G.C.E. (Advanced Level) Examination - August 2014

Combined Mathematics - I

New Syllabus - Three hours

Part A

- Answer all questions.

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$ for all $n \in \mathbb{Z}^+$.

2. Using a graphical method or otherwise, find all values of x satisfying the inequality $|x+1| > 3x+7$

3. Sketch the loci of the points represented by the complex numbers z satisfying

   (i) $\text{Arg}(z+1) = \frac{\pi}{3}$,

   (ii) $\text{Arg}(z-1) = \frac{5\pi}{6}$

   on the same Argand diagram and find the complex number represented by their point of intersection.

4. Let $n \in \mathbb{Z}^+$. The coefficient of x^{n-2} in the expansion of $\left(2+\frac{3}{x}\right)(1+x)^n$ is 120. Find the value of n.

(05) Show that $\lim_{x\to 0} \frac{\tan^2 2x}{x\left(1-\sqrt{1+x}\right)} = -8$

(06) Find the area of the region enclosed by the straight line $y = 2x$ and the curve $y = x^2$

(07) Let C be the curve given by $x = e^t + e^{-t}$, $y = e^t - e^{-t}$, where t is a real parameter. Find $\frac{dy}{dx}$ in terms of t and show that the equation of the tangent line at the point on C corresponding to $t = \ln 2$ is $5x - 3y - 8 = 0$.

(08) Let $\lambda \in \mathbb{R}$ and $\lambda \neq \pm 1$. The area of the region enclosed by the coordinate axes and the straight line $(1+\lambda)x - 2(1-\lambda)y - 2(1-\lambda) = 0$ is 4 square units. Find the values of λ.

(09) Find the equation of the circle which touches the y-axis at the point $(0, 3)$ and intersects the circle $x^2 + y^2 - 8x + 4y - 5 = 0$ orthogonally.

(10) Let $\tan\alpha = -1$ and $\sin\beta = \frac{1}{\sqrt{5}}$, where $\frac{3\pi}{2} < \alpha < 2\pi$ and $\frac{\pi}{2} < \beta < \pi$. Find the value of $\cos(\alpha+\beta)$.

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Part B

- Answer five questions only.

(11) (a) Let $a \in \mathbb{R}$ and let $f(x) = 3x^3 + 5x^2 + ax - 1$. It is given that $(3x - 1)$ is a factor of $f(x)$. Find the value of a. Express $f(x)$ in the form $(3x-1)(x+k)^2$, where k is a constant.
By writing $3x - 1$ in the above expression in the form $b(x+1) + c$, where b and c are constants, find the remainder when $f(x)$ is divided by $(x+1)^3$.

(b) Let $a, b, c \in \mathbb{R}$ and $ac \neq 0$ Show that zero is not a root of the equation $ax^2 + bx + c = 0$ Let α and β be the roots of this equation, and let $\lambda = \frac{\alpha}{\beta}$ Show that $ac(\lambda+1)^2 = b^2\lambda$.
Let $p, q, r \in \mathbb{R}$ and $pr \neq 0$. Also, let γ and δ be the roots of the equation $px^2 + qx + r = 0$, and let $\mu = \frac{\gamma}{\delta}$. Show that $\lambda = \mu$ or $\lambda = \frac{1}{\mu}$ holds if and only if $acq^2 = prb^2$.
It is given that the roots of the equations $kx^2 - 3x + 2 = 0$ and $8x^2 + 6kx + 1 = 0$ are in the same ratio, where $k \in \mathbb{R}$. Find the value of k.

(12) (a) Six schools participate in a Youth Sports Conference and each school is represented by three players comprising a criketer, a soccer player and a hockey player. It is required to select a committee of six members from among these players. Find the number of different ways in which this committee can be formed.

(i) if two players from each sport must be included.
(ii) if two players from each sport must be included, is such a way that all six schools are represented,
(iii) if two players each from two schools and one player each from two of the remaining schools must be included.

(b) Let $U_r = \dfrac{r^2 - r - 5}{r(r+1)(r+4)(r+5)}$ for $r \in \mathbb{Z}^+$

By comparing coefficients of r^n for $n = 0, 1, 2, 3$ show that there exist constants A and B such that $r^2 - r - 5 = A(r^2 - 1)(r + 5) - Br^2(r + 4)$ for $r \in \mathbb{Z}^+$ Find $f(r)$ such that $U_r = f(r) - f(r+1)$ for $r \in \mathbb{Z}^+$

Show that $\sum_{r=1}^{n} U_r = -\dfrac{n}{(n+1)(n+5)}$ for $n \in \mathbb{Z}^+$

Show further that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Hence, find $\sum_{r=3}^{\infty} 3U_r$

(13) (a) Let $a, b \in \mathbb{R}$ and let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & a \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b & 1 \\ 1 & 1 \end{pmatrix}$. Find the values of a and b such that $\mathbf{A}^T\mathbf{A} = \mathbf{B}$, where $\mathbf{A}^T$ denotes the transpose of the matrix $\mathbf{A}$.

Let $\mathbf{C} = \begin{pmatrix} 7 & 5 \\ 5 & 3 \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} u \\ u+1 \end{pmatrix}$, where $u \in \mathbb{R}$.
Also, let $\mathbf{CX} = \lambda\mathbf{BX}$. where $\lambda \in \mathbb{R}$. Find the value of λ and the value of u.

