G.C.E. (Advanced Level) Examination - April 2005

Combined Mathematics - II

Three hours

- Answer six questions only
- In this question paper, g denotes the acceleration due to gravity.

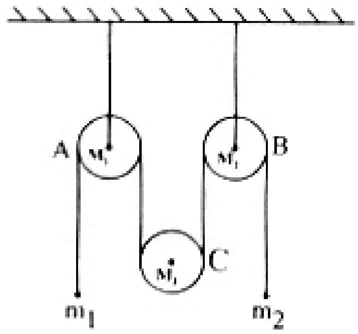
(01) (*a*) A particle A dropped from rest at a height h above the ground at time $t = 0$, falls vertically under gravity. At the same instant another partice B is projected vertically upwards from a point on the ground with velocity u. Draw the **velocity-time** graph for the motion of each particle on the same diagram.

Using the velocity-time graphs, show that the two particles are at the same height from the ground at time $\frac{h}{u}$.

(b) A ship sails with uniform velocity having components u and v estward and nothward respectively, relative to water. When the ship is at a distance d north from a submarine, a torpedo is fired from the submarine, with the intention of destroying the ship. Assuming that the torpedo moves uniformly with velocity w relative to water, show that, if the torpedo strikes the ship, then $w > u$, and find the time taken by the torpedo to move from the submarine to the ship.

(02) (*a*) Two smooth pulleys A and B of masses M_1 and M_2 respectively are fixed to a ceiling with two vertical light rods. A light inelastic string passes round A, B and a movable smooth pulley C of mass M_3, with particles of masses m_1 and m_2 attached to the ends of the string, as shown in the figure. The portions of the string not in contact with the pulleys are vercical.

Show that the tension in the string is $\dfrac{4m_1 m_2 M_3 g}{4m_1 m_2 + M_3(m_1 + m_2)}$, and find the force exerted by the system on the ceiling.



(b) A particle P of mass m is attached to a fixed point O by a light inestensible string. The particle, held with the string taut and making an angle $\alpha\left(<\frac{\pi}{2}\right)$ with the downward vertical, is given a velocity u, perpendiculat to the string, in the vertical plane through OP. Assuming that the particle is in circular motion, write down the equation of conservation of energy for the particle, by considering the general position where OP makes an angle θ with the downward vertical.

Show that the particle describes a circular arc until OP makes an angle $\cos^{-1}\left[\frac{1}{3}\left(2\cos\alpha - \frac{u^2}{ga}\right)\right]$ with the downward vertical and them begins to move freely under gravity, provided that

$$ga(3 + 2\cos\alpha) > u^2 > 2ga\cos\alpha.$$

(03) A and B are two points on the ground. A particle P of mass m is projected from the point A with velocity $u(>0)$ inclined at an angle $\alpha\left(0<\alpha<\frac{\pi}{2}\right)$ with AB, in the vertical plane through the horizontal line AB. Simultaneously, a second particle Q of mass λm is projected from the point B with velocity $v\ (>0)$ inclined at an angle $\beta\left(0<\beta<\frac{\pi}{2}\right)$ with BA, in the same vertical plane. If the two particles collide in mid air, show that $u\sin\alpha = v\sin\beta$, and that their vertical components of velocities remain equal until they collide.

If the particle P moves horizontally just before collision, deduce that the particle Q also moves horizontally at the same instant.

If further, the distance between the points A and B is $\frac{u^2 \sin 2\alpha}{g}$, and the particles coalesce after the collision,

Show that

(i) $u \cos \alpha = v \cos \beta$.

(ii) the composite particle begins to move horizontally with velocity $\left(\frac{1-\lambda}{1+\lambda}\right) u \cos\alpha$, and

(iii) the composite particle will fall to the ground at a distance $\frac{u^2 \sin 2\alpha}{(1+\lambda)g}$ from A

Show also that if the mass of the particle Q is negligible compared to that of P, the composite particle falls to the ground at B, and on the other hand if the mass of the particle P is negligible compared to that of Q, the composite paricle falls to the ground at A.

