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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2016 අගෝස්තු
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2016 ஓகஸ்ற்
General Certificate of Education (Adv. Level) Examination, August 2016

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

Index Number							

Instructions:

- *This question paper consists of two parts;*
  Part A (Questions 1 - 10) and **Part B** (Questions 11 - 17).
- **Part A:**
  *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
- **Part B:**
  *Answer **five** questions only. Write your answers on the sheets provided.*
- *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
- *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

For Examiners' Use only

(10) Combined Mathematics I		
Part	Question No.	Marks
A	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	
B	11	
	12	
	13	
	14	
	15	
	16	
	17	
	Total	
	Percentage	

Paper I	
Paper II	
Total	
Final Marks	

Final Marks

In Numbers	
In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

[see page two

Part A

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r(r+1) = \frac{n}{3}(n+1)(n+2)$ for all $n \in \mathbb{Z}^{+}$.

2. Sketch the graphs of $y = |x| + 1$ and $y = 2|x-1|$ in the same diagram. **Hence or otherwise**, find all real values of x satisfying the inequality $|x| + 1 > 2|x-1|$.

[see page three

3. Sketch on the same Argand diagram, the loci of points representing complex numbers z satisfying

(i) $|z - i| = 1$, (ii) $\text{Arg}\,(z - i) = \frac{\pi}{6}$

and find the complex number represented by the point of intersection of these loci in the form $r\,(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$.

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4. How many different numbers with five digits can be made from the digits 1, 2, 3, 4 and 5, if each digit is used only once?

How many of these numbers

(i) are even numbers?

(ii) have the digits 3 and 4 next to each other?

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[see page four

5. Let $\alpha > 0$. Find the value of α such that $\lim_{x \to 0} \frac{1-\cos(\alpha x)}{\sqrt{4+x^2}-\sqrt{4-x^2}} = 16$.

6. Show that the area of the region enclosed by the curves $y = x^2$ and $y = 2x - x^2$ is $\frac{1}{3}$ square units.

[see page five

7. A curve C is given by the parametric equations $x = 3\sin^2\frac{\theta}{2}$, $y = \sin^3\theta$ for $0 < \theta < \frac{\pi}{4}$. Show that $\frac{dy}{dx} = \sin 2\theta$.

If the gradient of the tangent at a point P on C is $\frac{\sqrt{3}}{2}$, find the value of the parameter θ corresponding to P.

8. Let l be the straight line that passes through the origin and the point of intersection of the straight lines $2x + 3y - k = 0$ and $x - y + 1 = 0$, where $k\ (\neq 0)$ is a constant. Find the equation of l in terms of k.

It is given that the two points $(1, 1)$ and $(3, 4)$ are on the same side of l. Show that $k < 18$.

[see page six

9. Let $A \equiv (1, 2)$, $B \equiv (-5, 4)$ and S be the circle with AB as a diameter. Find the equations of
   (i) the circle S, and
   (ii) the circle with centre $(1, 1)$ which intersects S orthogonally.

10. Solve the equation $\cos x + \cos 2x + \cos 3x = \sin x + \sin 2x + \sin 3x$ for $0 \leq x \leq \frac{\pi}{2}$.

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10 E I

PART B

* Answer five questions only.

11. (*a*) Let $a, b, c \in \mathbb{R}$ such that $a \neq 0$ and $a + b + c \neq 0$, and let $f(x) = ax^2 + bx + c$.

Show that 1 is **not** a root of the equation $f(x) = 0$.

Let α and β be the roots of $f(x) = 0$.

Show that $(\alpha - 1)(\beta - 1) = \frac{1}{a}(a + b + c)$ and that the quadratic equation with $\frac{1}{\alpha - 1}$ and $\frac{1}{\beta - 1}$ as the roots is given by $g(x) = 0$, where $g(x) = (a + b + c)\,x^2 + (2a + b)\,x + a$.

Now, let $a > 0$ and $a + b + c > 0$.

Show that the minimum value m_1 of $f(x)$ is given by $m_1 = -\frac{\Delta}{4a}$, where $\Delta = b^2 - 4ac$.
Let m_2 be the minimum value of $g(x)$. Deduce that $(a + b + c)\,m_2 = am_1$.

Hence, show that $f(x) \geq 0$ for all $x \in \mathbb{R}$ if and only if $g(x) \geq 0$ for all $x \in \mathbb{R}$.

(*b*) Let $p(x) = x^3 + 2x^2 + 3x - 1$ and $q(x) = x^2 + 3x + 6$. Using the remainder theorem, find the remainder when $p(x)$ is divided by $(x - 1)$ and the remainder when $q(x)$ is divided by $(x - 2)$.

Verify that $p(x) = (x - 1)\,q(x) + 5$, and find the remainder when $p(x)$ is divided by $(x - 1)(x - 2)$.

12. (*a*) Let $n \in \mathbb{Z}^+$. State, in the usual notation, the binomial expansion for $(1 + x)^n$.

Show, in the usual notation, that $\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n - r}{r + 1}$ for $r = 0, 1, 2, \ldots, n - 1$.

The coefficients of x^r, x^{r+1} and x^{r+2} taken in that order, in the binomial expansion of $(1 + x)^n$ are in the ratios $1 : 2 : 3$. In this case, show that $n = 14$ and $r = 4$.

