SUMSRI 2018: Ryser's conjecture

Teresa Flaitz, Yigal Kamel, Cody Newman, Hannah Sheats

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Abstract

We survey some results on Ryser's conjecture. We make the case that by phrasing Ryser's conjecture in the language of edge colored graphs, it is possible to ask a variety of interesting strengthenings and generalizations of the original conjecture.

1 Introduction

Let G be a hypergraph. We say that G is r-uniform if every edge of G contains exactly r vertices. We say that G is r-partite if there exists a partition of $V(G)$ into sets $\{V_1, \ldots, V_r\}$ such that for every edge e of G, $|e \cap V_i| \leq 1$ for all $i \in [r]$; we use bipartite to mean 2-partite. A matching in G is a set of pairwise disjoint edges. A vertex cover of G is a set of vertices A such that each edge of G contains a vertex from A. We denote the size of a largest matching in G by $\nu(G)$ and we denote the size of a minimum vertex cover of G by $\tau(G)$. Note that for every hypergraph G we have

$$\nu(G) \leq \tau(G),$$

since a minimum vertex cover must contain at least one vertex from each edge in a maximum matching.

The following is a seminal result in graph theory from 1931.

Theorem 1.1 (König [14]). *For every bipartite graph G, $\tau(G) \leq \nu(G)$.*

In the 1970's, Ryser made the following conjecture which would generalize König's theorem to r-partite hypergraphs.

Conjecture 1.2 (Ryser (see [13])). *For every r-partite hypergraph graph G, $\tau(G) \leq (r-1)\nu(G)$.*

Let $r \geq 2$ and let A be a r-dimensional $0,1$-matrix. The term rank of A , denoted $\nu(A)$, is the maximum number of 1's, such that no pair is in the same $(r-1)$-dimensional hyperplane. The covering number of A, denoted $\tau(A)$, is the minimum number of $(r-1)$-dimensional hyperplanes which contain all of the 1's of A. In this language, Ryser's conjecture says that if A is an r-dimensional $0,1$-matrix, then $\tau(A) \leq (r-1)\nu(A)$; indeed, this is how Ryser's conjecture was originally formulated.

1.1 Duality

A *component* of a graph G is a maximal connected subgraph of G. If A is a set of vertices, let $G[A]$ denote the subgraph of G induced by A. A set of vertices A in G is *independent* if $G[A]$ has no edges. The size of a maximum independent set of vertices in G is denoted by $\alpha(G)$. An *r-coloring* of the edges of a graph G is a set $\{G_1, ..., G_r\}$ of spanning subgraphs of G whose edge sets partition $E(G)$. We say an edge $e \in G$ is *color-i* if $e \in E(G_i)$. In an r-colored graph G, a *component cover* of G is a set of monochromatic components of G whose union contains V. We say $tc_r(G) \leq k$ if in every r-coloring of the edges of G, there exists a component cover of size at most k.

Gyárfás [9] noted that Ryser's conjecture is equivalent to the following statement about edge colored graphs.

Conjecture 1.3 (Ryser)**.** *For any graph G and any integer $r \geq 2$, $tc_r(G) \leq (r-1)\alpha(G)$.*

So in particular, König's theorem can be reformulated as follows.

Theorem 1.4 (König)**.** *For any graph G, $tc_2(G) \leq \alpha(G)$.*

1.2 Known results

Aside from the case $r = 2$ which is König's theorem, Ryser's conjecture has only been verified in the following cases: $r = 3$ by Aharoni [2]; $r = 4$ and $\alpha = 1$ by Tuza [17]; $r = 5$ and $\alpha = 1$ by Tuza [17].

The following table highlights the known cases.

α \ r	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25
6	$\downarrow$	$\downarrow$	18	24	30

1.3 Lower bounds

For a given r, if there exists an affine plane of order $r-1$, there exists an example which shows that Ryser's conjecture is best possible. Since it is known that an affine plane of order $r-1$ exists whenever $r-1$ is a prime power (it is not known whether there exists an affine plane of non-prime-power order), the following is known.

Example 1.5. *For all $r \geq 2$, if $r-1$ is a prime power, then there exists a graph G such that $tc_r(G) \geq (r-1)\alpha(G)$.*

Finding matching lower bounds when $r-1$ is not a prime power is an active area of research ([3], [1], [12]); however, it is still unknown whether for all $r \geq 2$ there exists a graph G such that $tc_r(G) \geq (r-1)\alpha(G)$.

