The Structure of Zero Divisor Sum Graphs

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Abstract

Let Σ_n be the graph whose vertex set is the set of non-zero zero divisors of $\mathbb{Z}_n$ where vw is an edge if $v+w$ is a non-zero zero divisor. We study various graph-theoretic properties of Σ_n including vertex degree, connectivity and cycles. Further investigation is also made into planar graphs and automorphisms of these kinds of graphs.

1 Preliminaries

1.1 Ring Structure

In order to fully understand the topic at hand it is important to review the properties of rings. A *ring* is a non-empty set R that has two binary operations, addition and multiplication, satisfying the following:

- $(\mathbf{R}, +)$ is an abelian group.
- Multiplication is associative and commutative.
- For all $a, b, c \in \mathbf{R}$, the distributive law, $a(b+c) = (ab) + (ac)$ holds.

We use the ring $\mathbb{Z}_n$, this is the set of integers $0, 1, ..., (n-1)$ under addition and multiplication modulo n. One of the most important concepts is that of a zero divisor.

Definition 1.1. *In a ring* $\boldsymbol{R}$*, a* zero divisor *is an element* $z \in \boldsymbol{R}$ *such that there exists* $x \in \boldsymbol{R},\ x \neq 0$ *where* $zx = 0$.

Theorem 1.2. *[4] In a ring* $\mathbb{Z}_n$*, the zero divisors are precisely those non-zero elements that are not relatively prime to* n.

In other words, k is a zero divisor in $\mathbb{Z}_n$ if and only if $\gcd(k, n) > 1$, a fact which we will use often throughout the paper. The other important class of elements are the units.

Definition 1.3. *A unit is an element $u \in R$ that has a multiplicative inverse in* R*, i.e. there exists a $v \in R$ such that $uv = 1$.*

Theorem 1.4. *[4] Let $n > 0$. Every element of $\mathbb{Z}_n$ is either a zero divisor or a unit.*

1.2 Graph Structure

We use a graph theoretic approach to study the non-zero zero divisors of the ring $\mathbb{Z}_n$.

Definition 1.5. *A graph G consists of a vertex set, $V(G)$, an edge set $E(G)$, and an association to each edge, $e \in E(G)$ of two vertices called the endpoints of e.*

Two vertices are *adjacent* if they share a common edge. If x and y are adjacent in a graph we denotes this $x \sim y$.

Two adjacent vertices are referred to as *neighbors* of each other. The set of neighbors of a vertex v, denoted $N(v)$ is called the *neighborhood* of v. An *edge* is incident at a vertex if that vertex is one of its endpoints. The *degree* of a vertex v is the number of edges incident at it, denoted by deg (v).

A *uv-walk* in a graph G is a sequence of vertices in G beginning with u and ending at v such that consecutive vertices in the sequence are adjacent. A *trail* in a graph G is a walk in which no edge is traversed more than once. A *uv-path* is a walk in a graph starting at a vertex u and ending at a vertex v in which no vertex is repeated except for maybe the first and the last. A *cycle* is a closed path.

Definition 1.6. *Let G be a graph. A subgraph of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*

1.3 Sum Graphs

Definition 1.7. *A sum graph is a graph whose vertices are labeled with integers where vertices i and j are joined by a line if and only if the vertex $i + j$ is in the graph.*

We investigate the graph Σ_n. The vertex set $V(\Sigma_n)$ consists of the non-zero zero divisors of $\mathbb{Z}_n$. Distinct vertices $x, y \in V(\Sigma_n)$ are adjacent if $x + y \in V(\Sigma_n)$
It is important to notice that no vertex v is adjacent to its inverse $-v$ because this would imply that 0 is in the graph.

Note that when n is prime, $V(\Sigma_n) = \emptyset$. For our study we will never consider this case.

Theorem 1.8. *If $h \mid n$, then Σ_h is a subgraph of Σ_n.*

Proof.
For any $k \in V(\Sigma_h)$, the $\gcd(h, k) > 1$. Since $h \mid n$, then $\gcd(n, k) > 1$. Thus, every vertex in Σ_h is a vertex in Σ_n.
For all edges xy in Σ_h, we know that $x, y, x + y \in V(\Sigma_h)$. From the statement above, we know that $x, y, x + y \in V(\Sigma_n)$. Therefore, Σ_h is a subgraph of Σ_n. ▲

2 Vertex Degrees

Definition 2.1. *A dominating vertex is a vertex in a graph that is adjacent to every other vertex of G.*

Proposition 2.2. *Let $n = 2x$ for some integer x. Then x is a dominating vertex in Σ_n if and only if x is even.*

Proof.
($\Leftarrow$) Let $x \in V(\Sigma_n)$ be even. We want to show for all $v \in V(\Sigma_n)$, with $v \neq x$, we have $v + x \in V(\Sigma_n)$.

Case I: Suppose v is even, or that $2 \mid v$.
Since $2 \mid x$, then $2 \mid v + x$. This implies that $\gcd(v + x, n) \geq 2 > 1$, therefore $v + x \in V(\Sigma_n)$.

Case II: Suppose v is odd, or that $2 \nmid v$.
Since v is a zero-divisor, there exists w such that $vw = 0 \in \mathbb{Z}_n$, or, viewing v and w as integers, $n \mid vw$. Since $2 \mid n$ and $2 \nmid v$, we know $2 \mid w$. Then

$$\begin{aligned} w(v + x) &= wv + wx \\ &= wv + w(n/2) \\ &= 0 + (w/2)n \\ &= 0 \end{aligned}$$

as we are in $\mathbb{Z}_n$ and $w/2$ is an integer.

Therefore, for all $v \in V(\Sigma_n)$, where $v \neq x$, it follows that $x \sim v$. Hence, x is a dominating vertex in Σ_n.

