The Varieties of One-Sided Loops of Bol-Moufang Type

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Abstract. We show that there are 20 different varieties of one-sided loops of Bol-Moufang type. We also show all inclusions between these varieties and give all necessary counterexamples. This work extends the research done by Phillips and Vojtěchovský which found the relationships among 26 different varieties of quasigroups and 14 different varieties of loops of Bol-Moufang type. We then show which varieties of one-sided loops of Bol-Moufang type have left and right inverse properties. The proofs and counterexamples are aided by the automated theorem prover, Prover9, and the finite model builder, Mace4.

1. Introduction

Bol-Moufang Identities

Let $*$ be a binary operation on a set G. Let $L_a : G \to G$ be defined by $L_a(x) = a * x$ and let $R_a : G \to G$ be defined by $R_a(x) = x * a$.

Definition 1. $(G, \cdot)$ is called a *quasigroup* if L_a and R_a are bijective.

Quasigroups have the cancellation property. Let $(G, \cdot)$ be a quasigroup with $x, y, z \in G$. Then if $x * z = y * z$, we have $x = y$, and if $z * x = z * y$, we have $x = y$.

Definition 2. A *loop* is a quasigroup $(G, \cdot)$ for which $*$ has a two-sided identity element e, such that $e * x = x * e = x$.

A left loop is a quasigroup $(G, \cdot)$ for which $*$ has a left identity element, and a right loop is a quasigroup $(G, \cdot)$ for which $*$ has a right identity element.

An identity is said to be of Bol-Moufang type if there are three variables on each side, with one of the variables occurring twice, and the other two occurring once. Furthermore, the order of the variables must be the same on both sides. For the rest of the paper, all identities of Bol-Moufang type are left or right loop identities with $\cdot$ as the binary operation.

Prover9 and Mace4

In our research we made considerable use of the automated theorem prover, Prover9[1] to assist us in finding implications between identities of Bol-Moufang type. In addition, our counterexamples were discovered using the finite model builder, Mace4[1]. The proofs in this paper are "humanized" proofs generated by Prover9. However, some proofs generated were too long to humanize and we have not succeeded in finding shorter and simpler versions of these proofs.

Our assumptions, when using Prover9, included the identities of an equational quasigroup.

Definition 3. An *equational quasigroup* is a set G together with three binary operations, $*, /, \backslash$ satisfying for all $x, y, \in G$,

$$x * (x\backslash y) = y, \quad x\backslash(x * y) = y, \quad (x * y)/y = x, \quad (x/y) * y = x.$$

Inspiration

Our work builds on that of Phillips and Vojtěchovský[2] who classified varieties of quasigroups and loops defined by identities of Bol-Moufang type. Phillips and Vojtěchovský created Hasse diagrams which showed the varieties of loops and quasigroups of Bol-Moufang type. We extend their work to create Hasse diagrams of the varieties of left and right loops of Bol-Moufang type.

Notation

In our research, we adopt the notation used by Phillips and Vojtěchovský[2] for the identification of identitites of Bol-Moufang type.

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	o(o(oo))
2	o((oo)o)
3	(oo)(oo)
4	(o(oo))o
5	((oo)o)o

For example, C34 is the identity $(xy)(yz) = (x(yy))z$.

For ease of reference, in the appendix we have included the Hasse diagrams of varieties of quasigroups (Figure 2) and loops (Figure 1) of Bol-Moufang type discovered by Phillips and Vojtěchovský.

2. Varieties of Left Loops of Bol-Moufang Type

Each of the boxes of the Hasse diagram in Figure 3 (Appendix) respresents a distinct variety of left loops. Only the propostions and proofs below (Propositions 1 - 10) are needed to show this. From Figures 1 and 2 (Appendix) we can identity relationships of quasigroups and loops of Bol-Moufang type, eliminating the need to test the relationships between many of the varieties. For example, after determining that LAQ and RNQ are varieties of left loops, we may want to check if one implies the other. Looking at Figure 1 (Appendix) we know that they are distinct varieties of loops, neither of which imply the other. Therefore, in a left loop, the varieties must remain distinct, without implications. Using Figures 1 and 2 (Appendix), this was the general strategy we used to create the Hasse Diagram to represent the varieties of left loops of Bol-Moufang type.

Proposition 1. *Each identity in* GR *in Figure 3 (Appendix) defines the same variety of left loops.*

Proof. The identites listed under GR in Figure 3 of the Phillips and Vojtěchovský paper [2] are all equivalent in quasigroups, and are therefore equivalent in left loops. In addition to these identities, D14, E14, and F34 are also equivalent identities. From Figure 2 (Appendix), A12 $\Rightarrow$ D14, E14, and F34. We will show that D14 $\Rightarrow$ A12, E14 $\Rightarrow$ A12, F34 $\Rightarrow$ A12. We must show that D14, E14, and F34 have associativity.

Assume $(G, \cdot)$ is a left loop. We first show that D14 $\Rightarrow$ A12. Assume D14, that is:

$$x(y(zx)) = (x(yz))x. \tag{1}$$

We will show D14 has associativity.
By substituting y/z for y in (1),

$$x((y/z)(zx) = (x((y/z)z))x = (xy)x. \tag{2}$$

By substituting (x, e, y) for (x, y, z) in equation (1) we obtain

$$x(e(yx)) = (x(ey))x$$

Thus,

$$x(yx) = (xy)x. \tag{3}$$

Combining equations (2) and (3) gives us

$$x((y/z)(zx)) = x(yx). \tag{4}$$

Cancellation gives us

$$(y/z)(zx) = yx. \tag{5}$$

By substituting (z, x, y) for (x, y, z) in equation (5) we get $(x/y)(yz) = xz$. Rewriting x as $(x/y)y$ on the right hand side, we get

$$(x/y)(yz) = ((x/y)y)z. \tag{6}$$

Substitute x for x/y in equation (6) to get $x(yz) = (xy)z$.
Therefore we have associativity. Hence, D14 $\Rightarrow$ A12.

We now show that E14 $\Rightarrow$ A12. Assume E14, that is:

$$x(y(zy)) = (x(yz))y. \tag{7}$$

We will show E14 has associativity..
Substituting $(z, z\backslash y)$ for (y, z) in (7):

$$x(z((z\backslash y)z)) = (x(z(z\backslash y)))z = (xy)z. \tag{8}$$

Substituting (e, x) for (x, z) into equation (8) we obtain $(ey)x = e(x((x\backslash y)x))$. Because e is a left identity, we have

$$yx = x((x\backslash y)x). \tag{9}$$

Using equations (8) and (9),

$$(xy)z = x(z((z\backslash y)z)) = x(yz)$$

This shows that E14 has associativity. Hence, E14 $\Rightarrow$ A12.

