Identities in Quasigroups and Loops

Hannah Hoganson and Marco Tapia

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1 What are Quasigroups and Loops?

A quasigroup is a set endowed with a binary operation, *, such that for any equation of the form $x * y = z$, where two of these variables are known, the third one is uniquely determined. This is known as the Latin Square Property. A loop is a quasigroup that also has a neutral element, which we will denote as e, which has the property that $x * e = e * x = x$ for every x.

An Equational Quasigroup is defined by these four identities:

1. $x * (x \backslash y) = y$
2. $(y/x) * x = y$
3. $x \backslash (x * y) = y$
4. $(y * x)/x = y$

There is a natural bijection between quasigroups and equational quasigroups. Defining a loop is as simple as adding the condition that there exist a neutral element in the set.

The three operations here are , $*$', known as star, /, known as right divide, and \, known as left divide. Both right and left divide are defined in terms of star. So if $x/y = z$ then $x = z * y$ and if $x \backslash y = z$ then $y = x * z$.

We decided to investigate what properties quasigroups and loops would have when they satisfy an additional identity. The identities we used had three distinct variables on each side. First we tested what happened when we had the same parenthesization pattern on each side, and just varied the operations. Then we went further and tested what happened when the operations were varied with different parenthesization patterns on each side. To do this testing we used the automated theorem prover program Prover9 to check which identities were equivalent and which implications were valid, and we used Mace4 to check which implications were not valid as well as to come up with relevant counterexamples.

2 Naming the Identities

We gave our identities the following naming structure; an A or a B to denote the parenthesization pattern on each side, and two of the numbers 1 through 9 to denote the operations used on each side.

For the parentheses we have:

A: $x(yz)$ and B: $(xy)z$

For the operations we have:

$$\begin{array}{lll} 1: ** & 2: */ & 3: /* \\ 4: *\backslash & 5: \backslash * & 6: // \\ 7: /\backslash & 8: \backslash / & 9: \backslash\backslash \end{array}$$

So for example, the equation B13 would read $(x*y)*z = (x/y)*z$; the equation A27 would read $x*(y/z) = x/(y\backslash z)$ and the equation A3B5 would read $x/(y*z) = (x\backslash y)*z$. When mixing the parenthesization patterns, we always use A on the left side and B on the right side, so we will sometimes drop the AB and just denote the equation by 35.

3 Varying the Operators

Before we begin proving the equivalences among our identities there are a few lemmas that are frequently used and should be proved first.

1. $x/(y\backslash x) = y$

We know $y*(y\backslash x) = x$ so right dividing by $y\backslash x$ gives $y = x/(y\backslash x)$.

2. $(x/y)\backslash x = y$

We know $x = (x/y)*y$ so left dividing by (x/y) gives $y = (x/y)\backslash x$.

3. $x*y = y*x \Leftrightarrow x/y = y\backslash x$

$(\Rightarrow)$ $x*y = y*x$ let $x = y\backslash z$ so $z = y*x$, then $(y\backslash z)*y = z$ and right division by y gives $y\backslash z = z/y$.

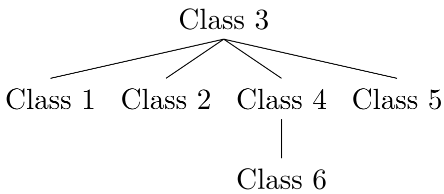
$(\Leftarrow)$ $x/y = y\backslash x$ right multiplication by y gives $x = (y\backslash x)*y$ let $y\backslash x = z$ so $x = y*z$, then $z*y = y*z$.

3.1 Identities in Each Class

The identities in quasigroups can be classified as follows:

Class 1	A12,A13,A16,A23,A26,A36,A47,A58,B12,B13,B16,B23,B26,B36,B47,B58
Class 2	A14,A15,A19,A28,A37,A45,A49,A59,B14,B15,B19,B28,B37,B45,B49,B59
Class 3	A17,A18,A25,A29,A34,A38,A39,A48,A56,A57,B17,B18,B25,B27,B29,B34,B38,B46,B48,B56
Class 4	A24,A35,A67,A68,A78,A79,A89,B24,B35,B67,B68,B78,B79,B89
Class 5	A27,A46,B39,B57
Class 6	A69,B69

In loops, the two identities in Class 6 become part of Class 4.
The classes are related as follows:



We will show all of this in the following sections.

3.2 Class 1

Theorem 1. *The identities A12, A13, A16, A23, A26, A36, A47, A58, B12, B13, B16, B23, B26, B36, B47, B58 are equivalent to* $x*y = x/y$.

Note that all of the proofs for identities in this class hold in loops and quasigroups. Also note that proofs in the other direction, assuming $x * y = x/y$, are trivial.

A12, A58, B13, B26, can all be proved by cancellation or multiplication.
A12: $x * (y * z) = x * (y/z)$ left dividing by x leaves $y * z = y/z$.
A58: $x\backslash(y * z) = x\backslash(y/z)$ left multiplying by x leaves $y * z = y/z$.
B13: $(x * y) * z = (x/y) * z$ right dividing by z leaves $x * y = x/y$.
B26: $(x * y)/z = (x/y)/z$ right multiplying by z leaves $x * y = x/y$.

A13, A26, A47, B12, B36, B58 can all be proved by combining two variables.
A13: $x * (y * z) = x/(y * z)$ let $y * z = w$, so $x * w = x/w$.
A26: $x * (y/z) = x/(y/z)$ let $y/z = w$, so $x * w = x/w$.
A47: $x * (y\backslash z) = x/(y\backslash z)$ let $y\backslash z = w$, so $x * w = x/w$.
B12: $(x * y) * z = (x * y)/z$ let $x * y = w$, so $w * z = w/z$.
B36: $(x/y)/z = (x/y) * z$ let $x/y = w$, so $w/z = w * z$.
B58: $(x\backslash y) * z = (x\backslash y)/z$ let $x\backslash y = w$, so $w * z = w/z$.

A16, A23, B16, B23 can be proved by slightly trickier substitutions.
A16: $x * (y * z) = x/(y/z)$ let $z = y\backslash y$ so $y = y * z$ then $x * y = x/[(y * z)/z] = x/y$. (Note: In a loop we can simply let $z = e$.)
A23: $x * (y/z) = x/(y * z)$ let $z = y\backslash y$ so $y = y * z$ and $y = y/z$, then $x/y = x * y$. (Note: In a loop we can simply let $z = e$.)
B16: $(x * y) * z = (x/y)/z$ let $y = x\backslash x$ so $x = x * y$, then $x * z = [x/(x\backslash x)]/z = x/z$. (Note: In a loop we can simply let $y = e$.)
B23: $(x * y)/z = (x/y) * z$ let $y = x\backslash x$ so $x = x * y$ and $x/y = x$, then $x/z = x * z$. (Note: In a loop we can simply let $y = e$.)

A36 and B47 can be proved with the definition of equational quasigroups.
A36: $x/(y*z) = x/(y/z)$ right multiplying by $(y*z)$ gives $x = [x/(y/z)]*(y*z)$, so $y/z = y*z$.
B47: $(x*y)\backslash z = (x/y)\backslash z$ left multiplying by $(x*y)$ gives $z = (x*y)*[(x/y)\backslash z]$, so $x*y = x/y$.

3.3 Class 2

Theorem 2. *The identities A14, A15, A19, A28, A37, A45, A49, A59, B14, B15, B19, B28, B37, B45, B49, B59 are equivalent to* $x * y = x\backslash y$.

Note that all of the proofs for identities in this class hold in loops and quasigroups. Also note that proofs in the other direction, assuming $x * y = x\backslash y$, are trivial.

A14, A59, B15, B28 can all be proved by cancellation or multiplication.
A14: $x * (y * z) = x * (y\backslash z)$ left dividing by x leaves $y * z = y\backslash z$.
A59: $x\backslash(y * z) = x\backslash(y\backslash z)$ left multiplying by x leaves $y * z = y\backslash z$.
B15: $(x * y) * z = (x\backslash y) * z$ right dividing by z leaves $x * y = x\backslash y$.
B28: $(x * y)/z = (x\backslash y)/z$ right multiplying by z leaves $x * y = x\backslash y$.

