The Orthogonal Josephus Problem

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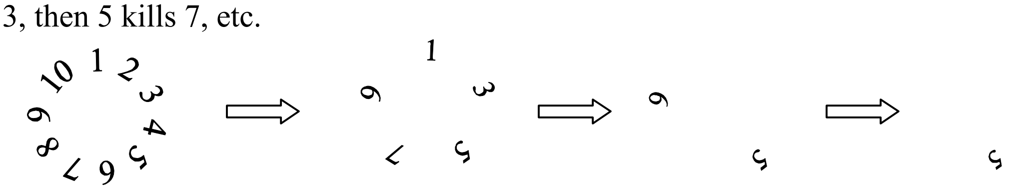
Abstract

We consider the Josephus Problem from a new perspective. $J(n,k)$ represents the position of the survivor when n people are eliminated with a skip factor of k. We demonstrate that there exists an explicit formula for $J(n,k)$ when n is fixed. We show that the set of all cycles generated by the orders of elimination (for a fixed n) is a group if and only if $1 \le n \le 5$.

1. Introduction

During the first century A.D. Flavius Josephus lived as a noted military general and historian. In his chronicle, *The Wars of the Jews* [2], Josephus detailed his heroism in battles against the Romans. In one particular battle against the Romans, Vespasian, a Roman conqueror, and his army seized the city of Jotapata. It was during this battle that Flavius Josephus and forty of his men were taken captive. Josephus' men chose to commit suicide rather than become slaves to the Romans. Josephus, not wishing to die, devised a method in which each person kills the other to ensure that just one person is left to kill himself. The idea was that each of the forty-one men sat in a circle whose seats are numbered in a clockwise orientation starting with a particular pre-chosen seat. Let us say, for simplicity's sake, we have only 10 people sitting in our circle and we wish to kill every other person starting from seat one. The person in seat 1 kills the person in seat 2, then the person in seat 3 kills seat 4, 5 kills 6 and so on until we cycle around again and 1 kills 3, then 5 kills 7, etc.



Order of Elimination for 10 people killing every 2nd person										
	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
n =10	2	4	6	8	10	3	7	1	9	5

This process of killing continues until just one person remains standing. In this case, we say the Josephus problem has a *skip factor* of two since we kill every second person. If we kill every third, ninety-ninth, or ten-thousand-and-seventh person we would say we

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have a skip factor of 3, 99 or 10,007, respectively. Let k denote the skip factor. In our example, person 5 is the survivor. We could also write $J(10, 2)=5$, where $n=10$ is the number of people in our circle and $k=2$ is the skip factor. We let $J(n, k)$ denote the position of the survivor in a circle of n people in which every k^{th} person is killed.

If we look at the case of Josephus and his men, we have $n=41$ and $k=2$. So, if we proceed killing every second person in the circle starting from seat one, then $J(41, 2)=19$. Therefore, we have found that Josephus chose seat 19 to sit in. How would Josephus be able to figure out where to sit if he had a circle of n people and a skip factor k? If we vary n, the number of people in the circle, and keep k, the skip factor, constant then there exists a recursive algorithm for determining $J(n, k)$, see [1].

1) Input n, k;
2) $D := 1$;
3) while $D \le (k-1)n$ do $D := \left\lceil \frac{k}{k-1} D \right\rceil$;
4) $J(n,k) := kn + 1 - D$;
5) Ouput $J(n, k)$,

where $\lceil x \rceil$ is the least integer greater than or equal to x. Note also that $\lfloor x \rfloor$ is the greatest integer less than or equal to x. When $k = 2$, we have a special case where the recursive formula is:

$$J(n, 2)=\begin{cases} 2J\left(\lfloor n/2 \rfloor\right)-(-1)^n, \text{if } n>1 \\ 1, \text{if } n=1 \end{cases}$$

So now we know that Josephus could easily survive if there would be a change in the number of people in the circle and if the skip factor remained constant. This is the usual way mathematicians and computer scientists have looked at the original Josephus problem. However, we have chosen to take a different approach to the Josephus problem.

Looking at the problem orthogonally, we want to see what happens when we vary the skip factor and keep the number of people in the circle constant. Using this approach, we want to know that if Josephus is sitting in seat m, and there are n people in the circle, is it possible for Josephus to name a skip factor k so that he always survives? We will show that it is possible. We will show there exists an explicit formula for $J(n,k)$ if we fix the number of people in the circle and vary the skip factor. Another interesting question is: can Josephus name a skip factor so that the people will be eliminated in a specific order? We will also show that it is not always possible for the people sitting in the circle to be eliminated in a particular order.

2. A formula for $J(n, k)$ when n is fixed.

We attack our research questions by first collecting data. For example, we want to see if we can find a pattern for the sequence $J(1, k)$, the position of the survivor when there is only one person in the circle, for any positive integer k. Similarly, we want to find a pattern for the sequence $J(2, k)$, $J(3, k)$ and so on, until we have confidence that we can predict the pattern for the sequence $J(n, k)$, for large n and k. It is time-consuming to collect the data by hand. So, we created a program in *Maple* to help us. Our program allows us to enter any positive integers n and k for $J(n, k)$ and outputs the position of the survivor. For simplicity, let n range from 1 to 3, and k from 1 to 20. Here is the Maple code for our program, which is an implementation of the algorithm mentioned above.

```
A: array (1..3, 1..3): for n from 1 to 3 do
for k from 1 to 20 do
c :=1; while c<= (k–1)*n do
c := ceil(k/ (k–1) *c) od:
A[n, k] := k*n+1–c; od;
```

This is the data that the Maple program gathers

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$J(1,k)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$J(2,k)$	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
$J(3,k)$	3	3	2	2	1	1	3	3	2	2	1	1	3	3	2	2	1	1	3	3

Based on this data, we observe the following:

Proposition 2.1: *Let J(n, k) represent the position of the survivor in a circle of n people with a skip factor of k.*

1. *For* $J(1, k)$, *the sequence is identically* 1.
2. *For* $J(2, k)$, *the sequence repeats* 2, 1.

$$J(2,k)=\begin{cases} 1, \ \text{if}\ k = 2m+2, \ \text{for}\ m \geq 0 \\ 2\ , \ \text{if}\ k = 2m+1, \ \text{for}\ m \geq 0 \end{cases}$$

3. *For* $J(3, k)$, *the sequence repeats* 3, 3, 2, 2, 1, 1.

$$J(3,k)=\begin{cases} 1, \text{if}\ \ k = 6m+5 \ \text{or}\ k = 6m+6 \ \text{for}\ m \geq 0 \\ 2, \text{if}\ \ k = 6m+3 \ \text{or}\ k = 6m+4 \ \text{for}\ m \geq 0 \\ 3, \text{if}\ \ k = 6m+1 \ \text{or}\ k = 6m+2 \ \text{for}\ m \geq 0 \end{cases}$$

Proof:

1. For n=1, the proof is trivial.

2. We show $J(2,k)=\begin{cases} 1, \ \text{if}\ k = 2m+2, \ \text{for}\ m \geq 0 \\ 2\ , \ \text{if}\ k = 2m+1, \ \text{for}\ m \geq 0 \end{cases}$ by cases. There are 2 different orders or ways to eliminate 2 people, either 1,2 or 2,1.