1.4 Large monochromatic components

The following result is known, but it is worth noting that it would be implied by Ryser's conjecture.

Theorem 1.6 (Furedi [8] (see Gyarfas [10]))**.** *In every r-partite hypergraph H with n edges, there exists a vertex of degree at least $\frac{n}{(r-1)\nu(H)}$.*

In other words, in every r-coloring of the edges of a graph G with n vertices, there exists a monochromatic component of order at least $\frac{n}{(r-1)\alpha(G)}$.

2 Special lower bounds

Example 2.1. *For all $r \geq 3$, and $n \geq 2r$ there exists an r-coloring of K_n such that every color class has at least r components. In particular, this implies that it is not possible to guarantee that the component covering in Ryser's conjecture consists of only one color.*

Proof. Partition the vertex set into a set X of order $r-1$ and a set Y of order $n-r+1$. Let $v_1, \dots, v_{|Y|}$ be the vertices in Y. For $i \in [r-1]$, color all edges from y_i to X with color i. Color all edges from y_r to X with color 1 and all edges from y_{r+1} to X with color 2. Color the remaining edges from Y to X arbitrarily from $[r-1]$ and color the edges inside X arbitrarily with colors from $[r-1]$. Finally, color all edges inside Y with color r. □

Problem 2.2. *Let $r \geq 4$. Does there exist r-coloring of K_n having the property that every minimal monochromatic component cover consists of $r-1$ components of different colors?*
A weaker question: Does there exist r-coloring of K_n having the property that every minimal monochromatic component cover consists of $r-t$ (more generally, just some function which grows with r) components of different colors?

Francetic, Herke, McKay, and Wanless disproved [6, Theorem 3.1] a conjecture of Aharoni (see [3]) by constructing a 13-coloring of K_n such that every color class has 13 components and no set of 12 components which cover $V(K_n)$ has non-trivial intersection. This does leave open the possibility that for some r there exists a r-coloring of K_n having the property that every set of $r-1$ components which cover $V(K_n)$ have distinct colors.

3 Aharoni's proof for $r = 3$

Given a hypergraph H and a matching M, let $\rho(M)$ be the minimum size of a set of edges F having the property that every edge in M intersects some edge in F. Let the *matching width* of H, denoted $mW(H)$, be the maximum value of $\rho(M)$ over all matchings in H. Note that $mw(H)$ is witnessed by a maximal matching.

Observation 3.1. *Given a hypergraph of rank at most r (that is, each edge has order at most r), $mw(H) \leq \nu(H) \leq r \cdot mw(H)$.*

Aharoni and Haxell [4] proved the following generalization of Hall's theorem [11] which we state here in its "defect form."

Theorem 3.2 (Aharoni, Haxell [4])**.** *Let $\mathcal{A}$ be a collection of hypergraphs. If for all $\mathcal{B} \subseteq \mathcal{A}$, we have $mw(\mathcal{B}) \geq |\mathcal{B}| - d$, then $\mathcal{A}$ has a partial system of disjoint representatives of order at least $|\mathcal{A}| - d$.*

Let G be an r-partite hypergraph and let L_v denote the link graph of v, which is an $(r-1)$-partite hypergraph. Letting $A = \{v_1, \ldots, v_m\} = V_1$, then we may apply Hall's theorem with $\mathcal{A} = \{L_{v_1}, \ldots, L_{v_m}\}$. Also note that in the case $r = 2$, this is just the ordinary Hall's theorem.

Assuming Theorem 3.2, we give a proof of Aharoni's theorem [2]. We state it in a more general language which both shows how to derive the $r = 2$ and $r = 3$ case in a common language, and shows that Ryser's conjecture would be true if it were the case that $\tau(H') \leq s \cdot mw(H')$ for all s-partite hypergraphs H'.

Observation 3.3. *Let $r \geq 2$. If $\tau(H') \leq (r-1) \cdot mw(H')$ for all $(r-1)$-partite hypergraphs H', then $\tau(H) \leq (r-1)\nu(H)$ for all r-partite hypergraphs H.*

Proof. For all $S \subseteq V_1$, let $L_S = \bigcup_{v \in S} L_v$. Let d be the largest non-negative integer such that $mw(L_B) = |B| - d$ for some $B \subseteq V_1$. Note that by Theorem 3.2, we have $\nu(G) \geq |V_1| - d$. Let $C \subseteq V \setminus V_1$ be a vertex cover of L_B. Note that $(V_1 \setminus B) \cup C$ is a vertex cover of G.

