G.C.E. (Advanced Level) Examination - April 2005
Combined Mathematics - I
Three hours

- Answer six questions only.

(01) (a) Let $f(x) = x^2 + bx + c$ and $g(x) = x^2 + qx + r$, where $b, c, q, r \in \mathbb{R}$ and $c \neq r$

Let α, β be the roots of $g(x) = 0$

Show that $f(\alpha)\,f(\beta) = (c-r)^2 - (b-q)(cq-br)$.

Hence, or otherwise, prove that if $f(x) = 0$ and $g(x) = 0$ have a common root, then $b - q$, $c - r$ and $cq - br$ are in Geometric Progression.

If α, γ are the roots of $f(x) = 0$, show that the quadratic equation whose roots are β, γ is

$$x^2 - \frac{(c+r)(q-b)}{(c-r)}x + \frac{cr(q-b)^2}{(c-r)^2} = 0.$$

(b) The remainders when $P(x) = ax^3 + bx + c$ is divided by $x + 1$, $x - 1$ and $x - 2$ are 4, 0 and 4 respectively. Find the values of a, b, c and determine all linear factors of $p(x)$.

(02) (a) A debating team consisting of 5 persons is to be chosen from a group of 7 boys and 5 girls. In how many ways can this team be formed so that it contains.

(i) any 5 persons of the group,
(ii) at least one girl
(iii) at least one girl and one boy?

(b) Find, in terms of k, the coefficient of x^3 in the expansion of $(1 + 2x + kx^2)^5$.

If this coefficient is zero find the value of k.

For this value of k, if a_n denotes the coefficient of x^n in the expansion of $(1 + 2x + kx^2)^5$, show that

(i) $a_0 + a_2 + a_4 + a_6 + a_8 + a_{10} = -121$

(ii) $a_1 + a_3 + a_5 + a_7 + a_9 = 122$

(03) (a) Using the Principle of Mathematical Induction, prove that

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \dots\dots + \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

for every positive interger n.

Find the smallest interger n, for which

$$\frac{1}{4} - \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} < \frac{1}{100}$$

(b) Find the set of real values of x for which

$$\frac{1}{2}|x-1| > |x-4|.$$

(04) (a) Let z_1 and z_2 be any two complex numbers. Construct the point representing the complex number $z_1 + z_2$ in the Argand diagram.

Draw a diagram illustrating the case when $|z_1 + z_2| = |z_1| + |z_2|$

Explain geometrically why, in general, $|z_1 + z_2| \leq |z_1| + |Z_2|$.

If $z_1 = -12 + 5i$ and $|z_2| = 5$, find the greatest value of $|z_1 + z_2|$.

If $|z_1 + z_2|$ has its greatest value and also $\frac{\pi}{2} < \arg z_2 < \pi$, express z_2 in the form $p + iq$.

(b) The points A, B, C and D represent the respective complex numbers z_1, z_2, z_3 and z_4 in the Argand diagram.

Show that if AB and CD intersect perpendicularly then $\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$ is purely imaginary.

(05) (a) If $y = \frac{1}{2}(\sin^{-1} x)^2$, show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 1 = 0 .$$

Find $\left(\frac{d^2y}{dx^2}\right)_{x=0}$, $\left(\frac{d^3y}{dx^3}\right)_{x=0}$ and $\left(\frac{d^4y}{dx^4}\right)_{x=0}$

(b) A rectangular box is to be made having a capacity of 256 cm^3, with a square base, but without a lid. Find the dimensions of the cheapest box if the meterial for rectangular sides cost 8 times as much per square centimetre as the material for the base.

(06) (a) By using the substitution $\tan\frac{x}{2} = t$, evalute the integral $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin x}$.

(b) By using integration by parts, evaluate the intergral $\int_0^1 15x^3\sqrt{1+x^2}\ dx$.

(c) Find $\int \frac{x^2-10x+13}{(x-2)(x^2-5x+6)}\ dx$.

(07) The vertices B and C of a triangle ABC lie on the line $4x - 3y = 0$ and the x-axis respectively. The side BC passes through $\left(\frac{2}{3}, \frac{2}{3}\right)$ and has slope m.

(i) Find the coordinates of B and C in terms of m.

(ii) Show that $OB = \left|\frac{10(m-1)}{3(3m-4)}\right|$ and $OC = \left|\frac{2(m-1)}{3m}\right|$,
Where O is the origin.

(iii) If ABOC is a rhombus, find the two possible values of m, and the corresponding coordinates of A.

(08) Show that the two circles with equations $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 - 4r^2 = 0$ never touch each other externally, but touch each other internally if $g^2 + f^2 = r^2$.

Find the coordinates of the point of contact in the latter case.

Show that there are two circles, which pass through the origin and the point $(a, 0)$, where $0 < a < 1$, and touch the circle whose equation is $x^2 + y^2 - 4 = 0$.

Find the coordinates of the points of contact.

Find also the equationof the cirle having these points as ends of a diameter.

(09) (a) Show that

(i) $8\cos^4\theta - 4\cos^3\theta - 8\cos^2\theta + 3\cos\theta - 1 = \cos 4\theta - \cos 3\theta$ for every θ, and

(ii) $\cos 4\theta = \cos 3\theta$ if 7θ is an interger multiple of 2π.

Deduce that

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

(b) State the *Sine Rule* for a triagle.
Let O be a point inside a triangle ABC, Such that

$$O\hat{A}B = O\hat{B}C = O\hat{C}A = \theta$$

Applying the *Sine Rule* to the triangles OBC and OAB, prove, in the usual notation, that

$$OB = \frac{a\sin(C-\theta)}{\sin C} = \frac{c\sin\theta}{\sin B}$$ and

deduce that $\cot\theta = \cot A + \cot B + \cot C$.