Escher tilings and ribbons: A mathematical look

Kelli Hall[1]

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Abstract

Using combinatorial methods, M.C. Escher created repeating patterns of tilings with decorated squares, hoping to find every possible pattern. In this paper, we will give an algebraic proof for his pictorial findings and then extend our mathematical approach to a few cases that involve ribbons.

1. Introduction

Dutch artist M.C. Escher is well-known, especially in the mathematics community, for his mathematically inspired art. A large portion of Escher's work focused on regular division of the plane, which led to his interest in counting the total number of possible patterns that could be created given certain restrictions. He decided to use a stamp with an asymmetric, geometric design called a *motif*. Then, in a 2×2 array (known as a *translation block*), he allowed the motif to be placed in 4 ways and considered different cases involving various restrictions on the ways that the different motif designs could be placed. Patterns are generated by translating the translation block in all directions.

Escher attempted his counting problems in an artistic way, by methodical drawing and checking of all possible patterns he could imagine. In this paper, we will use a mathematical approach to prove Escher's findings. As Escher discovered, most patterns are equivalent to others, i.e., they are rotations or reflections of other patterns. We want to find the number of distinct patterns. This is the same as finding the number of equivalence classes of the translation blocks. To do this, we will need a few tools.

1.1 Orbit-Stabilizer Theorem: *Let G be a finite group of permutations of a set S. Then, for any i from S,*

$$|G| = |orb_G(i)| \cdot |stab_G(i)|,$$

where $stab_G(i) = \{\phi \in G \mid \phi(i) = i\}$ *for each i in S is the stabilizer of i in G and* $orb_G(i) = \{\phi(s) \mid \phi \in G\}$ *for each s in S is the orbit of s under the actions of G.*

See [1, Theorem 7.3] for proof of this theorem.

1.2 Burnside's Theorem: *If G is a finite group of permutations on a set S, then the number of orbits of the actions of G on S is*

[1] Department of Mathematics, Marshall University, Huntington, WV 25755; hall142@marshall.edu

$$\frac{1}{|G|}\sum_{\phi\in G}|fix(\phi)| \ ,$$

where $fix(\phi) = \{i\in S \mid \phi(i) = i\}$ *for* $\phi\in G$.

Proof: Let n denote the number of pairs (ϕ, i), with $\phi\in G$, $i\in S$, and $\phi(i) = i$. We count the number of these pairs in two ways. First, hold each $\phi\in G$ fixed and sum as i goes through S. This is $|fix(\phi)|$. Thus,

$$n = \sum_{\phi\in G}|fix(\phi)|.$$

Next, hold each $i\in S$ fixed and sum as ϕ goes through G. This is $|stab_G(i)|$. Thus,

$$n = \sum_{i\in S}|stab_G(i)|.$$

When $s, t\in S$ are in the same orbit, then $orb_G(s) = orb_G(t)$ and $|stab_G(s)| = |stab_G(t)|$. Choose $s\in S$, sum over $orb_G(s)$, and apply the Orbit-Stabilizer Theorem, to obtain

$$\sum_{t\in orb_G(s)}|stab_G(t)| = |orb_G(s)|\cdot|stab_G(s)| = |G|.$$

Then, by summing over all $\phi\in G$ one orbit at a time, we have

$$\sum_{\phi\in G}|fix(\phi)| = \sum_{i\in S}|stab_G(i)| = |G|\cdot \textit{(number of orbits)}.$$

Therefore, if G is a finite group of permutations on a set S, then the number of orbits of the actions of G on S is

$$\frac{1}{|G|}\sum_{\phi\in G}|fix(\phi)|. \qquad \square$$

Burnside's Theorem will help us to count the number of equivalence classes of translation blocks.

In this paper, we will use the groups $\mathbf{D}_n$ (the dihedral group of order $2n$), $\mathbf{K_4}$ (the noncyclic group of order 4), and $\mathbf{Z_4}$ (the cyclic group of order 4).

2. Tilings: Escher's three cases.

Escher's Case I

Theorem: *There are 23 distinct patterns, or equivalence classes of translation blocks, in Escher's Case I, where the patterns are made only with rotations of the motif.*

Proof: This case requires using a translation block where the patterns are made only with rotations of the motif. Let the translation block of a tiling be composed of some tiles *W, X, Y, Z,* where each tile is a rotation of all the other tiles. Thus, there are 4^4=256 possible translation blocks of the tiles.

We need to determine the natural group to use. The actions on the tiles include the rotations of each tile and diagonal, horizontal, and vertical switching of tiles. Let *W’* be a rotation of a tile by 90°, *W’’* be a rotation by 180°, and *W’’’* be a rotation by 270°. We get the following group, where the diagrams are the images of the identity block under the actions:

W	*X*
Z	*Y*

e, Identity

Z’	*W’*
Y’	*X’*

R_{90}

Y’’	*Z’’*
X’’	*W’’*

R_{180}

X’’’	*Y’’’*
W’’’	*Z’’’*

R_{270}

Y	*Z*
X	*W*

F_D

Z	*Y*
W	*X*

F_H

X	*W*
Y	*Z*

F_V

W’	*Z’*
X’	*Y’*

$F_V R_{90}$

Y’	*X’*
Z’	*W’*

$F_H R_{90}$

X’	*Y’*
W’	*Z’*

$F_D R_{90}$

Z’’	*Y’’*
W’’	*X’’*

$F_V R_{180}$

X’’	*W’’*
Y’’	*Z’’*

$F_H R_{180}$

W’’	*X’’*
Z’’	*Y’’*

$F_D R_{180}$

Y’’’	*X’’’*
Z’’’	*W’’’*

$F_V R_{270}$

W’’’	*Z’’’*
X’’’	*Y’’’*

$F_H R_{270}$

Z’’’	*W’’’*
Y’’’	*X’’’*

$F_D R_{270}$

The subgroup $\{e, R_{90}, R_{180}, R_{270}\}$ is isomorphic to $\mathbf{Z_4}$. The subgroup $\{e, F_D, F_H, F_V\}$ is isomorphic to $\mathbf{K_4}$. Since we are working with $\mathbf{Z_4}$**,** the “flips” of $\mathbf{K_4}$ are just diagonal, horizontal, and vertical switching of tiles. Thus, the entire group is $G = \mathbf{K_4}\ \mathbf{Z_4}$**,** so
$|G| = 16$.

