The Traveling Salesman and a Tale of Four Cities[1]

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The traveling salesman problem is an application of graph theory in which a traveling salesman has to leave his home, visit a number of cities, and then return to his home. Each trip between cities has a value that represents the distance between the two cities, and the traveling salesman wants to minimize the distance traveled in visiting all of the cities. At this point a solution has not been found to the traveling salesman problem; that is, an efficient algorithm does not exist that determines the tour that visits all of the cities and minimizes the distance traveled [1].

This paper addresses a variation on the traditional version of the traveling salesman problem. Given a grid in the Cartesian plane, four points are chosen to represent cities. These four points are fixed in the plane. A fifth point is movable and represents the traveling salesman's home. Each tour that begins at the traveling salesman's home, visits all of the cities, and then returns home is given a colored symbol to represent the tour. At each lattice point in the grid, the shortest tour that begins and ends at that point is determined, and the point is colored the corresponding color. When each of the points in the grid is colored, various colored regions appear. Each colored region represents the tour that is the shortest that begins and ends from a home in that region. A Matlab program (see appendix) was used to graph the colored regions for a few positions of the fixed points.

Shown below is an example of the colored regions that appear when the four fixed points form a square. The coordinates of the points are: (1,2), (2,2), (2,1), (1,1).

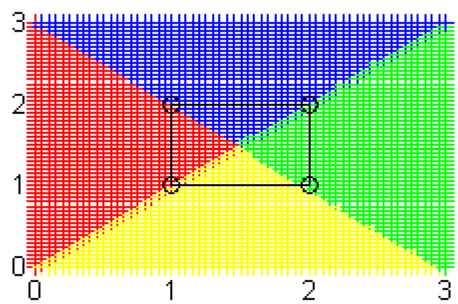


Figure 1

Given four points, there are 4! or twenty-four possible tours that begin at the home point, visit each point once, and then return to the home point. This is because when the traveling salesman is at his home, there are four cities to which he could travel first, then there are three cities to which he could travel next, then two cities, and then one city remaining. By the Multiplication Principle, there are $4 \times 3 \times 2 \times 1$ or twenty-four possible tours. In this list of possible tours, there are two tours that are essentially the

[1] This paper won a prize in the undergraduate student poster session at the Joint Mathematics Meetings in Washington, D.C. on January 21, 2000.

same: the tour that visits city 1 first and the other cities in numerical order ($0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 0$) and the tour that visits city 4 first and visits the cities in reverse numerical order ($0 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$). Even though the order in which the cities are visited is different, the distance to travel each tour is the same. Thus, the number of tours can be reduced from twenty-four to twelve. The number of tours can be reduced further as our first theorem shows.

Theorem 1. Consider our variation on the traveling salesman problem for four fixed points in the Cartesian plane. The greatest number of colors that will appear is less than or equal to six. That is, there are only six possible optimal tours for the traveling salesman.

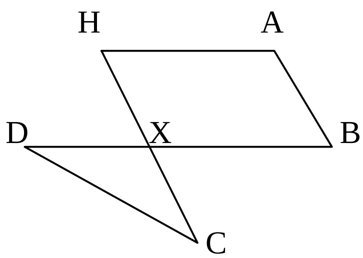
Proof: Consider the starting and ending city of the tour (ignoring the home city). For instance, suppose that the salesman wants to begin at city 1 and end at city 4. Between those two cities, there are two different tours that the salesman can take: $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 0$ and $0 \rightarrow 1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 0$. In the first tour the salesman visits city 2 before city 3, and in the second tour, the opposite is true. Regardless of the distance between the two cities, the length of one of the tours will be less than or equal to the other tour, so only one color would be needed to represent the tour that begins at city 1 and ends at city 4. Then each pair of cities gives rise to one color that represents the minimum tour. Therefore, there are at most $\binom{4}{2}$ or six different optimal tours that are possible which visit each of the cities only once. Thus, six is the greatest number of colors that will appear in any graph.

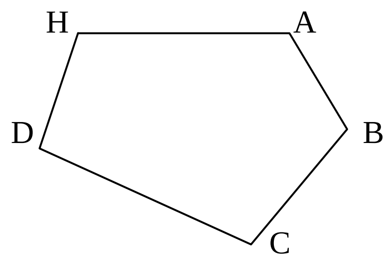
Four fixed points in $\Re^2$ can form one of two shapes: either a quadrilateral or a triangle, where the fourth point is in the interior of the triangle. The focus of this paper is what occurs when the four points form a quadrilateral in the Cartesian plane.

