Classifying Bol-Moufang Quasigroups Under a Single Operation

Ashley Arp	Black Hills State University
Jasmine Van Exel	Paine College
Michael Kaminski	Knox College
Davian Vernon	Morehouse College

Abstract

Given the binary operation $*$ of a quasigroup, we may define conjugate operations, $\backslash$ and $/$. We examine Bol-Moufang identities under these conjugate operations and then classify these identities according to logical equivalence. We find all implications among the varieties defined by these identities and all necessary counterexamples.

1 Introduction

A set M with a binary operation $*$ is a *quasigroup* if the equation $a * b = c$ has a unique solution whenever two of the elements $a, b, c \in M$ are known. This is referred to as the *quasigroup condition*. We may define operations $\backslash$ and $/$ such that $x\backslash y$ is the unique solution to the equation $x * b = y$ for some unknown $b \in M$ and y/x is the unique solution to $a * x = y$ for some unknown $a\ inM$. It can be shown that a multiplication table of a finite quasigroup is a Latin square. The purpose of this paper is to classify varieties of quasigroups of Bol-Moufang type under the conjugate binary operations $*$, $\backslash$ and $/$ and also to determine the relations among these varieties.

An element $e \in M$ is called a *left (right) identity element* of M if and only if $e*x = x$ $(x * e = x)$ for every $x \in M$. An element $e \in M$ is an *identity element* if and only if $e * x = x * e = x$ respectively, for every $x \in M$. We define a *left (right) loop* to be a quasigroup with a left (right) identity element. A *loop* is a quasigroup with an identity element.
We can also express the definition of a quasigroup equationally. The set $M(*, \backslash, /)$ is a quasigroup if the binary operations $*$, $\backslash$, and $/$ satisfy

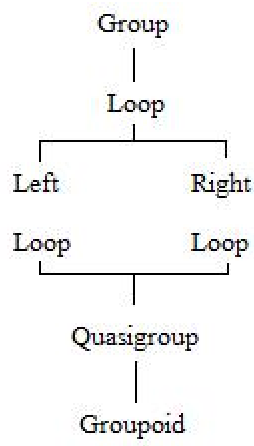


Figure 1: Hierarchy From Groupoids to Groups

$$x * (x \backslash y) = y, \quad (y/x) * x = y, \quad x \backslash (x * y) = y, \quad (y * x)/x = y$$

for all $x, y \in M$. This is known as the *equational definition of a quasigroup*. A more in-depth introduction to quasigroups and loops can be found in [5].

This characterization is essential when using automated theorem provers such as Prover9 and finite model builders such as Mace4.

An identity $\alpha = \beta$ is said to be of Bol-Moufang type if

(i) there is only one binary operation in the identity;
(ii) the same three variables appear on both sides of the identity;
(iii) one of the variables appears twice on each side;
(iv) the remaining two variables appear only once;
(v) the variables appear in the same order on both sides of the identity.

For example $x * ((y * z) * x) = (x * y) * (z * x)$ is an identity of Bol-Moufang type. A systematic notation of identities of Bol-Moufang type is found in [2], and will be reviewed and expanded upon in the next section.

Two identities of Bol-Moufang type are said to be *equivalent* if any quasigroup satisfying one identity necessarily satisfies the other. In this paper a *variety* refers to a collection of quasigroups satisfying a set of equivalent Bol-Moufang identities. However, all identities must be interpreted as holding in some underlying structure (i.e. groups, loops, etc.). In our case, we will work with quasigroups. The underlying idea behind our investigation (and that of [2]) is that when one property or identity holds in a quasigroup, it may imply the satisfaction of some other property or identity. For example, the imposition of the identity $(x*y)*(z*x) = x*((y*z)*x)$ on a quasigroup implies the existence of an identity element [3]. Therefore the quasigroup is in fact a loop.

The classification of quasigroups of Bol-Moufang type under the binary operation $*$ has already been done in [2] and we refer to their results throughout the paper. Their main result is illustrated in Figure 2, which shows there are 26 distinct varieties of quasigroups of Bol-Moufang type under $*$. The figure arranges the varieties in such a way that the variety pictured at the top, the variety for groups, is the strongest variety and implies each variety below it on the chart. This flow of implications continues with each variety implying the lower connecting varieties. The superscript in the variety name, R, L, 2, or 0, refers to the existence of a right, left, two sided, or no identity element respectively. These relations prove to be very helpful when we extend the classification of quasigroups of Bol-Moufang type to the operations $\backslash$ and $/$.

2 Systematic Notation

In our exploration of the Bol-Moufang identities we adopt the systematic notation used by Phillips and Vojtěchovský [2]. There are six formally distinct ways in which the variables in the identities can be ordered, and there are exactly five ways in which these expressions can be bracketed.

A	xxyz
B	xyxz
C	xyyz
D	xyzx
E	xyzy
F	xyzz

1	0(0(00))
2	0((00)0)
3	(00)(00)
4	(0(00))0
5	((00)0)0

For example, A13 under $*$ is shorthand for

$$x * (x * (y * z)) = (x * x) * (y * z).$$

The *dual* of a given identity is obtained by reading that identity from right to left. For instance the dual of A13 is the identity F35 of the form $(z*y)*(x*x) = ((z*y)*x)*x$.

For the purpose of our expansion of the Phillips and Vojtěchovský classification of varieties of Bol-Moufang quasigroups we use a generalized notation, as exemplified in the case of A13

$$SA13 \leftrightarrow x * (x * (y * z)) = (x * x) * (y * z).$$

$$LA13 \leftrightarrow x\backslash(x\backslash(y\backslash z)) = (x\backslash x)\backslash(y\backslash z).$$

$$RA13 \leftrightarrow x/(x/(y/z)) = (x/x)/(y/z).$$

In this notation S is used for the binary operation $*$ (star), L is used for the binary operation \ (left divide), and R is used for the binary operation / (right divide). Thus, when an identity is being referred to under one of the three operations of a quasigroup, (take our earlier example of A13), it will be written under the new notation in the above form.
In summary, the 60 identities of Bol-Moufang type can each be written in three different ways based on the choice of binary operation. This gives us 180 configurations to consider.

3 Conjugate Groupings of Implications

In this section we introduce a new notation for the previously discussed operations $*$, $\backslash$, and $/$. The notation is used to clarify upcoming discussions.
Given a quasigroup $(G, \odot)$ with *principal binary operation* $\odot$ we may define conjugate binary operations called the *left conjugate*, *right conjugate*, *reverse principal*, *double left conjugate*, and *double right conjugate* binary operations. The table below gives the notations and definitions for each of the six binary operations for $\odot$ given that $a \odot b = c$. Note that the six equations in the right-most column are, by definition equivalent statements.

