G.C.E. (Advanced Level) Examination - August 2013

Combined Mathematics - II

New Syllabus - Three hours

Part A

- **Answer all questions.**

(01) A particle is projected under gravity from a point O with a speed u at an angle $\frac{\pi}{3}$ to the horizontal. Let h be the vertical distance of the particle above the level of O when it has travelled a distance k horizontally.

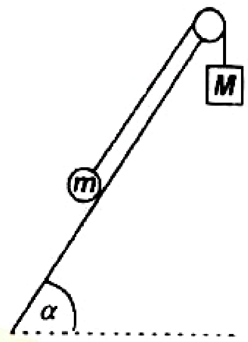
Show that $\sqrt{3}k = h + \frac{2gk^2}{u^2}$

(02) A van of width b is moving with uniform velocity u along a straight road parallel to the pavement almost touching it. A boy steps onto the road from the pavement at a distance d in front of the van and walks with uniform velocity $v(< u\sec\alpha)$ in the direction which makes an acute angle α with the direction of motion of the van. If the boy just escapes without being hit by the van,

show that $bu = (b\cos\alpha + d\sin\alpha)v$.

(03) A particle of mass m is at rest on a smooth horizontal table. Two particles, each of mass $2m$, moving on the table in opposite direction with speeds u and $2u$ towards the particle at rest, collide with it simultaneously and coalesce. Find the speed of the combined particle after collisions, and show that the loss of kinetic energy due to collisions is $\frac{23}{5}mu^2$.

(04) A particle of mass m rests on a fixed smooth plane inclined at an anlge α to the horizontal and is connected by a light inextensible string passing over a small smooth pulley at the top of the plane to a mass $M(M > m\sin\alpha)$ hanging freely. The system is released from rest with the mass M close to the pulley with the string taut and lying in a line of greatest slope of the plane, as shown in the diagram. Show that the speed v of the particle of mass m when it has moved a distance d upwards along the plane is given by $(M + m)v^2 = 2gd(M - m\sin\alpha)$.

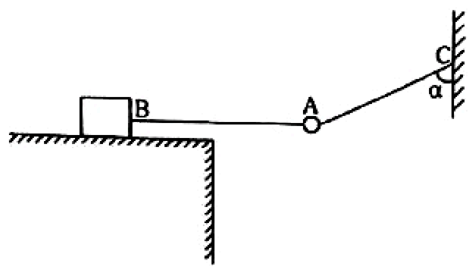


(05) In the usual notation, let $\mathbf{i}$ and $\mathbf{i} + \mathbf{j}$ be the position vectors of two points A and B respectively, with respect to a fixed origin O. Also, let C be a point on the straight line through A parallel to OB. Show that

$\overline{OC} = (1 + \lambda)\mathbf{i} + \lambda\mathbf{j}$, where λ is a real number.

Find the value of λ such that BC is perpendicular to OB.

(06) A wooden block of weight w_1 resting on a rough horizontal table is connected by a light inextensible string BC to a fixed small nail on a vertical wall as shown in the diagram. A particle of weight w_2 is tied at a point A of the string so that CA makes an angle α with the downward vertical. If the portion AB is horizontal, and the block is in limiting equilibrium, show that $\mu w_1 = w_2 \tan\alpha$, where μ is the coefficient of friction between the block and the table.



(07) Let A, B and C be three mutually exclusive and exhaustive events of a sample space Ω.

Is it possible to have the probabilities

$$P(A\cup B)=\frac{1}{2}, P(B\cup C)=\frac{1}{2} \text{ and } P(C\cup A)=\frac{2}{3}$$

simultaneously? Justify your answer.

(08) Let A and B be two events of a sample space Ω. Show that if $P(A|B) = P(A|B')$, then A and B are independent, where B' denotes the complementary event of B.

(09) The mean and the mode of the following eight observations are 4 and 6 respectively.

$$2, 3, 6, 2, 1, x, y, z$$

Here x, y and z are real numbers. Find the values of x, y and z, and calculate the standard deviation of the eight observations.

(10) A frequency table has five class intervals of equal width. The mid-point of the third class interval is 22.5. The upper class boundary of the fifth class interval is 40. The frequencies of the class intervals in order starting from the first class interval are 7, 19, 27, 15 and 2. Calculate the mode of the distribution.

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Part B

- **Answer five questions only.**

(11)(*a*) A particle is projected vertically upwards with a velocity u from a point on a fixed rigid horizontal floor. After moving under gravity it strikes the floor. The coefficient of restitution between the particle and the floor is $e(0 < e < 1)$.

(i) Sketch the velocity-time graph for the motion of the particle until the third impact.

