G.C.E. (Advanced Level) Examination - April 2003

10 - Combined Mathematics - II

Three hours

- Answer six questions only.

(01) (a) A particle, projected vertically upwards with speed U from a point O on the ground, moves under the action of gravity alone and falls back to O after a time T. Sketch a velocity-time graph for the motion of the particle.

Using the velocity-time graph only, show that

(i) the time taken by the particle for the upward motion is the same as the time taken for the downward motion and is equal to $\frac{U}{g}$.

(ii) the maximum height reached by the particle is $\frac{1}{2}\frac{U^2}{g}$.

(iii) if the particle is at the same heigh at times t_1 and t_2, then $t_1 + t_2 = T$.

(b) A ship A sailing with uniform velocity V in the direction of the North observes a steamboat B approaching it from the direction α degrees East of North. At the same instant, the ship A also observes another steamboat C approaching it from the direction α degrees West of South. Each of the boats B and C moves with uniform speed U in still water, and the boat B steers in the direction ϕ degrees West of South while the boat C steers in thd direction θ degrees East of North. If $0^\circ < \theta < \alpha < \phi < 90^\circ$, draw, in the same diagram, the velocity triangle for A and B and the velocity triangle for A and C.

Using the diagram, show that

(i) $\frac{U}{\sin\alpha} = \frac{V}{\sin(\alpha-\theta)} = \frac{V}{\sin(\phi-\alpha)}$

(ii) the velocity of B relative to C is of magnitude $2\sqrt{U^2 - V^2\sin^2\alpha}$.

(02) (a) A particle P is projected under gravity with initial speed U at an angle, $\alpha_1\left(<\frac{\pi}{2}\right)$ with the horizotal from a point O on the ground. If P is at a height $h\left(\le\frac{U^2}{2g}\sin^2\alpha_1\right)$ above the ground when it is at horizontal distances d_1 and d_2 from O, show that

$d_1 + d_2 = \frac{U^2}{g}\sin 2\alpha_1$ and $d_1 d_2 = \frac{2hU^2}{g}\cos^2\alpha_1$.

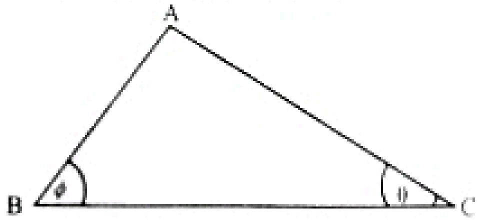
Deduce that the particle attains a maximum height after traversing a horizontal distance $OA = \frac{U^2}{2g}\sin 2\alpha_1$ from O.

Anothe particle Q is projected under gravity form O, with the same speed U in the vertical plane through OA at an angle $\alpha_2\left(<\frac{\pi}{2}\right)$ with the horizontal. If Q also clears a maximum height when it is at the same horizontal distance OA from O, find $\alpha_1 + \alpha_2$.

(b) The diagram gives a vertical cross-section ABC of a wedge of mass M with smooth faces AB and AC, inclined at angles ϕ and θ $(\sin 2\phi > \sin 2\theta)$ respectively to the horizontal.

Two particle P and Q, each of mass m, slide down along AB and AC respectively. If the wedge is fixed, find the accelerations of P and Q.

If the wedge is smooth and free to move on a fixed smooth horizontal plane, write down the equations to determine the acceleration of the wedge and the particles relative to the plane.



Show that the wedge moves with an acceleration

$$\frac{mg(\sin 2\phi - \sin 2\theta)}{2\left[M + m\left(\sin^2\theta + \sin^2\phi\right)\right]}$$

When $\theta = \phi$, show that the wedge moves with uniform velocity and hence, or otherwise, find the accelerations of P and Q.

(03) (a) Two smooth spheres P and Q of small radii and of equal mass are at a point A of a smooth horizontal circular groove of small width and of radius a. At time t = 0, the spheres P and Q are projected simultaneously with speeds U and V respectively along the groove in opposite directions. At what time t will the spheres P and Q first collide?

