G.C.E. (Advanced Level) Examination - August 2009
10 - Combined Mathematics - II
Three hours

- Answer six questions only.
- In the question paper, g denotes the acceleration due to gravity.

(01) (*a*) A balloon is rising with constant velocity U, relative to earth. At time $t = 0$, a particle P is projected from the balloon, vertically upwards, with velocity V relative to the ballon. At time $t = t_1$ another particle Q is projected from the balloon, vertically upwards, also with velocity V relative to the balloon. The two particles P and Q meet at time $t = t_2$

Sketch the velocity-time graphs, separately, for the motion of

(i) P relative to the balloon, during the interval $0 \le t \le t_1$,

and (ii) Q relative to P, during the interval $t_1 \le t \le t_2$

Hence, or otherwise, show that $t_2 = \frac{V}{g} + \frac{1}{2}t_1$

Show further that the velocities of Q and P when the two particles meet are $U \pm \frac{1}{2}gt_1$, respectively.

(b) A submarine which travels at a speed $u\ kmh^{-1}$ sights a ship at a distance $d\ km$ in a direction 30^0 West of South, in the sea. The ship is travelling due North with velocity $v\ km\ h^{-1}$, where $u < v < 2u$

By considering the motion of the submarine relative to the ship, show that, in order to intercept the ship the submarine may preceed in one of two directions, and find the angle between these two directions.

Show further that the corresponding times differ by $\dfrac{d\sqrt{4u^2 - v^2}}{v^2 - u^2}$ hours

(02) (*a*) The cross-section of a smooth wedge of mass 2m, through its centre of mass, is a triangle ABC right-angled at C. A small smooth pulley is fixed at the vertex A, the angle $B\hat{A}C$ being 60^0. A light inextensible string passes over the pulley and has particles P and Q, of masses 3m and m respectively attached to its ends. The wedge is placed on a smooth horizontal table with the face BC in contact with the table. The particle Q is held at rest vertically below A, in contact with the vertical face AC and the particle P lying on the inclined face AB. If Q is now set free, show that the acceleration of the wedge is $\frac{\sqrt{3}g}{23}$, and find the tension in the string.

(b) A simple pendulum of length l hangs at rest with the bob at a height $2l$ above a horizontal floor. A particle of mass equal to that of the bob, strikes the bob horizontally, and subsequently reaches the floor at a point whose horizontal distance from the initial line of the string is $\frac{l}{2}$. If the string turns through an acute angle α before coming instantneously to rest, show that the coefficient of restitution between the two particles is

$$\frac{8\sin\frac{\alpha}{2} - 1}{8\sin\frac{\alpha}{2} + 1}.$$

(03) A particle of mass m is attached to one end of a light inelastic string of length l. The other end of the string is attached to a fixed point O, and the particle is in equilibrium under gravity. The particle is then projected horizontally with speed u.

(i) Show that the tension in the string when it makes an angle θ with the downward vertical through O is $m\left(3g\cos\theta - 2g + \frac{u^2}{l}\right)$.

(ii) Find the least possible value of u so that the particle can sunsequently reach the horizontal level of O.

(iii) When the string first becomes horizontal, it comes into contact with a thin horizontal bar which is fixed perpendicular to the plane of motion of the string, at a distance $\frac{l}{2}$ from O. Show that if $2gl < u^2 < \frac{7}{2}gl$ the string becomes slack before the particle reaches the highest point, at a height $\frac{l}{2}$ above the level of the bar.

(04) A point P moves on the circle $x^2 + y^2 = a^2$ with uniform speed $a\omega$. If Q is the foot of the perpendicular from P on the y-axis, show that Q executes simple harmonic motion with period $\frac{2\pi}{\omega}$.

A light spiral spring of natural length l is fixed at the lower end with its axis vertical. A particle of mass m placed at the upper end can compress the spring a distance $d(< l)$, when it is at rest. If the same particle is dropped on the upper end of the spring from a height h, show that the particle will execute a simple harmonic motion with amplitude $a = \sqrt{d^2 + 2dh}$, provided $l \geq a - d$.

In this motion, if the particle remains on the spring for **at least** an interval of time $\frac{3\pi}{2}\sqrt{\frac{d}{g}}$, find the maximum value of $\left(\frac{h}{d}\right)$.

(05) (a) A uniform circular hoop of weight W rests on a fixed rough rail which is inclined at an angle 30° to the horizontal. The hoop and the rail are in the same vertical plane. The hoop is held in equilibrium by means of a string which leaves the hoop tangentially and is inclined at 30° to the rail, this angle being measured in the same sense as the angle of inclination of the rail. Find the tension in the string, and show that the coefficient of friction between the rail and the hoop is not less than $(2 - \sqrt{3})\sqrt{2}\cos 15^\circ$

(b) $ABCDEF$ is a regular hexagon of which each side is of length a metres. Forces P, 3P, 2P and 4P newtons act along BA, EB, DE and AD respectively, in directions indicated by the order of the letters. Find the magnitude and direction of the resultant of the system.

By taking moments about one vertex of the hexagon, also find the line of action of the resultant.

What couple in the plane of the hexagon, added to the system would reduce the system to a single force along $\overrightarrow{FE}$?

