G.C.E. (Advanced Level) Examination - August 2007
10 - Combined Mathematics - I
Three hours

- Answer six questions only.

(01) (a) α and β are the roots of the equation $x^2 + bx + c = 0$. Find the quadratic equation, in terms of b and c, whose roots are α^3 and β^3.

Hence, find the quadratic equation, in terms of b and c, whose roots are $\alpha^3 + \frac{1}{\beta^3}$ and $\beta^3 + \frac{1}{\alpha^3}$.

(b) $f(x)$ is a polunomial in x of degree greater than 3. When $f(x)$ is divided by $(x-1)$ $(x-2)$ and $(x-3)$, the remainders are a, b and c respectively. By repeated application of the Remainder Theorem, show that when $f(x)$ is divided by $(x-1)(x-2)(x-3)$, the remainder can be expressed as

$\lambda(x-1)(x-2) + \mu(x-1) + \upsilon$, where λ, μ and υ are constants.

Find λ, μ and υ in terms of a, b and c.

(02) (a) A candidate sitting an examination is required to answer six questions out of twelve given under three parts A, B and C, with each part containing four questions. Find the number of different ways the candidate can select six questions if,

(i) the first question in each part is compulsory.

(ii) he cannot answer more than three questions from any part,

(iii) it is compulsory to answer at least one question from each part.

(b) State the *binomial theorem* for a positive integral index.

Let a, b and d be integers such that $a = b + d$. Show that $a^n - b^{n-1}(b + nd)$ is divisible by d^2 for positive integral n.

If U is the n^{th} term of an arithmetic progression whose first term is a and the common difference is d, prove that $a^n - (a-d)^{n-1} U$ is divisible by d^2.

Deduce that $7^{60} - 3^{64}$ is divisible by 16.

(03) (a) By using the principle of Mathematical Induction, prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{34n}{105}$ is an interger, for positive integral n.

(b) Write down u_r, the r^{th} term of the series

$$\frac{3}{1.2}\left(\frac{1}{2}\right) + \frac{4}{2.3}\left(\frac{1}{2}\right)^2 + \frac{5}{3.4}\left(\frac{1}{2}\right)^3 + \cdots\cdots$$

Find $f(r)$ such that $u_r = f(r-1) - f(r)$.

Hence, find $S_n = \sum_{r=1}^{n} u_r$.

Evaluate $\lim_{n\to\infty} S_n$

(04) (a) The complex number $z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ and $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ are represented on an Argand diagram by the points A and B respectively. Find Arg z_1 and Arg z_2.

Given that OACB is a square in the Argand diagram, where O is the origin, find the modulus and the argument of the complex number represented by C.

(b) (i) Find the least and the greates values of $|z-3|$ subject to the condition

$$\left|z - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right| \leq 2.$$

(ii) Find the least value of $|z|$, subject to the condition $\arg(z-1) = \frac{\pi}{6}$.

(05) (a) (i) Show that $\frac{d^r}{dx^r}(xe^x) = (x+r)e^x$ for any positive integer r.

(ii) If $y = x^2 e^x$, prove that $\frac{dy}{dx} = 2xe^x + y$.

Deduce that $\frac{d^r y}{dx^r} - \frac{d^{r-1}y}{dx^{r-1}} = 2(x + r - 1)e^r$.

Hence, show that $\frac{d^n y}{dx^n} = n(2x + n - 1)e^x + y$,

for any positive interger n.

(b) The tangent at the point $P(at^2, at^3)$, to the curve $ay^2 = x^3$, where a is a constant, meets the curve again at Q. Find the coordinates of Q in terms of t.

(06) (a) Using partial fractions, find $\int \frac{x^3 + 1}{x(x-1)^3}\, dx$.

(b) Find A, B and C such that
$25 \cos x + 15 = A(3 \cos x + 4 \sin x + 5) + B(-3 \sin x + 4 \cos x) + C$.

Hence, find $\int \frac{25\cos x + 15}{3\cos x + 4\sin x + 5}\, dx$.

(c) Using the method of intergration by parts, show that

$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = \frac{5}{6} \int_0^{\frac{\pi}{2}} \sin^4 x\, dx = \frac{5.3}{6.4} \int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{5\pi}{32}$.

Hence, evaluate $\int_0^{\frac{\pi}{6}} \sin^6 3x\, dx$.

(07) Let ABC be a triangle such that $A \equiv (2, 4)$ and B and C lie $y = x + 1$. A line l, drawn parallel to BC cuts AB and AC at D and E respectively such that the areas of the triangles ABC and ADE are in the ratio 9 : 4. Let G be the foot of the perpendicular from A to l and M be the mirror image of G in the line AB.

(i) Find the coordinates of G and the equation of l.

(ii) Show that AM = AG.

Hence or otherwise, prove that, as the point B moves along the line $y = x + 1$, the point M moves on a circle which has the centre at A and the radius $\frac{\sqrt{2}}{3}$.

(08) Two circles are said to intersect orthogonally when the two tangents at each point of intersection are at right angles. Find the condition for the two circles
$x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$
to intersect orthogonally.

Prove that the equation

$x^2 + y^2 + 4x + 2\lambda y - 6 = 0$ ....(*)

Where λ is a parameter, represents a system of circle passing through the points $(-2 + \sqrt{10}, 0)$ and $(-2 - \sqrt{10}, 0)$.

S = 0 is a circle belonging to the system represented by (*). Show tht there exists a unique circle S' = 0 belonging to the same system which is orthogonal to S = 0

Find S' = 0 when $S \equiv x^2 + y^2 + 4x + 4y - 6 = 0$.

Find also the general equation of the circle orthogonal to both S = 0 and S' = 0.

(09) (a) State the ***sine rule***, in the usual notation.
P is a point inside a triangle ABC such that
$\angle PAB = \angle PBC = \angle PCA = \varphi$.

Prove that the area of the triangle ABC is

$\frac{abc}{2}\left(\frac{BP}{bc} + \frac{CP}{ac} + \frac{AP}{ab}\right)\sin\varphi$, in the usual notation.

Deduce that $\frac{1}{\sin^2\varphi} = \frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C}$

(b) Show that

(i) $2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{5}{12}\right)$

(ii) $2\tan^{-1}\left(\frac{5}{12}\right) = \tan^{-1}\left(\frac{120}{119}\right)$

(iii) $\tan^{-1}\left(\frac{120}{119}\right) - \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{239}\right)$

Deduce that $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \frac{\pi}{4}$.