

# Determining conditions sufficient for the existence of arc-disjoint hamiltonian paths and out-branchings in tournaments

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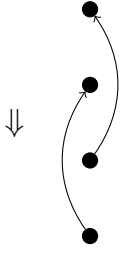
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## Abstract

We study problems closely related to the question raised by Bang-Jensen in [1]: *When does a tournament have a hamiltonian path and out-branching rooted at the same vertex, arc-disjoint from one another?* We answer this question under various circumstances, in some cases managing to show that the out-branching can be taken to be path-like or star-like, and in other cases showing that multiple out-branchings can be found that are arc-disjoint from each other.

## 1 Introduction and graph theoretic preliminaries

We begin our study with a discussion of general terms used in graph theory. Formally, for a set  $V$  and a set  $E$  of unordered pairs of elements of  $V$ , we define the ordered pair  $G = (V, E)$  to be a *graph* on the set  $V$ . The elements of  $V$  are called the *vertices* of  $G$ . If  $v_1v_2 \in E(G)$ , we call  $v_1v_2$  the *edge* in  $G$  joining  $v_1$  and  $v_2$ . The set  $E$  is the set of *edges* of  $G$ . For a particular graph  $G$  we often write  $V = V(G)$  and  $E = E(G)$ . A *complete graph* is a graph where given any distinct  $v, w \in V(G)$ ,  $vw \in E(G)$ .



**Figure 1:** A tournament on 4 vertices: in the right representation, the large arrow signifies that all arcs not shown emanate in the direction indicated.

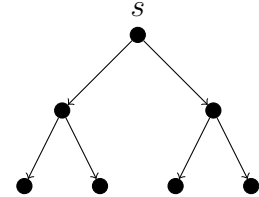
By contrast, in a *digraph* or *directed graph*  $D$ , each edge of  $D$  is an ordered pair and thus has an associated direction or orientation. The edges of a digraph  $D$  are called the *arcs* of  $D$ . For a particular digraph, we often write  $V = V(D)$  and  $A = A(D)$ . Given an arc  $a = vw$ , we call  $v$  the *tail* of  $a$  and  $w$  the *head* of  $a$ .

In this paper we focus on the class of digraphs called tournaments. A *tournament* is an oriented complete graph  $D$ , a special kind of digraph, in which for each distinct  $v, w \in V(D)$ , either  $vw \in A(D)$  or  $wv \in A(D)$ , but not both. We display an example tournament in Fig.1. We often use the letter  $T$  to denote a tournament digraph.

A *walk*  $W$  in a digraph  $D$  is an alternating sequence of vertices and edges  $v_0v_1 \dots v_{k-1}v_k$  such that  $v_{i-1}v_i \in A(D)$  for  $i = 1, 2, \dots, k-1$ . A *path*  $P$  is a walk with no repeated vertices. A *cycle*  $C$  is a walk with  $v_0 = v_k$  and no other pair of repeated vertices. We say that path  $P$  or cycle  $C$  is *hamiltonian* if  $V(D)$  is its vertex set, i.e. it visits each vertex in  $V(D)$ .

The *out-neighborhood*  $N^+(v)$  of vertex  $v$  in digraph  $D$  is the set of all vertices of  $D$  dominated by  $v$ . The *in-neighborhood*  $N^-(v)$  of vertex  $v$  is the set of all vertices of  $D$  which dominate  $v$ . The number of vertices in the out-neighborhood (in-neighborhood) of  $v$  is the *out-degree*  $d^+(v)$  (*in-degree*  $d^-(v)$ ) of  $v$ . A digraph  $D$  is called *k-regular* if  $d^+(v) = d^-(v) = k$  for every vertex  $v$  in  $D$ .

An *out-branching*  $F_s^+$  rooted at  $s$  in a digraph  $D$  is a connected spanning sub-digraph of  $D$  in which every vertex  $x \neq s$  has exactly one arc entering it and  $s$  has no arcs entering it. Note that a hamiltonian path is an out-branching. Fig 2 is an example of an out-branching.



**Figure 2:** An out-branching rooted at  $s$ .

Finally, we say that a digraph  $D$  is *strong* or *irreducible* if given any  $v, w \in V(D)$  there is a walk  $W$  from  $v$  to  $w$ . We say that  $D$  is *k-strong* if  $D$  has at least  $k + 1$  vertices and  $D - X$  is strong for every subset  $X$  of  $V(D)$  with  $|X| < k$ . If a tournament  $T$  not strong, then it is called *reducible*. The reader is referred to [1] for these and other standard notations.

## 1.1 Some motivating questions

With the above definitions in hand, we move now to discuss the primary questions of our research. We study several problems related to the following question raised by Bang-Jensen in [1].