A15, A28, A49, B14, B37, B59 can all be proved by combining two variables.
A15: $x * (y * z) = x\backslash(y * z)$ let $y * z = w$, then $x * w = x\backslash w$.
A28: $x * (y/z) = x\backslash(y/z)$ let $y/z = w$, then $x * w = x\backslash w$.
A49: $x * (y\backslash z) = x\backslash(y\backslash z)$ let $y\backslash z = w$, then $x * w = x\backslash w$.
B14: $(x * y) * z = (x * y)\backslash z$ let $x * y = w$, then $w * z = w\backslash z$.
B37: $(x/y) * z = (x/y)\backslash z$ let $x/y = w$, then $w * z = w\backslash z$.

B59: $(x\backslash y) * z = (x\backslash y)\backslash z$ let $x\backslash y = w$, then $w * z = w\backslash z$.

A19, A45, B19, B45 can all be proved by slightly trickier substitutions. (Note: In a loop we can simply let x=e to see $y * z = y\backslash z$)
A19: $x * (y * z) = x\backslash(y\backslash z)$ let $y = z/z$ so $z = y * z$ and $y\backslash z = z$, then $x * z = x\backslash z$.
A45: $x * (y\backslash z) = x\backslash(y * z)$ let $y = z/z$ so $z = y * z$ and $y\backslash z = z$, then $x * z = x\backslash z$.
B19: $(x * y) * z = (x\backslash y)\backslash z$ let $x = y/y$ so $x * y = y$ and $x\backslash y = y$, then $y * z = y\backslash z$.
B45: $(x * y)\backslash z = (x\backslash y) * z$ let $x = y/y$ so $x * y = y$ and $x\backslash y = y$, then $y * z = y\backslash z$.

A37 and B49 can be proved using the definition of an equational quasigroup.
A37: $x/(y * z) = x/(y\backslash z)$ right multiplying by $(y\backslash z)$ gives $[x/(y * z)] * (y\backslash z) = x$, so $y * z = y\backslash z$.
B49: $(x * y)\backslash z = (x\backslash y)\backslash z$ left multiplying by $(x * y)$ gives $z = (x * y) * [(x\backslash y)\backslash z]$, so $x * y = x\backslash y$.

3.4 Classes 4 and 6

Theorem 3. *The identities A24, A35, A67, A68, A78, A79, A89, B24, B35, B67, B68, B78, B79, B89 are equivalent to* $x/y = x\backslash y$.

In loops A69 and B69 are also members of Class 4. In quasigroups A69, B69 define Class 6. Proofs in the other direction, assuming $x/y = x\backslash y$, are trivial, and it is easy to see that Class 4 implies Class 6.

A24, A89, B35, B68 can be proved using cancellation or multiplication.
A24: $x * (y/z) = x * (y\backslash z)$ left dividing by x leaves $y/z = y\backslash z$.
A89: $x\backslash(y/z) = x\backslash(y\backslash z)$ left multiplying by x leaves $y/z = y\backslash z$.
B35: $(x/y) * z = (x\backslash y) * z$ right dividing by z leaves $x/y = x\backslash y$.
B68: $(x/y)/z = (x\backslash y)/z$ right multiplying by z leaves $x/y = x\backslash y$

A35, A68, A79, B24, B67, B89 can be proved by combining two variables.
A35: $x/(y * z) = x\backslash(y * z)$ let $y * z = w$ then $x/w = x\backslash w$.
A68: $x/(y/z) = x\backslash(y/z)$ let $y/z = w$ then $x/w = x\backslash w$.
A79: $x/(y\backslash z) = x\backslash(y\backslash z)$ let $y\backslash z = w$ then $x/w = x\backslash w$.
B24: $(x * y)/z = (x * y)\backslash z$ let $x * y = w$ then $w/z = w\backslash z$.
B67: $(x/y)/z = (x/y)\backslash z$ let $x/y = w$ then $w/z = w\backslash z$.
B89: $(x\backslash y)/z = (x\backslash y)\backslash z$ let $x\backslash y = w$ then $w/z = w\backslash z$.

A67 and B79 can be proved using the definition of equational quasigroups.
A67: $x/(y/z) = x/(y\backslash z)$ right multiplying by (y/z) gives $x = [x/(y\backslash z)] * (y/z)$, so $y\backslash z = y/z$.
B79: $(x/y)\backslash z = (x\backslash y)\backslash z$ left multiplying by (x/y) gives $z = (x/y) * [(x\backslash y)\backslash z]$, so $x/y = x\backslash y$.

A78 and B78 have more complicated proofs using the substitution $z = x$.
A78: $x/(y\backslash z) = x\backslash(y/z)$ let $z = x$, so

$$x/(y\backslash x) = x\backslash(y/x)$$

$$y = x\backslash(y/x)$$

$$x * y = y/x$$

$$(x * y) * x = y$$

let $x * y = w$ so $y = x\backslash w$, then substituting we get $w * x = x\backslash w$ but we also know $w * x = x/w$ from above, so $x/w = x\backslash w$.
B78: $(x/y)\backslash z = (x\backslash y)/z$ let $z = x$, so

$$(x/y)\backslash x = (x\backslash y)/x$$

$$y = (x\backslash y)/x$$

$$y * x = x\backslash y$$

$$x * (y * x) = y$$

let $y * x = w$ so $y = w/x$, substituting we get $x * w = w/x$ but we know $x * w = w\backslash x$ from above, so $w/x = w\backslash x$.

The proofs that A69 and B69 belong to Class 4 hinge on the fact that there is an identity, which is why when working in a quasigroup they form their own class.
A69 loop: $x\backslash(y\backslash z) = x/(y/z)$ let $y = e$ and $x = z$:

$$z\backslash(e\backslash z) = z/(e/z)$$

$$e = z/(e/z)$$

$$e/z = z$$

$$e = z * z$$

Note that at this point we can see that we have a boolean loop, where every element is its own inverse. Now let $z = e = y * y$ in A69:

$$x\backslash[y\backslash(y * y)] = x/(y/e)$$

$$x\backslash y = x/y$$

.
B69 loop: $(x/y)/z = (x\backslash y)\backslash z$ let $y = e$ and $x = e$, then

$$(e/e)/z = (e\backslash e)\backslash z$$

$$e/z = z$$

$$e = z * z$$

Note that at this point we can see that we have a boolean loop, where every element is its own inverse. Now let $y = e = x * x$ in B69:

$$(x/e)/z = [x\backslash(x * x)]\backslash z$$

$$x/z = x\backslash z$$

Quasigroup:
The proof for A69 $\Rightarrow$ B69 is provided in section 6. The proof for B69 $\Rightarrow$ A69 follows from an analogous proof.

3.5 Class 3

Theorem 4. *The identities A17, A18, A25, A29, A34, A38, A39, A48, A56, A57, B17, B18, B25, B27, B29, B34, B38, B46, B48, B56 are equivalent to* $x * y = x/y = x\backslash y$*.*

Because all the binary operations are the same Class 3 also implies every identity in Classes 1,2,4,5. Note that the proofs in the other direction, assuming $x * y = x/y = x\backslash y$, are trivial.

A17: $x * (y * z) = x/(y\backslash z)$ let $z = x$, then $x * (y * x) = x/(y\backslash x) = y$ Now let $y = x/x$ so $y * x = x$ and $x = y\backslash x$, then $(y\backslash x) * x = y$ so $y\backslash x = y/x$. Now we can rewrite A17 as $x * (y * z) = x/(y/z)$ which is A16 so we also have $x * y = x/y$.

A18: $x * (y * z) = x\backslash(y/z)$ let $z = y\backslash y$ so $y * z = y$ and $y/z = y$, then $x * y = x\backslash y$. Now we can rewrite A18 as $x * (y * z) = x * (y/z)$ which is A12 so $x * y = x/y$.

A25: $x * (y/z) = x\backslash(y * z)$ let $z = y\backslash y$ so $y * z = y$ and $y/z = y$, then $x * y = x\backslash y$. Now we can rewrite A25 as $x * (y/z) = x * (y * z)$ which is A12 so $x * y = x/y$.

A29: $x * (y/z) = x\backslash(y\backslash z)$ left divide by x to get $y/z = x\backslash[x\backslash(y\backslash z)]$ and apply A29 to the right side of this equation to get

$$y/z = x * [x/(y\backslash z)]$$

$$x\backslash(y/z) = x/(y\backslash z)$$

Which is A78 so $x/y = x\backslash y$ so A29 can be rewritten as $x * (y/z) = x/(y/z)$ which is A26 so $x * y = x/y$.