We now show that F34 $\Rightarrow$ A12. Assume

$$(xy)(zz) = (x(yz))z. \tag{10}$$

We will show $x(x(yz)) = x((xy)z)$.
By equation (10) and the equational quasigroup identity, $(xy)/y = x$, we obtain

$$((xy)(zz))/z = ((x(yz))z)/z = x(yz). \tag{11}$$

By substituting (e, x, y) for (x, y, z) into equation (10) we get $(ex)(yy) = (e(xy))y$. Because e is a left identity, we have

$$x(yy) = (xy)y. \tag{12}$$

Notice, this is the right alternative property.
Using equation (12) and the equational quasigroup identity, $(xy)/y = x$, we have

$$(x(yy))/y = ((xy)y)/y = xy. \tag{13}$$

Substituting (xy, z) for (x, y) in (13) gives us $((xy)(zz))/z = (xy)z = x(yz)$ and therefore we have associativity. Hence F34 $\Rightarrow$ A12.

Therefore D14, E14, F34 $\Rightarrow$ A12. Hence, D14, E14, F34, A12, and all other identities listed under GR are equivalent in a left loop.

$\square$

Proposition 2. *Each identity in RC1,* F15 *and* F23 *,defines the same variety of left loops.*

Proof. We will first show that F15 $\Rightarrow$ F23.
Assume $(G, \cdot)$ is a left loop. Assume

$$x(y(zz)) = ((xy)z)z. \tag{14}$$

We will show that $x((yz)z) = (xy)(zz)$.
We substitute (e, x, y) for (x, y, z) in equation (14) to get $e(x(yy)) = ((ex)y)y$. Since e is a left identity, we have

$$x(yy) = (xy)y. \tag{15}$$

By equation (15) we have $(xy)(zz) = ((xy)z)z$.
Therefore, by (14), $x((yz)z) = (xy)(zz)$.
We know from [2] that F23 $\Rightarrow$ F15 for quasigroups. Therefore, F15 and F23 are equivalent RC1 identities.

□

Propostion 3. *LBQ $\Rightarrow$ LAQ in a left loop.*

Proof. To show LBQ $\Rightarrow$ LAQ, we must show that B14 $\Rightarrow$ A13. Assume $(G, \cdot)$ is a left loop and

$$x(y(xz)) = (x(yx))z. \tag{16}$$

We will show that $x(x(yz)) = (xx)(yz)$.
Substituting (x, e, y) for (x, y, z) in equation (16) we obtain $x(e(xy)) = (x(ex))y$. Since e is a left identity, we have

$$x(xy) = (xx)y. \tag{17}$$

Notice, this is the left alternative property.
Then, by equation (17), we have $(xx)(yz) = x(x(yz))$. Therefore, B14 $\Rightarrow$ A13. Hence, LBQ $\Rightarrow$ LAQ in a left loop.

□

Proposition 4. *CQ $\Rightarrow$ LC1 in a left loop.*

Proposition 5. *CQ $\Rightarrow$ RC1 in a left loop.*

Proposition 6. *LC4 and LC1 are equivalent in a left loop.*

Proposition 7. *RC4 and RC2 are equivalent in a left loop.*

Note. Propositions 4 through 7 were proved using Prover9. We have not succeeded in finding shorter and simpler versions of these proofs.

Proposition 8. *RG1, RG2, and RG3 $\not\Rightarrow$ GR, RC2 and RBQ $\not\Rightarrow$ RAQ, and RC2 $\not\Rightarrow$ RC1, MNQ, or RNQ in a left loop.*
Example.

0	1	2
2	0	1
1	2	0

For RG1, RG2, and RG3 $\not\Rightarrow$ GR, let $x = 1, y = 0, z = 2$. Then $(xy)z \neq x(yz)$.
For RC2 and RBQ $\not\Rightarrow$ RAQ, let $x = 0, y = 1, z = 2$. Then $(x(yy))z \neq ((xy)y)z$.
For RC2 $\not\Rightarrow$ RC1, let $x = 0, y = 1, z = 2$. Then $x((yz)z) \neq (xy)(zz)$.
For RC2 $\not\Rightarrow$ MNQ, let $x = 0, y = 1, z = 2$. Then $x((yy)z) \neq (x(yy))z$.
For RC2 $\not\Rightarrow$ RNQ, let $x = 2, y = 0, z = 1$. Then $x(y(zz)) \neq (xy)(zz)$.

Proposition 9. *LC2 $\not\Rightarrow$ LC3 or LC1 in a left loop.*
Example.

0	1	2	3
2	0	3	1
1	3	0	2
3	2	1	0

For LC2 $\not\Rightarrow$ LC3, let $x = 1, y = 2, z = 3$. Then $x(x(yz)) \neq ((xx)y)z$.
For LC2 $\not\Rightarrow$ LC1, let $x = 1, y = 2, z = 3$. Then $(xx)(yz) \neq (x(xy))z$.

Proposition 10. *LC3 $\not\Rightarrow$ LC1 or LC2 in a left loop.*
Example.

0	1	2	3	4
2	3	1	4	0
3	0	4	2	1
4	2	0	1	3
1	4	3	0	2

For LC3 $\not\Rightarrow$ LC1, let $x = 1, y = 2, z = 3$. Then $(xx)(yz) \neq (x(xy))z$.
For LC3 $\not\Rightarrow$ LC2, let $x = 2, y = 3, z = 4$. Then $x(x(yz)) \neq (x(xy))z$.

3. Varieties of Right Loops of Bol-Moufang Type

Definition 4. The *dual* of an identity I, called $I^\vee$, is obtained by reading the identity backwards. That is, reading from right to left.

Proposition 11. *Let I,J be identities. If $I \Rightarrow J$ holds for left loops then $I^\vee \Rightarrow J^\vee$ holds for right loops.*

Proof. Assume $(G,\cdot)$ is a left loop. By the definition of duality, we know that I holds in $(G,\cdot)$ iff $I^\vee$ holds in $(G,\circ)$.

Assume $I \Rightarrow J$ in $(G,\cdot)$. We will show that $I^\vee \Rightarrow J^\vee$ in $(G,\circ)$.

