THE LINEAR CHROMATIC NUMBER OF A GRAPH

DANIEL CAPRONI, JOSHUA EDGERTON, MARGARET RAHMOELLER, MYCHAEL
SANCHEZ, ANNA TRACY

ABSTRACT. We study linear colorings and the linear chromatic number of graphs and describe a strategy we use to find linear colorings of graphs. We compute the linear chromatic number for specific graphs and briefly discuss linear N-graphs. We then find bounds on the linear chromatic number of graphs.

1. Graph Theoretic Preliminaries

A *graph* G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a collection of objects called *vertices* and $E(G)$ is a multiset of pairs of (not necessarily distinct) vertices. The sets $V(G)$ and $E(G)$ are called the *vertex set* and *edge set*, respectively, and elements of $E(G)$ are called *edges.* We denote the number of vertices in $V(G)$ by $n(G)$ and the number of edges in $E(G)$ by $e(G)$. If $e \in E(G)$, then $e = \{u, v\}$ for some vertices $u, v \in V(G)$. We call u and v the *endpoints* of e and we sometimes write $e = uv$. If u and v are the endpoints of an edge e, then we say that u and v are *adjacent* and that e is *incident* at u and at v. If two vertices u and v are adjacent, we write $u \leftrightarrow v$. The *degree* of a vertex v, denoted $\deg(v)$, is the number of edges incident at v. The maximum degree of a vertex in a graph is denoted $\Delta(G)$.

We call repeated elements in $E(G)$ *multiple edges*, and we call an edge a *loop* if it has only one endpoint. A *simple graph* is a graph with no loops and no multiple edges. For a vertex $v \in V(G)$, the *neighborhood* of v, denoted $\mathcal{N}(v)$, is the set $\{u \in V(G) \mid u \leftrightarrow v\}$. Note that $v \notin \mathcal{N}(v)$ unless v has a loop. We denote the set of all neighborhoods by $\mathcal{N}_G$. Note that if a graph is simple, then for each $v \in V(G)$, $\deg(v) = |\mathcal{N}(v)|$. An *isolated vertex* is vertex of degree zero.

A *coloring,* or more specifically, a *k-coloring* of a graph G is a surjective map $\pi : V(G) \to [k]$ where $[k] = \{1, \ldots, k\}$. Here, elements of $[k]$ serve as "colors" for the vertices of G. A coloring of G is *proper* if adjacent vertices do not receive the same color. Obviously, we can achieve a proper coloring of a graph by giving each vertex its own color; however, this would not optimize the use of our colors. The *chromatic number* of G, denoted $\chi(G)$, is the least k such that G has a proper k-coloring.

Finally, we must decide when two graphs are to be regarded as the same. If G and H are graphs, then an *isomorphism* between G and H is a bijective map $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say that G and H are *isomorphic* if there is an isomorphism between G and H.

2. Neighborhood Complexes and Linear colorings

Let X be a finite set. A *simplicial complex* on X is a family Γ of subsets such that

(1) $\{x\} \in \Gamma$ for every $x \in X$.

(2) If $F \in \Gamma$, then $H \in \Gamma$ for every $H \subseteq F$.

We call elements of Γ *faces*, and we call maximal faces *facets*. We denote the sets of all facets and all facets containing x in Γ, by $\mathcal{F}_\Gamma$ and $\mathcal{F}_\Gamma(x)$, respectively.

Just as we may color a graph G, we may also color a simplicial complex. Let Γ be a simplicial complex on a finite set X. Then a surjective map $\sigma : X \to [k]$ is a *k-linear coloring* of Γ if whenever $\sigma(x) = \sigma(y)$, we have $\mathcal{F}_\Gamma(x) \subseteq \mathcal{F}_\Gamma(y)$ or $\mathcal{F}_\Gamma(y) \subseteq \mathcal{F}_\Gamma(x)$. We call sets of the form $\sigma^{-1}(i)$ *color classes* and denote them C_i. Since every vertex in a color class receives the same color, we can order the vertices so that

$$\mathcal{F}_\Gamma(v_1) \subseteq \cdots \subseteq \mathcal{F}_\Gamma(v_n)$$

where $C_i = \{v_1, \ldots, v_n\}$. That is, the sets of facets containing the v_j are linearly ordered by inclusion. The *linear chromatic number* of Γ, denoted $\lambda(\Gamma)$, is the least k such that Γ has a k-linear coloring.

We are primarily interested in a simplicial complex defined on the vertex set of a graph. The *neighborhood complex* of a graph G, written $\mathcal{N}(G)$, is the simplicial complex on $V(G)$ where a set $F \subseteq V(G)$ is a face in $\mathcal{N}(G)$ if and only if

$$\bigcap_{\alpha \in F} \mathcal{N}(\alpha) \neq \emptyset.$$

A neighborhood complex can be visualized geometrically by "gluing" together the faces. An example of a neighborhood complex is illustrated in Figure 2.1. The faces of the tetrahedron in the figure are the faces in the neighborhood complex, the edges are the two-element subsets, and the vertices are the singletons.
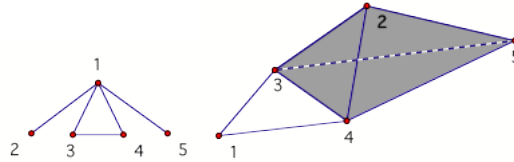

FIGURE 2.1. A Graph G and its Neighborhood Complex

Since we work only with neighborhood complexes, we denote the sets of all facets and all facets containing vertex x in $\mathcal{N}(G)$ by $\mathcal{F}_G$ and $\mathcal{F}_G(x)$, rather than $\mathcal{F}_{\mathcal{N}(G)}$ and $\mathcal{F}_{\mathcal{N}(G)}(x)$. Notice that the neighborhood complex is not well defined as a simplicial complex for a graph containing isolated vertices or loops. Multiple edges do not alter the neighborhood complex of a graph because each vertex in a facet is counted only once; however, considering only simple graphs eases computations with vertex degrees, eliminates special cases of theorems, and produces no real loss of generality. Henceforth, we focus our attention on simple graphs with no isolated vertices.

