Modular Representations of Graphs

Crystal Altamirano, Stephanie Angus, Lauren Brown,
Joseph Crawford, and Laura Gioco

July 2011

Abstract

A graph G has a representation modulo r if there exists an injective map $f : V(G) \to \{0, 1, ..., r-1\}$ such that vertices u and v are adjacent if and only if $f(u) - f(v)$ is relatively prime to r. The representation number of G, $rep(G)$, is the smallest integer such that G has a representation modulo r. In this paper we study the representation numbers of various graphs, such as the complete ternary tree and Harary graphs. We also give a sharp upper bound for the representation number of a connected graph G.

1 Introduction

We first define critical terms used in graph theory that are relevant to this paper.

Definition 1.1. *A graph G consists of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates to each edge two vertices, called its endpoints. If u and v are endpoints of an edge e, we say that u and v are adjacent and that e is incident at u and v. The degree of a vertex v, denoted $deg(v)$, is the number of edges incident at v.*

Definition 1.2. *A path, denoted P_n, is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.*

Definition 1.3. *A graph G is connected if for all $u, v \in V(G)$ there exists a $u, v-path$.*

Definition 1.4. *An independent set in a graph is a set of pairwise nonadjacent vertices.*

Definition 1.5. *A loop is an edge whose endpoints are equal. Multiple edges are edges which share the same pair of endpoints. A simple graph is a graph with no loops or multiple edges.*

Definition 1.6. *A cycle is a simple graph with an equal number of vertices and edges whose vertices may be placed around a circle in such a way that two vertices are adjacent if and only if they appear consecutively along the circle. A graph with no cycle is called acyclic.*

Definition 1.7. *A graph G is called bipartite if $V(G)$ is the union of two disjoint (possibly empty) independent sets.*

Definition 1.8. *A subgraph of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$ to indicate that H is a subgraph of G. The subgraph of G induced by S is the graph, denoted $G[S]$, whose vertex set is S and in which $xy \in E(G[S])$ if and only if $x, y \in S$ and $xy \in E(G)$.*

Definition 1.9. *The complement of G, denoted $\overline{G}$, is a graph with the same vertices such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.*

Definition 1.10. *A complete graph, denoted K_n, is a simple graph whose vertices are pairwise adjacent.*

Definition 1.11. *A discrete graph, denoted D_n, is a graph without edges.*

Definition 1.12. *A reduced graph, denoted $red(G)$, is a graph in which no two vertices have the same set of adjacencies.*

1.1 Dimensions and Representations

Definition 1.13. *Let G be a finite graph with vertices $\{v_1, ..., v_t\}$. A representation of G modulo r is an injective map $f : V(G) \to \{0, 1, ..., r-1\}$ such that vertices u and v are adjacent if and only if the $gcd(f(u) - f(v), r) = 1$. The representation number of G, denoted $rep(G)$, is the smallest integer r such that G has a representation modulo r.*

Definition 1.14. *A product representation of length t is the assignment of distinct vectors of length t to each vertex so that vertices u and v are adjacent if and only if their vectors differ in every coordinate position. The product dimension of a graph, denoted pdim (G), is the minimum length of a product representation of G [3].*

Erdős and Evans were the first to define graph representations in [1] and prove that any finite graph has a representation modulo some positive integer. Although modular representations have received much attention over the years, there are still numerous open problems in the topic [5]. Little is known about the representation numbers for many families of graphs, such as trees and Harary graphs.

As described in [3], there is a close relationship between product representations and modular representations. We can obtain a product representation of length t from a representation of a graph G modulo a product of primes $q_1, ..., q_t$. If $r = q_1, ..., q_t$ is a product of (not necessarily distinct) primes and $f : V(G) \to \{0, 1, ..., r-1\}$ is a mod r representation of G, we construct a product representation of length t as follows: for each i, $1 \leq i \leq t$, the ith coordinate of the vector assigned to $v \in V(G)$ is the unique integer $j(v), 0 \leq j(v) \leq q_i - 1$ such that $f(v) \equiv j(v) (\text{mod } q_i)$.

Conversely, a modular representation can be obtained given a product representation by choosing distinct primes for the coordinates, provided that the prime for each coordinate is larger than the number of values used in that coordinate. The numbers assigned to the vertices can then be obtained using the Chinese Remainder Theorem. The resulting modular representation generated from the product representation is called the coordinate representation.