A34: $x/(y * z) = x * (y\backslash z)$ let $y = z/z$ so $z = y * z$ and $z = y\backslash z$, then $x/z = x * z$. Now we can rewrite A34 as $x * (y * z) = x * (y\backslash z)$ which is A14 so $x * y = x\backslash y$.

A38: $x/(y * z) = x\backslash(y/z)$ let $z = y\backslash y$ so $y * z = y$ and $y/z = y$, then $x/y = x\backslash y$. Now we can rewrite A38 as $x/(y * z) = x/(y/z)$ which is A36 so $x * y = x/y$.

A39: $x/(y * z) = x\backslash(y\backslash z)$ let $y = z/z$ so $z = y * z$ and $z = y\backslash z$, then $x/z = x\backslash z$ so A39 can be rewritten as $x/(y * z) = x/(y/z)$ which is A36 so $x * y = x/y$.

A48: $x * (y\backslash z) = x\backslash(y/z)$ let $x = y$, then

$$y * (y\backslash z) = y\backslash(y/z)$$

$$z = y\backslash(y/z)$$

$$y * z = y/z$$

So A48 can be rewritten $x * (y\backslash z) = x\backslash(y * z)$ which is A45 so $x * y = x\backslash y$.

A56: $x\backslash(y * z) = x/(y/z)$ let $z = y\backslash y$ so $y * z = y$ and $y/z = y$, then $x\backslash y = x/y$. Now A56 can be rewritten as $x/(y * z) = x/(y/z)$ which is A36 so $x * y = x/y$.

A57: $x\backslash(y * z) = x/(y\backslash z)$ let $y = z/z$ so $y\backslash z = z$ and $y * z = z$, then $x\backslash z = x/z$. Now A57 can be rewritten as $x/(y * z) = x/(y/z)$ which is A36 so $x * y = x/y$.

B17: $(x * y) * z = (x/y)\backslash z$ let $y = x\backslash x$ so $x/y = x$ and $x = x * y$, then $x * z = x\backslash z$. Now B17 can be rewritten as $(x * y) * z = (x/y) * z$ which is B13 so $x * y = x/y$.

B18: $(x * y) * z = (x\backslash y)/z$ let $x = y/y$ so $x\backslash y = y$ and $y = x * y$, then $y * z = y/z$. Now B18 can be rewritten as $(x * y) * z = (x\backslash y) * z$ which is B15 so $x * y = x\backslash y$.

B25: $(x * y)/z = (x\backslash y) * z$ let $x = y/y$ so $y = x * y$ and $x\backslash y = w$, then $y/z = y * z$. Now B25 can be rewritten as $(x * y) * z = (x\backslash y) * z$ which is B15 so $x * y = x\backslash y$.

B27: $(x * y)/z = (x/y)\backslash z$ let $y = x\backslash x$ so $x/y = x$ and $x = x * y$, then $x/z = x\backslash z$. Now B27 can be rewritten as $(x * y)/z = (x/y)/z$ which is B26 so $x * y = x/y$.

B29: $(x * y)/z = (x\backslash y)\backslash z$ let $x = y/y$ so $x\backslash y = y$ and $y = x * y$, then $y/z = y\backslash z$. Now B29 can be rewritten as $(x * y)/z = (x/y)/z$ which is B26 so $x * y = x/y$.

B34: $(x/y) * z = (x * y)\backslash z$ let $y = x\backslash x$ so $x/y = x$ and $x = x * y$, then $x * z = x\backslash z$. Now B34 can be rewritten as $(x/y) * z = (x * y) * z$ which is B13 so $x * y = x/y$.

B38: $(x/y) * z = (x\backslash y)/z$ let $y = z$, then $x = (x\backslash z)/z$ so $x * z = x\backslash z$. Now B38 can be rewritten as $(x/y) * z = (x * y)/z$ which is B23 so $x * y = x/y$.

B46: $(x * y)\backslash z = (x/y)/z$ let $y = x\backslash x$ so $x/y = x$ and $x = x * y$, then $x\backslash z = x/z$. Now B46 can be rewritten as $(x * y)/z = (x/y)/z$ which is B26 so $x * y = x/y$.

B48: $(x * y)\backslash z = (x\backslash y)/z$ let $x = y/y$ so $x\backslash y = y$ and $y = x * y$, then $y\backslash z = y/z$. Now B48 can be rewritten as $(x * y)/z = (x/y)/z$ which is B26 so $x * y = x/y$.

B56: $(x\backslash y) * z = (x/y)/z$ right dividing by z we get $x\backslash y = [(x/y)/z]/z$ now we can apply B56 again to the right hand side to get

$$x\backslash y = [(x/y)\backslash z] * z$$

$$(x\backslash y)/z = (x/y)\backslash z$$

This is B78 so $x/y = x\backslash y$. Now B56 can be rewritten as $(x/y) * z = (x/y)/z$ which is B36 so $x/y = x * y$.

3.6 Class 5

Theorem 5. *The identities A27, A46, B39, B57 are all equivalent. This class implies commutativity.*

Unlike the other classes, Class 5 cannot be characterized by the coincidence of any of the binary operations. Quasigroups (and loops) in Class 5 are commutative; however there are commutative quasigroups (and loops) which do not belong to Class 5. Our strategy here is to show that A27 implies commutativity and the other three identities, then to show that each other identity implies A27.

A27 $\Rightarrow x * y = y * x$: $x * (y/z) = x/(y\backslash z)$ let $z = x$,

$$x * (y/x) = x/(y\backslash x) = y$$

Now left dividing by x gives $y/x = x\backslash y \Leftrightarrow x * y = y * x$.

A27 $\Rightarrow$ A46: $x*(y/z) = x/(y\backslash z)$ Let's use commutativity to rearrange each side of our equation.

$x * (y/z) = x * (z\backslash y)$ and $x/(y\backslash z) = x/(z/y)$, so $x * (z\backslash y) = x/(z/y)$ which is A46.

A27 $\Rightarrow$ B39: $x * (y/z) = x/(y\backslash z)$ Again use commutativity to rearrange each side of A27. $x * (y/z) = (y/z) * x$ and $x/(y\backslash z) = (y\backslash z)\backslash x$, so $(y/z) * x = (y\backslash z)\backslash x$ which is B39.

A27 $\Rightarrow$ B57: $x * (y/z) = x/(y\backslash z)$ Again use commutativity to rearrange each side of A27. $x * (y/z) = (y/z) * x = (z\backslash y) * x$ and $x/(y\backslash z) = (y\backslash z)\backslash x = (z/y)\backslash x$, so $(z\backslash y) * x = (z/y)\backslash x$ which is B57.

Now to prove that each of the other identities implies A27 we just need to show they imply commutativity and then can use each of the rearrangements above.

A46 ($\Rightarrow$) $x * (y\backslash z) = x/(y/z)$ let y=x,

$$x * (x\backslash z) = x/(x/z)$$

$$z * (x/z) = x$$

$$x/z = z\backslash x \Leftrightarrow x * z = z * x$$

B39 ($\Rightarrow$) $x * y = y * x$: $(x/y) * z = (x\backslash y)\backslash z$ let $z = y$, then

$$x = (x\backslash y)\backslash y$$

$$(x\backslash y) * x = y$$

$$x\backslash y = y/x \Leftrightarrow x * y = y * x$$

B57 ($\Rightarrow$) $x * y = y * x$: $(x\backslash y) * z = (x/y)\backslash z$ let $z = x$

$$(x\backslash y) * x = (x/y)\backslash x = y$$

$$x\backslash y = y/x \Leftrightarrow x * y = y * x$$

4 Varying the Operators and Parentheses

4.1 Groups

Theorem 6. *The identities A1B1, A2B2, A4B3, A5B5, A8B8 when added to a quasigroup form a group.*

The variety of quasigroups defined by any of the identities A1B1 etc. is the variety of groups. These proofs hold in quasigroups and loops and can be read backwards to get the proof in the opposite direction; this will be illustrated once, in the proof for A2B2. Also note that A1B1 is the definition of associativity, $x * (y * z) = (x * y) * z$.

A2B2: ($\Rightarrow$) $x * (y/z) = (x * y)/z$ right multiplying by z gives $[x * (y/z)] * z = x * y$, now let $y/z = w$, so $y = w * z$ then, $(x * w) * z = x * (w * z)$.
($\Leftarrow$) $(x * w) * z = x * (w * z)$ let $w = y/z$, so $w * z = y$, then $[x * (y/z)] * z = x * y$ now right division by z gives A2B2, $x * (y/z) = (x * y)/z$.