Lemma 2.1. *Every facet is the neighborhood of some vertex.*

Proof. Let G be a graph and let F be a facet in $\mathcal{N}(G)$. Then $\bigcap_{\alpha \in F} \mathcal{N}(\alpha) \neq \emptyset$. Hence, there is a vertex u such that $u \leftrightarrow \alpha$ for each $\alpha \in F$. Let $v \in F$. Then $v \leftrightarrow u$ and $v \in \mathcal{N}(u)$. Thus, $F \subseteq \mathcal{N}(u)$. Since $\mathcal{N}(u)$ is a face of $\mathcal{N}(G)$ and F is maximal, $F = \mathcal{N}(u)$. $\square$

It follows from this result that the number of facets in $\mathcal{N}(G)$ cannot exceed the number of vertices.

A linear coloring of $\mathcal{N}(G)$ is also a coloring of the underlying graph G. For this reason (and to simplify notation), we refer to a linear coloring of $\mathcal{N}(G)$ as a linear coloring of G. Further, we refer to the linear chromatic number of $\mathcal{N}(G)$ as the linear chromatic number of G and denote it $\lambda(G)$, rather than $\lambda(\mathcal{N}(G))$. It has been proved [1] that every linear coloring of a graph is proper. This gives us the lower bound

$$\chi(G) \leq \lambda(G)$$

for every graph G. We seek a better bound for $\lambda(G)$.

3. Linear N-Graphs

In general, equality does not hold for the above lower bound. A *linear N-Graph* is a graph G such that $\lambda(G) = \chi(G)$. For more details, see [2]. An example of linear N-graph is the paw, illustrated in Figure 3.1.
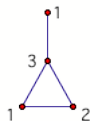

Figure 3.1. Paw

If two graphs are isomorphic, we suspect they have the same linear chromatic number.

Proposition 3.1. *If $G \cong H$, then $\lambda(G) = \lambda(H)$.*

Proof. Since $G \cong H$, there exists an isomorphism $f : V(G) \to V(H)$. Let $k = \lambda(G)$, and let $\kappa : V(G) \to [k]$ be a linear coloring of G. Define $\bar{\kappa} : V(H) \to [k]$ by $\bar{\kappa}(x) = \kappa(f^{-1}(x))$. We claim that $\bar{\kappa}$ is a linear coloring of H. Suppose $x, y \in V(H)$ with $\bar{\kappa}(x) = \bar{\kappa}(y)$. Then there exist unique $u, v \in V(G)$ such that $x = f(u)$ and $y = f(v)$. From the construction of $\bar{\kappa}$, we have $\kappa(u) = \kappa(v)$. Since κ is linear, $\mathcal{F}_G(u) \subseteq \mathcal{F}_G(v)$ or $\mathcal{F}_G(u) \subseteq \mathcal{F}_G(v)$. Without loss of generality, we may assume that $\mathcal{F}_G(u) \subseteq \mathcal{F}_G(v)$. Now let $F \in \mathcal{F}_H(x)$. Then $x \in F$ and

$$\bigcap_{\alpha \in F} \mathcal{N}(\alpha) \neq \emptyset.$$

Therefore, there is a vertex m such that α is adjacent to m for each $\alpha \in F$. Then $f^{-1}(m)$ is adjacent to β for each $\beta \in f^{-1}(F)$. This shows

$$\bigcap_{\beta \in f^{-1}(F)} \mathcal{N}(\beta) \neq \emptyset.$$

Hence, $f^{-1}(F)$ is a facet in $\mathcal{N}(G)$. Further, $u \in f^{-1}(F)$. Thus, $f^{-1}(F) \in \mathcal{F}_G(u)$. It follows that $f^{-1}(F) \in \mathcal{F}_G(v)$. Since $v \in f^{-1}(F)$, we have $f(v) \in F$. But $f(v) = y$, and F is a facet in $\mathcal{N}(H)$ containing y; that is, $F \in \mathcal{F}_H(y)$. Hence, $\mathcal{F}_H(x) \subseteq \mathcal{F}_H(y)$, and $\bar{\kappa}$ is linear. This proves $\lambda(H) \geq \lambda(G)$. An analogous argument proves that $\lambda(G) \geq \lambda(H)$. Therefore, $\lambda(H) = \lambda(G)$. $\square$

Corollary 3.2. *If G is a linear N-Graph and $G \cong H$, then H is a linear N-Graph.*

Proof. Since $G \cong H$, $\chi(G) = \chi(H)$ and $\lambda(G) = \lambda(H)$. Since G is a linear N-Graph, $\chi(G) = \lambda(G)$. Hence,

$$\begin{aligned}\lambda(H) &= \lambda(G) \\ &= \chi(G) \\ &= \chi(H).\end{aligned}$$

Therefore, G is a linear N-Graph. $\square$

A *path P_n* is a simple graph with n vertices whose vertex set can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. If u and v are vertices in a graph G, a *uv-path* is a path whose endpoints are u and v. The *length* of a path is the number of edges in it.

Let G be a graph. A subgraph of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that G is connected if for all $u, v \in V(G)$ there is a *uv*-path. The *components* of G are its maximal connected subgraphs.

Suppose that a graph has multiple components. In a proper coloring, vertices in different components can receive the same color because they are not adjacent. However, they cannot share a common neighbor; thus, they cannot receive the same color in a linear coloring. Apparently more colors are needed to color such a graph linearly than to color it properly. In fact, a much stronger assertion is true, as we see in the following sequence of results.