**Problem 1.** (Bang-Jensen [1]) “When does a tournament contain two arc-disjoint out-branchings  $F_{s,1}^+, F_{s,2}^+$  both rooted in the same vertex such that one of these is a hamiltonian path from  $s$ ?”.

**Problem 2.** When does a tournament contain a hamiltonian cycle and out-branching that are arc-disjoint?

From this point forward, we will let  $P$  denote a hamiltonian path, let  $C$  denote a hamiltonian cycle, and let  $F$  denote an out-branching in a tournament or digraph. We refer to the question posed in Problem 1 as finding a  $(P, F)$ -pair, and that question posed in Problem 2 as finding a  $(C, F)$ -pair. Before continuing, we mention some standard facts about tournaments.

**Fact 1.** A tournament  $T$  is *strong* if and only if it has a hamiltonian cycle.

**Fact 2.** Every tournament  $T$  has a unique decomposition into strong components  $T^1, T^2, \dots, T^{l-1}, T^l$ , where every vertex  $T^i$  dominates every vertex in  $T^j$  for  $j < i$ . Tournament  $T$  is *strong* when  $l = 1$ , *reducible* when  $l > 1$ .

## 2 Reducible Tournaments

It is well known that every tournament  $T$  has a hamiltonian path, seen as follows. If  $T$  is strong, then it has a hamiltonian cycle. If  $T$  is reducible, then each strong component has a hamiltonian path, and we may construct a hamiltonian path for the entire tournament by connecting the initial and final vertices of consecutive strong components.

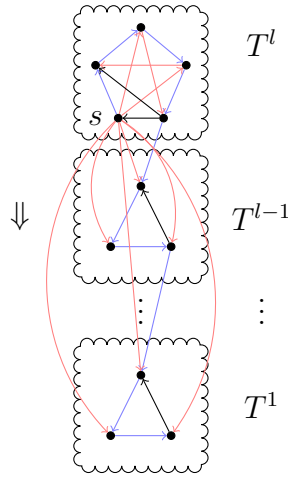
Since the existence of a hamiltonian path beginning at some vertex  $s$  is guaranteed for any tournament, the main concern in our study is as follows: given a hamiltonian path  $P$  rooted at  $s$  in tournament  $T$ , determine whether  $T$  has a  $(P, F)$ -pair for that  $s$  and that  $P$ . We begin our study by considering reducible tournaments, and we make the following proposition:

**Theorem 3:** Let  $T$  be a reducible tournament and let  $T$  be decomposed into strong components  $T^1, T^2, \dots, T^{l-1}, T^l$ , where  $T^i$  dominates every vertex in  $T^j$  for  $j < i$  and  $|T^1| > 1$ . Let  $s$  be a vertex in  $T$ . A  $(P, F)$ -pair rooted at  $s$  exists in  $T$  if and only if  $s$  is in  $T^l$  and a  $(P_l, F_l)$ -pair exists in  $T^l$ .

*Proof:*

It is easily verified that a hamiltonian path starting at  $s$  exists if and only if  $s$  is in  $T^l$ , so we assume that  $s$  is in  $T^l$ . It is also easily shown that if  $|T^l| = 1$  and  $|T^{l-1}| = 1$ , then a  $(P, F)$ -pair does not exist in  $T$ . Suppose that  $T$  has a  $(P, F)$ -pair based at  $s$ . The path  $P$  visits each vertex in  $T$  once and only once, so since  $P$  begins at  $s$ , it must be that  $P$  visits each vertex of  $T^l$  before visiting any vertices not in  $T^l$ . So we choose path  $P_l$  to be the partition of  $P$  within  $T^l$ . By similar reasoning, the out-branching  $F$  spans  $T^l$  and does so in a connected manner; no branch of  $F$  may pass through vertices in lower components and return to  $T^l$ , for no arcs from lower components to  $T^l$  exist. The restriction of  $F$  in this way yields  $F_l$ .

Now suppose that there is a  $(P_l, F_l)$ -pair in  $T^l$ . To find  $P$ , consider  $T - T_l$ . This sub-tournament also has the existence of a hamiltonian path. Connect the last vertex in  $P_l$  to



**Figure 3:** An example of a reducible tournament with a  $(P, F)$  - pair rooted at  $s$ .

the first vertex in our new path to create  $P$ . To obtain the out-branching  $F$ , simply take arcs emanating from  $s$  directly to each vertex in the lower components. Combining this tree with  $F_l$  yields an out-branching from  $s$  which is arc-disjoint from  $P$ . ■

The previous theorem shows that the existence of a  $(P, F)$ -pair in a reducible tournament is conditional upon the existence of a  $(P, F)$ -pair in the tournament's dominant strong component. We must therefore investigate properties of strong tournaments and determine which are sufficient to guarantee the existence of a  $(P, F)$ -pair.

