G.C.E. (Advanced Level) Examination - April 2006
Combined Mathematics - II
Three hours

- Answer six questions only.
- In this question paper, g denotes the acceleration due to gravity.

(01) (a) A man on a straight road sees a bus starting to move with constant acceleraion, from rest at a bus stop at a certain distance ahead of him. Immediately, he runs after the bus with constant velocity U ms^{-1} and just catches it in T seconds.

Draw the velocity-time graphs for the man and the bus on the same diagram.

Find, in terms of U and T, the acceleration of the bus and the initial distance of the man from the bus stop.

(b) Water flows in a river, with constant velocity U ms^{-1}, between two straight parallel banks which are d metres apart. A boat moving with speed $2U$ ms^{-1} relative to water, is required to take a straight course from a point A on one bank to a point B on the other bank and back to A. $\overline{AB}$ makes a certain acute angle α with the upstream direction of the river and the time from A to B is twice that from B to A.

Draw velocity-triangles for the journey from A to B and the return journey, and show that

(i) $\sin\alpha = \sqrt{\dfrac{5}{8}}$,

(ii) the velocity of the boat in its journey from A to B, relative to the banks, is of magnitude $U\sqrt{\dfrac{3}{2}}$.

Deduce the total time taken by the boat to complete the two journeys.

(02) (a) A smooth wedge of mass M, rests on a smooth horizontal table. Initially, a particle of mass m is gently placed on its face inclined at an angle α to the horizontal. Show, using the principle of conservation of momentum, or otherwise, that when the particle has acquired a celocity v relative to the wedge, the velocity of the wedge is $\dfrac{mv\cos\alpha}{M+m}$.

If, at this instant, the particle impinges on an inelastic obstacle fixed to the wedge and comes to rest relative to the wedge, find the velocity of the wedge and the impulse on the table.

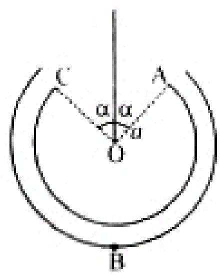
(b) A light inextensible string passes over a smooth fixed pulley. One end of the string carries a bucket of mass M and the other end carries a counterpoise of equal mass. A small ball of mass m is dropped vertically so as to strike the horizontal bottom of the bucket with velocity u. If e is the coefficient of restitution, show that the bucket begins to move with velocity $\dfrac{m(1+e)u}{2M+m}$, and find the impulse in the string.

Find, also, the time between the first and second impacts of the ball and the bucket.

(03) (a) The figure given below depicts a smooth narrow tube ABC bent into the form of a circular arc of centre O, radius a and angle $2(\pi-\alpha)$, where α is an acute angle. The tube is fixed in a vertical plane with the two open ends A, C uppermost and at the same horizontal level. A particle, placed at the lowest point B of the tube, is projected with a horizontal velocity u. The particle moves through the tube to the end A, then moves freely under gravity as a projectile, and enters the tube again at the other end C.

Find the velocity of the particle as it leaves the tube at A, and show that
$u^2 = ga\,[2(1+\cos\alpha) + \sec\alpha]$.

Show further that the greatest height reached by the paricle is $\dfrac{a}{2}(\cos\alpha + \sec\alpha)$ above O.



(b) A particle P is released from rest at a point A on the smooth outer surface of a fixed sphere, of centre O and radius a, where OA Makes an acute angle α with the upward vertical. Show that, when OP makes and angle θ with the upward vertical, with P still on the surface of the sphere, $a\dot{\theta}^2 = 2g(\cos\alpha - \cos\theta)$.

Find the value of θ at the point where the particle P leaves the surface.

(04) (a) An electric train works at power 3000 kW and maintains a constant speed of 160 km h^{-1}, on a level track. Calculate the resistance to its motion.

The train operates with the same power as before while the resistance to motion remains the same. Find the acceleration of the train, when travelling at a speed of 60km h^{-1}, up a track of inclination 1 in 70, given that the mass of the train is 450 metric tonnes.

[Take the acceleration due to gravity, $g = 9.8\text{m s}^{-2}$]

(b) An elastic string, of natural length l and modulus mg, is attached to a fixed point O on a smooth horizontal table, at a distance $2l$ from one edge. The other end of the string is attached to a particle P of mass m. A light inelastic string of lenght l joins the particle P to a second particle Q of mass m. Initially, with OP = PQ = l, the paricle Q is placed near the edge of the table and gently pushed over it, so that the system begins to move from rest. At time t, OP = $l + x$, the particle P remains on the table and the particle Q is at a depth x below the level of the table. Using the principle of conservation of mechanical energy, or otherwise, show that $\dot{x}^2 = \omega^2\left[l^2 - (l - x)^2\right]$, where $\omega^2 = \frac{g}{2l}$.

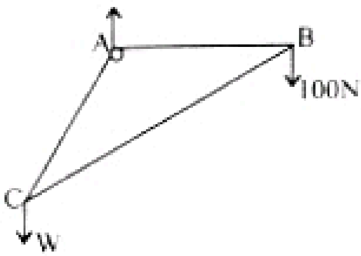
Find the centre and the amplitude of the resulting simple harmonic motion of P.