(*b*) Let $U_r = \frac{10r + 9}{(2r - 3)(2r - 1)(2r + 1)}$ and $f(r) = r(Ar + B)$ for $r \in \mathbb{Z}^+$, where A and B are real constants.

Find the values of constants A and B such that

$$U_r = \frac{f(r)}{(2r - 3)(2r - 1)} - \frac{f(r + 1)}{(2r - 1)(2r + 1)} \text{ for } r \in \mathbb{Z}^+.$$

Show that $\sum_{r=1}^{n} U_r = -3 - \frac{(n + 1)(2n + 3)}{(4n^2 - 1)}$ for $n \in \mathbb{Z}^+$.

Show further that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

[see page eight

13.(*a*) Let $\mathbf{A} = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Find real constants λ and μ such that $\mathbf{AX} = \lambda\mathbf{X}$ and $\mathbf{AY} = \mu\mathbf{Y}$.

Let $\mathbf{P} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$. Find $\mathbf{P}^{-1}$ and $\mathbf{AP}$, and show that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

(*b*) In an Argand diagram, the point A represents the complex number $2+i$. The point B is such that $OB = 2(OA)$ and $A\hat{O}B = \frac{\pi}{4}$, where O is the origin and $A\hat{O}B$ is measured counter-clockwise from OA. Find the complex number represented by the point B.

Also, find the complex number represented by the point C such that $OACB$ is a parallelogram.

(*c*) Let $z \in \mathbb{C}$ and $w = \frac{2}{1+i} + \frac{5z}{2+i}$. It is given that $\operatorname{Im} w = -1$ and $|w - 1 + i| = 5$.
Show that $z = \pm(2+i)$.

14.(*a*) Let $f(x) = \frac{(x-3)^2}{x^2-1}$ for $x \neq \pm 1$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \frac{2(x-3)(3x-1)}{(x^2-1)^2}$.

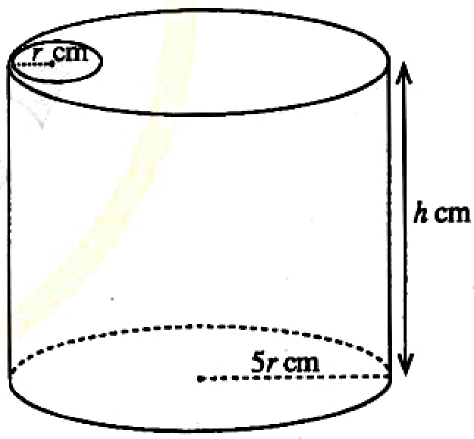
Write down the equations of the asymptotes of $y = f(x)$.

Find the coordinates of the point at which the horizontal asymptote intersects the curve $y = f(x)$.

Sketch the graph of $y = f(x)$ indicating the asymptotes and the turning points.

(*b*) A thin metal container, in the shape of a right circular cylinder of radius $5r$ cm and height h cm has a circular lid of radius $5r$ cm with a circular hole of radius r cm. (See the figure.) The volume of the container is given to be 245π cm^3. Show that the surface area S cm^2 of the container with the lid containing the hole is given by $S = 49\pi\left(r^2 + \frac{2}{r}\right)$ for $r > 0$.

Find the value of r such that S is minimum.



15.(*a*) (i) Find $\int \frac{dx}{\sqrt{3+2x-x^2}}$.

(ii) Find $\frac{d}{dx}\left(\sqrt{3+2x-x^2}\right)$ and hence, find $\int \frac{x-1}{\sqrt{3+2x-x^2}}\,dx$.

Using the above integrals, find $\int \frac{x+1}{\sqrt{3+2x-x^2}}\,dx$.

(*b*) Express $\frac{2x-1}{(x+1)(x^2+1)}$ in partial fractions and hence, find $\int \frac{(2x-1)}{(x+1)(x^2+1)}\,dx$.

(*c*) (i) Let $n \neq -1$. Using integration by parts, find $\int x^n(\ln x)\,dx$.

(ii) Evaluate $\int_1^3 \frac{\ln x}{x}\,dx$.

[see page nine

16. (*a*) The equation of the diagonal AC of a rhombus $ABCD$ is $3x - y = 3$ and $B \equiv (3, 1)$. Also, the equation of CD is $x + ky = 4$, where k is a real constant. Find the value of k and the equation of BC.

(*b*) Sketch the circles, C_1 and C_2 given by the equations $x^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 1$ respectively, indicating clearly their point of contact.

A circle C_3 touches C_1 internally and C_2 externally. Show that the centre of C_3 lies on the curve $8x^2 + 9y^2 - 8x - 16 = 0$.

17. (*a*) Write down the trigonometric identity for $\tan(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$.

Hence, obtain $\tan 2\theta$ in terms of $\tan\theta$, and show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

By substituting $\theta = \frac{5\pi}{12}$ in the last equation, verify that $\tan\frac{5\pi}{12}$ is a solution of $x^3 - 3x^2 - 3x + 1 = 0$.

Given further that $x^3 - 3x^2 - 3x + 1 = (x+1)(x^2 - 4x + 1)$, deduce that $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$.

(*b*) Show that $\tan^2\frac{A}{2} = \dfrac{1 - \cos A}{1 + \cos A}$ for $0 < A < \pi$.

In the usual notation, using the Cosine Rule for a triangle ABC, show that

$(a + b + c)(b + c - a)\tan^2\frac{A}{2} = (a + b - c)(a + c - b)$.

(*c*) Show that $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$.

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