Now we must find $|fix(\phi)|$ for all $\phi \in G$.

$\left| fix(e) \right| = 256$

$\left| fix(R_{90}) \right|$:

R_{90} applied to

W	*X*
Z	*Y*

is

Z'	*W'*
Y'	*X'*

Then, we have the equations *W =Z'*, *W' = X, X'= Y,* and *Y'= Z*. These simplify to *W =Z'*, *X =Z''*, *Y =Z'''*, *Z = Z*. We get:

Z'	*Z''*
Z	*Z'''*

There are 4 choices for *Z*, and, as a result, there is only one choice for each of the rotations. Thus, $\left| fix(R_{90}) \right| = 4$.

Henceforth, translation blocks will be written as a string, called a *signature*, starting with the tile in the upper left corner, and then moving clockwise. Our identity signature is *WXYZ*.

$\left| fix(R_{180}) \right|$:

If $WXYZ \in fix(R_{180})$, then since $R_{180}(WXYZ) = Y''\,Z''\,W''\,X''$, we get the equations $W = Y''$, $X = Z''$, $Y = W''$, and $Z = X''$. Thus $W = Y''$, $X = Z''$, $Y = Y$, and $Z = Z$. Hence, $WXYZ = Y''\,Z''YZ$, where there are 4 choices for *Y* and 4 choices for *Z*.
Thus, $\left| fix(R_{180}) \right| = 4^2 = 16$.

$\left| fix(F_V R_{90}) \right| = 0$

One of the equations for this action is *W =W'*, which is untrue. A tile and its rotations cannot be equal.

This process continues for the remaining elements, and we find the following results:

$\left| fix(R_{270}) \right| = 4$

$\left| fix(F_V) \right| = 16$

$\left| fix(F_H) \right| = 16$

$\left| fix(F_D) \right| = 16$

$\left| fix(F_H R_{90}) \right| = 0$

$\left| fix(F_V R_{180}) \right| = 16$

$\left| fix(F_H R_{180}) \right| = 16$

$\left| fix(F_D R_{180}) \right| = 0$

$\left| fix(F_V R_{270}) \right| = 0$

$\left| fix(F_H R_{270}) \right| = 0$

$\left| fix(F_D R_{90}) \right| = 4$ $\left| fix(F_D R_{270}) \right| = 4$

Now we apply Burnside's Theorem:

$$\frac{1}{|G|}\sum_{\phi\in G}\left| fix(\phi) \right| = \frac{1}{16}(256+4\cdot 14+6\cdot 16) = 23.$$

Thus, there are 23 equivalence classes of signatures. This means that there are 23 distinct patterns in Escher's Case I, where the patterns are made only with rotations of the motif.□

Escher's Case II

This case requires using two stamps: Block 1 and Block 2, where Block 2 is a reflection of Block 1.

Theorem (Subcase IIa): *There are 10 distinct patterns in Escher's Case IIa. The translation blocks will have two copies of Block 1 and two of Block 2, where the rotations of Block 1 are the same and the rotations of Block 2 are the same.*

Proof: Let $\overline{W},\overline{X},\overline{Y},\overline{Z}$ denote the vertical reflections of tiles *W, X, Y, Z*, respectively. When performing an action that requires both a reflection and a rotation, always perform the reflection first. Also, we have the following equalities:
$\overline{(X')} = \overline{X}'''$, $\overline{(X'')} = \overline{X}''$, and $\overline{(X''')} = \overline{X}'$.

Thus, the elements are:

$e(WXYZ) = WXYZ$

$R_{90}(WXYZ) = Z' W' X' Y'$

$R_{180}(WXYZ) = Y''Z''W''X''$

$R_{270}(WXYZ) = X''' Y''' Z''' W'''$

$F_{VB}(WXYZ) = \overline{X}\,\overline{W}\,\overline{Z}\,\overline{Y}$

$F_{HB}(WXYZ) = \overline{Z}''\overline{Y}''\overline{X}''\overline{W}''$

$F_{LB}(WXYZ) = \overline{W}'\overline{Z}'\overline{Y}'\overline{X}'$

$F_{RB}(WXYZ) = \overline{Y}'''\overline{X}'''\overline{W}'''\overline{Z}'''$

$F_V(WXYZ) = XWZY$

$F_V F_{LB}(WXYZ) = \overline{Z}'\overline{W}'\overline{X}'\overline{Y}'$

$F_V F_{RB}(WXYZ) = \overline{X}'''\overline{Y}'''\overline{Z}'''\overline{W}'''$

$F_H R_{90}(WXYZ) = Y' X' W' Z'$

$F_H R_{180}(WXYZ) = X'' W'' Z'' Y''$

$F_H R_{270}(WXYZ) = W''' Z''' Y''' X'''$

$F_H F_{VB}(WXYZ) = \overline{Y}\,\overline{Z}\,\overline{W}\,\overline{X}$

$F_H F_{HB}(WXYZ) = \overline{W}''\overline{X}''\overline{Y}''\overline{Z}''$

$F_H F_{LB}(WXYZ) = \overline{X}'\overline{Y}'\overline{Z}'\overline{W}'$

$F_H F_{RB}(WXYZ)\ \overline{Z}'''\overline{W}'''\overline{X}'''\overline{Y}'''$

$F_H(WXYZ) = ZYXW$

$F_D(WXYZ) = YZXW$

$F_V R_{90}(WXYZ) = W' Z' Y' X'$

$F_V R_{180}(WXYZ) = Z'' Y'' X'' W''$

$F_V R_{270}(WXYZ) = Y''' X''' W''' Z'''$

$F_V F_{VB}(WXYZ) = \overline{W}\ \overline{X}\ \overline{Y}\ \overline{Z}$

$F_V F_{HB}(WXYZ) = \overline{Y}''\ \overline{Z}''\ \overline{W}''\ \overline{X}''$