Show that P reaches the edge of the table at the instant $t = \pi\sqrt{\frac{l}{2g}}$, and find its speed at this instant.

(05) (a) Forces of magnitude aF, bF, aF, bF and cF act respectively along the sides BA, BC, DC, DA and the diagonal BD of a rectangle ABCD, where a = AB, b = BC and c = BD, in the directions specified by the order of letters. Show that the system is equivalent to a single force and find its magnitude, diretion and line of action.

If the magnitude of the force acting DA is increase to $2bF$ while other forces remain the same as before, show that the new system is equivalent to a force of magnitude aF along CD.

(b) In the figure given below, ABCis a triangular framework constring of three smoothly jointed light rods AB, BC, CA where AB = AC and $B\hat{A}C = 120^\circ$. The framework is in a vertical plane with AB horizontal. It is supported at A by a smooth peg, and carries loads of 100 newtons at B and W newtons at C.



Draw a stress diagram using Bow's notation, and from it, calculate the stresses in the rods, distinguishing between tensions and thrusts, and the value of W.

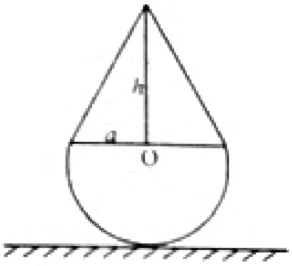
(06) A uniform ladder AB of length 2a amd weight W rests with one end A on a rough horizontal floor and the other end B against a rough vertical wall, μ being the coefficient of friction at both ends of the ladder. The ladder is inclined to the floor at an angle $\frac{\pi}{4}$, and a small cat of weight nW gently climbs up the ladder, starting from A. Show that, in the limiting position of equilibruim of the ladder, the cat has climbed a distance $\frac{a}{n(1+\mu^2)}\left[\mu^2(1+2n) + 2\mu(1+n) - 1\right]$ along the ladder.

Given further that $\mu = \frac{1}{2}$, show that the cat can reach the top of the ladder before the ladder slips, if n [illegible]

What happens if $n = \frac{1}{4}$?

(07) The body shown in the figure consists of a uniform solid hemisphere of centre O and radius a, and a uniform solid right circular cone of the same density with base radius a and height h, rigidly joined at the common base.

Find, by integration, the distances to the centres of mass of the cone and the hemisphere from the point O.



Hence show that the centre of mass of the composite body is at a distance $\frac{|h^2 - 3a^2|}{4(h+2a)}$ from O.

The composite body is placed with the hemispherical surface on a rough horizontal floor so that its axis of symmetry is vertical. It is slightly displaced from this position of equilibrium so that the axis of symmetry makes a small angle with the vertical. Show that the body will topple over, if $h > \sqrt{3}a$.

What happens if (i) $h < \sqrt{3}a$ (ii) $h = \sqrt{3}a$?

(08) (*a*) X and Y are two distinct events in a sample space S. State clealy what is meant by each of the following statements.

- X and Y are exhaustive events.
- X and Y are mutually exclusive events.
- X and Y are independent events.

A and *B* are two exhaustive and mutually exclusive events in *S*. If $P(A) = \frac{2}{5}$, find P(B).

A third event *C* in *S* is such that *A* and *C* *are independent and* $P(C) = \frac{1}{2}$.

$\bar{A}$ and $\bar{C}$ denote complementary events of *A* and *C* respectively.

(i) Calculate $P(A \cap C)$.

(ii) Find $P(A \cup C)$ and deduce $P(\bar{A} \cap \bar{C})$.

(iii) Are $\bar{A}$ and $\bar{C}$ independent? Justify your answer.

A fourth event *D* in S is such that *B* and *D* are mutually exclusice and $P(D) = \frac{1}{5}$.

$\bar{B}$ and $\bar{D}$ denote complementary events of *B* and *D* respectively. Are $\bar{B}$ and $\bar{D}$ mutually exclusive? Justify your answer.

(b) The probabilities that a government servant goes to work by car, bus or train on a certain day are $\frac{1}{10}$, $\frac{2}{5}$ and $\frac{1}{2}$ respectively. The probabilities of his being late for work by these modes of transport are $\frac{1}{5}$, $\frac{1}{2}$ and $\frac{3}{10}$ respectively. If he was late on this particular day, using Bayes' Theorem, calculate the probability that he travelled be train.

(09) (a) Let μ and σ denote the mean and standard deviation respectively, of the set of values $\{x_i : i = 1, 2, \dots\dots\dots n\}$.

Find the mean and standard deviation of each of the following sets of values.

(i) $\{x_i + \alpha : i = 1, 2, \dots\dots n\}$, where α is a constant.

(ii) $\{\beta x_i : i = 1, 2, \dots\dots n\}$, where β is a constant.

By using the above results, find the mean and standard deviation of the set of values $\{2x_i + 3 : i = 1, 2, \dots\dots, n\}$.

(b) The twelve numbers 3, 6, 9, 12, 4, 6, 8, 10, 12, 14, x, y have a mode of 6 and a mean of 8.

Find

(i) the values of x and y.

(ii) the median of the above twelve numbers.

When three additional numbers 8 - k, 8, 8 + k are included, the variance of the fifteen numbers is found to be 12. Find the values of k.