G.C.E. (Advanced Level) Examination - August 2013

Combined Mathematics - I

New Syllabus - Three hours

Part A

- **Answer all questions.**

(01) Using the **Principle of Mathematical Induction,**

Prove that $\sum_{r=1}^{n}(2r+1) = n(n+2)$ for all $n \in \mathbb{Z}^+$.

(02) Find all real values of x satisfying the inequality

$\frac{2x+1}{3x-1} \geq 1$

(03) The sum of the first n terms of an infinite series is given by $6 - \frac{2^{n+1}}{3^{n-1}}$ for all $n \in \mathbb{Z}^+$. Find the n^{th} term of this series and show that the series is a convergent geometric series.

(04) Let $a \in \mathbb{R}$. The term independent of x in the binomial expansion of $\left(x + \frac{a}{x^3}\right)^{20}$ is $\frac{969}{2}$.

Find the value of a.

(05) Show that $\lim_{x \to 0} \frac{1 - \cos x}{\sqrt{1+x^2} - \sqrt{1-x^2}} = \frac{1}{2}$.

(06) Show that

$$\frac{d}{dx}\left\{x \ln\left(x + \sqrt{x^2+1}\right)\right\} = \frac{x}{\sqrt{x^2+1}} + \ln\left[x + \sqrt{x^2+1}\right].$$

Hence, find $\int \ln\left(x + \sqrt{x^2+1}\right) dx$

(07) The image of the point (3, 1) on the straight line $x + 2y + a = 0$ is the point $\left(\frac{3}{5}, b\right)$, where a and b are constants. Find the values of a and b.

(08) Let C be the curve given by $x = 2\cos\theta$, $y = \sin\theta$, where θ is a parameter. The normal to the curve C at the point corresponding to $\theta = \frac{\pi}{4}$ meets the curve C again at the point corresponding to $\theta = \alpha$.

Show that $2\sin\alpha - 8\cos\alpha + 3\sqrt{2} = 0$.

(09) A circle C of radius 1 and the centre lying on the line, $x + y = 0$ intersects the circle $x^2 + y^2 + 4y + 3 = 0$ orthogonally. Find the coordinates of the centre of C.

(10) Show that if $\sin\theta = -\frac{1}{3}$ and $\pi < \theta < \frac{3\pi}{2}$,

then $\sin 2\theta = \frac{4\sqrt{2}}{9}$ and $\tan 2\theta = \frac{4\sqrt{2}}{7}$

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Part B

- **Answer five questions only.**

(11)(*a*) Let $f(x) = ax^3 + bx^2 - 11x + 6$, where $a, b \in \mathbb{R}$

If $(x - 1)$ is a factor of $f(x)$ and the remainder when $f(x)$ is divided by $(x - 4)$ is -6, find the values of a and b. Also, find the other two linear factors of $f(x)$

(*b*) Let α and β be the roots of the equation $x^2 + bx + c = 0$, and γ and δ be the roots of the equation $x^2 + mx + n = 0$, where $b, c, m, n \in \mathbb{R}$.

(i) Find $(\alpha - \beta)^2$ in terms of b and c, and **hence** write down $(\gamma - \delta)^2$ in terms of m and n.

Deduce that if $\alpha + \gamma = \beta + \delta$, then $b^2 - 4c = m^2 - 4n$.

(ii) Show that

$(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta) = (c-n)^2 + (b-m)(bn-cm)$.

Deduce that the equations $x^2 + bx + c = 0$ and $x^2 + mx + n = 0$ have a common root if and only if $(c-n)^2 = (m-b)(bn-cm)$

The equation $x^2 + 10x + k = 0$ and $x^2 + kx + 10 = 0$ have a common root, where k is a real constant. Find the values of k

(12)(*a*) A student council of 15 students consists of 3 Science students, 5 Arts students and 7 Commerce students. It is required to select 6 students from this student council to work on a project. Find the number of different ways in which this can be done, if

(i) all 15 students are eligible to be selected,

(ii) two particular students are not permitted to work together,

(iii) two students from each of the subject streams need to be selected.

Also, find the number of different ways in which a group selected under (iii) above can be seated around a circular table, if the two students from the Science stream in the group are not permitted to sit next to each other.

(*b*) Let $U_r = \dfrac{3(6r+1)}{(3r-1)^2(3r+2)^2}$ for $r \in \mathbb{Z}^+$ and let $S_n = \sum_{r=1}^{n} U_r$ for $n \in \mathbb{Z}^+$. Find the values of the constants A and B such that

$$U_r = \frac{A}{(3r-1)^2} + \frac{B}{(3r+2)^2} \quad \text{for } r \in \mathbb{Z}^+.$$

Hence, show that $S_n = \dfrac{1}{4} - \dfrac{1}{(3n+2)^2}$ for $n \in \mathbb{Z}^+$.

Is the infinite series $\sum_{r=1}^{\infty} U_r$ convergent? Justify your answer.

Find the smallest value of $n \in \mathbb{Z}^+$ such that $\left|S_n - \dfrac{1}{4}\right| < 10^{-6}$.