Lemma 3.3. *Let G be a graph. Then $\chi(G) = \max\{\chi(G_i) \,|\, G_i$ is a component of $G\}$.*

Proof. Let $M = \max\{\chi(G_i) \,|\, G_i$ is a component of $G\}$. Then $M \leq \chi(G)$. Since $\chi(G_i) \leq M$ for each i, the set $[M]$ is sufficient to color each G_i. Thus, $\chi(G) \leq M$. Therefore, $\chi(G) = M$, as claimed. $\square$

Theorem 3.4. *Let G be a graph with components $G_1, G_2, \ldots, G_n$. Then*

$$\lambda(G) = \sum_{i=1}^{n} \lambda(G_i).$$

Proof. We know that $\lambda(G) \leq \sum_{i=1}^{n} \lambda(G_i)$. Let $\kappa : V(G) \to [\lambda(G)]$ be a linear coloring. Let $u \in V(G_i)$ and $v \in V(G_j)$, where $i \neq j$. Since the components of G are disconnected, there is no uv-path. Then $u \notin \mathcal{F}_G(v)$ and $v \notin \mathcal{F}_G(u)$, because u and v cannot share a common neighbor. Thus, $\kappa(u) \neq \kappa(v)$. Hence, each G_i has a unique set κ_i of colors. It follows that $\lambda(G) \geq |\kappa_1 \cup \kappa_2 \cup \ldots \cup \kappa_n|$. Since the sets of colors for each component are pairwise disjoint, $|\kappa_1 \cup \kappa_2 \cup \ldots \cup \kappa_n| = \sum_{i=1}^{n} \lambda(G_i)$. Therefore, $\lambda(G) = \sum_{i=1}^{n} \lambda(G_i)$. $\qquad\square$

We now prove a necessary condition for the linearity of a graph.

Corollary 3.5. *If G is a linear N-graph, then G is connected.*

Proof. Let G be a graph with components $G_1, G_2, \ldots, G_m$ where $m \geq 2$. From Lemma 3.3, we have $\chi(G) = \max\{\chi(G_i) \,|\, G_i \text{ is a component of } G\}$. From Theorem 3.4, we have $\lambda(G) = \sum_{i=1}^{m} \lambda(G_i)$. Furthermore, $\lambda(G_i) \geq \chi(G_i)$, for all $1 \leq i \leq m$. Hence, $\lambda(G) \geq \sum_{i=1}^{m} \chi(G_i) > \chi(G)$. Therefore, G is not linear. $\qquad\square$

Remark. We would like to emphasize that even if a graph is linear, it is not necessary that every proper coloring of that graph is also a linear coloring. Observe the proper coloring of the bull, a linear N-graph, in Figure 3.2.
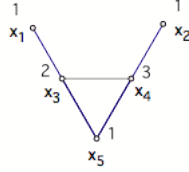

FIGURE 3.2. The Bull

We see that $\mathcal{F}_G(x_1) \nsubseteq \mathcal{F}_G(x_2)$ and $\mathcal{F}_G(x_2) \nsubseteq \mathcal{F}_G(x_1)$. However, x_1 and x_2 can share the same color in a proper coloring. For this reason, this proper coloring of the graph is not a linear coloring.

4. LINEAR CHROMATIC NUMBERS

In order to compute the linear chromatic number of graphs, we must have an efficient strategy for coloring a graph linearly.

How to Find a Linear Coloring

Let G be a graph with $\{v_1, v_2, v_3, \ldots, v_n\} = V(G)$.

(1) Compute $\mathcal{N}_G$.
(2) Delete $\mathcal{N}(v_i)$ whenever $\mathcal{N}(v_i) \subseteq \mathcal{N}(v_j)$, where $1 \leq i, j \leq n$ and $i \neq j$. The resulting set is $\mathcal{F}_G$.
(3) Find all $\mathcal{F}_G(v_i)$ where $1 \leq i \leq n$.
(4) Let $\mathcal{F}_G(v_s) = \min\{|\mathcal{F}_G(v_i)| \,|\, 1 \leq i \leq n\}$. If $\mathcal{F}_G(v_s) \subseteq \mathcal{F}_G(v_1) \subseteq \mathcal{F}_G(v_2) \subseteq \ldots \subseteq \mathcal{F}_G(v_m)$, where $v_i \in V(G)$, and $1 \leq m \leq n-1$, then assign each of these vertices $(v_1, v_2, \ldots, v_m)$ the same color. Repeat this step for the remaining vertices. This sequence ends after a finite number of steps.

For small graphs, this strategy can help compute the linear chromatic numbers. If the graph is larger, we can at least find an upper bound for the linear chromatic number.

A *complete graph*, denoted K_n, is a simple graph with n vertices that are pairwise adjacent. An example of a complete graph is depicted in Figure 4.1.
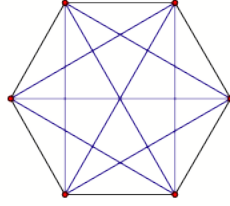

FIGURE 4.1. A Complete Graph

Example 4.1. $\lambda(K_n) = n$.

First note that $\lambda(K_n) \leq n$. Since the vertices in K_n are pairwise adjacent, we know $\chi(K_n) = n$. We also know that $\lambda(K_n) \geq \chi(K_n)$. Therefore, $\lambda(K_n) = n$.

We need additional tools to compute linear chromatic numbers. If G has a uv-path, the *distance* from u to v, denoted $d(u, v)$, is the length of a minimum uv-path. Since every linear coloring is proper, if two vertices receive the same color, they cannot be adjacent. Moreover, if two vertices receive the same color, then there is a facet containing both of them. Hence, their distance cannot be too large.

Lemma 4.2. *Let $\kappa : V(G) \to [k]$ be a linear coloring of a graph G. For all $u, v \in V(G)$, if $\kappa(u) = \kappa(v)$, then $d(u, v) = 2$.*

Proof. Let $u, v \in V(G)$ and suppose $\kappa(u) = \kappa(v)$. Since κ is linear, it is proper. Hence, $u \not\leftrightarrow v$. Thus, $d(u, v) \geq 2$. Since $\kappa(u) = \kappa(v)$, either $\mathcal{F}_G(u) \subseteq \mathcal{F}_G(v)$ or $\mathcal{F}_G(v) \subseteq \mathcal{F}_G(u)$. We may assume without loss of generality that $\mathcal{F}_G(u) \subseteq \mathcal{F}_G(v)$. It follows that there is a facet F in $\mathcal{N}(G)$ containing u and v. Thus, u and v share a common neighbor w. Then u, w, v is a uv-path in G, and therefore, $d(u, v) = 2$. $\square$

Next we compute the linear chromatic number of several graphs. Recall that the path P_n is the simple graph illustrated in Figure 4.2.

FIGURE 4.2. Path P_n

Example 4.3. If $n \geq 8$, then $\lambda(P_n) = n - 4$.