$F_D R_{90}(WXYZ) = X' Y' Z' W'$

$F_D R_{180}(WXYZ) = W'' X'' Y'' Z''$

$F_D R_{270}(WXYZ) = Z''' W''' X''' Y'''$

$F_D F_{VB}(WXYZ) = \overline{Z}\ \overline{Y}\ \overline{X}\ \overline{W}$

$F_D F_{HB}(WXYZ) = \overline{X}''\ \overline{W}''\ \overline{Z}''\ \overline{Y}''$

$F_D F_{LB}(WXYZ) = \overline{Y}'\ \overline{X}'\ \overline{W}'\ \overline{Z}'$

$F_D F_{RB}(WXYZ) = \overline{W}'''\overline{Z}'''\overline{Y}'''\overline{X}'''$

The subgroup {e, R_{90}, R_{180}, R_{270}, F_{VB}, F_{HB}, F_{LB}, F_{RB}} is isomorphic to $\mathbf{D_4}$, where the flips are with respect to the entire translation block. The subgroup {e, F_D, F_H, F_V} is isomorphic to $\mathbf{K_4}$, which are just diagonal, horizontal, and vertical switching of tiles. Thus, the entire group is $G = \mathbf{K_4}\ \mathbf{D_4}$, so $|G| = 32$.

In this subcase, we need to consider the 6 possible signatures for the restrictions given. They are:

A = $W\,W\,\overline{X}\ \ \overline{X}$

B = $\overline{X}\ \ W\,W\,\overline{X}$

C = $\overline{X}\ \ \overline{X}\ \ W\,W$

D = $W\,\overline{X}\ \ \overline{X}\ \ W$

E = $W\,\overline{X}\ \ W\,\overline{X}$

F = $\overline{X}\ \ W\,\overline{X}\ \ W$

Since A, B, and E are symmetric with C, D, and F, respectively, it is only necessary to find $|fix(\phi)|$ for each element with respect to A, B, and E (and then we will multiply each of those by two). We use $|fix(\phi : A)|$ to denote $|fix(\phi)|$ for subcase A

Now we must find $|fix(\phi)|$ for all $\phi \in G$.

$|fix(e)|$:

The identity fixes all signatures. For each of the 6 possible translation blocks there are 4 choices for W and 4 choices for $\overline{X}$. Thus, there are $6 \cdot 4 \cdot 4 = 96$ possible signatures.

$|fix(F_H F_{LB})|$:

A: $F_H F_{LB}(W\,W\,\overline{X}\ \ \overline{X}) = \overline{W}'\ X'\ X'\ \overline{W}'$

We have $X' = \overline{X}$, which is untrue because reflected tiles can only be equal to other reflected tiles.

Thus, $\left| fix(F_H F_{LB} : A) \right| = 0$

B: $F_H F_{LB}$ ($\overline{X}$ W W $\overline{X}$) = $\overline{W}$ ' $\overline{W}$ ' X' X'
We have $X' = \overline{X}$, which is untrue because reflected tiles can only be equal to other reflected tiles.
Thus, $\left| fix(F_H F_{LB} : B) \right| = 0$

E: $F_H F_{LB}$ (W $\overline{X}$ W $\overline{X}$) = X' $\overline{W}$ ' X' $\overline{W}$ '
$W = X'$ and $\overline{W}' = \overline{X}$. There are 4 choices for X', and 1 choice for $\overline{X}$.
Thus, $\left| fix(F_H F_{LB} : E) \right| = 4$. Hence, $\left| fix(F_H F_{LB} : F) \right| = 4$ also.

$\left| fix(F_H F_{LB}) \right| = 0 + 0 + 8 = 8.$

This process continues for the remaining elements, and the following results are found:

$\left| fix(R_{180}) \right| = 0$
$\left| fix(R_{270}) \right| = 0$
$\left| fix(F_{VB}) \right| = 16$
$\left| fix(F_{HB}) \right| = 16$
$\left| fix(F_{LB}) \right| = 0$
$\left| fix(F_{RB}) \right| = 0$
$\left| fix(F_V) \right| = 32$
$\left| fix(F_H) \right| = 32$
$\left| fix(F_D) \right| = 32$
$\left| fix(F_V R_{90}) \right| = 0$
$\left| fix(F_V R_{180}) \right| = 0$
$\left| fix(F_V R_{270}) \right| = 0$
$\left| fix(F_V F_{VB}) \right| = 0$
$\left| fix(F_V F_{HB}) \right| = 16$
$\left| fix(F_V F_{RB}) \right| = 8$

$\left| fix(F_H R_{90}) \right| = 0$
$\left| fix(F_H R_{180}) \right| = 0$
$\left| fix(F_H R_{270}) \right| = 0$
$\left| fix(F_H F_{VB}) \right| = 16$
$\left| fix(F_H F_{HB}) \right| = 0$
$\left| fix(R_{90}) \right| = 0$
$\left| fix(F_H F_{RB}) \right| = 8$
$\left| fix(F_D R_{90}) \right| = 0$
$\left| fix(F_D R_{180}) \right| = 0$
$\left| fix(F_D R_{270}) \right| = 0$
$\left| fix(F_D F_{VB}) \right| = 16$
$\left| fix(F_D F_{HB}) \right| = 16$
$\left| fix(F_D F_{LB}) \right| = 0$
$\left| fix(F_D F_{RB}) \right| = 0$
$\left| fix(F_V F_{LB}) \right| = 8$

Now we apply Burnside's Theorem:

$$\frac{1}{|G|}\sum_{\phi\in G}\left| fix(\phi)\right| = \frac{1}{32}(96 + 16\cdot 6 + 8\cdot 4 + 32\cdot 3) = 10.$$

Thus, there are 10 equivalence classes of signatures. This means that there are 10 distinct patterns in Escher's Case IIa, where the rotations of Block 1 are the same and the rotations of Block 2 are the same, for 2 tiles of Block 1 and 2 tiles of Block 2.□