For this value of λ, find the matrix $\mathbf{C} - \lambda\mathbf{B}$ and show that its inverse does not exist.

(b) Let $z \in \mathbb{C}$ Show that

(i) $|1-z|^2 = 1 - 2\operatorname{Re} z + |z|^2$ and

(ii) $\operatorname{Re}\left(\dfrac{1}{1-z}\right) = \dfrac{1 - \operatorname{Re} z}{|1-z|^2}$ for $z \neq 1$

Deduce that $\operatorname{Re}\left(\dfrac{1}{1-z}\right) = \dfrac{1}{2}$ if and only if $|z| = 1$ and $z \neq 1$.

Let S be the set consisting of complex numbers z satisfying both conditions $\mathrm{Re}\left(\frac{1}{1-z}\right)=\frac{1}{2}$ and $-\frac{\pi}{3}<Arg\, z<\frac{\pi}{3}$. Plot the points that represent the complex numbers in S in an Argand diagram.

Show that if z is in S and $\mathrm{Re}\, z+\mathrm{Im}\, z=\frac{1}{\sqrt{2}}$,

then $z=\cos\left(\frac{\pi}{12}\right)-i\sin\left(\frac{\pi}{12}\right)$.

(14) (a) Let $f(x)=\dfrac{8x}{(x+1)(x^2+3)}$ for $x\neq -1$

Show that $f'(x)=\dfrac{8(1-x)(2x^2+3x+3)}{(x+1)^2(x^2+3)^2}$

for $x\neq -1$.

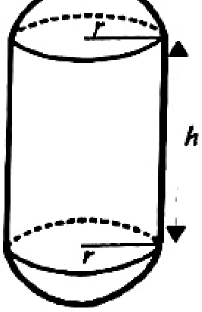
Sketch the graph of $y=f(x)$ indicating the turning point and the asymptotes.

Using the graph of $y=f(x)$ find **the number of** solutions of the equation $(x+1)(x^2+3)=16x$.

(b) A hollow composite body is to be formed by rigidly joining two hollow hemispheres of radius r metres to a right circular hollow cylinder with the same radius and height h metres, as shown in the figure. The total volume of the composite body is $36\pi m^3$ Show that $h=\dfrac{108-4r^3}{3r^2}$.

The cost of material for the cylindrical surface is 300 rupees per square metre and that for hemispherical surfaces is 1000 rupees per square metre. Show that the total cost of material C rupees, required to make this composite body is given by $C=800\pi\left(4r^2+\frac{27}{r}\right)$ for $0<r<3$

Find the value of r such that C is minimum.



(15) (a) Find $\int\dfrac{3x+2}{x^2+2x+5}dx$.

(b) Using integration by parts, show that $\int_1^{e^\pi}\cos(\ln x)dx=-\frac{1}{2}(e^\pi+1)$.

(c) Establish the formula $\int_0^a f(x)dx=\int_0^a f(a-x)dx$.

where a is a constant.

Let $p(x)=(x-\pi)(2x+\pi)$ and $I=\int_0^{\frac{\pi}{2}}\dfrac{\sin^2 x}{p(x)}dx$

Using the above result, show that

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{p(x)}dx.$$

Using the above two integrals for I, **deduce** that

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{p(x)}\,dx.$$

Hence, show that $I=\frac{1}{6\pi}\ln\left(\frac{1}{4}\right)$.

16. Let l_1 and l_2 be the straight lines given by $2x \quad y=5$ and $x+2y=4$ respectively. Show that the acute angle between l_1 and l_2 is $\tan^{-1}\left(\frac{3}{4}\right)$, and ind the equation of the bisector of this angle.

Let A be the point of intersection of l_1 and l_2, and let $R=\{(x,y): x+2y\leq 4 \text{ and } 2x+y\geq 5\}$.

Find the coordinates of the point A and shade the region R in the xy-plane.

Show that the equation of the circle S of radius $\sqrt{5}$ which lies in the region R and which touches both lines l_1 and l_2 is $x^2+y^2-14x+8y+60=0$.

Using the usual formula for the chord of contact, show that the equation of the chord of contact of the tangents drawn from the point A to the circle S is $x-y=10$.

Find the equation of the circle passing through the point A and the points of contact of S with l_1 and l_2.

17. (a) Let $f(x) = \frac{(1-\tan x)^2}{1+\tan^2 x}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Express $f(x)$ in the form $A\cos(2x+\alpha)+B$, where $A(>0)$, B and $\alpha\left(0<\alpha<\frac{\alpha}{2}\right)$ are constants to be determined.

Hence, solve the equation $f(x) = \frac{2+\sqrt{2}}{4}$.

Using the first expression given for $f(x)$,

show that $f(x) = \frac{2+\sqrt{2}}{4}$ can be written as

$2\tan^2 x + 4k\tan x - k^2 = 0$, where $k = 2-\sqrt{2}$.

Deduce that $\tan\frac{\pi}{24} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$

Also, sketch the graph of $y = 2f(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(b) In the usual notation, state the **Sine Rule** for a triangle.

Let ABC be a triangle. In the usual notation, it is given that $a:b:c = 1:\lambda:\mu$, where λ and μ are constants. Show that

$\mu^2(\sin 2A + \sin 2B + \sin 2C) = 4\lambda\sin^3 C$.