A4B3: $x * (y\backslash z) = (x/y) * z$ let $z = y * w$, so $y\backslash z = w$, then $x * w = (x/y) * (y * w)$. Now let $x/y = v$, so $x = v * y$, then $(v * y) * w = v * (y * w)$.

A5B5: $x\backslash(y * z) = (x\backslash y) * z$ let $y = x * w$, then

$$x\backslash[(x * w) * z] = [x\backslash(x * w)] * z = w * z$$

$$(x * w) * z = x * (w * z)$$

A8B8: $x\backslash(y/z) = (x\backslash y)/z$ let $y = w * z$, so $y/z = w$, then

$$x\backslash w = [x\backslash(w * z)]/z$$

$$(x\backslash w) * z = x\backslash(w * z)$$

Which is A5B5, so we're done.

4.2 Abelian Groups

Theorem 7. *The identities A3B6, A4B9, A5B7, A6B3, A6B9, A7B2, A9B4 when added to a quasigroup form an abelian group.*

The variety of quasigroups defined by any of the identities A3B6 etc. is the variety of abelian groups. The proofs in the opposite direction can easily be derived by following these proofs backwards.

A3B6: $x/(y * z) = (x/y)/z$ let $y = v * w$ and apply A3B6 three times:

$$x/[(v * w) * z] = [x/(v * w)]/z = [(x/v)/w]/z = (x/v)/(w * z) = x/[v * (w * z)]$$

So, $(v * w) * z = v * (w * z)$. Now that we have associativity we know we have an identity, e. To show commutativity let $z = x/y$ in A3B6:

$$x/[y * (x/y)] = (x/y)/(x/y) = e$$

$$x = y * (x/y)$$

$$y\backslash x = x/y \Leftrightarrow x * y = y * x$$

A4B9, A9B4: $x * (y\backslash z) = (x\backslash y)\backslash z$ left division by x gives $y\backslash z = x\backslash[(x\backslash y)\backslash z]$ let $y = x * w$ so $x\backslash y = w$, then $(x * w)\backslash z = x\backslash(w\backslash z)$ which is A9B4 (So these two identities are equivalent). Now let $y = x$ in A4B9, $x * (x\backslash z) = z = (x\backslash x)\backslash z$ and $(x\backslash x) * z = z$ so $x\backslash x$ is a left identity, call it e. Now let $y = e$ in A9B4, then

$$x\backslash(e\backslash z) = (x * e)\backslash z$$

$$e\backslash z = z = x * [(x * e)\backslash z]$$

So $x = x * e$ so e is also a right sided identity, and e is our neutral element. Now let $z = y$ in A4B9, then

$$x * (y\backslash y) = (x\backslash y)\backslash y$$

$$x = (x\backslash y)\backslash y$$

$$(x\backslash y) * x = y$$

$$x\backslash y = y/x \Leftrightarrow x * y = y * x$$

.

Now that we have commutativity let's begin again with A4B9: $x*(y\backslash z) = (x\backslash y)\backslash z$ so $(x\backslash y)[x * (y\backslash z)] = z$ let $z = y * w$, then $(x\backslash y) * (x * w) = y * w$. Now let $y = x * v$,

$$v * (x * w) = (x * v) * w = (v * x) * w$$

A5B7: $x\backslash(y * z) = (x/y)\backslash z$ let $z = x$, then

$$x\backslash(y * x) = x/(y\backslash x) = y$$

$$y * x = x * y$$

So we have commutativity. Now in A5B7 left multiplication by x along with the substitution $x = w * y$ gives $y * z = (w * y) * [w\backslash z]$. Let $z = w * v$, then

$$y * (w * v) = (w * y) * v = (y * w) * v$$

A6B3: $x/(y/z) = (x/y) * z$ let $y/z = w$ so $y = w * z$, then

$$x/w = [x/(w * z)] * z$$

$$(x/w)/z = x/(w * z)$$

This is A3B6, so we're done.

A6B9: $x/(y/z) = (x\backslash y)\backslash z$ left multiplication by $x\backslash y$ and right multiplication by y/z gives the equations

$$z = (x\backslash y)[x/(y/z)]$$

$$x = [(x\backslash y)\backslash z] * (y/z)$$

Let $y = w * z$ so $y/z = w$ and $z = w\backslash y$, substitution gives

$$z = [x\backslash(w * z)](x/w)$$

$$x = [[x\backslash(w * z)]\backslash z] * w$$

Now let $x = w$ to get $z = z * (w/w)$ and $w = (z\backslash z) * w$. Thus we have a neutral element, call it e. Now let $z = y$ in A6B9,

$$x/(y/y) = x = (x\backslash y)\backslash y$$

$$(x\backslash y) * x = y$$

$$x\backslash y = y/x \Leftrightarrow x * y = y * x$$

So we have commutativity. Now let $x\backslash y = w$, so $y = x * w$, in A6B9, then

$$x/[(x * w)/z] = w\backslash z$$

$$x = (w\backslash z) * [(x * w)/z] = [(x * w)/z] * (w\backslash z)$$

$$x/(w\backslash z) = (x * w)/z$$

Which is A7B2 (proved below) so we're done.

A7B2: $x/(y\backslash z) = (x * y)/z$ let $y\backslash z = w$ so $z = y * w$, then $x/w = (x * y)/(y * w)$. Now let $x * y = v$ so $x = v/y$ and we get $(v/y)/w = v/(y * w)$ which is A3B6 so we're done.

4.3 Boolean Groups

Theorem 8. *In loops, adding any of the identities A1B2, A1B3, A1B4, A1B5, A1B6, A1B7, A1B8, A1B9, A2B1, A2B3, A2B4, A2B5, A2B6, A2B7, A2B8, A2B9, A3B1, A3B2, A3B3, A3B4, A3B5, A3B7, A3B8, A3B9, A4B1, A4B2, A4B4, A4B5, A4B6, A4B7, A4B8, A5B1, A5B2, A5B3, A5B4, A5B6, A5B8, A5B9, A6B1, A6B2, A6B4, A6B5, A6B6, A6B7, A6B8, A7B1, A7B3, A7B4, A7B5, A7B6, A7B7, A7B8, A7B9, A8B1, A8B2, A8B3, A8B4, A8B5, A8B6, A8B7, A8B9, A9B1, A9B2, A9B3, A9B5, A9B6, A9B7, A9B8, A9B9 forms a boolean group.*

In quasigroups, only the identities A1B3, A1B4, A2B3, A2B6, A3B1, A3B2, A4B1, A4B5, A4B6, A5B4, A5B9, A6B7, A7B6, A7B8, A7B9, A8B7, A9B3, A9B5, A9B7 form a boolean group.

The variety of quasigroups defined by any of the identities A1B3 etc. is the variety of Boolean groups and the variety of loops defined by any of the identities A1B2 etc. is the variety of Boolean groups. A Boolean group is a group with the property that $x * x = e$, the neutral element, for every x. The identities defining any of the other classes are satisfied by Boolean groups, since they satisfy commutativity and associativity, and the three quasigroup operations all coincide.

The proofs for A6B7, A7B6, A7B8, A7B9, A8B7, A9B7 all follow the same strategy. We will first show that in each of these equations $x/y = x\backslash y$, from here we can rewrite the identity as $x/(y/z) = (x\backslash y)\backslash z$, A6B9. Now we have that these identities all imply an abelian group. Because we have a group we have a neutral element, e. Now we can write $z/e = z\backslash e$, so $z = z\backslash e$, and finally, $z * z = e$ thus giving us a boolean group.
To further simplify these proofs note that if $x * y = y\backslash x$ left multiplication by y gives us $y * (x * y) = x$. Now let $x * y = w$, so $x = w/y$, then we get $y * w = w/y$. We also know from our assumption that $y * w = w\backslash y$, so $w/y = w\backslash y$. An analogous proof holds for the assumption that $x * y = y/x$. Now to prove these identities imply a boolean group it will suffice to show that either $x * y = y/x$, $x * y = y\backslash x$, or $x/y = x\backslash y$.

