G.C.E. (Advanced Level) Examination - August 2010

Combined Mathematics I

Three hours

- Answer six questions only.

(01) (*a*) α and β are the roots of the quadratic equation $f(x) \equiv x^2 + px + q = 0$, where p and q are real and $2p^2 + q \neq 0$. If $y(p - x) = p + x$, substituting for x in $f(x) = 0$ or otherwise, show that
$g(y) \equiv (2p^2 + q)y^2 + 2(q - p^2)\,y + q = 0$, where $y \neq -1$.

Hence, find the roots of the equation $g(y) = 0$ in terms of α and β.

Express $\left(\dfrac{\alpha}{2\beta+\alpha}\right)^2 + \left(\dfrac{\beta}{2\alpha+\beta}\right)^2$ in terms of p and q.

(b) If a, b, c and m are constants such that $a + b + c = 0$ and $ab + bc + ca + 3m = 0$, prove that
$(y + ax)(y + bx)(y + cx) = y(y^2 - 3mx^2) + abcx^3$.

If $y = x^2 + m$, show that
$(x^2 + ax + m)(x^2 + bx + m)(x^2 + cx + m) = x^6 + abcx^3 + m^3$.

If $g(x) = x^6 + 16x^3 + 64$ has factors $(x^2 - 2x + m)$, $(x^2 + ax + m)$ and $(x^2 + bx + m)$, find the values of m, a and b.

Hence,

(i) Show that $g(x)$ is non-negative for all x.

(ii) Find the roots of the equation $g(x) = 0$.

(02)(*a*) Find how many different **four-digit** numbers can be formed from the seven d[illegible] 1, 2, 4, 5, 6, 8 and 9, if any digit is selected.

(i) with repetition,

(ii) without repetition.

In case (i), find how many of the four-digit numbers do not have any digit repeated more [illegible] two times.

In case (ii), find how many of the four-digit numbers have two odd digits and two even digits. Find how many of them are even.

(b) For all $x \in \mathbb{R}$, let, in the usual notation,
$(1 + x)^n = {}^nC_0 + {}^nC_1x + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$,
where n is a positive integer.

By considering the product of $(1 + x)^{n-1}$ and $(1 + x)$ show that
${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, for $r = 1, 2, \ldots\ldots, n - 1$.

Deduce that
${}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots + (-1)^{n-1}\,{}^nC_{n-1} + (-1)^n\,{}^nC_n = 0$.

Verify the above result by an **alternative method.**

If n is an even integer, **deduce** that
${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots + {}^nC_n = 2^{n-1}$.

(03) By the **Principle of Mathematical Idnuction** prove that
$4n^3 - 6n^2 + 4n - 1 = n^4 - (n - 1)^4$ for any positive integer n.

Hence, write down u_r so that
$u_r - u_{r-1} = 4r^3 - 6r^2 + 4r - 1$ for $r = 1, 2, \ldots$

Deduce that $\sum_{r=1}^{n} r^3 \left(\dfrac{n(n+1)}{2}\right)^2$.

[You may assume that $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$].

Write down V_r, the r^{th} term of the series
$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + (1^2 + 2^2 + 3^2 + 4^2) + \ldots$

Show that $\sum_{r=1}^{n} v_r = \dfrac{n(n+1)^2(n+2)}{12}$.

Is this series convergent? Justify your answer.

Let w_r be the r^{th} term of the series

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \ldots$$

Find $f(r)$ such that $w_r = f(r) - f(r + 1)$.

Hence, find $S_n = \sum_{r=1}^{n} w_r$.

Is this series convergent? Justify your answer.

(04)(*a*) Determine the locus of the complex number z which satisfies $|z - a| = |z + a|$, where a is a non-zero real number.

(b) Let z_1 and z_2 ($\neq 0$) be two complex numbers such that $|z_1 - 2z_2| = |z_1 + 2z_2|$.

Using part (a) or otherwise, prove that $\dfrac{iz_1}{z_2} = k$, where k is real.

(i) Show that $\left|\arg(z_1) - \arg(z_2)\right| = \dfrac{\pi}{2}$.

(ii) The two points P_1 and P_2, in the Argand diagram represent the complex numbers $z_1 + 2z_2$ and $z_1 - 2z_2$ respectively.

If OP_1 is not perpendicular to OP_2, show that

$P_1\hat{O}P_2 = \tan^{-1}\left(\dfrac{4|k|}{k^2 - 4}\right)$, where O is the origin of the Argand plane.