To see this, consider a path P_n of length $n \geq 8$, with endpoints u_1 and u_n. Then,

$$\mathcal{N}_{P_n} = \{\{u_2\}, \{u_1, u_3\}, \{u_2, u_4\}, \{u_3, u_5\}, \ldots, \{u_{n-3}, u_{n-1}\}, \{u_{n-2}, u_n\} \{u_{n-1}\}\}$$

and
$$\mathcal{F}_{P_n} = \{\{u_1, u_3\}, \{u_2, u_4\}, \{u_3, u_5\}, \ldots, \{u_{n-3}, u_{n-1}\}, \{u_{n-2}, u_n\}\}$$

We have

$$\mathcal{F}_{P_n}(u_1) = \{\{u_1, u_3\}\}$$
$$\mathcal{F}_{P_n}(u_2) = \{\{u_2, u_4\}\}$$
$$\mathcal{F}_{P_n}(u_3) = \{\{u_1, u_3\}, \{u_3, u_5\}\}$$
$$\mathcal{F}_{P_n}(u_4) = \{\{u_2, u_4\}, \{u_4, u_6\}\}$$

$$\vdots$$

$$\mathcal{F}_{P_n}(u_{n-3}) = \{\{u_{n-5}, u_{n-3}\}, \{u_{n-3}, u_{n-1}\}\}$$
$$\mathcal{F}_{P_n}(u_{n-2}) = \{\{u_{n-4}, u_{n-2}\}, \{u_{n-2}, u_n\}\}$$
$$\mathcal{F}_{P_n}(u_{n-1}) = \{\{u_{n-3}, u_{n-1}\}\}$$
$$\mathcal{F}_{P_n}(u_n) = \{\{u_{n-2}, u_n\}\}$$

We note that the vertices u_1, u_2, u_{n-1}, and u_n are located at the ends of P_n. Each set of facets containing one of these vertices only includes one facet. Every other vertex in P_n is contained in two facets.

For every $2 < i < n - 1$, we have $\mathcal{F}_{P_n}(u_i) = \{\{u_{i-2}, u_i\}, \{u_i, u_{i+2}\}\}$. Hence, if $u_i, u_j \in V(P_n)$, then $\mathcal{F}_{P_n}(u_i) \nsubseteq \mathcal{F}_{P_n}(u_j)$. Therefore, each one of these vertices must be assigned its own color. Hence, $\lambda(P_n) \geq n - 4$. Looking at the "special" vertices, u_1, u_2, u_{n-1}, and u_n, we notice that each set of facets containing these vertices is a subset of another set of facets.

$$\mathcal{F}_{P_n}(u_1) = \{\{u_1, u_3\}\} \subseteq \mathcal{F}_{P_n}(u_3) = \{\{u_1, u_3\}, \{u_3, u_5\}\}$$
$$\mathcal{F}_{P_n}(u_2) = \{\{u_2, u_4\}\} \subseteq \mathcal{F}_{P_n}(u_4) = \{\{u_2, u_4\}, \{u_4, u_6\}\}$$
$$\mathcal{F}_{P_n}(u_{n-1}) = \{\{u_{n-3}, u_{n-1}\}\} \subseteq \mathcal{F}_{P_n}(u_{n-3}) = \{\{u_{n-5}, u_{n-3}\}, \{u_{n-3}, u_{n-1}\}\}$$
$$\mathcal{F}_{P_n}(u_n) = \{\{u_{n-2}, u_n\}\} \subseteq \mathcal{F}_{P_n}(u_{n-2}) = \{\{u_{n-4}, u_{n-2}\}, \{u_{n-2}, u_n\}\}$$

Hence, u_1, u_2, u_{n-1}, and u_n may be assigned the same color as u_3, u_4, u_{n-3}, and u_{n-2}, respectively, so $\lambda(P_n) \leq n - 4$. Therefore, $\lambda(P_n) = n - 4$.
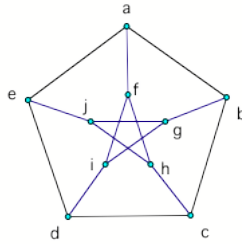

FIGURE 4.3. Petersen graph $\mathcal{P}$

A Petersen graph, illustrated in Figure 4.3, is a graph with ten vertices.

Example 4.4. Let $\mathcal{P}$ be the Petersen graph. Then $\lambda(\mathcal{P}) = 10$.

We have

$$\mathcal{N}_{\mathcal{P}} = \{\{b, f, e\}, \{c, a, g\}, \{b, d, h\}, \{c, e, i\}, \{d, a, j\},$$
$$\{a, h, i\}, \{b, i, j\}, \{c, f, j\}, \{d, f, g\}, \{e, g, h\}\}.$$

Since all neighborhood sets are the same size and no two neighborhoods are equal, $\mathcal{F}_{\mathcal{P}} = \mathcal{N}_{\mathcal{P}}$. Then

$\mathcal{F}_{\mathcal{P}}(a) = \{\{c, a, g\}, \{d, a, j\}, \{a, h, i\}\}$

$\mathcal{F}_{\mathcal{P}}(b) = \{\{b, f, e\}, \{b, d, h\}, \{b, i, j\}\}$

$\mathcal{F}_{\mathcal{P}}(c) = \{\{c, a, g\}, \{c, e, i\}, \{c, f, j\}\}$

$\mathcal{F}_{\mathcal{P}}(d) = \{\{b, d, h\}, \{d, a, j\}, \{d, f, g\}\}$

$\mathcal{F}_{\mathcal{P}}(e) = \{\{b, f, e\}, \{c, e, i\}, \{c, f, j\}\}$

$\mathcal{F}_{\mathcal{P}}(f) = \{\{b, f, e\}, \{c, f, j\}, \{d, f, g\}\}$

$\mathcal{F}_{\mathcal{P}}(g) = \{\{c, a, g\}, \{d, f, g\}, \{e, g, h\}\}$

$\mathcal{F}_{\mathcal{P}}(h) = \{\{b, d, h\}, \{a, h, i\}, \{e, g, h\}\}$

$\mathcal{F}_{\mathcal{P}}(i) = \{\{c, e, i\}, \{a, h, i\}, \{b, i, j\}\}$

$\mathcal{F}_{\mathcal{P}}(j) = \{\{d, a, j\}, \{b, i, j\}, \{c, f, j\}\}$

Note that for each $u, v \in V(\mathcal{P})$, we have $\mathcal{F}_{\mathcal{P}}(u) \neq \mathcal{F}_{\mathcal{P}}(v)$. Since each set of facets is the same size, each vertex must be assigned its own color. Therefore, $\lambda(\mathcal{P}) = n = 10$.

