G.C.E. (Advanced Level) Examination - August 2010

Combined Mathematics II

Three hours

- Answer six questions only.
- In the question paper, g denotes the acceleration due to gravity.

(01)(*a*) A particle P of mass M is projected at time $t = 0$, vertically upwards under gravity with velocity u from a point on the ground. Three particles P_1, P_2 and P_3 each of very small mass $m(<<M)$, are projected from the particle P horizontally in the same sense with velocities $2v$, $3v$ and $6v$ relative to the particle P at the times $t = \frac{u}{2g}$, $t = \frac{u}{g}$ and $t = \frac{3u}{2g}$ respectively.

Draw the velocity-time graph for the velocity of the particle P. Show that the velocity-time graphs for each of the vertical components of the velocities of the particles P_1, P_2 and P_3 coincide with portions of the velocity-time graph of the particle P and identify these portions.

In a separate diagram, draw the velocity-time graphs for each of the horizontal components of the velocities of the particles P_1, P_2 and P_3. Using the velocity-time graphs, show that

(i) the four particles reach the ground at the same time $t = \frac{2u}{g}$.

(ii) the three particles P_1, P_2 and P_3 fall on the ground at the same position.

(b) A man can swim with speed u in still water. A river of width d flows with speed $v(<u)$ relative to the ground. The man is at a point P on the bank of the river and wishes to swim to a point Q upstream on the other bank of the river and swim back to the point P. If the banks are straight and parallel to each other, and PQ makes an angle $\alpha, \left(0 < \alpha \le \frac{\pi}{2}\right)$ with upstream, drawing the velocity triangles of relative velocities on the same diagram or otherwise, show that the time taken by the man to swin to the point Q and then back to the point *P is*

$$\frac{2d\sqrt{u^2 \operatorname{cosec}^2\alpha - v^2}}{u^2 - v^2}$$

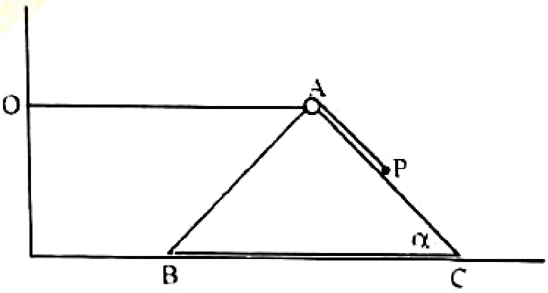
Deduce that

(i) There is no change in the total time taken if the point Q is downstream of the point *P* and *PQ* makes an angle $\alpha, \left(0 < \alpha \le \frac{\pi}{2}\right)$ with downstream.

(ii) the total time is minimum when the point Q is directly opposite to the point P on the other bank.

(02) A light inextensible string of length l attached to a point O on a verical wall passes over a smooth pulley fixed at the vertex A of the triangular vertical cross section ABC of a smooth wedge of mass M through its centre of mass, with the face through BC placed on a fixed smooth horizontal floor. A particle P of mass m is attached to the other end of the string and the string is kept taut in a vertical plane as shown in the diagram, with OA being horizontal.

If F is the magnitude of the acceleration of the wedge relative to the floor and f is the magnitude of the acceleration of the particle P relative to the wedge, show that $f = F$. If AC is inclined to the horizontal at an angle α, write down the equations of motion for the particle P along AC, and for the system horizontally.



Hence or otherwise, show that the wedge moves towards the wall with an acceleration $\frac{mg\sin\alpha}{M + 2m(1 - \cos\alpha)}$

The system is initially at rest with B at a horizontal distance d from the verticle wall. If PC is grater than d, show that B will strike the wall after a time $\sqrt{\dfrac{2d\{M+2m(1-\cos\alpha)\}}{mg\sin\alpha}}$ with speed $\sqrt{\dfrac{2dmg\sin\alpha}{M+2m(1-\cos\alpha)}}$. Show also that, the speed of the particle P relative to the floor, just before B strikes the wall, is $2\sqrt{\dfrac{dmg\sin\alpha(1-\cos\alpha)}{M+2m(1-\cos\alpha)}}$.

(03) A smooth particle P is projected with velocity u at an angle $\alpha, \left(0<\alpha<\dfrac{\pi}{2}\right)$ to the horizontal, under gravity.

The particle P, at the instance it moves horizontally, strikes another smooth particle Q of equal mass at rest hanging from one end of an inextensible string of length l, the other end of the string being attached to a point O on a horizontal rail, The rail is perpendicular to the vertical plane in which the path of the particle P and OQ lie. Show that, the horizontal distance between the two particles P and Q initially is $\dfrac{u^2\sin 2\alpha}{2g}$.

