) Find $\sum_{r=1}^{n} {}^nC_r$ and deduce that

$$\frac{2^n}{n} > \frac{(n-1)}{2} \text{ for positive intergral } n.$$

(ii) The r^{th} term, U_r of an infinite series is given by $\frac{2^{r-1}r}{(r+1)(r+2)}$. Find $f(r)$ such that $U_r = f(r) - f(r-1)$.

Hence, find $\sum_{r=1}^{n} U_r = S_n$

Does $\lim_{n\to\infty} S_n$ exist in $\mathbb{R}$? Justify your answer.

(04) By factorising $z^3 - 1$, solve the equation $z^3 - 1 = 0$. Show that if one of the complex roots of the above equation is ω, then the other is ω^2.

Show that $\text{Re}\left(\frac{1}{1+\omega^r}\right) = \frac{1}{2}$ for r = 1, 2, 3 and interpret the result geometrically.

z_1, z_2 and z_3 are three complex numbers satisfying the relation $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$.

Show that z_1 can be expressed either as

$z_1 = -\omega z_2 - \omega^2 z_3$ or $z_1 = -\omega^2 z_2 - \omega z_3$

Deduce that the three complex numbers z_1, z_2 and z_3 represent the vertices of an equilateral triangle.

(05) (a) Using first principles, find the derivative of the function $f(x) = \tan x$ with respect to x.

Differentiate $\tan(\sin^{-1} x)$, when $0 < x < 1$, with respect to x.

(b) Show that if y is a differentiable function of u and $u = \ln(\cos x)$ when $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then

$$\sin^3 x \frac{d^2y}{du^2} = \sin x \cos^2 x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx}.$$

(c) Let C be the curve given parametrically by $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$, where a is a non-zero constant and t is a non-zero parameter. Find the equation of the normal to the curve C at the point with parametric value t_0.

Show that four normals can be drawn from the point $(-13a, 0)$,to the curve C and find the parametric values of the feet of the four normals.

(06) (*a*) Using partial fractions, find $\int \frac{dx}{(x^2 - a^2)^2}$, where $a \neq 0$.

(b) (i) Show that $\frac{d}{dx}\left(\frac{2^x}{ln 2}\right) = 2^x$.

(ii) Find $\int 2^x dx$.

(iii) Utilising the method of integration by parts, evaluate $\int_{-1}^{1} 2^{\sqrt{x+1}} dx$.

(07) (a) Obtain the equations of the bisectors l_1 and l_2 of the angles between the straight lines given by $y = m_1x + c_1$ and $y = m_2x + c_2$ where $m_1 \neq m_2$.

Hence, verify that l_1 and l_2 are perpendicular.

(b) Let ABC be a triangle such that the base BC moves along the x - *axis* in the positive direction. AB = AC and A lies above the x - *axis*. The area of the triangle *ABC* is 9 square units and the length of the side *BC* is 6 units. Also, let B = (b, 0).

(i) Find the equations of the sides *AB* and *AC*.

(ii) Using the squations of the angle bisectors obtained in (*a*) above, find the equations of the internal bisectors of the angles *B* and *C* of the triangle *ABC*.

Hence, find the value of $tan\left(\frac{\pi}{8}\right)$.

(iii) Verify that the three internal bisectors of the angles of the triangle *ABC* meet at a point and determine its locus.

(08) (*a*) Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$.
$S = 0$ is a variable circle passing through a fixed point, and $S' = 0$ is a fixed circle. The circle $S = 0$ cuts the circle $S' = 0$ at the opposite ends of a diameter. Show that the centre of $S = 0$ lies on a fixed straight line.

(b) A and B are the two distinct point (x_1, y_1) and (x_2, y_2) respectively. Find the equation of the circle having AB as a diameter.

CD is the diameter perpendicular to AB. Show that the coordinates of C and D take the form

$\left(\frac{1}{2}(x_1 + x_2) + \lambda,\ \frac{1}{2}(y_1 + y_2) + \mu\right)$ and

$\left(\frac{1}{2}(x_1 + x_2) - \lambda,\ \frac{1}{2}(y_1 + y_2) - \mu\right)$, where λ and μ are to be determined.

(09) (a) State and prove the ***sine rule.***
P is a point inside the triangle ABC such that $P\hat{A}B = P\hat{B}C = P\hat{C}A = \phi$. Prove that

$$\frac{bc}{a}(\cot\phi - \cot A) = \frac{ac}{b}(\cot\phi - \cot B) = \frac{ab}{c}(\cot\phi - \cot C),$$

in the usual notation.

(b) Let x, y and z be any three non-negative real numbers such that $x + y + z = \pi$, $\cos x + \cos y = 1$ and $t = \sin x + \sin y$.

(i) $\tan^{-1}(t) = \frac{x + y}{2}$

(ii) $0 \leq t \leq \sqrt{3}$

Hence, find the values of x, y and z, when t attains its maximum value.