The Linear Chromatic Number of a Sperner Family

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Abstract

Let S be a finite set and $\mathcal{S}$ a complete Sperner family on S, i.e. a Sperner family such that every $x \in S$ is contained in some member of $\mathcal{S}$. The linear chromatic number of $\mathcal{S}$, defined by Cıvan, is the smallest integer n with the property that there exists a function $f : S \to \{1, \ldots, n\}$ such that if $f(x) = f(y)$, then every set in $\mathcal{S}$ which contains x also contains y or every set in $\mathcal{S}$ which contains y also contains x. We give an explicit formula for the number of complete Sperner families on S of linear chromatic number 2. We also prove tight bounds on the number of elements in a Sperner family of given chromatic number, and prove that complete Sperner families of maximum linear chromatic number are far more numerous those of lesser linear chromatic number.

1 Introduction

Let S be a set. A *Sperner family* on S is an antichain in the poset 2^S; that is, a family $\mathcal{S}$ of subsets of S which are pairwise incomparable by inclusion. For $x \in S$, write $\mathcal{S}_x = \{A \in \mathcal{S} : x \in A\}$; $\mathcal{S}$ is called *complete* if $\bigcup\limits_{A \in \mathcal{S}} A = S$, or equivalently $\mathcal{S}_x \neq \emptyset$ for all $x \in S$. Sperner families have long been of interest in extremal set theory, beginning in the 1930s with the seminal work of Sperner [6] on the maximum size of such a family.

As an adjunct to their study [2] of multicomplexes, Cıvan and Yalçın defined the notion of a *linear coloring* of a simplicial complex. This concept was later generalized by Cıvan [1] in the context of Sperner families: given a set S and integer $k > 0$, a *linear k-coloring* of a Sperner family $\mathcal{S}$ on S is a function $f : S \to [k] = \{1, \ldots, k\}$

such that if $f(x) = f(y)$, then $\mathcal{S}_x \subseteq \mathcal{S}_y$ or $\mathcal{S}_y \subseteq \mathcal{S}_x$. This is equivalent to requiring that for each i, $1 \leq i \leq k$, there exist a linear ordering (by refinement) of the set of families $\{\mathcal{S}_v : f(v) = i\}$. The *linear chromatic number* of $\mathcal{S}$, denoted $\lambda(\mathcal{S})$, is the smallest $k > 0$ for which a linear k-coloring exists.

The purpose of this article is to study linear colorings of complete Sperner families, addressing several of the questions raised by Cıvan in [1]. Our first result, proved in Section 3, is an exact formula for the number of complete Sperner families of linear chromatic number 2 on a finite set; this is an analogue (albeit in a more limited context) of the classical enumeration problem introduced by Dedekind. In Section 4 we study the maximum and minimum size of a complete Sperner family of specified chromatic number, generalizing Sperner's Theorem [6]. Finally, in Section 5 we prove that complete Sperner families of maximum linear chromatic number far outnumber those of lesser chromatic number. The common idea undergirding all the results in this article is that one may study a Sperner family of linear chromatic number m on a set of size n by associating to it to an antichain in the poset $L_{n_1} \times \ldots \times L_{n_m}$, where L_k is the poset $\{0, 1, \ldots, k\}$ with the usual order and $n = n_1 + \ldots + n_m$ is some partition of n into positive integer parts. The technical framework for establishing this correspondence is laid down in Section 2 and is used in each of the subsequent sections.

Throughout this article we write $Sp_n(S)$ for the set of Sperner families of linear chromatic number n on a set S and $Sp(S) = \cup_n Sp_n(S)$. We denote by $ls(m; n)$ the number of complete Sperner families of linear chromatic number n on a set of size m; it is easy to see that $ls(m; 1) = 1$ for any $m > 0$ and that $ls(m; n) = 0$ when $n > m$. We write $[a, b]$ to mean $\{n \in \mathbb{Z} : a \leq n \leq b\}$.

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2 Representations of Sperner Families

In this section we describe two convenient ways of representing Sperner families of specified linear chromatic number. The first involves antichains in chain products and applies to families of arbitrary chromatic number. The second applies only to families of linear chromatic number 2 but is particularly helpful in the solution of the enumeration problem addressed in Section 3.

2.1 Chain Products

We begin by defining chain products and establishing some of their basic properties; we then proceed to describe their signifiance in the context of linear colorings of Sperner families.

