G.C.E. (Advanced Level) Examination - April 2006
Combined Mathematics - I
Three hours

- Answer six questions only.

(01) (*a*) Find the condition for the quadratic equation $px^2 + qx + r = 0$ to have coincident roots, where p, q and r are real numbers.

Show that if a, b and c are real numbers, and the quadratic equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ has coincident roots, then $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$.

(b) Find the factors of the expression
$a^3(b-c) + b^3(c-a) + c^3(a-b)$

(02) (*a*) It is required to divide 12 children of different heights into two groups.
Find the number of ways in which this can be done
- (i) if one group consists of 7 children and the other group consists of 5 children,
- (ii) if each group consists of 6 children,
- (iii) if each group consists of 6 children and if the tallest and the shortest must be in the same group.

(b) State the binomial theorem for a positive integral index.

By choosing appropriate values for x and y in $3(x+y)^n$, show that 3^{2n+1} can be expressed as $7k + 3(2^n)$, where k and n are positive integers.

Hence, show that $3^{2n+1} + 2^{n+2}$ is divisible by 7 for positive integral n.

(03) (*a*) Let p be an interger. By using the Principle of Mathematical Induction, prove that $p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$ for all positive integral n

(b) Write down the r^{th} term U_r of the series

$$\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \cdots\cdots$$

(i) Show that $U_r = \frac{1}{2}\left\{f(r) - \frac{1}{1+r+r^2}\right\}$, where f(r) is a function of r which is to be determind.

(ii) Find f(r+1) and show that

$$U_r = \frac{1}{2}\{f(r) - f(r+1)\}.$$

(iii) Prove that the sum to n terms of the given series is $\frac{n(n+1)}{2(1+n+n^2)}$.

(04) (*a*) Show that

$$\frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} = \cos(\alpha-\beta) + i\sin(\alpha-\beta).$$

Let $z_1 = -1 + i$ and $z_2 = 1 + i\sqrt{3}$.

Find the real part and the imaginary part of $\frac{z_1}{z_2}$

Express each of z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \pi$.

Deduce that $\cos\frac{5\pi}{12} = \frac{1}{4}(\sqrt{6} - \sqrt{2})$.

(*b*) Let R be the region consisting of the points representing the complex number z in the Argand diagram satisfying the conditions.

$$0 \le \operatorname{Im} z \le \frac{\sqrt{3}}{2} \text{ and } |z - 2| \le 1.$$

Shade the region R and find the complex number z for which the principal argument 'Arg z' is grestest as the point representing z varies over the region R.

(05) (*a*) Let $y = (1 + 4x^2)\tan^{-1}(2x)$.
Show that

(i) $(1+4x^2)\dfrac{dy}{dx} - 8xy = 2(1+4x^2)$ and

(ii) $(1+4x^2)\dfrac{d^2y}{dx^2} - 8y = 16x$.

Find $\left(\dfrac{d^3y}{dx^3}\right)_{x=0}$.

(b) A closed right circular sylinder is to be made such that its volume is $1024\ \pi\ \text{cm}^3$. Find the radius of the cylinder that will make its total surface area a minimum.

(06) (*a*) By making an appropriate substitution, evaluate the integral.

$$\int_0^{\frac{\pi}{2}} \frac{dx}{3 + 2\cos x + \sin x}.$$

(b) By using integration by parts, find

$\int e^{4x} \sin 3x\ dx$.

(c) By using partial fractions, find $\int \dfrac{dx}{x^3+1}$.

(07) Express the coordinates of the image of the point (x_1, y_1) in the line $px + qy + r = 0$, in the form $(x_1 - p\lambda,\ y_1 - q\lambda)$, where λ is a constant to be determined.

Hence, find the image of the line $lx + my + n = 0$ in the line $px + qy + r = 0$.

The equations of the side AB and the diagonal AC of the rhombus ABCD are $3x - y + 6 = 0$ and $x - y + 8 = 0$ respectively. The vertex B has coordinates $(3, 15)$. Find the equations of the remaining three sides of the rhombus, without finding the coordinates of A, C and D *explicitly*.

(08) Obtain the equation of the chord of contact of tangents drawn to the circle $x^2 + y^2 = a^2$ from the external point (x_0, y_0).

A circle through the points $(1, 1)$ and $(-1, 0)$ intersects the circle $S = x^2 + y^2 - a^2 = 0$ at the distinct points P and Q. The tangents drawn at P and Q to the circle $S = 0$ meet at R. Show that the point R lies on the line $(2a^2 - 3)x + (a^2 - 1)y - a^2 = 0$.

(09) (*a*) (i) By solving the equation $\sin 3\theta = \cos 2\theta$

Show that $\sin 18^\circ = \dfrac{\sqrt{5} - 1}{4}$.

(ii) Show that

$\dfrac{\pi}{4} = 2\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7}$ and

$\tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{2}{11}$.

Deduce that $\dfrac{\pi}{4} = 2\tan^{-1}\dfrac{2}{11} + 3\tan^{-1}\dfrac{1}{7}$.

(b) State the *Sine Rule* and deduce the *Cosine Rule*.

With the usual notation in a triangle ABC, it is given that

$$\frac{b+c}{5} = \frac{c+a}{6} = \frac{a+b}{7}.$$

Show that

(i) $\dfrac{\sin A}{4} = \dfrac{\sin B}{3} = \dfrac{\sin C}{2}$

(ii) $\dfrac{\cos A}{-1} = \dfrac{4\cos B}{11} = \dfrac{2\cos C}{7}$