A TRAVELING SALESMAN PROBLEM FOR FIVE CITIES IN THREE-DIMENSIONAL SPACE

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1. Introduction

The traveling salesman problem requires a salesman to start at a given city, visit a number of other cities exactly once, and return to the starting city in such a way that the total distance traveled is as short as possible. Suppose all cities are represented by points in the x,y plane. A problem of interest is to divide the plane into a finite number of regions with the property that the optimal traveling salesman tours from any two points in a given region visit the cities in the same order. We will refer to the starting city as the home point and denote it by $H = (x_H, y_H)$. Suppose there are three other cities $C_1 = (x_1, y_1), C_2 = (x_2, y_2)$, and $C_3 = (x_3, y_3)$ that must be visited. In this case there are three possibilities for the optimal tour. They are:

$$\begin{array}{ll} (1) & H \to C_1 \to C_2 \to C_3 \to H, \\ (2) & H \to C_1 \to C_3 \to C_2 \to H, \\ (3) & H \to C_2 \to C_1 \to C_3 \to H. \end{array}$$

Any other tour is just the reverse of one of these, so it has the same length as one of these tours. Color the home point H red, blue, or green depending on whether tour 1, 2, or 3 is optimal. As H varies over the x, y plane this divides the plane into three colored regions. A problem of interest is to discover some properties of these colored regions.

Four Stationary Cities.

The problem considered here is a variation on the problem just formulated. The variation is that there are now four fixed cities instead of three, and these cities, as well as the movable home city are assumed to be points in three-dimensional space. We will use the following notation for these cities:

$$H = (x_H, y_H, z_H), C_1 = (x_1, y_1, z_1), C_2 = (x_2, y_2, z_2), C_3 = (x_3, y_3, z_3), C_4 = (x_4, y_4, z_4).$$

Theorem 2.1. *The problem we have posed for four stationary cities has exactly the following twelve distinct tours:*

$$\begin{array}{ll} (1) & H \to C_1 \to C_2 \to C_3 \to C_4 \to H, \\ (2) & H \to C_1 \to C_2 \to C_4 \to C_3 \to H, \\ (3) & H \to C_1 \to C_3 \to C_2 \to C_4 \to H, \\ (4) & H \to C_1 \to C_3 \to C_4 \to C_2 \to H, \\ (5) & H \to C_1 \to C_4 \to C_2 \to C_3 \to H, \\ (6) & H \to C_1 \to C_4 \to C_3 \to C_2 \to H, \\ (7) & H \to C_2 \to C_1 \to C_3 \to C_4 \to H, \\ (8) & H \to C_2 \to C_1 \to C_4 \to C_3 \to H, \\ (9) & H \to C_2 \to C_3 \to C_1 \to C_4 \to H, \\ (10) & H \to C_2 \to C_4 \to C_1 \to C_3 \to H, \\ (11) & H \to C_3 \to C_1 \to C_2 \to C_4 \to H, \end{array}$$

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$$(12)\ H \to C_3 \to C_2 \to C_1 \to C_4 \to H.$$

Proof. There are 24 permutations of the four cities C_1, C_2, C_3, C_4. They give rise to the following possible tours $t_i, i = 1, \ldots, 24$:

$$\begin{aligned}
t_1 &: H \to C_1 \to C_2 \to C_3 \to C_4 \to H,\\
t_2 &: H \to C_1 \to C_2 \to C_4 \to C_3 \to H,\\
t_3 &: H \to C_1 \to C_3 \to C_2 \to C_4 \to H,\\
t_4 &: H \to C_1 \to C_3 \to C_4 \to C_2 \to H,\\
t_5 &: H \to C_1 \to C_4 \to C_2 \to C_3 \to H,\\
t_6 &: H \to C_1 \to C_4 \to C_3 \to C_2 \to H,\\
t_7 &: H \to C_2 \to C_1 \to C_3 \to C_4 \to H,\\
t_8 &: H \to C_2 \to C_1 \to C_4 \to C_3 \to H,\\
t_9 &: H \to C_2 \to C_3 \to C_1 \to C_4 \to H,\\
t_{10} &: H \to C_2 \to C_3 \to C_4 \to C_1 \to H,\\
t_{11} &: H \to C_2 \to C_4 \to C_1 \to C_3 \to H,\\
t_{12} &: H \to C_2 \to C_4 \to C_3 \to C_1 \to H,\\
t_{13} &: H \to C_3 \to C_1 \to C_2 \to C_4 \to H,\\
t_{14} &: H \to C_3 \to C_1 \to C_4 \to C_2 \to H,\\
t_{15} &: H \to C_3 \to C_2 \to C_1 \to C_4 \to H,\\
t_{16} &: H \to C_3 \to C_2 \to C_4 \to C_1 \to H,\\
t_{17} &: H \to C_3 \to C_4 \to C_1 \to C_2 \to H,\\
t_{18} &: H \to C_3 \to C_4 \to C_2 \to C_1 \to H,\\
t_{19} &: H \to C_4 \to C_1 \to C_2 \to C_3 \to H,\\
t_{20} &: H \to C_4 \to C_1 \to C_3 \to C_2 \to H,\\
t_{21} &: H \to C_4 \to C_2 \to C_1 \to C_3 \to H,\\
t_{22} &: H \to C_4 \to C_2 \to C_3 \to C_1 \to H,\\
t_{23} &: H \to C_4 \to C_3 \to C_1 \to C_2 \to H,\\
t_{24} &: H \to C_4 \to C_3 \to C_2 \to C_1 \to H.
\end{aligned}$$

Tours $t_{24}, t_{18}, t_{22}, t_{12}, t_{16}, t_{10}, t_{23}, t_{17}, t_{20}, t_{14}, t_{21}$, and t_{19} are the exact reverse of $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{11}, t_{13}$, and t_5 respectively. Thus there are only 12 distinct tours as claimed. Moreover these 12 tours are the ones listed in the theorem. □

Theorem 2.2. *Of the 12 possible tours listed in Theorem 2.1 at most 6 can be optimal.*

Proof. (This proof is attributed to Thayer Morrill.) Observe tours 1 and 3. They begin and end with the same stationary points C_1 and C_4. Thus the tour which has the shortest path from C_1 to C_4 is always shorter than the other, so it will never be necessary to choose both of these paths. Let $T_1 = \min\{\textit{tour } 1,\ \textit{tour } 3\}$. T_1 is one of the tours that has the possibility of being optimal for a given starting point H.

In a similar way observe that the following pairs of tours have the same starting and ending stationary cities.

tour 2, tour 5
tour 4, tour 6
tour 7, tour 9
tour 8, tour 10
tour 11, tour 12

Define $T_2 = \min\{\textit{tour } 2,\ \textit{tour } 5\}$, $T_3 = \min\{\textit{tour } 4,\ \textit{tour } 6\}$, $T_4 = \min\{\textit{tour } 7,\ \textit{tour } 9\}$, $T_5 = \min\{\textit{tour } 8,\ \textit{tour } 10\}$, $T_6 = \min\{\textit{tour } 11,\ \textit{tour } 12\}$. Clearly the optimal tour is the shortest of the tours T_1, T_2, T_3, T_4, T_5, T_6. This completes the proof of the theorem. □

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\end{aligned}$$

Tours $t_{24}, t_{18}, t_{22}, t_{12}, t_{16}, t_{10}, t_{23}, t_{17}, t_{20}, t_{14}, t_{21}$, and t_{19} are the exact reverse of $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{11}, t_{13}$, and t_5 respectively. Thus there are only 12 distinct tours as claimed. Moreover these 12 tours are the ones listed in the theorem. □

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This shows that the points C_1, C_2, C_3, C_4 lie on a sphere centered at K. But we have seen that these points lie on a unique sphere and that sphere is centered at H^*. Thus $K = H^*$. This completes the proof.□

We now assign numbers to the colored regions defined by our coloring scheme.

Region 1: home points where T_1 is the shortest tour,
Region 2: home points where T_2 is the shortest tour,
Region 3: home points where T_3 is the shortest tour,
Region 4: home points where T_4 is the shortest tour,
Region 5: home points where T_5 is the shortest tour,
Region 6: home points where T_6 is the shortest tour.

