QUASI-AUTOMORPHISM GROUPS OF TYPE F_∞

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1. Introduction

It was shown by Ken Brown in 1987 that R. Thompson's groups F, T, and V are of type F_∞ [1], and thus have finite group presentations. More recently, Nucinkis and St. John-Green studied the quasi-automorphism variations of the groups, QF, QT, and QV, showing that the group QF is of type F_∞ as well as giving its finite presentation. They also introduce the groups $\widetilde{Q}T$ and $\widetilde{Q}V$, the disjoint union of $\tau_{2,c}$ and a single vertex ζ, which are also shown to be of type F_∞ and presented finitely [3].

This paper aims to show the property of F_∞ for the group QV as a preliminary result. The proof of the result for QT, as well as applying the method to confirm the property for QF, are expected to follow.

In section 3 we will prove directly, using the strategy of Ken Brown and identification with diagram groups, that the group QV is of type F_∞. We analyze the descending link at each vertex.

Following the demonstration that the groups are of type F_∞, we will use Brown's finiteness condition to prove that QF, QT, and QV have finite group presentations.

2. Background

2.1. **Descending Links.** We will begin this section by defining the most important element of the argument: the descending links.

Definition 2.1. Let $\mathrm{lk}_\downarrow(k,l)$ be a descending link for $k \geq 2$ and $l \geq 1$. Let C be a set of colors (denoted $a_1, a_2, ..., a_l$) such that $|C| = l$.

(1) For QF, define the vertex set $\mathcal{V}_{k,l}^{QF} = \{(m, m+1, a) \mid 1 \leq m \leq k-1 \text{ and } a \in C\}$ where C is a set of colors such that $|C| = l$. And let the set of simplexes defined as $\mathcal{S}_{k,l}^{QF} = \{V \subset \mathcal{V} \mid \text{if } v_1 = (m, m+1, b), v_2 = (n, n+1, c) \in V \text{ then } \{m, m+1\} \cap \{n, n+1\} = \emptyset \text{ and } b \neq c\}$.

(2) For QV, define the vertex set $\mathcal{V}_{k,l}^{QV} = \{(m, n, a) \mid 1 \leq m \leq k, 1 \leq n \leq k, \text{ and } n \neq m \text{ and } a \in C\}$ where C is a set of colors such that $|C| = l$. And let the set of simplexes defined as $\mathcal{S}_{k,l}^{QV} = \{V \subset \mathcal{V} \mid \text{if } v_1 = (m_1, n_1, b), v_2 = (m_2, n_2, c) \in V \text{ then } \{m_1, n_1\} \cap \{m_2, n_2\} = \emptyset \text{ and } b \neq c\}$.

(3) For QT, define the vertex set $\mathcal{V}_{k,l}^{QT} = \{(m, n, a) \mid 1 \leq m \leq k, n = m + 1 \text{ but if } m = k, n = 1, \text{ and } a \in C\}$ where C is a set of colors such that $|C| = l$. And let the set of simplexes defined as $\mathcal{S}_{k,l}^{QT} = \{V \subset \mathcal{V} \mid \text{if } v_1 = (m_1, n_1, b), v_2 = (n_1, n_2, c) \in V \text{ then } \{m_1, n_1\} \cap \{m_2, n_2\} = \emptyset \text{ and } b \neq c\}$.

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Looking at each vertex in the descending link, we can see that they are part of a larger structure. We can investigate this structure through two functions on a vertex v, the star and the link of the vertex.

Definition 2.2. Let $K = (\mathcal{V}, \mathcal{S})$ be a simplicial complex. Let v be a vertex in K. Define the function $\mathrm{st}(v) = \{S \in \mathcal{S} \mid S \text{ is a face of some } S' \in \mathcal{S} \text{ such that } v \in S'\}$.

The link can be defined easily once the star is defined.

Definition 2.3. Let $K = (\mathcal{V}, \mathcal{S})$ be a simplicial complex. Let v be a vertex in K. Define the function $\mathrm{lk}(v) = \{S \in \mathrm{st}(v) \mid v \in S\}$.

There are a couple of details that need to be taken care of before we can continue.