($\Rightarrow$) If x is a dominating vertex, then x is even.

Suppose towards a contradiction that x is odd. Because x is a dominating vertex, $x \sim 2$, so $x + 2 \in V(\Sigma_n)$. Thus $x + 2$ is odd and $\gcd(x + 2, n) > 1$. This forces $x + 2 = x$, since x is the only odd non-zero zero divisor. This is a contradiction because $x + 2 \neq x$, or $2 \neq 0$. Therefore x is even. Hence, x is a dominating vertex if and only if x is even.

▲

Definition 2.3. *Let R be a ring, and $a, b \in R$. We say that a is associate to b if a=ub for some unit u.*

Proposition 2.4. *If $g, g' \in V(\Sigma_n)$ and g is associate to g', then*

$$deg(g) = deg(g').$$

Proof.
We wish to show that if $g \sim x$ then there exists $y \in V(\Sigma_n)$ such that $g' \sim y$.

Suppose $g \sim x$. This implies that $g + x$ is a non-zero zero divisor. Since g is associate to g', we know $gk = g'$ for some unit k.

Let $y = xk$. Observe that x is associate to y. Since x is a zero divisor, there exists non-zero $s \in \mathbb{Z}_n$ such that $xs = 0$. Also observe:

$$\begin{aligned} ys &= (xk)s \\ &= (xs)k \\ &= 0 \end{aligned}$$

So y is a zero divisor. We now show that y is a non-zero zero divisor.

Assume that $y = 0$. Since $y = xk$, this implies that $xk = 0$. Since k is a unit, we can multiply by its inverse and end up with $x = 0$. This is a contradiction since x is a non-zero zero divisor. Therefore $y \neq 0$, and $y \in V(\Sigma_n)$.
Now, we show that $g' \sim y$, or that $g' + y$ is a non-zero zero divisor. Observe:

$$\begin{aligned} & g' + y \\ = & \; gk + xk \\ = & \; (g + x)k, \end{aligned}$$

and since $g + x$ is a zero divisor, there exists a non-zero $c \in \mathbb{Z}_n$ such that

$$(g + x)c = 0.$$

and so

$$[(g+x)k]c = 0$$

Since $kc \neq 0$, this implies that $g' + y$ is a non-zero zero divisor and therefore that $g' \sim y$.

Define the map Φ:$N(g) \mapsto N(g')$ by

$$\Phi(x) = xk.$$

This map is injective because if $xk = yk$, multiplying on the right side by k^{-1} gives

$$x = y.$$

Thus, for each neighbor of g we are able to find a unique neighbor of g'. Therefore, deg $(g') \geq$ deg (g). By symmetry, deg $(g) \geq$ deg (g'). Therefore, if g is associate to g',

$$\deg(g) = \deg(g').$$

▲

Definition 2.5. *The minimum degree of a graph G is the smallest degree of all the vertices in a graph, denoted $\delta(G)$.*

Proposition 2.6. Σ_n, $\delta(\Sigma_n) = 1$ *if and only if* $n = 4q$, *where q is prime.*

Proof.
($\Leftarrow$) Notice $V(\Sigma_n) = A \cup B$ where $A = \{2, 4, 6, ..., 4q-2\}$ and $B = \{q, 2q, 3q\}$. Each element in A is adjacent to every other element in A except its inverse. Thus deg $(v) > 1$ for all $v \in A$. Also $q \sim 2q$, but $q \not\sim 3q$(because they are inverses) and $q \not\sim v$ for all $v \in A \setminus \{2q\}$ because the sum of an even number and an odd number is odd. Therefore q will have degree 1.

($\Rightarrow$) Let $v \in V(\Sigma_n)$ such that deg $(v) = 1$.

Case I: $n = tv$ where $t \geq 5$.
Then, $v, 2v, 3v, 4v \in V(\Sigma_n)$. As a result, $v \sim 2v$ and $v \sim 3v$. So, deg $(v) > 1$, which is a contradiction.

Case II: $n = 2v$
This implies $v = -v$. Since deg $(v) = 1$, v is adjacent to some vertex, say x. This means

$$\begin{aligned}
&v + x \in V(\Sigma_n) \\
&\Rightarrow -v - x \in V(\Sigma_n) \\
&\Rightarrow v - x \in V(\Sigma_n) \text{ since } v = -v \\
&\Rightarrow v \sim -x
\end{aligned}$$

Suppose $x = -x$. Then, since $x < n$, we have $n = 2x$. Thus $v = \frac{n}{2} = x$. This is a contradiction because v is not adjacent to itself. So $x \neq -x$ which implies deg $(v) > 1$, a contradiction. Therefore $n \neq 2v$.

Case III: $n = 3v$
Subcase A:If $2 \mid n$, and since $3 \mid n$, this implies that $6 \mid n$. Since $6 \mid n$, this subcase has been reduced to Case 1.

Subcase B: If $2 \nmid n$, then 2 is a unit in $\mathbb{Z}_n$. Now $-2v = v$ (as $3v = 0$). Since deg $(v) = 1$, suppose $v \sim x$. This implies

$$\begin{aligned}
&v + x \in V(\Sigma_n) \\
&\Rightarrow (-2)(v + x) \in V(\Sigma_n)[\text{ as } -1 \text{ and } 2 \text{ are units.}] \\
&\Rightarrow v - 2x \in V(\Sigma_n) \\
&\Rightarrow v \sim -2x
\end{aligned}$$

Suppose $x = -2x$. Then, since $x < n$, we have $x = \dfrac{n}{3}$ or $x = \dfrac{2n}{3}$. Thus either $v = x$ or $v = -x$. This is a contradiction because v is not adjacent to itself or to its inverse. So $x \neq -2x$ which implies deg $(v) > 1$, a contradiction. Therefore $n \neq 3v$.