Theorem 2.4. *Regions 1, 2, and 3 intersect at C_1.*
Regions 3, 4, and 5 intersect at C_2.
Regions 2, 5, and 6 intersect at C_3.
Regions 1, 4, and 6 intersect at C_4.

Proof. The tours 1, 2, 3, 4, 5 and 6 in (2.1) have C_1 as a boundary point in the paths $C_1 \to C_{i_1} \to C_{i_2} \to C_{i_3}$ in (2.1). Therefore if $H = C_1$ all these tours have length $4a$. The remaining tours in (2.1) have C_1 as an interior point. They all therefore have length $5a$. Here we use the observation that each tour has 5 legs, and tours that start or end at C_1 have a leg of length 0. The tours T_1, T_2, T_3 are defined in terms of tours 1, 2, 3, 4, 5 and 6. Therefore these tours have equal lengths and are optimal. Thus C_1 belongs to Regions 1, 2 and 3.

Now consider C_2. C_2 is a boundary point of the paths connecting fixed cities in tours 7, 8, 9, 10, 4 and 6. Thus for $H = C_2$ these tours have length $4a$ and are optimal. T_3, T_4 and T_5 are defined in terms of these tours and are therefore optimal. Thus C_2 lies in Regions 3, 4 and 5.

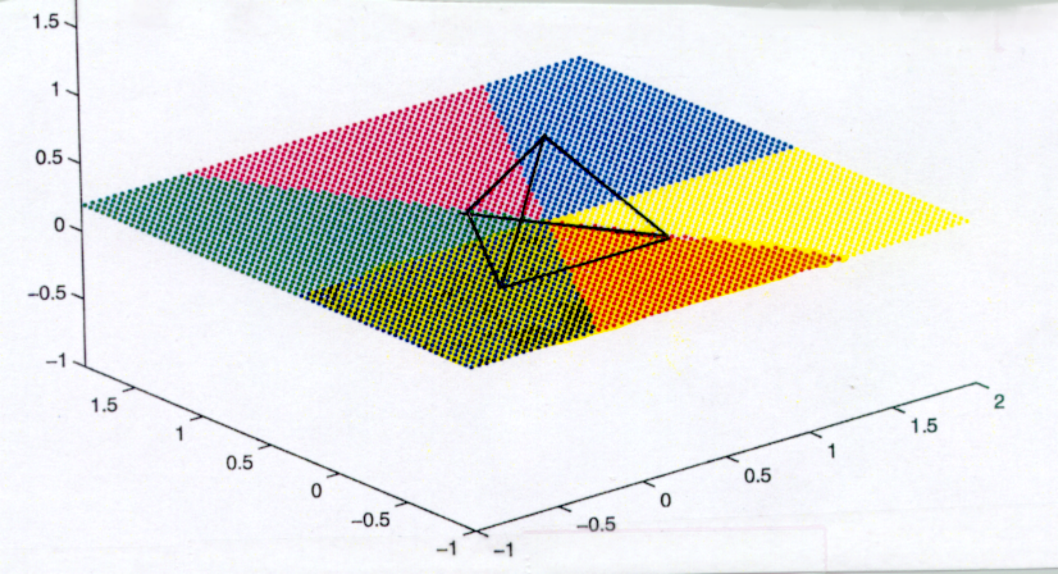
Now consider C_3. This city in a boundary point of the path through the fixed cities in tours 2, 5, 8, 10, 11, 12. Thus, by reasoning as above, T_2, T_5, and T_6 are optimal for $H = C_3$.

Finally, consider C_4. This city is in the boundary of tours 1, 3, 7, 9, 11 and 12. These tours define T_1, T_4 and T_6. Thus Regions 2, 4 and 6 contain C_4. This completes the proof of the theorem. □

3. Graphics.

In this section we show some graphics to demonstrate the points made in Theorems 2.3 and 2.4.

Here we color a plane through the point H^* described in Theorem 2.3. Indeed it has 6 colors.