Name	Notation	Definition
Principal	$\odot^{(*)}$	$a \odot^{(*)} b = c$
Left Conjugate	$\odot^{(\backslash)}$	$a \odot^{(\backslash)} c = b$
Right Conjugate	$\odot^{(/)}$	$c \odot^{(/)} b = a$
Reverse Principal	$\odot^{(\circ)}$	$b \odot^{(\circ)} a = c$
Double Right Conjugate	$\odot^{(//)}$	$c \odot^{(//)} a = b$
Double Left Conjugate	$\odot^{(\backslash\backslash)}$	$b \odot^{(\backslash\backslash)} c = a$

Now let $*$ be the particular principal binary operation of some quasigroup $(G, *)$. We have, as defined above, the left conjugate, right conjugate, reverse principal, double right conjugate, and double left conjugate binary operations. For this particular principal binary operation $*$ we give special notations to these conjugate binary operations. The notations are as follows:

$$a * b = a *^{(*)} b$$

$$a \backslash c = a *^{(\backslash)} c$$

$$c/b = c *^{(/)} b$$

$$b \circ a = b *^{(\circ)} a$$

$$c//a = c *^{(//)} a$$

$$b \backslash\backslash c = b *^{(\backslash\backslash)} c.$$

Here we see that our usual notions of the binary operations are expressed on the left while their equivalents are expressed on the right. It turns out that each of these conjugate binary operations induces a quasigroup on the set G.

Proposition 3.1. *If $(G, *)$ is a quasigroup then $(G, \backslash)$, $(G, /)$, $(G, \circ)$, $(G, \backslash\backslash)$, and $(G, //)$ are also quasigroups.*

Proof:
Suppose $(G, *)$ is a quasigroup with left conjugate operation $\backslash$, right conjugate operation $/$, reverse principal operation $\circ$, double left conjugate operation $\backslash\backslash$, and double right conjugate operation $//$. We consider the equation of the form $x \backslash y = z$. To show $(G, \backslash)$ is a quasigroup we must show that if two of $x, y, z \in G$ are known, then the third element is uniquely determined. We thus have three cases.

Case 1:
Suppose $x, y \in G$ are known. Let $x = a$ and $y = b$. Then we have the equation $a \backslash b = z$. Since $\backslash$ is a binary operation we see that z is uniquely determined by our choice of x and y.

Case 2:
Suppose $x, z \in G$ are known. Let $x = a$ and $z = c$. Then we have the equation $a \backslash y = c$. We may then left multiply both sides by a to obtain $a * (a \backslash y) = a * c$. By the equational definition of a quasigroup we find $y = a * c$. Since $*$ is a binary

operation we see that y is uniquely determined by our choice of x and z.

Case 3:
Suppose $y, z \in G$ are known. Let $y = b$ and $z = c$. Then we have the equation $x\backslash b = c$. We may then left multiply both sides by x to obtain $x * (x\backslash b) = x * c$. By the equational definition of a quasigroup we find $b = x * c$. We then right divide both sides by c to obtain $b/c = (x * c)/c$. By the equational definition of a quasigroup we find that $b/c = x$. Since $/$ is a binary operation we see that x is uniquely determined by our choice of y and z.

The proofs showing that $(G, /)$, $(G, \circ)$, $(G, \backslash\backslash)$, and $(G, //)$ are quasigroups are analogous. $\square$

We call the quasigroups induced by the conjugate binary operations the *left conjugate quasigroup*, the *right conjugate quasigroup*, the *reverse principal quasigroup*, the *double left conjugate quasigroup*, and the *double right conjugate quasigroup*. Further, we call the quasigroup induced under the principal binary operation the *principal quasigroup*. While we know that the conjugate quasigroups are indeed quasigroups, we do not yet know if they are necessarily of Bol-Moufang type. We ask, if the principal quasigroup is of Bol-Moufang type, are any of its conjugate quasigroups necessarily of Bol-Moufang type? A second question then arises. Suppose that the left or right conjugate quasigroup is of Bol-Moufang type; is the principal quasigroup or other conjugate quasigroup necessarily of Bol-Moufang type?

4 Iterated Conjugates

We know from the previous section that if $(G, *)$ is a quasigroup then $(G, \backslash)$ and $(G, /)$ are also quasigroups. We thus have quasigroups in which $\backslash$ and $/$ are the principal binary operation. We may thus look at their conjugate operations. Since these operations were defined in terms of $*$, it is no surprise that their conjugate binary operations relate to $*$.

Proposition 4.1. *The following are equivalent as binary operations:*

- $\backslash^{(*)}$ *and* $\backslash$
- $\backslash^{(\backslash)}$ *and* $*$

- $\backslash^{(/)}$ *and* $\backslash\backslash$
- $\backslash^{(\circ)}$ *and* $//$
- $\backslash^{(\backslash\backslash)}$ *and* $/$
- $\backslash^{(//)}$ *and* $\circ$

Proof:
Suppose $(G, \backslash)$ is a quasigroup and that we know that $a\backslash b = c$. We then have

$$a\backslash^{(\backslash)}c = b$$

$$c\backslash^{(/)}b = a.$$

Because we know that $a\backslash b = c$, by multiplying on the left by a it follows that

$$a * c = b.$$

This shows that $\backslash^{(\backslash)}$ and $*$ are equivalent as binary operations since for all $a, c \in G$, $a\backslash^{(\backslash)}c = a * c$. From the fact that $a * c = b$ we know by the definition of $\backslash\backslash$ that we have

$$c\backslash\backslash b = a.$$

This shows then that $\backslash^{(/)}$ and $\backslash\backslash$ are equivalent as binary operations since for all $b, c \in G$, $c\backslash^{(/)}b = c\backslash\backslash b$. The other equivalences follow analogously. $\square$

We note that the result here is simply a permutation of the six binary operations defined in terms of $*$. Similar proofs may be carried out on each of the conjugate quasigroups. The result is shown in Table 1.

Table 1: Iterated Conjugate Relations

Principal Quasigroup	$(G, *)$	$(G, \backslash)$	$(G, /)$	$(G, \circ)$	$(G, \backslash\backslash)$	$(G, //)$
Principal	$a * b = c$	$a\backslash b = c$	$a/b = c$	$a \circ b = c$	$a\backslash\backslash b = c$	$a//b = c$
Left Conjugate	$a\backslash c = b$	$a * c = b$	$a//c = b$	$a\backslash\backslash c = b$	$a \circ c = b$	$a/c = b$
Right Conjugate	$c/b = a$	$c\backslash\backslash b = a$	$c * b = a$	$c//b = a$	$c\backslash b = a$	$c \circ b = a$
Reverse Principal	$b \circ a = c$	$b//a = c$	$b\backslash\backslash a = c$	$b * a = c$	$b/a = c$	$b\backslash a = c$
Double Left Conjugate	$b\backslash\backslash c = a$	$b/c = a$	$b \circ c = a$	$b\backslash c = a$	$b//c = a$	$b * c = a$
Double Right Conjugate	$c//a = b$	$c \circ a = b$	$c\backslash a = b$	$c/a = b$	$c * a = b$	$c\backslash\backslash a = b$

5 Conjugate Groupings of Implications

Given a principal quasigroup $(G, *)$ we examine implications in which the hypothesis is that the principal quasigroup, or one of the conjugate quasigroups, is of Bol-Moufang type and the conclusion is that a different one of these quasigroups is of Bol-Moufang type. We may significantly reduce the number of implications which need to be checked via a grouping of equivalent implications. These equivalence classes arise from the relations among the conjugate operations. We may thus group identity implications into classes of implications from which only one implication needs to be checked to determine the truth value of every implication in the class.

