G.C.E. (Advanced Level) Examination - April 2002
10 - Combined Mathematics - I
Three hours

- Answer six questions only.

(01) Let $f(x) = x^2 + 2x + 9$; $x \in \mathbb{R}$.

(i) If α, β are the roots of $f(x) = 0$, obtain the quadratic equation whose roots are $\alpha^2 - 1$ iy $\beta^2 - 1$.

(ii) Find the value of a real onstant k for which the equation $f(x) = k$ has exactly one real root for x.

(iii) Find the greatest value of $\frac{1}{f(x)}$, giving the value of x for which it is attained.

(iv) Determine the set of values of a real constant λ for which the equation $f(x) = \lambda x$ has no real solution for x.

(02) (a) A school debating team consisting of exactly four students is to be selected from amongst twelve eligible students. Find the number of ways in which the team can be selected.

Anura and Bhavan are among the twelve eligible students. Find the number of ways in which the team can be selected for each of the cases,

(i) both Anura and Bhavan are in the team

(ii) either Anura or Bhavan is in the team,

(iii) neither Anura nor Bhavan is in the team.

(b) Consider the expansion of $\left(\frac{7}{6x} - \frac{6x}{7}\right)^{11}$

Show that

(i) the expansion does not contain even powers of x or even powers of $\frac{1}{x}$.

(ii) the coefficient of $\frac{1}{x}$ is 2002.

(03) (a) Power by using the Principle of Mathematical Induction that $n! \geq 2^{n-1}$ for every positive integer n.

Deduce that $\sum_{k=1}^{n} \frac{1}{k!} \leq 2 - \frac{1}{2^{n-1}}$

Hence show that $e \leq 3$, where e if the base of natural logarithms.

(b) Sketch in the same diagram, the graphs of $y = |3x - a|$ and $y = |bx - 2|$ where a and b are positive numbers.

Using the graphs or otherwise, find a and b if the set of all values of x satisfying the inequality

$|3x - a| < |bx - 2|$ is $\left\{x, x > \frac{4}{3}\right\}$

(04) The complex number z is given by $z = x + iy$, $x > 0$, $y > 0$. The points A, B and C in the Argand diagram corresponded to z, 2iz and $z + 2iz$ respectively. Plot the points A, B, C and determine $A\hat{O}B$ and $\tan A\hat{O}C$.

(i) Obtain a relation between x and y if C lies on the imaginary axis.

(ii) If $y = 2x$, Show that the point representing the complex number z^2 lies on the line OC.

(iii) Shade, in a separate diagram, the region consisting of the points representing the complex number z for which $|z| \leq 4$ and $\tan^{-1}\left(\frac{1}{2}\right) \leq \arg z \leq \tan^{-1}(2)$

Find the area of the shaded region.

(05) (a) If $y = e^{4x} \sin 3x$, show that

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 25y = 0$$

Find $\left(\frac{dy}{dx}\right)_{x=0}$, $\left(\frac{d^2y}{dx^2}\right)_{x=0}$ and $\left(\frac{d^3y}{dx^3}\right)_{x=0}$

(b) From a solid sphere is cut a right circular sylinder whose axis passes throuth the centre of the sphere. Prove that the valume of the cylinder cannot exceed $\frac{1}{\sqrt{3}}$ times that of the sphere

(06) (a) By making a suitable substitution, evalute the integral $\int_1^2 \frac{x^3}{\sqrt{x^2-1}}\,dx$.

(b) By using integration by parts, evaluate the integral $\int_0^1 x \tan^{-1} x \, dx$.

(c) Find $\int_1^2 \frac{5x-4}{(1-x+x^2)(2+x)}\,dx$

(07) $u_1 \equiv a_1x + b_1y + c_1 = 0$ and $u_2 \equiv a_2x + b_2y + c_2 = 0$ are two given non-parallel straight lines. Show that for every value of λ the straight line $u_1 + \lambda u_2 = 0$ passes through a fixed point.

The equations of the perpendiculars draws through B, C to the opposite sides of a triangle ABC are $x - 4y + 5 = 0$ and $2x - y + 3 = 0$ respectively. If the coordinates of A are taken as $(k, -k)$, find the equations of the lines AB and AC, and the coordinates of B and C in terms of k.

Prove that as k varies, the centroid of the triangle ABC lies on the line $x + 5y - 4 = 0$.

(08) Find a condition that the circles

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

may touch and prove that, if they touch, the point of contact lies on each of the lines

$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$ and

$(f_1 - f_2)x - (g_1 - g_2)y + f_1g_2 - f_2g_1 = 0$

Show that the circles $x^2 + y^2 - 2x + 4y = 0$ and $x^2 + y^2 - 10x + 20 = 0$ thoch each other externally, and find the coordinates of A, the point of contact of the two circles.

P is a point such that the length of the tangent from P to the first circle is k (a constant) times that of the tangent from P to the second. Prove that, if $k^2 \neq 1$ the locus of P is a circle through A and find its equation in terms of k.

(09) ABC is a triangle such that b > c. D and E are points on BC such that AD is the median through A and the lines AD, AE trisect the angle A. Applying the Sine rule to two suitably chosen triangles prove that $\cos\frac{A}{3} = \frac{b}{2c}$

If DE : EB = 1 : k show that $\cos\frac{A}{3}$ is also equal to $\frac{(2+k)c}{2kb}$.

Deduce that $A = 90°$ if $k = 1$ and that $A = 135°$ if $k = 2$

In each case,. determine b and c in terms of a