In general, a poset P is called *ranked* (or *graded*) if there exists a *ranking function* $r : P \to \mathbb{Z}$ such that $r(y) = r(x) + 1$ whenever y covers x. The ith *level* of P is then defined by $N_i = \{x \in P : r(x) = i\}$. If the ranking function has range $[0, m]$, P is called *rank-symmetric* if $|N_i| = |N_{m-i}|$ for all i and *rank-unimodal* if there exists $\mu \in [0, m]$ such that $|N_i| \leq |N_j|$ for $0 \leq i \leq j \leq \mu$ and $|N_i| \geq |N_j|$ for $\mu \leq i \leq j \leq m$. Finally, P is said to have the *Sperner property* if the maximum size of an antichain in P is equal to the size of the largest level of P.

For any positive integer n, we denote by L_n the usual linear order with underlying set $\{0, 1, \dots, n\}$. Now let $n_1, \dots, n_s$ be positive integers; a poset of the form $C_{n_1,\dots,n_s} = L_{n_1} \times \dots L_{n_s}$ is called a *chain product*. Note that $C_{n_1,\dots,n_s}$ has the natural structure of a ranked poset via the function $r(a_1, \dots, a_s) = a_1 + \dots + a_s$. We record the following useful fact.

Proposition 2.1. *[4, p. 230] Chain products are rank-symmetric, rank-unimodal, and have the Sperner property.*

We now describe a natural connection between linear colorings of Sperner families and antichains in chain products. An *ordered partition* of a set S is an n-tuple $\mathcal{P} = (S_1, \dots, S_n)$ such that $\{S_1, \dots, S_n\}$ is a partition of S. Given an ordered partition $\mathcal{P}$ as above, a *ranking* of $\mathcal{P}$ is an n-tuple $R = (r_1, \dots, r_n)$ of bijective functions $r_i : S_i \to \{1, \dots, |S_i|\}$, $i = 1, \dots, n$. Now let $M_n(S)$ denote the set of ordered triples $(\mathcal{P}, R, \mathcal{C})$ where $\mathcal{P} = (S_1, \dots, S_n)$ is an ordered partition of S, $R = (r_i)_{i=1}^n$ is a ranking of $\mathcal{P}$, and $\mathcal{C}$ is antichain in $C_{|S_1|,\dots,|S_n|}$. Then there is a function

$$\Phi : M_n(S) \to Sp_n(S)$$

defined as follows: given a triple $(\mathcal{P}, R, \mathcal{C}) \in M_n(S)$, each element $c \in \mathcal{C}$ corresponds to an n-tuple $(c_1, \dots, c_n)$, where for each i, $0 \leq c_i \leq |S_i|$. Now define $c_S = \bigcup_{i=1}^{n} \{s_i \in S_i : r_i(s_i) \leq c_i\}$. It is easy to check that $\Phi(\mathcal{P}, R, \mathcal{C}) = \{c_S : c \in \mathcal{C}\}$ is a Sperner family in S and that the coloring $\lambda : S \to \{1, \dots, n\}$ defined by sending $s \in S$ to the (unique) i such that $s \in S_i$ is an n-linear coloring of this Sperner family.

Proposition 2.2. Φ *is surjective; moreover, if* $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$*, then* $|\mathcal{C}| = |\mathcal{S}|$*.*

Proof.
Let $\mathcal{S}$ be an n-linearly colorable Sperner family on S. Then there is a partition of S into color classes $S_1, \ldots, S_n$, and for each $i = 1, \ldots n$ there exists a bijection $r_i : S_i \to [1, \ldots, |S_i|]$ such that $\mathcal{S}_x \subseteq \mathcal{S}_y$ if $r_i(y) \leq r_i(x)$. Now consider the element of $M_n(S)$ described by the partition $\mathcal{P} = (S_1, \ldots, S_n)$, the ranking $R = (r_i)_{i=1}^n$, and the antichain $\mathcal{C} = \{C(A) : A \in \mathcal{S}\}$ in $C_{|S_1|,\ldots,|S_n|}$, where $C(A) = (c_1, \ldots, c_n)$ is defined by $c_i = \max\{0\} \cup r_i(A \cap S_i)$. Clearly $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$, and the equality $|\mathcal{C}| = |\mathcal{S}|$ is obvious from the construction. $\square$

2.2 Directed bipartite graph representation

In this section we introduce a convenient way of representing a Sperner family of linear chromatic number 2 by a particular labeled, directed bipartite graph. Since the results of Section 2.1 imply that each such Sperner family is represented by a unique triple in $M_2(S)$, we show how to associate a directed graph to each such triple and vice versa.