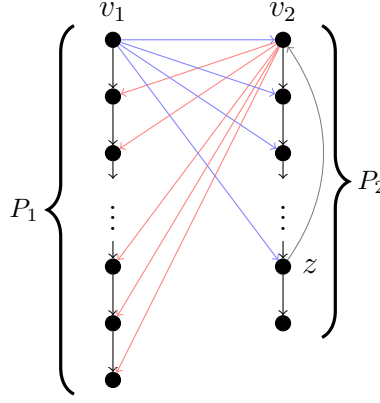
### 3 Strong Tournaments

We see now that the existence of a  $(P, F)$ -pair in a reducible tournament is equivalent to the existence of a  $(P, F)$ -pair in the tournament's dominant strong component. We must therefore investigate properties of strong tournaments and determine which are sufficient to guarantee the existence of a  $(P, F)$ -pair. We make use of the Gallai-Milgram theorem, a proof of which can be found in [2]. In a path partition, the paths used must be vertex disjoint and collectively span the vertex set of the graph containing them. We mention also that  $\alpha(D)$  is the independence number of a digraph  $D$ , the size of the largest set of vertices of  $D$  between which no arc exists.

**Theorem 2:** Every digraph  $D$  has an  $\alpha(D)$ -path partition.

Rather than seek a  $(P, F)$ -pair, we find it convenient to show that a  $(C, F)$ -pair, implying the existence of a  $(P, F)$ -pair. We start with an arbitrary hamiltonian cycle  $C$ . When the arcs of  $C$  are discarded from a tournament  $T$ , the leftover arcs form a digraph. The Gallai-Milgram Theorem says that this leftover digraph contains a structure not much different from a hamiltonian path. In order to obtain an out-branching, we show that the paths promised by the Gallai-Milgram theorem have connecting arcs allowing us to join the paths into one structure.

**Theorem 3:** Let  $T$  be a strong tournament with at least 5 vertices such that  $d^-(v) > 1$  for all  $v \in V(T)$ . Then  $T$  has a  $(C, F)$ -pair for each hamiltonian cycle  $C$  in  $T$ .



**Figure 4:** Constructing the out-branching  $F$ .

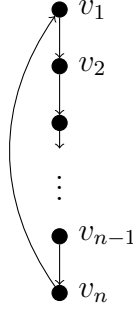
*Proof:* Let  $C$  be a hamiltonian cycle of strong tournament  $T$ . Consider the digraph  $D = T - A(C)$ . Note that  $\alpha(D) = 2$ . It follows by the Gallai-Milgram theorem that  $D$  has vertex disjoint paths,  $P_1, P_2$  (with  $P_2$  possibly empty) such that  $V(D) = V(P_1) \cup V(P_2)$ . If  $P_2$  is empty, then we are done via the chose  $F = P_1$ , so we may assume that  $PP_2$  is non-empty, as diagrammed in Fig. 1. We will show that there must exist arcs between  $P_1$  and  $P_2$  such that an out-branching  $F$  of  $D$  may be constructed.

To begin, consider arcs between  $v_1$  and  $v_2$ . If either  $v_1v_2$  or  $v_2v_1$  is in  $A(D)$ , then we make  $v_1$  or  $v_2$ , respectively, the root of  $F$ . If no such arc exists in  $A(D)$ , then assume without loss of generality that  $v_1v_2$  is in  $C$ . In this case, we consider arcs between  $P_1$  and  $v_2$ ; if for some  $x \in V(P_1)$  the arc  $xv_2 \in A(D)$ , then we make  $v_1$  the root of  $F$  and we are done. Similarly, we are done if for some  $y \in V(P_2)$  the arc  $yv_1 \in A(D)$ .