2 The Complete Ternary Tree

A tree is a connected acyclic graph. In a tree, we can designate a particular vertex to be called the root of the tree. The parent of a vertex is the vertex adjacent to it on the path to the root; every vertex except the root has a unique parent. If vertex v is a parent of vertex u, then u is a child of v. A leaf is a vertex with degree one.

We define T_n recursively as follows: let T_1 consist of a single vertex. If $n > 1$, T_n is constructed from T_{n-1} by adding three new neighbors to each leaf of T_{n-1}. So, T_n is a complete ternary tree on n levels.

2.1 A Lower Bound for the Product Dimension of T_n

In this section we obtain a lower bound of the product dimension of the complete ternary tree.

Akhtar [2] presents a way to label certain vertices of the complete binary tree to obtain a lower bound on its product dimension using a result of Lóvasz et. al. We will apply the same method to a complete ternary tree to construct a lower bound for $pdim(T_n)$. While this labeling method is more suitable for a complete binary tree (since by definition a ternary tree has more vertices, hence, more edges), we hope that the lower bound we obtain assists us in understanding the representation number of T_n.

The following lemma is a result of Lóvasz et. al. [4], which proved helpful in the case of a complete binary tree.

Lemma 2.1. *Let $u_1, ..., u_r$ and $v_1, ..., v_r$ be two lists of vertices (not necessarily distinct) in a graph G. If u_i is adjacent to v_j when $i = j$ and u_i is not adjacent to v_j when $i < j$ then $pdim\ G \geq \lceil log_2\ r \rceil$.*

Theorem 2.2. *For $n \geq 2$, $pdim\ (T_n)\ \geq \lceil -log_2 13 + (n+1) log_2 3 \rceil$.*

Proof. We must first cleverly select which u_i and v_j vertices should be labeled in T_n in order to satisfy the conditions given above. The leaves of T_n are denoted as level 1 vertices, the parents of the leaves are denoted as level 2 vertices, and so forth.

We first label the leaves of T_n, and work up to the root of the tree. For T_n with levels $n \geq 4$, we must ignore every third level. Essentially, ignore level 3 after the vertices in levels 1 and 2 have been labeled. Then, start labeling again with the vertices in levels 4 and 5. Now, ignore level 6, starting again with levels 7 and 8, and so forth.

Consider the following process. For each i between 0 and $k = \lfloor \frac{n-1}{3} \rfloor$, denote the number of vertices on level $3i+2$ as m_i. Now, let $w_{i,0}, ..., w_{i,m_i}$ be the vertices on level $3i+2$. For each j, let $u_{i,2j+1}$ be the left child of w_{ij}, and $v_{i,2j+2}$, the middle child of w_{ij}. We define $u_{i,2j+2}$ and $v_{i,2j+1}$ to be w_{ij}. Then order the vertices u_{ij} as follows:

$$(u_{0,1}), ..., (u_{0,2m_{0+1}}), (u_{1,1}), ..., (u_{1,2m_{1+1}}), ..., (u_{k,1}), ..., (u_{k,2m_{k+1}}).$$

A similar order of vertices follows for the v_{ij}.

We now may apply the result of Lóvasz et. al. to construct the lower bound for T_n. Recall if the conditions of the lemma are met then the pdim $G \geq \lceil log_2\ r \rceil$. By observation, we find that two-thirds edges are used in each of the levels of vertices that were chosen to be labeled in T_n. With careful calculation, we find that the $r = \frac{1}{13}(3^{n+1} - 1)$. So, by Lemma 2.1, pdim $G \geq \lceil -log_2 13 + (n+1) log_2 3 \rceil$ for T_n. $\square$

2.2 An Upper Bound on Representation

According to Narayan, the product dimension for a graph with $r \geq 3$ vertices is at most $r-1$ [3]. This upper bound can be improved in the case of ternary trees. First, let us define the function π^* as follows: if p is prime,

$$\pi^*(n) = \sum_{p<n} 1.$$

Let us further define $S_n = t_0, t_1, ..., t_{n-1}$ as a sequence of n tuples, each of length $\pi^*(n)$, where $t_i = (i \pmod{p_1}, i \pmod{p_2}, ..., i \pmod{p_{\pi^*(n)}})$ and p_i is the ith prime number.