Given our traveling salesman problem, for four fixed points in the Cartesian plane, it has been shown that at most six colors will appear, but what is the fewest number of colors that can appear? Before the theorem is proved that shows the minimum number of colors possible, a lemma needs to be proved. Given four points in the plane, it has been shown that at most six of the tours are relevant. Four of the possible tours visits the cities in order around the quadrilateral. For example, when the points are labeled 1, 2, 3, and 4 in a clockwise manner around the quadrilateral, four of the possible tours are $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 0$, $0 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 0$, $0 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 2 \rightarrow 0$, and $0 \rightarrow 4 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$. None of these tours cross each other. The other two possible tours do cross each other. The following lemma shows that there exists a tour that is shorter that a tour that contains a crossed path.

Lemma: Given five points in the Cartesian plane, there exists a traveling salesman tour that is shorter than a tour that contains a crossed path.

Proof: Let A, B, C, and D represent four fixed cities, and let H denote the home point. Consider the following traveling salesman tour: $H \rightarrow A \rightarrow B \rightarrow D \rightarrow C \rightarrow H$. Label the point of intersection X, as shown in the diagram below:





Then
$$\begin{aligned} HA + AB + BD + DC + CH &= HA + AB + (BX + XD) + DC + (CX + XH) \\ &= HA + AB + (BX + XC) + CD + (DX + XH) \\ &> HA + AB + BC + CD + DH \end{aligned}$$
Then the tour that travels sequentially around the quadrilateral is shorter than a tour that crosses itself.

This lemma is then used to prove the following theorem.

Theorem 2. Given our traveling salesman problem for four fixed points in $\Re^2$ that form a quadrilateral, at least four colors will be present.

Proof: Let A, B, C, and D be four fixed points in $\Re^2$ that form a quadrilateral. Consider a home point that is on the line segment AB. Using the lemma from above, it can be shown that the shortest tour would be one that visits all the points in order around the quadrilateral. Let the point be colored red. Similarly, the shortest tour that begins at a home point that is on the line segment BC, would be the tour that travels around the edge of the quadrilateral. Let the point be colored yellow. A similar argument can be given for points on the line segment CD and points on the line segment AD. Thus, when the four fixed points form a quadrilateral, at least four colors will be present.

Below are two examples of quadrilaterals where four colors are present. The quadrilateral in Figure 2 has coordinates (1,2), (2.5,2), (2.5,1), (1,1) and the quadrilateral in Figure 3 has coordinates (1,2), (3,3), (2,1), (1,1).

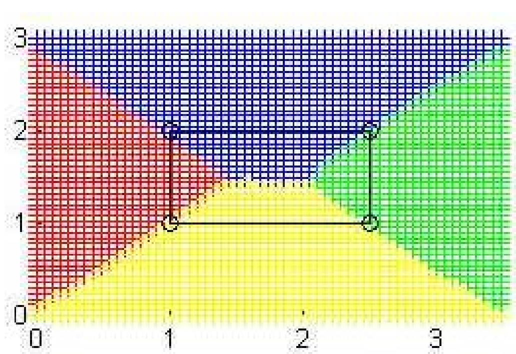


Figure 2

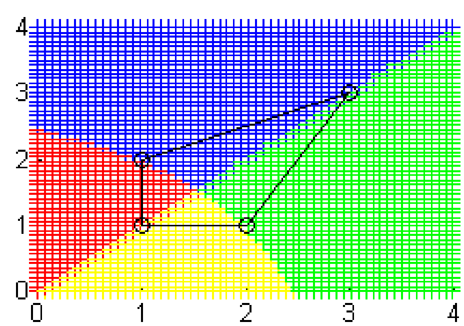


Figure 3

Now consider a quadrilateral with the following general coordinates: (4-t,h), (4+t,h), (8,0), (0,0).

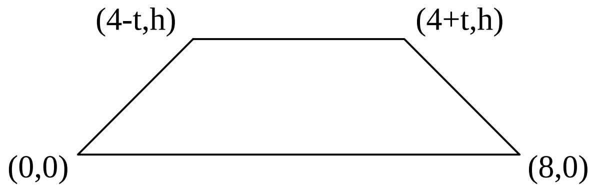


Figure 4

When h=1 there is a range of t-values for which the colored regions in our problem form very interesting pictures. When t=4, the quadrilateral is a rectangle and four colors are present as can be seen in Figure 2.