Case 1: We eliminate people in seat order 1,2.

Since we wish to eliminate person number one first, we know that k mod 2=1. Hence, our survivor is person in position two. Solving for k we get: k=lcm(2,1)m +1 = 2m+1. Therefore, if k=2m+1, then $J(2,k)$=2.

Case 2: We eliminate people in seat order 2,1
Since we wish to eliminate person number 2 first, we know that k mod 2=2.
Hence, our survivor is person in position one. Solving for k we get: k=lcm(2,1)m +2 = 2m+2. Therefore, if k=2m+2, then $J(2,k)$=1.

2. We show $J(3,k)= \begin{cases} 1, \text{if } k = 6m+5 \text{ or } k = 6m+6 \text{ for } m \geq 0 \\ 2, \text{if } k = 6m+3 \text{ or } k = 6m+4 \text{ for } m \geq 0 \\ 3, \text{if } k = 6m+1 \text{ or } k = 6m+2 \text{ for } m \geq 0 \end{cases}$ by cases. There are six different orders or ways to eliminate three people and they are listed below.

1,2,3	2,1,3	3,1,2
1,3,2	2,3,1	3,2,1

Case 1: We eliminate people in seat order 1,2,3
Since we wish to eliminate person number one first, we know that k mod 3=1. Now renumber the remaining seats starting from the position to the right of the person just eliminated. In this case seats 2 and 3 get renumbered 1 and 2, respectively. Since we wish to eliminate person 2, whose new position is 1, then we know k mod 2 =1. Hence, our survivor is the person originally numbered 3. Solving the system of equations for k:

$$\begin{cases} k \bmod 3 = 1 \\ k \bmod 2 = 1 \end{cases}$$

we get that k = lcm(3,2)m +1 = 6m+1. Therefore, if k=6m+1, then $J(3,k)$=3.

Case 2: We eliminate people in seat order 1,3,2
Since we wish to eliminate person number 1 first, we know that k mod 3=1. Now renumber the remaining seats starting from the position to the right of the person just eliminated. In this case seats 2 and 3 get renumbered 1 and 2, respectively. Since we wish to eliminate person 3 next, whose new position is 2, then we know k mod 2 =2. Hence, our survivor is the person originally numbered 2. Solving the system of equations for k:

$$\begin{cases} k \bmod 3 = 1 \\ k \bmod 2 = 2 \end{cases}$$

we get that k = lcm(3,2)m +4 = 6m+4. Therefore, if k=6m+4, then $J(3,k)$=2.

Case 3: We eliminate people in seat order 2,1,3
Since we wish to eliminate person number 2 first, we know that k mod 3=2. Now renumber the remaining seats starting from the position to the right of the person just

eliminated. In this case seats 1 and 3 get renumbered 2 and 1, respectively. Since we wish to eliminate person 1 next, whose new position is 2, then we know k mod 2 =2. Hence, our survivor is the person originally numbered 3. Solving the system of equations for k:

$$\begin{cases} k \bmod 3 = 2 \\ k \bmod 2 = 2 \end{cases}$$

we get that k = lcm(3,2)m +2 = 6m+2. Therefore, if k=6m+2, then J(3,k)=3.

Case 4: We eliminate people in seat order 2,3,1.
Since we wish to eliminate person number 2 first, we know that k mod 3=2. Now renumber the remaining seats starting from the position to the right of the person just eliminated. In this case seats 1 and 3 get renumbered 2 and 1, respectively. Since we wish to eliminate person 3 next, whose new position is 1, then we know k mod 2 =1. Hence, our survivor is the person originally numbered 1. Solving the system of equations for k:

$$\begin{cases} k \bmod 3 = 2 \\ k \bmod 2 = 1 \end{cases}$$

we get that k = lcm(3,2)m +5 = 6m+5. Therefore, if k=6m+5, then J(3,k)=1.

Case 5: We eliminate people in seat order 3,1,2
Since we wish to eliminate person number 3 first, we know that k mod 3=1. Now renumber the remaining seats starting from the position to the right of the person just eliminated. In this case seats 1 and 2 get renumbered 1 and 2, respectively. Since we wish to eliminate person 1 next, whose new position is 1, then we know k mod 2 =1. Hence, our survivor is the person originally numbered 2. Solving the system of equations for k:

$$\begin{cases} k \bmod 3 = 3 \\ k \bmod 2 = 1 \end{cases}$$

we get that k = lcm(3,2)m +3 = 6m+3. Therefore, if k=6m+3, then J(3,k)=2.

Case 6: We eliminate people in seat order 3,2,1
Since we wish to eliminate person number 3 first, we know that k mod 3=3. Now renumber the remaining seats starting from the position to the right of the person just eliminated. In this case seats 1 and 2 get renumbered 1 and 2, respectively. Since we wish to eliminate person 2 next, whose new position is 2, then we know k mod 2 =2. Hence, our survivor is the person originally numbered 1. Solving the system of equations for k:

$$\begin{cases} k \bmod 3 = 3 \\ k \bmod 2 = 2 \end{cases}$$

we get that k = lcm(3,2)m +6 = 6m+6. Therefore, if k = 6m+6, then J(3,k)=1.□

This Proposition gives the intuition for the more general result below.

Theorem 2.2: *If there is a solution, k, for the system of modular equations*

$$\begin{cases} k \bmod n = m_n \\ k \bmod (n-1) = m_{n-1} \\ \quad \vdots \\ k \bmod 3 = m_3 \\ k \bmod 2 = m_2 \end{cases}$$

then it is in the form $k = [\operatorname{lcm}(n, n-1, \ldots, 3, 2)]m + j$ *where* $0 \le j < \operatorname{lcm}(n, n-1, \ldots, 3, 2)$ *and where j is also a solution to the system.*

Proof: Suppose a solution *k* exists. Then *k* satisfies each of the equations

$$\begin{cases} k \bmod n = m_n \\ k \bmod (n-1) = m_{n-1} \\ \quad \vdots \\ k \bmod 3 = m_3 \\ k \bmod 2 = m_2. \end{cases}$$

Since *k* is a solution to the system, then *k* mod lcm(n, n-1,…, 3, 2) = *j* where $0 \le j < \operatorname{lcm}(n, n-1, \ldots, 3, 2)$.

Therefore, $k = [\operatorname{lcm}(n, n-1, \ldots, 3, 2)]m + j$.