If $I^\vee$ holds in $(G,\circ)$, then I holds in $(G,\cdot)$. By our assumption we know that J holds in $(G,\cdot)$ which implies that $J^\vee$ holds in $(G,\circ)$. Hence $I^\vee \Rightarrow J^\vee$. Therefore $I^\vee \Rightarrow J^\vee$ holds for right loops.

$\square$

Because of our proof of Propostion 11, the following rules apply:

$$A^\vee = F, \quad B^\vee = E, \quad C^\vee = C, \quad D^\vee = D, \quad 1^\vee = 5, \quad 2^\vee = 4, \quad 3^\vee = 3.$$

Because of duality, we can easily obtain the Hasse diagram for the varieties of right loops of Bol-Moufang type (Appendix, Figure 4).

4. Right and Left Inverse Properties

Definition 5. A quasigroup $(G,\cdot)$ has the

(1) *Left Inverse Property (LIP)* if there exists a bijection $J_\lambda : G \to G$ such that for all $x \in G$

$$J_\lambda(a)\cdot(a\cdot x) = x.$$

(2) *Right Inverse Property (RIP)* if there exists a bijection $J_\rho : G \to G$ such that for all $x \in G$

$$(x\cdot a)\cdot J_\rho(a) = x.$$

(3) *Two-Sided Inverse Property (IP)* if G has both LIP and RIP.

To show a left loop has LIP, it is enough to show that there exists a map J_λ such that $J_\lambda(a)\cdot(a\cdot x) = x$ holds for all x. Bijectivity follows by the quasigroups axioms as shown below.

Injectivity: Let $J_\lambda(a) = J_\lambda(b)$. Then $J_\lambda(a)\cdot(a\cdot x) = J_\lambda(b)\cdot(b\cdot x)$. By cancellation, $a\cdot x = b\cdot x$ and therefore $a = b$.

Surjectivity: Let $b \in G$. Fix some $g \in G$. Then $b\cdot(y\cdot g) = g$ has a unique solution y since we are in a quasigroup. Then we also know that $J_\lambda(y)\cdot(y\cdot g) = g$. Therefore, $J_\lambda(y) = b$.

An analogous argument holds for RIP.

For the following propostions, we have included Hasse diagrams displaying the inverse properties of one-sided loops in Figures 5 and 6 (Appendix).

Proposition 12. *The left loops defined by RBQ and RC2 have RIP.*

Proof. Assume $(G,\cdot)$ is a left loop. We will first show that the left loops in RBQ have RIP. Assume

$$x((yz)y) = ((xy)z)y. \tag{18}$$

Substitute x/y for x in (18):

$$(((x/y)y)z)y = (x/y)((yz)y) = (xz)y. \tag{19}$$

Assume, by way of contradiction, that $(f(x)c)x \neq f(x)$.
This means f is a function of x, that is, for some c, there is a particular value for x such that $(f(x)c)x \neq f(x)$. This is the negation of RIP.
By equation (19), we have that

$$(f(x)/x)((xc)x) = (f(x)c)x \neq f(x). \tag{20}$$

By equations (19) and (20) it follows that

$$(f(xy)/(xy))((x/c)((cy)c)(xy)) \neq f(xy). \tag{21}$$

Then, it follows that $(f(x(c\backslash y))/(x(c\backslash y)))(((x/c)(yc))(x(c\backslash y))) \neq f(x(c\backslash y))$. Substituting w for $x(c\backslash y)$ in the previous equation gives us that $x = w/(c\backslash y)$. Then

$$(f(w)/w)(((w/(c\backslash y))/c)(yc))(w) \neq f(w). \tag{22}$$

Now let $y = e$. Then

$$(f(w)/w)(((w/(c\backslash e))/c)(ec))(w) \neq f(w).$$

Substitute $c\backslash e$ for w to obtain $(f(c\backslash e)/(c\backslash e))((((c\backslash e)/(c\backslash e))/c)(ec))(c\backslash e) = (f(c\backslash e)/(c\backslash e))(c\backslash e) \neq f(c\backslash e)$.
By letting $f(c\backslash e) = x$ and $c\backslash e = y$ in the previous equation we get $(x/y)y \neq x$, contradicting the definition of an equational quasigroup.
Therefore, E25 has the right inverse property. Hence, the left loops in the variety RBQ have RIP.

We will now show that the left loops in the variety RC2 have RIP. Assume

$$x((yy)z) = ((xy)y)z. \tag{23}$$

Using two equational quasigroup assumptions, we get

$$(x/y)\backslash x = y. \tag{24}$$

By equation (23) we get

$$x((yy)(((xy)y)\backslash z)) = ((xy)y)((xy)y)\backslash z = z. \tag{25}$$

Using equation (25), we have $(x((yy)(((xy)y)\backslash z)))\backslash x = (yy)(((xy)y))\backslash z = z\backslash x$. Substituting (y,x) for (x,y) gives $(xx)(((yx)x)\backslash z) = y\backslash z$.
Now substituting x/y for y, and y for x, we get

$$(yy)(((x/y)y)y)\backslash z = ((yy)(xy))\backslash z = (x/y)\backslash z. \tag{26}$$

We now can substitute (y,x,y) for (x,y,z) in equation (26) and use equation (24) to obtain

$$((xx)(yx))\backslash y = (y/x)\backslash y = x.$$

Multiply on the left by u/x to obtain $(u/x)(((xx)(yx))\backslash y) = (u/x)((yx)\backslash y) = (u/x)x = u$. It follows that $x = (xy)((zy)\backslash z)$. Therefore, C25 has the right inverse property. Hence, the left loops in the variety RC2 have RIP.

□

Proposition 13. *The left loops defined by RG2, RG3, LC2, LC3, and LBQ have LIP.*

Proof. Assume $(G,\cdot)$ is a left loop. We will first show that RG2 has the left inverse property in a left loop. Assume

$$x((xy)z) = (xx)(yz). \tag{27}$$

Then

$$x\backslash((xx)(yz)) = x\backslash(x((xy)z)) = (xy)z. \tag{28}$$

Substituting (e,y) for (y,z) into equation (27) we have $(xx)(ey) = x((xe)y)$
Since e is a left identity we get

$$(xx)y = x((xe)y). \tag{29}$$

Left dividing by x we get

$$x\backslash((xx)y) = (xe)y. \tag{30}$$

Using equations (27) and (30) and making the substitution yz for y gives us

$$(xe)(yz) = x\backslash((xx)(yz)) = x\backslash x((xy)z) = (xy)z. \tag{31}$$

By way of contradiction, assume $x(cf(x)) \neq f(x)$, similar to the proof of Proposition 12. Using equation (31) and substituting xe for x gives

$$f(xe) \neq (xe)(cf(xe)) = (xc)f(xe). \tag{32}$$

Now substitute x/c for x. This gives us $f((x/c)e) \neq ((x/c)c)f((x/c)e))) = x(f((x/c)e))$. Let $x = e$. Then $e(f((e/c)e)) \neq f((e/c)e)$. This contradicts our left identity assumption, and therefore the left loops in RG2 have LIP.