When t=3, the quadrilateral is a trapezoid and six colors are present as can be seen in Figure 5.

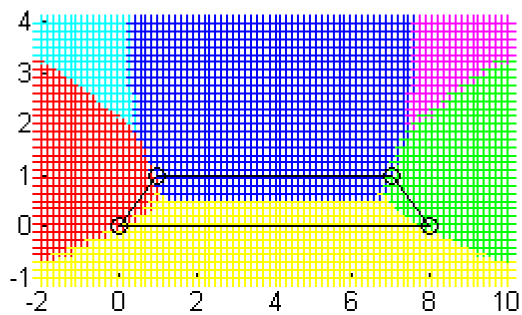


Figure 5

When t=2, the quadrilateral is still a trapezoid and six colors are present, but one of the colored regions has split. The color that appears directly below the trapezoid also appears at a sufficient distance above the trapezoid.

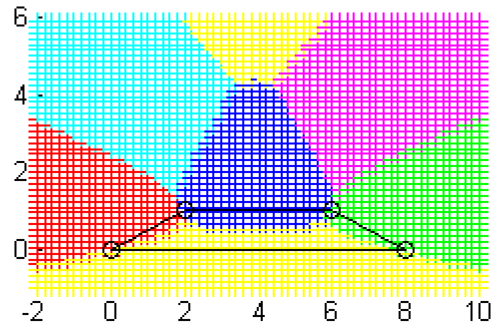


Figure 6

When t=0.2, the quadrilateral is still a trapezoid, but only four colors are present; one of the colored regions has split.

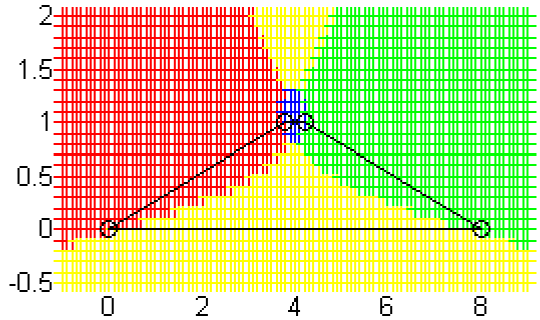


Figure 7

When t=0, the figure is now a triangle and three colors are present.

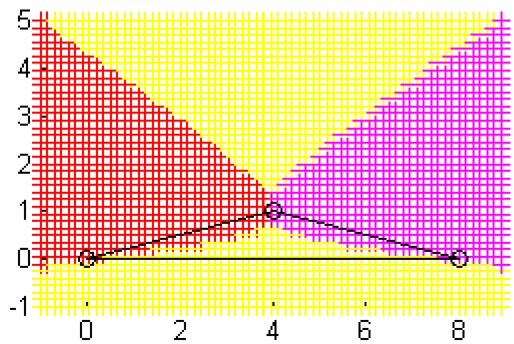


Figure 8

After experimenting with a variety of *h* and *t*-values, the critical point where there are six colors with one of the colors appearing in two regions was determined. Regardless of the *h*-values, when the *t*-value yields a trapezoid with angles of 45°, 135°, 135°, and 45°, the colored region below the trapezoid does not split, but when *t* is increased by $\varepsilon > 0$, the colored region below the trapezoid splits and a portion of it appears above the trapezoid. This result is a consequence of the following theorem.

Theorem 3. If $h+\sqrt{(4-t)^2+h^2}<8$ and $h+2t<\sqrt{(4-t)^2+h^2}$, the colored region below the trapezoid will split, and the color that is below the trapezoid will appear above it.