To begin, we remark that we need not consider implications involving the operations $\circ$, $\backslash\backslash$, or $//$ in any identity, as any identity of Bol-Moufang type with one of these operations may be expressed as an identity of Bol-Moufang type involving one of the three other binary operations. For example the identity

$$x//(x//(y//z)) = x//((x//y)//z)$$

of the form A12 is equivalent to

$$(x\backslash(y\backslash z))\backslash z = ((x\backslash y)\backslash z)\backslash z$$

an identity of the form F45, by the definition of $//$. Thus we have six different implication types corresponding to the possible combinations of the binary operations $*$, $\backslash$, and $/$ for the assumption identity and the conclusion identity.

Proposition 5.1. *Let $(G, *)$ be a quasigroup. $P \Longrightarrow Q$ be an implication in which P is a Bol-Moufang identity under $\backslash$ or $/$ and Q is a Bol-Moufang identity under $*$, $\backslash$, or $/$. Then $P \Longrightarrow Q$ if and only if $P' \Longrightarrow Q'$ where P' is the Bol-Moufang identity P with operation replaced by $*$ and Q' is the Bol-Moufang identity whose operation is specified by the following table:*

Proof:

Case 1:
Suppose P is a Bol-Moufang identity under $\backslash$ and Q is a Bol-Moufang identity under $*$. Since $\backslash$ and $*$ are conjugates of $\backslash$ by Table 1, we know that we may write P in more explicit terms under $\backslash^{(*)}$ and may write Q in more explicit terms under $\backslash^{(\backslash)}$. Since $\backslash$ induces a quasigroup on our set G and we have expressed our identities under

P	Q	Q'
$\backslash$	$*$	$\backslash$
$\backslash$	$\backslash$	$*$
$\backslash$	$/$	$\backslash\backslash$
$/$	$*$	$/$
$/$	$\backslash$	$//$
$/$	$/$	$*$

conjugates of the principal binary operation $\backslash$. The implication under consideration is one in which a Bol-Moufang identity under the principal binary operation, $\backslash^{(*)}$, is satisfied for the premise and a Bol-Moufang identity under a conjugate binary operation, namely $\backslash^{(\backslash)}$, is satisfied for the conclusion. However, this is simply $P' \Longrightarrow Q'$.

Case 2:
Suppose P is a Bol-Moufang identity under $\backslash$ and Q is a Bol-Moufang identity under $\backslash$. Since $\backslash$ is a conjugate of $\backslash$ by Table 1, we know that we may write P in more explicit terms under $\backslash^{(*)}$ and may write Q in more explicit terms under $\backslash^{(*)}$. Since $\backslash$ induces a quasigroup on our set G and we have expressed our identities under conjugates of the principal binary operation $\backslash$. The implication under consideration is one in which a Bol-Moufang identity under the principal binary operation, $\backslash^{(*)}$, is satisfied for the premise and a Bol-Moufang identity under a conjugate binary operation, namely $\backslash^{(*)}$, is satisfied for the conclusion. However, this is simply $P' \Longrightarrow Q'$.

Case 3:
Suppose P is a Bol-Moufang identity under $\backslash$ and Q is a Bol-Moufang identity under $/$. Since $\backslash$ and $/$ are conjugates of $\backslash$ by Table 1, we know that we may write P in more explicit terms under $\backslash^{(*)}$ and may write Q in more explicit terms under $\backslash^{(\backslash\backslash)}$. Since $\backslash$ induces a quasigroup on our set G and we have expressed our identities under conjugates of the principal binary operation $\backslash$. The implication under consideration is one in which a Bol-Moufang identity under the principal binary operation, $\backslash^{(*)}$, is satisfied for the premise and a Bol-Moufang identity under a conjugate binary operation, namely $\backslash^{(\backslash\backslash)}$, is satisfied for the conclusion. However, this is simply $P' \Longrightarrow Q'$.

The remaining three cases follow analogously. $\square$

We note that in the above proposition we find $\backslash$ or $/$ is changed to $\backslash\backslash$ or $//$. We opt to convert identities with these operations to identities with $\backslash$ and $/$. To do this we simply recall that:

$$a//b = b\backslash a \text{ and } b\backslash\backslash a = a/b$$

We note that that upon carrying out the conversion from $\backslash\backslash$ or $//$ we obtain the dual of the identity and change the binary operation to $/$ or $\backslash$ respectively. Hence the manner in which identity implications are transformed into equivalent implications can be summarized as seen in Table 2. The left-hand column has forms of identity implications whereas the right-hand column contains the forms of equivalent implications.

Table 2: Conversion of Identities for Equivalence Classes

Before			After		
S Identity 1	$\Longrightarrow$	S Identity 2	S Identity 1	$\Longrightarrow$	S Identity 2
S Identity 1	$\Longrightarrow$	L Identity 2	S Identity 1	$\Longrightarrow$	L Identity 2
S Identity 1	$\Longrightarrow$	R Identity 2	S Identity 1	$\Longrightarrow$	R Identity 2
L Identity 1	$\Longrightarrow$	S Identity 2	S Identity 1	$\Longrightarrow$	L Identity 2
L Identity 1	$\Longrightarrow$	L Identity 2	S Identity 1	$\Longrightarrow$	S Identity 2
L Identity 1	$\Longrightarrow$	R Identity 2	S Identity 1	$\Longrightarrow$	R Dual of Identity 2
R Identity 1	$\Longrightarrow$	S Identity 2	S Identity 1	$\Longrightarrow$	R Identity 2
R Identity 1	$\Longrightarrow$	L Identity 2	S Identity 1	$\Longrightarrow$	L Dual of Identity 2
R Identity 1	$\Longrightarrow$	R Identity 2	S Identity 1	$\Longrightarrow$	S Identity 2

We have an immediate corollary from Proposition 5.1 concerning the validity of implications where the premise and conclusion involve the same operation.

Corollary 5.2. *The relationships among Bol-Moufang quasigroups under $\backslash$ with Bol-Moufang quasigroups under $\backslash$ and among Bol-Moufang quasigroups under $/$ with Bol-Moufang quasigroups under $/$ mimic the structure of those relations among Bol-Moufang quasigroups under $*$ with Bol-Moufang quasigroups under $*$*

6 Quasigroup Relations

In this section we examine the implications between principal quasigroups of Bol-Moufang type and their conjugate quasigroups and between conjugate quasigroups of Bol-Moufang type and the principal and second conjugate quasigroups. Variety names are provided rather than individual identities. Varieties refer to those listed in Table 3 with the structure of Figure 2 both reproduced from [2].