Consider, for example, when $n = 5$ and $\pi^*(n) = 2$. $S = (0,0), (1,1), (0,2), (1,0), (0,1)$. Observe how every tuple has at least one coordinate in common with every other tuple except the one immediately proceeding and the one immediately following.

Sublemma 2.3. *For every $i \leq n$ and $j \leq i-2$, t_i will have at least one coordinate in common with t_j and no coordinates in common with t_{i-1}.*

Proof. Let $k = i - j$; thus, $t_j = t_{i-k}$. Because k is an integer greater than or equal to two, k must have at least one prime divisor; let it be called p_s. Since $k < i \leq n$, $p_s < n$ and, as a result, $i \pmod{p_s}$ is an element in t_i. Since $k \equiv 0 \pmod{p_s}$ and $i \equiv i - k \pmod{p_s}$, t_{i-k} also contains the element $i \pmod{p_s}$, so they have the same value in the s coordinate.

Now, suppose t_i and t_{i-1} have a coordinate in common. Then for some prime p, $i \equiv i-1 \pmod{p}$, implying $0 \equiv -1 \equiv p-1 \pmod{p}$, which is a contradiction. Therefore, t_i will never have a coordinate in common with t_{i-1}. $\square$

We can use this to begin labeling T_n by assigning t_0 to the root and each subsequent element in S_n to every vertex in each subsequent level of the tree. More coordinates, however, are required to produce a product labeling of T_n.

Lemma 2.4. *For $n \geq 3$, $pdim(T_n) \leq \pi^*(n) + 2n - 3$.*

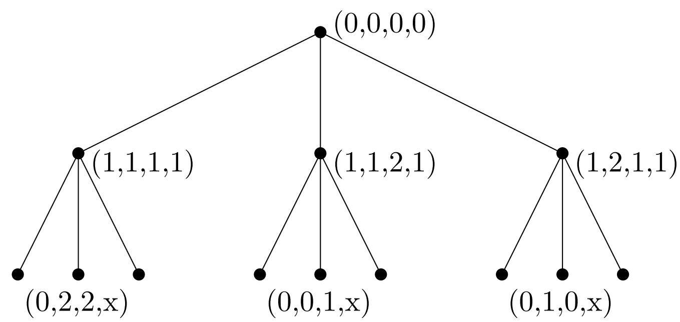


Figure 1: A 3-coordinate product representation of T_3, where $x \in \{0, 2, 3\}$

Proof. Observe that a 4-coordinate labeling of T_3 exists (see fig. 2.2). Supposing a labeling exists for T_n with $\pi^*(n) + 2n - 3$ coordinates, and consider T_{n+1}. Assign the first $\pi^*(n + 1)$ digits from S_n; that is, t_0 is assigned to the root, t_1 is assigned to each child, and so on until t_n is assigned to each leaf. For all vertices except the leaves, assign the next $2n-3$ coordinates to match the last $2n - 3$ digits of the labeling of T_n, and the final two digits to match the last digit of the T_n labeling.

Now every vertex in T_{n+1} has a (fully defined) labeling except for the leaves, which only have $\pi^*(n+1)$ coordinates assigned. Each leaf has to share a coordinate with every vertex on the level above except its parent. The next two coordinates match the vertices whose lowest common parent is the root, and therefore a third of the leaves will have the same two coordinates. The next two coordinates match the vertices whose lowest common parent is one level down from the root, and so forth. The last vertex will be the first three nonnegative integers that don't match the last digit of the parent, so that each leaf has a distinct label. □

Corollary 2.5. *For $n \geq 3$, $pdim(red(T_n)) \leq \pi^*(n) + 2n - 4$.*

Proof. Begin by giving $red(T_n)$ the labeling for T_n. Recall from our theorem that the last coordinate is to give distinction to the leaves in the same neighborhood. Since no two leaves in $red(T_n)$ are in the same neighborhood, this coordinate is extraneous and can be removed. □

Converting a product representation into a modular representation would give us a square-free modulus. However, the representation number of T_n does not have to be square-free. Suppose the second coordinate of each labeling is listed mod 9 instead of mod 3.