Proof. The theorem is only true for the strict inequality, so when equality holds, the region does not split. We will now demonstrate this. Suppose that $4-t=h$. Then consider the first condition when equality holds. Using some algebraic manipulation the equation can be simplified as follows:

$$
\begin{aligned}
h + \sqrt{(4-t)^2 + h^2} &= 8 \\
\sqrt{(4-t)^2 + h^2} &= 8 - h \\
(4-t)^2 + h^2 &= 64 - 16h + h^2 \\
(4-t)^2 &= 64 - 16h
\end{aligned}
$$

Since $4 - t = h$, a substitution yields the following quadratic equation:

$$h^2 + 16h - 64 = 0$$

Using the quadratic formula yields the following values for h:

$$h = -8 \pm 8\sqrt{2}$$

Since h represents a length, specifically a height, only the positive value for h is relevant. After some manipulation, the value for h is:

$$h = 8(-1+\sqrt{2}) = 8(\sqrt{2}-1)\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{8}{\sqrt{2}+1}$$

Then a substitution and some manipulation yields the following value for t:

$$t = 4 - h = 4 - \frac{8}{\sqrt{2}+1} = \frac{4(\sqrt{2}+1)-8}{\sqrt{2}+1} = \frac{4\sqrt{2}-4}{\sqrt{2}+1} = 4\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

Now consider the second condition of the theorem when the equality holds.

$$h + 2t = \sqrt{(4-t)^2 + h^2}$$

Substituting the values for h and t into the left side of the equation yields the following value:

$$h + 2t = \frac{8}{\sqrt{2}+1} + 8\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \frac{8\sqrt{2}}{\sqrt{2}+1} = \left(\frac{8}{\sqrt{2}+1}\right)\sqrt{2}$$

The substitution of h for $4-t$ into the right side of the equation yields the following value:

$$\sqrt{(4-t)^2 + h^2} = \sqrt{h^2 + h^2} = \sqrt{2h^2} = h\sqrt{2} = \left(\frac{8}{\sqrt{2}+1}\right)\sqrt{2}$$

The two values are equal; therefore, the two equations are true when $h = \dfrac{8}{\sqrt{2}+1}$ and $t = 4\left(\dfrac{\sqrt{2}-1}{\sqrt{2}+1}\right)$. Thus, when $4 - t = h$ and the conditions of the theorem hold with equality, the angles of the trapezoid are 45º, 135º, 135º, and 45º which are the critical values for the trapezoid. The following figure shows the colored regions corresponding to the trapezoid in Figure 4 with $h = \dfrac{8}{\sqrt{2}+1}$ and $t = 4\left(\dfrac{\sqrt{2}-1}{\sqrt{2}+1}\right)$.

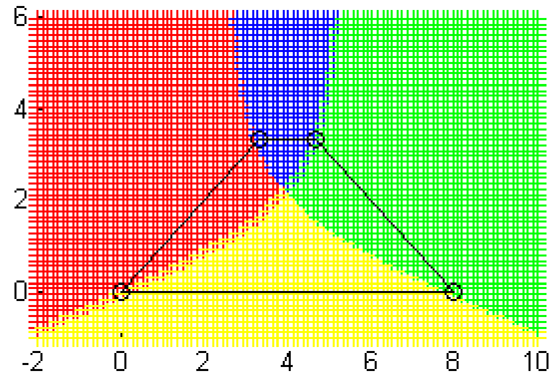


Figure 9

Now consider the following configuration of cities located at the points (0,0), (4-t,h), (4+t,h), and (8,0).

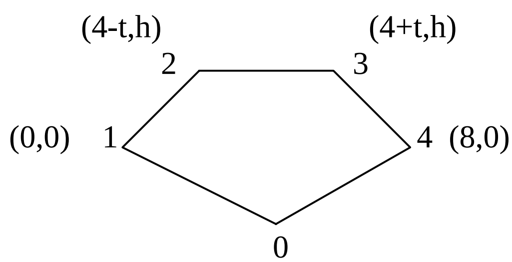


Figure 10

The numbers 1, 2, 3, and 4 in Figure 10 represent the four fixed cities, and 0 represents home. The optimal tour that visits cities 1, 2, 3, and 4 is 0→4→3→2→1→0. We are going to show that when the conditions of the theorem are satisfied, this tour is also optimal for 0 sufficiently high above the trapezoid with vertices (4-t,h), (4+t,h), (8,0) and (0,0) as in Figure 12 below.