If OP_1 is perpendicular to OP_2, determine the two possible values of k.

(05)(*a*) Evaluate $\lim\limits_{x\to 0} \dfrac{1 - \cos 4x + x\sin 3x}{x^2}$.

(b) (i) Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2} - 1}{x}\right)$ and $z = \tan^{-1} x$. Find $\dfrac{dy}{dz}$.

(ii) Let $y = e^{m\sin^{-1}x}$, Where m is a constant.

Show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - m^2y = 0$.

Find the value of $\dfrac{d^3y}{dx^3}$ at $x = 0$.

(c) A wire of given length l is cut into two portions. One portion is bent into the shape of a circle and the other portion into the shape of a square. Show that $A(x)$, the sum of the areas of the circle and the square is given by $A(x) = \dfrac{x^2}{4\pi} + \dfrac{(l - x)^2}{16}$ square units.

where x, $(0 \le x \le l)$ is the length of the portion of the wire that is bent into the form of the circle.
Hence, show that the area $A(x)$ is minimum when the side of the square is equal to the diameter of the circle.

(06)(a) Using **partial fractions**, find

$$\int \frac{2x}{(1 + x^2)(1 + x)^2}\,dx.$$

(b) Let $I = \int e^{ax}\cos bx\,dx$ and

$J = \int e^{ax}\sin bx\,dx$, where a and b are non-zero real numbers.
Show that
(i) $bI + aJ = e^{ax}\sin bx$,
(ii) $aI - bJ = e^{ax}\cos bx$.
Hence, find I and J.

(c) By using the substitution $x^3t + 1 = 0$ or otherwise, show that $\displaystyle\int_{-1}^{-\frac{1}{2}} \frac{dx}{x(x^3 - 1)} = \frac{1}{3}\ln\left(\frac{9}{2}\right)$.

(07)(*a*) Show that the equations of the bisectors of the angle between the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

(b) The equation of a straight line through a point (x_0, y_0), is given in the parametric form

$\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = t$, where $a^2 + b^2 = 1$ and t is a parameter.

Show that $|t|$ is the distance from the point (x_0, y_0) to the point (x, y) measured along the line

(c) $ABCD$ is a rhombus that entirely lies in the first quadrant. The equations of AB and AD are $x - 2y + 5 = 0$ and $2x - y + 1 = 0$ respectively. The angle BAD is acute and $AC = 2\sqrt{2}$. Using parts (a) and (b) or otherwise, find the equations of AC, and the two remaining sides of the rhombus. If E is the point of intersection of the diagonals of the rhombus, find the length of DE and hence, find the area of the rhombus.

(08) State the conditions for two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ to touch each other internally or externally.

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a circle and $P_1(x_1, y_1)$ be a point which lies outside the circle $S = 0$. Show that the length of a tangent drawn from the point P_1 to the circle $S = 0$ is given by

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

Prove that the two circles
$S_1 \equiv x^2 + y^2 + 4x - 2y - 5 = 0$ and
$S_2 \equiv x^2 + y^2 - 8x - 6y + 15 = 0$ touch each other externally.

Find the coordinates of A, the point of contact of the two circles $S_1 = 0$ and $S_2 = 0$.

Let P be a point such that the length of a tangent from the point P to the circle $S_1 = 0$ is equal to k times the length of a tangent from the point P to the circle $S_2 = 0$.

Prove that the locus of the point P,

(i) if $k = 1$, is a line through the point A perpendicular to the line joining the centres of the two circles $S_1 = 0$ and $S_2 = 0$.

(ii) If $k \neq 1$, is a circle through the point A.

Write down the equation of the locus of P when $k = \frac{1}{2}$ and show that it touches one of the two circles $S_1 = 0$ and $S_2 = 0$ externally and the other internally at the point A.

(09)(a) State and prove the ***Cosine rule*** for a triangle ABC, in the usual notation.

In the usual notation for a triangle ABC, show that

(i) $2\left(\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}\right) = \frac{a^2 + b^2 + c^2}{abc}$,

(ii) if $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$, then the angle C is $\frac{\pi}{3}$.

(b) Express $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$, where R and α are real.

Hence, find the general solution of the equation

$$\sqrt{3}\cos^2\theta + (1 - \sqrt{3})\sin\theta\cos\theta - \sin^2\theta - \cos\theta + \sin\theta = 0.$$

(c) Show that, $\cos^{-1}(-x) = \pi - \cos^{-1}x$, for $-1 \leq x \leq 1$