Thus we have shown that the minimum degree of Σ_n is 1 when $n = 4v$. However, we must still show v is prime. Suppose towards contradiction $v = st$ where $s, t > 1$. Then $n = 4st$. We see

$$v \sim s \text{ as } s(t+1) \text{ is a zero divisor and } st + s \neq n \text{ and}$$
$$v \sim 2s \text{ as } s(t+2) \text{ is a zero divisor and } st + 2s \neq n.$$

Thus deg $(v) > 1$, which is a contradiction. So v is prime.

▲

Proposition 2.7. *There exists an isolated vertex, $\delta(\Sigma_n) = 0$, if and only if $n = 3p$ or $n = 2p$, where p is prime.*

Proof. We will consider the two cases, $n = 2p$ or $n = 3p$.
($\Leftarrow$)Assume $n = 3p$.
The vertex set of Σ_{3p} is of the form: $A \cup B$ where

$$A = \{3, 6, ..., 3p - 3\}$$

$$\textit{and}$$

$$B = \{p, 2p\}$$

Notice, $p \not\sim 2p$ because $p + 2p \notin V(\Sigma_{3p})$.

Let $a \in A$. For all these vertices, $p \not\sim a$ because the vertex set A contains only multiples of 3.And p is not a multiple of 3. So $p + a \notin A$ and $2p + a \notin A$. Also, for every $b \in B$ we have $p + b \neq 2p$ and $2p + b \neq p$, because b is not a multiple of p. Lastly, $p \not\sim 2p$ because $p + 2p = 3p$, and $3p \notin V(\Sigma_{3p})$.

Therefore p and $2p$ are isolated vertices in Σ_{3p}.

Note: If p=3, then n=9, which consists of two vertices, which are isolated.

($\Leftarrow$) Assume $n = 2p$.
Consider Σ_n where $n = 2p$ and p is prime. In this case, the vertices of Σ_n are of the form:

$$V = \{p, 2, 4, 6, ..., 2p - 2\}.$$

Suppose towards contradiction that $x \sim p$; therefore $x + p \in V(\Sigma_n)$. The vertex x corresponds to an even integer.
Because the sum of an even and an odd number is odd, and p is the only odd vertex,

$$x + p \equiv p (\text{mod} \quad 2p)$$

This implies that $x = 0$. However, the element 0 is not in the graph and therefore we have a contradiction. Hence, p is an isolated vertex of the graph.

Note: If p=2, then n=4, which is an isolated vertex.

($\Rightarrow$) Case I: Suppose $n = tp$, where $t \geq 4$ and t is not prime.

Let $A = \{p, 2p, ..., (t-1)p\}$. Since $t \geq 4$, there are at least three elements in A. Let r be such that $r \mid t$ and let $B = \}r, 2r, ..., (p-1)r\}$. Since there are at least three elements in r, deg $(b) > 0$ for $b \in B$. There are at least $p - 1$ elements in B. If $p \geq 5$, there are at least 3 elements in the set, and thus we satisfy our conditions. If $p < 5$, consider the following two cases.

Case II: Let $p = 2$ so that $n = 2t$, where t is not prime.
Since t is not prime, it can be said that $t = mp$, where p is prime and $m \geq 2$. Then

by substitution, $n = 2mp$. Since $m \geq 2$, this implies that $2m \geq 4$, which was treated in Case 1.

Case III: Let $p = 3$ so that $n = 3t$, where t is not prime.
Since t is not prime, it can be said that $t = mp$, where p is prime and $m > 1$. Then by substitution, $n = 3mp$. Since $m > 1$, this implies that $3m \geq 3$, which was treated in Case 1.

Case IV: Suppose $n = tp$ where t is prime, where $t, p > 3$.
Notice $V(\sigma(tp))$ can pe partitioned into two sets:

$$A = \{p, 2p, ..., (q-1)p\}$$

$$B = \{q, 2q, ..., (p-1)q\}$$

All the elements of A are adjacent to everything except for itself and its inverse. All the elements of B are adjacent to everything except for itself and its inverse. Since t and p are primes greater than three, the degree of any element in A and B will be greater than zero.

▲

Proposition 2.8. *Let $p > 3$ be prime. Then the graph Σ_{p^k} for $k \in \mathbb{Z}$, has $p^{k-1} - 1$ vertices, each of degree $p^{k-1} - 3$. In particular, all vertices have even degree.*

Proof.
Observe that

$$V(\Sigma_{p^k} = \{p, 2p, 3p...(p-1)p, p^2, ..., (p^{k-1} - 1)p\}.$$

In particular there are $p^{k-1} - 1$ vertices in the graph. Each vertex is adjacent to all others except itself and its inverse in $\mathbb{Z}_{p^k}$.

We show that no vertex can be its own inverse. If $x \in V(\Sigma_{p^k})$ and $x = -x$, then $2x \equiv 0 \pmod{p^k}$. Because $p > 3$ is prime, $\gcd(2, p^k) = 1$. Since 2 is a unit in $\mathbb{Z}_{p^k}$, we can multiply by the inverse of 2, which yields $x \equiv 0 \pmod{p^k}$, a contradiction.

Since there are $p^{k-1} - 3$ neighbors of each vertex, the degree of each vertex in Σ_{p^k} is $p^{k-1} - 3$. ▲

3 Connectedness

Recall that uv-path is a walk in a graph from a vertex u to a vertex v in which no vertices are repeated.

Definition 3.1. *A graph G is connected if there exists a uv-path between all pairs of distinct vertices u and v of G.*

Definition 3.2. *The distance between $u, v \in V(G)$ is the smallest length of a uv-path in G and is denoted $d(u, v)$.*

Definition 3.3. *The diameter of a graph is the greatest distance between any two vertices of a connected graph G and is denoted $diam(G)$.*

Theorem 3.4. *The graph Σ_n is disconnected if and only if:*

$$n = 9 \text{ or}$$
$$n = pq \text{ where } p \text{ and } q \text{ are distinct primes.}$$

Proof.
($\Leftarrow$) The graph Σ_9 has two vertices and no edges, hence $Sigma_9$ is disconnected.