If the coefficient of the restitution between the two particles is e, show that, the particles P and Q begin to move horizontally with velocities $\dfrac{(1-e)u\cos\alpha}{2}$ and $\dfrac{(1+e)u\cos\alpha}{2}$ respectively just after the collision.

When OQ makes an angle θ with the downward vertical, write down the component of the equation of motion of the particle Q along OQ, and the equation of conservationof mechanical energy for the particle Q.

Deduce that the particle Q completes circular motion if $u\cos\alpha \geq \dfrac{2\sqrt{5gl}}{1+e}$.

Show that, the horizontal distance travelled by the particle P is $\dfrac{(3-e)u^2\sin 2\alpha}{4g}$.

Deduce that if $e = 3$, the particle P comes back to the point of projection.

(04) A particle P of mass m is attached to one end of an elastic string of natural length l with the other end of the string being attached to a fixed point O of a ceiling. If λ is the modulus of elasticity of the string, show that, when the particle P hangs in equilibrium, the extension a of the string is given by $a=\dfrac{mgl}{\lambda}$.

The string is now stretched by a further length $b(>a)$ such that OP is vertical and equal to $l+a+b$, and the particle P is released from rest. When the length of the string is $l+a+x$, where $-a\leq x\leq b$, write down the equation of motion of the particle P and show that $\ddot{x}+\dfrac{g}{a}x=0$, in the usual notation.

Assuming that the solution of the above equation is of the form $x=A\cos\sqrt{\dfrac{g}{a}}t+B\sin\sqrt{\dfrac{g}{a}}t$, find the constants A and B.

Show that, the particle P performs simple harmonic motion for a time $\sqrt{\dfrac{a}{g}}\left(\dfrac{\pi}{2}+\alpha\right)$, where $\alpha=\sin^{-1}\left(\dfrac{a}{b}\right)$ and the velocity of the particle P at the time when it leave simple harmonic motion is $\sqrt{\dfrac{g}{a}\left(b^2-a^2\right)}$ upwards.

Show also that, the particle P thereafter moves under gravity and will strike the ceiling with non-zero velocity if $b>a\sqrt{1+\dfrac{2\lambda}{mg}}$.

(05) (*a*) Forces of magnitude 2P, P, 2P, 3P, 2P and P newtons act along the sides AB, BC, CD, ED, EF and AF of a regular hexagon ABCDEF of side $2a$ metres, in the directions indicated by the order of the letters.
Prove that the system is equivalent to a resultant force of $2\sqrt{3}P$ newtons acting along AC together with a couple of magnitude $\sqrt{3}Pa$ newton metres.

If the system is equivalent to a single resultant force, find the point of intersection of the line of action of this resultant force and *FA* (produced if necessary)

Hence, find the magnitude and the direction of the single force that should be introduced to the system in order to keep the system in equlibrium.

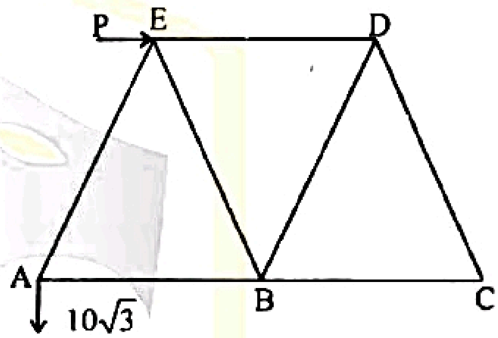
(b) Two uniform rods AB and BC of equal length and of weights W and w (W > w) respectively are freely joined at B. The rods rest in equilibrium in a verticle plane with $A\hat{B}C = \frac{\pi}{2}$ and the ends A and C on a rough horizontal ground. If μ is the coefficient of friction between the rods and the ground, show that the least possible value of μ is $\frac{W+w}{W+3w}$ in order to preserve the equilibrium.

If $\mu = \frac{W+w}{W+3w}$, prove that the slipping is about to occur at C but not at A.

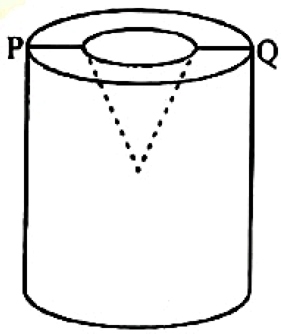
(06)(*a*) Four uniform rods AB, BC, CD and DE each of length $2a$ are smoothly joinred at B, C and D. The weight of each of the rods AB and DE is 2W and the weight of each of the rods BC and CD is W. The rods are suspended in a vertical plane from the points A and E on the same horizontal level and the system is in equilibrium with the rods AB and BC making angles α and β respectively, with the vertical.
Show that, $\tan\beta = 4\tan\alpha$.