By the assumption, we have $|C| = \tau(L_B) \leq (r-1) \cdot mw(H) \leq (r-1)(|B| - d)$ and thus

$$\tau(G) \leq |V_1| - |B| + |C| \leq |V_1| - |B| + (r-1)(|B| - d) \leq (r-1)(|V_1| - d) \leq (r-1)\nu(G).$$

$\square$

Theorem 3.4. *Let r be a positive integer and let H be an r-partite hypergraph. If $r \leq 3$, then $\tau(H) \leq (r-1)\nu(H)$.*

Proof. By the previous observation, all that remains to be checked is that $\tau(H') \leq s \cdot mw(H')$ for all s-partite hypergraphs with $s \leq 2$. When $s = 1$, this is trivial and when $s = 2$, we have $\tau(H') = \nu(H') \leq 2 \cdot mw(H')$. $\square$

We close with the following problem, which as mentioned above, would imply Ryser's conjecture.

Problem 3.5. *Is it true that $\tau(H') \leq s \cdot mw(H')$ for all s-partite hypergraphs H'?*

4 Tuza's proofs

The *closure* of a graph G with respect to a given coloring is a multigraph $C(G)$ on the vertices of G with edge set defined as follows: there is a color-i edge between v_1 and v_2 in $C(G)$ if and only if there is a path of color i between v_1 and v_2 in G.

We begin with some general observations.

Let the edges of a graph G be r-colored. Take the closure of G with respect to this coloring. Note that $tc_r(G) \leq tc_r(C(G))$, since given a component cover of $C(G)$, the corresponding monochromatic components of G form a component cover of G. Hence, we will refer to $C(G)$ simply as G. Let $G_{i,j}$ be the subgraph of G induced by the edges of colors i and j. By Theorem 1.4,

$$tc_r(G) \leq tc_2(G_{i,j}) \leq \alpha(G_{i,j}). \tag{1}$$

Therefore, Ryser's Conjecture holds whenever there exist colors $i, j \in [r]$ such that

$$\alpha(G_{i,j}) \leq (r-1)\alpha(G).$$

4.1 $r = 4$, $\alpha = 1$

The following theorem is due to Tuza. We reprove the Theorem in the language of 4-edge-colored graphs instead of the language of 4-partite hypergraphs. We also state the Theorem in such a way which highlights a stronger conclusion.

Theorem 4.1. *For every 4-coloring of K_n and every distinct $i, j \in [4]$, there exists a monochromatic component cover of size at most three that either consists of only colors in $\{i, j\}$ or only colors in $[4] \setminus \{i, j\}$.*

Proof. Let the edges of K_n be 4-colored. Take the closure of K_n with respect to this coloring. Note that $tc_r(K_n) \leq tc_r(C(K_n))$, since given a component cover of $C(K_n)$, the corresponding monochromatic components of K_n form a component cover of K_n. Let $G_{i,j}$ be the graph induced by the edges of colors i and j. If $\alpha(G_{i,j}) \leq 3$, then by Proposition 1, $tc_4(K_n) \leq tc_2(G_{i,j}) \leq \alpha(G_{i,j}) \leq 3$. This gives a component cover of K_n of size at most three consisting of the selected colors. Therefore, assume $\alpha(G_{i,j}) \geq 4$. Let $X = \{x_1, x_2, x_3, x_4\}$ be an independent set of vertices in $G_{i,j}$. Since the no edge between vertices in X are color-i or color-j, X induces a 2-colored K_4 in the non-selected colors. Therefore, by Proposition 1, the vertices in X are covered by a single monochromatic component A, say of color $k \neq i, j$. Let $l \neq i, j, k$ be the remaining color.

Now, we consider two cases. First, assume that there is an edge of color-l between two vertices $x_1, x_2 \in X$. Let B denote the color-l component containing x_1x_2, and let C denote the color-l component containing x_3. Note that B may be equal to C. Let $X' = X \setminus \{x_4\}$ and $v \notin X'$. Since X is independent in $G_{i,j}$, there is at most one color-s edge from v to X' for $s = i, j$. Thus, vx_t is color-k or color-l for some $t \in [3]$. Then v is covered by either A, B, or C. Next, assume that there is no edge of color l between any two vertices in X. Suppose $v \notin X$. Then there is at most one edge of color-s from

v to X for $s \in \{i, j, l\}$. Thus, vx_t is color-k for some $t \in [4]$. So $v \in A$. In either of these cases, we obtain a component cover of size at most three consisting of non-selected colors.

