# G.C.E. (Advanced Level) Examination - August 2011

## Combined Mathematics I

## Three hours

**Instructions :**

This question paper consists of two parts;

Part A (Questions 1 - 10) and Part B (Questions 11 - 17)

## PART - A

- Answer all equstions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.

(01) Using th ePrinciple of Mathematical Induction, Prove that $n^3 + 5n$ is divisible by 3 for every $n \in \mathbb{Z}^+$.

(02) Find how many numbers between 2000 and 4000 can be formed using the digits 1, 2, 3 and 4, if repetitions of the digits are (i) not allowed, (ii) allowed.

(03) Using the binomial expansion for a positive integral index, show that $\left(1+\sqrt{3}\right)^6 + \left(1-\sqrt{3}\right)^6 = 416$.

**Hence,** find the integer part of $\left(1+\sqrt{3}\right)^6$.

(04) Show that $\lim\limits_{x\to 0} \dfrac{\sqrt{4+3\sin x}-\sqrt{4-3\sin x}}{2x} = \dfrac{3}{4}$.

(05) Find constants A and B such that $\dfrac{d}{dx}\left\{e^{2x}\left(A\sin 3x + B\cos 3x\right)\right\} = 13e^{2x}\sin 3x$.

**Hence, find** $\int e^{2x}\sin 3x\,dx$.

(06) Find the equation of the strainght line parallel to the straight line $3y + 2x + 5 = 0$, and passing through the point that divides the straight line joining the points (2, 3) and (-1, 2) externally in the ratio 3 : 2.

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(07) A curve is given by $x = 3t, y = \frac{3}{t}$, where $t$ is a non-zero parameter. Show that the equation of the tangent to the curve at the point $\left(3t, \frac{3}{t}\right)$ is $x + t^2 y = 6t$.

Deduce that, as $t$ varies, the area of the triangular region bounded by the coordinate axes and this tangent is a constant.

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(08) Find the equations of the two circles, each of radius $\sqrt{2}$, touching the straight line $x + y + 1 = 0$ and having the centres on the y-axis.

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(09) The length of the tangent from a point $P$ to the circle $x^2 + y^2 - 12x = 0$ is twice the length of the gangent from the point $P$ to the circle $x^2 + y^2 - 9 = 0$. Show that the point $P$ lies on the circle $x^2 + y^2 + 4x - 12 = 0$.

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(10) The sides of a triangle are $p - 1$, $p$ and $p + 1$, where $P$ is a real number such that $p > 1$. If the largesst angle of the triangle is twice the smallest angle of the triangle, using the Sine rule and the cosine rule find the value of $p$.

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# PART - B

- Answer five questions only. Write your answers on the sheets provided.
- At the end of the time allotted, tie the answers of the two parts together so that **Part A** is on top of Part **B** before handing them over to the supervisor.
- You are permitted to remove only **Part B** of the question paper from the Examination Hall.

(11) (a) Let $\alpha$ and $\beta$ be the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a$, $b$ and c are real numbers. Show that $\alpha$ and $\beta$ are both

(i) real, if and only if $b^2 - 4ac \geq 0$.

(ii) purely inaginary, if and only if $b = 0$ and $ac > 0$.'

Find the quadratic equation whose roots are $\alpha^2$ and $\beta^2$. Show that the roots of this quadratic equation are both real, if and only if either $\alpha$ and $\beta$ are both real or $\alpha$ and $\beta$ are both purely inaginary.

(b) Let $f(x) = x^3 - 3abx - (a^3 + b^3)$, where $a$ and $b$ are real numbers. Show that $(x-a-b)$ is a factor of $f(x)$. Find the other factor of $f(x)$ in quadratic form.

Hence or otherwise, show that if $a$ and $b$ are distinct, then $f(x) = 0$ has only one real root.

Deduce that $x^3 - 9x - 12 = 0$ has only one real root and find it.

(12) (a) Let $u_r = \dfrac{1}{(2r-1)(2r+1)(2r+3)}$ for $r \in \mathbb{Z}^+$.

Find $\dfrac{u_{r+1}}{u_r}$ in terms of r.

Hence, show that $(2r-1)u_r - (2r+1)u_{r+1} = 4u_{r+1}$ for $r = 1, 2, 3, \ldots\ldots$

Deduce that $\sum_{r=1}^{n} u_r = \dfrac{1}{12} - \dfrac{1}{4(2n+1)(2n+3)}$

Is the series $\sum_{r=1}^{\infty} u_r$ convergent? Justify your answer.