Proposition 6.1. *The variety* MNQ *under $*$ implies the variety* LNQ *under $\backslash$.*

Proof:
We first note that MNQ under $*$ is characterized by the identity SC24 and LNQ under $\backslash$ is characterized by the identity LA35. This gives us the implication

$$x * ((y * y) * z) = (x * (y * y)) * z$$

which implies

$$(x\backslash x)\backslash(y\backslash z) = ((x\backslash x)\backslash y)\backslash z.$$

We note that by Figure 2 that MNQ has a two-sided identity. Thus we know that there is an element e such that $e * x = x * e = x$ for all x. If we take the equation $x * e = x$ and left divide by x on both sides we find that $e = x\backslash x$. Further, if we consider the equation $e*x = x$ and left divide both sides by e we find that $x = e\backslash x$.

We thus find that

$$y\backslash z = y\backslash z$$

$$e\backslash(y\backslash z) = (e\backslash y)\backslash z$$

$$(x\backslash x)\backslash(y\backslash z) = ((x\backslash x)\backslash y)\backslash z.$$

This is the conclusion identity. Thus the implication is proven. □

Proposition 6.2.

- *The variety* MQ *under* $*$ *implies the variety* LNQ *under* $\backslash$.
- *The variety* MNQ *under* $*$ *implies the variety* RNQ *under* $/$.
- *The variety* MQ *under* $*$ *implies the variety* RNQ *under* $/$.
- *The variety* RG1 *under* $*$ *implies the variety* RG3 *under* $\backslash$.
- *The variety* LG1 *under* $*$ *implies the variety* LG3 *under* $/$.

Proof:
All proofs given by Prover9. □

The following corollary stems from the relationships among identities involving the same operation as seen in Figure 2.

Corollary 6.3. *(a) The varieties* LC1, RC1, EQ, *and* GR *under* $*$ *imply the variety* LNQ *under* $\backslash$.

(b) The varieties LC1*,* RC1*,* EQ*, and* GR *under* $*$ *imply the variety* RNQ *under* $/$*.*

(c) The variety GR *under* $*$ *characterized by the identity* SA12 *implies the variety* RG3 *under* $\backslash$*.*

(d) The variety GR *under* $*$ *characterized by the identity* SA12 *implies the variety* LG3 *under* $/$*.*

Proof:
(a) Each of the varieties LC1, RC1, EQ, and GR is a parent variety of MNQ under $*$. This means each of these varieties imply MNQ under $*$. Thus by the previous proposition, these varieties imply the variety LNQ under $\backslash$.

(b-d) The proofs are analogous. □

Corollary 6.4. *(a) Each of the varieties* MNQ*,* LC1*,* RC1*,* EQ*,* GR*, and* MQ *under* $\backslash$ *imply the variety* LNQ *under* $*$*. Additionally, each of the varieties* MNQ*,* LC1*,* RC1*,* EQ*,* GR*, and* MQ *under* $/$ *imply the variety* RNQ *under* $\backslash$*.*

(b) Each of the varieties MNQ*,* LC1*,* RC1*,* EQ*,* GR*, and* MQ *under* $/$ *imply the variety* RNQ *under* $*$*. Additionally, each of the varieties* MNQ*,* LC1*,* RC1*,* EQ*,* GR*, and* MQ *under* $\backslash$ *imply the variety* LNQ *under* $/$*.*

(c) Both of the varieties RG1 *and* GR *under* $\backslash$ *imply the variety* RG3 *under* $*$*. Additionally, both of the varieties* RG1 *and* GR *under* $/$ *imply the variety* LG3 *under* $\backslash$*.*

(d) Both of the varieties LG1 *and* GR *under* $/$ *imply the variety* LG3 *under* $*$*. Additionally, both of the varieties* LG1 *and* GR *under* $\backslash$ *imply the variety* RG3 *under* $\backslash$*.*

Proof:
(a) All of these implications follow from applying conjugacy of implications. We first note that LNQ under $*$ is characterized by the identity SA35. The first set of implications are all of the form:

$$\text{L Identity} \Longrightarrow \text{SA35}$$

Via the conjugacy of implications we find that implications of this form hold if and only if the implications:

$$\text{S Identity} \Longrightarrow \text{LA35}$$

hold. But these implications were proven in the previous propositions.
We now note that the variety RNQ under \ is characterized by the identity LF13. The second set of implications are all of the form:

$$\text{R Identity} \Longrightarrow \text{LF13}$$

Via the conjugacy of implications we find that implications of this form hold if and only if the implications:

$$\text{S Identity} \Longrightarrow \text{Dual of LF13}$$

hold. We find that the dual of LF13 is LA35. Hence we know these implications hold if and only if the implications:

$$\text{S Identity} \Longrightarrow \text{LA35}$$

hold. But again, these were already shown in the previous propositions.

(b-d) The proofs are analogous. □

7 Counterexamples

For all other implications, there are counterexamples. In the interest of brevity, not all of the counterexamples are listed here. For each counterexample, we have a list of implications, where the assumption is always $SA12$. The given multiplication table holds true for $SA12$, but fails under the conclusions. The corresponding x, y, z values produce the contradictions in the conclusion identities.

We elect to assume SA12 because the variety it represents, that of groups, implies each other variety under the binary operation $*$. In this manner, finding a counterexample for an implication of the form $SA12 \Longrightarrow I$, where I is a particular identity, also suffices as a counterexample for all implications of the form $S \Longrightarrow I$, where S is any identity under $*$. Note that similar reasoning works for assumptions which are weaker than that of groups (See Figure 2).

For the conclusions $LE14, LC15, LF34, LA23, LA34, RF34, LA25$, and $RD14$ use the values

$$x = 0 \quad y = 1 \quad z = 0.$$

For the conclusions $LA14, LC24, LF13, LF15, LF25, LB14, LA13, LB45, LC45$, and $LE25$ use the values

$$x = 0 \quad y = 0 \quad z = 1.$$

$*$:	0	1	2	3	4	5
0	2	5	3	0	1	4
1	4	3	5	1	0	2
2	3	4	0	2	5	1
3	0	1	2	3	4	5
4	5	2	1	4	3	0
5	1	0	4	5	2	3

$/$:	0	1	2	3	4	5
0	3	5	2	0	1	4
1	5	3	4	1	0	2
2	0	4	3	2	5	1
3	2	1	0	3	4	5
4	1	2	5	4	3	0
5	4	0	1	5	2	3

$\backslash$:	0	1	2	3	4	5
0	3	4	0	2	5	1
1	4	3	5	1	0	2
2	2	5	3	0	1	4
3	0	1	2	3	4	5
4	5	2	1	4	3	0
5	1	0	4	5	2	3