Remark 2.4. The $\mathrm{st}(v)$ is contractable since $\mathrm{st}(v) \cong C(\mathrm{lk}(v))$ where $C(\mathrm{lk}(v))$ is a cone on the link of v.

This is immensely helpful later, since contractable spaces are ∞-connected. We will also utilize a lemma from an earlier paper that allows us to compare the connectedness of the link and the star:

Lemma 2.5. [2] *Let K be a finite flag complex.*

(1) *K is $(n-1)$-connected provided, for any collection of vertices $S \subseteq K^{(0)}$, the intersection $\bigcap_{v \in S} \mathrm{lk}(v)$ is $(n - |S|)$-connected.*
(2) *If S is any collection of vertices of K and $\bigcap_{v \in S} \mathrm{lk}(v)$ is n-connected, then so is $\bigcap_{v \in S} \mathrm{st}(v)$.*

This lemma allows us to investigate the links of the vertices instead of the stars, which allows us to use our theorem inductively because if the intersection of t links is n-connected, then the intersection of t stars in the cover is also n-connected (for the same set of vertices). We must then also show what happens when we intersect links. This is because we can use the above lemma to deal with the much simpler intersection of links.

Theorem 2.6. *Let $v_1, ..., v_t$ be vertices of a descending link $\mathrm{lk}_{\downarrow}(k, l)$. For $1 \leq i \leq t$, $v_i = (n_i, m_i, a_i)$. Define the sets N and A as follows:*

$$N = \{n_i, m_i \mid v_i = (n_i, m_i, a_i) \text{ for } 1 \leq i \leq t\} \tag{1}$$

$$A = \{a_i \mid v_i = (n_i, m_i, a_i) \text{ for } 1 \leq i \leq t\} \tag{2}$$

Then

(1) *If the descending link is in QV, $\bigcap_1^t \mathrm{lk}(v_i) \cong \mathrm{lk}_{\downarrow}(k - |N|, l - |A|)$.*
(2) *If the descending link is in QF, then we must acknowledge a special condition where one of the $v_i = (2, 3, a)$. Since there is no way to form a vertex in QF using 1 as the first term, then 1 is also eliminated. Thus if $v = (2, 3, a)$, then $N = \{1, 2, 3\}$ and $\mathrm{lk}(v) \cong \mathrm{lk}_{\downarrow}(k - |N|, l - |A|)$.*
(3) *If the descending link is in QT, then $\bigcap_1^t \mathrm{lk}(v_i) \cong \mathrm{lk}_{\downarrow}(k - |N|, l - |A|)$. We must note that this descending link that is given by the intersection is not in QT, but in QF.*

Proof. (1) Let $\mathrm{lk}_{\downarrow}(k, l)$ be a descending link for $k \geq 2$ and $l \geq 1$. Consider a single $\mathrm{lk}(v)$ for some $(n, m, a) \in \mathcal{V}$. This is $\mathrm{lk}(v) \cong \mathrm{lk}_{\downarrow}(k - 2, l - 1)$, since it contains no vertex (x, y, b) such that $\{n, m\} \cap \{x, y\} \neq \emptyset$ or $a = b$. Thus we have effectively "removed" two possible numbers (m, n) and one color (a). Consider now the intersection of α vertices. Let N and A be defined as in our hypothesis. N is the collection of all numbers that appear in the n vertices, while A is the collection of all colors that appear in the n vertices. This means that the cardinality of N and A shows us how many of the numbers and colors are disallowed in the intersection of the link. Thus the corresponding link of the intersection of n vertices is $\mathrm{lk}_{\downarrow}(k-|N|, l-|A|)$.

(2) The proof for QF is identical, except we must deal with the case of having a $v_i = (2,3,a)$ where a is any color in C. All vertices of the descending link of QF are of the form $v = (m, m+1, a)$. Thus, if there exists a $v_i = (2,3,a)$, then there is no way we can create a $v_j = (1, m, a)$ that is in the link of v_i. We have therefore effectively eliminated the 1 as well as the 2 and 3, and thus the vertex $(2,3,a)$ adds 3 elements to N, namely $\{1,2,3\}$.