Construct the following distance matrix where d_{ij} is the distance between the ith and jth cities and $0 \le i \le 4$ and $0 \le j \le 4$:

$$\begin{bmatrix} 0 & d_{01} & d_{02} & d_{03} & d_{04} \\ d_{10} & 0 & d_{12} & d_{13} & d_{14} \\ d_{20} & d_{21} & 0 & d_{23} & d_{24} \\ d_{30} & d_{31} & d_{32} & 0 & d_{34} \\ d_{40} & d_{41} & d_{42} & d_{43} & 0 \end{bmatrix}$$

The following matrix can be used to represent a tour where $0 \le i \le 4$ and $0 \le j \le 4$ and

$$x_{ij} = \begin{cases} 1 & \text{if go from } i \text{ to } j \text{ directly} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} x_{00} & x_{01} & x_{02} & x_{03} & x_{04} \\ x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \\ x_{20} & x_{21} & x_{22} & x_{23} & x_{24} \\ x_{30} & x_{31} & x_{32} & x_{33} & x_{34} \\ x_{40} & x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Then every tour corresponds to a transversal, which is a set of entries from the matrix such that no two entries are in the same row or column. For example, the tour 0→4→3→2→1→0, which is optimal for the cities in Figure 10 can be represented by a matrix where $x_{04} = x_{43} = x_{32} = x_{21} = x_{10} = 1$, and $x_{ij} = 0$ for all other values of i and j. Note that the entries in the transversal have value 1 and all other entries in the matrix have value 0. Since only one entry from each row and column represents part of a tour, only one entry in each row and column will have a value of one, thus the sum of the elements in any row or column is one. That is,

$$\sum_j x_{ij} = x_{i0} + x_{i1} + x_{i2} + x_{i3} + x_{i4} = 1.$$

Similarly, $\sum_i x_{ij} = 1$. Now let each column of the distance matrix be assigned a value v_j and each row be assigned a value u_i. We will write this in the following way:

$$\begin{array}{c c} & \begin{array}{ccccc} v_0 & v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} & \begin{bmatrix} 0 & d_{01} & d_{02} & d_{03} & d_{04} \\ d_{10} & 0 & d_{12} & d_{13} & d_{14} \\ d_{20} & d_{21} & 0 & d_{23} & d_{24} \\ d_{30} & d_{31} & d_{32} & 0 & d_{34} \\ d_{40} & d_{41} & d_{42} & d_{43} & 0 \end{bmatrix} \end{array}$$

Figure 11

Considering the tour from above with distances $d_{04}, d_{43}, d_{32}, d_{21}$, and d_{10}, suppose that

$$u_i + v_j \le d_{ij} \quad \forall i, j.$$

If the following equations are true

$$\begin{aligned} u_0 + v_4 &= d_{04} \\ u_4 + v_3 &= d_{43} \\ u_3 + v_2 &= d_{32} \\ u_2 + v_1 &= d_{21} \\ u_1 + v_0 &= d_{10} \end{aligned}$$

(1)

then the tour with distances $d_{04}, d_{43}, d_{32}, d_{21}$, and d_{10} form the optimal tour. To prove this statement, it needs to be shown that $d_{04} + d_{43} + d_{32} + d_{21} + d_{10}$ is shorter than any other tour.

Note that

$$\begin{aligned} d_{04} + d_{43} + d_{32} + d_{21} + d_{10} &= d_{04}x_{04} + d_{43}x_{43} + d_{32}x_{32} + d_{21}x_{21} + d_{10}x_{10} \\ &= \sum_{i=0}^{4}\sum_{j=0}^{4} d_{ij}x_{ij} \end{aligned}$$

Denote an arbitrary tour by x^*. Recall that since x^* is a tour, the matrix $[x_{ij}^*]$ is a transversal. It needs to be shown that the length of the tour 0→4→3→2→1→0 is shorter than any other tour; that is

$$d_{04} + d_{43} + d_{32} + d_{21} + d_{10} = \sum_{j=0}^{4}\sum_{i=0}^{4} d_{ij}x_{ij} \le \sum_{j=0}^{4}\sum_{i=0}^{4} d_{ij}x_{ij}^* .$$