Next, we want to show that *j* is also a solution

$$\begin{cases} m_2 = k \bmod 2 \\ \quad = ([\operatorname{lcm}(n, n-1, \ldots, 3, 2)]m + j) \bmod 2 \\ \quad = [\operatorname{lcm}(n, n-1, \ldots, 3, 2)m \bmod 2 + j \bmod 2 \\ \quad = j \bmod 2 \quad \text{since } 2 \mid \operatorname{lcm}(n, n-1, \ldots, 3, 2). \\ m_3 = k \bmod 3 \\ \quad = ([\operatorname{lcm}(n, n-1, \ldots, 3, 2)]m + j) \bmod 3 \\ \quad = [\operatorname{lcm}(n, n-1, \ldots, 3, 2)m \bmod 3 + j \bmod 3 \\ \quad = j \bmod 3 \\ \qquad \vdots \\ m_n = k \bmod n \\ \quad = ([\operatorname{lcm}(n, n-1, \ldots, 3, 2)]m + j) \bmod n \\ \quad = [\operatorname{lcm}(n, n-1, \ldots, 3, 2)]m \bmod n + j \bmod n \\ \quad = j \bmod n \end{cases}$$

Therefore, *j* must be a solution since it satisfies the modular system as above.□

3. The Josephus Groups

Referring back to Josephus and his men, would it have been possible for Josephus to find a way to save himself and an accomplice by eliminating the people from the circle in a specific order? No, we will show that the people in the circle cannot always be eliminated in a particular order.

Definition 3.1: Let $J_{n,k}$ denote the order of elimination with n people and a skip factor of k. Then let the disjoint cycles of $J_{n,k}$ be denoted by $j_{n,k}$.

We wrote a computer program in *Maple* (see appendix) that determines the permutations for $J_{n,k}$, where k ranges from 1 to lcm(1,2,3,..,n). The program then converts $J_{n,k}$ into disjoint cycles, $j_{n,k}$. The program generates data that helps us determine if the permutations form a subgroup of S_n, where S_n is the *symmetric group* of degree n or the set of n! permutations of the set $\{1,2,\ldots,n\}$.

Definition 3.2: A *perpendicular cycle set* is the set of all disjoint cycles generated by eliminating n people and varying the skip factor; that is $G_n = \{j_{n,k} | k \in Z^+\}$ for each $n \geq 1$.

We have shown that $G_2 = S_2$, $G_3 = S_3$ (see Appendix A). However, we found that some of the permutations of $\{1,2,\ldots,n\}$ do not correspond to the order in which we wish to eliminate n people since there does not exist a k that will eliminate them in the specified order. For example, $G_4 = A_4$, where A_4 is the set of *even* permutations in S_4, that is, the permutations which can be written as a product of an even number of 2-cycles. For example, the permutation (1,2,4,5,3) is even since it can be written as (1,3)(1,5)(1,4)(1,2). Similarly, a permutation is called *odd* if it can be written as a product of an odd number of 2-cycles. For example the permutation (1,4)(2,3,6) is odd since it can be written as (1,4)(2,6)(2,3). We also know that $G_5 = A_5$.

When n=6 and n=7 the sets of permutations for which we can find a k to kill the people in the circle in a specific order do not form a group. In order for the set of permutations to be called a *group* the set must satisfy the following:

1. the set must have a binary operation.
2. the set is closed under the binary operation.
3. the binary operation must be associative.
4. the set must have an identity element.
5. each element in the set must have an inverse.

However, certain permutations have no inverse in the subset we are considering. For example, $(2\,4\,3) \in G_6$, but $(2\,3\,4) \notin G_6$. And $(1\,7\,2\,6\,3)(4\,5) \in G_7$, but $(4\,5)(1\,3\,6\,2\,7) \notin G_7$. See the examples below.

Example 3.3 a) $(2\,4\,3) \in G_6$, but $(2\,3\,4) \notin G_6$

Order of elimination:	*Disjoint cycle:*
1 4 2 3 5 6	(2 4 3)

$$\text{Let } A = \begin{cases} k \bmod 6 = 1 \\ k \bmod 5 = 3 \\ k \bmod 4 = 3 \\ k \bmod 3 = 1 \\ k \bmod 2 = 1 \end{cases} \Rightarrow \begin{cases} k = 1 \bmod 6 \\ k = 3 \bmod 5 \\ k = 3 \bmod 4 \\ k = 1 \bmod 3 \\ k = 1 \bmod 2 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} k = 3 \bmod 5 \\ k = 1 \bmod 3 \\ k = 1 \bmod 2 \end{cases}$$

Using the Chinese Remainder Theorem:
$m = 5 \cdot 3 \cdot 2 = 30$. $M_1 = 30/5 = 6$; $M_2 = 30/3 = 10$; $M_3 = 30/2 = 15$. Then:

$$\begin{array}{l} 6y_1 = 1 \bmod 5 \\ 10y_2 = 1 \bmod 3 \\ 15y_3 = 1 \bmod 2 \end{array} \Longrightarrow \begin{array}{l} 1y_1 = 1 \bmod 5 \\ 1y_2 = 1 \bmod 3 \\ 1y_3 = 1 \bmod 2 \end{array} \Longrightarrow \begin{array}{l} y_1 = 1 \\ y_2 = 1 \\ y_3 = 1 \end{array}$$

And we obtain $k = 3 \cdot 6 \cdot 1 + 1 \cdot 10 \cdot 1 + 1 \cdot 15 \cdot 1 = 43$. Since 43 mod 30 = 13, we have that $k=30m + 13$. Next, we want to test if 13 is a solution to the system A above. We find that 13 is not a solution since 13 mod 4 $\neq 3$. Now, we try $k = 43$. This k value works since 43 mod 6 = 1 and 43 mod 4 =3. So, $k = 43$ is the smallest solution to the system A. Hence, if we add any multiple of lcm(2, 3, 4, 5, 6) = 60, we have $k = 60m + 43$ as a solution. We can now conclude that (2 4 3)$\in G_6$.

Now, we want to show that the inverse of (2 4 3), which is (2 3 4), is not in G_6. That is, we want to show (2 3 4)$\notin G_6$.

Order of elimination: 1 3 4 2 5 6

Disjoint cycle: (2 3 4)

$$\text{Let } A = \begin{cases} k \bmod 6 = 1 \Rightarrow k = 6m + 1 \Rightarrow k \bmod 3 = 1 \\ k \bmod 5 = 2 \\ k \bmod 4 = 1 \\ k \bmod 3 = 3 \\ k \bmod 2 = 1 \end{cases}$$

Note that we also have k mod 3 = 3. Hence there is no k that can satisfy both equations. Therefore, (2 3 4)$\notin G_6$.