A6B7: $x/(y/z) = (x/y)\backslash z$ let $z = x$, then

$$x/(y/x) = (x/y)\backslash x = y$$

$$x = y(y/x)$$

$$y\backslash x = y/x$$

A7B6: $x/(y\backslash z) = (x/y)/z$ let $z = x$, then

$$x/(y\backslash x) = (x/y)/x$$

$$y = (x/y)/x$$

$$y * x = x/y$$

A7B8: $x/(y\backslash z) = (x\backslash y)/z$ let $z = x$, then

$$x/(y\backslash x) = (x\backslash y)/x$$

$$y = (x\backslash y)/x$$

$$y * x = x\backslash y$$

A7B9: $x/(y\backslash z) = (x\backslash y)\backslash z$ let $z = x$, then

$$x/(y\backslash x) = (x\backslash y)\backslash x$$

$$y = (x\backslash y)\backslash x$$

$$(x\backslash y) * y = x$$

$$x\backslash y = x/y$$

A8B7: $x\backslash(y/z) = (x/y)\backslash z$ let $z = x$, then

$$x\backslash(y/x) = (x/y)\backslash x = y$$

$$y/x = x * y$$

A9B7: $x\backslash(y\backslash z) = (x/y)\backslash z$ let $z = x$, then

$$x\backslash(y\backslash x) = (x/y)\backslash x = y$$

$$y\backslash x = x * y$$

A1B3: $x * (y * z) = (x/y) * z$ let $y = z$, then $x * (z * z) = (x/z) * z = x$, so $z * z$ is a right identity, call it e. Now let $z = e$ in A1B3, $x * (y * e) = (x/y) * e$ so $x * y = x/y$. Now we can rewrite A1B3 as $x * (y * z) = (x/y) * z = (x * y) * z$ so we have a group and our right identity is a 2-sided identity so we also have a boolean group.

A2B3: $x * (y/z) = (x/y) * z$ let $z = y$, then $x(y/y) = (x/y) * y = x$ so y/y is a right identity, call it e. Now let $z = e$ in A2B3, $x * (y/e) = (x/y) * e$ so $x * y = x/y$ and we can rewrite A2B3 as $x * (y * z) = (x/y) * z$ which is A1B3 so we're done.

A2B6, A3B1: $x * (y/z) = (x/y)/z$ let $y = w * z$ so $y/z = w$, then

$$x * w = [x/(w * z)]/z$$

$$(x * w) * z = x/(w * z)$$

which is A3B1, and because we can follow this substitution backwards to see A3B1 $\Rightarrow$ A2B6 the two are equivalent. In A2B6 substitute $x = y * y$, then $(y * y) * (y/z) = [(y * y)/y]/z = y/z$ so $y * y$ is a left identity, call it e. Now let $x = y = z$ in A3B1,

$$(x * x) * x = x/(x * x)$$

$$e * x = x/e$$

$$x = x/e$$

$$x * e = x$$

So e is a 2-sided identity element and A2B6, A3B1 define a boolean loop. Now let $y = z$ in A3B1, $(x * z) * z = x/(z * z) = x/e = x$ so $x * z = x/z$ and A2B6 and A3B1 can both be rewritten as $x * (y * z) = (x * y) * z$, so we have a boolean group.

A4B1: $x * (y\backslash z) = (x * y) * z$ let $z = y * w$ so $y\backslash z = w$, then $x * w = (x * y) * (y * w)$. Now let $x * y = v$ so $x = v/y$, then $(v/y) * w = v * (y * w)$ which is A1B3 so we're done.

A4B5: $x * (y\backslash z) = (x\backslash y) * z$ let $y = x$, then $x * (x\backslash z) = z = (x\backslash x) * z$ so $x\backslash x$ is a left identity element, call it e: $x\backslash x = e$, so $x = x * e$ thus e is actually a 2-sided identity element. Now let $x = e$ in A4B5, $e * (y\backslash z) = (e\backslash y) * z$, so $y\backslash z = y * z$. Now A4B5 can be rewritten as associativity, so we have a group. Now if we let $x = z = e$ in A4B5 we see that $y\backslash e = y * e = y$ so $e = y * y$ and we have a boolean group.

A4B6: $x * (y\backslash z) = (x/y)/z$ let $x = y$

$$x * (x\backslash z) = (x/x)/z$$

$$z = (x/x)/z$$

$$z * z = x/x$$

$$(z * z) * x = x$$

So $z * z = e$, a left identity element. Let $y = e$ in A4B6, then

$$x * z = (x/e)/z$$

$$(x * z) * z = x/e$$

Let $x = z$, then

$$(z * z) * z = z/e$$

$$e * z = z/e$$

$$z = z/e$$

$$z * e = z$$

So e is also a right identity element. Now let $y = e$ in A4B6, then $x(e\backslash z) = (x/e)/z$, $x * z = x/z$ so A4B6 can be rewritten $x * (y\backslash z) = (x/y)/z = (x * y) * z$, which is A4B1, so we're done.

A5B4: $x\backslash(y * z) = (x * y)\backslash z$ let $y = x$,

$$x\backslash(x * z) = (x * x)\backslash z$$

$$z = (x * x)\backslash z$$

$$(x * x) * z = z$$

So $x * x$ is a left identity element, call it e. Now let $x = e$ in A5B4, so $e\backslash(y * z) = (e * y)\backslash z$, so $y * z = y\backslash z$. Now we can rewrite A5B4 as $x * (y\backslash z) = (x * y) * z$, which is A4B1, so we're done.

A5B9: $x\backslash(y * z) = (x\backslash y)\backslash z$ let $x = y$,

$$x\backslash(x * z) = (x\backslash x)\backslash z$$

$$z = (x\backslash x)\backslash z$$

$$(x\backslash x) * z = z$$

So $x\backslash x$ is a left identity, call it e. Let $x = e$ in A5B9, $e\backslash(y * z) = (e\backslash y)\backslash z$, so $y * z = y\backslash z$. Now A5B9 can be rewritten as $x * (y\backslash z) = (x * y) * z$ which is A4B1, so we're done.

A1B4, A9B3, A9B5: Here we will show these three identities are equivalent and show they imply a boolean group.
Start with A9B3: $x\backslash(y\backslash z) = (x/y) * z$ let $z = y * w$ so $y\backslash z = w$ then, $x\backslash w = (x/y) * (y * w)$.

Now let $x/y = v$, $x = v * y$, then $(v * y)\backslash w = v * (y * w)$, which is A1B4, and this proof can be followed backwards to see that A1B4 $(\Rightarrow)$ A9B3.
Now look at A9B5: $x\backslash(y\backslash z) = (x\backslash y) * z$ let $x\backslash y = w$, $y = x * w$ then $x\backslash[(x * w)\backslash z] = w * z$ so $(x * w)\backslash z = x * (w * z)$, which is A1B4 and again the proof can be followed in reverse.
So now that we know A9B3 and A9B5 are equivalent to A1B4, then they imply B35; $(x/y) * z = (x\backslash y) * z$ so $x/y = x\backslash y$. Now we can rewrite A9B5 as $x/(y\backslash z) = (x\backslash y) * z$ and let $z = x$,

$$x/(y\backslash x) = (x\backslash y) * x$$

$$y = (x\backslash y) * x$$

$$y/x = x\backslash y \Leftrightarrow x * y = y * x$$

We can also use this to rewrite A1B4 as $x * (y * z) = (x * y)/z$ and letting $z = y$ we get $x * (y * y) = (x * y)/y = x$ so $y * y$ is a right identity element, call it e.
Now that we have commutativity and a right identity element we can rewrite A9B3 as $x\backslash(z/y) = (x/y) * z$ let $y = e$ and we get $x\backslash(z/e) = (x/e) * z$, so $x\backslash z = x * z$ and we can rewrite all of our equations as associativity, so our right identity becomes a 2-sided identity and we have a boolean group.

The rest of the Identities imply a boolean group when working in a Loop but form totally different classes when working with a quasigroup. Once we have an identity the Loop proofs become very easy so we will just give a few representative proofs below.