Notice that every vertex in the Petersen graph has degree three. More generally, a *k-regular* graph G, is a graph such that for all $v \in V(G)$, we have $\deg(v) = k$. The following lemmata make the calculation of $\lambda(\mathcal{P})$ much easier.

Lemma 4.5. *If G is a k-regular graph, then for all $x \in V(G)$ the set $\mathcal{N}(x)$ is a facet in $\mathcal{N}(G)$.*

Proof. Suppose not. Then there is a vertex u such that $\mathcal{N}(u) \notin \mathcal{F}_G$. Since G is k-regular, $\deg(u) = k = |\mathcal{N}(u)|$. Further, since $\mathcal{N}(u)$ is only a face, it must be the case that $\mathcal{N}(u)$ is not maximal. Thus, there is a vertex $v \notin \mathcal{N}(u)$ such that $\mathcal{N}(u) \cup \{v\}$ is a face in $\mathcal{N}(G)$. Then every vertex in $\mathcal{N}(u) \cup \{v\}$ is adjacent to some vertex w. It follows that $\deg(w) \geq |\mathcal{N}(u) \cup \{v\}| = k + 1 > k$. This is a contradiction. $\square$

Lemma 4.6. *Let $\kappa : V(G) \to [\lambda(G)]$ be a linear coloring of a k-regular graph G, and let $u, v \in V(G)$. If $\kappa(u) = \kappa(v)$, then $\mathcal{N}(u) = \mathcal{N}(v)$.*

Proof. Suppose $\kappa(u) = \kappa(v)$. Without loss of generality, $\mathcal{F}_G(u) \subseteq \mathcal{F}_G(v)$. If $x \in \mathcal{N}(u)$, then by Lemma 4.5, $\mathcal{N}(x) \in \mathcal{F}_G(u)$; hence, $\mathcal{N}(x) \in \mathcal{F}_G(v)$, and so $x \in \mathcal{N}(v)$. Hence, $\mathcal{N}(u) \subseteq \mathcal{N}(v)$, and since G is regular, $\mathcal{N}(u) = \mathcal{N}(v)$. $\square$

Let G be a graph. Then the *reduction of G*, denoted G_{red}, is the graph obtained from G by identifying vertices $u, v \in V(G)$ whenever $\mathcal{N}(u) = \mathcal{N}(v)$. A graph G is *reduced* if $G = G_{\text{red}}$. If a graph G is reduced and k-regular, then for all $u \neq v \in V(G)$,

we know $\mathcal{N}(u) \neq \mathcal{N}(v)$. Since G is k-regular, the sets of facets are the same size and are incomparable by inclusion. Therefore, every vertex must receive its own color in a linear coloring. This proves the following theorem.

Theorem 4.7. *If G is k-regular and reduced, then $\lambda(G) = n(G)$.*

We can use Theorem 4.7 to easily find $\lambda(G)$ for graphs G that are k-regular and reduced.

As depicted in Figure 4.4, the cycle C_n is a 2-regular, connected graph with n vertices and may be drawn as a circle of vertices linked by edges.
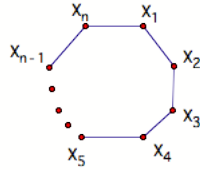

FIGURE 4.4. Cycle C_n

Example 4.8. If $n > 4$, then $\lambda(C_n) = n$.

Since $n > 4$, C_n is 2-regular and reduced. The result follows.

The computation of $\lambda(C_n)$ could have been made directly, similar to our treatment of the Petersen graph. However, it is much simpler to compute using the properties of k-regular, reduced graphs.

As illustrated in Figure 4.5, the *wheel,* denoted W_n, is a combination of the cycle C_n and a vertex x_c where $x_i \leftrightarrow x_c$ for all $x_i \in V(C_n)$. Note that $n \neq n(W_n)$, but rather, $n = n(C_n)$ and $n(W_n) = n + 1$. In the following example, we find the linear chromatic number for all wheels when $n \geq 5$.
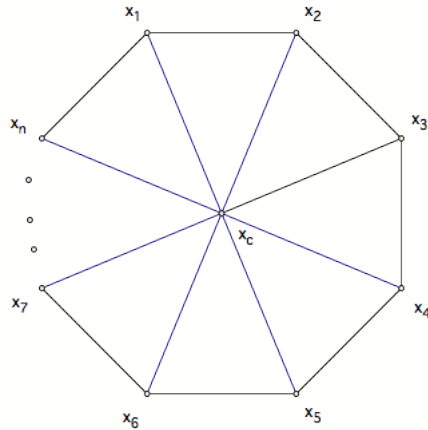

FIGURE 4.5. Wheel W_n

Example 4.9. If $n \geq 5$, then $\lambda(W_n) = n + 1$.

Let W_n be depicted as in Figure 4.5. Then

$$\mathcal{F}_{W_n} = \mathcal{N}_{W_n} = \{\{x_i, x_{i+2}, x_c\} \mid 1 \leq i \leq n\} \cup \{\{x_1, x_2, \ldots, x_n\}\},$$