So, we may assume that there are no such arcs. By our assumption that  $d^-(v) > 1$  for all  $v \in T$ , then there must exist an arc  $zv_2 \in A(D)$  for some  $z \in P_2$ . Also, all arcs between  $v_1$  and  $P_2$  in  $A(D)$  have  $v_1$  as their tails. Thus there is an arc  $v_1y \in A(D)$  for some  $y$  between  $v_2$  and  $z$  inclusive along  $P_2$ . We construct  $F$  from  $P_1 \cup P_2 \cup v_1y$  by making  $v_1$  the root and deleting the arc lying in  $P_2$  having head at  $y$ . This completes the proof. ■

Ideally, we want to show every strong tournament has a  $(C, F)$ -pair. However, we point out that if a strong tournament has more than one vertex  $v$  with  $d^-(v) = 1$ , then no  $(C, F)$ -pair exists, as follows. Suppose,  $d^-(v_1) = d^-(v_2) = 1$ . If  $v_1$  is selected as the root vertex,  $s$ , then  $v_1v_2$  is the only available arc into  $v_2$ . Thus, no  $(C, F)$ -pair exists since each of  $C$  and  $F$  must reach  $v_2$  from  $v_1$ . Similarly, if any other vertex is chosen as  $s$ ,  $v_nv_1$  is the only way into  $v_1$ . Refer to Figure 4 to see this effect as applied to the “One Major Upset” Tournament resulting from a transitive tournament by reversing the arc between the vertices of highest and lowest out-degree.

In the proof of the previous theorem, note that the out-branching  $F$  produced has at most three leaves. We prove basically the same theorem without the use of the Gallai-Milgram theorem. This way the proof is more intuitive and based solely on the in-degree requirements. Also, the out-branching it produces can chosen so at to have many leaves.



**Figure 5:** The One Major Upset Tournament on  $n$  vertices

**Theorem 4:** Every strong tournament  $T$  has a  $(C, F)$ -pair, provided there is at most one vertex  $v$  such that  $d^-(v) = 1$ .

*Proof:* Since  $T$  is strong and has at most one vertex of in-degree one, note that  $T$  has at least five vertices. Let  $C$  be a hamiltonian cycle in  $T$ , and consider  $D = T - A(C)$ . Define  $F_s$  to be an out-branching spanning as many vertices as possible in  $D$ , letting  $s$  denote its root vertex. So there are two vertices, denoted  $y_1$  and  $y_2$ , in  $T$  such that in cycle  $C$  vertex  $y_1$  defeats  $s$  and  $s$  defeats  $y_2$ . Suppose for contradiction that some vertex  $y_3 \neq y_1, y_2$  is not contained in  $F_s$ . Then, either  $sy_3 \in A(D)$  or  $y_3s \in A(D)$ . In the first case,  $F_s$  can be expanded by including  $sy_3$ , and in the second case  $F_s$  can be expanded by including  $y_3s$  and letting  $y_3$  be the new root vertex. Thus, we reach a contradiction either way, so  $F_s$  contains all vertices of  $D$  except possibly  $y_1$  and  $y_2$ .

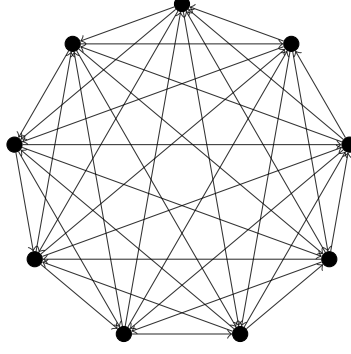
By our assumption, there is at most one vertex  $v$  with  $d^-(v) = 1$ . If there is such a  $v$  with  $d^-(v) = 1$ , we make this vertex  $s$ . Therefore, neither  $y_1$  nor  $y_2$  has in-degree one. Besides  $sy_2$ , a second way into  $y_2$  cannot come from a vertex other than  $y_1$  else we would expand  $F_s$ . Therefore,  $y_1y_2 \in A(D)$ . So, there exists a vertex  $x \neq s, y_2$  such that  $xy_1 \in A(D)$ . Thus,  $y_1 \in V(F_s)$ . Since  $y_1y_2 \in A(D)$ , we also have that  $y_2 \in F_s$ . So,  $F_s$  spans  $V(D)$ .

Thus, every strong  $T_n$  has a  $(C, F)$ -pair. ■

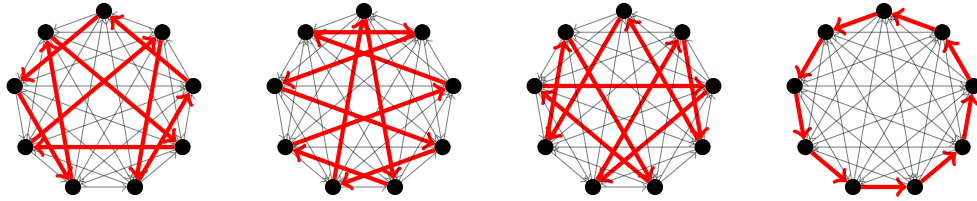
Note: In selecting  $F_s$  we could have selected a vertex  $v$  of average or higher out-degree (so  $d^-(v) \geq (n-1)/2$ ) and chosen  $F_s$  as having as many vertices as possible in  $D$ , while including all arcs out of  $v$  other than its out arc in  $C$ . So, the resulting out-branching  $F_s$  would have a vertex with  $(n-3)/2$  or more arcs out of it in  $F_s$  making  $F_s$  more star-like than path-like.

