Betti Numbers of Splittable Graphs

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Abstract

We examine the edge and secant ideals of two types of splittable graphs - the wheel and the complete tripartite graph. With use of the computer software `Macaulay2` and a splitting function we are able to compute equations for the betti numbers of the edge and secant ideals of wheels and secant ideals of tripartite graphs.

1 Introduction

We begin this paper as we began our research, with an introduction to important terms useful in understanding our results. Knowledge of *modules*, *exact sequences*, *resolutions*, *betti numbers*, and *graph theory*, among a few other concepts, will prove to be very useful. We will apply this background knowledge to the types of graphs we chose to focus on: the *wheel graph* and *complete tripartite graph*. We will look at the *edge ideals* and *secant ideals* generated by these graphs and will use patterns discovered in *betti diagrams*, along with a *splitting function*, to formulate generalized equations for the betti numbers of the wheel and complete tripartite graph. It is important to note here that all rings discussed in this paper are commutative and have an identity element.

1.1 Modules

All of the following definitions can be found in any graduate-level algebra text, for example [2] or [3].

Definition 1.1. *If R is a ring, then an **R-module** is an abelian group $(M,+)$ together with an action of R, that is, a map $R \times M \longrightarrow M$, written $(r,m) \longmapsto rm$, satisfying for all $r,s \in R$ and $m,n \in M$:*

(i.) $r(sm) = (rs)m$

(ii.) $r(m+n) = rm + rn$

(iii.) $(r+s)m = rm + sm$

(iv.) $1m = m$

Definition 1.2. *Let R be a ring and let M be an R-module. An R**-submodule** of M is a subgroup N which is closed under the action of ring elements, i.e., $rn \in N$, for all $r \in R$, $n \in N$*

Definition 1.3. *Let M be an R-module. A subset $S \subseteq M$ **spans** M if every $m \in M$ can be written $m = \sum_{i=1}^{n} r_i s_i$ for some $n \geq 1$, $r_i \in R$ and $s_i \in S$, $i = 1, \ldots, n$. A subset $S \subseteq M$ is called **linearly independent** if whenever $r_1, \ldots, r_n \in R$ and $s_1, \ldots, s_n \in S$ satisfy, $r_1 s_1 + \ldots + r_n s_n = 0$, it follows that $r_1 = \ldots = r_n = 0$. A **basis** for M is a subset that is both linearly independent and spans M. An R-module M is called **free** if it has a basis.*

Example $\mathbb{Z}/4\mathbb{Z}$ is not a free module because no subset of it is linearly independent and therefore $\mathbb{Z}/4\mathbb{Z}$ has no basis. However, R^n is a free R-module for any ring R and any integer n because it has a basis, namely $\{(1,0,...,0),(0,1,0,...,0),...,(0,0,...,1)\}$.

Definition 1.4. *Let R be a ring and let M and N be R-modules. A map $\phi : M \longrightarrow N$ is an **R-module homomorphism** if it respects the R-module structures of M and N, i.e.,*

(a) $\phi(x+y) = \phi(x) + \phi(y)$, *for all $x,y \in M$ and*

(b) *$\phi(rx) = r\phi(x)$, for all $r \in R$, $x \in M$.*

*The **kernel** of some ϕ is defined by ker $\phi = \{m \in M\colon \phi(m) = 0\}$. The **image** of some ϕ is defined by im $\phi = \{n \in N\colon n = \phi(m) \text{ for } m \in M\}$.*

1.2 Exact Sequences and Resolutions

Definition 1.5. *The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be **exact** (at Y) if image α = ker β. A sequence*

$$\cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots$$

*of homomorphisms is said to be an **exact sequence** if it is exact at every X_n between a pair of homomorphisms.*

Definition 1.6. *A **homomorphism of exact sequences** is a collection $\{\psi_i\}_{i\in\mathbb{Z}}$ of module homomorphisms such that each square in the following diagram commutes:*

$$\begin{tikzcd}
... \arrow[r] & R \arrow[r] \arrow[d, "\psi_1"'] & R_1 \arrow[r] \arrow[d, "\psi_2"'] & ... \arrow[r] & R_n \arrow[r] \arrow[d, "\psi_n"'] & 0 \\
... \arrow[r] & R' \arrow[r] & R_1' \arrow[r] & ... \arrow[r] & R_n' \arrow[r] & 0
\end{tikzcd}$$

Definition 1.7. *Let R be a ring and let N be an R-module. A **resolution** of N is an exact sequence*

$$\cdots \xrightarrow{\phi_2} M_1 \xrightarrow{\phi_1} M_0 \xrightarrow{\phi_0} N \to 0$$

of R-modules and R-module homomorphisms.