□

4.2 $r = 5$, $\alpha = 1$

The following theorem is also due to Tuza. We reprove the Theorem in the language of 5-edge-colored graphs instead of the language of 5-partite hypergraphs. We also state the Theorem in such a way which highlights a stronger conclusion and we simplify the original proof.

Theorem 4.2. *For every 5-coloring of K_n and every distinct $i, j \in [5]$, there exists a monochromatic component cover of size at most four that either consists of only colors in $\{i, j\}$ or only colors in $[5] \setminus \{i, j\}$.*

Proof. Let K_n be five colored and let $G_{i,j}$ be the graph induced by colors i and j. If $\alpha(G_{i,j}) \leq 4$ for any $i, j \in [5]$, then, since, by Proposition 1, $tc_5(K_n) \leq tc_2(G_{i,j}) \leq \alpha(G_{i,j}) \leq 4$, we are done. So assume that for all i, j we have $\alpha(G_{i,j}) \geq 5$. Let $X = \{x_1, ..., x_5\}$ be an independent set of vertices in $G_{4,5}$.

Property I There exists a set T of at most four monochromatic components in K_n whose intersections with $G[X]$ contain all monochromatic components in $G[X]$ of two of the colors and those of the third color with size at least 3.

Property II There exists a set T of at most four monochromatic components in K_n whose intersections with $G[X]$ contain all monochromatic components in $G[X]$ of one color and those of the remaining colors with size at least 2.

If properties I or II hold, T is the desired component cover. There are only two cases which are not covered by the properties above.

Case I We have:

- two color-1 components, A_1 and B_1, whose intersections with X have sizes 4 and 1, respectively,
- two color-2 components, A_2 and B_2, whose intersections with X have sizes 3 and 2, respectively,
- two color-3 components, A_3 and B_3, whose intersections with X have sizes 3 and 2, respectively.

Let $x_5 \in B_1 \cap X$. The edges from x_5 to the vertices in $A_1 \cap X$ must be of color 2 or 3. Therefore, we obtain the following situation:

$$A_1 \cap X = \{x_1, x_2, x_3, x_4\},\ B_1 \cap X = \{x_5\},$$

$$A_2 \cap X = \{x_1, x_2, x_5\},\ B_2 \cap X = \{x_3, x_4\},$$

$$A_3 \cap X = \{x_3, x_4, x_5\},\ B_3 \cap X = \{x_1, x_2\}.$$

Suppose that the sets $\{A_1, B_1, A_2, B_2\}$ and $\{A_1, B_1, A_3, B_3\}$ are not covers. Then, there exist two vertices $w_2, w_3 \in V(K_n)$ such that $w_2 \in A_2 \setminus A_3$, and $w_3 \in A_3 \setminus A_2$. It follows that the edges w_2x_3, w_2x_4, w_3x_1, and w_3x_2 are of color 4 or 5. Therefore, the edge w_2w_3 is color-1. Let A be the color-1 component containing w_2w_3, and let $W = \{w \in V(K_n) : w \notin A_1, B_1, A_2, A_3\}$. If $w \in W$ then ww_2 or ww_3 is color-1. Consequently, $W \subseteq V(a)$, so $\{A_1, B_1, A_2, A\}$ is a cover of K_n.

Case II We have:

- two color-1 components, A_1 and B_1, whose intersections with X have sizes 3 and 2, respectively,
- two color-2 components, A_2 and B_2, whose intersections with X have sizes 3 and 2, respectively,
- two color-3 components, A_3 and B_3, whose intersections with X have sizes 3 and 2, respectively.

