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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2018 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2018 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2018

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

06.08.2018 / 0830 – 1140

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

අමතර කියවීම් කාලය - මිනිත්තු 10 යි
மேலதிக வாசிப்பு நேரம் - 10 நிமிடங்கள்
Additional Reading Time - 10 minutes

Use **additional reading time** to go through the question paper, select the questions and decide on the questions that you give priority in answering.

Index Number							

Instructions:

* *This question paper consists of two parts;* **Part A** (Questions 1 - 10) and **Part B** (Questions 11 - 17).
* **Part A:** *Answer* ***all*** *questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
* **Part B:** *Answer* ***five*** *questions only. Write your answers on the sheets provided.*
* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
* *You are permitted to remove* ***only*** **Part B** *of the question paper from the Examination Hall.*

For Examiners' Use only

(10) Combined Mathematics I

Part	Question No.	Marks
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
	15	
	16	
	17	
	Total	
	Percentage	

Paper I	
Paper II	
Total	
Final Marks	

Final Marks

In Numbers	
In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

[see page two

Part A

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{Z}^+$.

2. Sketch the graphs of $y = 3 - |x|$ and $y = |x-1|$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $|x| + |x-1| \le 3$.

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3. Sketch, in an Argand diagram, the locus of the points that represent complex numbers z satisfying $\operatorname{Arg}(z-3i) = -\frac{\pi}{3}$.

Hence or otherwise, find the minimum value of $|z-1|$ such that $\operatorname{Arg}(\bar{z}+3i) = \frac{\pi}{3}$.

4. The coefficients of x and x^4 of the binomial expansion of $\left(x^2 + \frac{3k}{x}\right)^8$ are equal. Find the value of the constant k.

5. Show that $\lim\limits_{x \to 0} \dfrac{1-\cos\left(\dfrac{\pi x}{4}\right)}{x^2(x+1)} = \dfrac{\pi^2}{32}$.

6. Show that the area of the region enclosed by the curves $y = e^{2x}$, $y = e^{3-x}$, $x = 0$, $x = 3$ and $y = 0$ is $\frac{3}{2}\left(e^2 - 1\right)$ square units.

7. A curve C is given by the parametric equations $x = \ln\left(\tan\frac{t}{2}\right)$ and $y = \sin t$ for $\frac{\pi}{2} < t < \pi$.

Show that $\frac{dy}{dx} = \cos t \sin t$.

Deduce that the gradient of the tangent line drawn to the curve C at the point corresponding to $t = \frac{2\pi}{3}$ is $-\frac{\sqrt{3}}{4}$.

8. Let l_1 be the straight line $x + y - 5 = 0$. Find the equation of the straight line l_2 passing through the point $P \equiv (3, 4)$ and perpendicular to l_1.

Let Q be the point of intersection of l_1 and l_2, and let R be the point on l_2 such that $PQ : QR = 1 : 2$. Find the coordinates of R.

9. Let $P \equiv (1, 2)$ and $Q \equiv (7, 10)$. Write down the values of the constants a and b such that the equation of the circle with points P and Q as the ends of a diameter is $S \equiv (x-1)(x-a) + (y-2)(y-b) = 0$.

Let $S' \equiv S + \lambda(4x - 3y + 2) = 0$, where $\lambda \in \mathbb{R}$. Show that the points P and Q lie on the circle $S' = 0$, and find the value of λ such that this circle passes through the point $R \equiv (1, 4)$.

10. Show that $\sec^3 x + 2\sec^2 x \tan x + \sec x \tan^2 x = \dfrac{\cos x}{(1-\sin x)^2}$ for $x \neq (2n+1)\dfrac{\pi}{2}$, where $n \in \mathbb{Z}$.

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2018 ஓகஸ்ற்
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සංයුක්ත ගණිතය I
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Part B

* Answer five questions only.

11. (a) Let $a, b \in \mathbb{R}$. Write down the discriminant of the equation $3x^2 - 2(a+b)x + ab = 0$ in terms of a and b, and **hence**, show that the roots of this equation are real.

Let α and β be these roots. Write down $\alpha + \beta$ and $\alpha\beta$ in terms of a and b.