Theorem (Subcase IIb): *There are 39 distinct patterns in Escher's Case IIb. The translation blocks will have two copies of Block 1 and two of Block 2, where the rotations of Block 1 are differnt and the rotations of Block 2 are different.*

Proof: We want to find the number of distinct patterns when the Block 1's have different rotations and the Block 2's have different rotations. The group G = $\mathbf{K_4}$ $\mathbf{Z_4}$ is the appropriate group to use. In this subcase we need to consider the 6 possible rotation blocks for the restrictions given. In all of the following cases, W is not equal to X and Y is not equal to Z. They are:

A = $WX\overline{Y}\,\overline{Z}$

B = $\overline{Z}\,WX\overline{Y}$

C = $\overline{Y}\,\overline{Z}\,WX$

D = $X\overline{Y}\,\overline{Z}\,W$

E = $W\overline{Y}\,X\overline{Z}$

F = $\overline{Z}\,W\overline{Y}\,X$

Since A, B, and E are symmetric with C, D, and F, respectively, it is only necessary to find $\left| fix(\phi)\right|$ for each element with respect to A, B, and E (and then we will multiply each of those by two).

Now we must find $\left| fix(\phi)\right|$ for all $\phi \in G$.

$\left| fix(e)\right|$:

The identity fixes all possible patterns. Each translation block has 4 choices for W, 3 choices for X (Block 1's have different rotations therefore $W \neq X$), 4 choices for $\overline{Z}$, and 3 choices for $\overline{Y}$. Thus, $\left| fix(e)\right| = 6\cdot(4\cdot 3)\cdot(4\cdot 3) = 864$.

$\left| fix(F_{VB})\right|$:

A: $F_{VB}(WX\overline{Y}\,\overline{Z}) = \overline{X}\;\overline{W}\;Z\,Y$

We have $W = \overline{X}$, which is untrue because reflected tiles can only be equal to other reflected tiles.

Thus, $\left| fix(F_{VB} : A)\right| = 0$

B: $F_{VB}(\overline{Z}\,WX\overline{Y}) = \overline{W}\,ZY\overline{X}$

With equations $W = Z$ and $Y = X$, the translation block becomes $\overline{Z}\,ZX\overline{X}$.

There are 4 choices for X and 3 choices for Z.
Thus, $\left| fix(F_{VB} : B) \right| = 12$

E: $F_{VB}(W\overline{Y}\,X\overline{Z}\,) = Y\,\overline{W}\;\;Z\;\overline{X}$
The translation block becomes $Y\overline{Y}\,Z\overline{Z}$, with 4 choices for Y and 3 choices for Z.
Thus, $\left| fix(F_{VB} : E) \right| = 12$

Therefore, $\left| fix(F_{VB}) \right| = 12(2) + 12(2) = 48.$

This process continues for the remaining elements, and the following results are found:

$\left| fix(R_{90}) \right| = 0$
$\left| fix(R_{180}) \right| = 32$
$\left| fix(R_{270}) \right| = 0$
$\left| fix(F_{HB}) \right| = 48$
$\left| fix(F_{LB}) \right| = 0$
$\left| fix(F_{RB}) \right| = 0$
$\left| fix(F_{V}) \right| = 0$
$\left| fix(F_{H}) \right| = 0$
$\left| fix(F_{D}) \right| = 0$
$\left| fix(F_V R_{90}) \right| = 0$
$\left| fix(F_V R_{180}) \right| = 32$
$\left| fix(F_V R_{270}) \right| = 0$
$\left| fix(F_V F_{VB}) \right| = 0$
$\left| fix(F_V F_{HB}) \right| = 48$
$\left| fix(F_V F_{RB}) \right| = 0$

$\left| fix(F_H R_{90}) \right| = 0$
$\left| fix(F_H R_{180}) \right| = 32$
$\left| fix(F_H R_{270}) \right| = 0$
$\left| fix(F_H F_{VB}) \right| = 48$
$\left| fix(F_H F_{HB}) \right| = 0$
$\left| fix(F_H F_{LB}) \right| = 0$
$\left| fix(F_H F_{RB}) \right| = 0$
$\left| fix(F_D R_{90}) \right| = 0$
$\left| fix(F_D R_{180}) \right| = 0$
$\left| fix(F_D R_{270}) \right| = 0$
$\left| fix(F_D F_{VB}) \right| = 48$
$\left| fix(F_D F_{HB}) \right| = 48$
$\left| fix(F_D F_{LB}) \right| = 0$
$\left| fix(F_D F_{RB}) \right| = 0$
$\left| fix(F_V F_{LB}) \right| = 0$

Now, we can apply Burnside's Theorem:

$$\frac{1}{|G|}\sum_{\phi \in G}\left| fix(\phi) \right| = \frac{1}{32}(864 + 32(3) + 48(6)) = 39.$$

Therefore there are 39 distinct patterns in Escher's Case IIb, where the Block 1's have different rotations and the Block 2's have different rotations, for 2 tiles of Block 1 and 2 tiles of Block 2.□

Subcase IIc:

Theorem: *The total number of distinct patterns without restrictions on rotations in Escher's Case II is 67.*

Proof: We want to find the number of distinct patterns when there are no restrictions on rotations. We have already determined the number when the Block 1's have the same rotation and the Block 2's have the same rotation. We also know the answer when Block 1's have different rotations and Block 2's have different rotations. Now all we have to find is the number of distinct patterns when the Block 1's have the same rotation and the Block 2's have different rotations, and vice versa. The group $G = \mathbf{K_4}\ \mathbf{Z_4}$ is the group to use. (If we were using $G = \mathbf{K_4}\ \mathbf{D_4}$, the reflections of the translation block would not give us a permutation of this subset of signatures, i.e. there would be no closure.) In this subcase, we need to consider the 6 possible signatures for the restrictions given. In all of the following cases, Y is not equal to Z. They are:

A = $WW\overline{Y}\,\overline{Z}$

B = $\overline{Z}\,WW\overline{Y}$

C = $\overline{Y}\ \overline{Z}\,WW$

D = $W\overline{Y}\ \overline{Z}\,W$

E = $W\overline{Y}\,W\overline{Z}$

F = $\overline{Z}\,W\overline{Y}\,W$

Since A, B, and E are symmetric with C, D, and F, respectively, it is only necessary to find $|fix(\phi)|$ for each element with respect to A, B, and E.