(b) Seven light rods AB, BC, CD, DE, EA, EB and BD of equal length are smoothly jointed at their ends to form a framework as shown in the figure. The framework is smoothly hinged at C and carries a weight of $10\sqrt{3}$ newtons at A. The framework is held in a vertical plane, with AC horizontal, by a horizontal force P at E.

(i) Evaluate the magnitude of the force P at E.
(ii) Find the magnitude and the direction of the reaction at C.
(iii) Using Bow's notation, draw a stress diagram for the framework and find the stresses in all the rods, distinguishing between tensions and thrusts.



(07) Show that the centre of gravity of a uniform solid right circular cone of height h is on its axis of symmetry at a distance $\frac{1}{4}h$ from the base of the cone.



A mould for a right circular cone with base radius r and height h is produced by making a conical hollow in a uniform solid right circular cylindrical block of radius R(> r) and height H(> h). The axis of sysmmetry of the conical hollow coincides with that of the cylindrical block. The mould made is as shown in the figure. Find the distance to the centre of gravity of the mould from the diameter PQ.

If $R = 2r$ and the centre of gravity of the mould is at the vertex of the conical hollow,

deduce that $h = 2\left(4 - \sqrt{14}\right)H$

The mould with $R = 2r$ is suspended from the point P and hangs freely in equilibrium. Further, if $H = 3r$, find the inclination of PQ with the downward vertical.

(08) Let A and B be any two events. Let A' and B' be the complementary events of A and B respectively.

Prove that, $P(A \cap B') = P(A) - P(A \cap B)$.

Hence, show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

If A and B are independent events, show that

(i) A and B'

(ii) A' and B'

are independent.

Past information reveals that the regular batsman X or the regular bowler Y of the Sri Lankan team has a chance of sustaining an injury prior to an international one-day series. The probability of X sustaining such an injury is 0.2 and that for Y is 0.1. The injuries may occur independently of each other. The events N, A, B and AB are defined as follows:

N : neither X not Y sustains an injury.

A : only X sustains an injury.

B : only Y sustains an injury.

AB : X and Y sustain injuries.

Show that, P(N) = 0.72, P(A) = 0.18, P(B) = 0.08 and P(AB) = 0.02.

The conditional probabilities of Sri Lankan team winning a series, losing a series or ending a series in a draw for given event N, A, B or AB are shown in the table, where the cell (U, V) in the table represents P(V|U), the conditional probability of V given U.

(i) Drawing an appropriate tree diagram or otherwise find the probability that the Sri Lankan team will win the forthcoming series.

(ii) Given that the Sri Lankan team has lost a series, find the conditional probability that Y had an injury prior to that series.

Event (U)	Result of a series (V)		
	Win	Lose	Draw
N	0.9	0.08	0.02
A	0.5	0.4	0.1
B	0.7	0.2	0.1
AB	0.3	0.6	0.1

(09)(a) Let $\{x_1, x_2, \ldots, x_n\}$ be a set of n observations obtained from a certain study.

Define the mean and the variance of this set of data.

The amount of active ingredient in a certain tablet is supposed to lie between 52 mg and 67mg. The mean and the variance of a random sample of 40 tablets tested for the amount of active ingredient contained were 58 mg and 3.2mg^2 respectively. It was found when data were rechecked that the two values 63 mg and 55mg of two tablets had been erroneously taken as 65mg and 53mg.

Show that

(i) the mean is not affected due to this error,

(ii) the variance is reduced due to the correction.

(b) A ferry, supposed to transport passengers across the Kelani River at a certain town, is designed for a maximum tare of 1500 kg approximately. Since it is not safe to exceed this weight limit, the local authority of the area desires to conduct a survey to find out the weight distribution of the passengers who are supposed to use this ferry service. A random sample of 200 passengers from this population is taken. The weights of these 200 passengers are given in the grouped frequency distribution.

(i) Find the mean, median and mode of the weight distribution. The local authority expects to express the weight limit of the ferry in terms of the maximum number of passengers who can be transported safely at a time. Based on the above information, find the maximum number of passengers who can be transported safely at a time.

(ii) Find the standard deviation andthe coeficient of skewness of the distribution and obtain the shape of the distribution.

Class interval (weight in kg)	Frequency
0 - 10	10
10 - 20	27
20 - 30	33
30 - 40	35
40 - 50	38
50 - 60	30
60 - 70	19
70 - 80	8