A1B2: $x * (y * z) = (x * y)/z$ First let $y = e$, then $x * (e * z) = (x * e)/z$ so $x * z = x/z$, now $x * (y * z) = (x * y)/z = (x * y) * z$, so we have associativity. Now let $x = y = e$ in A1B2 to see that it is boolean,

$$e * (e * z) = (e * e)/z$$

$$z = e/z$$

$$z * z = e$$

A4B8: $x * (y\backslash z) = (x\backslash y)/z$ First let $x = e$, then $y\backslash z = y/z$. Now rewrite as $x * (y\backslash z) = (x/y)/z$ and let $y = e$, then $x * z = x/z$. Now we can rewrite A4B8 as associativity, $x * (y * z) = (x * y) * z$. Now let $x = y = e$ in A4B8 to see that it is boolean,

$$e * (e\backslash z) = (e\backslash e)/z$$

$$z = e/z$$

$$z * z = e$$

A6B6: $x/(y/z) = (x/y)/z$ First let $y = e$,

$$x/(e/z) = x/z$$

$$x = (x/z) * (e/z)$$

$$(x/z)\backslash x = e/z$$

$$z = e/z$$

$$z * z = e$$

So we have a boolean loop. Now let $y = z$ in A6B6,

$$x/(y/y) = (x/y)/y$$

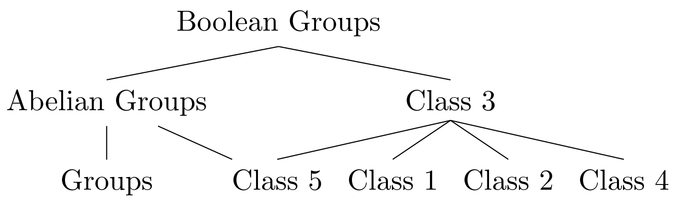
$$x = (x/y)/y$$

$$x * y = x/y$$

Now using $x * y = x/y$ and A6B6 we can get associativity,

$$x * (y * z) = x/(y/z) = (x/y)/z = (x * y) * z$$

In loops the classes are related as follows:



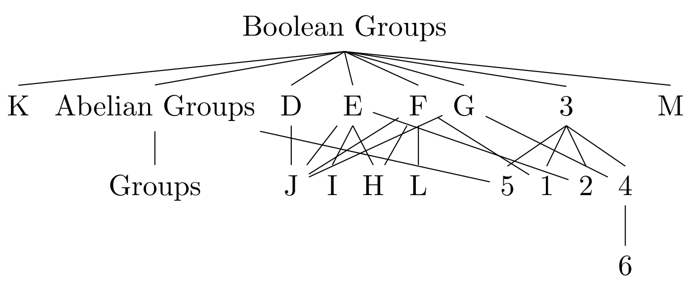
4.4 In Quasigroups

The proofs of classification in a quasigroup were done using Prover9, but we have human proofs of their implications.

The identities in quasigroups form 13 different classes, labeled as follows:

Boolean Groups	13,14,23,26,31,32,41,45,46,54,59,67,76,78,79,87,93,95,97
Abelian Groups	36,49,57,63,69,72,94
Groups	11,22,43,55,88
D	27,34,75,96
E	12,21,24,29,37,48,58,73,84,85,92,98
F	15,28,35,38,47,51,56,65,74,82,83,86
G	16,17,19,61,71,91
H	25,52,77
I	33,42,66,89
J	18,81
K	39
L	44,53,68,99
M	64

Note that all the classes denoted by letters in a quasigroup are actually in the Boolean Group class when working in a loop. The Abelian Groups and Groups classes are the same in both quasigroups and loops.



Definition. A right Boolean loop is a loop with a right identity, e_R, where $x * x = e_R$, $\forall x$.

Definition. A left Boolean loop is a loop with a left identity, e_L, where $x * x = e_L$, $\forall x$.

Class E $\Rightarrow$ Right Boolean Loop:
Start with A1B2, $x * (y * z) = (x * y)/z$ and let $z = y$, then $x * (y * y) = (x * y)/y = x$. So $y * y$ is a right identity, and we have a right boolean loop.

Class E $\Rightarrow$ Class 2:
We know Class E contains A2B1, $x * (y/z) = (x * y) * z$, and A2B4, $x * (y/z) = (x * y)\backslash z$ so we know $(x * y) * z = (x * y)\backslash z$ which is B14, so $x * y = x\backslash y$ $\forall x, y$.

Class I $\Rightarrow$ Right Boolean Loop:
Start with A3B3, $x/(y * z) = (x/y) * z$, and let $y = z$,

$$x/(y * y) = (x/y) * y = x$$

$$x = x * (y * y)$$

So $y * y$ is our right identity, and we can see that members of Class I are Right Boolean Loops.

Class L $\Rightarrow$ Left Boolean Loop:
Start with A4B4, $x * (y\backslash z) = (x * y)\backslash z$, and let $x = y$,

$$x * (x\backslash z) = (x * x)\backslash z$$

$$z = (x * x)\backslash z$$

$$(x * x) * z = z$$

So $x * x$ is our left identity, and we can see members of Class L are Left Boolean Loops.

Class F $\Rightarrow$ Class 1:
Start with A7B4, $x/(y\backslash z) = (x * y)\backslash z$ and let $z = x$,

$$x/(y\backslash x) = (x * y)\backslash x$$

$$y = (x * y)\backslash x$$

$$(x * y) * y = x$$

$$x * y = x/y$$

Class F $\Rightarrow$ Left Boolean Loop:
Start again with A7B4 but rewrite the left hand side as A4, $x/(y\backslash z) = x * (y\backslash z) = (x * y)\backslash z$. So we have that A4B4 holds (but note it is not equivalent) so we have a Left Boolean Loop.

Class D $\Rightarrow$ Commutativity:
Start with A2B7, $x * (y/z) = (x/y)\backslash z$, and let $z = x$,

$$x * (y/x) = (x/y)\backslash x$$

$$x * (y/x) = y$$

$$y/x = x\backslash y \Leftrightarrow x * y = y * x$$

Class G $\Rightarrow$ Class 4:
Start with A1B7, $x*(y*z) = (x/y)\backslash z$, an let $x = z$,

$$x*(y*x) = (x/y)\backslash x$$

$$x*(y*x) = y$$

$$y*x = x\backslash y$$

Now look at A7B1, $x/(y\backslash z) = (x*y)*z$, let $x = z$,

$$x/(y\backslash x) = (x*y)*x$$

$$y = (x*y)*x$$

$$y/x = x*y$$

Now combining these two results we get that $x/y = x\backslash y$, Class 4.

Class 4 $\Rightarrow$ Flexible Law $[(x*y)*x = x*(y*x)]$:
Start with the Equational Quasigroup property $(x*y)/y = x$, in class 4 we can rewrite this as $(x*y)\backslash y = x$ so $y = (x*y)*x$. Now let's do the same thing with $y\backslash(y*x) = x$, use class 4 to rewrite this as $y/(y*x) = x$ so $y = x*(y*x)$. Combining these results we can see that $(x*y)*x = x*(y*x)$. Note that because of this implication the Flex Law also holds in Class G.

Commutativity $\Rightarrow$ Flexible Law:
Assume we are working in a commutative quasigroup, then $x*(y*x) = (y*x)*x = (x*y)*x$. So the Flex Law holds in Abelian Groups, Boolean Groups, Class D, and Class 5.

5 Examples and Counterexamples

What follows are examples of quasigroups and loops that show that one class does not imply another. For example we know that Class A implies Class D, but Class D does not imply Class A, so we have an example of something in Class D that is not in Class A. The ordered triple at the end are the values of x,y and z that give us our counterexample.

5.1 In Quasigroups

*:	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

/:	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

\:	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Table 1: D not in A : (0,0,1)

*:	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

/:	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

\:	0	1	2
0	2	0	1
1	0	1	2
2	1	2	0

Table 2: E not in A : (0,0,0)

*:	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

/:	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

\:	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Table 3: 1 not in E : 2 not in D : 3 not in A : 4 not in F : 5 not in B : (0,0,1)

*:	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

/:	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

\:	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Table 4: Class 1 not in 3 : (0,1,0)

*:	0	1	2
0	1	2	0
1	2	0	1
2	0	1	2

/:	0	1	2
0	2	1	0
1	0	2	1
2	1	0	2

\:	0	1	2
0	2	0	1
1	1	2	0
2	0	1	2

Table 5: 5 not in 3 : B not in A : (0,0,0)

*:	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

/:	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

\:	0	1	2
0	2	0	1
1	0	1	2
2	1	2	0

Table 6: J not in C,D,F : H not in D: (0,0,0)

*:	0	1	2
0	1	0	2
1	2	1	0
2	0	2	1

/:	0	1	2
0	2	0	1
1	0	1	2
2	1	2	0

\:	0	1	2
0	1	0	2
1	2	1	0
2	0	2	1

Table 7: J not in E : H not in E : 2 not in 3 : (0,0,0)

*:	0	1	2	3
0	1	3	0	2
1	3	1	2	0
2	0	2	1	3
3	2	0	3	1

/:	0	1	2	3
0	2	3	0	1
1	0	1	2	3
2	3	2	1	0
3	1	0	3	2

\:	0	1	2	3
0	2	0	3	1
1	3	1	2	0
2	0	2	1	3
3	1	3	0	2

Table 8: C not in A : (0,0,0)