(3) The proof for QT is identical to QV, except we must point out that the intersection of links of vertices of QT gives a descending link in QF. Given any vertex, the link of that vertex will be in QF because it would be impossible for the link to "wrap around" (i.e., have a $v_1 = (m,n,a), v_2 = (i,m,b)$ such that i is the maximum number in the link) in the same way that links do in QT. Thus the corresponding link of the intersection of n vertices is $\mathrm{lk}_{\downarrow}(k-|N|, l-|A|)$, where $\mathrm{lk}_{\downarrow}(k-|N|, l-|A|)$ is a descending link in QF. □

2.2. **Connectivity of the Descending Link.** It is necessary when determining whether or not a group is F_∞ to analyze the descending link at each vertex. As k and l get very large, we want the descending link to approach ∞-connectedness.

Definition 2.7. A topological space X is said to be k-connected if and only if for every continuous function $f : S^j \mapsto X$ for every $j \leq k$ extends to a map of a ball $\hat{f} : B^{j+1} \mapsto X$.

In order to achieve this, we first consider the connectivity of the nerve of a descending link, defined as follows.

Definition 2.8. Given a topological space X and a set $\mathcal{U} = \{U_1, U_2, ..., U_n\}$ of sets contained in X that cover X, the nerve of the cover $\mathcal{N}(\mathcal{U})$ is the set of finite subsections of $\mathcal{U}$ such that $\bigcap_{j=1}^{k} U_i \neq \emptyset$ where $0 \leq k \leq n$.

We then apply the nerve theorem to determine the connectivity of the descending link, once the connectivity of the nerve has been established.

Theorem 2.9. *(Nerve Theorem.) Let Δ be a simplicial complex and $(\Delta_i)_{i \epsilon I}$ be a family of simplicial complexes such that $\bigcup_{i \epsilon I} \Delta_i = \Delta$. If every nonempty finite intersection $\Delta_1 \cap \Delta_2 \cap ... \cap \Delta_k$ is $(k+1)$-connected, then Δ is k-connected if and only if the nerve of Δ_i is k-connected.*

2.3. **Brown's Finiteness Condition.** Ken Brown has determined the necessary criteria for showing that a group is of type F_∞ and thus finitely presented.

Theorem 2.10. [1] *(Brown's Finiteness Criterion.) Given a CW complex X with a group G acting on X such that*

(1) *X is contractible*
(2) *G acts cellularly on X*
(3) *$X_1 \subseteq X_2 \subseteq ... \subseteq X_n \subseteq X$ such that*
  (a) *G acts cocompactly on X_i and leaves each X invariant*

(b) *G acts with finite cell stabilizers*
(c) *For every $k \geq 0$, there exists an N such that X_n is k-connected for every $n \geq N$*

G is of type F_∞.

This paper shows that the first condition is satisfied; a subsequent version will expose the satisfaction of the other two conditions for QF, QT, and QV.

3. Connectivity of the Descending Link in the Group QV

In this section we will use the nerve theorem in order to show the sufficient conditions for a descending link of QV to be n-connected. We will start by showing that the cover by stars is n-connected under certain conditions on the word of the descending link, and then use the Nerve Theorem to show that the descending links are n-connected.

3.1. **Nerve Connectedness.**

We must first define what exactly the nerve is in this case before we can continue with the proof.

Definition 3.1. Let X be a topological space. Let $\mathcal{U}$ be a cover of X. Define $\mathcal{V} = \{u \mid u \in \mathcal{U}\}$ as the set of vertices. Define $\mathcal{S} = \{V \subset \mathcal{V} \mid \bigcap^t u_i \neq \emptyset \text{ for some } t \in \mathbb{N}\}$ as the set of simplicies. Define a simplicial complex $\mathcal{N}(\mathcal{U}) = (\mathcal{V}, \mathcal{S})$ as the nerve of the cover.

Depending on which group we talk about, we can define the cover differently. For QV, we must cover the descending link by letting $\mathcal{U} = \{\mathrm{st}(v) \mid v \in \mathcal{V}_{k,l}^{QV}\}$. For QF we can define $\mathcal{U} = \{\mathrm{st}(v) \mid v = (1,2,a) \text{ or } v = (2,3,a) \text{ where } a \in C\}$. For QT, we can define $\mathcal{U} = \{\mathrm{st}(v) \mid= v = (1,2,a) \text{ or } v = (2,3,a) \text{ or } v = (n,1,a) \text{ where } a \in C\}$.