Example 3.3 b) If $n = 7$, then we get the following system of equations. So that we may apply the Chinese Remainder Theorem, we reduce the system to one where the moduli are pairwise relatively prime.

$$\text{Let } A = \begin{cases} k = 7 \bmod 7 \\ k = 6 \bmod 6 \\ k = 1 \bmod 5 \\ k = 4 \bmod 4 \\ k = 3 \bmod 3 \\ k = 2 \bmod 2 \end{cases} \Rightarrow \begin{cases} k = 7 \bmod 7 \\ k = 6 \bmod 6 \\ k = 1 \bmod 5 \end{cases}$$

Using the Chinese Remainder Theorem:
$m = 7 \cdot 6 \cdot 5 = 210$. $M_1 = 210/7 = 30$, $M_2 = 21/6 = 35$, $M_3 = 210/5 = 42$. Then:

$$\begin{matrix} 30y_1 = 1 \bmod 7 \\ 35y_2 = 1 \bmod 6 \\ 42y_3 = 1 \bmod 5 \end{matrix} \Longrightarrow \begin{matrix} 2y_1 = 1 \bmod 7 \\ 5y_2 = 1 \bmod 6 \\ 2y_3 = 1 \bmod 5 \end{matrix} \Longrightarrow \begin{matrix} y_1 = 4 \\ y_2 = 5 \\ y_3 = 3 \end{matrix}$$

Thus, $k = 4 \cdot 7 \cdot 30 + 5 \cdot 6 \cdot 35 + 3 \cdot 1 \cdot 42 = 2{,}016$. Since 2016 mod 210 = 126, we have k=210m+126. Next, we want to test if 126 is a solution to the system A above. We find that 126 is not a solution since 205 mod 4 $\neq 4$. Now, we try k = 336. This k value works since 336 mod 4 = 4; 336 mod 3 = 3; and 336 mod 2 = 2. So, k = 336 is the smallest solution to the system A. Hence, if we add any multiple of lcm(2,3,4,5,6,7) = 420, we have k = 420m + 336 as a solution. The system gives this order of elimination 7, 6, 1, 5, 4, 3, 2, which corresponds to cycle (1 7 2 6 3)(4 5). We can now conclude that (1 7 2 6 3)(4 5)$\in G_7$.

Looking at the inverse, (4 5)(1 3 6 2 7), we get an order of elimination 3, 7, 6, 5, 4, 2, 1 in the following system of equations:

$$\text{Let } A' = \begin{cases} k = 3 \bmod 7 \\ k = 4 \bmod 6 \Rightarrow k = 6m + 4 \Rightarrow k = 1 \bmod 3 \\ k = 5 \bmod 5 \\ k = 4 \bmod 4 \\ k = 3 \bmod 3 \\ k = 2 \bmod 2 \end{cases}$$

Note that we also have k = 3 mod 3. Hence, there is no k that can satisfy both equations. Therefore, the disjoint cycle (4 5)(1 3 6 2 7)$\notin G_7$.

To reiterate, the examples above demonstrate that G_6 and G_7 are not groups, and lead us to prove that G_n is not a group for $n \geq 6$. The following lemmas are the keys to obtaining this result.

Lemma 3.4: *Let* $n \in Z^+$*, then* gcd*(n, n*–1*)* = 1.

Proof: Assume by way of contradiction that d = gcd(n, n–1) > 1, then $d|n$ and $d|(n–1)$. So, $n = dq$, for any positive integers q. It follows that $n–1 = dq–1$. Since $d|(n–1)$, we know that $d|(dq–1)$. This means $(dq–1)\text{mod } d = 0$. So, $dq \text{ mod } d=1$. This is the contradiction since $d|dq$; that is, $dq \text{ mod } d = 0$. Hence, $d = \gcd(n, n-1) = 1$.

Lemma 3.5: *If* $n \in Z^+$ *is not prime and if* $n>2$*, then there exists a prime p such that* $1 < p \le n-1$ *and p does not divide n.*

Proof: Suppose n is not prime and $n>2$. Consider any positive prime p that divides n–1, that is, $p|(n–1)$ and hence $1<p\le n–1$. Then by Lemma 1, any prime that divides n–1 will not divide n. Therefore, p does not divide n.□

Lemma 3.6: *Let* $p \in Z^+$ *be the largest prime such that* $1< p< n$ *and p does not divide n and assume that n is even. The following system has no solution:*

$$A = \begin{cases} k \bmod n = n - p + 1 \\ k \bmod (n/2) = 0 \end{cases}$$

Proof: Suppose $p \in Z^+$ be the largest prime such that $1< p< n$ and p does not divide n and assume that n is even and assume by way of contradiction that k is a solution of system A. If $k \bmod n = n - p + 1$, then $k = nm + (n–p+1)$, for some $m \in Z^+$. Substituting for k in the equation $k \bmod n/2 = 0$ we get

$0 = (nm + (n - p + 1)) \bmod (n/2) = (n(m+1) + (-p+1)) \bmod (n/2) = (-p+1) \bmod (n/2)$,

since $(n(m+1)) \bmod (n/2)$=0.

Thus, $(n/2)|(-p+1)$ since if $x \bmod y$ =0 then $y|x$.

But then $(-p+1)= (n/2)r$ for some $r\in Z^+$ and $p = 1 - (n/2)r$.

We consider two cases that lead to contradictions.

Case 1: Assume that $(n/2)r \ge 1$. We have that $p = 1 - (n/2)r$. In this case, we conclude that $p \le 0$. This is a contradiction since we assume that $1 < p < n$.

Case 2: Assume that $(n/2)r < 1$. Since $p = 1 - (n/2)r$, we obtain $0 < p < 1$. This is a contradiction since we assume that $1 < p < n$.

Thus, system A has no solution.□

Lemma 3.7: *If n>5 is an integer and 1<p<n is the largest prime that does not divide n, then* $p > \frac{n+1}{2}$.

Proof: We proceed by Mathematical Induction.

Basis step: If n=6, then p=5 and $5 > \frac{6+1}{2} = \frac{7}{2}$.

Induction step: Assume that if $p_k < k$ is the largest prime such that p_k does not divide k, then $p_k > \frac{k+1}{2}$. Our goal is to show that if $p_{k+1} < k+1$ is the largest prime such that p_{k+1} does not divide k+1, then $p_{k+1} > \frac{k+2}{2}$. First note that $p_{k+1} \geq p_k$. We have two cases to consider:

Case 1: If $\frac{k+1}{2}$ is odd, then $p_{k+1} \geq p_k \geq \frac{k+1}{2} + 2$ since p_k is odd and $p_k > \frac{k+1}{2}$.
But, $\frac{k+1}{2} + 2 = \frac{k+5}{2} > \frac{k+2}{2}$.

Case 2: If $\frac{k+1}{2}$ is even, then $p_{k+1} \geq p_k \geq \frac{k+1}{2} + 1$ since p_k is odd and $p_k > \frac{k+1}{2}$.
But, $\frac{k+1}{2} + 1 = \frac{k+3}{2} > \frac{k+2}{2}$.
Therefore, $p > \frac{n+1}{2}$, by the Principle of Mathematical Induction.□

Lemma 3.8: *Let* $1 < p < n$ *be largest prime such that p does not divide n and assume that n is an odd number. Show that the following system has no solution:*

$$B = \begin{cases} k \bmod n = n - p + 1 \\ k \bmod (n-1) = p - 1 \\ k \bmod [(n-1)/2] = 0 \end{cases}$$

Proof: Let $1 < p < n$ be largest prime such that p does not divide n and assume that n is an odd number. Suppose, by way of contradiction, that k is a solution and consider the following two modular equations from system B:

$$B' = \begin{cases} k \bmod (n-1) = p - 1 \\ k \bmod [(n-1)/2] = 0 \end{cases}$$

From system B' we have $k \bmod (n-1) = p-1$, so $k = (n-1)m + (p-1)$, for some $m \in Z^+$.
Substituting for k in the equation $k \bmod (n-1)/2 = 0$ we get

$$0 = ((n-1)m + (p-1)) \bmod (n-1)/2 = (p-1) \bmod (n-1)/2,$$

since $((n-1)m) \bmod (n-1)/2$=0. Thus $((n-1)/2) | (p-1)$, since if x mod y =0 then $y|x$.
Hence, $(p-1) = ((n-1)/2)r$ for some $r \in Z^+$. There are some cases to consider:

Case 1: If $r > 2$ then $p-1 > n-1$ so $p>n$ and this contradicts our assumption that $1<p<n$.