Now let $n = pq$, where p and q are distinct primes. We can partition the vertices into two sets:

$$A = \{p, 2p, ..., (q-1)p\}$$
$$B = \{q, 2q, ..., (p-1)q\}$$

There are no edges between vertices in A and vertices in B because no vertex that is a multiple of q is adjacent to a vertex that is a multiple of p. Therefore Σ_n has at two connected components and so it is disconnected.

In the case where $n = 6$, there are three isolated vertices- and therefore disconnected.

($\Rightarrow$) We now show that for every n not of this form, Σ_n is connected. So we considered all of the possible prime factorizations of n that are not of the form 3^2 or pq.

Case I: Let $n = p^k$ where $k > 1$ and $n \neq 9$. In order to show that Σ_n is connected, it suffices to show that there exists a uv-path between all vertices u and v in Σ_n.

Let $u \in V(\Sigma_{p^k})$. We know from Proposition 2.8 that the graph of Σ_{p^k} has $p^{k-1} - 1$ vertices each of degree $p^{k-1} - 3$. For distinct $u, v \in V(\Sigma_{p^k})$, if $u \not\sim v$, then u and v are inverses. Therefore, there must be another vertex r such that $u \sim r$ and $v \sim r$. Since inverses are unique, r cannot be the inverse of either u or v. Therefore there exists an uv-path through r. Hence, Σ_{p^k} is connected with diameter 2.

Case II: Let $n = p^a q^b$ where p, q are distinct primes and $a > 1$ or $b > 1$. Without loss of generality, assume that $a > 1$.

In this case, $pq \not\sim pq$ and $pq \not\sim (n - pq)$. However, pq is adjacent to every other vertex in the graph; therefore $pq \sim p$. We know that $p \sim (n - pq)$, so $pq, p, (n - pq)$ is a path. Therefore the graph is connected and has diameter 2.

Case III: Let $n = {p_1}^{e_1} {p_2}^{e_2} ... {p_r}^{e_r}$ for $r \geq 3$ where the p_i are distinct primes. Let

$$a_i = n/p_i = {p_1}^{e_1} {p_2}^{e_2} ... {p_i}^{e_i - 1} ... {p_r}^{e_r}$$

For any $i, j \in \mathbb{Z}$, $a_i \sim a_j$ because all a_i are multiples of p_k, and therefore $a_i + a_j$ is a multiple of p_k where $k \neq i, j$ and $k \in \mathbb{N}$. The vertex a_j is of the form:

$$a_j = {p_1}^{e_1} {p_2}^{e_2} ... {p_j}^{e_j - 1} ... {p_r}^{e_r}.$$

The inverse of a_i is $-a_i = {p_1}^{e_1} {p_2}^{e_2} ... ({p_i}^{e_i} - {p_i}^{e_i - 1}) ... {p_r}^{e_r}$, so $a_j \neq -a_i$, and since a_j is a multiple of p_k, it is true that $a_i \sim a_j$ for all i, j. So, $a_1, a_2, ..., a_r$ form a clique and are therefore in the same component.

Now we want to show that for all $v \in V(\Sigma_n)$, $v \sim a_s$ for all s except maybe one. Suppose $v = -a_i$ for some i. Then obviously, $v + a_j \neq 0$ for all $j \neq i$. So we have shown that $v + a_j$ is non-zero.

Now we need to show that $\gcd(n, v) > 1$. We know that v is a multiple of p_k for all $k \neq i$, and that a_j is a multiple of all p_k for $k \neq j$. Since $r \geq 3$ we know we can pick $k \neq i, j$. Therefore p_k is a factor of a_j and v and since a_j and v are multiples of p_k, it is true that $a_j + v$ is a multiple of p_k and hence, a zero divisor. So, $v \sim a_k$ for all $k \neq i$ if $v = -a_i$.

Now, suppose v is not of the form a_i for any i. Then there exists i such that $\gcd(v, p_i) > 1$, therefore v is a multiple of p_i. For all $s \neq i$, we know a_s is a multiple of p_i, since v is a multiple of p_i and a_s is a multiple of p_i, it is true that $v + a_s$ is a multiple of p_i and therefore is a zero divisor. Here, their sum is obviously not 0 because v is not the inverse of any a_i.

Therefore $v \sim a_s$ for all $s \neq i$.

Finally, we need to show that for $u, v \in V(\Sigma_n)$, there exists a uv-path. Find $i, j \in V(\Sigma_n)$ such that $\gcd(p_i, u) > 1$ and $\gcd(p_j, v) > 1$. Since $r > 3$, we find $k \neq i, j$, therefore $u \sim a_k$ and $v \sim a_k$ and thus there exists a uv-path of distance 2, so Σ_n is connected.

Therefore, a Σ_n is disconnected if and only if $n = 9$ or $n = pq$. ▲

Corollary 3.5. *If Σ_n is connected, its diameter is less than or equal to 2.*

Proof. From the proof of Proposition 3.4 we can see that $\mathrm{diam}(\Sigma_n) \leq 2$ when Σ_n is connected. ▲

Definition 3.6. *A uv-trail in a graph is a uv-walk in which no edge is traversed more than once.*

Definition 3.7. *An Eulerian graph is a graph which contains a closed trail containing every edge.*

Theorem 3.8. *[3] A nontrivial connected graph is Eulerian if and only if every vertex of G has even degree.*

Combining Propositions 2.8 and Case I of 3.4, we develop the following corollary.