Now we obtain the following situation:

$$A_1 \cap X = \{x_1, x_2, x_3\},\ B_1 \cap X = \{x_4, x_5\},$$

$$A_2 \cap X = \{x_1, x_2, x_4\},\ B_2 \cap X = \{x_3, x_5\},$$

$$A_3 \cap X = \{x_1, x_2, x_5\},\ B_3 \cap X = \{x_3, x_4\}.$$

If neither $\{A_1, A_2, A_3, B_1\}$ nor $\{A_2, B_2, A_3, B_3\}$ is a cover of K_n then there exist two vertices, v_1 and v_2, such that $v_1 \in (B_2 \cap B_3) \setminus A_1$ and $v_2 \in A_1 \setminus (B_2 \cup B_3)$. Furthermore, v_1 and v_2 are in separate color-4 and color-5 components, so v_1 and v_2 do not share any monochromatic components; a contradiction. $\square$

5 General properties of a minimal counterexample

Recall that to prove $\mathrm{tc}_r(G) \leq (r-1)\alpha(G)$ it suffices to consider the closure of the given edge coloring of G, which means that we have an r-coloring of a multigraph in which every monochromatic component is a clique. In the closure of G with respect to a given coloring, we call an edge e of color i and multiplicity 1 an *essential edge* of color i.

Theorem 5.1. *Suppose there exists positive integers r and n, a multigraph G on n vertices with $\alpha := \alpha(G)$, and an r-coloring $c : E(G) \to [r]$ in which every monochromatic component is a clique such that G cannot be covered by at most $(r-1)\alpha$ monochromatic components. Choose such a graph and a coloring which (i) minimizes r, (ii) minimizes α, (iii) minimizes n, (iv) minimizes $e(G)$. Let Δ denote the number of vertices in a largest monochromatic component. Then G has the following properties:*

(i) *$r \geq 4$ and if $\alpha = 1$, then $r \geq 6$*

(ii) *Each color class contains at least $(r-1)\alpha + 1$ components.*

(iii) *Every component of color i contains an essential edge of color i.*

(iv) *Each vertex is incident with an edge of every color. In particular, every monochromatic component has at least 2 vertices.*

(v) *Every vertex is contained in at most* $\min\{\alpha(G), r\}$ *components of order 2.*

(vi) $\mathrm{tc}_r(G) = (r-1)\alpha + 1$

(vii) *Any set of* r *components intersect in at most one vertex.*

(viii) *Any set of* $r-1$ *components intersect in at most one vertex.*

(ix) *Any set of* $r-2$ *components intersect in at most two vertices.*

(x) *For all* $S \subseteq [r]$ *with* $s := |S| \geq 2$, $\alpha(G_S) > \frac{(r-1)\alpha}{s-1}$, *where* G_S *is the colored graph induced by edges having a color in* S. *In particular, for any two colors* i, j, *we have* $\alpha(G_{i,j}) \geq (r-1)\alpha(G) + 1$.

(xi) $\Delta \geq 3$; *and if* $\alpha = 1$, *then* $\Delta \geq 4$.

(xii) *If every edge has multiplicity 1,* $\Delta \leq (r-1)\alpha$; *and if in addition* $\alpha = 1$, *then* $\Delta \leq r-2$.

Proof. *(i)* This follows from Theorem (Aharoni) and Theorem (Tuza)

(ii) This follows since every color class is a component cover.

(iii) Let v be any vertex. Consider the graph G' obtained by removing all of the vertices in monochromatic components containing v. Since $\alpha(G') \leq \alpha(G) - 1$, and G is a minimal counterexample, we have $\mathrm{tc}_r(G') \leq (r-1)(\alpha-1)$ and thus $(r-1)\alpha + 1 \leq \mathrm{tc}_r(G) \leq (r-1)(\alpha-1) + r = (r-1)\alpha + 1$.

(iv) If not, we may remove the edges of color i corresponding to this component and call the resulting graph G'. Note that $e(G') < e(G)$, but $\alpha(G') = \alpha(G)$, so by minimality, we have $\mathrm{tc}_r(G) \leq \mathrm{tc}_r(G') \leq (r-1)\alpha$.

(v) If there exists v such that v is incident with edges of at most $r-1$ colors, then consider the graph G' obtained by removing all of the vertices in monochromatic components containing v. Since $\alpha(G') \leq \alpha(G)-1$, and G is a minimal counterexample, we have $\mathrm{tc}_r(G') \leq (r-1)(\alpha-1)$ and thus $\mathrm{tc}_r(G) \leq (r-1)(\alpha-1)+(r-1) = (r-1)\alpha$, a contradiction.

(vi) If $r \leq \alpha$, then each vertex is only contained in r components total; so suppose $r > \alpha$. Let v be a vertex and let U be the set of vertices which are in components of order 2 and are adjacent to v. If there is an edge xy with both endpoints in U such that xy is in a component H, then let G' be the graph obtained by deleting every vertex, except x and y which are in a monochromatic component containing v together with every vertex in H. Note that this is $r-1$ components in total and $\alpha(G') \leq \alpha - 1$. By minimality, we have $\mathrm{tc}_r(G) \leq \mathrm{tc}_r(G') + r - 1 \leq (r-1)(\alpha-1) + r - 1 = (r-1)\alpha$, a contradiction. Thus U is an independent set which implies $|U| \leq \alpha$.