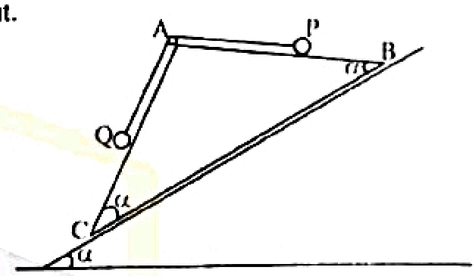
(ii) Show that the time taken by the particle until the third impact is $\frac{2u}{g}\left(1+e+e^2\right)$.

(iii) Show further that the total time taken by the particle to come to rest is $\frac{2u}{g(1-e)}$.

(*b*) A train of total mass 300 metric tons moves down a straight track of inclination $\sin^{-1}\left(\frac{1}{98}\right)$ to the horizontal, at a constant speed with its engine turned off. If the magnitude of the frictional resistance for the upward motion remains at the same constant value as for the downward motion, show that the power needed to pull the train up the same track at a constant speed 54 km h^{-1} is 900 kW.

Assuming that the engine is working at this power when the train is travelling on a straight horizontal track at a speed 18 km h^{-1}, with a resistance of same magnitude as before, find the acceleration of the train. [Take the acceleration due to gravity $g = 9.8$ m s^{-2}]

(12)(*a*) The triangle ABC is a vertical cross-section through the centre of gravity of a uniform smooth wedge of mass M. The line AC and BC are lines of greatest slope of the respective faces, and the lines BA and AC make equal angles $\alpha\left(0 < \alpha < \frac{\pi}{4}\right)$ with BC. The wedge is placed with the face containing BC on a fixed smooth plane of inclination α to the horizontal, with AB horizontal as shown in the figure. Two particles P and Q of masses m_1 and m_2 respectively, placed on AB and AC respectively are connected by a light inextensible string which passes over a small smooth pulley at the vertex A. The system is released from rest with the string taut.



Write down the equations of motion, for the particle P along BA, for the particle Q along AC and for the whole system along BC, in order to determine the acceleration of each particle relative to the wedge and the acceleration of the wedge.

Show that if $m_1 = m_2$, the acceleration of each particle relative to the wedge is zero, and the magnitude of the acceleration of the wedge is $g \sin \alpha$.

(b) A particle P of mass m is placed at the highest point of the smooth outer surface of a fixed sphere of radius a and centre O. Another particle Q of mass $2m$ moving horizontally with velocity u collides directly with P. The coefficient of restitution between P and Q is $\frac{1}{2}$. Find the velocity of P just after the collision.

Assuming that the particle P is still in contact with the sphere when the radius OP has turned through an angle θ, Show that the magnitude of the reaction on the particle P from the sphere is $\frac{m}{a}\left[ga(3\cos\theta - 2) - u^2\right]$.

Also, show that if $u = \sqrt{ga}$, then the particle P leaves the surface of the sphere just after the collision with Q.

(13)A particle of mass m is attached to one end of a light elastic string of natural length l and the other end of the string is attached to a fixed point O. When the particle hangs in equilibrium the **extension** of the string $\frac{l}{3}$. Find the modulus of elasticity of the string.

The particle is held at the point distant $\frac{l}{2}$ vertically below O, and is released from rest. Find the velocity of the particle when it first reaches the point A distant l vertically below O. Let B be the lowest point reached by the particle. Show that, for the motion of the particle from A to B, *the* **extension** x of the string satisfies the equation $\ddot{x}+\frac{3g}{l}\left(x-\frac{l}{3}\right)=0$.

Assuming that the solution of the above equation is of the form $x=\frac{l}{3}+\alpha\cos\omega t+\beta\sin\omega t$, find the values of the constants α, β and ω.

Hence, find the centre and amplitude of the simple harmonic motion performed by the particle from A to B.

Show that the particle reaches the point B after a time $\sqrt{\frac{l}{g}}\left\{1+\frac{2\pi}{3\sqrt{3}}\right\}$ from the instant of release.

Find the tension of the string when the particle is at B.

(14)(a) Let OABC be a quadrilateral, and let D and E be the mid-points of the diagonals OB and AC respectively. Also, let F be the mid-point of DE. By taking the position vectors of the points A, B and C with respect to O to be $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, show that $\overline{OF}=\frac{1}{4}(a+b+c)$.

Let P and Q be the mid-points of the sides OA and BC respectively. Show that the points P, F and Q are collinear and find the ratio $PF : FQ$.

(b) Let ABCD be a rhombus with sides of length $2l$ and BD = $2l$. Diagonals of the rhombus meet at the point O. Forces of magnitude $2P$, $6P$, $4P$, $8P$ and $6P$ newtons act along AB, BC, DC, DA and BD respectively, in the directions indicated by the order of the letters. Resolve the system of forces in the directions of $\overline{OC}$ and $\overline{OD}$, and show that the line of action of the resultant is parallel to BC.