Corollary 3.9. *Let $p > 3$ be prime, then Σ_{p^k} is Eulerian.*

Proof.
From Proposition 2.8 we see that the vertices of the graph of Σ_{p^k} are all of even degree, and from Proposition 3.4 we see that Σ_{p^k} is connected. Therefore by Theorem 3.8, Σ_{p^k} is Eulerian. ▲

4 Cycles

Theorem 4.1. *Let n be composite, then Σ_n contains a cycle if and only if $n > 9$.*

Proof.
($\Leftarrow$) Let $n > 9$ be such that $n = pm$ where p is the smallest prime that divides n and $m \in \mathbb{N}$. Note that this imples $m \geq 4$.
If $n = 4p$, then $p \neq 2$ because $n > 9$. However, if $p \geq 3$, then 2,4,6 form a cycle.
If $n = 5p$ then $2p \not\sim 3p$, but $p, 2p, 4p, 3p$ form a cycle.
If n is not of the form $4p$ or $5p$, then $p, 2p, 3p$ form a cycle.

($\Rightarrow$) Now we show by brute force that for $n \leq 9$ the graph of Σ_n does not contain a cycle.
For $n = 1, 2, 3, 5, 7$, n is prime, so Σ_n obviously contains no cycle because it is the empty set.
For $n = 4$, the graph contains only the vertex 2, so Σ_4 does not contain a cycle either.
For $n = 6$, the graph contains 3 vertices, but 3 is an isolated vertex, so there is no cycle.
For $n = 8$, the graph contains 3 vertices, but 2 and 6 are not adjacent, so there is no cycle.
Therefore, for composite n, the graph Σ_n contains a cycle if and only if $n > 9$. ▲

Definition 4.2. *If a graph G has a cycle, then the girth of G is the length of the shortest cycle in G. The girth of a graph G is denoted $gr(G)$.*

Corollary 4.3. *If $n \neq 5p$ and $n > 9$ is composite, then $gr(\Sigma_n) = 3$.*

Proof. This is clear from the proof of Proposition 4.1. ▲

Corollary 4.4. *If $n = 5p$ for $p = 2$ or 3, then $gr(\Sigma_n) = 4$.*

Proof.
Recall that p is the smallest prime that divides n where $n > 9$. Since $5 \mid n$, and $p \leq 5$, there are only 3 cases to consider, $n = 10$, $n = 15$, and $n = 25$. In each case, inspection shows that Σ_n contains no 3-cycles.

If $n = 10$ a cycle is formed by 2,4,8,6 and $gr(\Sigma_{10}) = 4$.

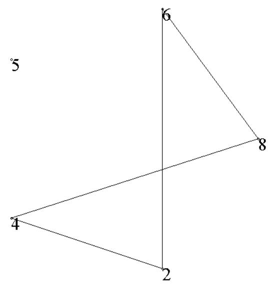

Figure 1: This is a picture of Σ_{10}; note that there are no three cycles.

If $n = 15$ a cycle is formed by 3,6,12,9 and $gr(\Sigma_{15}) = 4$.

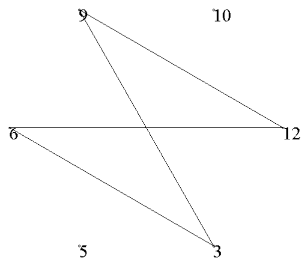

Figure 2: This is a picture of Σ_{15}; note that there are no three cycles.

If $n = 25$ a cycle is formed by 5, 10, 20, 15 and again $gr(\Sigma_{25}) = 4$.

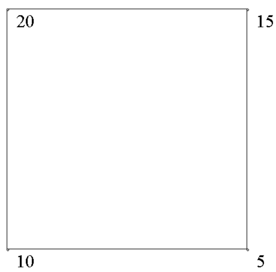

Figure 3: This is a picture of Σ_{25}; note that there are no three cycles.

▲

Thus, $gr(\Sigma_n) = 4$ in all 3 cases.

5 Planarity

Definition 5.1. *A graph is planar if it can be drawn in the plane with no edge crossings.*

To prove our next lemma, we use the Handshaking Theorem[5]:

Theorem 5.2. *The sum of the degrees of the vertices of a graph is equal to twice the number of edges.*

We denote the number of vertices as m and the number of edges as e to obtain the equation

$$\Sigma_{v \in V} deg(v) = 2e.$$

Lemma 5.3. *For all graphs, $\delta m/2 \leq e$.*

Proof.
For every $v \in V(\Sigma_n)$, $\delta \leq deg(v)$. Therefore, $\delta m \leq \Sigma_{v \in V} deg(v)$. By the handshaking theorem, $\delta m \leq 2e$. Dividing by 2, $\delta m/2 \leq e$. ▲

Definition 5.4. *A face is a portion of a plane drawing of a planar graph which is bounded by edges and has no edge running through the interior.*

Theorem 5.5. *[5] (Euler's Formula) For any connected plane graph G, $m-e+f=2$, where f denotes the number of faces in G.*

Lemma 5.6. *If Σ_n is planar, then $e \leq 3m-6$.*

Proof.
Every face in Σ_n has length ≥ 3. Let $F_1, F_2, \ldots, F_k$ be the faces in Σ_n and $l(F_i)$ be the length of the boundary of the face F_i. Because every edge e is part of the boundary of two faces, $2e = \Sigma_1^k l(F_i) \geq \Sigma_1^k 3 = 3f$. So $f \leq 2e/3$. Using Euler's formula, we know

$$\begin{gathered} f = 2 - m + e \leq 2e/3 \\ e/3 \leq m - 2 \\ e \leq 3m - 6 \end{gathered}$$