If, after the collision, P and Q move with speeds U_1 and V_1 respectively along the groove and if e (< 1) is the coefficient of restitution between the spheres, write down the equations to determine U_1 and V_1

If U > V ,

(i) Show that, after the collision, Q will move in the direction opposite to its earlier direction of motion.

(ii) and if the two spheres move im opposite directions after the collision, show that $e > \frac{U - V}{U + V}$.

If e satisfies the condition given in (ii), show that

P and Q will collide again when $t = \frac{2\pi a(1+e)}{e(U+V)}$.

(b) A particle P of mass m is suspended from a fixed point O by a light inelastic string of lenght a. initially P is at rest with the string taut and an impulse I is applied to P in a direction perpendicular to OP. In the ensuing motion, if v is the velocity of P and T is the tension in the string when OP makes an angle θ with the downward vertical, show that

$$v^2 = \frac{I^2}{m^2} - 2ga + 2ga\cos\theta$$

and $$T = \frac{I^2}{ma} - 2mg + 3mg\cos\theta$$

Deduce that,

(i) if the particle describes a complete circle, then $I > m\sqrt{5ag}$

and (ii) if the particle leaves circular motion when OP makes an acute angle α with the upward vertical,

then $m\sqrt{2ga} < I < m\sqrt{5ga}$ and $\cos\alpha = \frac{I^2}{3m^2ga} - \frac{2}{3}$

(04) A particle P of mass m is attached to the end A of an elastic string AB of natural length l and modulus of elasticity 4mg, while the end B is attached to a fixed point at a height, greater than $2l$, above the ground. The particle P is held at B and released from rest.

Using the principle of conservation of energy,

(i) show that the maximum length of the string is $2l$.

and (ii) find the velocity of P when the string is just stretched

Let x (> l) be the length of the string at time t. Write down an equation to determine the velocity $\dot{x}$ of P. Show that, that equation yields an euqation of the form

$\ddot{y} + \frac{4g}{l}y = 0;\ y \geq -\frac{l}{4}$ where $y = x - \frac{5l}{4}$.

Assuming a solution for y in the form $y = A\cos\omega t + B\sin\omega t$, find the constants a, b and ω.

Hence,

(iii) determine the maximum value of y and thus, obtain the maximum length of the string and

(iv) find the greatest speed of P.

(05) (a) $ABCDEF$ is a regulat hexagon of side 2 metres. Forces of magnitude 4, 3, 2, 5 and 6 newtons act along the sides AB, BC, CD, DE and EF respectively, the forces acting in the directions indicated by the order of the letters. Also, another force P of magnitude P newtons acts at F in the plane of the hexagon, in a direction making an angle θ with FC.

(i) Determine P and θ if the system of forces reduces to a couple only and find the magnitude of the couple.

(ii) If the force P acts along AF and $P = 7$, show that the system reduces to a sisngle force and find the point of intersection of its line of action with AB, produced if necessary.

(b) A thin smooth hemispherical bowl of radius a is fixed with its rim uppermost and horozontal. *A* smooth uniform rod *AB* of weight *W* and length $2l\ (> 2a)$ rests with the end A on the inner surface of the bowl and a point C of the rod in contact with the rim. Indicate the forces acting on the rod.

By taking moments about A, show that the reaction R at C is of magnitude $\frac{Wl}{2a}$.

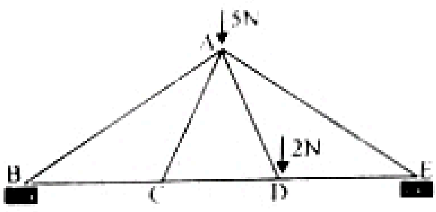
Also, obtain another relation between R and W.

Hence, show that the length CB is $\frac{1}{4}\left(7l - \sqrt{l^2 + 32a^2}\right)$.

(06) (a) Five equal unifrom rods, each of weight W are hinged freely at their ends to form a regular pentagon ABCDE.The pentagon is placed in a vertical plane with CD resting on a horizontal plane and the regular pentagonal form is maintained by means of a light rod connecting the mid points of BC and DE. Indicate the forces acting on the rods AB and BC.

Also, prove that the tension in the light rod is $\left(\cot\frac{\pi}{5} + 3\cot\frac{2\pi}{5}\right)W$.