For the conclusions $RC15, RB25, RB34$, and $RA23$ use the values

$$x = 0 \quad y = 0 \quad z = 1.$$

$*$:	0	1	2	3	4	5
0	2	3	0	1	5	4
1	4	5	1	0	3	2
2	0	1	2	3	4	5
3	5	4	3	2	1	0
4	1	0	4	5	2	3
5	3	2	5	4	0	1

$/$:	0	1	2	3	4	5
0	2	4	0	1	5	3
1	4	2	1	0	3	5
2	0	5	2	3	4	1
3	5	0	3	2	1	4
4	1	3	4	5	2	0
5	3	1	5	4	0	2

For the conclusions $RA14, RA15, RA35, RC24, RF15, RF25, RB14, RA13, RB45, RC45$, and $RE25$ use the values

$$x = 0 \quad y = 1 \quad z = 0.$$

$*:$	0	1	2	3	4	5
0	2	3	0	1	5	4
1	5	4	1	0	2	3
2	0	1	2	3	4	5
3	4	5	3	2	0	1
4	3	2	4	5	1	0
5	1	0	5	4	3	2

$/:$	0	1	2	3	4	5
0	2	5	0	1	3	4
1	5	2	1	0	4	3
2	0	4	2	3	1	5
3	4	0	3	2	5	1
4	3	1	4	5	2	0
5	1	3	5	4	0	2

All other necessary counterexamples are listed below and can easily be verified by using Mace4 [4].

$SD14 \Longrightarrow LA35$	$SF34 \Longrightarrow LA35$	$SA23 \Longrightarrow LA35$	$SA25 \Longrightarrow LA35$
$SB45 \Longrightarrow LA35$	$SC14 \Longrightarrow LA35$	$SA13 \Longrightarrow LA35$	$SC45 \Longrightarrow LA35$
$SC25 \Longrightarrow LA35$	$SA15 \Longrightarrow LA35$	$SA35 \Longrightarrow LA35$	$SF13 \Longrightarrow LA35$
$SF15 \Longrightarrow LA35$	$SD14 \Longrightarrow RF13$	$SF34 \Longrightarrow RF13$	$SA23 \Longrightarrow RF13$
$SA25 \Longrightarrow RF13$	$SB45 \Longrightarrow RF13$	$SC14 \Longrightarrow RF13$	$SA13 \Longrightarrow RF13$
$SC45 \Longrightarrow RF13$	$SC25 \Longrightarrow RF13$	$SA15 \Longrightarrow RF13$	$SA35 \Longrightarrow RF13$
$SF13 \Longrightarrow RF13$	$SF15 \Longrightarrow RF13$	$SD14 \Longrightarrow LB25$	$SF34 \Longrightarrow LB25$
$SB23 \Longrightarrow LB25$*	$SA23 \Longrightarrow LB25$	$SE25 \Longrightarrow LB25$	$SB25 \Longrightarrow LB25$
$SC25 \Longrightarrow LB25$	$SA25 \Longrightarrow RE14$	$SA23 \Longrightarrow RE14$	$SB23 \Longrightarrow RE14$*
$SF34 \Longrightarrow RE14$	$SB14 \Longrightarrow RE14$	$SE14 \Longrightarrow RE14$	$SC14 \Longrightarrow RE14$

*These counterexamples are shown in the 16/1 Quasigroup Case

In the Phillips and Vojtěchovský paper [1], the authors refer to a construction known as $M(G,2)$ that is used to build various counterexamples. As sets, $M(G,2) = G \times \{0,1\}$, where G is any non-abelian group. The case in which $G = D_4$, where D_4 is the dihedral group of order 8, is referred to as the 16/1 Quasigroup. This special quasigroup is defined by the following:

1) $(g,0) * (h,0) = (gh,0)$

2) $(g,0) * (h,1) = (hg,1)$

3) $(g,1) * (h,0) = (gh^{-1},1)$

4) $(g,1) * (h,1) = (h^{-1}g,0)$

for all $g,h \in D_4$.

Note that the conditions for this quasigroup are under the operation called $*$. In order to study identities involving $\backslash$ and $/$, it is necessary to derive analogous formulas in which $*$ is replaced by $\backslash$ and $/$.

Proposition 7.1. *The following formulas hold under* $(G, \backslash)$*:*

1) $(g, 0)\backslash(h, 0) = (gh^{-1}, 0)$

2) $(g, 0)\backslash(h, 1) = (hg^{-1}, 1)$

3) $(g, 1)\backslash(h, 0) = (gh^{-1}, 1)$

4) $(g, 1)\backslash(h, 1) = (h^{-1}g, 0)$.

Proposition 7.2. *The following conditions hold under* $(G, /)$*:*

1) $(g, 0)/(h, 0) = (gh^{-1}, 0)$

2) $(g, 0)/(h, 1) = (hg, 1)$

3) $(g, 1)/(h, 0) = (gh, 1)$

4) $(g, 1)/(h, 1) = (h^{-1}g, 0)$.

Now that we have the quasigroup conditions for each of the separate binary operations we encounter, we can begin constructing the counterexample. We will be utilizing the group D_4 to disprove LB25 $\Longrightarrow$ RE14 and show that these identities do not hold for all elements $x, y, z \in M(D_4, 2)$. In the argument, D_4 is generated by elements F and R which are subject to the relations $F^2 = R^4 = 1$ and $FRF = R^{-1}$.

$SB23 \Longrightarrow LB25$ Counterexample:

We want to show LB25 does not hold for all $x, y, z \in D_4$. Let $x = (R, 1)$, $y = (R, 0)$, and $z = (F, 1)$.

$$\begin{aligned}
x\backslash[[y\backslash x]\backslash z] &= (R,1)\backslash[[(R,0)\backslash(R,1)]\backslash(F,1)] \\
&= (R,1)\backslash[(RR^{-1},1)\backslash(F,1)] \\
&= (R,1)\backslash[(e,1)\backslash(F,1)] \\
&= (R,1)\backslash(F^{-1},1) \\
&= (RF,1)
\end{aligned}$$

$$\begin{aligned}
[[x\backslash y]\backslash x]\backslash z &= [[(R,1)\backslash(R,0)]\backslash(R,1)]\backslash(F,1) \\
&= [(RR^{-1},1)\backslash(R,1)]\backslash(F,1) \\
&= [(e,1)\backslash(R,1)]\backslash(F,1) \\
&= (R^{-1},0)\backslash(F,1) \\
&= (FR,1)
\end{aligned}$$

$(RF,1) \neq (FR,1)$. Therefore LB25 does not hold true in this quasigroup.