1.5
1
0.5
0
−0.5
−1
1.5
1
0.5
0
−0.5
−1
2
1.5
1
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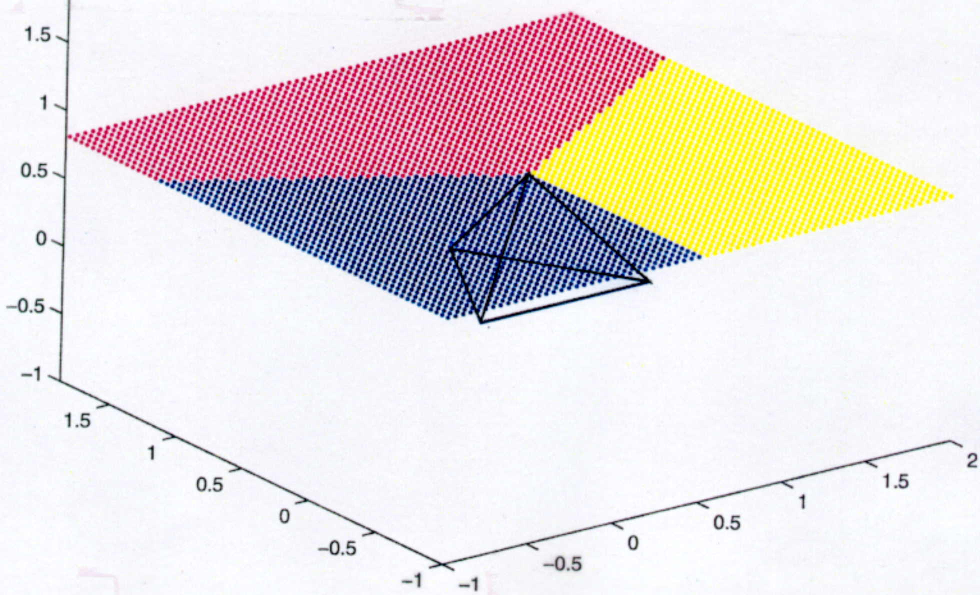
0
−0.5
−1

This figure shows a plane passing through C_4. We see that three colors intersect at that point as Theorem 2.4 shows.



1.5
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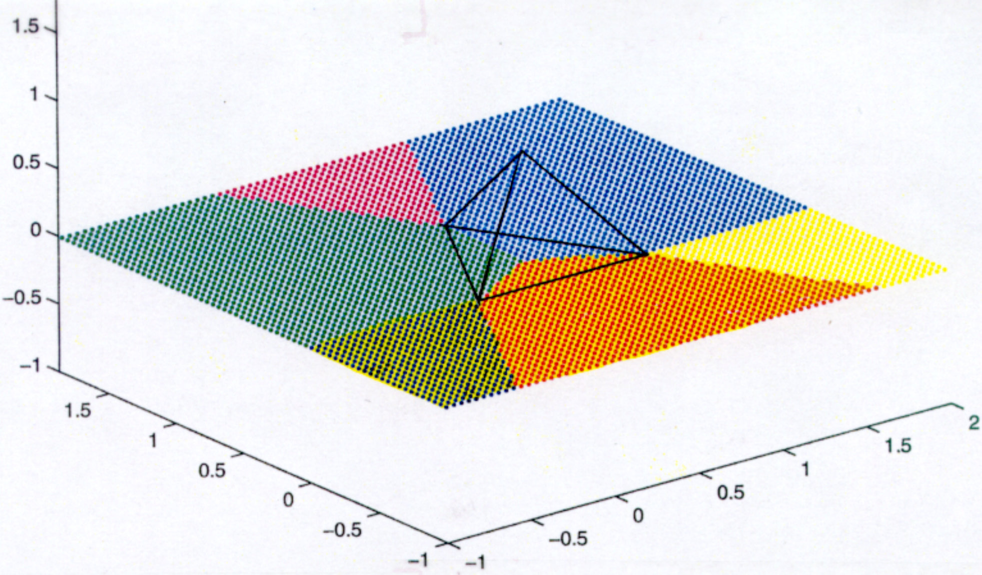
Here we color a plane through the three cities C_1, C_2, C_3. As Theorem 2.4 shows, each of these vertices have three colors.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