## 4 $(C, F)$ -pairs and -tuples in regular tournaments

In this section we consider arc-disjoint hamiltonian cycles and out-branchings in *regular* tournaments—tournaments in which every vertex has equal in-degree and out-degree. We say that a tournament is *k-regular* if every vertex has in-degree and out-degree equal to  $k$ .



**Figure 6:** A 4-regular tournament



**Figure 7:** The tournament's four arc-disjoint hamiltonian cycles

Note that there are no regular tournaments on an even number of vertices<sup>1</sup>, for then there are  $\frac{n(n-1)}{2}$  arcs in the tournament, but  $n$  does not divide  $\frac{n(n-1)}{2}$ .

It is easily verified that every  $k$ -regular tournament has exactly  $2k + 1$  vertices. It is also easy to show that every  $k$ -regular tournament is strong, and therefore has at least one hamiltonian cycle. However, observe that in the 4-regular tournament in Figure 6 we can find several hamiltonian cycles that are arc-disjoint. It is much more difficult to prove that regular tournaments have multiple arc-disjoint hamiltonian cycles. In this direction we have the well-known Kelly conjecture, first proposed in 1968:

**Conjecture 1.** (Kelly 1968) Every regular tournament on  $n$  vertices has  $\frac{n-1}{2}$  arc-disjoint hamiltonian cycles. Or alternatively, every  $k$ -regular tournament has  $k$  arc-disjoint hamiltonian cycles.

Rather than trying to make progress on this more long-standing difficult problem of proving the existence of many arc-disjoint hamiltonian cycles, we work on the more promising direction of showing that there must exist many arc-disjoint out-branchings among which one of them is a pre-specified cycle.

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<sup>1</sup>Some authors allow in regular tournaments that the in-degree and out-degree of each vertex differ by at most one, so that regular tournaments on an even number of vertices are permitted.

## 4.1 Edmonds' branching theorem

Edmonds' branching theorem is particularly useful in our investigation because it presents a condition equivalent to the existence of any specified number of arc-disjoint out-branchings rooted at a specified vertex in any digraph. It is also useful in that its proof is algorithmic. For a full proof, see that given in [2].

**Theorem 1:** Let  $D = (V, A)$  and fix vertex  $s$  of  $D$ . Then  $D$  has  $k$  arc-disjoint out-branchings rooted at  $s$  if and only if  $d^-(X) \geq k$  for all nonempty subsets  $X$  of  $V - s$ .

Note that, in some sense, Edmonds' branching theorem answers Problem 1: given a particular hamiltonian path  $P$  in tournament  $T$ , to see whether there exists an out-branching in  $T$  that is arc-disjoint from  $P$  we 'need only' consider each choice of vertex  $s$  and determine whether in  $T - A(P)$  it is true for each nonempty subset  $X$  of  $V(T) - s$  that there exists at least one arc of  $T - A(P)$  having head in  $X$  and tail not in  $X$ . On the one hand, it is not as daunting as it may sound to check  $T - A(P)$  in this fashion for a given  $P$ , since implicit in Edmonds' theorem is an algorithm for finding such an out-branching when one exists. On the other hand, however, one may need to check separately for each choice of root vertex and/or each choice of hamiltonian path  $P$ , so Edmonds' theorem hardly gives us a complete answer.

## 4.2 $(C, F)$ -pairs and -tuples in $k$ -regular tournaments

While we saw in Section 3 that every strong tournament with  $d^-(v) > 1$  for every  $v \in V(T)$  has a hamiltonian path and arc-disjoint from it is an out-branching rooted at some vertex  $s$ , we did not resolve the question when we insist that  $s$  is specified. With the additional condition that a tournament is  $k$ -regular, we find that every  $s \in V(T)$  is the root of a hamiltonian path and out-branching that are arc-disjoint. In fact, in a  $k$ -regular tournament we obtain the stronger result that every  $s \in V(T)$  is the root of a hamiltonian path and  $k - 1$  out-branchings that are arc-disjoint. We begin with the weaker result:

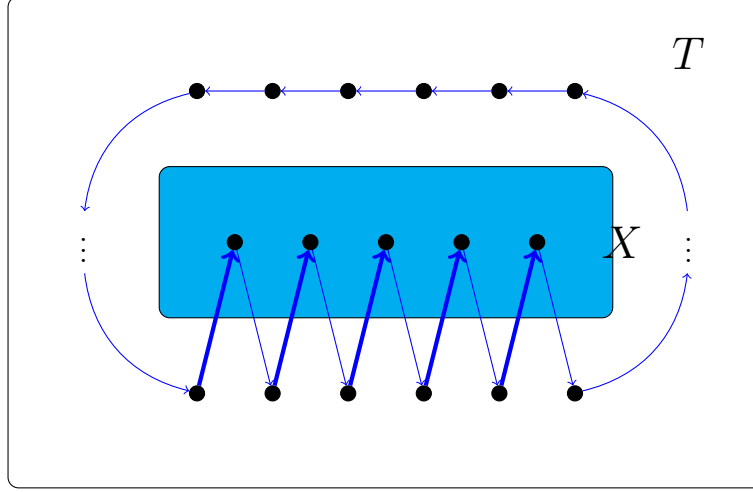
**Theorem 5:** Let  $T$  be a  $k$ -regular tournament on  $2k + 1$  vertices with  $k \geq 2$ . Then for each vertex  $s$  in  $T$  there exist arc-disjoint out-branchings  $F_{s,1}^+, F_{s,2}^+$  such that  $F_{s,1}^+$  is a hamiltonian path.

*Proof:* Let  $C$  be a hamiltonian cycle of  $T$  and consider the remaining digraph  $D = T - A(C)$ . We will show that every vertex of  $D$  is the root of an out-branching of  $D$ .

To begin, suppose for contradiction that some vertex  $s$  of  $D$  is not a root of any out-branching of  $D$ . It follows by Edmonds' branching theorem that there exists a nonempty subset  $X$  of  $V(T) - s$  such that  $d^-(X) = 0$ . Let  $X$  be a minimal such set.

Since  $k \geq 2$ , we have  $|X| \geq 2$ . Consider the set  $X - v$  for some  $v \in X$ . Since  $X$  is minimal, we must have  $d^-(X - v) \geq 1$ . No arcs enter  $X$  from  $V(T) - X$  by definition, so  $v$  dominates at least one vertex in  $X - v$ . For the same reason, we also have  $N^-(v) \subseteq X - v$ . Since  $|N_D^-(v)| = k - 1$ , and no vertex can both dominate and be dominated by  $v$ , it must be





**Figure 8:** Determining the in-degree of a subset  $X$  in a  $k$ -regular tournament  $T$

that  $|X - v| \geq k$ . This gives us  $|X| \geq k + 1$ , and it follows that  $|\overline{X}| \leq k$ . Note that  $s \in \overline{X}$ , and also  $N_D^+(s) \subseteq \overline{X}$ , so that  $|\overline{X}| \geq k$ . Therefore  $|\overline{X}| = k$ .

Now choose some  $w \in N_D^+(s)$ . We have  $|N_D^+(w)| = k - 1$  with  $N_D^+(w) \subseteq \overline{X}$ . But  $s$  dominates  $w$  and  $w \notin N_D^+(w)$ , and there are only  $k - 2$  other vertices in  $\overline{X}$ , not enough to contain  $k - 1$  out-neighbors of  $w$ . We have reached a contradiction and hence  $s$  is a root of an out-branching of  $D$ . ■

The previous theorem shows that in a  $k$ -regular tournament every vertex is the root of an arc-disjoint hamiltonian path and out-branching. However, the requirement that a tournament be  $k$ -regular is hugely restrictive, and it is natural to expect that we can get a stronger result.

Note that the above proof relied on the fact that in a  $k$ -regular tournament  $T$  we know exactly the in-degree and out-degree of each vertex. This information is indeed very powerful, as it allows us to determine the in-degree  $d^-(X)$  of any nonempty subset  $X$  of  $V(T)$ . To see this, note that since  $T$  is  $k$ -regular, each vertex of  $X$  has in-degree  $k$ , so there are  $|X|k$  arcs entering vertices of  $X$  collectively. However, each arc between vertices of  $X$  is not included in  $d^-(X)$ . Since  $X$  may be viewed as a sub-tournament of  $T$ , there are  $\binom{|X|}{2} = \frac{(|X|-1)|X|}{2}$  arcs in  $X$ . Therefore we compute the in-degree of  $X$  to be

$$d^-(X) = |X|k - \frac{(|X| - 1)|X|}{2}.$$

The next theorem uses Edmonds' branching theorem to find for any vertex of a  $k$ -regular tournament  $T$  a hamiltonian cycle and  $k - 1$  out-branchings, the set of  $k$  of these being arc-disjoint.

**Theorem 7:** Let  $T$  be a  $k$ -regular tournament with  $k \geq 2$ . Then every vertex has a  $(C, F_1, \dots, F_{k-1})$ -tuple.