Case 2: If $r = 2$ then $p-1 = n-1$. Thus, $p=n$. This contradicts our assumption that $1<p<n$.

Case 3: If $r = 0$ then $p-1 = 0$. This contradicts our assumption that $1< p< n$.

Case 4: If $r = 1$ then $p-1=(n-1)/2$ which means that $p = (n+1)/2$.
By Lemma 4 this contradicts the fact that $p > (n+1)/2$.□

We are now prepared to prove the main theorem of this section. Recall that $G_n = \{j_{n,k} | k \in Z^+\}$ for each $n \geq 1$.

Theorem 3.9: *G_n is not a group if $n \geq 6$.*

Proof: To demonstrate that G_n is not a group, we want to construct a permutation f of $\{1,\ldots,n\}$ such that $f \in G_n$ but $f^{-1} \notin G_n$.

Here is how to construct $f \in G_n$. Let $n > 5$, then find the largest prime p such that $1<p<n$ and p does not divide n. Consider the following system:

$$\begin{cases} k = m \bmod m, \text{ if } m \neq p \text{ where } 2 \leq m \leq n \\ k = 1 \bmod p \end{cases}$$

This system has a solution, k, by the construction which may be obtained using the Chinese Remainder Theorem. Hence, the system gives an order of elimination that corresponds to a permutation $f \in G_n$. The order of elimination we get is:

$$J_{n,k} = \{n, n-1, n-2, \ldots, 1, p, p-1, p-2, \ldots, 2\}.$$

We want to show that $f^{-1} \notin G_n$. The inverse order of elimination is

$$n-p+1, n, n-1, n-2, \ldots, n-p+2, n-p, \ldots, 3, 2, 1,$$

and gives us the following system of equations:

$$A = \begin{cases} k = (n-p+1) \bmod n \\ k = (p-1) \bmod (n-1) \\ k = m \bmod m, \text{ if } m \leq n-2 \end{cases}$$

Case 1: If n is even, then since p is the largest prime such that p does not divide n and $1<p< n$ and n is even then by Lemma 3.6, the system A does not have a solution. Hence, $f^{-1} \notin G_n$.

Case 2: If n is odd, then since p is the largest prime such that p does not divide n and $1<p< n$ and n is odd then by Lemma 3.8, the system A does not have a solution. Hence, $f^{-1} \notin G_n$.
Therefore, G_n is not a group if $n \geq 6$.□

Now we want to show that the procedure used in the proof for Theorem 3.9 does not contradict what we know for $n = 3, 4$, and 5.

Example 3.10 a) If n=5, then we get the following system of equations. So that we may apply the Chinese Remainder Theorem we reduce the system to one where the moduli are pairwise relatively prime:

$$\begin{cases} k = 5 \bmod 5 \\ k = 4 \bmod 4 \\ k = 1 \bmod 3 \\ k = 2 \bmod 2 \end{cases} \Longrightarrow \begin{cases} k = 5 \bmod 5 \\ k = 4 \bmod 4 \\ k = 1 \bmod 3 \end{cases}$$

Using the Chinese Remainder Theorem:
$m = 5 \cdot 4 \cdot 3 = 60$. $M_1 = 60/5 = 12$, $M_2 = 60/4 = 15$, $M_3 = 60/3 = 20$. Then:

$$\begin{matrix} 12y_1 = 1 \bmod 5 \\ 15y_2 = 1 \bmod 4 \\ 20y_3 = 1 \bmod 3 \end{matrix} \Longrightarrow \begin{matrix} 2y_1 = 1 \bmod 5 \\ 3y_2 = 1 \bmod 4 \\ 2y_3 = 1 \bmod 3 \end{matrix} \Longrightarrow \begin{matrix} y_1 = 3 \\ y_2 = 3 \\ y_3 = 5 \end{matrix}$$

And we obtain
$k = 3 \cdot 5 \cdot 12 + 3 \cdot 4 \cdot 15 + 5 \cdot 1 \cdot 20 = 460 \bmod 60 = 40$.

The smallest solution to the original system is $k = 40$. Hence, if we add any multiple of lcm(2,3,4,5) = 60, we have $k = 60m + 40$ as a solution. The system gives this order of elimination 1, 3, 4, 2, 5, 6, 7, 8, which corresponds to the cycle (1 5 2 4 3). We can now conclude that (1 5 2 4 3)$\in G_5$.

Looking at the inverse, (1 3 4 2 5), we get an order of elimination of 3, 5, 4, 2, 1, and the following system of equations which we reduce in order to apply the Chinese Remainder Theorem:

$$\begin{cases} k = 3 \bmod 5 \\ k = 2 \bmod 4 \\ k = 3 \bmod 3 \\ k = 2 \bmod 2 \end{cases} \Longrightarrow \begin{cases} k = 3 \bmod 5 \\ k = 2 \bmod 4 \\ k = 3 \bmod 3 \end{cases}$$

Using the Chinese Remainder Theorem:
$m = 5 \cdot 4 \cdot 3 = 60$. $M_1 = 60/5 = 12$, $M_2 = 60/4 = 15$, $M_3 = 60/3 = 20$. Then:

$12y_1 = 1 \bmod 5$ ⟹ $2y_1 = 1 \bmod 5$ ⟹ $y_1 = 3$
$15y_2 = 1 \bmod 4$ ⟹ $3y_2 = 1 \bmod 4$ ⟹ $y_2 = 3$
$20y_3 = 1 \bmod 3$ ⟹ $2y_3 = 1 \bmod 3$ ⟹ $y_3 = 5$

Therefore, $k = 3\cdot 3\cdot 12 + 3\cdot 2\cdot 15 + 5\cdot 3\cdot 20 = 498 \bmod 60 = 18$. The smallest solution to the original system is $k = 18$. Hence, if we add any multiple of lcm(2,3,4,5) = 60, we have $k = 60m + 18$ as a solution. We can now conclude that $(1\ 3\ 4\ 2\ 5)\in G_5$. Therefore, the procedure described in Theorem 3.9 does not contradict our conclusion that $G_5 = A_5$ is a group.