Now, let $\beta = \alpha + 2$. Show that $a^2 - ab + b^2 = 9$ and **deduce** that $|a| \leq \sqrt{12}$, and find b in terms of a.

(b) Let $c (\neq 0)$ and d be real numbers, and let $f(x) = x^3 + 4x^2 + cx + d$. The remainder when $f(x)$ is divided by $(x + c)$ is $-c^3$. Also, $(x - c)$ is a factor of $f(x)$. Show that $c = -2$ and $d = -12$.

For these values of c and d, find the remainder when $f(x)$ is divided by $(x^2 - 4)$.

12. (a) A committee of six members has to be selected from among the members of two groups, each having three boys and two girls, such that the number of girls in the committee is at most two. Find the number of different such committees that can be formed if,

(i) even number of members from each group are to be selected for the committee,

(ii) only one girl is to be selected for the committee.

(b) Let $f(r) = \dfrac{1}{(r+1)^2}$ and $U_r = \dfrac{(r+2)}{(r+1)^2 (r+3)^2}$ for $r \in \mathbb{Z}^+$.

Show that $f(r) - f(r+2) = 4U_r$ for $r \in \mathbb{Z}^+$.

Hence, show that $\displaystyle\sum_{r=1}^{n} U_r = \frac{13}{144} - \frac{1}{4(n+2)^2} - \frac{1}{4(n+3)^2}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\displaystyle\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Let $t_n = \displaystyle\sum_{r=n}^{2n} U_r$ for $n \in \mathbb{Z}^+$.

Show that $\displaystyle\lim_{n \to \infty} t_n = 0$.

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13.(*a*) Let $\mathbf{A}=\begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & -1 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} 3 & 2a \\ -1 & 0 \\ 1 & 3a \end{pmatrix}$, where $a\in\mathbb{R}$.

Find the matrix $\mathbf{P}$ defined by $\mathbf{P}=\mathbf{AB}$ and show that $\mathbf{P}^{-1}$ **does not exist** for any value of a.

Show that if $\mathbf{P}\begin{pmatrix} 1 \\ 2 \end{pmatrix}=5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then $a=2$.

With this value for a, let $\mathbf{Q}=\mathbf{P}+\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

Write down $\mathbf{Q}^{-1}$ and find the matrix $\mathbf{R}$ such that $\mathbf{AA}^{\mathrm{T}}-\frac{1}{2}\mathbf{R}=\left(\frac{1}{5}\mathbf{Q}\right)^{-1}$.

(*b*) Let $z=x+iy$, where $x,y\in\mathbb{R}$. Define the modulus $|z|$ and the conjugate $\bar{z}$ of z.

Show that (i) $z\bar{z}=|z|^2$,

(ii) $z+\bar{z}=2\,\mathrm{Re}\,z$ and $z-\bar{z}=2i\,\mathrm{Im}\,z$.

Let $z\neq 1$ and $w=\dfrac{1+z}{1-z}$. Show that $\mathrm{Re}\,w=\dfrac{1-|z|^2}{|1-z|^2}$ and $\mathrm{Im}\,w=\dfrac{2\,\mathrm{Im}\,z}{|1-z|^2}$.

Show further that if $z=\cos\alpha+i\sin\alpha\ (0<\alpha<2\pi)$, then $w=i\cot\dfrac{\alpha}{2}$.

(*c*) In an Argand diagram, the points A and B represent the complex numbers $-3i$ and 4 respectively. The points C and D lie in the first quadrant such that $ABCD$ is a rhombus and $B\hat{A}D=\theta$, where $\theta=\sin^{-1}\left(\frac{7}{25}\right)$. Find the complex numbers represented by the points C and D.

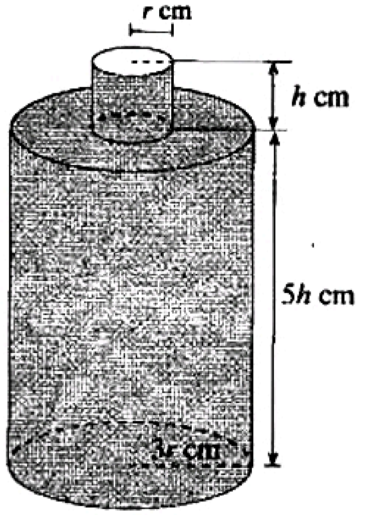
14.(*a*) Let $f(x)=\dfrac{16(x-1)}{(x+1)^2(3x-1)}$ for $x\neq -1,\frac{1}{3}$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{-32x(3x-5)}{(x+1)^3(3x-1)^2}$ for $x\neq -1,\frac{1}{3}$.