1.3 Graded Rings and R-Modules

Definition 1.8. *A **graded ring** is a ring R together with a direct sum decomposition,*

$$R \cong R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

as abelian groups such that if $x \in R_i$ and $y \in R_j$ then $xy \in R_{i+j}$ for all $i, j \geq 0$.

Example Let $\mathbb{K}$ be a field. Then the polynomial ring $R = \mathbb{K}[x_0, \ldots, x_n]$ is a graded ring, in which R_i consists of homogeneous elements of total degree i, i.e. sums of monomials, each of total degree i.

Definition 1.9. *If R is a ring, a **graded R-module** is an R-Module M together with a decomposition,*

$$M \cong \bigoplus_{n=0}^{\infty} M_n$$

*as abelian groups, such that $R_i \cdot M_j \subseteq M_{j+i}$ for all i and j. If M and N are graded R-modules, **a graded R-module homomorphism** is an R-module homomorphism $\phi : M \longrightarrow N$ such that $\phi(M_i) \subseteq N_i$ for all i.*

Definition 1.10. *Let R be a graded ring and M be a graded R-module. For an integer j we define the **jth twist** of M, denoted $M(j)$, by $M(j)_n = M_{j+n}$. This is read: "M twisted by j."*

Example Let $R = \mathbb{K}[x_1, ..., x_n]$. If $M = R(-3)$ then the twist causes the constants in M_0 to have degree 3, $x_1, ..., x_n \in M_1$ to have degree 4, $x_1^2, ..., x_n^2 \in M_2$ to have degree 5, $x_i x_j \in M_2$ to also have degree 5 and so on.

1.4 Betti Numbers and Betti Diagrams

Definition 1.11. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $\mathfrak{m} = (x_1, \ldots, x_n)$ its unique maximal ideal. A free resolution of an R-module M*

$$\cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

*is called **minimal** if for all $i \geq 1$ $\varphi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$.*

Theorem 1.12. *[3] Let S be the polynomial ring in finitely many variables over a field $\mathbb{K}$, and M a finitely generated graded S-module. Any two minimal free resolutions of M are isomorphic.*

This theorem leads us to defining *betti numbers* and *betti diagrams*. Both are key to our research as well as useful when looking at minimal free resolutions.

Definition 1.13. *Let M be an R-module and*

$$\ldots \to F_i \to F_{i-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

*its minimal free resolution. Since each resolvent F_i is a free R-module, it may be written $F_i \cong \bigoplus_j R(-j)^{\beta_{i,j}}$ for some integers $\beta_{i,j}$ known as **betti numbers**.*

Betti numbers are important later on in this paper as we begin discussing our research. They give us a way to generalize resolutions.

Definition 1.14. *The **betti diagram**, introduced by Bayer and Stillman[1], is a table displaying the betti numbers of a minimal free resolution in the following format:*

$$\begin{array}{cccc}
\beta_{0,0} & \beta_{1,1} & \cdots & \beta_{i,i} \\
\beta_{0,1} & \beta_{1,2} & \cdots & \beta_{i,i+1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{0,j} & \beta_{1,j+1} & \cdots & \beta_{i,j+1}
\end{array}$$

Such diagrams helped us to discover patterns in the betti numbers of minimal free resolutions.