Note that $x_{ij}(u_i + v_j - d_{ij}) = 0$. This is because $x_{ij} = 0$ whenever the tour does not include the distance between the ith and jth cities and $u_i + v_j - d_{ij} = 0$ whenever the tour does include the distance between the ith and jth cities. Then

$$\sum_{j=0}^{4}\sum_{i=0}^{4} x_{ij}(u_i + v_j - d_{ij}) = 0$$

So,

$$\begin{aligned} \sum_{j=0}^{4}\sum_{i=0}^{4} d_{ij}x_{ij} &= \sum_{j=0}^{4}\sum_{i=0}^{4} (u_i + v_j)x_{ij} = \sum_{i=0}^{4} u_i \sum_{j=0}^{4} x_{ij} + \sum_{j=0}^{4} v_j \sum_{i=0}^{4} x_{ij} \\ &= \sum_{i=0}^{4} u_i \sum_{j=0}^{4} x_{ij}^* + \sum_{j=0}^{4} v_j \sum_{j=0}^{4} x_{ij}^* \\ &= \sum_{j=0}^{4}\sum_{i=0}^{4} (u_i + v_j)x_{ij}^* \\ &\le \sum_{j=0}^{4}\sum_{i=0}^{4} d_{ij}x_{ij}^* \end{aligned}$$

and the condition $u_i + v_j \le d_{ij} \quad \forall i, j$. Thus x is optimal if it possible to find u_i's and v_j's that satisfy (1). We will show how to choose the u_i's and v_j's such that these conditions are satisfied for city 0 high above the trapezoid as in Figure 12 below.

Given a home point 0 that is located below the quadrilateral, the optimal tour 0→4→3→2→1→0. See Figure 10.

Looking at Figure 12 below, it will be shown that for y sufficiently large, the home point 0 with coordinates (4,y) is such that the optimal tour is 0→4→3→2→1→0.

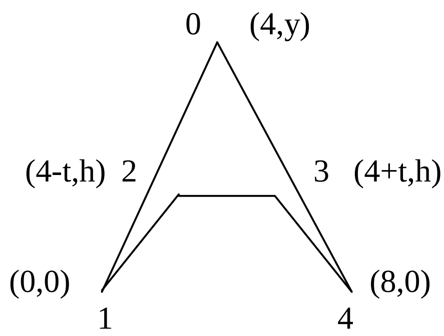


Figure 12

Let the following u_i's and v_j's be chosen:

$$\begin{array}{cc}
 & \begin{matrix} \sqrt{t^2+(y-h)^2}-\sqrt{(4-t)^2+h^2} & 0 & 2t-\sqrt{(4-t)^2+h^2} & \sqrt{t^2+(y-h)^2}-\sqrt{16+y^2} & 0 \end{matrix} \\
\begin{matrix} \sqrt{16+y^2} \\ \sqrt{16+y^2}-\sqrt{t^2+(y-h)^2}+\sqrt{(4-t)^2+h^2} \\ \sqrt{(4-t)^2+h^2} \\ \sqrt{(4-t)^2+h^2} \\ \sqrt{16+y^2}-\sqrt{t^2+(y-h)^2}+\sqrt{(4-t)^2+h^2} \end{matrix} &
\begin{bmatrix}
M & \sqrt{16+y^2} & \sqrt{t^2+(y-h)^2} & \sqrt{t^2+(y-h)^2} & \sqrt{16+y^2} \\
\sqrt{16+y^2} & M & \sqrt{(4-t)^2+h^2} & \sqrt{(4+t)^2+h^2} & 8 \\
\sqrt{t^2+(y-h)^2} & \sqrt{(4-t)^2+h^2} & M & M & \sqrt{(4+t)^2+h^2} \\
\sqrt{t^2+(y-h)^2} & \sqrt{(4+t)^2+h^2} & 2t & M & \sqrt{(4-t)^2+h^2} \\
\sqrt{16+y^2} & 8 & \sqrt{(4+t)^2+h^2} & \sqrt{(4-t)^2+h^2} & M
\end{bmatrix}
\end{array}$$

Figure 13

This notation means that

$$v_0 = \sqrt{t^2 + (y-h)^2} - \sqrt{(4-t)^2 + h^2}$$
$$v_1 = 0$$
$$v_2 = 2t - \sqrt{(4-t)^2 + h^2} \qquad \text{and}$$
$$v_3 = \sqrt{t^2 + (y-h)^2} - \sqrt{16 + y^2}$$
$$v_4 = 0$$
$$u_0 = \sqrt{16 + y^2}$$
$$u_1 = \sqrt{16 + y^2} - \sqrt{t^2 + (y-h)^2} + \sqrt{(4-t)^2 + h^2}$$
$$u_2 = \sqrt{(4-t)^2 + h^2}$$
$$u_3 = \sqrt{(4-t)^2 + h^2}$$
$$u_4 = \sqrt{16 + y^2} - \sqrt{t^2 + (y-h)^2} + \sqrt{(4-t)^2 + h^2}$$