(if $i = n-1$ or $i = n$, then we interpret $i+2$ as 1 or 2, respectively, i.e. using modular arithmetic). Thus,

$$\mathcal{F}_{W_n}(x_1) = \{\{x_1, x_3, x_c\}, \{x_1, x_{n-1}, x_c\}, \{x_1, x_2, \ldots, x_n\}\}$$
$$\mathcal{F}_{W_n}(x_2) = \{\{x_2, x_4, x_c\}, \{x_2, x_n, x_c\}, \{x_1, x_2, \ldots, x_n\}\}$$
$$\vdots$$
$$\mathcal{F}_{W_n}(x_{n-1}) = \{\{x_{n-1}, x_1, x_c\}, \{x_{n-1}, x_{n-3}, x_c\}, \{x_1, x_2, \ldots, x_n\}\}$$
$$\mathcal{F}_{W_n}(x_n) = \{\{x_n, x_2, x_c\}, \{x_n, x_{n-2}, x_c\}, \{x_1, x_2, \ldots, x_n\}\}$$

and

$$\mathcal{F}_{W_n}(x_c) = \{\{x_i, x_{i+2}, x_c\} \mid 1 \leq i \leq n\}.$$

We know that in order for $u, v \in V(W_n)$ to be assigned the same color, either $\mathcal{F}_{W_n}(u) \subseteq \mathcal{F}_{W_n}(v)$ or $\mathcal{F}_{W_n}(v) \subseteq \mathcal{F}_{W_n}(u)$. The center vertex x_c cannot share its assigned color with any other vertex in W_n because $x_c \leftrightarrow x_i$, for $1 \leq i \leq n$. The $\mathcal{F}_{W_n}(x_i)$'s, where $1 \leq i \leq n$, are incomparable by inclusion. Thus, the x_i's must be assigned distinct colors. Therefore, every vertex in a wheel must be assigned its own color, and hence, $\lambda(W_n) = n + 1$.

The *hypercube*, denoted Q_n, is a simple graph whose vertices are n-tuples with entries in $\{0, 1\}$ where $v \leftrightarrow u$ whenever u differs from v in exactly one entry. The hypercube is illustrated in Figure 4.6.
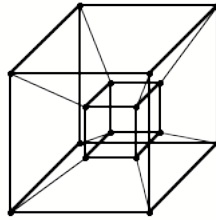

FIGURE 4.6. Hypercube Q_n

Lemma 4.10. *For all $x \neq y \in V(Q_n)$, where $n \geq 3$, $\mathcal{N}(x) \neq \mathcal{N}(y)$.*

Proof. Let $x \in V(Q_n)$. After relabeling the vertices, we may assume that $x = (0, 0, \ldots, 0)$. Suppose that there is a $y \in V(Q_n)$ such that $\mathcal{N}(y) = \mathcal{N}(x)$. Then $y \not\leftrightarrow x$ and $d(x, y) = 2$. Hence, y differs from x by exactly two entries and y has exactly two entries equal to 1. Since $n \geq 3$, choose $w \in V(Q_n)$ such that $w \leftrightarrow y$ and w has exactly three entries equal to 1. Then $w \in \mathcal{N}(y)$. But w differs from x by

three entries; therefore, $w \notin \mathcal{N}(x)$. This is a contradiction. Therefore, $\mathcal{N}(x) \neq \mathcal{N}(y)$ as claimed. □

Example 4.11. If $n \geq 3$, then $\lambda(Q_n) = 2^n$.

By Lemma 4.10, Q_n is reduced. Since Q_n is n-regular, by Theorem 4.7, $\lambda(Q_n) = 2^n$.

A *complete bipartite graph*, denoted $K_{m,n}$, is a graph with partite sets $M \subseteq V(K_{m,n})$ and $N \subseteq V(K_{m,n})$ of size m and n respectively, such that no two vertices within each set are adjacent. Further, every vertex in M is adjacent to every vertex in N. An example of a complete bipartite graph is illustrated in Figure 4.7.
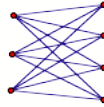

FIGURE 4.7. Complete Bipartite Graph

We compute $\lambda(K_{m,n})$ using the following lemma whose proof can be found in [1].

Lemma 4.12. *A coloring* $\kappa : V(G) \to [k]$ *of* G *is a k-linear coloring of* $\mathcal{N}(G)$ *if either* $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ *or* $\mathcal{N}(u) \subseteq \mathcal{N}(v)$ *holds for every* $x, y \in V(G)$ *with* $\kappa(x) = \kappa(y)$.

Example 4.13. Let $K_{m,n}$ be a complete bipartite graph. Then $\lambda(K_{m,n}) = 2$.

Note that $\chi(G) = 2$. Since $\chi(G) \leq \lambda(G)$, we have $\lambda(G) \geq 2$. Further, since every vertex in a partite set has the same neighborhood, by Lemma 4.12, we may give them the same color. Thus, $\lambda(G) \leq 2$. Therefore, $\lambda(G) = 2$.
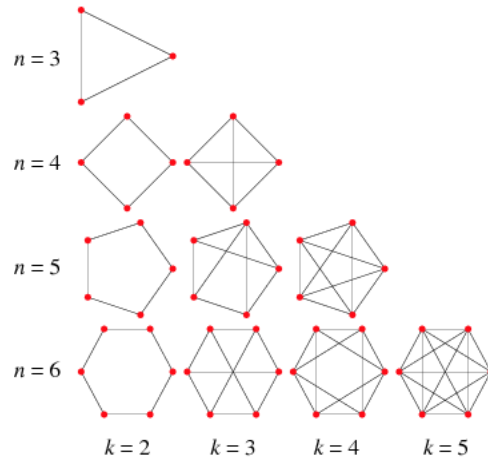

FIGURE 4.8. Harary Graph

Next, we study the *Harary graph* $H_{k,n}$, illustrated in Figure 4.8. Given $2 \leq k < n$, place n vertices around a circle, equally spaced. If k is even, form $H_{k,n}$ by making

each vertex be adjacent to the nearest $\frac{k}{2}$ vertices in each direction around the circle. If k is odd and n is even, form $H_{k,n}$ by making each vertex be adjacent to the nearest $\frac{k-1}{2}$ vertices in each direction and to the diametrically opposite vertex. When k and n are both odd, index the vertices by the integers modulo n. Construct $H_{k,n}$ from $H_{k-1,n}$ by adding the edges $\{i, i + \frac{n-1}{2}\}$ for $0 \leq i \leq \frac{n-1}{2}$. We compute $\lambda(H_{k,n})$ when at least one of k and n is even.