1.5 Graph Theory

Definition 1.15. *A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a set, called the* ***vertices*** *of G, and $E(G)$ is a set of unordered pairs of distinct vertices of $V(G)$, called the* ***edges*** *of G.*

Definition 1.16. *If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are graphs, H is a* ***subgraph*** *of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(H)$. Given $S \subseteq V(G)$, the subgraph of G* ***induced*** *by S, denoted G_S, is the graph defined by $V(G_S) = S$ and $xy \in E(G_S)$ if and only if $x, y \in S$ and $xy \in E(G)$.*

Definition 1.17. *An edge e is* ***incident*** *to vertices u and v if e is and edge uv. Two vertices are* ***adjacent*** *if connected by an edge. A* ***walk*** *is an alternating sequence of vertices and edges in which each vertex is incident to the edge before and after it in the sequence. A* ***cycle*** *is a walk that starts and ends at the same vertex with no other repeated vertices or repeated edges.*

A *complete bipartite graph*, denoted $K_{n,m}$, has two sets of n and m vertices in which each vertex in one disjoint set is adjacent to every vertex in the other set, but is not adjacent to any other vertex in the same set. A *star*, S_j, is the complete bipartite graph $K_{1,j}$ with one disjoint set containing one central vertex and the other set containing j outer vertices. A *cycle* of n vertices, C_n, is a graph having only one cycle of length n. A *wheel*, W_n, is obtained by adding to C_n a new vertex which is adjacent to all other vertices.

Definition 1.18. ***Vertex coloring*** *of a graph is an assignment of "colors" to each vertex of the graph so that no two adjacent vertices share the same color. The* ***chromatic number*** *of a graph G, denoted $\chi(G)$, is the minimum number of colors required to color the vertices of a graph. A graph is* ***r-colorable*** *if its vertices can be colored with r colors.*

1.6 Edge and Secant Ideals

Definition 1.19. *Let G be a graph with vertex set $\{x_0, x_1, \cdots x_n\}$. The* ***edge ideal*** *of G, denoted $I(G)$, is the ideal of $\mathbb{K}[x_0, \ldots, x_n]$ generated by $x_i x_j$ where x_i and x_j are adjacent vertices of G and $0 \leq i < j \leq n$.*

Definition 1.20. *[8] The* ***join*** *of I and J is the ideal $I * J$ in $\mathbb{K}[x_0, \ldots, x_n, y_0, \ldots, y_n, z_0, \ldots, z_n]$ defined by taking the ideal generated by $I(\mathbf{y})$, $J(\mathbf{z})$ and $x_i - y_i - z_i$, $i = 0, \ldots, n$ and intersecting it with the subring $\mathbb{K}[x_0, \ldots, x_n]$. Here $I(\mathbf{y})$ (respectively, $J(\mathbf{z})$) denotes the ideal I (respectively, J) with each x_i replaced by y_i (respectively, z_i). The* ***rth secant ideal*** *$I \subset \mathbb{K}[x_1, ..., x_n]$ is defined by*

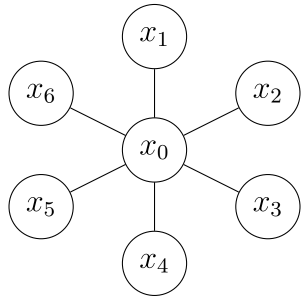

Figure 1: Star S_6

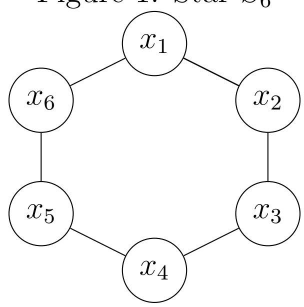

Figure 2: Cycle C_6

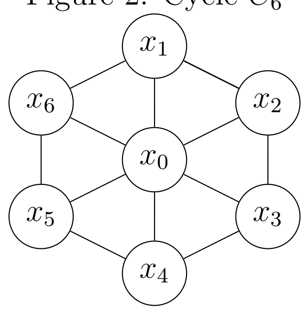

Figure 3: Wheel W_6

(1) $I^{\{1\}} = I$.

(2) $I^{\{r\}} = I * I^{\{r-1\}}$ *for* $r > 1$.

We can relate the rth secant ideal of a graph to the graph's colorability. Computing the generators of the secant ideal is made easy by the following result.

Let G be a graph, V be a subset of the vertices of G, and G_V be the induced subgraph of G with vertex set V. Also, let $\chi(G)$ be the chromatic number of the vertex coloring of G.