Now that we have defined a cover for QV, we will now show that the nerve of any descending link in QV is n-connected under certain conditions.

Theorem 3.2. *For $n \geq 0$ the nerve of the cover of $\mathrm{lk}_\downarrow(k,l)$ is guaranteed to be n-connected when $k \geq 2(n+3)$ and $l \geq n+3$.*

Proof. Let $\mathrm{lk}_\downarrow(k,l)$ be the descending link where $k \geq 2(n+3)$ and $l \geq n+3$. We can cover this space by the $\mathrm{st}(v)$ for every vertex v of the the descending link. Denote this collection of $\mathrm{st}(v)$ as $\mathcal{U}$. We will show that any $n+2$ elements in $\mathcal{N}(\mathcal{U})$ form a simplex (i.e. they have a non-empty intersection). Let $v_1, v_2, ..., v_{n+2}$ be vertices in the diagram. Consider the worst case of the $n+2$-fold intersection where all the vertices are distinct. By theorem 2.6 we see that $|N| = (2n+2)$ and $|A| = n$, so then $\bigcap_1^{n+2} \mathrm{lk}(v_i) = \mathrm{lk}_\downarrow(k-2n-2, l-n-2)$. Since $k \geq 2n+6$ and $a \geq n+3$ this means that we have at least one vertex that can be made from the remaining possible numbers and the remaining possible $a \in C$. This new vertex, v_α, is completely distinct from all $v_1, v_2, ..., v_{j+2}$ because neither of the numbers nor the color used in the vertex have been used before. Thus $v_\alpha \in \bigcup_1^{n+2} \mathrm{st}(v_i)$. Therefore, $\mathcal{N}(\mathcal{U})$ is the $n+1$-skeleton of a high dimensional simplex, and is n-connected. □

3.2. **Connectedness of the descending link of a word w.**

We can now prove that the descending link is highly connected in QV.

Theorem 3.3. *For $n \geq 0$, the descending link $\mathrm{lk}_\downarrow(k,l)$ is n-connected when $k \geq 4n+5$ and $l \geq 2n+3$.*

Proof. We will use induction to show that this is true for any $n \geq 0$.

Base Case ($n = 0$)

Let $\mathrm{lk}_{\downarrow}(k,l)$ be the descending link for $k \geq 5$ and $l \geq 3$. Let v_1, v_2 be vertices in the descending link. We will show that any single $\mathrm{st}(v)$ is 0-connected, and that the intersection of $\mathrm{st}(v_1) \cap \mathrm{st}(v_2)$ is non-empty. Note that we must throw away empty intersections, so this requirement is vacuous. Each $\mathrm{st}(v)$ is contractible, and is thus ∞-connected, which means it is also 0-connected. We cannot use our previous theorem [**?**] because $k < 6$. However, it is easy to see that the nerve is path connected.

Consider two elements in the nerve of the descending link, $\mathrm{st}(v_1), \mathrm{st}(v_2)\mathcal{N}(\mathcal{U})$. If v_1 and v_2 are non-disjoint vertices, then $\mathrm{st}(v_1) \cap \mathrm{st}(v_2) \neq \emptyset$, and thus there is an edge between them. If v_1 and v_2 are disjoint vertices that they are not connected by an edge. However, consider v_3 such that $v_3 \in \mathrm{st}(v_1)$. This means that v_3 is disjoint from v_1, and that there is an edge between $\mathrm{st}(v_3)$ and $\mathrm{st}(v_1)$. Since there are only 5 possible numbers, v_3 must be non-disjoint with v_2, and thus $\mathrm{st}(v_3) \cap \mathrm{st}(v_1) \neq \emptyset$ and thus have an edge between them. Thus between any two points we can see that there exists a path between said points. Thus the nerve of this descending link is 0-connected.

Therefore, by the Nerve theorem, $\mathrm{lk}_{\downarrow}(k,l)$ is 0-connected for $k \geq 5$ and $l \geq 3$.