(b) The framework shown in the diagram consists of seven light rods AB, AC, AD, AE, BC, CD and DE. All the rods except AB, AE are of equal length. The framework is in equilibrium in a vertical plane with two supports at B and E at the same horizontal level. The joints A and D carry loads of 5 newtons and 2 newtons respectively. Draw a stress diagram, using Bow's notation and determine the stresses in AB and AE, indication whether each stress is a tension or a thrust.



(07) Show that the centre of gravity G of a uniform triangular lamina ABC is at the point of intersection of its medians.

Show also that G coincides with the centre of gravity of three particles of equal mass placed at A, B and C. A uniform triangular lamina ABC, obtuse angled at C stands in a vertical plane with the side AC in contact with a horizontal table. Show that the largest weight which if suspended from vertex B will not overturn the lamina is $\frac{1}{3}W\frac{a^2+3b^2-c^2}{c^2-a^2-b^2}$, where W is the weight of the triangle and a, b, c have their usual meanings.

(08) (a) If A and B are two random events, define the independence of A and B. Give an expression for $P(A \cup B)$, in the usual notation in terms of the probabilities of the events A, B and $A \cap B$.

The random variable X takes values 0 and 1 only, with equal probability. Y is another random variable, also taking values 0 and 1 only with equal probability.

Let the two random events A and B be defined as follows:

$A : X = 0$ and $\bar{A} : X = 1$

and $B : Y = 0$ and $\bar{B} : Y = 1$

Let $U = X + Y$, show that U takes values 0, 1 and 2 and express the events $U = 0, 1, 2$ in terms of $A, \bar{A}, B, \bar{B}$.

Taking A and B to be independent,

(i) find $P(U = r)$; $r = 0, 1, 2$.

(ii) if $V = XY$, find the corresponding probabilities of V.

(b) A certain illness X has only one of the two sysmptoms A and B. It is known that, in the usual notation, $P(X \mid A) = 0.2$ and $P(X \mid B) = 0.3$

In a certain population, 40% have sysmptom A and the remaining 60% have sysmptom B. Calculate the probability that a randomly picked person has illness X.

Also, show that the probability of the sysmptom B being shown, given that a patient is suffering from the illness X is equal to $\frac{6}{7}$.

Has the presence of illness X increased or decteased the probability of the appearance of sysmpton B? Give reasons.

(09) Define the mean, median and mode of a raw data set. Consider the variance

$$S^2 = \frac{1}{N}\left\{\sum_{i=1}^{N} x_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2\right\}$$

of a raw data set $x_1, x_2, \ldots\ldots x_n$; $N \geq 2$. The deviation d_i of the i^{th} observation x_i from the mean $\bar{x}$ is defined by.

$d_i = x_i - \bar{x},\ i = 1,\ldots N$.

Show that $\frac{1}{N}\sum_{i=1}^{N} d_i^2 = S^2$.

The ages in years of five ladies, employed in a certain bank, are x_1, x_2, x_3, x_4 and x_5. Except for the youngest, each of the others is reluctant to disclose her age. However, the youngest one who is 31 years old reveals that the mean and the median of the ages of the five ladies are 35 and 36 years respectively. If the mode is not equal to the median, show that there are two sets of values of the ages satisfying the conditions given above.

If the youngest one further reveals that S^2, the variance of the ages is 5.2, determine which of those two sets of ages gives the correct ages, by calculating S^2 from the values $d_i = x_i - \bar{x},\ i = 1, 2, \ldots\ldots, 5$.

Also, calculate coefficient of skewness of the ages.

The age of retirement in their service is 55 years and let $y_i = 55 - x_i$; $i = 1, 2, \ldots.5$ be the remaining period of service in years.

Show that, in the usual notaion, $\bar{y} = 55 - \bar{x}$.

Show further that the deviation of y_i from $\bar{y}$ is equal to $-d_i$ ($i = 1, 2, \ldots, 5$).

Hence, or otherwise, show that the variance of ages and the variance of the remaining periods of service are equal.

Also, write down the value of the coefficient of skewness of the remaining periods of service.