Let $\mathcal{B}$ denote the set of pairs (D, h): here D is a directed bipartite graph of maximum degree 1 with bipartite sets $X = [0, c]$ and $Y = [0, d]$, in which $c + d = |S|$; moreover, all edges originate in X and terminate in Y, and we require that for any pair $x_1 \to y_1, x_2 \to y_2$ of distinct directed edges, $x_1 - x_2$ and $y_1 - y_2$ have opposite signs. The map $h : (X - \{0\}) \cup (Y - \{0\}) \to S$ is required to be bijective; in other words, h is a labeling by elements of S of all vertices in D except $0 \in X$ and $0 \in Y$.

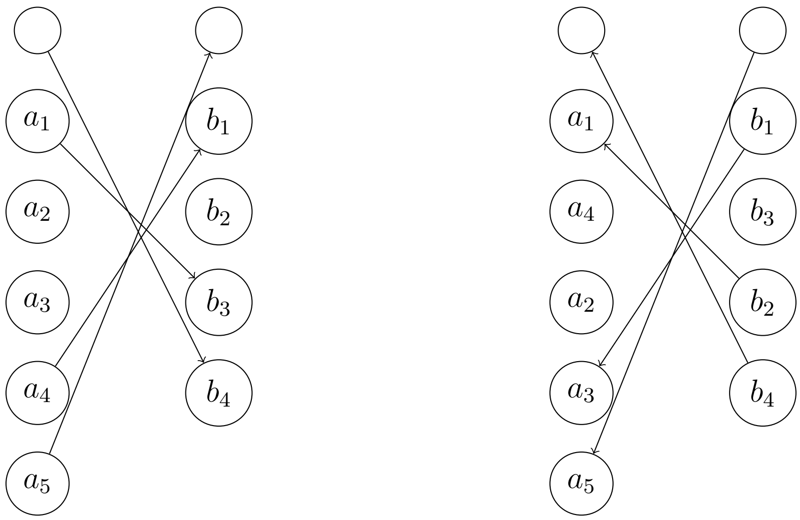
We now establish a bijection

$$\alpha : M_2(S) \to \mathcal{B}$$

Given a triple $t = (\mathcal{P}, R, \mathcal{C}) \in M_2(S)$, set $\alpha(t) = (D, h)$, where D is the directed bipartite graph with partite sets $X = [0, |S_1|]$ and $Y = [0, |S_2|]$ and a directed edge from $x \in X$ to $y \in Y$ if and only if $(x, y) \in \mathcal{C}$; we define the labeling $h : (X - \{0\}) \cup (Y - \{0\}) \to S$ by $h(z) = r_1^{-1}(z)$ if $z \in X - \{0\}$ or $h(z) = r_2^{-1}(z)$ if $z \in Y - \{0\}$.
To construct an inverse $\beta : \mathcal{B} \to M_2(S)$ for α, associate to a pair $(D, h) \in \mathcal{B}$ the triple $(\mathcal{P}, R, \mathcal{C}) \in M_2(S)$ where $\mathcal{P} = (h(S_1), h(S_2))$, S_1 being the partite set of D containing (all) the initial vertices of the edges and S_2 the other partite set. The ranking $R = (r_1, r_2)$ is defined by $r_1(x) = h^{-1}(x)$ when $x \in h(S_1)$ and $r_2(y) = h^{-1}(y)$ when $y \in h(S_2)$, and the antichain $\mathcal{C}$ is defined by $(x, y) \in \mathcal{C} \Leftrightarrow h^{-1}(x) \to h^{-1}(y) \in E(D)$.

It is clear that α and β are inverse to each other. This visual representation of elements of $M_2(\mathcal{S})$ is particularly convenient in that it is easy to read off the members of the

Figure 1: Representations of the 2-colorable Sperner family $\mathcal{S} = \{\{b_1, b_2, b_3, b_4\}, \{a_1, b_1, b_2, b_3\}, \{a_1, a_2, a_3, a_4, b_1\}, \{a_1, a_2, a_3, a_4, a_5\}\}$ on $S = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4\}$.



associated Sperner family $\Phi(\beta(D,h))$: to a directed edge $x \to y$ of D we associate the member of $\Phi(\beta(D,h))$ given by $\{h(z) : z \in X, 1 \le z \le x\} \cup \{h(z) : z \in Y, 1 \le z \le y\}$.