Now we will show that the left loops in RG3 have LIP. By duality, it suffices to show that LG3 has the right inverse property in a right loop. Assume $(G, \cdot)$ is right loop. Assume

$$x(y(zy)) = (x(yz))y. \tag{33}$$

Substitute $(x, z, (z\backslash y))$ for (x, y, z) to get

$$(x(z(z\backslash y)))z = x(z((z\backslash y)z)) \iff (xy)z = x(z((z\backslash y)z)). \tag{34}$$

Now left divide by x to get

$$x\backslash((xy)z) = z((z\backslash y)z). \tag{35}$$

By way of contradiction, assume $(f(x)c)x \neq f(x)$.
Using equation (34) and substituting $(f(x), c, x)$ for (x, y, z) gives us

$$(f(x)c)x = (f(x))(x((x\backslash c)x)) \neq f(x) \iff (f(x))(x((x\backslash c)x)) \neq f(x). \tag{36}$$

Using equation (35) we have $(f(x))((y\backslash((yc)x))) = (f(x))(x((x\backslash c)x)) \neq f(x)$.
Now, substituting y/c for y we get

$$(f(x))((y/c)\backslash((y/c)c)x) = (f(x))((y/c)\backslash(yx)) \neq f(x). \tag{37}$$

Substitute $(x\backslash y, x, c)$ for (x, y, c) in equation (37) to get $(f(x\backslash y))((x/c)\backslash(x(x\backslash y))) = (f(x\backslash y))((x/c)y) \neq f(x\backslash y)$. Substitute $(x, ((x/c)y), c)$ for (x, y, c) in the previous equation to obtain $(f(x\backslash((x/c)y)))((x/c)\backslash((x/c))y) = (f(x\backslash((x/c)y)))y \neq f(x\backslash((x/c)y))$. Simplifying gives us

$$(f(x\backslash((x/c)y)))y \neq f(x\backslash((x/c)y)). \tag{38}$$

Finally, let $y = e$.

$$f(x\backslash(x/c)) \neq f(x\backslash(x/c)).$$

This contradiction gives us that LG3 has the right inverse property in a right loop. Hence the left loops in the variety RG3 have LIP.

Now we will show that the left loops in LC2 have LIP. By duality it suffices to show that the right loops in RC2 have RIP. Assume $(G, \cdot)$ is a right loop. Assume

$$x((yz)z) = ((xy)z)z. \tag{39}$$

Thus,

$$(x((yz)z))/z = (((xy)z)z)/z = (xy)z. \tag{40}$$

Substituting (x, e, y) for (x, y, z) into our assumption, and since e is our right identity we have

$$((xe)y)y = x((ey)y) \iff (xy)y = x((ey)y). \tag{41}$$

Now substitute $x/((ey)y)$ for x into equation (41).
This gives $(x/(((ey)y))y)y = (x/((ey)y))((ey)y) = x$.
Right divide by y twice to get

$$x/((ey)y) = (x/y)/y. \tag{42}$$

Renaming the variables, we get $y/((ez)z) = (y/z)/z$.
Using equation (40) gives us $(x(y/((ez)z)))z = (x(((y/((ez)z)))z)z)/z$.
Substituting $(y/z)/z$ for $y/((ez)z)$ and simplifying using the definition of $/$, we obtain $(x(((y/z)/z)z)z)/z = (xy)/z$.
Therefore,

$$(x(y/((ez)z)))z = (xy)/z. \tag{43}$$

Substitute y for z in equation (43) to get

$$(xy)/y = (x(y/((ey)y)))y)y \iff x = (x(y/((ey)y)))y)y.$$

Right divide by y to obtain $x/y = (x(y/((ey)y))y)$.
Now right dividing by $y/((ey)y)$ gives $(x/y)/(y/((ey)y)) = x = (x/y)y$.
Substitute x for x/y. This gives $x/(y/((ey)y)) = xy$.
Multiplication gives us

$$(xy)(y/((ey)y)) = x.$$

This shows we have a right inverse and therefore the right loops in RC2 have RIP. Hence the left loops in LC2 have LIP.

Now we will show that the left loops in LC3 have LIP. Assume $(G, \cdot)$ is a left loop. Assume

$$x(x(yz)) = ((xx)y)z. \tag{44}$$

By substituting $(x, (xx)\backslash y, z)$ for (x, y, z) we get

$$x(x((xx\backslash y)z)) = ((xx)(xx\backslash y))z = yz. \tag{45}$$

Left divide by x to obtain

$$x\backslash(yz) = x\backslash(x((xx\backslash y)z)) \iff x\backslash(yz) = x((xx\backslash y)z). \tag{46}$$

Substitute x for y into equation (46) to get

$$x\backslash(xy) = y = x((xx\backslash x)y) \iff x\backslash y = ((xx)\backslash x)y.$$

Now substitute xy for y to get $x\backslash(xy) = ((xx)\backslash x)(xy)$.
Therefore,

$$y = ((xx)\backslash x)(xy).$$

Therefore we have a left inverse and the left loops in LC3 have LIP.