Theorem 2.6. *For $n \geq 3$,*

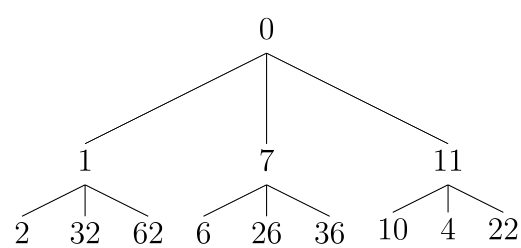
$$rep(T_n) \leq 3 \prod_{1}^{d} p_i,$$

where $d = \pi^*(n) + 2n - 4$.

Proof. Begin by giving T_n the labeling for $red(T_n)$, where each leaf not in $red(T_n)$ is assigned the label of the labeled vertex that shares its neighborhood. Now each set of 3 vertices that share a neighborhood share a labeling. Suppose the second coordinate is labeled mod 9 instead of mod 3. Without loss of generality, the leftmost leaf's label is unchanged, the middle leaf's second coordinate is raised by 3, and the rightmost leaf's second coordinate is raised by 6. Now all three leaves have distinct labels, but with the same relationships to the other vertices. □

2.3 The Representation of T_3

The ternary tree may be represented modulo $90 = 2 \cdot 3^2 \cdot 5$ as shown below.



Theorem 2.7. $rep(T_3) = 90$.

Assume that $pdim(T_3) = 2$. Let u represent the root of T_3, and w, x, y represent its children. Without loss of generality, let u be labeled $(0, 0)$ and let w, x, y be labeled $(1, n)$ for some number n. Now let a child of w be z, and have a labeling of $(0, a)$ for some a. Since x and z are not adjacent and their labels differ in the first coordinate, their labels would need to agree in the second coordinate. So, the label for x would be $(1, a)$. For the same reasons, y would be labeled $(1, a)$ which contradicts the definition of product representation. This means the only possible product representations less than 90 are of the form $2 \cdot p_1 \cdot p_2$ or $2^2 \cdot 3 \cdot p$ for some primes p, p_1, p_2.

In the following, let u denote the root of the tree and v_1, v_2, v_3 be its children.

Case 1: Suppose $r = 2^2 \cdot 3 \cdot p$ for some prime p and consider any representation modulo r. Without loss of generality, we may assume that u is labeled $(0,0,0)$. For $i = 1,2,3$, let (a_i, b_i, c_i) be the label on v_i. Since all the v_i are neighbors of u, we must have $a_i \equiv 1\,(mod\,2)$ and $b_i \not\equiv 0\,(mod\,3)$. Hence we may assume without loss of generality that $b_1 \equiv b_2 \equiv 1\,(mod\,3)$. Now consider the label (x, y, z) on any child w of v_1. Since w is not adjacent to v_2, we must have $z \equiv c_2\,(mod\,p)$, and since w is not adjacent to v_3, we must either have $y \equiv b_3\,(mod\,3)$ or $z \equiv c_3\,(mod\,p)$. If the former is the case for all three choices of w, then since there are only two choices for (x, y, z) satisfying all the said conditions, we are led to a contradiction. Thus we must have $z \equiv c_2 \equiv c_3\,(mod\,p)$. Now consider a child w of v_2, and let (x, y, z) be its label. Since w is not adjacent to v_1, we must have $z \equiv c_1\,(mod\,p)$. Since w is not adjacent to v_3, we must have $y \equiv b_3\,(mod\,3)$. Finally, since $x \equiv 0\,(mod\,2)$, there are only two choices for the label, which is a contradiction since there are three choices for w.

Case 2: Suppose $r = 2 \cdot p_1 \cdot p_2$ for some primes p_1 and p_2. Again, we may assume that u is labeled $(0,0,0)$; let (a_i, b_i, c_i) denote the label on v_i where $i = 1,2,3$. If $b_2 \equiv b_3\,(mod\,p_1)$, consider the label (x, y, z) on any child of v_2. Any such vertex is not adjacent to v_3; hence we must have $z \equiv c_3\,(mod\,p_2)$. However, it is also not adjacent to v_1, so $y \equiv b_1\,(mod\,p_1)$. Since obviously we must have $x \equiv 0\,(mod\,2)$, the label is uniquely determined, and hence there are not enough labels of this type to assign to all three children of v_1. A similar argument applies if $c_2 \equiv c_3\,(mod\,p_2)$; hence we assume $b_2 \not\equiv b_3\,(mod\,p_1)$ and $c_2 \not\equiv c_3\,(mod\,p_2)$. Now consider the label (x, y, z) on any child of v_1. Such a vertex is adjacent neither to v_2 nor to v_3, so either $y \equiv b_2\,(mod\,p_1)$ and $z \equiv c_3\,(mod\ p_2)$ or $y \equiv c_2\,(mod\,p_1)$ and $z \equiv b_3\,(mod\,p_2)$. Since $x \equiv 0\,(mod\,2)$, there is not a unique choice for this label and we have a contradiction. $\square$

