G.C.E. (Advanced Level) Examination - August 2015
Combined Mathematics - I
Three hours

Part A

- **Answer all questions**

(01) Using the principle of Mathematical Induction, prove that $8^n - 3^n$ is an integral multiple of 5 for all $n \in \mathbb{Z}^+$

(02) Find all real values of x satisfying the inequality $|x| < 2 - x^2$

(03) Sketch the locus C of the point represented by the complex number z satis fying the equation $|z - 3 + 4i| = 2$ on an Argand diagram. Hence find the greatest and the least values of $|z + 4i|$ for z on C.

(04) Let $n \in \mathbb{Z}^+$ and $n \geq 5$ The coefficient of x^{n-10} in the binemial expansion of $\left(3x + \frac{2}{x}\right)^n$ is less than 100. Find the valve of n.

(05) Using the result $\lim\limits_{y \to a} \frac{y^n - a^n}{y - a} = na^{n-1}$ for $n \in \mathbb{Z}^+$, or otherwise, show that $\lim\limits_{x \to 0} \frac{(x + \sqrt{2})^4 - 4}{\sin 4x} = 2\sqrt{2}$

(06) Sketch the two curves $y = e^x$ and $y = e^{-x}$ on the same diagram. Show that the area of the region enclosed by the x- axis, the curve $y = e^x$ in the range $-1 < x < 0$ and the curve $y = e^{-x}$ in the range $0 < x < 1$ is $2(1 - {}^1/e)$

(07) A curve C in the xy- plane is given in terms of a real parameter θ, by the equations $x = 2 + \cos 2\theta$, $y = 4\sin\theta$. Find the derivative $\frac{dy}{dx}$ in terms of θ. and show that the equation of the normal drawn to the curve C at the point where $\theta = {}^{\pi}/4$ is $x - \sqrt{2}y + 2 = 0$.

(08) Show that the straight line joining the points A (10, 0) and B (0, 5) is the perpendicular bisecter of the line segment CD joining the points C (1, 2) and D (3, 6) show further that the area of the quadrilateral ACBD is 25 square units.

(09) Find the centre and the radius of the circle which passes through the origin O and the two points where the line $y = 1$ intersects the circle $x^2 + y^2 - 2x - 2y + 1 = 0$

(10) Let $\sin\alpha + \sin\beta = 1$ and $\cos\alpha + \cos\beta = \sqrt{3}$, where α and β are acute angles. Find the value of $\alpha + \beta$.

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Part B

- **Answer five questions only**

(11) (a) Polynomials F (x), G (x) and H(x) of degree 4 in x are given as follows:

$F(x) \equiv (x^2 - \alpha x + 1)(x^2 - \beta x + 1)$, where α and β are real constants;

$G(x) \equiv 6x^4 - 35x^3 + 62x^2 - 35x + 6,$

$H(x) \equiv x^4 + x^2 + 1.$

(i) If both F(x) = 0 and G(x) = 0 have the same roots, show that the quadratic equation with α and β as its roots is $6x^2 - 35x + 50 = 0$.

Hence find all the roots of the equation G(x) = 0.

(ii) If F$(x) \equiv$ H(x), find possible values of α and β, and show that the roots of the equation H(x) = 0 are not real.

(b) (i) Let $f(x) \equiv 2x^4 + \gamma x^3 + \delta x + 1$, where γ and δ are real constants. Given that $f\left(-\frac{1}{2}\right) = 0$ and $f(-2) = 21$, find the two real linear factors of $f(x)$.

(ii) Find the two linear expressions P(x) and Q(x) satisfying the equation $(x^2 - x + 1)$ P(x) + $(x^2 - 1)$ Q(x) = $3x$, for all real x.

(12).(a) A panel of four members is to be formed in order to serve as judges in a talent show competition. This panel is to be selected from a group consisting of three sportswomen, two sportsmen, six female singers, five male singers, two actresses and four actors. The head judge has to be a sportman or a sportswoman. Other three members of the panel have to be selected from the group excluding sportsmen and sportswomen. Find the number of different ways in which the panel can be formed, in each of the following cases.

(i) if at least one female singer and one male singer must be included in the panel,

(ii) if two males and two females, including the head judge, must be in the panel,

(iii) if the head judge must be a sportswoman.

(b) Find the values of the constants A, B and C such that $A(r+5)^2 - B(r+1)^2 = r + C$ for $r \in \mathbb{Z}^+$.

Hence show that the r^{th} term $U_r = \dfrac{8}{(r+1)^2(r+3)(r+5)^2}$

of an infinite series can be expressed as $f(r) - f(r+2)$, where $f(r)$ is a function to be determined.

Find the sum of the series $\sum_{r=1}^{n} U_r$, and **deduce that the** series $\sum_{r=1}^{\infty} U_r$ converges to the sum $\frac{1}{8^2} + \frac{1}{15^2}$

(13).(a) Three matrices **A**, **B** and **C** are given by

$\mathbf{A} = \begin{pmatrix} 0 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{pmatrix}$

(i) Show that $\mathbf{AC} = \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Also find the product **CA**.

(ii) Find the values of a, b, c and d such that $\mathbf{BC} = \mathbf{I}_2$.