(vii) If the components are not of distinct colors, then their intersection is empty; so suppose the colors are distinct. If there are at least two vertices u, v in the

intersection, replace them with a vertex w such that for all $x \notin \{u, v\}$, wx is an edge colored with every color appearing on ux or on vx; call this new graph G'. A covering of G' gives a covering of G since any component in the covering of G' which contains w, contains u and v in G.

(viii) Let $H_1, \dots, H_{r-1}$ be a set of $r-1$ components of distinct colors which have at least two vertices u, v in the intersection. Since every vertex is incident with an edge of every color and u and v are not in the same component, there exists components $H_r(u)$ and $H_r(v)$ color r such that $u \in H_r(u)$ and $v \in H_r(v)$. Let $u \neq u' \in H_r(u)$ and $v \neq v' \in H_r(v)$. Note that $u'v$ and uv' cannot have color r, so they are colored with a color from $[r-1]$ and thus u' is in some component H_i with $i \in [r-1]$ and v' is contained in some component H_j with $j \in [r-1]$.

(ix) Let $H_1, \dots, H_{r-2}$ be a set of $r-2$ components of distinct colors and suppose there are 3 vertices $U = \{u_1, u_2, u_3\}$ in their intersection. Each of the vertices $u_j \in U$ is contained in two components $H_{r-1}(u_j), H_r(u_j)$ of colors $r-1$ and r respectively. If any three vertices from U were contained in the same component of color $r-1 \leq h \leq r$, then we would have a set U' of two vertices and a set $H_1, \dots, H_{r-2}, H_h$ of $r-1$ components whose intersection contains U'.

(x) If $\alpha(G_S) \leq \frac{(r-1)\alpha(G)}{s-1}$, then by minimality, we have $\mathrm{tc}_r(G) \leq \mathrm{tc}_s(G_S) \leq (s-1)\alpha(G_S) \leq (r-1)\alpha(G)$.

(xi) Let A be a maximal independent set in G. Since every vertex is in A or adjacent to a vertex in A, we have $n \leq \alpha((\Delta-1)r+1)$, with equality if and only if every monochromatic component intersecting A has Δ vertices. Let G_i be a color class containing a component of order Δ. Then we have $n \geq 2(r-1)\alpha+\Delta$, with equality if and only if all of the other components of G_i are size 2. Since $r > 1$, combining the above inequalities and using $r \geq 3$ shows $\Delta \geq \frac{3\alpha(r-1)}{\alpha r-1} > 2$. If $\alpha = 1$, then we have $\Delta = \frac{3\alpha(r-1)}{\alpha r-1} = 3$ with equality if and only if every component is order Δ and all but one component in color class i has order 2, which is not possible, so $\Delta \geq 4$ in this case.

(xii) Let X be a color-i component of order Δ and let $G' = G - X$. If $\alpha(G') \leq \alpha - 1$, we would have a contradiction, so suppose there is an independent set $A \subseteq V(G) \setminus V(X)$ of order α. Since every edge has multiplicity 1 and every vertex from X must send an edge to some vertex in A, none of color i, we have $\Delta \leq (r-1)\alpha$.

If $\alpha = 1$ and $\Delta = r-1$, then let $H_1, \dots, H_{r-1}$ be all of the components of color j for some $j \neq i$ which intersect X. Since we are assuming $H_1, \dots, H_{r-1}$ does not form a component cover, then there exists some vertex which sends no edges of color i or j to X and thus $|X| \leq r-2$.

$\square$

6 Partitioning

Given $r \in \mathbb{Z}^+$ and a graph G, let $t := tp_r(G)$ be the smallest integer so that in every r-coloring of the edges of G there exists at most t monochromatic connected subgraphs whose vertex sets *partition* $V(G)$. Erdős, Gyárfás, Pyber [5] and later Fujita, Furuya, Gyárfás, Tóth [7] made the following strengthening of Ryser's conjecture.