Theorem 1.21. *(Sturmfels-Sullivant)[8] The rth secant* $I(G)^{\{r\}}$ *of an edge ideal I(G)* *is generated by the squarefree monomials $\prod_{i \in V} x_i$ for any V whose subgraph G_V is not r-colorable:*

$$I(G)^{\{r\}} = \langle \prod_{i \in V} x_i : \chi(G_V) > r \rangle.$$

The minimal generators of $I(G)^{\{r\}}$ are those monomials $\prod_{i \in V} x_i$ such that G_V is not r-colorable but G_U is r-colorable for every proper subset $U \subset V$.

Remark Note that an n-cycle has chromatic number 2 when n is even and 3 when n is odd. For W_n, when n is even, the subgraph induced by exclusively outer vertices is minimally 2-colorable. Any induced subgraph containing the central vertex and any other two vertices must contain a triangle so it is minimally 3-colorable. For W_n, when n is odd, the subgraph induced by all outer vertices, as well as any induced subgraph containing the central vertex and any other two vertices, is *minimally* 3-colorable.

2 Splitting the Edge Ideal of W_n

To standardize notation for the wheel graph, we will henceforth refer to the central vertex as x_0, and the outside vertices as x_1 through x_n consecutively and clockwise.

Definition 2.1. ***(Eliahou-Kervaire)*** *[4] Let I be a monomial ideal in a polynomial ring R, i.e., an ideal whose generators are monomials. I is splittable if I is the sum of two nonzero monomial ideals J and K, that is $I = J + K$, such that*

(1) *The minimal generating set $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$, and*

(2) *there is a splitting function*

$$\mathcal{G}(J \cap K) \longrightarrow \mathcal{G}(J) \times \mathcal{G}(K)$$
$$w \longmapsto (\phi(w), \psi(w))$$

such that:

(i) *for all $w \in \mathcal{G}(J \cap K), w = lcm(\phi(w), \psi(w))$, and*

(ii) *for every subset $S \subset \mathcal{G}(J \cap K)$, both $lcm(\phi(S))$ and $lcm(\psi(S))$ strictly divide $lcm(S)$.*

Theorem 2.2. ***(Fatabbi)*** *[5] Suppose I is a splittable monomial ideal with splitting $I = J + K$. Then*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

$$\textit{for all } i, j \geq 0, \textit{where } \beta_{i-1,j}(J \cap K) = 0 \textit{ when } i = 0.$$

Armed with Definition 2.1, we will show that our wheel can be split into a star graph and an n-cycle. We let $J = \langle x_0x_1, x_0x_2, \ldots, x_0x_n\rangle$ and $K = \langle x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\rangle$. To compute $J \cap K$, we note that $x_0x_ix_{i+1}$ is in both J and K by absorption. Thus $J \cap K = \langle x_0x_1x_2, x_0x_2x_3, \ldots, x_0x_{n-1}x_n, x_0x_nx_1\rangle$. However, we can rewrite

$$J \cap K = x_0\langle x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\rangle = x_0K$$

Lemma 2.3. *$J \cap K = x_oK$ is isomorphic to $K(-1)$.*

Proof: Define the map $f : K(-1) \longrightarrow J \cap K$ where $f(a) = x_0a$ for all $a \in K(-1)$. Note that any $a \in K(-1)$ can be written rx_ix_{i+1} for some $r \in R$. If there exists $f(a) = f(b)$, then $x_0a = x_0b$. Because x_0 is not a zero-divisor, we can cancel, and see that, whenever $f(a) = f(b)$, $a = b$. Then consider any $y \in J \cap K$. Then $y = x_0a$, for some $a \in K(-1)$. Thus $y = f(a)$ for all $y \in J \cap K$, so f is surjective.
We can note that $f(a + b) = x_0(a + b) = x_0a + x_0b = f(a) + f(b)$. Similarly, for any $r \in R$, $f(ra) = x_0ra = rx_0a = r \cdot f(a)$.
Finally, consider an element $r \in K(-1)_d$. Then $f(r) = x_0r$, which is an element of $J \cap K_d$. Therefore, f is a graded R-module isomorphism between K and $J \cap K$. $\square$