Now we must find $|fix(\phi)|$ for all $\phi \in G$.

$|fix(e)|$:

Each translation block has 4 choices for W, 4 choices for $\overline{Z}$, and 3 choices for $\overline{Y}$.
Thus, $|fix(e)| = 6(4 \cdot 4 \cdot 3) = 288$.

The other elements:
When looking at the elements in G, one can see that the W in the upper left corner of the translation block maps to other tiles such that no tile is fixed. In some instances, W is mapped to a rotation of itself, which does not yield a fixed signature. At other times, W is mapped to a reflected tile, which also fails to give a fixed signature. There are even some instances where Y is equal to Z, which is also not allowed.

In the end, the identity is the only element that fixes any translation blocks in this subcase.

Thus, when we apply Burnside's Theorem, we find:

$$\frac{1}{|G|}\sum_{\phi\in G}|fix(\phi)| = \frac{1}{16}(288) = 18.$$

Therefore, there are 18 distinct patterns when the two Block 1's have the same rotation, but the two Block 2's have different rotations for 2 tiles of Block 1 and 2 tiles of Block 2.

Consequently, the total number of distinct patterns without restrictions on rotations in Escher's Case II is 10 + 39 +18 = 67.□

<u>Escher's Case III</u>

Theorem: *There 154 distinct patterns in the totally unrestricted case. This case removes all restrictions on rotations and number of tiles of Block 1 and number of tiles of Block 2 used.*

Proof: This proof entails finding the number of distinct patterns when there are:

i. 4 copies of Block 1 and no copies of Block 2, which we found to be 23,

ii. 2 copies of Block 1 and 2 copies of Block 2, with either the copies of Block 1 having the same rotation and the copies of Block 2 having the same rotation or the copies of Block 1 having different rotations and the copies of Block 2 having different rotations, which we found to be 49,

iii. 2 Block 1's and 2 Block 2's, with either the copies of Block 1 having the same rotation and the copies of Block 2 having different rotations or the copies of Block 1 having different rotations and the copies of Block 2 having the same rotation, which we found to be 18, and

iv. 3 copies of Block 1 and 1 copy of Block 2.

We know parts i, ii, and iii. Now we must examine part iv.
This situation requires G=$\mathbf{K_4Z_4}$. (If we used G=$\mathbf{K_4D_4}$, the reflections of the translation block would fail to permute the signatures in this subset.)

Now we must find $|fix(\phi)|$ for all $\phi \in G$.

$|fix(e)|$:
There are 4 choices for each tile in the translation block. There are 4 possible rotations of the translation block, therefore the identity fixes $4(4^4) = 4^5 = 1024$ possible signatures.

The other elements:
Every $\phi \in G$ (other than e) rotates and/or reflects all of the tiles within the translation block.Therefore the reflected tile will either be moved to a non-reflected tile or it will be unreflected. Thus, each $\phi \in G$ (other than e) fixes nothing.

Applying Burnside's Theorem:

$$\frac{1}{|G|}\sum_{\phi \in G}|fix(\phi)| = \frac{1}{16}(1024) = 64.$$

Hence the total number of distinct patterns for the totally unrestricted case is
23 + 49 + 18 + 64 = 154.□

3. Ribbons

Escher also experimented with other patterns, including ribbons. He considered ribbons as continuous strands where some strands are woven under others. Escher used 4 types of ribbon tiles:

i. the original ribbon motif (Q),
ii. reflections of the original ($\overline{Q}$),
iii. switching the over-under actions of the original (Q_u), and
iv. reflections of the switch of the over-under actions ($\overline{Q_u}$).

<u>Case I</u>

This case considers pairs of the above types of stamps. These will be similar to Escher's cases I and II, where there are two tiles for each type. The possibilities are:

A: i and ii
B: i and iii
C: i and iv
D: ii and iii
E: ii and iv
F: iii and iv

Subcase A is analogous to Escher's Case III, where there are 154 distinct patterns. (Subcase F also produces the same result.) In this situation, 4 copies of Block 1 with no copies of Block 2 is equivalent to no copies of Block 1 with 4 copies of Block 2, 3 copies of Block 1 with 1 copy of Block 2 is equivalent to 1 copy of Block 1 with 3 copies of Block 2, and 2 copies of Block 1 having the same rotation with 2 copies of Block 2 having different rotations is equivalent to 2 copies of Block 1 having different rotations with 2 copies of Block 2 having the same rotation. For subcase B, these equivalencies do not hold and thus are counted distinctly. Therefore, the total for subcase B, with all restrictions removed, is 23 + 23 + 64 + 64 + 10 + 39 = 259. Subcase E is equivalent to B, so the total for E is also 259. The remaining cases, C and D, are also equivalent. The calculations for 4 copies of Block 1 with no copies of Block 2, 3 copies of Block 1 with 1 copy of Block 2, and 2 copies of Block 1 having the same rotation with 2 copies of Block 2 having different rotations will remain the same as in Escher's cases. However, when there are 2 copies of Block 1 having the same rotation with 2 copies of Block 2 having the same rotation and 2 copies of Block 1 having different rotations with 2 copies of Block 2 having different rotations, the group G = $\mathbf{K_4}\ \mathbf{D_4}$ cannot be used because the translation block would fail to permute the signatures in this subset. Thus these two situations must be examined using $G = \mathbf{K_4}\ \mathbf{Z_4}$.