Example 3.10 b) If n=4, then we have $(1\ 4\ 2)\in G_4$. So we get the following system of equations that we reduce for the Chinese Remainder Theorem:

$$\begin{cases} k = 4 \bmod 4 \\ k = 1 \bmod 3 \\ k = 2 \bmod 2 \end{cases} \Longrightarrow \begin{cases} k = 4 \bmod 4 \\ k = 1 \bmod 3 \end{cases}$$

Using the Chinese Remainder Theorem:
$m = 4\cdot 3 = 12$. $M_1 = 12/4 = 3$, $M_2 = 12/3 = 4$. Then

$3y_1 = 1 \bmod 4$ ⟹ $y_1 = 3$
$4y_2 = 1 \bmod 3$ ⟹ $y_2 = 1$

Thus, $k = 3\cdot 4\cdot 3 + 1\cdot 1\cdot 4 = 40 \bmod 12 = 4$. The smallest solution to the original system is $k = 4$. Hence, if we add any multiple of lcm(2,3,4) = 12, we have $k = 12m + 4$ as a solution. The system gives this order of elimination 4, 1, 3, 2, which corresponds to the cycle (1 4 2). We can now conclude that $(1\ 4\ 2)\in G_4$.

Looking at the inverse, (1 2 4), we get an order of elimination of 2, 4, 3, 1, and the following system of equations which we reduce in order to apply the Chinese Remainder Theorem:

$$\begin{cases} k = 2 \bmod 4 \\ k = 2 \bmod 3 \\ k = 2 \bmod 2 \end{cases} \Longrightarrow \begin{cases} k = 2 \bmod 4 \\ k = 2 \bmod 3 \end{cases}$$

Using the Chinese Remainder Theorem:
$m = 4\cdot 3 = 12$. $M_1 = 12/4 = 3$, $M_2 = 12/3 = 4$. Then:

$3y_1 = 1 \bmod 4$ ⟹ $y_1 = 3$
$4y_2 = 1 \bmod 3$ ⟹ $y_2 = 1$

Therefore, $k = 3 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 4 = 26 \bmod 12 = 2$. The smallest solution to the original system is $k = 2$. Hence, if we add any multiple of lcm(2,3,4) = 12, we have $k = 12m + 2$ as a solution. We can now conclude that $(1\ 2\ 4) \in G_5$. Therefore, the procedure described in Theorem 3.9 does not contradict our conclusion that $G_4 = A_4$ is a group.

Example 3.10 c) If n=3, then we have $(1\ 3\ 2) \in G_3$. So we get the following system of equations:

$$\begin{cases} k = 3 \bmod 3 \\ k = 1 \bmod 2 \end{cases}$$

Using the Chinese Remainder Theorem:
$m = 3 \cdot 2 = 6$. $M_1 = 6/3 = 2$, $M_2 = 6/2 = 3$. Then:

$$\begin{matrix} 2y_1 = 1 \bmod 3 \\ 3y_2 = 1 \bmod 2 \end{matrix} \Longrightarrow \begin{matrix} y_1 = 2 \\ y_2 = 1 \end{matrix}$$

And we obtain, $k = 2 \cdot 3 \cdot 2 + 1 \cdot 1 \cdot 3 = 15 \bmod 6 = 3$. The smallest solution to the original system is $k = 3$. Hence, if we add any multiple of lcm(2,3) = 6, we have $k = 6m + 3$ as a solution. The system gives this order of elimination 3, 1, 2, which corresponds to the cycle (1 3 2). We can now conclude that $(1\ 3\ 2) \in G_3$.

Looking at the inverse, (1 2 3), we get an order of elimination of 2, 3, 1, and the following system of equations:

$$\begin{cases} k = 2 \bmod 3 \\ k = 1 \bmod 2 \end{cases}$$

Using the Chinese Remainder Theorem:
$m = 3 \cdot 2 = 6$. $M_1 = 6/3 = 2$, $M_2 = 6/2 = 3$. Then:

$$\begin{matrix} 2y_1 = 1 \bmod 3 \\ 3y_2 = 1 \bmod 2 \end{matrix} \Longrightarrow \begin{matrix} y_1 = 2 \\ y_2 = 1 \end{matrix}$$

Thus, $k = 2 \cdot 2 \cdot 2 + 1 \cdot 1 \cdot 3 = 11 \bmod 6 = 5$. The smallest solution to the original system is $k = 5$. Hence, if we add any multiple of lcm(2,3) = 6, we have $k = 6m + 5$ as a solution.
We can now conclude that $(1\ 2\ 3) \in G_3$. Therefore, the procedure described in Theorem 3.9 does not does not contradict our conclusion that $G_3 = S_3$ is a group.

4. Conclusion

We have shown that G_n is not a group when $n \geq 6$ which means that it is not always possible to eliminate the people in a specific order. For this reason, we know Josephus would not be able to eliminate the people in the circle in any order he wanted. We know it is always possible for Josephus to save himself since he can easily determine the position of the survivor. We have shown there exists an explicit formula for determining the position of the survivor, $J(n, k)$ when we fix n and vary the skip factor k. We encourage further study into developing a nice way to express this general formula.

Appendix A

Table A.1: when $n = 3$.

order of elimination:	disjoint cycles:	system of modular equations:	if $k =$	then $J(n,k)$=
1,2,3	( )	$k \bmod 3 = 1$ $k \bmod 2 = 1$	$6m+1$	3
2,1,3	(1,2)	$k \bmod 3 = 2$ $k \bmod 2 = 2$	$6m+2$	3
3,1,2	(1,3,2)	$k \bmod 3 = 3$ $k \bmod 2 = 1$	$6m+3$	2
1,3,2	(2,3)	$k \bmod 3 = 1$ $k \bmod 2 = 2$	$6m+4$	2
2,3,1	(1,2,3)	$k \bmod 3 = 2$ $k \bmod 2 = 1$	$6m+5$	1
3,2,1	(1,3)	$k \bmod 3 = 3$ $k \bmod 2 = 2$	$6m+6$	1

Table A.2: when $n = 4$

order of elimination:	disjoint cycles:	system of modular equations:	if $k =$	then $J(n,k)$=
1,2,3,4	( )	$k \bmod 4 = 1$ $k \bmod 3 = 1$ $k \bmod 2 = 1$	$12m+1$	4
2,4,3,1	(1,2,4)	$k \bmod 4 = 2$ $k \bmod 3 = 2$ $k \bmod 2 = 2$	$12m+2$	1
3,2,4,1	(1,3,4)	$k \bmod 4 = 3$ $k \bmod 3 = 3$ $k \bmod 2 = 1$	$12m+3$	1
4,1,3,2	(1,4,2)	$k \bmod 4 = 4$ $k \bmod 3 = 1$ $k \bmod 2 = 2$	$12m+4$	2
1,3,4,2	(2,3,4)	$k \bmod 4 = 1$	$12m+5$	2