3 Connected Graphs

Theorem 3.1. *Let G be a graph with n vertices. With the exceptions of*

$$pdim\ (K_1) = 1, \qquad pdim\ (D_2) = 2, \qquad pdim\ (P_2) = 2,$$

$$pdim\ (K_2) = 2, \qquad pdim\ (D_3) = 2, \qquad pdim\ (K_2 + D_2) = 3,$$

and

$$pdim(K_n + K_1) = n \qquad \text{for } n \geq 2,$$

we always have

$$pdim\ (G) \leq n - 2.$$

Definition 3.2. *An equivalence on a graph G is a spanning subgraph of G whose components are complete graphs.*

Proposition 3.3. *[6, Proposition 8.4.7] The product dimension of a graph G is the minimum number of equivalences on $\overline{G}$, $E_1, \ldots, E_t$, such that $\bigcup E_i = \overline{G}$ and $\bigcap E_i = \emptyset$.*

Theorem 3.4. *Let G be a simple connected graph with r vertices. Then $rep(G)$ is $\leq \pi_{r-1}^{r-2}$, which is the product of the first $r-2$ primes starting at $r-1$.*

Proof. Let G be a simple connected graph of with r vertices. We know that $pdim(G) \leq r-2$. We can describe the set of equivalences $E_1, \ldots, E_{r-2}$. For each vertex, v, construct a product representation such that the *ith* coordinate of v contains an integer that corresponds to the component in E_i that contains v. Since this product representation has $pdim(G) \leq r-2$ and each coordinate has values between 0 and $r-2$, $rep(G) \leq \pi_{r-1}^{r-2}$.

Next, we will show that this upper bound is tight by considering a specific case. Let $G = K_{n-1} + K_1 + \{e\}$, so that K_{n-1} is connected to K_1 by a single edge, e. Let $V(K_{n-1}) = \{v_0, v_1, \ldots, v_{n-2}\}$ and $V(K_1) = \{w\}$. Without loss of generality, assume that e has endpoints w and v_0. Give v_i a coordinate representation of $(\underbrace{i, i, \ldots, i}_{n-2})$. Label w as $(1, 2, \ldots, n-2)$. This gives us a smallest product representation with pdim (G) $= n-2$. Therefore, $rep(G) = \pi_{r-1}^{r-2}$.

Thus, $rep(G) \leq \pi_{r-1}^{r-2}$ is a tight upper bound for connected graphs. $\square$

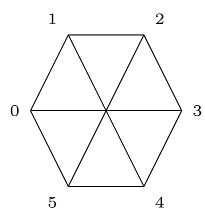
4 Harary Graphs

We will now look at a special type of Harary graph, $H_{3,2n}$, where 3 is the maximum degree and $2n$ is the number of vertices, with a specified construction. First we draw a $2n$-cycle. For each vertex, v_i, where $1 \leq i \leq n$, add an edge between v_i and v_{i+n}.

In this section, we describe representation numbers of $H_{3,6}, H_{3,8}$, and $H_{3,10}$, as well as a lower bound for the representation number of any $H_{3,2n}$.

4.1 Representation of $H_{3,6}$

Proposition 4.1. *$rep(H_{3,6}) = 8$.*



Proof. First, note that since $H_{3,6}$ is not a reduced graph, $rep(H_{3,6})$ is not necessarily square-free. Second, note that $H_{3,6}$ has a representation modulo 8. Third, note that $H_{3,6}$ has six vertices. Since each vertex needs a distinct label in a modular representation, $rep(H_{3,6}) \geq 6$.

If $rep(H_{3,6}) = 6$, then in any representation of $H_{3,6}$ modulo 6, some vertex must be labeled 0. Hence, its neighbors must receive labels from $\{1, 5\}$. However, every vertex in $H_{3,6}$ has three neighbors, so a unique labeling is impossible.