At what point does the composite particle fall to the ground, if the masses of the particles P and Q are equal? Justify your answer.

(04) (a) A lorry of mass M kg, with its engine working at power H kW, has a maximum speed u ms^{-1} on a horizontal road. With the engine working at the same power, the maximum speed of the lorry up a road of inclination α to the horozontal is v ms^{-1}. If the resistance is unchanged, find the value of H.

(b) A particle P of mass m hangs in equilibrium at one end of a light elastic string, of natural lenght l, whose other end is attached to a fixed point. In the vertical equilibrium position, the extension in the string is c. Find the modulus of elasticity of the string.

When the particle P is at rest in equilibrium, another particle Q, *of equal mass, falls from rest at a height* c *vertically above* P, impinges on P and adheres to it. Show that, at time t after the impact, the extension x in the string satisfies the equation $\ddot{x} + \omega^2(x-2c) = 0$; where $\omega^2 = \frac{g}{2c}$.

Find constants a and b *so that* $x = 2c + a \cos \omega t + b \sin \omega t$.

Hence show that the composite particle comes to instantaneous rest at time $\frac{3\pi}{4}\sqrt{\frac{2c}{g}}$ after the impact, and find the extension in the string at this instant.

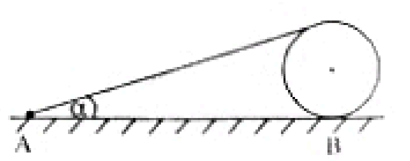
(05) (a) Forces P, Q, R, P, 2P, 3P newtons act along the sides AB, BC, CD, DE, EF, FA respectively, of a coplanar regular hexagon ABCDEF, of side $2a$ metres, in the sense indicated by the order of the letters.

(i) If the system is equivalent to a couple, find Q and R in terms of P, and calculate the moment of the couple.

(ii) If the system is equivalent to a single force along AD, find Q and R in terms of P.

(b) A particle A, of weight w, resting on a rough horizontal floor is attached to one end of a light inextensible string wound round a right circular cylinder of radius a and weight W, that rests on the floor, touching it along a generator through a point B. The other end of the string is fastened to the cylinder. The vertical plane through the string is perpendicular to the axis of the cylinder, passes through the centre of gravity of the cylinder and intersects the floor along AB, as shown in the figure. The string is just taut and makes an angle α with AB. The floor is rough enough to prevent the cylinder from moving at B. A couple of moment G is applied to the cylinder so that the particle is in limiting equilibrium. If μ is the coefficient of friction between the particle and the floor, show that the tension in the string is $\frac{\mu W}{(\cos\alpha + \mu \sin\alpha)}$.

By taking moments about B, find the value of G.

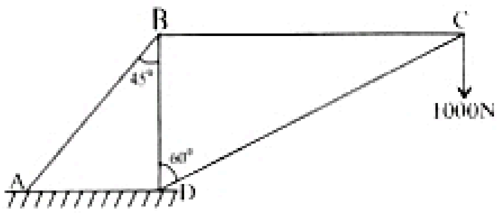


(06) (a) AB and BC ate two unifrom rods, of equal length $2a$ and of weights W and $2W$ respectively. They are smoothly hinged together at B and also hinged at A and C to a fixed horizontal beam. The rods are in equilibrium in a vertical plane with B below AC and $C\hat{A}B = \alpha$.

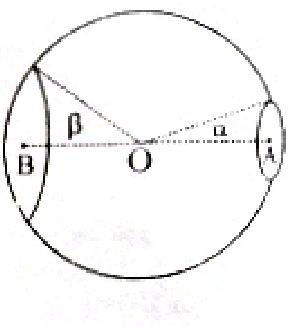
(i) Show that the horizontal component of the reaction of the hinge at B is $\frac{3}{4}W\cot\alpha$, and find the vertical component of this reaction.

