G.C.E. (Advanced Level) Examination - August 2012

Combined Mathematics - I

Three hours

PART - A

(01) Using the Principle of Mathematical Induction. Prove that $1+2+\dots\dots\dots+n = \dfrac{n(n+1)}{2}$ for any Positive integer n.

(02) Find the number of arrangements that can be made by using all the letters of the word ADDING find. in how many of these arrangements the vowels are separated

(03) If the coefficient of x and the coefficient of x^2 in the binomial expansion of $(1+px)^{12}$, where P is a non - zero constant. are $-q$ and $11q$ respectively, find the values of p and q

(04) show that $\displaystyle\lim_{x\to 0}\frac{x\sin x}{2\sin^2 3x - x^2\cos x} = \frac{1}{17}$

(05) Find constant A and B such that

$2e^x + 3e^{-x} = A(2e^x - e^{-x}) + B(2e^x + e^{-x})$

Hence, find $\displaystyle\int\frac{2e^x + 3e^{-x}}{2e^x + e^{-x}}\,dx$

(06) Let l be the straight line passing through the points (4,0) and (0,2) and m be the straight line passing through the points (2, 0) and (0,3) Find the equations of the straight lines l and m Hence find the equation of the straight line through the origin and the point of intersection of l and m.

(07) A curve C is given by the equation $y = 4 - 4x + 3x^2 - x^3$ Find the equation of the tangent drawn to the curve C at the point (1,2) show that this tangent is perpendicular to the tangent drawn to the curve $y^2 = 4x$ at the point (1,2). The gradient of the tangent drawn to the curve C at the point (1, 2)

(08) show that the equation of any circle through the points (2,0) and (0,2) can be written as $x^2+y^2-4+\lambda(x+y-2)=0$ where λ is a parameter Find the centre and the radius of this circle in terms of λ

(09) Find the equation of the circle S with AB, where A = (1,3) and B = (2,4) as a diameter also find the equation of the circle with centre (-1,2) which cuts the circle S orthogonally

(10) Taking $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ show that $\tan\left(\frac{\pi}{12}\right) = 2-\sqrt{3}$ Deduce the value of $\tan\left(\frac{23}{12}\pi\right)$

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PART - B

(11) (*a*) $f(x) \equiv x^2 + 2kx + k + 2$, where k is a real constant.

(i) Express $f(x)$ in the form $(x - a)^2 + b$, where a and b are constants to be determined in terms of k.

Find the turning point of $f(x)$ **without using calculus** and show that this point is a minimum point.

Find the minimum value of $f(x)$ in terms of k.

Hence, show that the curve $y = f(x)$

(α) lies entirely above the x-axis if $-1 < k < 2$,

(β) touches the x-axis if $k = -1$ or $k = 2$,

(γ) cuts the x-axis in two distinct points if $k < -1$ or $k > 2$.

(ii) Prove that the straight line $y = mx$ intersects the curve $y = f(x)$ in two real and distinct points for all real and finite values of m if and only if $k < -2$.

(b) Let $g(x) \equiv x^4 + 4x^3 + 7x^2 + 6x + 2$.

Using Remainder theorem repeatedly show that $(x + 1)^2$ is a factor of $g(x)$.

Express $g(x)$ in the form $(x-a)^2(x^2 + bx + c)$, where a, b and c are constants to be determined.

Deduce that $g(x) \geq 0$ for all real values of x.

(12) (*a*) Find constants A and B such that

$12x^2 + 1 \equiv A(2x - 1)^3 + B(2x + 1)^3$ for all $x \in IR$.

Hence, determine $f(r)$ for $r \in IR^+$, such that

$U_r = f(r) - f(r + 1)$, where $u_r = \dfrac{12r^2 + 1}{(2r-1)^3(2r+1)^3}$

Show that $\displaystyle\sum_{r=1}^{n} u_r = \frac{1}{2} - \frac{1}{2(2n+1)^3}$

Show that the series $\displaystyle\sum_{r=1}^{\infty} u_r$ is convergent and find the value of $\displaystyle\sum_{r=1}^{\infty} u_r$.

(*b*) Sketch, **in the same figure**, the graphs of $y = |2x - 1|$ and $y = |x| + \dfrac{5}{3}$.