Moreover, with the cities 0, 1, 2, 3, and 4 situated as in Figure 12, we clearly have $d_{01} = \sqrt{16 + y^2}$, $d_{02} = \sqrt{t^2 + (y-h)^2}$, $d_{03} = \sqrt{t^2 + (y-h)^2}$, and so on. Then the entries in the array in Figure 13 are just the distance d_{ij} so, except for the positions occupied by M, Figure 13 is the same as Figure 11 for the configuration of cities in Figure 12. Observe that the numbers in Figure 13 satisfy

$$u_0 + v_4 = d_{04}$$
$$u_4 + v_3 = d_{43}$$
$$u_3 + v_2 = d_{32}$$
$$u_2 + v_1 = d_{21}$$
$$u_1 + v_0 = d_{10}$$

(2)

It, therefore, follows from our discussion about Figure 11 that the tour
0→4→3→2→1→0 is optimal

for the cities in Figure 12, if we can show that

$$u_i + v_j \le d_{ij}$$

for all $0 \le i \le 4$ and $0 \le j \le 4$.

To simplify the proof of these inequalities, we will replace distances that can never occur in an optimal tour by an arbitrarily large number M. For example, the salesman never goes from a city to that city again. So the distances

d_{00}, d_{11}, d_{22}, d_{33}, and d_{44} are never used in an optimal tour. So we can replace each of them with a large number M without changing the problem. We can also argue that there is no need to use the distance d_{23}. The reason for this is that if there is a tour such as 0→1→2→3→4→0 that uses the distance d_{23}, there is also a tour, namely the reverse tour 0→4→3→2→1→0, which is just as good and does not use the distance d_{23}. Thus, we can find the best tour without using d_{23}. Consequently, in Figure 13, we have replaced d_{23} with the large number M.

Now observed that in addition to equations (2), from Figure 12 we have

$$\begin{aligned} u_0 + v_1 &= d_{01} \\ u_0 + v_3 &= d_{03} \\ u_2 + v_0 &= d_{20} \\ u_3 + v_0 &= d_{30} \\ u_3 + v_4 &= d_{34} \\ u_4 + v_0 &= d_{40} \end{aligned}$$

Also, since $t \ge 0$, we have

$$\begin{aligned} u_3 + v_1 &\le d_{31} \\ u_1 + v_3 &\le d_{13} \\ u_2 + v_4 &\le d_{24} \end{aligned}$$

and since M is as large as we like, we have

$$u_i + v_i \le d_{ii} = M \text{ and } u_2 + v_3 \le d_{23} = M$$

So, to complete the proof of the theorem, we only need to show that

$$\begin{aligned} u_0 + v_2 &\le d_{02} \\ u_1 + v_2 &\le d_{12} \\ u_4 + v_2 &\le d_{42} \\ u_1 + v_4 &\le d_{14} \\ u_4 + v_1 &\le d_{41} \end{aligned}$$

When these inequalities are written out explicitly, using the values from Figure 13, we see that the second and third inequalities hold if the first holds, and the fourth and fifth inequalities are identical. Thus, it suffices to prove the first and fourth inequalities.

Consider the first inequality. When written out explicitly, it states that

$$\sqrt{16+y^2} + 2t - \sqrt{(4-t)^2+h^2} \le \sqrt{t^2+(y-h)^2}\,.$$

We can rewrite this as

$$y\sqrt{1+\frac{16}{y^2}} + 2t - \sqrt{(4-t)^2+h^2} \le (y-h)\sqrt{1+\left(\frac{t}{y-h}\right)^2}$$

For y large we have $\sqrt{1+\frac{16}{y^2}} \approx 1$ and $\sqrt{1+\left(\frac{t}{y-h}\right)^2} \approx 1$. Thus this inequality is satisfied for y sufficiently large if

$$y + 2t - \sqrt{(4-t)^2+h^2} < y - h,$$

or equivalently, if

$$h + 2t < \sqrt{(4-t)^2 + h^2}\,.$$

But this condition is satisfied by the hypothesis of the theorem.