Theorem 2.4. *The edge ideal I of the wheel graph is splittable.*

To prove I is splittable, we now define our splitting function:

$$\mathcal{G}(J \cap K) \longrightarrow \mathcal{G}(J) \times \mathcal{G}(K)$$

$$x_0x_ix_{i+1} \longmapsto (x_0x_i, x_ix_{i+1}) \text{ for } 1 \leq i \leq n-1$$

$$x_0x_nx_1 \longmapsto (x_0x_1, x_nx_1) \text{ for } i = n$$

Proof: It is clear that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. For $w = x_0x_ix_{i+1}$ when $i \neq n$, $lcm(\phi(w), \psi(w)) = lcm(x_0x_i, x_ix_{i+1}) = w$. When $w = x_0x_nx_1$, $lcm(\phi(w), \psi(w)) = lcm(x_0x_1, x_nx_1) = w$.
For any $S \subset \mathcal{G}(J \cap K)$, we have $S = \{x_0x_{i_1}x_{i_1+1}, ..., x_0x_{i_k}x_{i_k+1}\}$. We can order these without loss of generality so that $i_1 < i_2 < ... < i_k$. We can then see that $lcm(S) = x_0lcm(x_{i_1}x_{i_1+1}, ..., x_{i_k}x_{i_k+1})$. If $x_0x_nx_1 \in S$, then $i_k = n$, so x_n divides $lcm(S)$, while it does not appear in $lcm(\phi(S)) = x_0x_{i_1}...x_{k_1-1}$, and $lcm(\phi(S))$ strictly divides $lcm(S)$. When $x_0x_nx_1 \notin S$, then x_{i_k+1} is found in $lcm(S)$, while it is not present in $lcm(\phi(S)) = x_0x_{i_1}...x_{i_k}$. Thus, in this case also, $lcm(\phi(S))$ also divides $lcm(S)$. In addition, for any $S \subset J \cap K$, x_0 is found in $lcm(S)$, while it is not in $lcm(\psi(S))$. Therefore, $lcm(\psi(S))$ strictly divides $lcm(S)$. With this, our function fulfills the criteria for a splitting function, and our ideal I is splittable. $\square$

The betti numbers for both J, the edge ideal of the star graph S, and K, the edge ideal of the n-cycle, are computed using Jacques' [7] formulae.

Theorem 2.5. *1. The edge ideal K of the n-cycle C_n has betti numbers, for $j < n$ and $2i \geq j$:*

$$\beta_{i,j}(K) = \frac{n}{n-2(j-i)}\binom{j-i}{2i-j}\binom{n-2(j-i)}{2i-j}$$

Remark *In the case that $j = n$, we have:*

$$\beta_{2m+1,n}(K) = 1 \text{ if } n = 3m+1$$

$$\beta_{2m+1,n}(K) = 1 \text{ if } n = 3m+2$$

$$\beta_{2m,n}(K) = 2 \text{ if } n = 3m$$

2. The edge ideal J of the star graph S_{n+1} has betti numbers:

$$\beta_{i,j}(J) = \binom{n}{i}$$

Combining these two formulae from Jacques [2.5] and adjusting Fatabbi's formula [2.2] for $\beta_{i,j}(J \cap K) = \beta_{i,j}(K(-1)) = \beta_{i,j-1}(K)$, we can thus compute the betti numbers for our edge ideal, where $2i + j > n$ and $i > 1$:

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j-1}(K)$$

$$= \binom{n}{i} + \frac{n}{n-2(j-i)}\binom{j-i}{2i-j}\binom{n-2(j-i)}{2i-j}$$

$$+\frac{n}{n-2(j-i)}\binom{j-i}{2i-j-1}\binom{n-2(j-i)}{2i-j-1}$$

3 Betti Numbers for secant ideal of W_n when n is even

Theorem 3.1. *When n is even the secant ideal of W_n is isomorphic as a graded R-module to the edge ideal of C_n.*

Proof: Let A be the secant ideal of W_n and B be the edge ideal of C_n. The secant ideal in this case is generated by $A = \langle x_0x_1x_2, x_0x_2x_3, \ldots, x_0x_nx_1\rangle$ and the edge idea of C_n $B = \langle x_1x_2, x_2x_3, \ldots, x_nx_1\rangle$

Now a bijective map can be produced between A and B.
Take

$$\phi : B \to A$$

$$y \longmapsto x_0 y$$

for all $y \in B$
The proof of bijectivity is very straightfoward.