*:	0	1	2	3
0	1	2	0	3
1	2	1	3	0
2	0	3	1	2
3	3	0	2	1

/:	0	1	2	3
0	2	3	0	1
1	0	1	2	3
2	1	0	3	2
3	3	2	1	0

\:	0	1	2	3
0	2	0	1	3
1	3	1	0	2
2	0	2	3	1
3	1	3	2	0

Table 9: H doesn't imply J : (0,0,0)

*:	0	1	2	3
0	1	3	2	0
1	2	0	1	3
2	0	2	3	1
3	3	1	0	2

/:	0	1	2	3
0	2	1	3	0
1	0	3	1	2
2	1	2	0	3
3	3	0	2	1

\:	0	1	2	3
0	3	0	2	1
1	1	2	0	3
2	0	3	1	2
3	2	1	3	0

Table 10: J doesn't imply H : (0,0,0)

*:	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

/:	0	1	2	3
0	0	3	1	2
1	2	1	3	0
2	3	0	2	1
3	1	2	0	3

\:	0	1	2	3
0	0	3	1	2
1	2	1	3	0
2	3	0	2	1
3	1	2	0	3

Table 11: 4 not in 3 : F not in A : (0,0,1)

*:	0	1	2	3	4	5
0	2	3	0	1	5	4
1	4	5	1	0	3	2
2	0	1	2	3	4	5
3	5	4	3	2	1	0
4	1	0	4	5	2	3
5	3	2	5	4	0	1

/:	0	1	2	3	4	5
0	2	4	0	1	5	3
1	4	2	1	0	3	5
2	0	5	2	3	4	1
3	5	0	3	2	1	4
4	1	3	4	5	2	0
5	3	1	5	4	0	2

\:	0	1	2	3	4	5
0	2	3	0	1	5	4
1	3	2	5	4	0	1
2	0	1	2	3	4	5
3	5	4	3	2	1	0
4	1	0	4	5	2	3
5	4	5	1	0	3	2

Table 12: G not in B : (0,0,1)

*:	0	1	2	3	4	5
0	2	3	0	5	1	4
1	4	2	1	0	5	3
2	0	1	2	4	3	5
3	5	0	3	2	4	1
4	1	5	4	3	2	0
5	3	4	5	1	0	2

/:	0	1	2	3	4	5
0	2	3	0	1	5	4
1	4	2	1	5	0	3
2	0	1	2	3	4	5
3	5	0	3	4	2	1
4	1	5	4	2	3	0
5	3	4	5	0	1	2

\:	0	1	2	3	4	5
0	2	4	0	1	5	3
1	3	2	1	5	0	4
2	0	1	2	4	3	5
3	1	5	3	2	4	0
4	5	0	4	3	2	1
5	4	3	5	0	1	2

Table 13: I not in D : (0,0,1)

*:	0	1	2	3	4	5
0	2	3	5	4	1	0
1	3	2	1	0	5	4
2	0	1	2	3	4	5
3	4	5	3	2	0	1
4	1	0	4	5	2	3
5	5	4	0	1	3	2

/:	0	1	2	3	4	5
0	2	4	5	1	3	0
1	4	2	1	5	0	3
2	0	1	2	3	4	5
3	1	0	3	2	5	4
4	3	5	4	0	2	1
5	5	3	0	4	1	2

\:	0	1	2	3	4	5
0	5	4	0	1	3	2
1	3	2	1	0	5	4
2	0	1	2	3	4	5
3	4	5	3	2	0	1
4	1	0	4	5	2	3
5	2	3	5	4	1	0

Table 14: L not in E : (0,1,0)

*:	0	1	2	3	4	5	6	7
0	1	4	7	0	6	5	2	3
1	5	2	3	6	0	1	4	7
2	0	7	4	1	5	6	3	2
3	6	3	2	5	1	0	7	4
4	4	1	0	7	3	2	5	6
5	3	6	5	2	4	7	0	1
6	7	0	1	4	2	3	6	5
7	2	5	6	3	7	4	1	0

/:	0	1	2	3	4	5	6	7
0	2	6	4	0	1	3	5	7
1	0	4	6	2	3	1	7	5
2	7	1	3	5	6	4	0	2
3	5	3	1	7	4	6	2	0
4	4	0	2	6	5	7	1	3
5	1	7	5	3	2	0	4	6
6	3	5	7	1	0	2	6	4
7	6	2	0	4	7	5	3	1

\:	0	1	2	3	4	5	6	7
0	3	0	6	7	1	5	4	2
1	4	5	1	2	6	0	3	7
2	0	3	7	6	2	4	5	1
3	5	4	2	1	7	3	0	6
4	2	1	5	4	0	6	7	3
5	6	7	3	0	4	2	1	5
6	1	2	4	5	3	7	6	0
7	7	6	0	3	5	1	2	4

Table 15: M not in A : (0,0,0)

*:	0	1	2	3	4	5	6	7
0	1	3	6	2	0	5	7	4
1	5	4	2	6	7	1	0	3
2	0	6	3	4	1	7	5	2
3	3	1	0	7	6	4	2	5
4	7	2	4	3	5	0	1	6
5	6	0	1	5	3	2	4	7
6	4	5	7	0	2	3	6	1
7	2	7	5	1	4	6	3	0

/:	0	1	2	3	4	5	6	7
0	2	5	3	6	0	4	1	7
1	0	3	5	7	2	1	4	6
2	7	4	1	0	6	5	3	2
3	3	0	2	4	5	6	7	1
4	6	1	4	2	7	3	5	0
5	1	6	7	5	4	0	2	3
6	5	2	0	1	3	7	6	4
7	4	7	6	3	1	2	0	5

\:	0	1	2	3	4	5	6	7
0	4	0	3	1	7	5	2	6
1	6	5	2	7	1	0	3	4
2	0	4	7	2	3	6	1	5
3	2	1	6	0	5	7	4	3
4	5	6	1	3	2	4	7	0
5	1	2	5	4	6	3	0	7
6	3	7	4	5	0	1	6	2
7	7	3	0	6	4	2	5	1

Table 16: K not in A : (0,0,0)

*:	0	1	2	3	4	5	6	7	8
0	1	5	3	4	0	2	6	8	7
1	3	6	8	0	7	5	4	2	1
2	0	8	7	5	6	3	2	1	4
3	4	3	0	2	5	1	8	7	6
4	5	7	6	3	8	0	1	4	2
5	2	0	5	1	3	4	7	6	8
6	7	2	1	6	4	8	5	3	0
7	6	1	4	8	2	7	3	0	5
8	8	4	2	7	1	6	0	5	3

/:	0	1	2	3	4	5	6	7	8
0	2	5	3	1	0	4	8	7	6
1	0	7	6	5	8	3	4	2	1
2	5	6	8	3	7	0	2	1	4
3	1	3	0	4	5	2	7	6	8
4	3	8	7	0	6	5	1	4	2
5	4	0	5	2	3	1	6	8	7
6	7	1	4	6	2	8	0	5	3
7	6	4	2	8	1	7	5	3	0
8	8	2	1	7	4	6	3	0	5

\:	0	1	2	3	4	5	6	7	8
0	4	0	5	2	3	1	6	8	7
1	3	8	7	0	6	5	1	4	2
2	0	7	6	5	8	3	4	2	1
3	2	5	3	1	0	4	8	7	6
4	5	6	8	3	7	0	2	1	4
5	1	3	0	4	5	2	7	6	8
6	8	2	1	7	4	6	3	0	5
7	7	1	4	6	2	8	0	5	3
8	6	4	2	8	1	7	5	3	0

Table 17: 6 not in 4 : (0,0,0)

5.2 In Loops

In loops, for the counterexample the first of the ordered quadruplet corresponds to the neutral element, and the next three correspond to x,y and z. So the ordered quadruplet looks like (e,x,y,z).