(ii) If, further, the lines of action of the reactions ar A and C are perpendicular to each other, show that $\tan\alpha = \frac{3}{\sqrt{35}}$.

(b) The figure shows a crane composed of four freely jointed loght rods AB, BC, CD and BD. The rod BC is horozontal, while the rod BD is vertical. The Crane is fixed to the horizontal ground at A and D, and there is a load of 1000 N hanging at C. Use Bow's notation to find the forces in the rods, distinguishing between tensions and thrusts.



(07) Out of a uniform spherical shell, of radius a, centre O and surface density σ, a zone is cut off by two parallel planes at distances $a\cos\alpha$, $a\cos\beta$ from O (on either side of O), where $0 < \alpha < \beta < \frac{\pi}{2}$, as shown in the figure.



Show, by integration, that

(i) the mass of the zone is $2\pi a^2\sigma(\cos\alpha + \cos\beta)$.

(ii) the centre of mass of the zone lies on the axis of sysmmetry midway between its two ends A, B with the end A at a distance $a\cos\alpha$ from O.

A thin uniform circular disc of the same surface density σ and radius $a\sin\beta$ is now fastened to the larger circular edge of the zone, so that the centre of the disc is at B. Show that the composite body can rest in equilibrium with any point of the spherical surface on a horizontal floor, provided that $\sin\alpha = \sin\beta\sqrt{1-\cos\beta}$.

(08) The probability of a certain driver making a parking offence whenever he parks his vehicle in a city is p. The probability that he is fined whenever he makes a parking offence is q.

(a) In a particular day the driver parks his vehicle in the city.

(i) Write down the sample space corresponding to the above situation.

(ii) Draw the tree diagram, and hence obtain the probability of eah possible outcome.

(b) In a particular day, the driver parks his vehicle in the city twice.

(i) Draw the tree diagram corresponding to the above situation.

(ii) Find the probability that he is fined on both occations.

(iii) Find the probability that he is fined only once, given that he makes parking offences on both occasions.

(iv) Find the probability that he is fined, given that he makes a parking offence only on one occasion.

(09) (a) Define the mean and the variance of a set of n observations.

Let $\{x_1, x_2, \ldots\ldots x_n\}$ be a set of n observations with the mean $\bar{x}$ and the variance σ_x^2.

Let $\{y_1, y_2, \ldots\ldots y_m\}$ be another set of m observations with the mean $\bar{y}$ and the variance σ_y^2.

Let $\bar{z}$ and σ^2 respectively be the mean and the variance of the combined set of observations.

Show that

(i) $\bar{z} = \dfrac{n\bar{x} + m\bar{y}}{n + m}$.

(ii) $\sum_{i=1}^{n} (x_i - \bar{z})^2 = n\left(\sigma_1^2 + d_1^2\right)$, where

$d_1 = \bar{x} - \bar{z}$,

(Hint : ($x_i - \bar{z} = x_i - \bar{x} + \bar{x} - \bar{z}$)

(iii) $\sigma^2 = \dfrac{1}{n + m} \left\{ n\left(\sigma_1^2 + d_1^2\right) + m\left(\sigma_2^2 + d_2^2\right) \right\}$,

where $d_2 = \bar{y} - \bar{z}$.

(b) A group of 100 students sat for a certain test paper in mathematics. The pass mark of the test paper is 30. The distribution of the marks of the candidates who pass, is given in the following table:

Marks	Number of Students
30 - 34	5
35 - 39	10
40 - 44	15
45 - 49	30
50 - 54	5
55 - 59	5

(i) Find the mean and the variance of the distribution of marks of the candidates who pass.

(ii) The mean and the standard deviation of the marks of all 100 studnets are 38 and 12 respectively.
Find the mean and the variance of the distribution of marks of the candidates who fail.