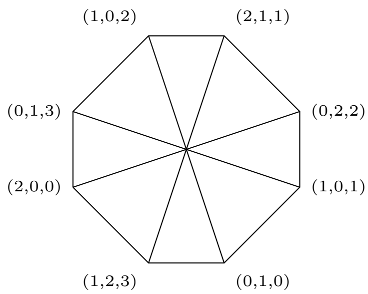
If $rep(H_{3,6}) = 7$, then $rep(H_{3,6})$ is a prime number. This, however, is not possible since the only graphs representable by prime numbers are complete graphs, and $H_{3,6}$ is not a complete graph.

Therefore, $rep(H_{3,6}) = 8$.

□

4.2 Representation of $H_{3,8}$

Proposition 4.2. *$rep(H_{3,8}) = 105$.*



Proof.

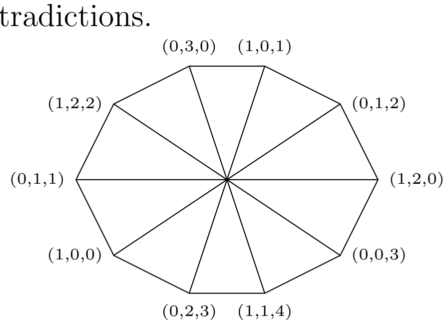
Let $P \subseteq H_{3,8}$, where P is a 5-cycle and an induced subgraph of $H_{3,8}$. From [5], $rep(P) = 105$. This gives a lower bound for $rep(H_{3,8})$. Since $H_{3,8}$ can be labeled with a representation modulo105, we also have an upper bound (see the figure above). So, $rep(H_{3,8}) = 105$.

□

4.3 Representation of $H_{3,10}$

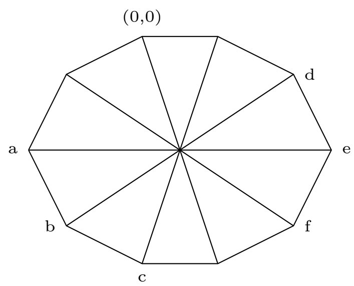
Proposition 4.3. *$rep(H_{3,10}) = 70$.*

Proof. We will prove that $rep(H_{3,10}) = 70$ in two steps. First, we will show that pdim $(H_{3,10}) \geq 3$. Then, we will show that the only two cases where pdim $(H_{3,10}) \geq 3$ and the representation number is less than 70 lead to contradictions.



Suppose not. That is, suppose $rep(H_{3,10}) < 70$.

Step 1. Suppose pdim $(H_{3,10}) < 3$. Without loss of generality, pick a vertex v and label it $(0, 0)$. There are six vertices not adjacent to vertex v. Each of the six vertices must have a 0 as one of its coordinates. Without loss of generality, vertices a, c, d, and f could have 0 in the first coordinate while vertices b and e have a 0 in the second coordinate (see figure below). Vertices b and e cannot have a 0 in the first coordinate because they are adjacent to vertices a, c, d and f. So, vertices b and e are forced to have a 0 in the second coordinate. In contrast, vertices a, c, d, and f could have a 0 in the second coordinate while vertices b and e have a 0 in the first coordinate. Then, if vertex a is assigned an x in its remaining coordinate, vertex b can be assigned a y in its other coordinate since it must be distinct from its neighbor vertex a. Vertex c can be assigned a z in its remaining coordinate, since it must be distinct from its neighbor vertex b and unique from vertex a. The label of vertex e must have a common coordinate with vertex c since they're not adjacent. Since vertices c and e have 0 in different coordinates, and this would force one of them to have the label $(0,0)$. Vertex v already has this label, so a labeling with two coordinates forces a contradiction. Therefore, pdim $(H_{3,10}) \geq 3$.