$SB23 \Longrightarrow RE14$ Counterexample:

We want to show that RE14 does not hold for all $x,y,z \in D_4$. Let $x = (R,1)$, $y = (R,0)$, and $z = (F,1)$.

$$\begin{aligned}
x/[y/(z/y)] &= (R,1)/[(R,0)/[(F,1)/(R,0)]] \\
&= (R,1)/[(R,0)/(FR,1)] \\
&= (R,1)/(FR^2,1) \\
&= (R^2FR,0) \\
&= (FR^3,0)
\end{aligned}$$

Note that $R^2 \in Z(G)$. Therefore, R^2 commutes with every element of G.

$$
\begin{aligned}
[x/[y/z]]/y &= [(R,1)/[(R,0)/(F,1)]]/(R,0) \\
&= [(R,1)/(FR,1)]/(R,0) \\
&= (R^3FR,0)/(R,0) \\
&= (R^3FRR^{-1},0) \\
&= (R^3F,0)
\end{aligned}
$$

$(R^3F,0) \neq (FR^3,0)$. Thus, RE14 does not hold in this quasigroup.

8 Results

Given the aforementioned proofs and counterexamples, we have thus found 52 new varieties of quasigroups of Bol-Moufang type. While Phillips and Vojtěchovský [2] classified Bol-Moufang identities of a binary operation into 26 varieties, we have found that each variety induced by the left and right conjugate operations are distinct from the original 26 varieties and each other. Furthermore, we have found all implications among varieties. These relationships are illustrated in Figures 3, 4, and 5.

In Figures 3, 4, and 5 the varieties connected to the tail ends of an arrow refer to varieties of the same operation for which the figure is for, while the variety connected to the head of the arrow is under the binary operation specified by the color of the arrow. The variety connected to the head of the arrow is implied by the varieties connected to the tails of the arrow. These figures show all the implications between varieties.

variety	abbrev.	defining identity	its name
groups	GR	$x(yz) = (xy)z$	
RG1-quasigroups	RG1	$x((xy)z) = ((xx)y)z$	A25
LG1-quasigroups	LG1	$x(y(zz)) = (x(yz))z$	F14
RG2-quasigroups	RG2	$x(x(yz)) = (xx)(yz)$	A23
LG2-quasigroups	LG2	$(xy)(zz) = (x(yz))z$	F34
RG3-quasigroups	RG3	$x((yx)z) = ((xy)x)z$	B25
LG3-quasigroups	LG3	$x(y(zy)) = (x(yz))y$	E14
extra q.	EQ	$x(y(zx)) = ((xy)z)x$	D15
Moufang q.	MQ	$(xy)(zx) = (x(yz))x$	D34
left Bol q.	LBQ	$x(y(xz)) = (x(yx))z$	B14
right Bol q.	RBQ	$x((yz)y) = ((xy)z)y$	E25
C-quasigroups	CQ	$x(y(yz)) = ((xy)y)z$	C15
LC1-quasigroups	LC1	$(xx)(yz) = (x(xy))z$	A34
LC2-quasigroups	LC2	$x(x(yz)) = (x(xy))z$	A14
LC3-quasigroups	LC3	$x(x(yz)) = ((xx)y)z$	A15
LC4-quasigroups	LC4	$x(y(yz)) = (x(yy))z$	C14
RC1-quasigroups	RC1	$x((yz)z) = (xy)(zz)$	F23
RC2-quasigroups	RC2	$x((yz)z) = ((xy)z)z$	F25
RC3-quasigroups	RC3	$x(y(zz)) = ((xy)z)z$	F15
RC4-quasigroups	RC4	$x((yy)z) = ((xy)y)z$	C25
left alternative q.	LAQ	$x(xy) = (xx)y$	
right alternative q.	RAQ	$x(yy) = (xy)y$	
flexible q.	FQ	$x(yx) = (xy)x$	
left nuclear square q.	LNQ	$(xx)(yz) = ((xx)y)z$	A35
middle nuclear square q.	MNQ	$x((yy)z) = (x(yy))z$	C24
right nuclear square q.	RNQ	$x(y(zz)) = (xy)(zz)$	F13

Table 3: Types of Varieties: Taken from [2]