Label the vertices of $H_{k,n}$ sequentially along the outer cycle by $0, \ldots, n-1$. If either k or n or both are even, then $H_{k,n}$ is regular. So for any two vertices $u, v \in V(H_{k,n})$ to have the same color, $\mathcal{F}_{H_{k,n}}(u) = \mathcal{F}_{H_{k,n}}(v)$. Hence, by Lemma 4.6, $\mathcal{N}(u) = \mathcal{N}(v)$. If κ is a linear coloring and $\kappa(u) = \kappa(v)$, then $v \notin \mathcal{N}(u)$ and $u \notin \mathcal{N}(v)$. If, however, k and n are both odd, $H_{k,n}$ is not regular so these properties no longer hold.

Example 4.14. If k is even, $\lambda(H_{k,n}) = n$, unless $k = n - 2$ and n is even, in which case

$$\lambda(H_{k,n}) = \frac{n}{2}.$$

Let $\kappa : V(H_{k,n}) \to [\lambda(H_{k,n})]$ be a linear coloring. If $\kappa(a) = \kappa(b)$ for some vertices a and b with $0 \leq a < b \leq n - 1$, then by Lemma 4.6, $\mathcal{N}(a) = \mathcal{N}(b)$. In particular, $b + 1$ and $b - 1$ (interpreted modulo n) are neighbors of a, but b is not. Thus,

$$b - 1 - a \equiv \frac{k}{2} \bmod n$$

and

$$a - (b + 1) \equiv \frac{k}{2} \bmod n.$$

Hence, $-2 \equiv k \bmod n$. Since $2 \leq k < n$, we have $k = n - 2$.

Example 4.15. If k is odd and n is even, then $\lambda(H_{k,n}) = n$, unless $k = 3$ and $n = 6$, in which case $\lambda(H_{k,n}) = 2$.

Let $\kappa : V(H_{k,n}) \to \lambda(H_{k,n})$ be a linear coloring. If $\kappa(a) = \kappa(b)$ for some vertices a and b with $0 \leq a \leq b \leq n - 1$, then by Lemma 4.6, $\mathcal{N}(a) = \mathcal{N}(b)$. Note that if $k = n - 1$, then $H_{k,n}$ is a complete graph and the result is obvious, so (recalling that k is odd and n is even) we assume henceforth that $k \leq n - 3$. Suppose first that $n \geq 8$. By construction, there exists a unique vertex $c \in \mathcal{N}(a)$ such that neither $c - 1$ nor $c + 1$ (interpreted modulo n) is a neighbor of a; in fact,

$$d \equiv b + \frac{n}{2} \bmod n.$$

Similarly, there exists a unique vertex $d \in \mathcal{N}(b)$ such that neither $d - 1$ nor $d + 1$ (interpreted modulo n) is a neighbor of b, with

$$d \equiv b + \frac{n}{2} \bmod n.$$

Since $\mathcal{N}(a) = \mathcal{N}(b)$, we have $c = d$, and hence, $a \equiv b \bmod n$. Since $0 \leq a \leq b \leq n - 1$, we have $a = b$. Thus, $\lambda(H_{k,n}) = n$. The only remaining case is $n = 6$, $k = 3$; in this case, $H_{k,n}$ is a complete bipartite graph, and so, $\lambda(H_{3,6}) = 2$.

After several computations we notice lower and upper bounds for linear chromatic numbers. We prove a lower bound using the following lemma.

Lemma 4.16. *Let* $\kappa : V \to [\lambda(G)]$ *be a linear coloring of a graph* G. *Then for any color class* C_k, $|C_k| \le \Delta(G)$.

Proof. Let $\kappa : V \to [\lambda(G)]$ be a linear coloring of G and let C_k be a color class. Since κ is linear, we have

$$\mathcal{F}_G(v_1) \subseteq \cdots \subseteq \mathcal{F}_G(v_m)$$

where $C_k = \{v_1, v_2, \ldots, v_m\}$. Since, $\mathcal{F}_G(v_1) \subseteq \mathcal{F}_G(v_i)$ for $1 \le i \le m$, there is a facet F containing each v_i. By Lemma 2.1 , $F = \mathcal{N}(u)$ for some $u \in V(G)$. Hence, $m \le \deg(u)$. Therefore, $|C_k| = m \le \deg(u) \le \Delta(G)$. $\qquad \square$

We now prove a lower bound for $\lambda(G)$.

Theorem 4.17. *Let* G *be a simple graph. Then*

$$\frac{n(G)}{\Delta(G)} \le \lambda(G).$$

Proof. Let $\kappa : V \to [\lambda(G)]$ be a linear coloring of G. For each color class C_k, we have $|C_k| \le \Delta(G)$. There are $\lambda(G)$ color classes. Thus,

$$\sum_{k=1}^{\lambda(G)} |C_k| \le \lambda(G)\Delta(G).$$

Since every vertex belongs to exactly one color class, $\sum_{k=1}^{\lambda(G)} |C_k| = n(G)$. Therefore, $n(G) \le \lambda(G)\Delta(G)$ and

$$\frac{n(G)}{\Delta(G)} \le \lambda(G).$$

$\qquad \square$

Theorem 4.18. *Let* $u, v \in V(G)$ *such that* $d(u,v) = diam(G)$. *If* P *is a path of length* $d(u,v)$, *then* $\lambda(G) \ge \lambda(P)$.

Proof. Suppose not, and let $\kappa : V(G) \to [\lambda(G)]$ be linear coloring. Let $V(P) = \{x_1, \ldots, x_n\}$. Since $\lambda(G) < \lambda(P)$, we may color P in G linearly with fewer colors than our optimal linear coloring of P. We can do no worse than our optimal coloring in Example 4.3, so assume that $n \ge 6$ and $\kappa(x_i) = \kappa(x_j)$ where $2 < i < j < n - 1$. Then either $\mathcal{F}_G(x_i) \subseteq \mathcal{F}_G(x_j)$ or $\mathcal{F}_G(x_j) \subseteq \mathcal{F}_G(x_i)$. Thus, $x_i \leftrightarrow x_j$, and there exists a vertex, w, such that $x_i \leftrightarrow w$, $x_j \leftrightarrow w$, and $w \notin P$. Otherwise, x_i, x_j, and, w are contained in our optimal coloring and P, and cannot use fewer colors. Since P is a u, v path of distance $diam(G)$, $w \nleftrightarrow t$ for all $t \in P$. Further, by 4.2, x_i and x_j have exactly one vertex $z \in V(P)$ between them. There are two cases; either $w \leftrightarrow z$ or $w \nleftrightarrow z$. In each case, $x_i \leftrightarrow w$, $x_j \leftrightarrow w$, $x_i \leftrightarrow z$, and $x_j \leftrightarrow z$.
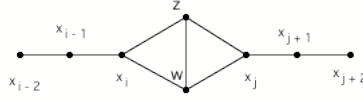
FIGURE 4.9. Illustration of Possible Cases