In Figure 1 we exhibit two representations for the Sperner family $\mathcal{S} = \{\{b_1, b_2, b_3, b_4\}, \{a_1, b_1, b_2, b_3\}, \{a_1, a_2, a_3, a_4, b_1\}, \{a_1, a_2, a_3, a_4, a_5\}\}$ on the set $S = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4\}$. The directed graph on the left corresponds to the triple $(\mathcal{P}, (r_1, r_2), \mathcal{C})$ where $\mathcal{P} = (\{a_1, a_2, a_3, a_4, a_5\}, \{b_1, b_2, b_3, b_4\})$, $r_1(a_i) = i$ for $1 \le i \le 5$ and $r_2(b_j) = j$ for $1 \le j \le 4$, and $\mathcal{C} = \{(0,4), (1,3), (4,1), (5,0)\}$. The directed graph of the right corresponds to the triple $(\mathcal{P}', (r_1', r_2'), \mathcal{C}')$ where $\mathcal{P}' = (\{b_1, b_2, b_3, b_4\}, \{a_1, a_2, a_3, a_4, a_5\})$ and $\mathcal{C}' = \{(0,5), (1,4), (3,1), (4,0)\}$; the ranking functions are defined by $r_1'(b_1) = 1$, $r_1'(b_2) = 3$, $r_1'(b_3) = 2$, $r_1'(b_4) = 4$ and $r_2'(a_1) = 1$, $r_2'(a_2) = 3$, $r_2'(a_3) = 4$, $r_2'(a_4) = 2$, $r_2'(a_5) = 5$.

3 On $ls(m;2)$

The goal of this section is to give a formula for $ls(m;2)$, the number of complete Sperner families of linear chromatic number 2 on a set of size m. Our first step is to reduce this calculation to a simpler problem, namely that of enumerating the so-called primitive complete Sperner families. To this end, let S be a set and $\mathcal{S}$ a Sperner family on S. We define the set of *base points* of $\mathcal{S}$ to be the subset of elements of S contained in every set in the family $\mathcal{S}$; that is, $B(\mathcal{S}) = \{x \in S : \mathcal{S}_x = \mathcal{S}\}$. If $B(\mathcal{S}) = \emptyset$, $\mathcal{S}$ is called *primitive*.

It is easy to see that completeness and primitivity are dual to each other, in the sense that a Sperner family $\mathcal{S}$ on a set S is complete if and only if the Sperner family $\mathcal{S}^C = \{S \setminus A : A \in \mathcal{S}\}$ is primitive. Of course, we also have $((\mathcal{S})^C)^C = \mathcal{S}$.

Lemma 3.1. *Let S be a set and $T \subseteq S$ a subset. Then there is a bijective correspondence:*

$$h : \{\mathcal{S} \in Sp(S) : B(\mathcal{S}) = T\} \to \{\mathcal{T} \in Sp(S \setminus T) : \mathcal{T} \textit{ is primitive}\}$$

Moreover, h preserves linear chromatic number.

Proof.
Given a Sperner family $\mathcal{S}$ on S with $B(\mathcal{S}) = T$, it is easy to check that $h(\mathcal{S}) = \{A \setminus T : A \in \mathcal{S}\}$ is a primitive Sperner family on $S \setminus T$ and that an inverse function is given by sending a primitive Sperner family $\mathcal{T}$ on $S \setminus T$ to $j(\mathcal{T}) = \{A \cup T : A \in \mathcal{T}\}$, which is a Sperner family on S whose set of base points is T. Now suppose $f : S \to \{1, \ldots, k\}$ is a k-linear coloring of $\mathcal{S}$; that is, for each i, $\{\mathcal{S}_x : f(x) = i\}$ is linearly ordered by refinement. Then $\{[h(\mathcal{S})]_x : x \in S \setminus T, f(x) = i\}$ is a set of families of subsets of $S \setminus T$, also linearly ordered by refinement. This implies that $f|_{S\setminus T} : S \setminus T \to \{1, \ldots, k\}$ is a k-linear coloring of $h(\mathcal{S})$, and so $\lambda(h(\mathcal{S})) \leq \lambda(\mathcal{S})$. Conversely, if $g : S \setminus T \to \{1, \ldots, k\}$ is a k-linear coloring of a primitive Sperner family $\mathcal{T}$ on $S \setminus T$, then by extending g in any way to a function $f : S \to \{1, \ldots, k\}$, we obtain a k-linear coloring of $j(\mathcal{T})$; this is so because $[j(\mathcal{T})]_x = j(\mathcal{T})$ for any $x \in T$. Hence, $\lambda(j(\mathcal{T})) \leq \lambda(\mathcal{T})$. Since j and h are inverse to each other, the claim is proven. □

Denoting by $ls_p(m;n)$ the number of complete *primitive* Sperner families of linear chromatic number n on a set of cardinality m, the following easy consequence of Lemma 3.1 permits the reduction of our enumeration problem to the primitive case.