Sketch the graph of $y=f(x)$ indicating the asymptotes and the turning points.

Using the graph, find the values of $k\in\mathbb{R}$ such that the equation $k(x+1)^2(3x-1)=16(x-1)$ has exactly one root.

(*b*) A bottle with a volume of $391\pi\,\mathrm{cm}^3$ is to be made by removing a disc of radius r cm from the top face of a closed hollow right circular cylinder of radius $3r$ cm and height $5h$ cm, and fixing an open hollow right circular cylinder of radius r cm and height h cm, as shown in the figure. It is given that the total surface area $S\,\mathrm{cm}^2$ of the bottle is $S=\pi r(32h+17r)$. Find the value of r such that S is minimum.



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15.(*a*) (i) Comparing the coefficients of x^2, x^1 and x^0, find the values of the constants A, B and C such that $Ax^2(x-1)+Bx(x-1)+C(x-1)-Ax^3=1$ for all $x\in\mathbb{R}$.

Hence, write down $\dfrac{1}{x^3(x-1)}$ in partial fractions and find $\displaystyle\int\frac{1}{x^3(x-1)}\,dx$.

(ii) Using integration by parts, find $\displaystyle\int x^2\cos 2x\,dx$.

(*b*) Using the substitution $\theta=\tan^{-1}(\cos x)$, show that $\displaystyle\int_0^{\pi}\frac{\sin x}{\sqrt{1+\cos^2 x}}\,dx=2\ln\left(1+\sqrt{2}\right)$.

Using the formula $\displaystyle\int_0^{a}f(x)\,dx=\int_0^{a}f(a-x)\,dx$, where a is a constant, find $\displaystyle\int_0^{\pi}\frac{x\sin x}{\sqrt{1+\cos^2 x}}\,dx$.

16. Let $A\equiv(-2,\,-3)$ and $B\equiv(4,\,5)$. Find the equations of the lines l_1 and l_2 through the point A such that the acute angle made by each of the lines l_1 and l_2 with the line AB is $\frac{\pi}{4}$.

The points P and Q are taken on l_1 and l_2 respectively such that $APBQ$ is a square.

Find the equation of PQ, and the coordinates of P and Q.

Also, find the equation of the circle S through the points A, P, B and Q.

Let $\lambda>1$. Show that the point $R\equiv(4\lambda,5\lambda)$ lies outside the circle S.

Find the equation of the chord of contact of the tangents drawn from the point R to the circle S.

As $\lambda\,(>1)$ varies, show that these chords of contact pass through a fixed point.

17. (*a*) Solve $\cos 2\theta+\cos 3\theta=0$ for $0\le\theta\le\pi$.

Write down $\cos 2\theta$ and $\cos 3\theta$ in terms of $\cos\theta$, and show that

$$\cos 2\theta+\cos 3\theta=4t^3+2t^2-3t-1,\ \text{where } t=\cos\theta.$$

Hence, write down the three roots of the equation $4t^3+2t^2-3t-1=0$ and show that the roots of the equation $4t^2-2t-1=0$ are $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$.

Deduce that $\cos\frac{3\pi}{5}=\frac{1-\sqrt{5}}{4}$.

(*b*) Let ABC be a triangle and let D be the point on BC such that $BD:DC=m:n$, where $m,n>0$. It is given that $B\hat{A}D=\alpha$ and $D\hat{A}C=\beta$. Using the Sine Rule for the triangles BAD and DAC, show that $\dfrac{mb}{nc}=\dfrac{\sin\alpha}{\sin\beta}$, where $b=AC$ and $c=AB$.

Hence, show that $\dfrac{mb-nc}{mb+nc}=\tan\left(\dfrac{\alpha-\beta}{2}\right)\cot\left(\dfrac{\alpha+\beta}{2}\right)$.

(c) Show that $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{4}{3}\right)=\frac{\pi}{2}$.

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