Now, we will show that the left loops in LBQ have LIP. By duality it suffices to show that the right loops in RBQ have RIP. Assume $(G, \cdot)$ is a right loop. Assume

$$x((yz)y) = ((xy)z)y. \tag{47}$$

Substitute x/y for x in equation (47) to get

$$(x/y)((yz)y) = (((x/y)y)z)y = (xz)y. \tag{48}$$

Now, substitute (e, y) for (y, z) in equation (47) to obtain

$$((xe)y)e = x((ey)e) \iff xy = x(ey) \iff y = ey. \tag{49}$$

Notice we now have a two-sided identity element, e.
Substitute (e, x, y) for (x, y, z) in equation (48) to get

$$(ey)x = (e/x)((xy)x) \iff yx = (e/x)((xy)x). \tag{50}$$

Substituting $x\backslash y$ for y gives us $(x\backslash y)x = (e/x)((x(x\backslash y))x) = (e/x)(yx)$.
Now right divide by x to obtain

$$x\backslash y = ((e/x)(yx))/x. \tag{51}$$

By way of contradiction, assume $(f(x)c)x \neq f(x)$. Substitute $(f(x), x, c)$ for (x, y, z) in equation (48). Then

$$((f(x))c)x = (f(x)/x)((xc)x) \neq f(x). \tag{52}$$

Substitute $f(xy)$ for $f(x)$ into equation (52) to get

$$(f(xy)/(xy))(((xy)c)(xy)) = (f(xy))/(xy)((x/c)((cy)c))(xy) \neq f(xy).$$

Substitute $c\backslash y$ for y. This gives $f(x(c\backslash y)) \neq (f(x(c\backslash y)))/(x(c\backslash y))((x/c))((c(c\backslash y))c)(x(c\backslash y)) \iff$

$f(x(c\backslash y)) \neq (f(x(c\backslash y)))/(x(c\backslash y))((x/c))((yc))(x(c\backslash y))$.
Now let $x(c\backslash y) = x$. We have $(f(x)/x)((x/(c\backslash y))/c(yc))x \neq f(x)$.
Now let $y = e$. Then

$$(f(x)/x)((x/(c\backslash e))/c(ec))x \neq f(x) \iff (f(x)/x)((x/(c\backslash e))x \neq f(x). \tag{53}$$

Using equation (51) we obtain $c\backslash e = ((e/c)(ec))/c = ((e/c)(c))/c = e/c$.
Rewriting equation (53) with $c\backslash e = e/c$ we have

$$(f(x)/x)((x/(e/c))x \neq f(x). \tag{54}$$

Now substitute e/c for x in equation (55) to get $(f(e/c)/(e/c))(((e/c)/(e/c)))(e/c) \neq f(e/c) \iff (f(e/c)/(e/c))(e(e/c)) \neq f(e/c) \iff (f((e/c))/(e/c))(e/c) \neq f(e/c)$.
This contradicts the equational quasigroup assumption $(x/y)y = x$. Therefore,the right loops in RBQ have RIP. Hence, the left loops in LBQ have LIP.

□

Propostion 14. *The left loops defined by FQ, MNQ, and LNQ do not have LIP.*
Example.

0	1	2	3	4
1	0	3	4	2
2	4	0	1	3
3	2	4	0	1
4	3	1	2	0

Propostion 15. *The left loops defined by RC1 and RC2 do not have LIP.*
Example.

0	1	2	3	4	5
1	0	3	5	2	4
2	5	0	4	1	3
3	4	1	0	5	2
4	3	5	2	0	1
5	2	4	1	3	0

Propostion 16. *The left loops defined by RBQ does not have LIP.*
Example.

0	1	2	3	4	5	6	7
1	0	3	5	2	4	7	6
2	3	0	1	6	7	4	5
3	2	1	7	0	6	5	4
4	5	6	0	7	1	2	3
5	4	7	6	1	0	3	2
6	7	4	2	5	3	0	1
7	6	5	4	3	2	1	0

Propostion 17. *The left loops defined by LAQ does not have LIP.*
Example.

0	1	2	3	4	5
1	2	3	5	0	4
2	3	5	4	1	0
3	0	4	2	5	1
4	5	1	0	2	3
5	4	0	1	3	2

For propositions 14 through 17, the element that does not have a left inverse is 1.

Propostion 18. *The left loops defined by RG2 does not have RIP.*
Example.

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

Propostion 19. *The left loops defined by RG3 does not have RIP.*
Example.

0	1	2	3	4	5
2	0	4	5	3	1
3	4	5	0	1	2
1	5	0	4	2	3
4	2	3	1	5	0
5	3	1	2	0	4

For propositions 18 and 19, the element that does not have a right inverse is 0.

Propostion 20. *The left loops defined by LC1 does not have RIP.*
Example.

0	1	2	3	4	5
1	0	3	2	5	4
2	4	0	5	1	3
3	5	4	0	2	1
4	3	5	1	0	2
5	2	1	4	3	0

Propostion 21. *The left loops defined by FQ, MNQ, RNQ do not have RIP.*
Example.

0	1	2	3	4
1	0	3	4	2
2	4	0	1	3
3	2	4	0	1
4	3	1	2	0

Propostion 22. *The left loops defined by LBQ does not have RIP.*
Example.

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	5	1	7	0	6	2	4
4	2	6	0	7	1	5	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

Propostion 23. *The left loops defined by RAQ does not have RIP.*
Example.

0	1	2	3	4	5
1	2	4	5	0	3
2	4	5	1	3	0
3	0	1	2	5	4
4	5	3	0	2	1
5	3	0	4	1	2

For propositions 20 through 23, the element that does not have a right inverse is 1.