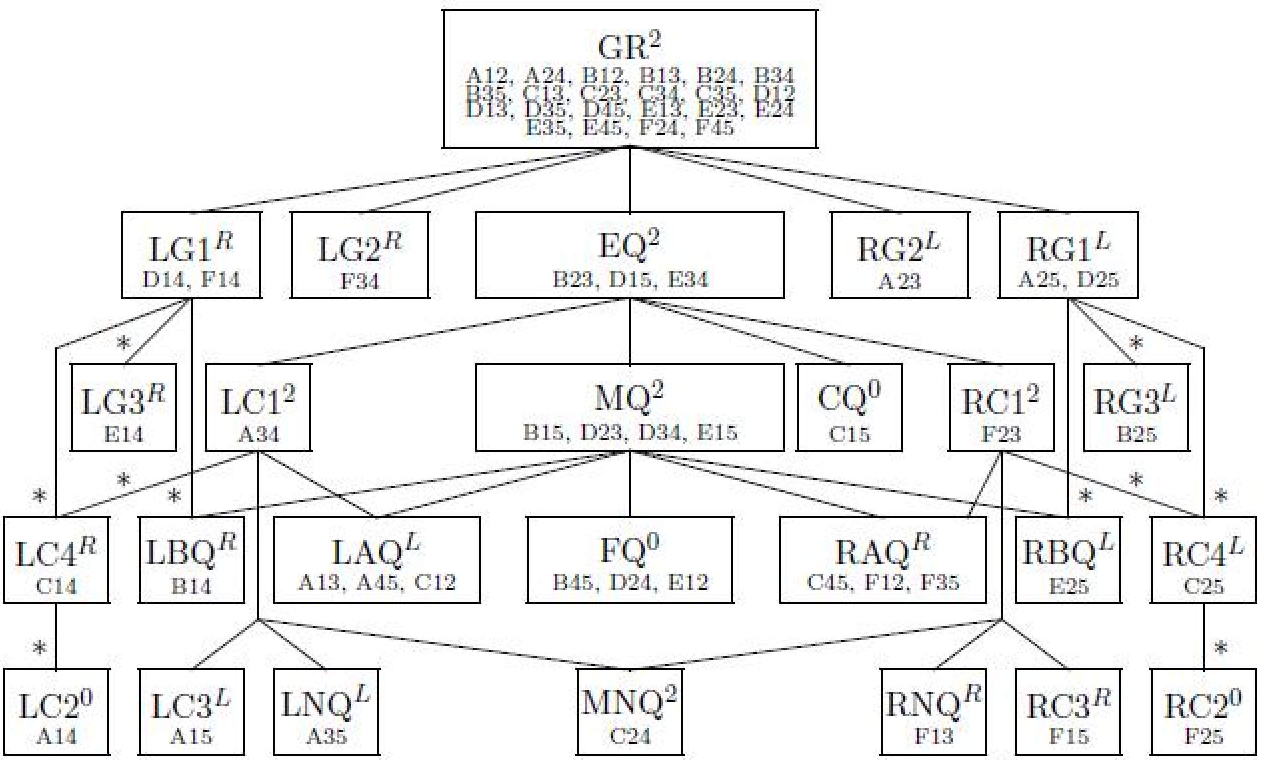
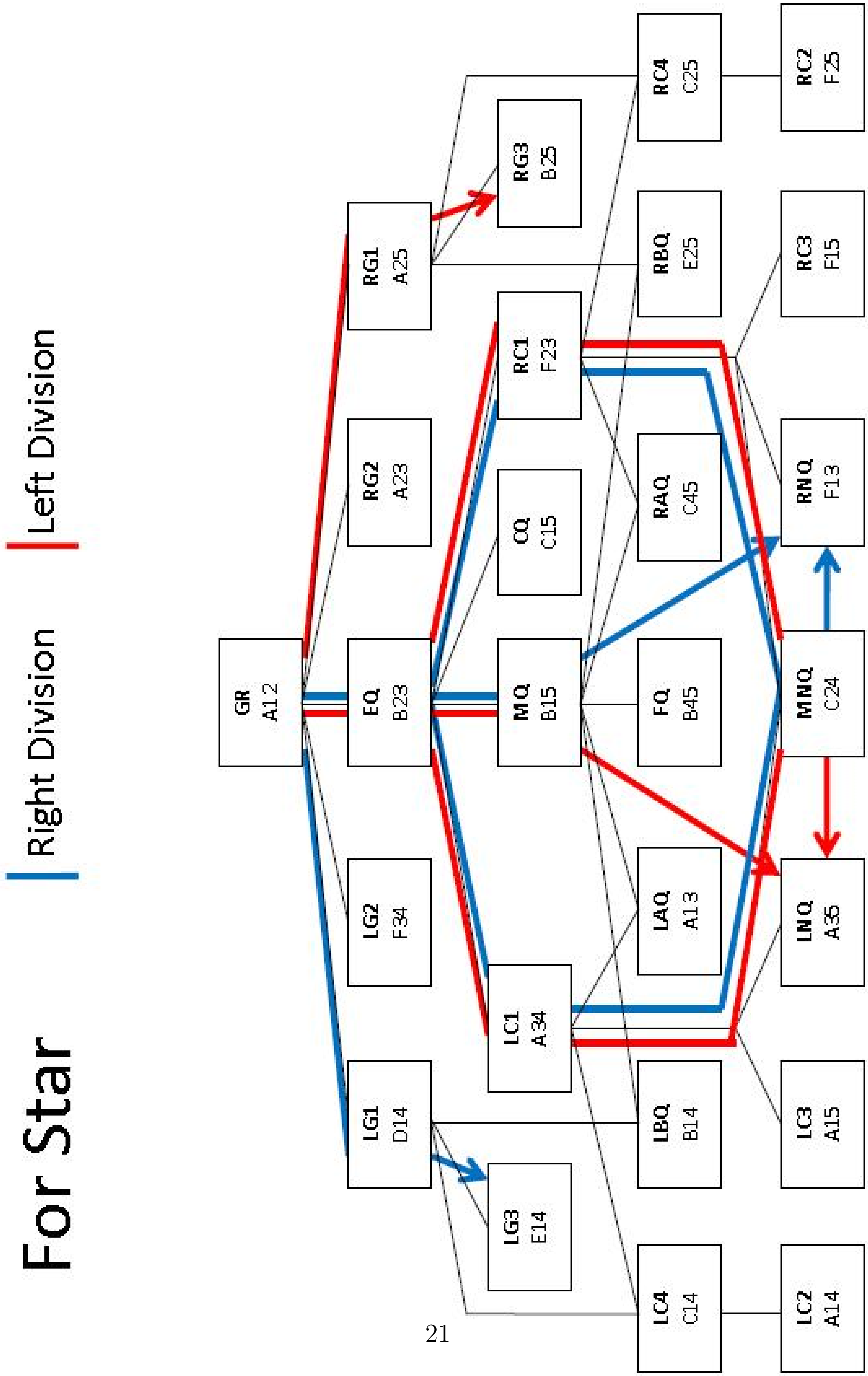


Figure 2: Varieties of Quasigroups of Bol-Moufang Type from [2]

Figure 3:



For Star
Right Division
Left Division
GR A12
RG1 A25
RG2 A23
EQ B23
LG2 F34
LG1 D14
RG3 B25
RC1 F23
CQ C15
MQ B15
LC1 A34
LG3 E14
RC4 C25
RBQ E25
RAQ C45
FQ B45
LAQ A13
LBQ B14
LC4 C14
RC2 F25
RC3 F15
RNQ F13
MNQ C24
LNQ A35
LC3 A15
LC2 A14

Figure 4:

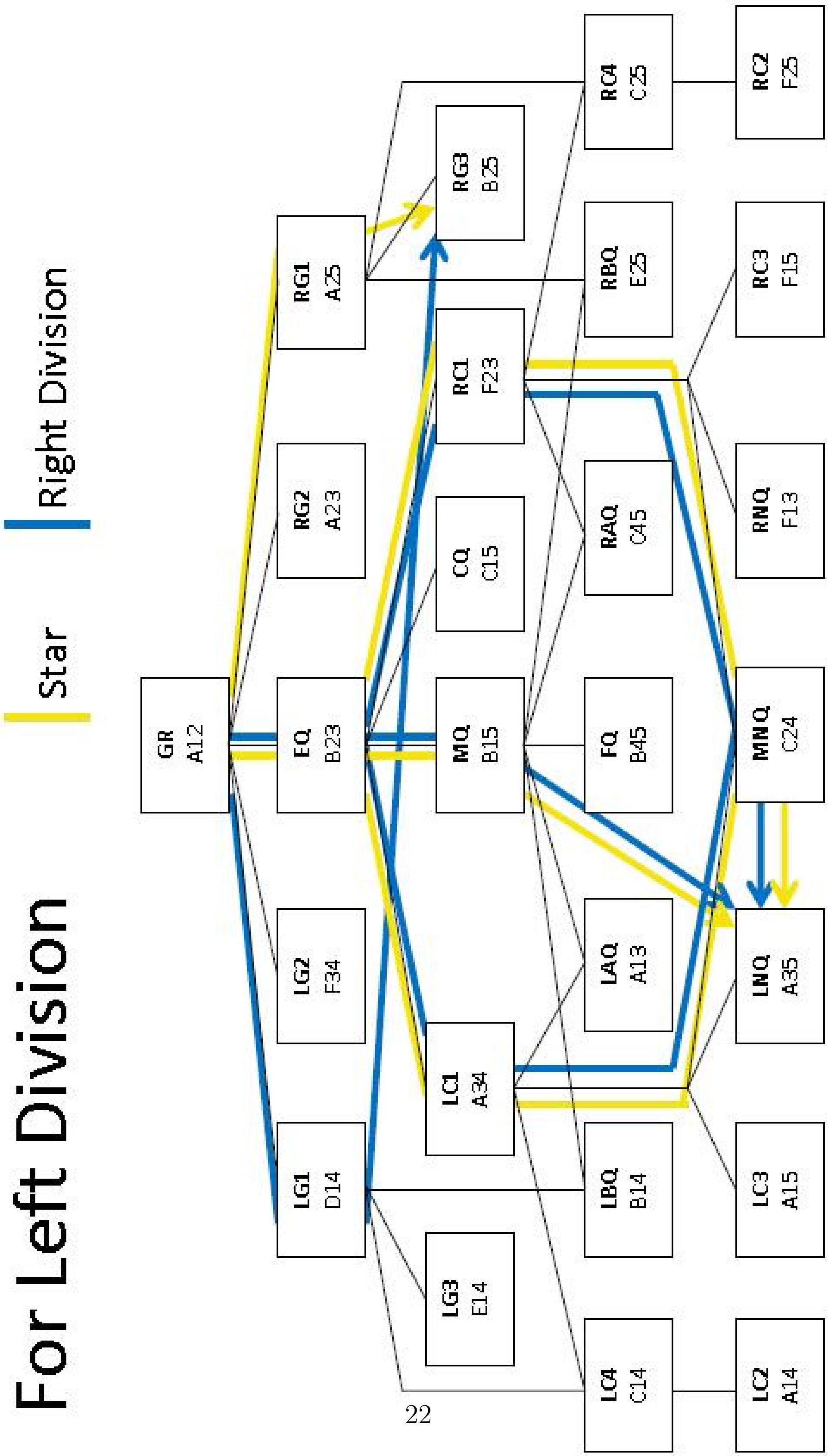
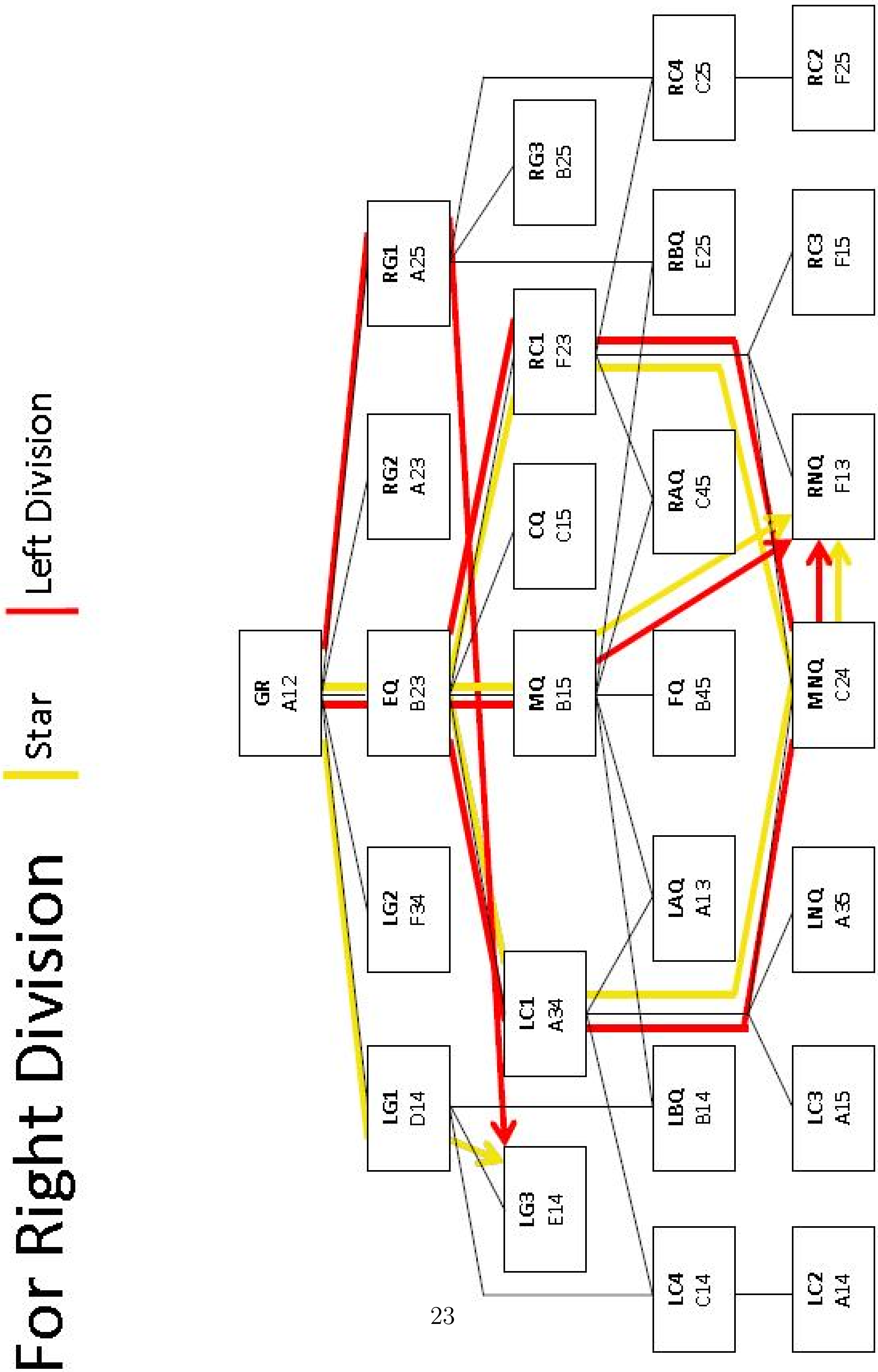

For Left Division
Star
Right Division
GR A12
RG1 A25
RG2 A23
EQ B23
LG2 F34
LG1 D14
RG3 B25
RC1 F23
CQ C15
MQ B15
LC1 A34
LG3 E14
RC4 C25
RBQ E25
RAQ C45
FQ B45
LAQ A13
LBQ B14
LC4 C14
RC2 F25
RC3 F15
RNQ F13
MNQ C24
LNQ A35
LC3 A15
LC2 A14

Figure 5:


For Right Division
Star
Left Division
GR A12
EQ B23
MQ B15
FQ B45
MNQ C24
RG1 A25
RG2 A23
RG3 B25
RC1 F23
CQ C15
RC4 C25
RBQ E25
RAQ C45
RC2 F25
RC3 F15
RNQ F13
LG2 F34
LG1 D14
LG3 E14
LC1 A34
LAQ A13
LBQ B14
LC4 C14
LNQ A35
LC3 A15
LC2 A14

9 Acknowledgements

The classification of the varieties of quasigroups of Bol-Moufang type was started by J.D. Phillips and Petr Vojtěchovský [1] following the work that the pair did concerning loops of Bol-Moufang type. For clarification on specific topics, we referred to Pflugfelder's text on loops and quasigroups [5]. The motivation for this particular study arose after examining the work of Phillips and Vojtěchovský and inquiring what kind of results would be obtained after changing the operation in the various identities.

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