		k mod 3 =2 k mod 2 =1		
2,1,4,3	(1,2)(3,4)	k mod 4 =2 k mod 3 =3 k mod 2 =2	12m+6	3
3,4,1,2	(1,3)(2,4)	k mod 4 =3 k mod 3 =1 k mod 2 =1	12m+7	2
4,2,1,3	(1,4,3)	k mod 4 =4 k mod 3 =2 k mod 2 =2	12m+8	3
1,4,2,3	(2,4,3)	k mod 4 =1 k mod 3 =3 k mod 2 =1	12m+9	3
2,3,1,4	(1,2,3)	k mod 4 =2 k mod 3 =1 k mod 2 =2	12m+10	4
3,1,2,4	(1,3,2)	k mod 4 =3 k mod 3 =2 k mod 2 =1	12m+11	4
4,3,2,1	(1,4)(2,3)	k mod 4 =4 k mod 3 =3 k mod 2 =2	12m+12	1
1,2,4,3	(3,4)	k mod 4 =1 ← odd k mod 3 =1 k mod 2 =2 ← even	impossible to find k	3 cannot be the survivor
1,3,2,4	(2,3)	k mod 4 =1 ← odd k mod 3 =2 k mod 2 =2 ← even	impossible to find k	4 cannot be the survivor
1,4,3,2	(2,4)	k mod 4 =1 ← odd k mod 3 =3 k mod 2 =2 ← even	impossible to find k	2 cannot be the survivor
2,1,3,4	(1,2)	k mod 4 =2 ← even k mod 3 =3 k mod 2 =1 ← odd	impossible to find k	4 cannot be the survivor
2,3,4,1	(1,2,3,4)	k mod 4 =2 ← even k mod 3 =1 k mod 2 =1 ← odd	impossible to find k	1 cannot be the survivor
2,4,1,3	(1,2,4,3)	k mod 4 =2 ← even k mod 3 =2 k mod 2 =1 ← odd	impossible to find k	3 cannot be the survivor
3,1,4,2	(1,3,4,2)	k mod 4 =3 ← odd k mod 3 =2 k mod 2 =2 ← even	impossible to find k	2 cannot be the survivor
3,2,1,4	(1,3)	k mod 4 =3 ← odd	impossible	4 cannot be the

		k mod 3 =3 k mod 2 =2 ← even	to find k	survivor
3,4,2,1	(1,3,2,4)	k mod 4 =3 ← odd k mod 3 =1 k mod 2 =2 ← even	impossible to find k	1 cannot be the survivor
4,1,2,3	(1,4,3,2)	k mod 4 =4 ← even k mod 3 =1 k mod 2 =1 ← odd	impossible to find k	3 cannot be the survivor
4,2,3,1	(1,4)	k mod 4 =4 ← even k mod 3 =2 k mod 2 =1 ← odd	impossible to find k	1 cannot be the survivor
4,3,1,2	(1,4,2,3)	k mod 4 =4 ← even k mod 3 =3 k mod 2 =1 ← odd	impossible to find k	2 cannot be the survivor

Table A.3: when $n = 5$

order of elimination:	disjoint cycles:	if k =	then $J(n,k)$=
1,2,3,4,5	( )	60m+1	5
2,4,1,5,3	(1,2,4,5,3)	60m+2	3
3,1,5,2,4	(1,3,5,4,2)	60m+3	4
4,3,5,2,1	(1,4,2,3,5)	60m+4	1
5,1,3,4,2	(1,5,2)	60m+5	2
1,3,2,5,4	(2,3)(4,5)	60m+6	4
2,5,1,3,4	(1,2,5,4,3)	60m+7	4
3,2,5,4,1	(1,3,5)	60m+8	1
4,5,3,1,2	(1,4)(2,5)	60m+9	2
5,2,3,1,4	(1,5,4)	60m+10	4
1,4,2,3,5	(2,4,3)	60m+11	5
2,1,5,4,3	(1,2)(3,5)	60m+12	3
3,4,5,1,2	(1,3,5,2,4)	60m+13	2
4,1,3,2,5	(1,4,2)	60m+14	5
5,3,2,4,1	(1,5)(2,3)	60m+15	1
1,5,2,4,3	(2,5,3)	60m+16	3
2,3,5,1,4	(1,2,3,5,4)	60m+17	4
3,5,4,2,1	(1,3,4,2,5)	60m+18	1
4,2,3,5,1	(1,4,5)	60m+19	1
5,4,2,1,3	(1,5,3,2,4)	60m+20	3
1,2,5,3,4	(3,5,4)	60m+21	4
2,4,5,3,1	(1,2,4,3,5)	60m+22	1
3,1,4,5,2	(1,3,4,5,2)	60m+23	2
4,3,2,1,5	(1,4)(2,3)	60m+24	5
5,1,2,3,4	(1,5,4,3,2)	60m+25	4

1,3,5,4,2	(2,3,5)	60*m*+26	2
2,5,4,1,3	(1,2,5,3,4)	60*m*+27	3
3,2,4,1,5	(1,3,4)	60*m*+28	5
4,5,2,3,1	(1,4,3,2,5)	60*m*+29	1
5,2,1,4,3	(1,5,3)	60*m*+30	3
1,4,5,2,3	(2,4)(3,5)	60*m*+31	3
2,1,4,3,5	(1,2)(3,4)	60*m*+32	5
3,4,2,5,1	(1,3,2,4,5)	60*m*+33	1
4,1,2,5,3	(1,4,5,3,2)	60*m*+34	1
5,3,1,2,4	(1,5,4,2,3)	60*m*+35	4
1,5,4,3,2	(2,5)(3,4)	60*m*+36	2
2,3,4,5,1	(1,2,3,4,5)	60*m*+37	1
3,5,2,1,4	(1,3,2,5,4)	60*m*+38	4
4,2,1,3,5	(1,4,3)	60*m*+39	5
5,4,1,3,2	(1,5,2,4,3)	60*m*+40	2
1,2,4,5,3	(3,4,5)	60*m*+41	3
2,4,3,1,5	(1,2,4)	60*m*+42	5
3,1,2,4,5	(1,3,2)	60*m*+43	5
4,3,1,5,2	(1,4,5,2,3)	60*m*+44	2
5,1,4,2,3	(1,5,3,4,2)	60*m*+45	3
1,3,4,2,5	(2,3,4)	60*m*+46	5
2,5,3,4,1	(1,2,5)	60*m*+47	1
3,2,1,5,4	(1,3)(4,5)	60*m*+48	4
4,5,1,2,3	(1,4,2,5,3)	60*m*+49	3
5,2,4,3,1	(1,5)(3,4)	60*m*+50	1
1,4,3,5,2	(2,4,5)	60*m*+51	2
2,1,3,5,4	(1,2)(4,5)	60*m*+52	4
3,4,1,2,5	(1,3)(2,4)	60*m*+53	5
4,1,5,3,2	(1,4,3,5,2)	60*m*+54	2
5,3,4,1,2	(1,5,2,3,4)	60*m*+55	2
1,5,3,2,4	(2,5,4)	60*m*+56	4
2,3,1,4,5	(1,2,3)	60*m*+57	5
3,5,1,4,2	(1,3)(2,5)	60*m*+58	2
4,2,5,1,3	(1,4)(3,5)	60*m*+59	3
5,4,3,2,1	(1,5)(2,4)	60*m*+60	1

