G.C.E. (Advanced Level) Examination - August 2008
10 - Combined Mathematics - II
Three hours

- Answer six questions only.
- In this question paper, g denotes the acceletation due to gravity.

(01) (a) A train normally runs in a straight track at a uniform velocity $V\ km\ h^{-1}$. Due to a repair ahead in the track, it slows down to a velocity $U\ km\ h^{-1}$, at a uniform retardation over a distance $d\ km$. Next, the train moves with uniform velocity U over a distance $2d\ km$ of the track under repair. Then, moving with uniform acceleration over a distance $3d\ km$, it regains the velocity V.

Sketch the velocity-time graph for the motion of the train.

Show that the time lost due to the track repair, as compared with normal running of the train is $\dfrac{2d(V-U)(V+3U)}{UV(U+V)}$ hours.

(b) A helicopter, whose speed relative to wind is v km h^{-1} flies around a square track ABCD of side a km, in the sense indicated by the order of the letters. A wind blows with velocity $w(<v)$ km h^{-1} in a direction making an acute angle θ with the side AB. Assuming that no time is lost in turning round the corners of the track and drawing velocity triangles, show that the sum of the time taken from A to B and the of the track and drawing velocity triangles, show that the sum of the time taken from A to B and the time taken from C to D is $\dfrac{2a\sqrt{v^2 - w^2\sin^2\theta}}{(v^2 - w^2)}$ hours.

Hence calculate the total time T taken for one complete path, and find the value of θ which makes T a maximum.

(02) (a) A smooth wedge of mass M rests on a smooth horizontal table. A particle of mass m is placed on a face of the wedge, of inclination α to the horizontal and is projected with velocity V up a line of greatest slope of the face. Show that the magnitude of the acceleration of the wedge and the magnitude of the acceleration of the particle relative to the wedge are in a constant ratio.

Show furter that the particle returns to its starting point on the wedge after a time $\dfrac{2V(M + m\sin^2\alpha)}{(M+m)g\sin\alpha}$.

(b) Three small smooth spheres A, B, C of equal radii and of masses a, b, c respectively, are placed separately on a smooth horizontal table, in that order, so that their centres lie on a straight line.

The sphere A is projected with a velocity u along the line of centres so as to strike B which in turn strikes C. The coefficient of restitution for each pair of spheres is e. Show that C moves off with velocity

$$\frac{(1+e)^2 u}{\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)}.$$

Given further that A and B are brough to rest after the first and second impacts respectively, find the ratios $a : b : c$ and express the kinetic energy retained in the system as a fraction of the original kinetic energy.

(03) A bowl is made up from a fixed smooth spherical shell with centre O and radius a by removing the upper part cut off by the horizontal plane at a distance $\frac{a}{4}$ above O.

A particle P is projected horizontally from the lowest point inside the bowl with speed u.

(i) Find the speed of the particle and the magnitude of the reaction between the bowl and the particle, when OP makes an angle θ with the **upward vertical.**

(ii) Show that the particle will leave the edge of the bowl, provided that $u^2 > \dfrac{11ga}{4}$.

(iii) Show also that, in the subsequent free motion under gravity after the particle leaves the bowl, will not fall back into the bowl, provided that $u^2 > \dfrac{13ga}{2}$.

(04) Two particles P and Q of masses m and $3m$ respectively, hang together in equilibrium at one end of a light elastic string of natural length l, extending it to a length $l + 4a$, the other end of the string being attached to a fixed point O. The particle Q suddenly falls off. If the length of the string after a time t is $l + x$, obtain the equation $\frac{d^2x}{dt^2} + \frac{g}{a}(x - a) = 0$, for $x > 0$.

Given that $x = a + b\sin\omega t + c\cos\omega t$, where $\omega^2 = \frac{g}{a}$, is the solution of the above equation, find the values of the constants b and c.

Find the maximum height reached by the particle P above the initial position and show that the time taken to reache this height is $\sqrt{\frac{a}{8}}\left\{\pi - \alpha + 2\sqrt{2}\right\}$, where α is the acute angle $\cos^{-1}\left(\frac{1}{3}\right)$.

(05) (a) A uniform solid hemisphere of weight W is placed with its curved surface on a rough plane inclined at an angle α to the horizontal. It is in limiting equilibrium with its plane face horizontal, when a small weight w is attached to a point on the circumference of its plane face. Show that if μ is the coefficient of friction, then

$$\mu = \frac{w}{\sqrt{W(W + 2w)}} = \tan\alpha$$

(b) A smooth hollow right circular cylinder H of radius a is fixed with its axis horizontal. Two equal smooth uniform right circular cylinders A and B, each of radius $b\left(< \frac{a}{2}\right)$ and weight W are placed symmetrically inside H so that they are in equilibrium with their axes parallel to that of H. Show that the reaction between A and B is $\frac{bW}{\sqrt{a(a-2b)}}$.