Let $a, b \in B$ and $\phi(a) = \phi(b)$. Therefore $x_0 a = x_0 b$, so a=b. Thus this map is injective

Let $x_0 y \in A$. So if $y \in B$, $\phi(y) = x_0 y$. So this proves it is surjective.

Now all that is left is to prove that this map is a graded R-module homomorphism. So let $a, b \in B$. Then,

1. $\phi(a + b) = x_0(a + b) = x_0 a + x_0 b = \phi(a) + \phi(b)$.

2. If $r \in R$, $r\phi(a) = r x_0 a = \phi(ra)$.

So if $a \in B_d$ then $\phi(a) = x_0 a \in A(-1)_d$. This proves that the secant ideal of W_n is isomorphic as a graded R-module to the edge ideal of C_n when n is even. Now we can use [7] result to produce this theorem.

Theorem 3.2. $\beta_{i,j}(A) = \beta_{i,j}(B(1)) = \beta_{i,j+1}(B) = \frac{n}{n-2(j+1-i)}\binom{j+1-i}{2i-j+1}\binom{n-2(j+1-i)}{j+1-i}$

4 Splitting the Secant Ideal of W_n for the Odd Case

Given the definition of splittable function, we can see that the secant ideal is splittable for W_n, in the case that n is odd. Observe by Theorem 1.21 that when n is odd

$$I^{\{2\}} = \langle x_0x_1x_2, \ldots, x_0x_nx_1, x_1x_2\ldots x_n\rangle.$$

Then, we can split the secant ideal by letting $J' = \langle x_0x_1x_2, \ldots, x_0x_nx_1\rangle$ and $K' = \langle x_1x_2\ldots x_n\rangle$. Therefore, $J' \cap K' = \langle x_0x_1x_2\ldots x_n\rangle$.

Theorem 4.1. *The secant ideal $I^{\{2\}}$ of the wheel W_n is splittable when n is odd.*

Proof: Given J' and K', let $w \in \mathcal{G}(J' \cap K')$, so $w = x_0x_1x_2\ldots x_n$. Then, $\phi(w) \in \mathcal{G}(J')$ and $\psi(w) \in \mathcal{G}(K')$. So let $\phi(w) = x_0x_1x_2$ and $\psi(w) = x_1x_2\ldots x_n$. Then,

$$lcm(\phi(w), \psi(w)) = x_0x_1\ldots x_n = w.$$

Hence, the first condition of the splitting function is satisfied.

Now let $S \subseteq \mathcal{G}(J' \cap K')$. Note that in this case, S only contains one element. That is, $S = \{x_0x_1x_2 \ldots x_n\}$. Then, $lcm(S) = x_0x_1x_2 \ldots x_n$. We can see that $lcm(\phi(S)) = x_0x_1x_2$, which strictly divides $lcm(S)$. Also, $lcm(\psi(S)) = x_1x_2 \ldots x_n$, strictly divides $lcm(S)$. Thus, the second condition is satisfied, and so the secant ideal is splittable when n is odd. $\square$

Now that we know our secant ideal is splittable for n being odd, we can apply Theorem 2.2. Since $I^{\{2\}}$ is split into J' and K', then

$$\beta_{i,j}(I^{\{2\}}) = \beta_{i,j}(J') + \beta_{i,j}(K') + \beta_{i-1,j}(J' \cap K').$$

Note that $J' = \langle x_0x_1x_2, \ldots, x_0x_nx_1 \rangle = J \cap K = x_0K$, where J and K are our monomial ideals from Section 2. Thus, we have

$$\begin{aligned} \beta_{i,j}(J') = \beta_{i,j}(J \cap K) &= \beta_{i,j}(K(-1)) \\ &= \beta_{i,j-1}(K) \\ &= \frac{n}{n-2(j-i-1)} \binom{j-i-1}{2i-j-1} \binom{n-2(j-i-1)}{2i-j-1} \end{aligned}$$

for all $i, j \geq 1$.

Now, we must find the formulae for $\beta_{i,j}(K')$ and $\beta_{i-1,j}(J' \cap K')$.

As we explored the secant ideal of W_n for n being odd, we found isomorphisms between the polynomial ring R and our ideals.