Table A.4: when $n = 6$

order of elimination:	disjoint cycles:	if k =	then $J(n,k)$=
1,2,3,4,5,6	( )	60*m*+1	6
2,4,6,3,1,5	(1,2,4,3,6,5)	60*m*+2	5
3,6,4,2,5,1	(1,3,4,2,6)	60*m*+3	1
4,2,1,3,6,5	(1,4,3)(5,6)	60*m*+4	5

5,4,6,2,3,1	(1,5,3,6)(2,4)	60*m*+5	1
6,1,3,2,5,4	(1,6,4,2)	60*m*+6	4
1,3,6,2,4,5	(2,3,6,5,4)	60*m*+7	5
2,5,4,1,6,3	(1,2,5,6,3,4)	60*m*+8	3
3,1,2,6,4,5	(1,3,2)(4,6,5)	60*m*+9	5
4,3,6,1,5,2	(1,4)(2,3,6)	60*m*+10	2
5,6,3,1,2,4	(1,5,2,6,4)	60*m*+11	4
6,2,1,5,4,3	(1,6,3)(4,5)	60*m*+12	3
1,4,5,6,2,3	(2,4,6,3,5)	60*m*+13	3
2,6,3,5,4,1	(1,2,6)(4,5)	60*m*+14	1
3,2,6,5,1,4	(1,3,6,4,5)	60*m*+15	4
4,5,3,6,2,1	(1,4,6)(2,5)	60*m*+16	1
5,1,2,4,6,3	(1,5,6,3,2)	60*m*+17	3
6,3,5,4,2,1	(1,6)(2,3,5)	60*m*+18	1
1,5,3,4,6,2	(2,5,6)	60*m*+19	2
2,1,6,4,3,5	(1,2)(3,6,5)	60*m*+20	5
3,4,5,2,6,1	(1,3,5,6)(2,4)	60*m*+21	1
4,6,2,3,1,5	(1,4,3,2,6,5)	60*m*+22	5
5,2,6,3,41	(1,5,4,3,6)	60*m*+23	1
6,4,3,2,1,5	(1,6,5)(2,4)	60*m*+24	5
1,6,2,3,4,5	(2,6,5,4,3)	60*m*+25	5
2,3,5,1,6,4	(1,2,3,5,6,4)	60*m*+26	4
3,5,2,1,4,6	(1,3,2,5,4)	60*m*+27	*6*
4,1,6,2,5,3	(1,4,2,3,6)	60*m*+28	3
5,3,4,1,2,6	(1,5,2,3,4)	60*m*+29	6
6,5,2,1,4,3	(1,6,3,2,5,4)	60*m*+30	3
1,2,5,6,3,4	(3,5)(4,6)	60*m*+31	4
2,4,3,6,5,1	(1,2,4,6)	60*m*+32	1
3,6,1,5,2,4	(1,3)(2,6,4,5)	60*m*+33	4
4,2,5,6,3,1	(1,4,6)(3,5)	60*m*+34	1
5,4,2,6,1,3	(1,5)(2,4,6,3)	60*m*+35	3
6,1,5,4,3,2	(1,6,2)(3,5)	60*m*+36	2
1,3,4,5,6,2	(2,3,4,5,6)	60*m*+37	2
2,5,1,4,3,6	(1,2,5,3)	60*m*+38	6
3,1,5,4,6,2	(1,3,5,6,2)	60*m*+39	2
4,3,2,5,1,6	(1,4,5)(2,3)	60*m*+40	6
5,6,1,3,4,2	(1,5,4,3)(2,6)	60*m*+41	2
6,2,4,3,1,5	(1,6,5)(3,4)	60*m*+42	5
1,4,2,3,5,6	(2,4,3)	60*m*+43	6
2,6,5,3,1,4	(1,2,6,4,3,5)	60*m*+44	4
3,2,4,1,5,6	(1,3,4)	60*m*+45	6
4,5,1,2,6,3	(1,4,2,5,6,3)	60*m*+46	3
5,1,4,2,3,6	(1,5,3,4,2)	60*m*+47	6
6,3,2,1,5,4	(1,6,4)(2,3)	60*m*+48	4
1,5,6,2,3,4	(2,5,3,6,4)	60*m*+49	4

2,1,4,6,5,3	(1,2)(3,4,6)	60*m*+50	3
3,4,1,6,2,5	(1,3)(2,4,6,5)	60*m*+51	5
4,6,5,1,3,2	(1,4)(2,6)(3,5)	60*m*+52	2
5,2,3,6,1,4	(1,5)(4,6)	60*m*+53	4
6,4,1,5,3,2	(1,6,2,4,5,3)	60*m*+54	2
1,6,4,5,2,3	(2,6,3,4,5)	60*m*+55	3
2,3,1,5,4,6	(1,2,3)(4,5)	60*m*+56	6
3,5,6,4,1,2	(1,3,6,2,5)	60*m*+57	2
4,1,3,5,2,6	(1,4,5,2)	60*m*+58	6
5,3,1,4,6,2	(1,5,6,2,3)	60*m*+59	2
6,5,4,3,2,1	(1,6)(2,5)(3,4)	60*m*+60	6

Appendix B

The Maple program that generates this data:

```
> restart;   with(group):
> n:=6:  maxk:=lcm($2..n):                      # number of people, range of k
> J:=array(1..n):
> store:=array(1..maxk,1..3):                   # define array to store cycles
> for i from 1 to maxk do store[i,1]:=i: od:    # no. cycle positions
>
> for k from 1 to maxk do                  # LOOP to calculate death order and cycles for  k
> f:=[$1..n];                                   # makes f = [1,2,3,...,m]
> m:=1:                                         # each time we store the nth killing
> for L from n by -1 to 2 do
> usek := k mod L;                              # of seat to kill (0=last person)
> twofs:=[seq(f[i],i=1..L), seq(f[i],i=1..L)];  # vector with two copies of f
> usethisk:= usek + L;                          # of seat to kill from twof vector
> J[m]:=twofs[usethisk];                        # record the seat of most recent
            #killed
> f:=[seq(twofs[i],i=(usek+1)..(usethisk-1))];  # change f so that living remain
                                                 #and the person after most recent
                                                 #killed is in first position
> m:=m+1:  od:                                   #increment n and end loop for this k
> J[n]:=f[1]:                                    #last person to die is the last one living
> tempJ:=[seq(J[i],i=1..n)]:                     # turn J into a list
> newJ:=convert(tempJ,'disjcyc');                #death order becomes disjoint cycles
> store[k,2]:=tempJ: store[k,3]:=newJ;           #store cycles in store array
> od:
```

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References

[1] Graham, Knuth, & Patashinik, *Concrete Mathematics: A Foundation for Computer Science,* 2nd edition, Addison-Wesley: Reading, 1994.

[2] Josephus, *Josephus: The Complete Works,* (William Whiston, A.M., translator), Nelson: Nashville, 1998.