Conjecture 6.1 (Erdős, Gyárfás, Pyber [5]; Fujita, Furuya, Gyárfás, Tóth [7]). $\mathrm{tp}_r(K_n) \leq r-1$ *and in general,* $\mathrm{tp}_r(G) \leq (r-1)\alpha(G)$

Theorem 6.2 (Erdős, Gyárfás, Pyber [5]). $\mathrm{tp}_3(K_n) = 2$

We present the following beautiful proof of Fujita, Furuya, Gyárfás, Tóth (which was originally written in a more general form).

Theorem 6.3 (Fujita, Furuya, Gyárfás, Tóth [7]). $\mathrm{tp}_2(G) \leq \alpha(G)$

Proof. We are done if $\alpha(G) = 1$, so suppose $\alpha(G) \geq 2$ and the statement holds for all G' with $\alpha(G') < \alpha(G)$.

We know that $\mathrm{tc}_2(G) \leq \alpha(G)$. Let $R_1, \dots, R_p$ be the red components and let $B_1, \dots, B_q$ be the blue components in such a covering. Note that

$$p + q \leq \alpha(G).$$

Let $R = \bigcup_{i=1}^{p} V(R_i)$ and $B = \bigcup_{i=1}^{q} V(B_i)$. Let $R' = R \setminus B$ and $B' = B \setminus R$. Note that we are done unless $R' \neq \emptyset$ and $B' \neq \emptyset$. Also note that there are no edges between R' and B' so

$$\alpha(G[R']) + \alpha(G[B']) \leq \alpha(G).$$

Thus $\alpha(G[R']) < \alpha(G)$ and $\alpha(G[B']) < \alpha(G)$. By induction we have $p' := \mathrm{tp}_2(G[R']) \leq \alpha(G[R'])$ and $q' := \mathrm{tp}_2(G[B']) \leq \alpha(G[B'])$. Let $C_1, \dots, C_{p'}$ be the component partition of $G[R']$ and let $D_1, \dots, D_{q'}$ be the component partition of $G[B']$. Note that

$$p' + q' \leq \alpha(G[R']) + \alpha(G[B']) \leq \alpha(G).$$

Since $p' + q + p + q' \leq 2\alpha(G)$, we have say $p' + q \leq \alpha(G)$. So $C_1, \dots, C_{p'}, B_1, \dots, B_q$ is the desired monochromatic connected subgraph partition. $\square$

The following table highlights the known values of this stronger version of Ryser's conjecture.

α \ r	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25
6	$\downarrow$	12	18	24	30

7 Covering with monochromatic subgraphs of bounded diameter

For vertices v_1, v_2, let $d_i(v_1, v_2)$ denote the length of the shortest v_1, v_2-path. If there is no v_1, v_2-path, we write $d_i(v_1, v_2) = \infty$. The diameter of a graph G, denoted $\mathrm{diam}(G)$, is the smallest integer δ such that $d(u, v) \leq \delta$ for all $u, v \in V(G)$. If G is not connected, we say $\mathrm{diam}(G) = \infty$.

Proposition 7.1 (Folklore)**.** *In every 2-coloring of K_n, if $diam(G_R) \geq 4$, then $diam(G_B) \leq 2$; and if $diam(G_R) \geq 3$, then $diam(G_B) \leq 3$. Furthermore this is best possible when $n \geq 4$.*

Proof. To see this is best possible, partition $V(K_n)$ as $\{V_1, V_2, V_3, V_4\}$ and color all edges from V_i to V_{i+1} red for all $i \in [3]$ and color all other edges blue. Both G_R and G_B have diameter 3. □

Let $dc_r^\delta(G)$ be the smallest integer t such that in every r-coloring of the edges of G, there exists $t' \leq t$ monochromatic connected subgraphs $C_1, \ldots, C_{t'}$ such that $\bigcup_{i \in [s']} V(C_i)$ where $diam(C_i) \leq \delta$ for all $i \in [t']$. For $r \geq 1$ and a graph G, let $D(r, G)$ be the smallest δ such that $dc_r^\delta(G) \leq \mathrm{tc}_r(G)$. For instance $D(2, K_n) = 3$.

Milićević conjectured the following strengthening of Ryser's conjecture.

Conjecture 7.2 (Milićević [16])**.** *Let K be a complete graph. For all $r \geq 2$, there exists δ such that* $\mathrm{dc}_r^\delta(K) \leq r - 1$.

Milićević [15] proved that $\mathrm{dc}_3^8(K) \leq 2$; i.e. $D(3, K) \leq 8$.