Lemma 4.2. *A principal ideal, I, generated by an element of degree k is isomorphic to the graded R-module $R(-k)$.*

Proof: Let $R(-k)$ be a graded R-module of the polynomial ring, R and I be a principal ideal that is generated by an element of degree k, say z. That is, $I = \langle z \rangle$. Define a map f by

$$f : R(-k) \longrightarrow I$$
$$r \longmapsto r(z)$$

where $r \in R(-k)$. Let $r, s \in R(-k)$. Then, $f(r) = r(z)$ and $f(s) = s(z)$. Assume $f(r) = f(s)$. Then, $r(z) = s(z)$. Since z is not a zero-divisor, by cancellation $r = s$. Thus, f is injective.
Consider $r(z) \in I$. Then, it is clear there exists $r \in R(-n)$, such that $f(r) = r(z)$, so f is surjective.
Take $r, s \in R(-k)$ and $t \in R$. Then,

1. $f(r+s) = (r+s)(z) = r(z) + s(z) = f(r) + f(s)$

2. $f(tr) = (tr)(z) = t(r(z)) = tf(r)$.

Therefore, f is a graded R-module homomorphism.
Thus, $I \cong R(-k)$. □

Note that $K' = \langle x_1 x_2 \ldots x_n \rangle$ is generated by an element of degree n, thus by Lemma 4.2, K' is isomorphic to $R(-n)$. Since $R(-n)$ has a trivial minimal free resolution, the module in the first position is twisted by $-n$ and has an exponent of 1. Thus,

$$\beta_{1,n} = 1$$

where n is odd. Because $K' \cong R(-n)$, we now have a formula for the betti numbers of K'. That is,

$$\beta_{i,j}(K') = \begin{cases} 1, & \text{if } i = 1, j = n \\ 1, & \text{if } i \geq 1, j \neq n. \end{cases}$$

Finally, we need to find the formula for the betti numbers of $J' \cap K'$. Note that $J' \cap K' = \langle x_0 x_1 x_2 \ldots x_n \rangle$ is generated by an element of degree $n+1$, thus by Lemma 4.2, $J' \cap K'$ is isomorphic to $R(-n-1)$. The module in the first position of the minimal free resolution of $R(-n-1)$ is twisted by $-n-1$ and has an exponent of 1. Thus,

$$\beta_{1,n+1} = 1$$

where n is odd. Because $J' \cap K' \cong R(-n-1)$, we now have a formula for the betti numbers of $J' \cap K'$. That is,

$$\beta_{i,j}(J' \cap K') = \begin{cases} 1, & \text{if } i = 1, j = n+1 \\ 1, & \text{if } i \geq 1, j \neq n+1. \end{cases}$$

We now have a formula for each of our monomial ideals, J', K', and $J' \cap K'$, so we can apply Theorem 2.2.

Theorem 4.3. *Suppose $I^{\{2\}}$ is a splittable monomial ideal with $I^{\{2\}} = J' + K'$. Then, for all $i, j \geq 1$, where n is odd,*

$$\beta_{i,j}(I^{\{2\}}) = \beta_{i,j}(J') + \beta_{i,j}(K') + \beta_{i-1,j}(J' \cap K').$$
$$= \frac{n}{n - 2(j-i-1)} \binom{j-i-1}{2i-j-1} \binom{n-2(j-i-1)}{2i-j-1} + \epsilon_{i,j}$$

where $\epsilon_{i,j} = 1$ when $i = 1$ and $j = n$, or $i = 2$ and $j = n+1$, or $\epsilon_{i,j} = 0$ otherwise.

5 Complete Tripartite

Definition 5.1. *A complete tripartite graph, denoted $K_{(l,m,n)}$, has 3 disjoint sets of l, m, n vertices in which each vertex in one disjoint set is connected to every vertex in the other 2 sets, but has no connections to vertices in its own set.*