▲

We use Lemma 5.3 and Lemma 5.6 to obtain the following inequalities:

$$
\begin{aligned}
\delta m/2 &\leq 3m - 6 \\
\delta m &\leq 6m - 12 \\
\delta &\leq 6 - 12/m
\end{aligned}
$$

Theorem 5.7. *Let p be prime and $k > 1$. If Σ_{p^k} is planar, then $k \leq 3$. Further, $p \leq 7$ for $k \leq 2$ and $p = 2$ for $k = 3$.*

Proof.
By the result above, $\delta \leq 6 - 12/m$. Let i be any integer such that $1 < i \leq k$ and let c be any integer such that $1 \leq c \leq p - 1$. Since p is prime, $cp^i \in V(\Sigma_{p^k})$ is adjacent to every vertex in $V(\Sigma_{p^k})$ except for itself and $p^k - cp^i$. Therefore, $\delta = m - 2$, and we have

$$
\begin{aligned}
m - 2 &\leq 6 - 12/m \\
m - 8 + 12/m &\leq 0 \\
m^2 - 8m + 12 &\leq 0 \\
(m - 6)(m - 2) &\leq 0 \\
m &\leq 6
\end{aligned}
$$

But we know that $m = p^{k-1} - 1$, so

$$
\begin{aligned}
m = p^{k-1} - 1 &\leq 6 \\
p^{k-1} &\leq 7
\end{aligned}
$$

If $k = 2$, then $p \leq 7$.
If $k = 3$, then $p^2 \leq 7$, and $p = 2$.
If $k \geq 4$, then $p^{k-1} \leq 7$, and $p < 2$, which is impossible. ▲

Definition 5.8. *A subdivision of an edge $e = uv$ occurs when a new vertex w is placed along e and the edge uv is replaced by the path uwv of length 2.*

Definition 5.9. *Let G and G' be graphs. G is homeomorphic to G' if there exists a graph H such that G and G' both result from subdivisions of $E(H)$.*

Definition 5.10. *The graph $K_{3,3}$ has six vertices. Three of the vertices a, b, and c are all adjacent to the other three vertices d, e, and f. Also, $a \nsim b, c$; $b \nsim c$; $d \nsim e, f$; $e \nsim f$.*

Definition 5.11. *The graph K_5 has five vertices, each of which are adjacent to the other four.*

Theorem 5.12. *[5] (Kuratowski's Theorem) A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Lemma 5.13. *Every subgraph of a planar graph is planar.*

Proof.
Suppose not. Let G be a planar graph and H be a non-planar subgraph of G. Then H contains a subgraph homeomorphic to K_5 or $K_{3,3}$. So G itself must contain a subgraph that is homeomorphic to K_5 or $K_{3,3}$. ▲

Definition 5.14. *Let G be a graph and $S \subseteq V(G)$. The subgraph induced by S, denoted $G[S]$ is the graph defined by $V(G[S]) = S$ and $e \in E(G[S])$ if $e \in E(G)$ and both endpoints of e are elements of S.*

Theorem 5.15. *Let $p_1^{e_1}, p_2^{e_2}, \ldots, p_r^{e_r}$ be primes. If $\Sigma_{p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}}$ is planar, then $p_i \leq 7$ for every i.*

Proof.
Let I be the subgraph of $\Sigma_{p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}}$ induced by $p_i, 2p_i, \ldots, (p_j - 1)p_i$ for some $1 \leq i \leq r$ and $1 \leq j \leq r$, $i \neq j$. By Lemma 5.13, if $\Sigma_{p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}}$ is planar, then I must be planar. Since the vertices of $\Sigma_{p_j^2}$ are $\{p_j,\ 2p_j, \ldots,\ (p_j - 1)p_j\}$, the mapping $qp_i \mapsto qp_j$ defines an isomorphism from I to $\Sigma_{p_j^2}$. Since I is planar, then $\Sigma_{p_j^2}$ must be planar. Therefore, by Proposition 5.7, $p_j \leq 7$ for all $1 \leq j \leq r$. ▲

Corollary 5.16. *Σ_n is only planar for*

$$n = 4, 6, 8, 9, 10, 12, 14, 15, 21, 25, 35, 49.$$

For any planar graph, each component is isomorphic to one of the following:

Figure 4: Σ_4

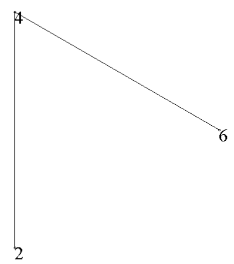

Figure 5: Σ_8

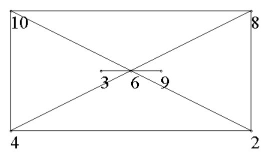

Figure 6: Σ_{12}

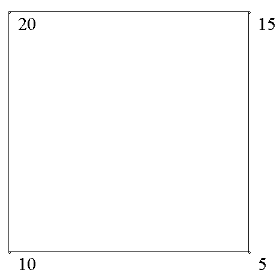

Figure 7: Σ_{25}

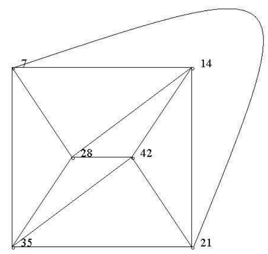

Figure 8: Σ_{49}

Definition 5.17. *A planar graph is outerplanar if every vertex of the graph lies on the boundary of the exterior region.*

Definition 5.18. *The graph $K_{2,3}$ has five vertices. Two of the vertices a and b are adjacent to the three other vertices in the graph, c, d, and e. Also, $a \nsim b$, $c \nsim d$, $c \nsim e$, and $d \nsim e$.*

Theorem 5.19. *Let G be a graph. If G has a subgraph homeomorphic to $K_{2,3}$, then G is not outerplanar.*