Step 2. For $i = 0, 1, 2, ..., 9$, let v_i denote the vertices of $H_{3,10}$ in a clockwise direction. Suppose the graph is represented by the product of $2 \cdot 3 \cdot p$ for some prime p, and assume, without loss of generality, that at least 4 of the vertices have 0 as a common second coordinate. Let v_0 be such a vertex and be labeled $(0,0,0)$, and v_i, v_j, and v_k be the other three. Now consider two subgraphs, one being $[v_0, v_1, v_2, v_3, v_4, v_5]$, and the other being $[v_0, v_5, v_6, v_7, v_8, v_9]$. One of these subgraphs contains at least two of v_i, v_j and v_k. Without loss of generality, suppose that v_0, v_2 and v_4 share a common second coordinate. This allows us to assume that either v_6 or v_8 also share this common second coordinate. Choose the coordinate to be v_6, so that v_i, $i = 0, 2, 4, 6$ can have the form $(0, 0, n)$ where n is some integer. Consider v_3. Since its neighbors are v_2 and v_4, it has the labeling $(1, 1, a)$, where a is some number not equal to the last coordinate of v_2 or v_4. Since v_3 is not adjacent to v_0 and v_6, both v_0 and v_6 must have the last coordinate labeled a, which leads to a contradiction as that would result in v_0 and v_6 not being unique. Therefore, $rep(H_{3,10}) = 70$.

$\square$

4.4 Lower Bound for $H_3, 2n$

Proposition 4.4. *$rep(H_{3,2n}) \geq rep(P_n)$*

Proof. Observe $H_{3,2n}$ has an induced subgraph P_n. It has been proven that $rep(P_n) = \pi_2^{\lceil log_2(n-1) \rceil}$, when $n \geq 5$. Therefore, if $n \geq 5$, $rep(H_{3,2n}) \geq \pi_2^{\lceil log_2(n-1) \rceil}$.

□

Proposition 4.5. *If $rep(C_{n+1})$ is known, $rep(H_{3,2n}) \geq rep(C_{n+1})$.*

Proof. Observe $H_{3,2n}$ has an induced subgraph C_{n+1}.

In most cases $rep(C_{n+1})$ is known, however there is one case where only an upper and lower bound exist for $rep(C_{n+1})$.

Case 1. Let $k = (n+1)/2$. When $n+1$ is even and $k \geq 3$, $rep(C_{n+1}) = \pi_2^{\lceil log_2(k-1) \rceil + 1}$ according to [5]. Thus, $rep(H_{3,2n}) \geq rep(C_{n+1}) = \pi_2^{\lceil log_2(k-1) \rceil + 1}$, when $k \geq 3$ and $n+1$ is even.

Case 2. Let $k = n/2$. When $n+1$ is odd, the representation numbers are completely determined by the next two formulas when k is odd or k is an odd power of 2.

In [5], if $k \geq 4$ and k is not a power of 2 then $rep(C_{2k+1}) = \pi_3^{\lceil log_3(k) \rceil}$. Thus, $rep(H_{3,2n}) \geq \pi_3^{\lceil log_3(k) \rceil}$ when k is not a power of 2 and is greater than 4.

When $2k+1 = 4^t + 1$ and $t \geq 2$, $rep(C_{2k+1}) = \pi_3^{\lceil log_3(2t+1) \rceil}$. Thus $rep(H_{3,2n}) \geq \pi_3^{\lceil log_3(2t+1) \rceil}$ when $t \geq 2$ and $2k+1 = 4^t + 1$.

Case 3. Let $k = n/2$. When k is an even power of 2, in [5] there is an upper and a lower bound on $rep(C_{2k+1})$. This lower bound is $\pi_3^{\lceil log_2(2k-1) \rceil}$. Then, $rep(H_{3,2n}) \geq \pi_3^{\lceil log_2(2k-1) \rceil}$ when $n+1 = 2k+1$ and k is an even power of 2.

□

5 Acknowledgments

We thank our seminar director, Dr. Reza Akhtar, for all his support, guidance, and time throughout this program. We would also like to thank our graduate assistant, Katherine Benson, for her encouragement, time, and countless explanations. Our appreciation also goes to all the faculty members who played a part in making SUMSRI an unforgettable experience. We also like to give thanks to the National Security Agency, the National Science Foundation, and Miami University for all their support of this program.

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Crystal Altamirano
Bryn Mawr College
Bryn Mawr, PA
caltamiran@brynmawr.edu

Stephanie Angus
Whittier College
Whittier, CA
sangus@poets.whittier.edu

Lauren Brown
Hanover College
Hanover, IN
brownl12@hanover.edu

Joseph Crawford
Morehouse College
Atlanta, GA
jcrawfo5@morehouse.edu

Laura Gioco
Fairfield University
Fairfield, CT
laura.gioco@student.fairfield.edu