(13)(*a*) Let $\mathbf{Q} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Find the value of $\lambda \in \mathbb{R}$ such that $\mathbf{Q}^T\mathbf{Q} = \lambda\mathbf{I}$ where $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$ and $\mathbf{I}$ is the 2×2 identity matrix.

Hence, find the inverse of the matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Let **A** be a 2×2 matrix such that $\mathbf{AP} = \mathbf{PD}$, where $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$. Find **A**.

(b) Let $z = x + iy$ be a complex number, where $x, y \in \mathbb{R}$. Define the modulus $|z|$ of z and the complex conjugate $\bar{z}$ of z.

Show that $|z|^2 = z\bar{z}$ and $z - \bar{z} = 2i\,\mathrm{Im}\,z$.

Hence, show that $|z-3i|^2 = |z|^2 - 6\,\mathrm{Im}\,z + 9$ and **that** $|1+3iz|^2 = 9|z|^2 - 6\,\mathrm{Im}\,z + 1$.

Deduce that $|z-3i| > |1+3iz|$ if and only if $|z| < 1$.
Plot the points that represent the complex numbers z satisfying the conditions $|z-3i| > |1+3iz|$ and $Arg\, z = \frac{\pi}{4}$ on an Argand diagram.

(14)(*a*) Let $f(x) = \dfrac{x^2}{x^3-1}$ for $x \neq 1$.

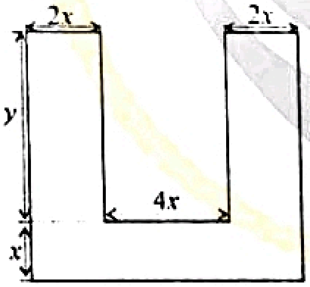
Show that $f'(x) = \dfrac{-x(x^3+2)}{(x^3-1)^2}$ for $x \neq 1$, and

deduce that the graph of $y = f(x)$ has turning points at $(0, 0)$ and $\left(-2^{1/3}, -\dfrac{4^{1/3}}{3}\right)$.

Sketch the graph of $y = f(x)$ indicating the turning points and the asymptotes.

(b) A garden whose boundary consists of eight straight line segments meeting at right angles is shown in the diagram. The dimensions in metres of the garden are indicated there. The area of the garden is given to be 800 m². Express y in terms of x and show that the perimeter P of the garden, measured in metres, is given by $P = \dfrac{800}{x} + 10x$ and this formula for the perimeter is valid only for $0 < x < 10$.

Hence, find the minimum value of the perimeter of the garden.



(15)(*a*) Using integration by parts, find $\int x^2 \sin^{-1}x \, dx$.

(b) Using partial fractions, find

$$\int \frac{x^2+3x+4}{(x^2-1)(x+1)^2}\,dx.$$

(c) Let $a, b \in \mathbb{R}$, such that $a^2 + b^2 > 1$ and

let $I = \displaystyle\int_0^{\pi/2} \frac{a+\cos x}{a^2+b^2+a\cos x+b\sin x}\,dx$ and

$$J = \int_0^{\pi/2} \frac{b+\sin x}{a^2+b^2+a\cos x+b\sin x}\,dx.$$

Show that $aI + bJ = \frac{\pi}{2}$.

By considering $bI - aJ$, find the values of I and J.

(16) Find the coordinates of the centre and the radius of the circle S whose equation is given by $x^2 + y^2 - 2x - 2y + 1 = 0$ and sketch the circle S in the xy-plane.

Let P be the point on the circle S furtherest from the origin O. Write down the coordinates of the point P and show that the equation of the tangent line l to the circle S at the point P is given by $x + y = 2 + \sqrt{2}$.

A circle S' which touches the line l also touches the circle S externally at a point distinct from P. Let (h, k) be the coordinates of the centre of the circle S'. By considering the positions of O and the centre of S' with respect to the line l, show that $h + k < 2 + \sqrt{2}$.

Show further that the coordinates of the centre of S' satisfy the equation

$$h^2 - 2hk + k^2 + 4\sqrt{2}(h+k) = 8(\sqrt{2}+1)$$

(17)(*a*) Prove the identity

$$\cos\alpha + \cos\beta - \cos\gamma - \cos(\alpha+\beta+\gamma)$$
$$= 4\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\gamma+\alpha)$$

(b) let $f(x) = 2\sin^2\frac{x}{2} + 2\sqrt{3}\sin\frac{x}{2}\cos\frac{x}{2} + 4\cos^2\frac{x}{2}$

Express $f(x)$ in the form $a\sin(x+\theta) + b$, where $a(>0)$, b and θ $\left(0 < \theta < \frac{\pi}{2}\right)$ are constants to be determined.

Deduce that $1 \leq f(x) \leq 5$.

Sketch the graph of $y = f(x)$ for $-\frac{\pi}{6} \leq x \leq \frac{11\pi}{6}$.

(c) Let $p > 2q > 0$.
The sides BC, CA and AB of a triangle ABC are of lenghts $p+q$, p and $p-q$ respectively.
Show that $\sin A - 2\sin B + \sin C = 0$ and deduce that $\cos\frac{A-C}{2} = 2\cos\frac{A+C}{2}$.