($n = 1$)

Let $\mathrm{lk}_{\downarrow}(k,l)$ be the descending link where $k \geq 9$ and $l \geq 5$. Let v_1, v_2 be vertices of the descending link. We will show that a single v is 1-connected, and that the intersection of $\mathrm{st}(v_1) \cap \mathrm{st}(v_2)$ is 0-connected. Since $\mathrm{st}(v)$ is contractible, it is ∞-connected, and is thus 1-connected. Now consider the worst case for the two-fold intersection where v_1 and v_2 are completely distinct vertices. Using our lemma, we can investigate the intersection of the links, rather than the intersection of the stars. By theorem 2.6 the intersection of $\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2) \cong \mathrm{lk}_{\downarrow}(k-4, l-2)$ since $|N| = 4$ and $|A| = 2$. Note that because $k - 4 \geq 9 - 4 \geq 5$ and $l - 2 \geq 5 - 2 \geq 3$, this means we have shown that $\mathrm{lk}_{\downarrow}(k-4, l-2)$ is a 0-connected space in our original case, so $\mathrm{lk}(v_1) \cap \mathrm{lk}(v_2)$ is 0-connected, and by lemma 2.5 $\mathrm{st}(v_1) \cap \mathrm{st}(v_2)$ is 0-connected. Using theorem 3.2 we see that the nerve of this link is 1-connected when $k \geq 8$ and $l \geq 4$. Thus we can use the nerve theorem to state that $\mathrm{lk}_{\downarrow}(k,l)$ is 1-connected when $k \geq 9$ and $l \geq 5$.

Assume that the statement is true for all $n \leq j$. We will now show that it holds true for $j + 1$.

Let $\mathrm{lk}_{\downarrow}(k,l)$ be a descending link where $k \geq 4(j+1)+5$ and $l \geq 2(j+1)+3$. Let $v_1, \ldots, v_{j+1}$ be vertices in the descending link. We will show that a single v is $(j+1)$-connected, a 2-fold intersection is j-connected, a 3-fold intersection is $(j-1)$-connected, and that an t-fold intersection is $(j+1-t)$ connected.

Since each individual $\mathrm{st}(v)$ is contractible, they are ∞-connected.

We now must ask, is this link $j+1-t$-connected for all $1 \leq t \leq j+1$? Notice that in the worst case, we get that $|N| = 2t$ and $|A| = t$, so by theorem 2.6 $\bigcap_1^t \mathrm{lk}(v_i) = \mathrm{lk}_{\downarrow}(k-2t, l-t)$. Consider now that, when plugging into our hypothesis, we get:

$$k - 2t \geq 4(j+1) + 5 - 2t = 4(j+1-\frac{1}{2}t) + 5 \geq 4(j+1-t) + 5 \tag{3}$$

$$l - t \geq 2(j+1) + 3 - t = 2(j + 1 - \frac{1}{2}t) + 3 \geq 2(j + 1 - t) + 3 \tag{4}$$

which satisfied our requirements for the link to be $(j+1-t)$-connected. Thus by lemma 2.5, $\bigcap_1^t \mathrm{st}(v_i)$ is $(j+1-t)$-connected.

Therefore any intersection of t vertices for $1 \leq t \leq j+1$ is $(j+1-t)$-connected. Since the nerve of this link is also $(j+1)$-connected by theorem 3.2, since $k \geq 4(j+1)+5 \geq 2(j+1+3)$ and $l \geq 2(j+1)+3 \geq j+1+3$. By the Nerve theorem $\mathrm{lk}_{\downarrow}(k,l)$ for $k \geq 4(j+1)+5$ and $l \geq 2(j+1)+3$ is $(j+1)$-connected. $\square$

References

[1] Kenneth S. Brown. Finiteness properties of groups. *J. Pure Appl. Algebra*, 44(1-3):45–75, 1987.

[2] Daniel S. Farley. Homological and finiteness properties of picture groups. *Trans. Amer. Math. Soc.*, 357(9):3567–3584 (electronic), 2005.

[3] Brita E.A. Nucinkis and Simon St. John-Green. Quasi-automorphisms of the infinite rooted 2-edge-coloured binary tree. *arXiv:1412.7715*.