Now consider the fourth inequality which states that

$$\sqrt{16 + y^2} - \sqrt{t^2 + (y-h)^2} + \sqrt{(4-t)^2 + h^2} \le 8\,.$$

We can write this condition as

$$y\sqrt{1 + \frac{16}{y^2}} - (y-h)\sqrt{1 + \left(\frac{t}{y-h}\right)^2} + \sqrt{(4-t)^2 + h^2} \le 8\,.$$

Since $\sqrt{1 + \frac{16}{y^2}} \approx 1$ and $\sqrt{1 + \left(\frac{t}{y-h}\right)^2} \approx 1$ for large values of y, and since

$$h + \sqrt{(4-h)^2 + h^2} < 8$$

the above inequality holds for y sufficiently large. This completes the proof of the theorem.

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References

[1] Chartrand, Gary. *Graphs as Mathematical Models*. Boston: Prindle, Weber, & Schmidt, Incorporated, 1977.

Appendix

Matlab program to graph the colored regions when coordinates form a square

```
P=[1 2;2 2;2 1;1 1;0 0];
i=1;
x1=P(1,1);
y1=P(1,2);
x2=P(2,1);
y2=P(2,2);
x3=P(3,1);
y3=P(3,2);
x4=P(4,1);
y4=P(4,2);
x(i)=P(5,1);
y(i)=P(5,2);
a=0;
b=3;
c=0;
d=3;
axis([a b c d])
hold on
x=a:.1.b;
y=c:.1.d;
for i=1:length(x)
    for j=1:length(y)
        X=[x(i),y(i)];
        A=[x1,y1];
        B=[x2,y2];
        C=[x3,y3];
        D=[x4,y4];
        clear d;
        d(1)=norm([X-A])+norm([A-B])+norm([B-C])+norm([C-D])+norm([D-X]);
        d(2)=norm([X-A])+norm([A-B])+norm([B-D])+norm([D-C])+norm([C-X]);
        d(3)=norm([X-A])+norm([A-C])+norm([C-B])+norm([B-D])+norm([D-X]);
        d(4)=norm([X-A])+norm([A-C])+norm([C-D])+norm([D-B])+norm([B-X]);
        d(5)=norm([X-A])+norm([A-D])+norm([D-B])+norm([B-C])+norm([C-X]);
        d(6)=norm([X-A])+norm([A-D])+norm([D-C])+norm([C-B])+norm([B-X]);
        d(7)=norm([X-B])+norm([B-A])+norm([B-C])+norm([C-D])+norm([D-X]);
        d(8)=norm([X-B])+norm([B-A])+norm([A-C])+norm([C-D])+norm([D-X]);
        d(9)=norm([X-B])+norm([B-C])+norm([C-A])+norm([A-D])+norm([D-X]);
        d(10)=norm([X-B])+norm([B-D])+norm([D-A])+norm([A-C])+norm([C-X]);
        d(11)=norm([X-C])+norm([C-A])+norm([A-B])+norm([B-D])+norm([D-X]);
        d(12)=norm([X-C])+norm([C-B])+norm([B-A])+norm([A-D])+norm([D-X]);
```

```
if d(1)<d(3)
   d(3)=d(1);
end
if d(1)>d(3)
   d(1)=d(3);
end
if d(2)<d(5)
   d(5)=d(2);
end
if d(2)>d(5)
   d(2)=d(5);
end
if d(4)<d(6)
   d(6)=d(4);
end
if d(4)>d(6)
   d(4)=d(6);
end
if d(7)<d(9)
   d(9)=d(7);
end
if d(7)>d(9)
   d(7)=d(9);
end
if d(8)<d(10)
   d(10)=d(8);
end
if d(8)>d(10)
   d(8)=d(10);
end
if d(11)<d(12)
   d(12)=d(11);
end
if d(11)>d(12)
   d(11)=d(12);
end
[m,k]=min(d);
if k==1
   plot(x(i),y(i), ‘r:+’)
end
if k==2
   plot(x(i),y(i), ‘m:+’)
end
if k==4
   plot(x(i),y(i), ‘b:+’)
end
```

```
        if k==7
          plot(x(i),y(i), 'c:+')
        end
        if k==8
          plot(x(i),y(i), 'g:+')
        end
        if k==11
          plot(x(i),y(i), 'y:+')
        end
    end
end
v1=[x1 x2 x3 x4 x1];
v2=[y1 y2 y3 y4 y1];
plot(v1,v2, 'k')
plot(x1, y1, 'k:o')
plot(x2, y2, 'k:o')
plot(x3, y3, 'k:o')
plot(x4, y4, 'k:o')
```