We strengthen Milićević's result and as corollary show that $3 \leq D(3, K_n) \leq 4$.

Example 7.3. *For every complete graph K on at least 7 vertices, there exists a 3-coloring of K such that if H_1 and H_2 are monochromatic subgraphs which cover $V(K)$, then $diam(H_i) \geq 3$ for some $i \in [2]$. In particular, $D(3, K) \geq 3$.*

Proof. Color K_7 with 3 colors so that each color class is a 7-cycle. Then take the blow-up of this example and color the edges inside the sets arbitrarily. If H_1 and H_2 are monochromatic subgraphs which cover $V(K)$, then for some $i \in [2]$, H_i contains vertices from at least four different sets which implies $\mathrm{diam}(H_i) \geq 3$. □

Theorem 7.4. *In every 3-coloring of a complete graph K there exists at most two monochromatic trees, H_1 and H_2, such that $V(H_1) \cup V(H_2) = V(K)$ and $diam(H_i) \leq 4$ for $i \in [2]$.*

Proof. Let $x \in V(K)$. For $i \in [3]$, let A_i be the neighbors of x of color i. If $A_i = \emptyset$ for some $i \in [3]$, then $H_1 \subseteq G_j[\{x\} \cup A_j]$ and $H_2 \subseteq G_k[\{x\} \cup A_k]$ $(i \neq j \neq k)$ satisfy the theorem with $\mathrm{diam}(H_i) \leq 2$ for $i \in [2]$. So we assume $A_i \neq \emptyset$ for all $i \in [3]$.

For $i, j \in [3]$ with $i \neq j$, define B_{ij} to be the set of vertices $v \in A_i$ such that vu is not color-j for all $u \in A_j$. If $B_{ij} = \emptyset$ for some $i, j \in [3]$, then $H_1 \subseteq G_j[\{x\} \cup A_j \cup A_i]$ and

$H_2 \subseteq G_k[\{x\} \cup A_k]$ cover $V(K)$, where $\mathrm{diam}(H_1) \leq 4$ and $\mathrm{diam}(H_2) \leq 2$. So suppose $B_{ij} \neq \emptyset$ for all $i, j \in [3]$. Note that $[B_{ij}, B_{ji}]$ is a complete bipartite graph of color k. We consider two cases.

First, assume there exist $i, j, k \in [3]$ such that $B_{ij} \setminus B_{ik} \neq \emptyset$. Let $z \in B_{ij} \setminus B_{ik}$. Then there is a vertex $u \in A_k$ such that zu is color-k. Since every z, B_{ji}-edge is color-k, $H_1 \subseteq G_k[\{x\} \cup A_k \cup \{z\} \cup B_{ji}]$ and $H_2 \subseteq G_i[\{x\} \cup A_i \cup (A_j \setminus B_{ji})]$ with $\mathrm{diam}(H_i) \leq 4$ for $i \in [2]$. So assume $B_{ij} \setminus B_{ik} = \emptyset$ for all $i, j, k \in [3]$.

Then $B_{ij} = B_{ik} =: B_i$ for all $i, j, k \in [3]$. If there exists $i \in [3]$ such that $A_i \neq B_i$, then $H_1 \subseteq G_i[\{x\} \cup A_i \cup (A_j \setminus B_j) \cup (A_k \setminus B_k)]$ with $\mathrm{diam}(G) \leq 4$, and $H_2 \subseteq G_i[B_j \cup B_k]$ with $\mathrm{diam}(H_2) \leq 2$. If, on the other hand, $A_i = B_i$ for all $i \in [3]$, then for any i, $H_1 \subseteq G_i[\{x\} \cup A_i]$ with $\mathrm{diam}(H_1) \leq 2$, and $H_2 \subseteq G_i[A_j \cup A_k]$ with $\mathrm{diam}(H_2) \leq 3$. $\square$

Note that it may be possible to improve the previous result by covering with two monochromatic *subgraphs* of diameter at most 3, but we cannot hope to cover with two monochromatic *trees* of diameter at most 3 (consider a random three coloring of the edges; no two double stars will cover the entire vertex set).

Problem 7.5. *In every 3-coloring of K there exists at most two monochromatic subgraphs, H_1 and H_2, such that $V(H_1) \cup V(H_2) = V(K)$ and diam$(H_i) \leq 3$ for $i \in [2]$.*

8 Conclusion

The results in this report are currently being extended to a full-length paper.

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