Hence, find the set of values of x for which $3|x| \geq |6x - 3| - 5$.

By considering the graph of $y = |x| - k$, for any $k \in IR$, **in the same figure** find for what value of l the equation $3|x| = |6x - 3| + l$ has only one real solution.

(13) (*a*) Let $A = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$ be a 2×2 matrix.

Show that $\mathbf{A^2 - 3A + 2I = 0}$, where I is the 2×2 identity matrix and $\mathbf{O}$ is the 2×2 zero matrix.

Hence, find A^{-1}.

Let $B = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ be a 2×2 matrix.

Show that BA = B.

Hence, or otherwise find a non-zero 2×2 matrix **C such that BC = O.**

(*b*) **Let z be a complex number.**

Prove that $|z|^2 = z\bar{z}$ and $|z| \geq \text{Re}\,z$

Hence, show that $|z_1| - |z_2| \leq |z_1 - z_2|$ for any two complex numbers z_1 and z_2.

Deduce that $|z_1 + z_2| \leq |z_1| + |z_2|$.

If $|z - i| < \dfrac{1}{2}$ then, show that $\dfrac{1}{2} < |z| < \dfrac{3}{2}$.

Shade the region R consisting the set of points in the Argand diagram which represent the complex number z for which $|z - i| \leq \dfrac{1}{2}$ and $\dfrac{\pi}{2} \leq \arg z \leq \dfrac{2\pi}{3}$.

(14) (*a*) By considering only the first derivative find the minimum and maximum values of $\dfrac{x^3}{x^4+27}$.

Sketch the graph of $y=\dfrac{x^3}{x^4+27}$.

Hence, find for what values of k, the equation $kx^4 - x^3 + 27k = 0$, where k is real, has

(i) two coincident real roots,
(ii) three coincident real roots,
(iii) two distinct real roots,
(iv) no real roots.

(b) Consider a rectangle $ABCD$ with $AB = a$ and $BC = b\ (< a)$. Let P be a movable point on CD. The length of $AP + PB$ is $L(x)$, Where $DP = x$.

Show that $L(x)=\sqrt{x^2+b^2}+\sqrt{(a-x)^2+b^2}$.

Find the minimum length of $L(x)$ and the position of P on CD corresponding to this minimum length.

Also, find the maximum length of $L(x)$

(15) (*a*) Show that $\displaystyle\int_0^{\pi}\left(\sin^3 x-\cos^3 x\right)dx=\frac{4}{3}$

(b) Using intergration by parts, or otherwise find $\displaystyle\int x^3\tan^{-1}x\,dx$

(c) Using Partial fractions *find* $\displaystyle\int\frac{2x^2-3}{(x-2)^2\,(x^2+1)}dx$

(16) (*a*) Find the equations of the bisectors of the angles between two non parallel straight lines $l_1 \equiv a_1x+b_1y+c_1=0$ and $l_2 \equiv a_2x+b_2y+c_2=0$.

Show that the bisector of the acute angle between two straight lines given by $2x - 11y - 10=0$ and $10x+5y-2=0$ is the bisector of the obtuse abgle be tween two straight lines given by $4x - 7y - 8=0$ and $8x + y - 4=0$.

(*b*) Show that, for all values of g and f the circle $x^2+y^2+2gx+2fy-r^2=0$ bisects the circumference of the circle $x^2+y^2-r^2=0$.

Show that two circles can be drawn through the point $(1, 1)$, touching the straight line $y+5=0$ and bisecting the circumference of the circle

$x^2+y^2-4=0$.

Find the equations of these two circles.

(17) (*a*) For a triangle ABC, prove in the usual notation, that $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

Deduce that $a=(b-c)\cos\dfrac{A}{2}\operatorname{cosec}\dfrac{B-C}{2}$

(*b*) Show that, for any real value of θ, the expression $\tan\theta-2\tan\left(\theta-\dfrac{\pi}{4}\right)$ cannot take any value between -7 and 1.

(*c*) Express $5\cos^2\theta+18\cos\theta\sin\theta+29\sin^2\theta$ in the form of $a+b\cos(2\theta+\alpha)$, where a and b are constants and α is an angle independent of θ.

Hence or otherwise find the general solution of the equation

$8(\cos x+\sin x)^2+2(\cos x+5\sin x)^2=19$.