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Appendix

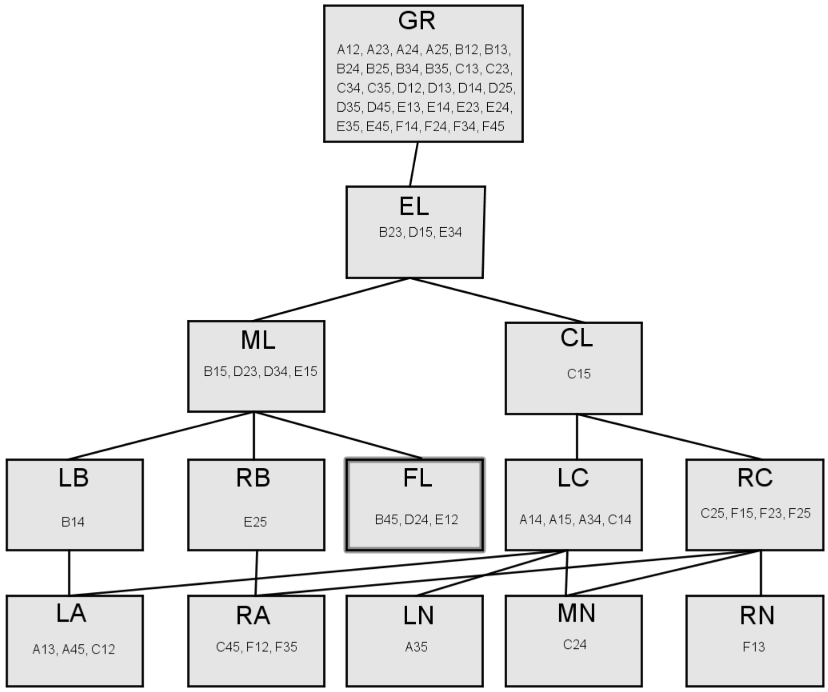

Figure 1: Varieties of Loops of Bol-Moufang Type

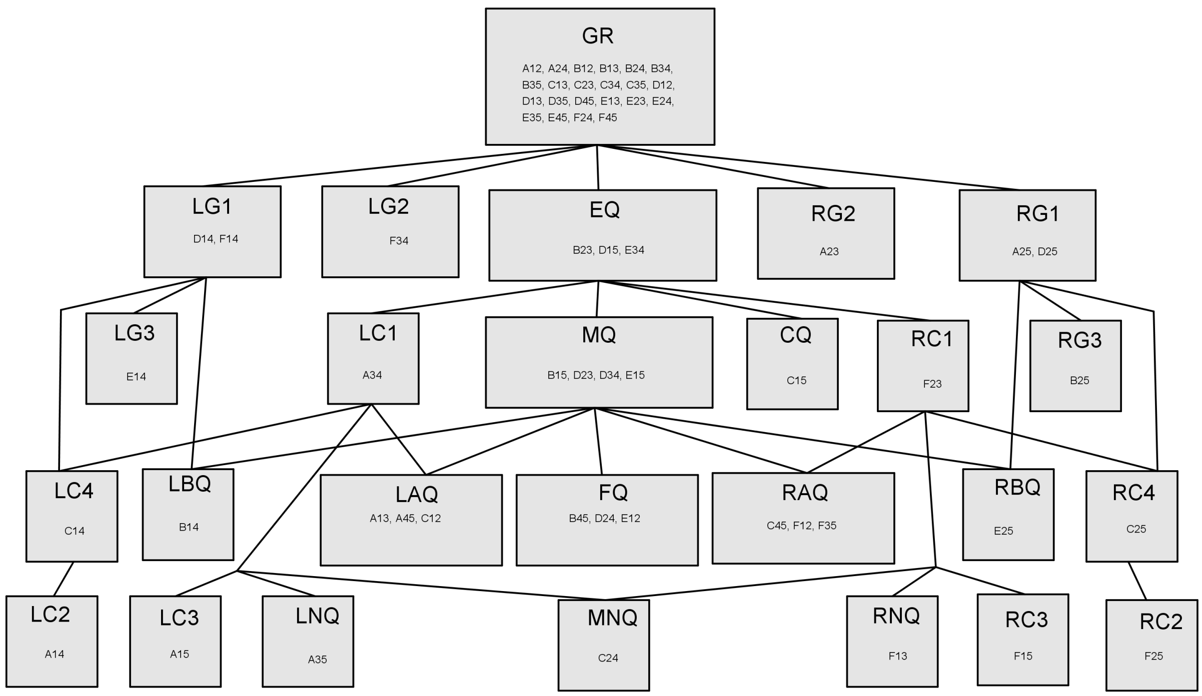

Figure 2: Varieties of Quasigroups of Bol-Moufang Type

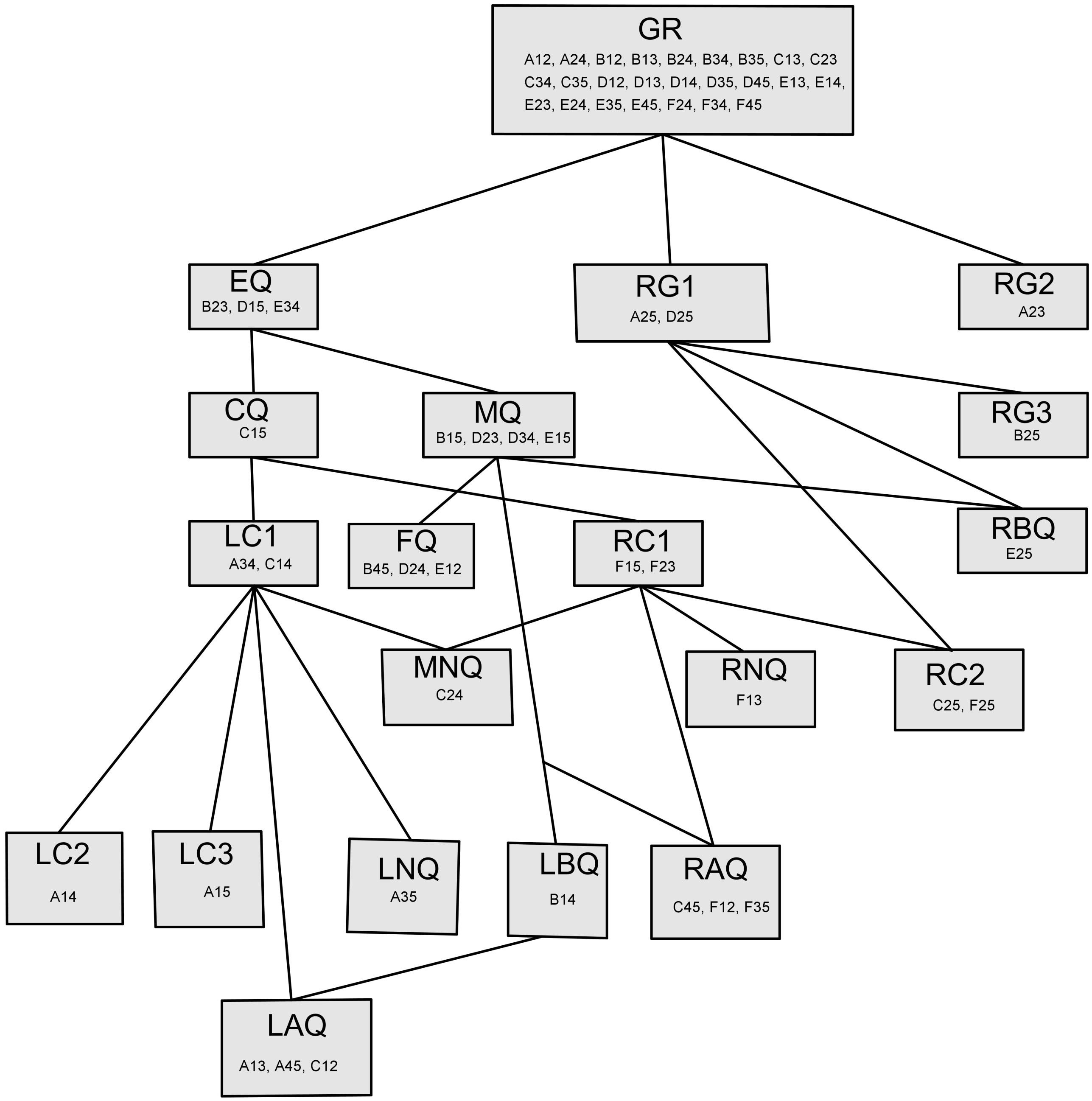

Figure 3: Varieties of Left Loops of Bol-Moufang Type