Proof.
For a contradiction, assume G has a subgraph homeomorphic to $K_{2,3}$ and G is outerplanar. Then it is possible to introduce a new vertex v inside the exterior region such that v is adjacent to c, d, and e and the resulting graph is planar. This would imply that $K_{3,3}$ is planar, which is not true by Kuratowski's Theorem. ▲

Proposition 5.20. *The graph Σ_n is outerplanar if and only if*

$$n = 4, 6, 8, 9, 15, 25.$$

Proof.
If Σ_n is outerplanar, it must be planar. So we have a limited number of values for n to investigate. The graphs of Σ_4, Σ_8, and Σ_{25} are clearly outerplanar. The graph of Σ_{12} is not outerplanar because the vertices 10 and 2 are each adjacent to $8, 4,$and 6. Thus it has a subgraph isomorphic to $K_{2,3}$. The graph of Σ_{49} is not outerplanar because the vertices 7 and 42 are each adjacent to $14, 21,$ and 28. Thus it has a subgraph isomorphic $K_{2,3}$. ▲

6 Automorphisms

Proposition 6.1. *Let $s \in \mathbb{Z}_n$. Then the map $M_s : V(\Sigma_n) \to V(\Sigma_n)$ defined by $M_s(x) = sx$ induces an automorphism of Σ_n if and only if $s \in \mathbb{Z}_n^*$.*

Proof.
($\Leftarrow$) Suppose $u \in \mathbb{Z}_n$ is a unit and x, y are adjacent vertices. Then $x + y$ is a vertex, so $x + y \neq 0$. Since u is a unit,

$$\begin{gathered} ux + uy \\ = u(x + y) \\ \neq 0 \end{gathered}$$

Thus, $ux \sim uy$. This shows that the map M_u preserves edges.

We must show M_u is one-to-one. Suppose that there exists $x, y \in V(\Sigma_n)$ such that $ux = uy$. Since $u \in \mathbb{Z}_n*$, we have $u^{-1}(ux) = u^{-1}(uy)$. Therefore, M_u is one-to-one, since $x = y$.

We do not need to show M_u is onto because any one-to-one map from a finite set to itself is automatically onto.

Therefore M_u a one-to-one correspondence between the vertices which preserves edges.

($\Rightarrow$) To prove the contrapositive, assume z is not a unit. Thus z is a zero divisor. Now suppose $z \in \mathbb{Z}_n$ is a zero divisor. Then there exists $t \neq 0$, such that $zt = 0$. Since $t \in V(\Sigma_n)$, $zt = 0$, which is not a vertex. Therefore M_z is not an automorphism.

▲

The following table shows all 50 automorphisms of Σ_16. Keep in mind that 8 is a dominating vertex and is adjacent to every other vertex in the graph. Since no other vertex is adjacent to everything else, then 8 will always map to itself. For that reason we will not include in the table $8 \mapsto 8$.