(b) Draw the graph of $y = |2x-8|$.

Hence, draw the graph of $y = -|2x-8|$.

Draw the graphs of $y = 4 - |2x-8|$, and $y = |2x-10|$ in the same figure.

Hence or otherwise, find the set of real values of $x$ satisfying the inequality $|2x-10| + |2x-8| \leq 4$.

(13) (a) Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $\lambda, \mu \in \mathbb{R}$. Find the values of $\lambda$ and $\mu$ such that $A(\lambda A + \mu I) = I$, where I is the $2 \times 2$ identity matrix.

Hence, find $A^{-1}$.

(b) Let p, Q and R be three distinct point which represent complex numbers $z_0$, $z_1$ and $z_2$ respectively in the Argand diagram.

If PQ = PR and $\theta$ is the angle measured from PQ to PR in the anti-clockwise sense, show that $z_2 - z_0 = (z_1 - z_0)(\cos\theta + i\sin\theta)$.

The points A, B, C and D, taken in the anti-clockwise sense, form a square in the Argand diagram.

Let $1 - i$ and z be the complex numbers represented by the points A and B respectively. Find the complex numbers represented by the points C and D in terms of z.

If C varies such that AC = 2, find the locus of B in the Argand diagram.

(14) (a) Let $f(x) = 2x^3 + ax^2 + bx$ for $x \in \mathbb{R}$. where $a$ and $b$ are real constants. Suppose that $f'(3) = 12$ and $f''(3) = 18$, where $f'$ and $f''$ have the usual meaning.

Find the values of $a$ and $b$.

For these values of $a$ and $b$, sketch the graph of $y = f(x)$ indicating the turning points.

Hence, find the number of solutions of the equation $2x^2 + ax + b = \dfrac{3}{x}$.

**(b)** A closed rectangular box with a square base is made of thin cardboard. The volume of the box is $8192 cm^3$. Let the length of a side of the square base be $4x cm$. A circular hole of radius $x$ $cm$ is cut out from the top square face. Show that the surface area $A cm^2$ of the box with the hole is given by $A = (32 - \pi)x^2 + \frac{8192}{x}$.

Hence, show that A is minimum when $x = \frac{16}{\sqrt[3]{32 - \pi}}$.

**(15)(a)** Using the method of **Integration by Parts,** evaluate $\int_1^e x^{\frac{3}{2}} \ln x \, dx$.

**(b)** Let $t = \tan x$

Show that $\cos 2x = \frac{1 - t^2}{1 + t^2}$, $\sin 2x = \frac{2t}{1 + t^2}$

and $\frac{dx}{dt} = \frac{1}{1 + t^2}$.

Hence, show that $\int_0^{\frac{\pi}{4}} \frac{1}{4\cos 2x + 3\sin 2x + 5} dx = \frac{1}{12}$

**(c)** Let $a$ and $b$ be distinct real numbers.

Find constants A and B such that

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \text{ for } x \in \mathbb{R} - \{a, b\}$$

By replacing $x$, $a$ and $b$ appropriately in the above equation, write down $\frac{1}{(x^2 + a^2)(x^2 + b^2)}$.

**in partial fractions and hence, find**

$$\int \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx$$

**(16) (a)** Show that the equations of the two straight lines drawn through the origin perpendicular to each other so as to form an isosceles right angled triangle with the straight line $lx + my + 1 = 0$ are $(l - m)x + (l + m)y = 0$ and $(l + m)x - (l - m)y = 0$.

**(b)** Show that, if the circle

$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ cuts the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at the ends of a diameter of the circle $S = 0$, then $2g^2 + 2f^2 - c = 2gg' + 2ff' - c'$.

A variable circle cuts each of the circles

$S_1 \equiv x^2 + y^2 - 25 = 0$ and

$S_2 \equiv x^2 + y^2 - 2x - 4y - 11 = 0$ at the ends of a diameter. Show that the centre of the variable circle lies on the straight line $x + 2y + 2 = 0$.

**(17) (a)** Using the identity $\cos^2\theta + \sin^2\theta = 1$ or otherwise, determine the real constants $a$ and $b$ such that $\cos^6\theta + \sin^6\theta = a + b\cos 4\theta$.

Hence or otherwise.

**(i)** Sketch the graph of $y = 8(\cos^6 x + \sin^6 x)$,

**(ii)** find the general solution of the equation

$$\cos^6 x + \sin^6 x = \frac{5}{4} + \frac{1}{2}\sin 4x.$$

**(b)** Slove the equation

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$