Also, find the moment of the system about O.

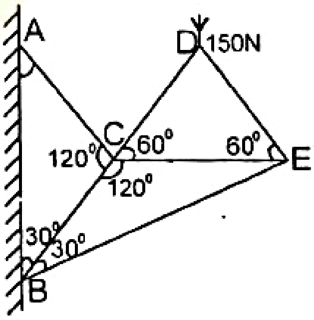
If the line of action of the resultant meets AB produced at the point E, show that $BE = 2l$.

Now, additional forces of magnitude αP, βP, γP and αP newtons are introduced to the system along EB, CE, CA and DC respectively in the directions indicated by the order of the letters. If the whole system is in equilibrium, find the values of α, β and γ.

(15)(a) Three uniform rods AB, BC and CA, each of length $2a$ and weight w are smoothly jointed at their ends to form an equilateral triangle ABC. The vertex A is smoothly hinged to a fixed point so that the triangle is free to rotate in a vertical plane. The triangle is held with AB horizontal and C below AB by a force P applied to the triangle at B perpendicular to BC in the plane of the triangle. Find the value of P.

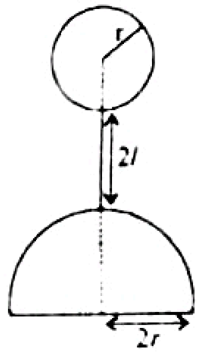
Also, find the horizontal and the vertical components of the force exerted on BC by AC at C.

(b) The adjoining figure represents a framework of six light rods smoothly jointed at the ends. It is smoothly hinged to a vertical wall at A and B, and carries a load 150N at D. Draw a stress diagram using Bow's notation and **hence**, determine the stresses in the rods, indicating whether they are tensions or thrusts.



(16)Show that the centre of mass of a uniform solid hemisphere of radius a is on its axis of symmetry, at a distance $\frac{3a}{8}$ from the centre of the base.

A composite body is made by rigidly joining a solid hemisphere and a solid sphere, made of the same uniform material, to the two ends of a uniform rod of length $2l$ and mass m in such a way that the axis of symmetry of the hemisphere, the rod and the centre of the sphere are all lying on the same line, as shown in the figure. The sphere is of radius r and mass m, and the hemisphere is of radius $2r$. Show that the centre of mass of the composite body is at a distance $\frac{1}{6}(8r + 3l)$ from the centre of the base of the hemisphere



This composite body is placed on a fixed plane inclined at an angle θ to the horizontal with the base of the hemisphere touching the plane. Assuming that the plane is rough enough to prevent slipping, show that the composite body will not topple if

$\tan\theta < \frac{12r}{8r + 3l}$

Show that if $l = \frac{4r}{3}$ and $\theta = \frac{\pi}{6}$, then the composite body will not topple and find the magnitude of the normal reaction exerted on the composite body by the inclined plane

(17)(a) According to a survey of 100 students of a school who sat for a certain examination it was revealed that 48 students have passed the examination. Also it was revealed that out of these 100 students, 50 students have participated in sports activities in the school, 30 students have participated in musical activities in the school and none of the students have participated in both sports and musical activities. Furthermore, of those who have participated in both sports and musical activities. Furthermore, of those who have participated in sports activities in the school 60% have passed the examination and of those who have not participated in sports activities or musical activities in the school 30% have passed the examination

A student is selected at random from the above 100 students

Find the probability that this student

(i) has passed the examination given that he has participated in musical activities in the school

(ii) has participated in sports activities in the school given that he has passed the examination.

(b) A frequency distribution of diameters of a set consisting of 50 small metal balls is given in the following table.

Diameter (cm)	Number of small balls
0.80 - 0.81	1
0.81 - 0.82	3
0.82 - 0.83	9
0.83 - 0.84	20
0.84 - 0.85	14
0.85 - 0.86	2
0.86 - 0.87	1

Calculate the first quartile of the distribution of diameters.

The mean and standard deviation of the diameters of this set of 50 metal balls are given to be 0.835 cm and 0.01 cm respectively. Also, for another set of 100 small metal balls, it is given that the mean of he diameters is the same as that of the first set of 50 metal balls and the standard deviation is 0.015 cm.

Find the mean and variance of the diameters of the combined set of 150 metal balls.

It is subsequently discovered that the measuring instrument used for the second set of 100 metal balls was faulty and the diameter of each ball has been underestimated by 0.015 cm. Find the true mean and true standard deviation of the diameters of these 100 metal balls