$2 \mapsto 2$	$14 \mapsto 14$	$4 \mapsto 10$	$12 \mapsto 6$	$10 \mapsto 4$	$6 \mapsto 12$
	$14 \mapsto 14$	$4 \mapsto 10$	$12 \mapsto 6$	$10 \mapsto 12$	$6 \mapsto 4$
$2 \mapsto 4$	$14 \mapsto 12$	$4 \mapsto 2$	$12 \mapsto 14$	$10 \mapsto 10$	$6 \mapsto 6$
	$14 \mapsto 12$	$4 \mapsto 2$	$12 \mapsto 14$	$10 \mapsto 6$	$6 \mapsto 10$
	$14 \mapsto 12$	$4 \mapsto 6$	$12 \mapsto 10$	$10 \mapsto 2$	$6 \mapsto 14$
	$14 \mapsto 12$	$4 \mapsto 6$	$12 \mapsto 10$	$10 \mapsto 14$	$6 \mapsto 2$
	$14 \mapsto 12$	$4 \mapsto 10$	$12 \mapsto 6$	$10 \mapsto 2$	$6 \mapsto 14$
	$14 \mapsto 12$	$4 \mapsto 10$	$12 \mapsto 6$	$10 \mapsto 14$	$6 \mapsto 2$
	$14 \mapsto 12$	$4 \mapsto 14$	$12 \mapsto 2$	$10 \mapsto 10$	$6 \mapsto 6$
	$14 \mapsto 12$	$4 \mapsto 14$	$12 \mapsto 2$	$10 \mapsto 6$	$6 \mapsto 10$
$2 \mapsto 6$	$14 \mapsto 10$	$4 \mapsto 12$	$12 \mapsto 4$	$10 \mapsto 14$	$6 \mapsto 2$
	$14 \mapsto 10$	$4 \mapsto 12$	$12 \mapsto 4$	$10 \mapsto 2$	$6 \mapsto 14$
	$14 \mapsto 10$	$4 \mapsto 2$	$12 \mapsto 14$	$10 \mapsto 4$	$6 \mapsto 12$
	$14 \mapsto 10$	$4 \mapsto 2$	$12 \mapsto 14$	$10 \mapsto 12$	$6 \mapsto 4$
	$14 \mapsto 10$	$4 \mapsto 14$	$12 \mapsto 2$	$10 \mapsto 4$	$6 \mapsto 12$
	$14 \mapsto 10$	$4 \mapsto 14$	$12 \mapsto 2$	$10 \mapsto 12$	$6 \mapsto 4$
	$14 \mapsto 10$	$4 \mapsto 4$	$12 \mapsto 12$	$10 \mapsto 14$	$6 \mapsto 2$
	$14 \mapsto 10$	$4 \mapsto 4$	$12 \mapsto 12$	$10 \mapsto 2$	$6 \mapsto 14$
$2 \mapsto 10$	$14 \mapsto 6$	$4 \mapsto 4$	$12 \mapsto 12$	$10 \mapsto 2$	$6 \mapsto 14$
	$14 \mapsto 6$	$4 \mapsto 4$	$12 \mapsto 12$	$10 \mapsto 14$	$6 \mapsto 2$
	$14 \mapsto 6$	$4 \mapsto 2$	$12 \mapsto 14$	$10 \mapsto 12$	$6 \mapsto 4$
	$14 \mapsto 6$	$4 \mapsto 2$	$12 \mapsto 14$	$10 \mapsto 4$	$6 \mapsto 12$
	$14 \mapsto 6$	$4 \mapsto 12$	$12 \mapsto 4$	$10 \mapsto 2$	$6 \mapsto 14$
	$14 \mapsto 6$	$4 \mapsto 12$	$12 \mapsto 4$	$10 \mapsto 14$	$6 \mapsto 2$
	$14 \mapsto 6$	$4 \mapsto 14$	$12 \mapsto 2$	$10 \mapsto 4$	$6 \mapsto 12$
	$14 \mapsto 6$	$4 \mapsto 14$	$12 \mapsto 2$	$10 \mapsto 12$	$6 \mapsto 4$
$2 \mapsto 12$	$14 \mapsto 4$	$4 \mapsto 2$	$12 \mapsto 14$	$10 \mapsto 10$	$6 \mapsto 6$
	$14 \mapsto 4$	$4 \mapsto 6$	$12 \mapsto 10$	$10 \mapsto 14$	$6 \mapsto 2$
	$14 \mapsto 4$	$4 \mapsto 6$	$12 \mapsto 10$	$10 \mapsto 2$	$6 \mapsto 14$
	$14 \mapsto 4$	$4 \mapsto 10$	$12 \mapsto 6$	$10 \mapsto 2$	$6 \mapsto 14$
	$14 \mapsto 4$	$4 \mapsto 10$	$12 \mapsto 6$	$10 \mapsto 14$	$6 \mapsto 2$
	$14 \mapsto 4$	$4 \mapsto 14$	$12 \mapsto 2$	$10 \mapsto 10$	$6 \mapsto 6$
$2 \mapsto 14$	$14 \mapsto 2$	$4 \mapsto 4$	$12 \mapsto 12$	$10 \mapsto 10$	$6 \mapsto 6$
	$14 \mapsto 2$	$4 \mapsto 4$	$12 \mapsto 12$	$10 \mapsto 6$	$6 \mapsto 10$
	$14 \mapsto 2$	$4 \mapsto 12$	$12 \mapsto 4$	$10 \mapsto 6$	$6 \mapsto 10$
	$14 \mapsto 2$	$4 \mapsto 12$	$12 \mapsto 4$	$10 \mapsto 10$	$6 \mapsto 6$
	$14 \mapsto 2$	$4 \mapsto 6$	$12 \mapsto 10$	$10 \mapsto 12$	$6 \mapsto 4$
	$14 \mapsto 2$	$4 \mapsto 6$	$12 \mapsto 10$	$10 \mapsto 4$	$6 \mapsto 12$
	$14 \mapsto 2$	$4 \mapsto 10$	$12 \mapsto 6$	$10 \mapsto 4$	$6 \mapsto 12$
	$14 \mapsto 2$	$4 \mapsto 10$	$12 \mapsto 6$	$10 \mapsto 12$	$6 \mapsto 4$

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8 A Appendix: Glossary of Terms

In the following definitions, let $G = \Sigma_n$ be the graph with vertex set $V(\Sigma_n)$ and edge set $E(\Sigma_n)$.

Adjacent: Two vertices $u, v \in V(G)$ are said to be *adjacent* if $uv \in E(G)$.

Associate: Let R be a ring, and $a, b \in R$. We say that a is associate to b if $a = ub$ for some unit u.

Connected: A graph G is connected if there exists a uv-path between all pairs of distinct vertices u and v of G.

Degree: The degree of a vertex v is the number of edges incident at it.

Diameter: The diameter of a graph is the greatest distance between any two vertices of a connected graph G and is denoted $\text{diam}(G)$.

Distance: The distance between $u, v \in V(G)$ is the shortest length of a uv-path in G.

Dominating vertex: Any vertex, $v \in V(\Sigma_n)$, that is adjacent to every other vertex in Σ_n.

Eulerian Graph: A graph which has a closed trail containing every edge.

Face: A face is a portion of the graph which is bounded by edges and such that there is no edge running through the interior.

Girth: If a graph G has a cycle, then the girth of G is the length of the shortest cycle in G. The girth of a graph G is denoted $gr(G)$.

Graph: A graph G consists of a vertex set, $V(G)$, an edge set $E(G)$, and an association to each edge, $e \in E(G)$ of two vertices, called the endpoints of e.

Incident: An edge is incident at a vertex if that vertex is one of its endpoints.

Minimum degree: The minimum degree of a graph G is the smallest degree of all the vertices in a graph.

Outerplanar: A planar graph is outerplanar if every vertex of the graph lies on the boundary of the exterior region.

Path:: A uv-path is a walk in a graph starting at a vertex u and ending at a vertex v in which no vertex is repeated
Planar Graph: A graph is planar if it can be drawn in the plane with no edge crossings.

Subgraph: Let Σ_n be a graph. A subgraph of Σ_n is a graph H such that $V(H) \subseteq V(\Sigma_n)$ and $E(H) \subseteq E(\Sigma_n)$.

Unit: A unit is an element $u \in R$ that has a multiplicative inverse in R, i.e. there exists a $v \in R$ such that $uv = 1$ and $vu = 1$.

Walk: A walk in a graph G is a sequence of vertices such that consecutive vertices in the sequence are adjacent.

Zero divisor: In a ring R, a zero divisor is an element $z \in R$ for which there exists $x \in R,\ x \neq 0$ such that $zx = 0$.

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