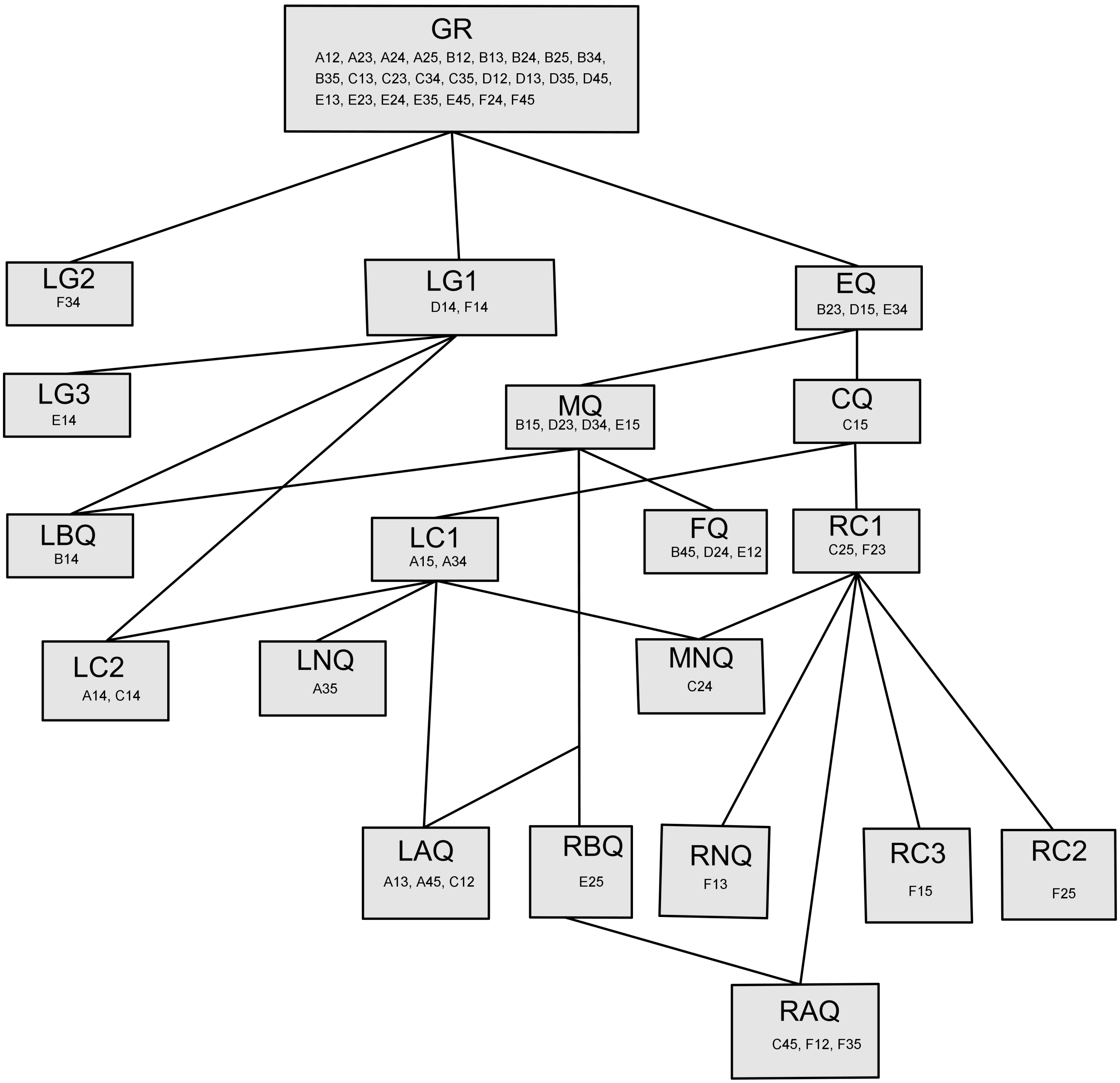

Figure 4: Varieties of Right Loops of Bol-Moufang Type

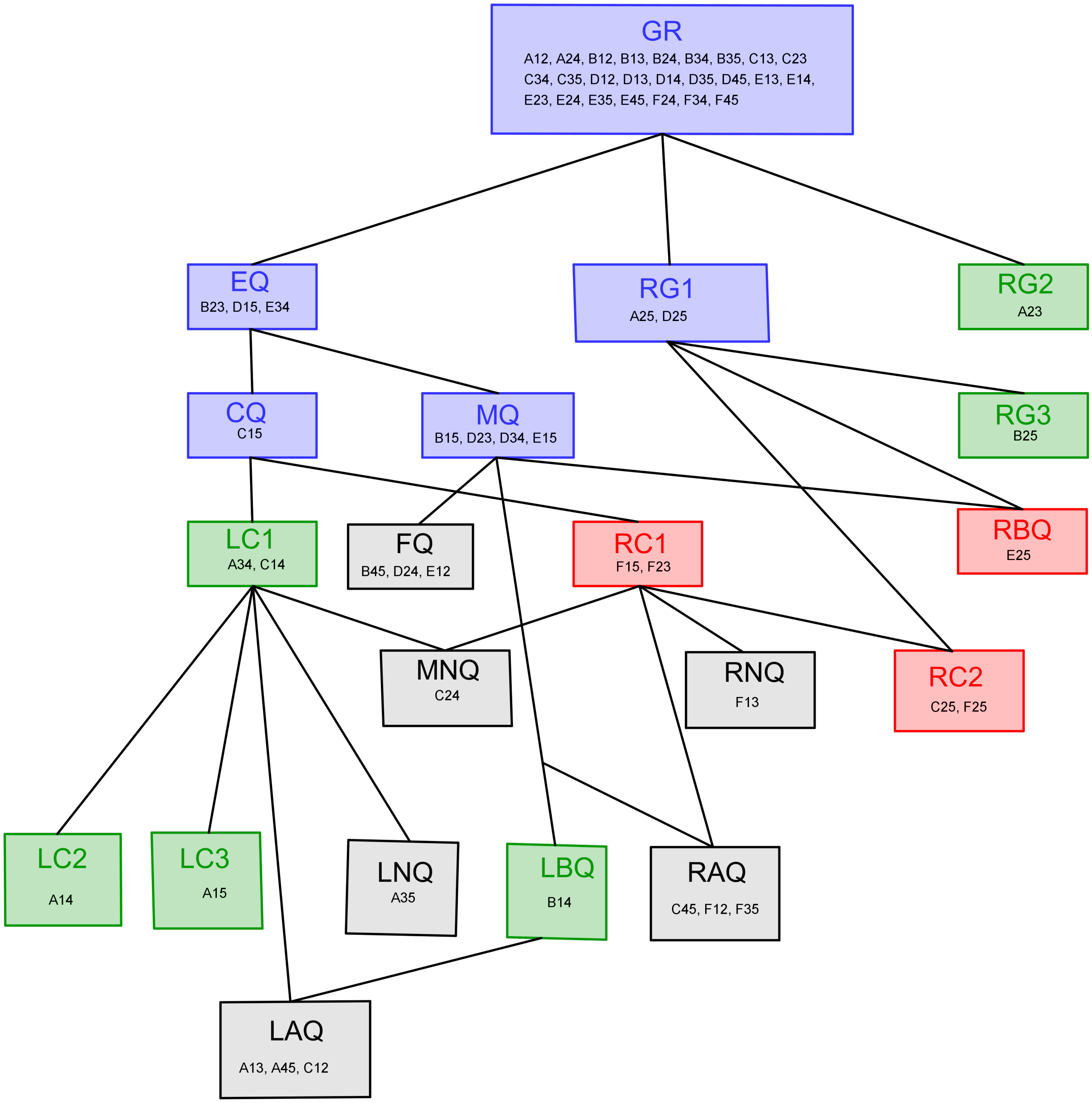

Figure 5: Varieties of Left Loops of Bol-Moufang Type with Inverse Properties

The varieties colored red represent those with RIP. The varieties colored green represent those with LIP. The varietes colored blue represent those with IP.