*Proof:* Let  $C$  be a cycle of  $T$  and consider the remaining digraph  $D = T - A(C)$ . Fix a vertex  $s \in V(T)$ . Since  $T$  is  $k$ -regular, for all nonempty subsets  $X$  of  $T$  we have  $d^-(X) = |X|k - \frac{(|X|-1)|X|}{2}$ , as discussed above. At most  $|X|$  arcs entering  $X$  are used in  $C$  if  $|X| < k+1$ , and  $|\bar{X}| = 2k+1 - |X|$  arcs if  $|X| \geq k+1$ . Therefore we have

$$d_D^-(X) \geq \begin{cases} |X|k - \frac{(|X|-1)|X|}{2} - |X| & \text{if } |X| < k+1 \\ |X|k - \frac{(|X|-1)|X|}{2} - (2k+1 - |X|) & \text{if } |X| \geq k+1. \end{cases}$$

Note the  $d_D^-(X) \geq k-1$ . It follows by Edmonds' branching theorem that  $s$  is the root of  $k-1$  arc-disjoint out-branchings in  $D$ .  $\blacksquare$

### 4.3 Some improvements of previous results for $k$ -regular tournaments

A number of results have been obtained showing the existence of multiple arc-disjoint hamiltonian cycles in  $k$ -regular tournaments. As an example, we have the following theorem of C. Thomassen for  $k$ -regular tournaments on a large number of vertices.

**Theorem 8:** (C. Thomassen) Every  $k$ -regular tournament has at least  $\left\lfloor \sqrt{\frac{2k+1}{1000}} \right\rfloor$  arc-disjoint hamiltonian cycles.

The proof of Theorem 7 is quite malleable in that it can be altered and appended to many of the results like Theorem 8. The idea is that if we can find  $n$  arc-disjoint hamiltonian cycles in a  $k$ -regular tournament, then we can find  $k-n$  arc-disjoint out-branchings from any vertex.

**Theorem 9:** Let  $T$  be a  $k$ -regular tournament such that  $T$  has  $n$  arc-disjoint hamiltonian cycles. Then every vertex  $s$  of  $T$  is the root of  $n$  hamiltonian cycles and  $k-n$  out-branchings that are arc-disjoint.

*Proof:* Suppose that  $k$ -regular tournament  $T$  has  $n$  arc-disjoint hamiltonian cycles, which we denote by  $C_1, \dots, C_n$ . Consider the digraph  $D = T - A(\bigcup_{i=1}^n C_i)$  and fix a vertex  $s \in V(T)$ . By an argument similar to the one used in Theorem 6, we have

$$d_D^-(X) \geq \begin{cases} |X|k - \frac{(|X|-1)|X|}{2} - n|X| & \text{if } |X| < k+1 \\ |X|k - \frac{(|X|-1)|X|}{2} - (2k+1 - n|X|) & \text{if } |X| \geq k+1. \end{cases}$$

We get that  $d_D^-(X) \geq k-n$ , so by Edmonds' branching theorem  $s$  is a root of  $k-n$  arc-disjoint out-branchings of  $D$ .  $\blacksquare$

Thus, we have the following improvement of Theorem 8.

**Corollary 9:** Let  $T$  be a  $k$ -regular tournament. Then every vertex  $s$  of  $T$  is the root of  $\left\lfloor \sqrt{\frac{2k+1}{1000}} \right\rfloor$  hamiltonian cycles and  $k - \left\lfloor \sqrt{\frac{2k+1}{1000}} \right\rfloor$  out-branchings that are arc-disjoint.

It is also noted that recent progress has been made toward a proof of the Kelly conjecture. In 2012, Kühn and Osthus provided a proof in [4] for sufficiently large tournaments. The proof is of an alternative formulation of the Kelly conjecture: every regular tournament  $T$  has a *hamiltonian decomposition*—a set of arc-disjoint hamiltonian cycles which covers all the edges of the underlying complete graph of  $T$ .

**Theorem 10:** (Kühn and Osthus 2012) For every  $\varepsilon > 0$  there exists  $n_0$  such that every  $k$ -regular oriented graph  $G$  on  $n \geq n_0$  vertices with  $k \geq 3n/8 + \varepsilon n$  has a hamiltonian decomposition. In particular, there exists  $n_0$  such that every regular tournament on  $n \geq n_0$  vertices has a hamiltonian decomposition.