*:	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

/:	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

\:	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Table 18: 5 not in 3 : (0,1,0,0) : A.G not in B.G : (0,0,1,0)

*:	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	8	9	6	7
2	2	3	0	1	6	9	4	8	7	5
3	3	2	1	0	7	8	9	4	5	6
4	4	5	6	7	0	1	2	3	9	8
5	5	4	9	8	1	0	7	6	3	2
6	6	8	4	9	2	7	0	5	1	3
7	7	9	8	4	3	6	5	0	2	1
8	8	6	7	5	9	3	1	2	0	4
9	9	7	5	6	8	2	3	1	4	0

/:	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	8	9	6	7
2	2	3	0	1	6	9	4	8	7	5
3	3	2	1	0	7	8	9	4	5	6
4	4	5	6	7	0	1	2	3	9	8
5	5	4	9	8	1	0	7	6	3	2
6	6	8	4	9	2	7	0	5	1	3
7	7	9	8	4	3	6	5	0	2	1
8	8	6	7	5	9	3	1	2	0	4
9	9	7	5	6	8	2	3	1	4	0

\:	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	8	9	6	7
2	2	3	0	1	6	9	4	8	7	5
3	3	2	1	0	7	8	9	4	5	6
4	4	5	6	7	0	1	2	3	9	8
5	5	4	9	8	1	0	7	6	3	2
6	6	8	4	9	2	7	0	5	1	3
7	7	9	8	4	3	6	5	0	2	1
8	8	6	7	5	9	3	1	2	0	4
9	9	7	5	6	8	2	3	1	4	0

Table 19: 3 not in B.G : (0,1,2,4)

*:	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	5	1	3	0
3	3	5	4	0	2	1
4	4	2	1	5	0	3
5	5	3	0	4	1	2

/:	0	1	2	3	4	5
0	0	1	5	3	4	2
1	1	0	4	2	5	3
2	2	4	0	1	3	5
3	3	5	1	0	2	4
4	4	2	3	5	0	1
5	5	3	2	4	1	0

\:	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	5	3	0	4	1	2
3	3	5	4	0	2	1
4	4	2	1	5	0	3
5	2	4	5	1	3	0

Table 20: G not in A.G : (0,0,1,2)

*:	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	8	9	6	7
2	2	3	0	1	6	7	4	5	9	8
3	3	2	1	0	7	8	9	4	5	6
4	4	5	6	7	2	9	0	8	1	3
5	5	4	7	8	9	3	1	6	0	2
6	6	8	4	9	0	1	2	3	7	5
7	7	9	5	4	8	6	3	1	2	0
8	8	6	9	5	1	0	7	2	3	4
9	9	7	8	6	3	2	5	0	4	1

/:	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	6	8	4	9	5	7
1	1	0	3	2	8	6	5	7	4	9
2	2	3	0	1	4	9	6	8	7	5
3	3	2	1	0	9	5	7	6	8	4
4	4	5	6	7	0	1	2	3	9	8
5	5	4	7	8	1	0	9	2	3	6
6	6	8	4	9	2	7	0	5	1	3
7	7	9	5	4	3	2	8	0	6	1
8	8	6	9	5	7	3	1	4	0	2
9	9	7	8	6	5	4	3	1	2	0

\:	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	8	9	6	7
2	2	3	0	1	6	7	4	5	9	8
3	3	2	1	0	7	8	9	4	5	6
4	6	8	4	9	0	1	2	3	7	5
5	8	6	9	5	1	0	7	2	3	4
6	4	5	6	7	2	9	0	8	1	3
7	9	7	8	6	3	2	5	0	4	1
8	5	4	7	8	9	3	1	6	0	2
9	7	9	5	4	8	6	3	1	2	0

Table 21: 5 not in A.G : (0,1,2,4)

*:	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	5	2	4
2	2	5	0	4	1	3
3	3	4	1	0	5	2
4	4	3	5	2	0	1
5	5	2	4	1	3	0

/:	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	5	2	4
2	2	5	0	4	1	3
3	3	4	1	0	5	2
4	4	3	5	2	0	1
5	5	2	4	1	3	0

\:	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	2	5	3
2	2	4	0	5	3	1
3	3	2	5	0	1	4
4	4	5	3	1	0	2
5	5	3	1	4	2	0

Table 22: 1 not in 3 : (0,0,1,2)

*:	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	4	0	2	1
4	4	3	5	1	0	2
5	5	2	1	4	3	0

/:	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	5	4	2	3
2	2	5	0	1	3	4
3	3	4	1	0	5	2
4	4	2	3	5	0	1
5	5	3	4	2	1	0

\:	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	4	0	2	1
4	4	3	5	1	0	2
5	5	2	1	4	3	0

Table 23: 2 not in 3 : (0,1,2,0)

*:	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

/:	0	1	2	3	4
0	0	1	2	3	4
1	1	0	4	2	3
2	2	3	0	4	1
3	3	4	1	0	2
4	4	2	3	1	0

\:	0	1	2	3	4
0	0	1	2	3	4
1	1	0	4	2	3
2	2	3	0	4	1
3	3	4	1	0	2
4	4	2	3	1	0

Table 24: 4 not in 3 : (0,0,1,2)

6 A69 and B69 in Quasigroups

We will prove the implication $A69 \Rightarrow B69$. The reverse implication follows by an analogous.

Assume G is a quasigroup such that for any $x, y, z \in G$, we have

$$x\backslash(y\backslash z) = x/(y/z) \tag{1}$$

Note that we can rearrange this expression to obtain

$$y\backslash z = x * (x/(y/z)) \tag{2}$$

We will refer to this identity numerous times throughout the proof.

In any quasigroup, the equation

$$y/(z\backslash y) = z \tag{3}$$

holds. Hence,

$$x/(y/(z\backslash y) = x/z.$$

Performing the substitution $(x, y, z) \mapsto (x, y, z\backslash y)$ in (1), we can transform the left side of this equation to obtain

$$x\backslash(y\backslash(z\backslash y)) = x/z$$

Now we apply (1) to the term $y\backslash(z\backslash y)$ above, which yields

$$x\backslash(y/(z/y)) = x/z \tag{4}$$

Performing the substitition $z \mapsto z * y$ in (4) gives

$$x\backslash(y/z) = x/(z * y) \tag{5}$$

For the next step, we note that by (4), the term y/z in (5) can be rewritten as

$$y\backslash(u\backslash(z\backslash u))$$

where u is simply a new variable. Thus, (5) becomes

$$x\backslash(y\backslash(u/(z/u)) = x/(z * y)$$

Finally, performing the substitution $(x, y, z) \mapsto (x, y, u/(z/u))$ in (1), we can rewrite the left side of the above equation to obtain

$$x/(y/(u/(z/u)) = x/(z * y) \tag{6}$$

Next, perform the substitution $(x, y, z) \mapsto (z, x, u/(y/u))$ in (2) to obtain

$$x\backslash(u/(y/u)) = z * (z/(x/(u/(y/u))))$$

By (4), the left side of this equation reduces to x/y. By (6), the term $z/(x/(u/(y/u)))$ on the right becomes $z/(y*x)$. Hence, we obtain

$$x/y = z*(z/(y*x)) \tag{7}$$

We note that (3) implies

$$(x\backslash y)/(z\backslash(x\backslash y)) = z$$

By our assumption (1), applied to the term $z\backslash(x\backslash y)$, we obtain

$$(x\backslash y)/(z/(x/y)) = z \tag{8}$$

Substituting $z*(x/y)$ for z in (8) yields

$$(x\backslash y)/z = z*(x/y) \tag{9}$$

Now, assume toward a contradiction that there exist $a, b, c \in G$ such that

$$(a/b)/c \neq (a\backslash b)\backslash c. \tag{10}$$

Performing the substitution $(x, y, z) \to (x, (a\backslash b), c)$ in (2), we obtain

$$x*(x/((a\backslash b)/c)) = (a\backslash b)\backslash c$$

By (10) it follows that

$$x*(x/((a\backslash b)/c)) \neq (a/b)/c \tag{11}$$

However, by (9), we have

$$(a\backslash b)/c = c*(a/b).$$

Substituting this expression into (11) above, we obtain

$$x*(x/(c*(a/b))) \neq (a/b)/c \tag{12}$$

Performing the substitution $(x, y, z) \mapsto ((a/b), c, x)$ in (7) yields the desired contradiction, and the proof is complete.

7 References

[1] **W. McCune**, *Prover9, equational reasoning tool and Mace4, finite model builder.* Available at http://www.cs.unm.edu/ mccune/mace4/.
[2] **J.D. Phillips and P. Vojtěchovský**, *The varieties of quasigroups of Bol-Moufang type: An equational reasoning approach.* J. Algebra **293** (2005), 17-33.
[3] **H.O. Pflugfelder**, *Quasigroups and Loops: Introduction* Heldermann (1990)

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