Theorem: *There are 12 distinct patterns of ribbons when there are 2 tiles and 2 tiles with reflections of the switched over-under actions where each pair has the same rotation.*

Proof: Our possibilities for 2 copies of Block 1 having the same rotation with 2 copies of Block 2 having the same rotation are:

H = $XX\overline{W_u}\ \overline{W_u}$ K = $\overline{W_u}\ \overline{W_u}\ XX$

I = $\overline{W_u}\ XX\overline{W_u}$ L = $X\overline{W_u}\ \overline{W_u}\ X$

J = $X\overline{W_u}\ X\overline{W_u}$ M = $\overline{W_u}\ X\overline{W_u}\ X$

Since H, I, and J are symmetric with P, L, and M, respectively, it is only necessary to find $\left|fix(\phi)\right|$ for each element with respect to H, I, and J.

Now we must find $\left|fix(\phi)\right|$ for all $\phi \in G$, where $G =$ **K4 Z4.**

$\left|fix(e)\right|$:

There are 4 choices for W and 4 choices for $\overline{W_u}$. Taking into account the 6 possibilities, we find that there are $6 \cdot 4 \cdot 4 = 96$ possible signatures.

Looking back at Escher's Case IIa, we find:

$\left|fix(R_{90})\right| = 0$

$\left|fix(R_{180})\right| = 0$

$\left|fix(R_{270})\right| = 0$

$\left|fix(F_V)\right| = 32$

$\left|fix(F_H)\right| = 32$

$\left|fix(F_D)\right| = 32$

$\left|fix(F_V R_{90})\right| = 0$

$\left|fix(F_V R_{180})\right| = 0$

$\left|fix(F_V R_{270})\right| = 0$

$\left|fix(F_H R_{90})\right| = 0$

$\left|fix(F_H R_{180})\right| = 0$

$\left|fix(F_H R_{270})\right| = 0$

$\left|fix(F_D R_{90})\right| = 0$

$\left|fix(F_D R_{180})\right| = 0$

$\left|fix(F_D R_{270})\right| = 0$

Thus, when we apply Burnside's Theorem, we find:

$$\frac{1}{|G|}\sum_{\phi\in G}\left|fix(\phi)\right| = \frac{1}{16}(96 + 32 + 32 + 32) = 12.$$

Therefore, there are 12 distinct patterns of ribbons when there are 2 tiles and 2 tiles with reflections of the switched over-under actions where each pair has the same rotation.□

Theorem: ***T****he total number of distinct patterns for subcases C and D is 282.*

Proof: Our possibilities for 2 copies of Block 1 having different rotations with 2 copies of Block 2 having different rotations are:

$N = WX\overline{Y_u}\ \overline{Z_u}$ $S = \overline{Z_u}\ \overline{Y_u}\ WX$

$O = \overline{Z_u}\ WX\overline{Y_u}$ $T = W\overline{Z_u}\ \overline{Y_u}\ X$

$P = W\overline{Y_u}\ X\overline{Z_u}$ $V = \overline{Z_u}\ W\overline{Y_u}\ X$

Since N, O, and P are symmetric with S, T, and V, respectively, it is only necessary to find $|fix(\phi)|$ for each element with respect to H, I, and J.

Now we must find $|fix(\phi)|$ for all $\phi \in G$, where $G = \mathbf{K_4}\ \mathbf{Z_4}$.

$|fix(e)|$:

There are 4 choices for W, 4 for $\overline{Y_u}$, 3 for X, and 3 for $\overline{Z_u}$ for each of the 6 possible translation blocks. Thus there are $6 \cdot (4 \cdot 3) \cdot (4 \cdot 3) = 864$ possible signatures.

Looking back at Escher's Case IIb, we find:

$|fix(R_{90})| = 0$

$|fix(R_{180})| = 32$

$|fix(R_{270})| = 0$

$|fix(F_V)| = 0$

$|fix(F_H)| = 0$

$|fix(F_D)| = 0$

$|fix(F_V R_{90})| = 0$

$|fix(F_V R_{180})| = 32$

$|fix(F_V R_{270})| = 0$

$|fix(F_H R_{90})| = 0$

$|fix(F_H R_{180})| = 32$

$|fix(F_H R_{270})| = 0$

$|fix(F_D R_{90})| = 0$

$|fix(F_D R_{180})| = 0$

$|fix(F_D R_{270})| = 0$

Thus, when we apply Burnside's Theorem, we find:

$$\frac{1}{|G|}\sum_{\phi \in G} |fix(\phi)| = \frac{1}{16}(864 + 32 + 32 + 32) = 60.$$

Therefore, the total number of distinct patterns for subcases C and D is
$23 + 23 + 64 + 64 + 18 + 18 + 12 + 60 = 282.$ □

Case II

Theorem: *For translation blocks with 3 different types of ribbon tiles, there are 48 distinct patterns when the repeated type of block has the same rotation and 192 distinct patterns when the repeated type of block has different rotations.*

Proof: We consider when there are 3 different types of ribbon tiles within the translation block. There are 12 subcases to consider where each subcase has 12 different translation blocks to check. Then those 12 different cases need to be examined when the repeated blocks have the same rotation or when they have different rotations. As in Case III of the tilings, only the identity fixes elements. We need to use G = **K₄ Z₄** for closure. The 12 subcases to consider are:

A: $WX\overline{Y}Z_u$

B: $W\overline{Y}\ \overline{Z}X_u$

C: $W\overline{Z}X_uY_u$

D: $WX\overline{Y}\ \overline{Z_u}$

E: $W\overline{X}\ \overline{Y}\ \overline{Z_u}$

F: $W\overline{X}\ \overline{Y_u}\ \overline{Z_u}$

G: $WXY_u\overline{Z_u}$

H: $WX_uY_u\overline{Z_u}$

I: $WX_u\overline{Y_u}\ \overline{Z_u}$

J: $\overline{W}\ \overline{X}\ Y_u\overline{Z_u}$

K: $\overline{W}X_uY_u\overline{Z_u}$

L: $\overline{W}X_u\overline{Y_u}\ \overline{Z_u}$

Now we must find $|fix(\phi)|$ for the subcases.

A: Repeated blocks have same rotation: $WW\overline{X}Y_u$

4 choices for W, 4 for X, 4 for Y_u, thus there are $12 \cdot 4^3 = 768$ possible signatures.