Theorem 5.2. *If I is the edge ideal of $K_{l,m,n}$, then the $\mathbb{N}$-graded Betti numbers of $I^{\{2\}}$ are given by*

$$\beta_{i,i+1}(I^{\{2\}}) = \sum_{d+e=i+2,\ d,e\geq 1} [\binom{l}{d}\binom{m}{e} + \binom{m}{d}\binom{n}{e} - \binom{n}{d}\binom{l}{e} - \binom{m}{d}\binom{l+n}{e}]$$

$$+2 \sum_{d+e+f=i+2,\ d,e,f\geq 1} \binom{l}{d}\binom{m}{e}\binom{n}{f}$$

$\beta_{i,j}(I^{\{2\}}) = 0$ *for all* $j \neq i+1$

Proof: Consider the edge ideal, I, of a complete tripartite graph with partition x of size l, y of size m, and z of size n, then $I = \langle x_i y_j,\ y_j z_h,\ z_h x_i \mid 1 \leq i \leq l,\ 1 \leq j \leq m,\ 1 \leq h \leq n\rangle$ Our goal is to split I into ideals J and K with known betti numbers such that $J \cap K = I^{\{2\}}$.
Let

$$J = \langle x_i y_j \ |1 \leq i \leq l,\ 1 \leq j \leq m\rangle$$

$$K = \langle x_i z_h,\ y_j z_h | 1 \leq i \leq l,\ 1 \leq j \leq m,\ 1 \leq h \leq n\rangle$$

Then $J \cap K = \langle x_i y_j z_h \ |1 \leq i \leq l,\ 1 \leq j \leq m,\ 1 \leq h \leq n\rangle = I^{\{2\}}$ because the only minimal induced subgraphs of I that are not 2-colorable are triangles with one vertex in each partition.

It is easy to see that the generators of I are a disjoint union of the generators of J and K. Define:

$$\phi : \mathcal{G}(J \cap K) \to \mathcal{G}(J) \quad \psi : \mathcal{G}(J \cap K) \to \mathcal{G}(K)$$

$$x_i y_j z_h \mapsto x_i y_j \qquad\qquad x_i y_j z_h \mapsto x_i z_h$$

1. $x_i y_j z_h = lcm(x_i y_j, x_i z_h)$
2. If $S = \{x_{i_1} y_{j_1} z_{h_1} ... x_{i_t} y_{j_t} z_{h_t}\}$ then $lcm(S) = lcm(x_{i_1}, ..., x_{i_t}) lcm(y_{j_1}, ..., y_{j_t}) lcm(z_{h_1}, ..., z_{h_t})$
   (a) $\phi(S) = \{x_{i_1} y_{j_1}, ..., x_{i_t} y_{j_t}\}$ so $lcm[\phi(S)] = lcm(x_{i_1}, ..., x_{i_t}) lcm(y_{j_1}, ..., y_{j_t})$
   (b) $\psi(S) = \{x_{i_1} z_{h_1}, ..., x_{i_t} z_{h_t}\}$ so $lcm[\phi(S)] = lcm(x_{i_1}, ..., x_{i_t}) lcm(z_{h_1}, ..., z_{h_t})$

So $lcm[\phi(S)]$ and $lcm[\psi(S)]$ both strictly divide the $lcm(S)$ and $\phi \times \psi$ is a splitting function of I.

Now we can shift the indices in Fatabbi's Theorem 2.2 to say that $\beta_{i,j}(I^{\{2\}}) = \beta_{i+1,j}(I) - \beta_{i+1,j}(J) - \beta_{i+1,j}(K)$. Note that J is a complete bipartite graph with partitions of size l and n, and that K is also a complete bipartite graph with partitions of size m and $l+n$. Sean Jacques [7] provides the following formula for the betti numbers of a complete bipartite graph with partitions of size x and y:

$$\beta_{i,i+1}(K_{x,y}) = \sum_{d+e=i+1,\ d,e\geq 1} \binom{x}{d}\binom{y}{e}$$

$$\beta_{i,j}(K_{x,y}) = 0 \text{ for all } j \neq i+1$$

And for complete tripartite graphs with partitions of size x, y, and z:

$$\beta_{i,i+1}(K_{x,y}) = \sum_{d+e=i+1,\ d,e\geq 1} \binom{x}{d}\binom{y}{e} + 2 \sum_{d+e+f=i+1,\ d,e,f\geq 1} \binom{x}{d}\binom{y}{e}\binom{z}{f}$$

$$\beta_{i,j}(K_{x,y,z}) = 0 \text{ for all } j \neq i+1$$

Combining these formula we obtain the desired result. □

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