Recall that P is a path for which we earlier showed the optimal coloring. Now $\{\{x_{i-2}, x_i\}\} = \mathcal{F}_G(x_i)$. But $x_j \neq x_{i-2}$. Therefore, $\mathcal{F}_G(x_i) \not\subseteq \mathcal{F}_G(x_j)$. An analogous argument shows that $\mathcal{F}_G(x_j) \not\subseteq \mathcal{F}_G(x_i)$. This is a contradiction. $\square$

To prove an upper bound, we need more tools. A partially ordered set (or poset) consists of a set S together with a relation $\preceq$ that is reflexive, antisymmetric, and transitive. A chain in a poset is a subset $C \subseteq S$ such that for all $x, y \in C$ either $x \preceq y$ or $y \preceq x$.

An important example of a poset is the power set $\mathcal{P}(X)$ of a set X, with order relation defined by containment; that is, $S \preceq T$ if and only if $S \subseteq T$. An often useful property of this poset is the following.

Lemma 4.19. ([3]) *Let X be a finite set of size n. There exists a partition of $\mathcal{P}(X)$ into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains.*

With this lemma, we prove an upper bound for $\lambda(G)$.

Theorem 4.20. *Let G be a graph. Then*

$$\lambda(G) \leq \binom{|\mathcal{F}_G|}{\lfloor \frac{|\mathcal{F}_G|}{2} \rfloor}.$$

Proof. Let $\mathcal{F}_G = \{F_1, \ldots, F_k\}$. By Lemma 4.19, $\mathcal{P}(\mathcal{F}_G)$ can be partitioned into chains $C_1, \ldots, C_m$, where $m = \binom{k}{\lfloor \frac{k}{2} \rfloor}$. In particular, for each subset $S \subseteq \mathcal{F}_G$ there exists a unique index $i(S)$, $1 \leq i(S) \leq m$, such that S belongs to the chain C_i. Then the coloring $\alpha : V(G) \to [k]$ defined by $\alpha(v) = i(\mathcal{F}_G(v))$, is linear. Therefore, $\lambda(G) \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$. $\square$

The upperbound on $\lambda(G)$ is tight, as we note in Example 4.21.
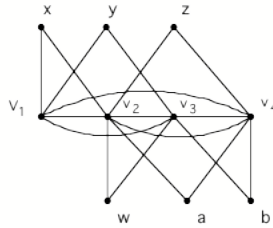
Example 4.21. See Figure 4.

We note that

$$\mathcal{F}_G = \{\{v_2, v_3, v_4, x, y, z\}, \{v_1, v_3, v_4, x, w, a\}, \{v_1, v_2, v_4, y, w, b\}, \{v_1, v_2, v_3, b, a, z\}\}.$$

Thus,

$$\mathcal{F}_G(v_1) = \{\{v_1, v_3, v_4, x, w, a\}, \{v_1, v_2, v_4, y, w, b\}, \{v_1, v_2, v_3, b, a, z\}\}$$

$$\mathcal{F}_G(v_2) = \{\{v_2, v_3, v_4, x, y, z\}, \{v_1, v_2, v_4, y, w, b\}, \{v_1, v_2, v_3, b, a, z\}\}$$

$$\mathcal{F}_G(v_3) = \{\{v_2, v_3, v_4, x, y, z\}, \{v_1, v_3, v_4, x, w, a\}, \{v_1, v_2, v_3, b, a, z\}\}$$

$$\mathcal{F}_G(v_4) = \{\{v_2, v_3, v_4, x, y, z\}, \{v_1, v_3, v_4, x, w, a\}, \{v_1, v_2, v_4, y, w, b\}\}$$

$$\mathcal{F}_G(x) = \{\{v_2, v_3, v_4, x, y, z\}, \{v_1, v_3, v_4, x, w, a\}\}$$

$$\mathcal{F}_G(y) = \{\{v_2, v_3, v_4, x, y, z\}, \{v_1, v_2, v_4, y, w, b\}\}$$

$$\mathcal{F}_G(z) = \{\{v_2, v_3, v_4, x, y, z\}, \{v_1, v_2, v_3, b, a, z\}\}$$

$$\mathcal{F}_G(w) = \{\{v_1, v_3, v_4, x, w, a\}, \{v_1, v_2, v_4, y, w, b\}\}$$

$$\mathcal{F}_G(a) = \{\{v_1, v_3, v_4, x, w, a\}, \{v_1, v_2, v_3, b, a, z\}\}$$

$$\mathcal{F}_G(b) = \{\{v_1, v_2, v_4, y, w, b\}, \{v_1, v_2, v_3, b, a, z\}\}.$$

The sets of facets containing x, y, z, w, a and b are incomparable by inclusion. Therefore, each of these vertices must be assigned its own color in a linear coloring. Hence, $\lambda(G) \geq 6$. Then by the previous theorem,

$$\lambda(G) \leq \binom{|\mathcal{F}_G|}{\lfloor \frac{|\mathcal{F}_G|}{2} \rfloor} = \binom{4}{2} = 6.$$

References

[1] Y. Civan, E. Yalcin, *Linear Colorings of Simplicial Complexes and Collapsing*, to appear in J. Combin. Theory, Ser. A, 2007.
[2] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.
[3] N. G. de Bruijn, C. A. v. E. Tengbergen, and D. Kruyswijk. On the set of divisors of a number. *Nieuw Arch. Wisk.* **23** (1951), 191-193.
[4] G. Baser, D. Taylan, *Linear N-Graphs*, Rose-Hulman Institute of Technology, Undergraduate Math Journal, Vol. 8, Issue 1, 2007.
[5] West, Douglas. *Graph Theory*. Prentice-Hall, 2000.
[6] Weisstein, Eric W. "Harary Graph." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/HararyGraph.html.