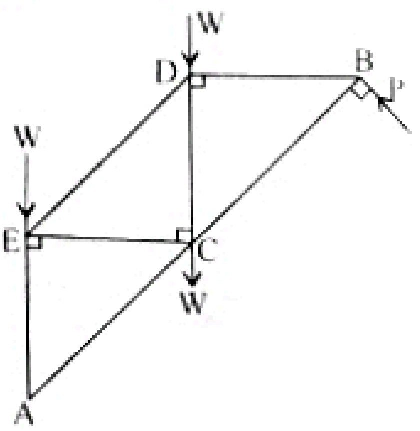
(06) (a) Two smooth uniform rods, AB, BC, each of length $2a$ and weight W, *are freely hingedat* B, and are suspended by two light inextensible strings AO, CO each of length $2a$, tied to a fixed point O. A uniform sphere of weight W and radius $\frac{a}{3}$ rests in contact with the rods and is supported by them. Show that, in the position of equilibrium, each rod makes with the vertical an angle θ given by $\cot^3\theta + \cot\theta - 30 = 0$.

Find the only possible value of cot θ, and hence show that the reaction at the hinge B is W.

(b) In the framwork of light rods shown in the figure horizontal and vertical rods are equal in length and all angles are 90° or 45°. It is in a vertical plane, smoothly pivoted at A and supported at B by a force P perpendicular to AB and is loaded with weights W newtons at C, D, E. Find the value of P in terms of W.

Given further that the stress in the rod CD is zero, draw a stress diagram, using Bow's notation in order to find the stresses in the rods BD, BC and DE.

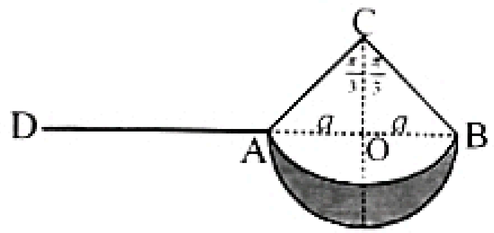
Find these stresses and state whether these stresses are tensions or thrusts.



(07) Show that the centre of mass of a uniform circular arc of a circle of radius r subtending an angle 2α at the centre, is at a distance $\frac{r\sin\alpha}{\alpha}$ from the centre.

Hence, show that the centre of mass of a uniform circular sector of a circle of radius a, subtending an angle 2α at the centre, is at a distance $\frac{2a \sin \alpha}{3\alpha}$ from the centre.

A crescent shaped uniform lamina is bounded by a semicircle with centre O and radius a and a circular arc subtending an angle $\frac{2\pi}{3}$ at its centre C as shown in the figure. Show that the centre of mass of this lamina is at a distance ka from C, where $k = \frac{3\sqrt{3}\pi}{\pi + 6\sqrt{3}}$



Let M be the mass of the lamina. The end A of a thin uniform straight rod AD of length $2a$ and mass m is rigidly fixed to the crescent at A along the extended line BA, forming a sickle as shown in the figure. The sickle is then placed on a horizontal floor with the plane of the lamina vertical and the semicircle and the free end D of the rod touching the floor. If it stays in equilibrium in this position, show tha $M\left(\sqrt{3}k - 1\right) < 4\sqrt{6}m$.

(08) Let A and B be two events with $P(A) > 0$.

Define $P(B|A)$, the conditional probability of B given A.

For three events A, B and C show that $P(A \cap B \cap C) = P(A)\,P(B|A)\,P(C|(A \cap B))$ provided $P(A) > 0$ and $P(A \cap B) > 0$.

Let $\{B_1, B_2, B_3\}$ be a partition of a sample space Ω and let A be any event of Ω .

Show that

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

for $i = 1, 2, 3$.

Vehicles approaching a crossroad must go in one of the three directions: left, right or straight on. Observations of traffic engineers reveal that of vehicles approaching from the West : 50% turn left, 20% turn right and the rest go straight on. Assuming that the driver of each vehicle chooses direction independently, find the probability that of the next three vehicles, approaching the crossroad from the West,

(i) all go straight on,
(ii) all go in the same direction,
(iii) two turn right and one turns left,
(iv) all go in different directions.

If the three consecutive vehicles, all go in the same direction, show that most of the time they all turn left.

(09) (a) Let the values of a random sample of size n taken from a population be $x_1, x_2, \ldots, x_n$.

Prove that $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$; where $\bar{x}$ is the sample mean.

The number of printing errors, x on each of first 200 pages of a book containing 250 pages was observed and the following details were found.

The total number of printing errors 920;
the sum of squares of printing errors 5032.

Find the mean and the standard deviation of the number of printing errors per page.

The mean and the standard deviation of the number of printing errors per page in the last 50 pages were found later as 4.4 and 2.2 respectively. Find, using first principles and correct to two decimal places, the mean and the standard deviation of the number of printing errors per page, in the book.

(b) The mean of the marks for pure Mathematics obtained by a group of students in an examintaion is 45. These marks are scaled linearly to give a mean of 50 and a standard deviation of 15. It is also given that the scaled mark of 80 correponds to an original mark of 70.

Calculate

(i) the linear scale,
(ii) the standard deviation of the original marks.
(iii) the mark which is not changed by the scaling

Given that the least and the greatest scaled marks are 2 and 92 respectively, find the corresponding original marks.