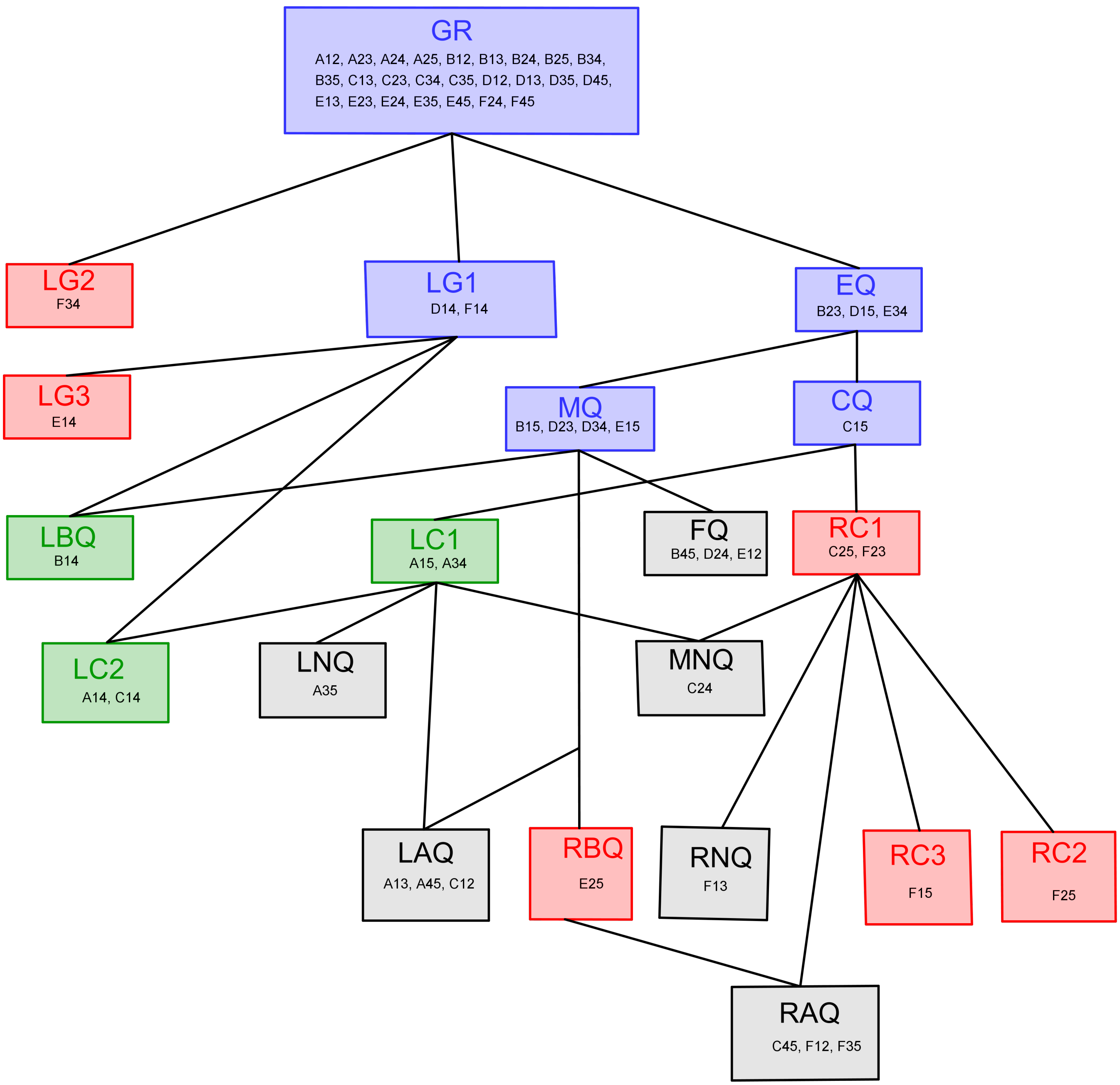

Figure 6: Varieties of Right Loops of Bol-Moufang Type with Inverse Properties

The varieties colored red represent those with RIP. The varieties colored green represent those with LIP. The varietes colored blue represent those with IP.

References

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