Burnside's Theorem: $\frac{1}{16}(768) = 48.$

Repeated blocks have different rotations: $WX\overline{Y}Z_u$

4 choices for W, 4 for X, 4 for $\overline{Y}$, 4 for Z_u, thus there are $12 \cdot 4^4 = 3072$ possible signatures.

Burnside's Theorem: $\frac{1}{16}(3072) = 192.$

B: Repeated blocks have same rotation: $\overline{W}\ \overline{W}\ XY_u$

4 choices for W, 4 for X, 4 for Y_u, thus there are $12 \cdot 4^3 = 768$ possible signatures.

Burnside's Theorem: $\frac{1}{16}(768) = 48.$

Repeated blocks have different rotations: $\overline{W}\ \overline{X}\ YZ_u$

4 choices for $\overline{W}$ 4 for $\overline{X}$, 4 for Y, 4 for Z_u, thus there are $12 \cdot 4^4 = 3072$

possible signatures.

Burnside's Theorem: $\frac{1}{16}(3072) = 192$.

This process continues for the remaining subcases. For each subcase, there are 48 distinct patterns when the repeated type of block has the same rotation and 192 distinct patterns when the repeated type of block has different rotations.□

Case III

Theorem: *For translation blocks with 4 different types of ribbon tiles where each tile may have a different rotation, there are 132 distinct patterns of ribbons.*

Proof: For the last case that is considered in this paper, we examine the situation where each tile in the translation block is a different type and each tile has a different rotation. There are 24 ways in which to order the four different tiles, so there are 24 translation blocks to consider. Since we have introduced the switching of the over-under actions of the ribbons, we will need to use a new group, $G = \mathbf{K_4}\,\mathbf{D_4} \oplus \mathbf{Z_2}$, so $|G| = 64$. The elements are:

$e = WXYZ$

$R_{90}(WXYZ) = Z'\,W'\,X'\,Y'$

$R_{180}(WXYZ) = Y''\,Z''\,W''\,X''$

$R_{270}(WXYZ) = W'''\,Y'''\,Z'''\,W'''$

$F_{VB}(WXYZ) = \overline{X}\,\overline{W}\,\overline{Z}\,\overline{Y}$

$F_{HB}(WXYZ) = \overline{Z}''\,\overline{Y}''\,\overline{X}''\,\overline{W}''$

$F_{LB}(WXYZ) = \overline{W}'\,\overline{Z}'\,\overline{Y}'\,\overline{X}'$

$F_{RB}(WXYZ) = \overline{Y}'''\,\overline{X}'''\,\overline{W}'''\,\overline{Z}'''$

$F_V(WXYZ) = XWZY$

$F_H(WXYZ) = ZYXW$

$F_D(WXYZ) = YZXW$

$F_VF_{LB}(WXYZ) = \overline{Z}'\,\overline{W}'\,\overline{X}'\,\overline{Y}'$

$F_VF_{RB}(WXYZ) = \overline{X}'''\,\overline{Y}'''\,\overline{Z}'''\,\overline{W}'''$

$F_HR_{90}(WXYZ) = Y'\,X'\,W'\,Z'$

$F_HR_{180}(WXYZ) = X''\,W''\,Z''\,Y''$

$F_HR_{270}(WXYZ) = W'''\,Z'''\,Y'''\,X'''$

$F_HF_{VB}(WXYZ) = \overline{Y}\,\overline{Z}\,\overline{W}\,\overline{X}$

$F_HF_{HB}(WXYZ) = \overline{W}''\,\overline{X}''\,\overline{Y}''\,\overline{Z}''$

$F_HF_{LB}(WXYZ) = \overline{X}'\,\overline{Y}'\,\overline{Z}'\,\overline{W}'$

$F_HF_{RB}(WXYZ) = \overline{Z}'''\,\overline{W}'''\,\overline{X}'''\,\overline{Y}'''$

$F_DR_{90}(WXYZ) = X'\,Y'\,Z'\,W'$

$F_DR_{180}(WXYZ) = W''\,X''\,Y''\,Z''$

$F_VR_{90}(WXYZ) = W' Z' Y' X'$

$F_DR_{270}(WXYZ) = Z''' W''' X''' Y'''$

$F_VR_{180}(WXYZ) = Z'' Y'' X'' W''$

$F_DF_{VB}(WXYZ) = \overline{Z}\ \overline{Y}\ \overline{X}\ \overline{W}$

$F_VR_{270}(WXYZ) = Y''' X''' W''' Z'''$

$F_DF_{HB}(WXYZ) = \overline{X}''\ \overline{W}''\ \overline{Z}''\ \overline{Y}''$

$F_VF_{VB}(WXYZ) = \overline{W}\ \overline{X}\ \overline{Y}\ \overline{Z}$

$F_DF_{LB}(WXYZ) = \overline{Y}'\ \overline{X}'\ \overline{W}'\ \overline{Z}'$

$F_VF_{HB}(WXYZ) = \overline{Y}''\ \overline{Z}''\ \overline{W}''\ \overline{X}''$

$F_DF_{RB}(WXYZ) = \overline{W}'''\ \overline{Z}'''\ \overline{Y}'''\ \overline{X}'''$

$U(WXYZ) = W_uX_uY_uZ_u$

$UR_{90}(WXYZ) = Z_u' W_u' X_u' Y_u'$

$UR_{180}(WXYZ) = Y_u''Z_u''W_u''X_u''$

$UR_{270}(WXYZ) = X_u'''Y_u'''Z_u'''W_u'''$

$UF_{VB}(WXYZ) = \overline{X_u}\,\overline{W_u}\,\overline{Z_u}\,\overline{Y_u}$

$UF_{HB}(WXYZ) = \overline{Z_u}''\ \overline{Y_u}''\ \overline{X_u}''\ \overline{W_u}''$

$UF_{LB}(WXYZ) = \overline{W_u}'\ \overline{Z_u}'\ \overline{Y_u}'\ \overline{X_u}'$

$UF_{RB}(WXYZ) = \overline{Y_u}'''\ \overline{X_u}'''\ \overline{W_u}'''\ \overline{Z_u}'''$