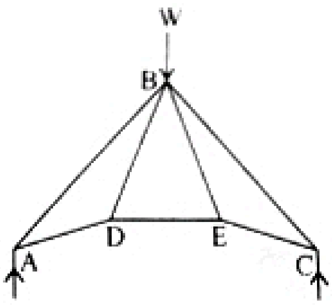
Another cylinder C equal to each of A and B is gently placed symmetrically on them, with its axis parallel to that of H. Show that equilibrium is possible with A and B in contact, only if $a < b(1 + 2\sqrt{7})$.

(06) (a) A rhombus $ABCD$ of side $2a$ consists of four smoothly jointed equal light rods, and lies on a smooth horizontal table. The rod AB is fixed. The mid-points of the rods BC and CD are joined by a light inextensible string which is kept taut by a couple of moment M applied to the rod DA, in the plane of the rhombus. If the angle $A\hat{B}C$ is 2θ, show that

(i) the reaction at the joint C is parallel to the string, and

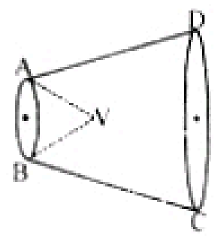
(ii) the tension in the string is $\frac{M}{a\sin\theta}$.

(b) The figure below represents a framework of freely jointed light rods with a load W at B, and it is supported vertically at A and C at the same level. The angle $A\hat{B}C$ is a right and is trisected by BD and BE. The angles $B\hat{A}D$ and $B\hat{C}E$ are each 30° and $BA = BC$.



Draw a stress diagram, using Bow's notation. Hence find the stress in each of the rods AD, AB, DE and DB, stating whether it is a tension or a thrust

(07) In the figure below, $ABCD$ represents a uniform solid body of density ρ in the form of a frustum of height h of a right circular cone. The diameters of its circular plane faces are $AB = 2\lambda a$ and $CD = 2a$, where λ is a parameter and $0 < \lambda < 1$.



Show, by intergration, that its mass is

$\frac{1}{3}\rho\pi a^2 h(1+\lambda+\lambda^2)$, and that its centre of mass G is at a distance $\frac{h}{4}\frac{(3+2\lambda+\lambda^2)}{(1+\lambda+\lambda^2)}$ from the centre of the smaller face.

Deduce the mass and the position of the centre of mass of a uniform right circular solid cone of base radius a and height h.

A solid body J is obtained from the frustum $ABCD$ by scooping out a right circular solid cone VAB of base radius λa and height $\frac{h}{2}$. Find the position of the centre of mass G_1 of J and verify that G_1 cannot coincide with V.

The body J is suspended freely from a point of the circumference of the larger face. Show that, in the position of equilibrium the axis of symmetry of J makes an acute angle β with the vertical given by

$$\tan\beta = \frac{8a}{h}\frac{(2+2\lambda+\lambda^2)}{(4+8\lambda+5\lambda^2)}.$$

(08) (a) Let A and B be two events.

Define each of the following statements.

(i) Events A and B are independents:

(ii) Events A and B mutually exclusive,

(iii) Events A and B are exhaustive.

Let the complementary events of A and B be denoted by A^I and B^I respectively.

Show that $P(A \cap B) + P(A \cap B^I) = P(A)$

Given that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B^I) = \frac{1}{2}$, find the value of $P(A^I \cap B)$ and the value of $P(A^I \cap B^I)$.

(b) Let A and B be two events with $P(B) > 0$.

State the connection between the conditional prpbability of A given B, denoted by $P(A|B)$, with $P(A \cap B)$ and $P(B)$.

A student goes to school either by bicycle or by bus. The probability that he will arrive early or on time is $\frac{19}{28}$. The probability that he will arrive late given that he cycled to school, is twice the probability that he will arrive late given that he travelled by bus. Whenever he travels to school by bus, the probability that he will arrive early or on time is $\frac{3}{4}$. Find the probability that, on a randomly chosen day

(i) he cycles to school,

(ii) he will have travelled by bus, given that he arrives late.

(09) The set of n numbers $\{x_1, x_2, \ldots, x_n\}$ with mean $\bar{x}$ and standard deviation S_x is transformed to the set of n numbers $\{y_1, y_2, \ldots, y_n\}$ by means of the formulae $y_i = ax_i + b$ for $i = 1, 2, \ldots\ldots\ldots, n$, where a and b are constants.

Let the mean and the standard deviation of the set of n numbers $\{y_1, y_2, \ldots, y_n\}$ be $\bar{y}$ and S_y respectively.

Show that

(i) $\bar{y} = a\bar{x} + b$ and

(ii) $S_y = |a| S_x$.

The table below shows the means and standard deviations of the marks in Geography and History obtained by the candidates who sat a certain examination.

	Mean	Standard deviation
Geography	m	12
History	53	s

Suppose that the marks in each subject were scaled linearly to have a mean of 50 and a standard deviation of 15. The original and the scaled marks of a particular candidate are shown below.

	Original Mark	Scaled mark
Geography	40	40
History	61	56

Find the value of m and the value of s.

The candidates were allowed to apply for rescrutiny of their answer scripts. After rescrutiny, History marks of 0.1% of the total number of candidates who sat for History were changed. The mean of the History marks of the candidates whose marks were changed increased from 65 to 68.

Find the mean of the marks, after rescrutiny, of all candidates who sat for History.