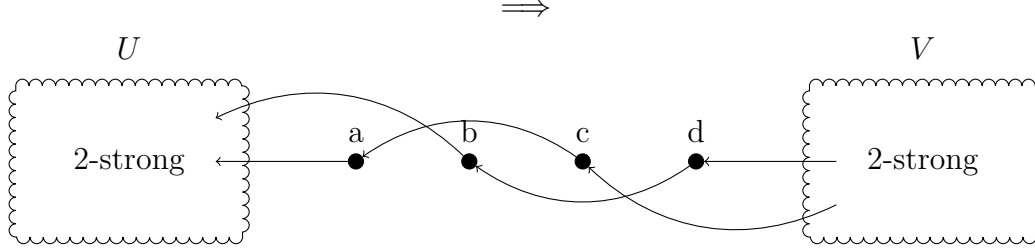
The proof is highly involved and relies on a more general result for digraphs, obtained by introducing the concept of a particular class of digraphs called *robust out-expanders*. In crude terms, we say that a digraph is a *robust out-expander* if for every set  $S$  which is not too small and not too large, the set of vertices which receive many edges from  $S$  is larger than  $S$ . Their main result states that every regular robust out-expander has a hamiltonian decomposition. They also prove that every digraph on  $n$  vertices with minimum in-degree and out-degree  $3n/8 + \varepsilon n$  is a robust out-expander. Thus, the  $k$ -regular tournaments in Theorem 10 are regular robust out-expanders, and the Kelly conjecture is proved for such  $k$ -regular tournaments. We direct the interested reader to [3], [4], and [5] for further information.

## 5 Future Research

We have determined conditions which guarantee for a tournament  $T$  both the existence of a  $(P, F)$ -pair for some vertex of  $V(T)$  and the existence of a  $(P, F)$ -pair for any vertex of  $V(T)$ . In the latter, however, we needed the tournament to be regular, a requirement which greatly restricts the structure of  $T$ , and we obtain many more out-branchings than originally desired. It is natural to ask if a weaker condition will guarantee the existence of a hamiltonian path and a single out-branching rooted at any vertex of  $T$  that are arc-disjoint.

In [1] Bang-Jensen notes that the infinite family of 2-strong tournaments displayed in Fig. 9 illustrates that 2-strong connectivity is not sufficient to guarantee the existence of a  $(P, F)$ -pair rooted at any vertex of  $T$ . To see this, draw five vertical lines: one between  $U$  and vertex  $a$ , between  $a$  and  $b$ , between  $b$  and  $c$ , between  $c$  and  $d$ , and between  $d$  and  $V$ . For two arc-disjoint paths from  $V$  to  $U$  to exist, we must use all of the arcs shown in detail in Fig. 3. Those arcs then form two paths called  $X$  and  $Y$ . Noticing this and using Edmonds' branching theorem, we can show that  $s$  cannot be rooted in  $V$  because neither  $X$  nor  $Y$  is hamiltonian. In order to find a hamiltonian path, we must enter on  $X$  and exit on  $Y$  or vice versa. Doing so would destroy any out-branching rooted in  $s$ .

Furthermore, in [6] Thomassen notes that no degree of arc-strong connectivity is sufficient to guarantee a  $(P, F)$ -pair from every vertex. Interestingly, arc-strong connectivity does guarantee arc-disjoint out-branchings that are path-like; consider the following theorem of Las Vergnas (see [2]), which is a generalization of the Gallai-Milgram theorem.



**Figure 9:** Each of the two boxes represents a 2-strong tournament, and all arcs except for those shown go from left to right.

**Theorem 11:** (Las Vergnas) Every digraph  $D$  of finite out-radius has an out-branching with at most  $\alpha(D)$  vertices of out-degree zero.

Theorem 11 implies that in a 2-arc-strong tournament every vertex is the root of two arc-disjoint out-branchings, each with at most two leaves. With regards to arc-disjoint hamiltonian cycles, Thomassen makes the additional suggestion that high connectivity may guarantee the existence of multiple arc-disjoint hamiltonian cycles.

**Conjecture 2.** (Thomassen [6]). For every  $k \geq 2$  there exists  $f(k) \in \mathbb{N}$  so that every  $f(k)$ -strong tournament has  $k$  arc-disjoint hamiltonian cycles.

It should be mentioned that the above conjecture represents a considerable weakening of the Kelly conjecture. Thomassen reports that  $f$  is not bounded above by any linear function, and also conjectures that  $f(2) = 3$ ; that is, every 3-strong tournament has two arc-disjoint hamiltonian cycles. A further conjecture of Erdős proposes that all tournaments have  $\min\{\delta^+(T), \delta^-(T)\}$  arc-disjoint hamiltonian cycles, where  $\delta^+(T)$  and  $\delta^-(T)$  are the minimum in-degree and out-degree in  $T$ , respectively.

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