$UF_V(WXYZ) = X_u W_u Z_u Y_u$

$UF_H(WXYZ) = Z_u Y_u X_u W_u$

$UF_D(WXYZ) = Y_u Z_u W_u X_u$

$UF_VR_{90}(WXYZ) = W_u' Z_u' Y_u' X_u'$

$UF_VR_{180}(WXYZ) = Z_u'' Y_u'' X_u'' W_u''$

$UF_VR_{270}(WXYZ) = Y_u'''X_u'''W_u'''Z_u'''$

$UF_VF_{VB}(WXYZ) = \overline{W_u}\,\overline{X_u}\,\overline{Y_u}\,\overline{Z_u}$

$UF_VF_{HB}(WXYZ) = \overline{Y_u}''\ \overline{Z_u}''\ \overline{W_u}''\ \overline{X_u}''$

$UF_VF_{LB}(WXYZ) = \overline{Z_u}'\ \overline{W_u}'\ \overline{X_u}'\ \overline{Y_u}'$

$UF_HR_{90}(WXYZ) = Y_u' X_u' W_u' Z_u'$

$UF_VF_{RB}(WXYZ) = \overline{X_u}'''\ \overline{Y_u}'''\ \overline{Z_u}'''\ \overline{W_u}'''$

$UF_HR_{180}(WXYZ) = X_u'' W_u'' Z_u'' Y_u''$

$UF_HR_{270}(WXYZ) = W_u''' Z_u''' Y_u''' X_u'''$

$UF_HF_{VB}(WXYZ) = \overline{Y_u}\,\overline{Z_u}\,\overline{W_u}\,\overline{X_u}$

$UF_HF_{HB}(WXYZ) = \overline{W_u}''\ \overline{X_u}''\ \overline{Y_u}''\ \overline{Z_u}''$

$UF_HF_{LB}(WXYZ) = \overline{X_u}'\ \overline{Y_u}'\ \overline{Z_u}'\ \overline{W_u}'$

$UF_HF_{RB}(WXYZ) = \overline{Z_u}'''\ \overline{W_u}'''\ \overline{X_u}'''\ \overline{Y_u}'''$

$UF_DR_{90}(WXYZ) = X_u' Y_u' Z_u' W_u'$

$UF_DR_{180}(WXYZ) = W_u'' X_u'' Y_u'' Z_u''$

$UF_DF_{VB}(WXYZ) = \overline{Z_u}\,\overline{Y_u}\,\overline{X_u}\,\overline{W_u}$

$UF_DR_{270}(WXYZ) = \overline{Z_u}'''\ \overline{W_u}'''\ \overline{X_u}'''\ \overline{Y_u}'''$

$UF_DF_{HB}(WXYZ) = \overline{X_u}''\ \overline{W_u}''\ \overline{Z_u}''\ \overline{Y_u}''$

$UF_DF_{RB}(WXYZ) = \overline{W_u}'''\,\overline{Z_u}'''\,\overline{Y_u}'''\,\overline{X_u}'''$ $UF_DF_{LB}(WXYZ) = \overline{Y_u}'\,\overline{X_u}'\,\overline{W_u}'\,\overline{Z_u}'$

The 24 subcases that need to be considered are all permutations of $W\overline{X}\ Y_u\overline{Z_u}$.

$|fix(e)|$:

There are 4 choices for each tile in the translation block and 24 subcases. Thus $|fix(e)|$ = $24 \cdot 4^4 = 6144$.

The other elements:

ϕ	$\lvert fix(\phi)\rvert$
R_{90}	0
R_{180}	0
R_{270}	0
F_{VB}	128
F_{HB}	128
F_{LB}	0
F_{RB}	0
F_V	0
F_H	0
F_D	0
F_VR_{90}	0
F_VR_{180}	0
F_VR_{270}	0
F_VF_{VB}	0
F_VF_{HB}	128
F_VF_{LB}	0
F_VF_{RB}	0
F_HR_{90}	0
F_HR_{180}	0
F_HR_{270}	0
F_HF_{VB}	128

ϕ	$\lvert fix(\phi)\rvert$
F_HF_{HB}	0
F_HF_{LB}	0
F_HF_{RB}	0
F_DR_{90}	0
F_DR_{180}	0
F_DR_{270}	0
F_DF_{VB}	128
F_DF_{HB}	128
F_DF_{LB}	0
F_DF_{RB}	0
UR_{90}	0
UR_{180}	128
UR_{270}	0
UF_{VB}	128
UF_{HB}	128
UF_{LB}	0
UF_{RB}	0
UF_V	128
UF_H	128
UF_D	128
UF_VR_{90}	0

ϕ	$\lvert fix(\phi)\rvert$
UF_VR_{180}	128
UF_VR_{270}	0
UF_VF_{VB}	0
UF_VF_{HB}	128
UF_VF_{LB}	0
UF_VF_{RB}	0
$U\,F_HR_{90}$	0
UF_HR_{180}	128
UF_HR_{270}	0
UF_HF_{VB}	128
UF_HF_{HB}	0
UF_HF_{LB}	0
UF_HF_{RB}	0
UF_DR_{90}	0
UF_DR_{180}	0
UF_DR_{270}	0
UF_DF_{VB}	128
UF_DF_{HB}	128
UF_DF_{LB}	0
UF_DF_{RB}	0

Thus, when we apply Burnside's Theorem, we find:

$$\frac{1}{|G|}\sum_{\phi\in G}|fix(\phi)| = \frac{1}{64}(6144 + 128(18)) = 132.$$

Therefore there are 132 distinct patterns of ribbons using 4 different types of tiles, where all of the tiles have different rotations.□

4. Further Considerations

There is much future research that may be done in the study of ribbons. For instance, the previous case may be considered where all of the tiles, two of the tiles, or three of the tiles have the same rotation. Within each of those subcases, one can vary which tiles have the same rotation – types i and ii; i, iii, and iv, etc. Escher's Cases may also be extended into three dimensions where instead of four square tiles there would be eight cubes in a translation block.

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