G.C.E. (Advanced Level) Examination - April 2004
10 - Combined Mathematics - I
Three hours

- Answer six questions only.

(01) (*a*) Let $\lambda \in \mathbb{R}$ and $P(x) = x^2 - 2\lambda(x-1) - 1$

Show that the roots of $p(x) = 0$ are real.

Find all the value of λ such that the sum of the roots of $p(x) = 0$ is equal to the sum of the squares of the roots.

(b) A quadratic polynomial $p(x)$ has remainders $1, \frac{1}{2}$ and $\frac{1}{3}$ when divided by $(x-1)$, $(x-2)$ and $(x-3)$ respectively.

Show that $(x-1)$, $(x-2)$ and $(x-3)$ are factors of the polynomial $Q(x)$ given by $Q(x) = x\,P(x) - 1$
Hence, find $Q(x)$.

(02) (a) I a certain examination you are required to answer six out of nine questions.

Find the number of ways that you can choose the six questions.

Also, find the number of ways that you can choose the six questions,

(i) if the first three questions are compulsory,

(ii) if at least four should be selected from the first five questions.

(b) Consider the binomial expansion of $(1 + 7x)^{23}$ in accending powers of x.

(i) Find the greatest numerical coefficient of the expansion and the terms of the expansion corresponding to it.

(ii) Given that x is positive, find the range of values of x for which the greatest term of the expansion is the fourth term.

(03) (a) Using the Principle of Mathematical Induction, prove that

$$\frac{1}{2^2-1} + \frac{1}{3^2-1} + \dots + \frac{1}{(n+1)^2-1} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

for every positive interger n

Deduce that the series $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$ is convergent and find its sum.

(b) Find the set of all real values of x satisfying the inequality $|x-1| - \left|\frac{1}{2}x + 1\right| < 1$.

Deduce the greatest integral value in the solution set.

(04) Let z be the complex number $\frac{1}{2}\left(1 + i\sqrt{3}\right)$

Find the modulus and argument of each of the complex numbers $2z^2$ and $\frac{3}{z^2}$.

In an Argand diagram O represents the origin, A represents the complex number $2z^2$ and B represents the complex number $\frac{3}{z^2}$.

Does the point representing z lie on the line passing through O and B? Justify your answer.

The point C is chosen so that OACB is a parallelogram.

Determine, in the Cartesian form $p + iq$, the complex number represented by C.

Find the lengths of the diagonals of OACB.

(05) (a) Let $y = e^{-x}(\cos 2x + \sin 2x)$

Show that $\frac{dy}{dx} + y = 2e^{-x}(\cos 2x - \sin 2x)$.

Determine two number p and q such that $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$.

Find $\left(\frac{d^3y}{dx^3}\right)_{x=0}$.

(b) **A rectangular poster is to be made so that it displays a rectangular printed region of area 972cm^2 surrounded by a left and right margins of 6cm each, and a top and bottom margins of 8cm each.**

Find the dimensions of the poster with the least area.

(06) (a) Using a suitable substitution, evaluate

$$\int_{[illegible]}^{[illegible]} \frac{dx}{(x+1)\sqrt{2x+3}}$$

(b) Using integration by parts, find $\int e^{3x} \cos 4x dx$.

(c) Find $\int \sin^4 2x dx$.

(07) Let u and v be two parallel lines passing through the points $A \equiv (5, 0)$ and $B \equiv (-5, 0)$ respectively. Let the line $4x + 3y = 25$ meet u at P and v at Q.

If the length of PQ is 5 units, show that there are two possibilities for the pair of parallel lines u and v.

Write down the equations of all four lines determined above.

Find the equations of the diagonals of the parallelogram formed by these four lines.

Also, find the area of the above parallelogram.

(08) Let $S_1 \equiv x^2 + y^2 - 2 = 0$ and
$S_2 \equiv x^2 + y^2 + 3x + 3y - 8 = 0$.

Show that $S_1 = 0$ and $S_2 = 0$ touch internally, and find the coordinates of the point of contact P.

A straight line drawn through the point P cuts $S_1 = 0$ and $S_2 = 0$ again at the points Q and R respectively.

Show that the mid-point of QR lies on the circle

$$x^2 + y^2 + \frac{3}{2}x + \frac{3}{2}y - 5 = 0$$

(09) (a) Show that if $0 < \theta < \frac{\pi}{2}$,

then $\sin\theta \tan\theta > 2(1 - \cos\theta)$.

(b) Using the expansions of $\sin(A - B)$ and $\cos(A - B)$, show that

$$\sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4} \text{ and } \cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Show that $\tan x = \frac{1 - \cos 2x}{\sin 2x}$ for $0 < x < \frac{\pi}{2}$ and

deduce that $\tan\frac{\pi}{24} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$.

(c) State the *Sine Rule* for a triagle.
Prove, in the usual notation for a triangle ABC,

that $\frac{a^2 - b^2}{c^2} = \frac{\sin(A - B)}{\sin(A + B)}$.