(iii) If $(\lambda\mathbf{A} + \mu\mathbf{B})\mathbf{C} = \mathbf{I}_2$ obtain an equation connecting λ and μ. Express the matrix $\mathbf{D} = \begin{pmatrix} -3 & 8 & -6 \\ 2 & -5 & 4 \end{pmatrix}$ in terms of **A** and **B**, and **hence** find the product **DC**.

(b) A complex number z is given as $z = \cos\theta + i\sin\theta$, where $\theta\ (-\pi < \theta \leq \pi)$ is a real parameter. Find the locus C of the point representing z, on an Argand diagram.

Obtain expressions for $\cos\theta$ and $\sin\theta$, in terms of z and $\frac{1}{z}$.

Let $w = \dfrac{2z}{z^2 + 1}$ and $t = \dfrac{z^2 - 1}{z^2 + 1}$, where z is on C such that $z \neq \pm i$.

(i) Show that Im $(w) = 0$ and Re $(t) = 0$. **Hence**, or otherwise, show further that $w^2 + t^2 = 1$.

(ii) Find the complex numbers z satisfying the equation $w = 2$.

(iii) Find the complex numbers z satisfying the equation $t = i$.

(14).(a) Let $y = x \sin \frac{1}{x}$ for $x \neq 0$. Show that

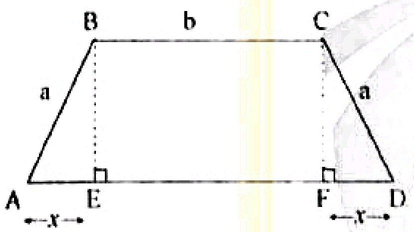
(i) $x \frac{dy}{dx} = y - \cos \frac{1}{x}$ and

(ii) $x^4 \frac{d^2y}{dx^2} + y = 0$

(b) Let $f(x) = \frac{2x^2 + 1}{(x - 1)^2}$ for $x \neq 1$.

Find the first derivative and the turning point of $f(x)$. Sketch the graph of $y = f(x)$, indicating the turning point and the asymptotes.

(c) In the given figure, ABCD is a trapezium with parallel sides BC and AD. Lengths of its sides, measured in centimetres are given by AB = CD = a, BC = b and AD = b + 2x, where $0 < x < a$. BE and CF are the perpendiculars drawn from the vertices B and C, respectively, on to the side AD.



Show that the area $S(x)$ of the trapezium ABCD is given by $S(x) = (b + x)\sqrt{a^2 - x^2}$ in square centimetres. If $a = \sqrt{6}$ and $b = 4$, show further that $S(x)$ is maximum for a certain value of x, and find this value of x and the maximum area of the trapezium.

(15).(a) Show that $\int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(\pi - x)\,dx$

Show also that $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{4}$

Hence show that $\int_0^{\pi} x \sin^2 x\,dx = \frac{\pi^2}{4}$

(b) Using a suitable substitution and the method of integration by parts, find $\int x^3 e^{x^2}\,dx$.

(c) Find the values of the constants A, B and C such that $\frac{1}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$ Hence integrate $\frac{1}{x^3 - 1}$ with respect to x.

(d) Using the substitution $t = \tan \frac{x}{2}$, show that

$$\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\cos x + 3\sin x} = \frac{1}{6}$$

(16) Let the equations of two circles be $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$. If these circles intersect orthogonally, show that $2gg' + 2ff' = c + c'$

Show that the circle C, with the equation $x^2 + y^2 - 8x - 6y + 16 = 0$ touches the x-axis.

Two circles, C_1 of radius r and C_2 of radius R (> r), with common centre at the origin O, touch the circle C at the points A and B, respectively. Find the values of r and R, and the coordinates of A and B.

Let S be a circle which intersects both the circles C and C_1 orthogonally and which touches the y-axis. Find the two possible equations for S.

The common tangent drawn to the two circles C and C_2 at the point B, meets the x-axis at P and the y-axis at Q. Show that the equation of the common tangent is $4x + 3y = 40$, and that the equation of the circle with the line segment PQ as a diameter is $3(x^2 + y^2) - 30x - 40y = 0$.

(17)(a) Show that $\cos^2(\alpha + \beta) + \cos^2\alpha + \cos^2\beta - 2\cos(\alpha + \beta)\cos\alpha\cos\beta = 1$

(b) Let $f(x) = \cos 2x + \sin 2x + 2(\cos x + \sin x) + 1$. Express $f(x)$ in the form $k(1 + \cos x)\sin(x + \alpha)$, where K and a are constants to be determined.

Let $g(x)$ be such that $\frac{f(x)}{1 + \cos x} = \sqrt{2}\{g(x) - 1\}$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Sketch the graph of $y = g(x)$ and hence show that equation $f(x) = 0$ has only one solution in the range given above.

(c) in the usual notation, using the Sine Rule for a trianlge ABC, show that

$$a(b - c)\operatorname{cosec}\frac{A}{2}\cot\frac{A}{2} = (b + c)^2\tan\left(\frac{B - C}{2}\right)\sec\left(\frac{B - C}{2}\right)$$