Corollary 3.2. $ls(m;n) = \displaystyle\sum_{k=n}^{m} \binom{m}{k} ls_p(k;n)$.

We now describe the computation of $ls_p(m;2)$. The main idea is to use the surjectivity of the map Φ of Section 2.1 to argue that each complete primitive Sperner family of linear chromatic number 2 is equal to $\Phi(t)$ for some $t \in M_2(S)$, and then to calculate $|\Phi^{-1}(\Phi(t))|$ by exploiting bijectivity of the map α of Section 2.2.

To begin, we establish that the color classes in any 2-linear coloring of a complete primitive Sperner family $\mathcal{S}$ of linear chromatic number 2 are uniquely determined by $\mathcal{S}$ itself.

Lemma 3.3. *With notation as above, suppose $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$.*

- *If $\mathcal{S}$ is complete, then $(|S_1|, b) \in \mathcal{C}$ for some b, $0 \leq b \leq |S_2|$ and $(a, |S_2|) \in \mathcal{C}$ for some a, $0 \leq a \leq |S_1|$.*
- *If $\mathcal{S}$ is primitive, then $(0, b) \in \mathcal{C}$ for some b, $0 \leq b \leq |S_2|$ and $(a, 0) \in \mathcal{C}$ for some a, $0 \leq a \leq |S_1|$.*
- *If $\mathcal{S}$ is both complete and primitive, then $(0, |S_2|), (|S_1|, 0) \in \mathcal{C}$.*

Proof.
If $(|S_1|, b) \notin \mathcal{C}$ for all b, $0 \leq b \leq |S_2|$, then (by construction of Φ), no subset in $\mathcal{S}$ contains $r_1^{-1}(|S_1|)$ and $\mathcal{S}$ is not complete; a similar argument establishes that $(a, |S_2|) \in \mathcal{C}$ for some a if $\mathcal{S}$ is complete. If $(0, b) \notin \mathcal{C}$ for all b, $0 \leq b \leq |S_2|$, then $r_1^{-1}(1)$ is a base point of $\mathcal{S}$ and so $\mathcal{S}$ is not primitive; a similar argument shows that $(a, 0) \in \mathcal{C}$ for some a if $\mathcal{S}$ is primitive. If $\mathcal{S}$ is both complete and primitive, then the first two statements together with $\mathcal{C}$ being an antichain force $(0, |S_2|), (|S_1|, 0) \in \mathcal{C}$. $\square$

Definition 3.4. *Let S be a set and $\mathcal{S}$ a Sperner family on S. For $x, y \in S$, we say that $\mathcal{S}$* separates *x from y if $\mathcal{S}_x \nsubseteq \mathcal{S}_y$. We say that x and y are* separated *by $\mathcal{S}$ if $\mathcal{S}_x \nsubseteq \mathcal{S}_y$ and $\mathcal{S}_y \nsubseteq \mathcal{S}_x$.*

Observe that in any linear coloring of $\mathcal{S}$, elements separated by $\mathcal{S}$ must lie in different color classes.

Lemma 3.5. *Let S be a set and $\mathcal{S}$ a complete primitive Sperner family on S of linear chromatic number 2. Then there exists a partition of S into subsets S_1 and S_2 such that:*

1. *$S_1, S_2 \in \mathcal{S}$.*
2. *$\mathcal{S}$ separates two elements x and y if and only if one of these elements lies in S_1 and the other in S_2.*

3. *Every linear coloring of $\mathcal{S}$ with two colors has color classes S_1 and S_2, and so S_1 and S_2 are uniquely determined (up to interchanging).*

Proof.
Fix a linear coloring $f : S \to \{1,2\}$; let $S_1 = f^{-1}(1) = \{x_1, \ldots, x_p\}$, $S_2 = f^{-1}(2) = \{y_1, \ldots, y_q\}$, where $\mathcal{S}_{x_i} \supseteq \mathcal{S}_{x_j}$ and $\mathcal{S}_{y_i} \supseteq \mathcal{S}_{y_j}$ for all $i < j$. By definition, no element of S_1 is separated by $\mathcal{S}$ from any other, and similarly for S_2. Since $\mathcal{S}$ is primitive, y_1 is not a base point of $\mathcal{S}$, so there exists some $A \in \mathcal{S}$ such that $y_1 \notin A$. This implies that $y_j \notin A$ for all $1 \leq j \leq q$, so we must have $A \subseteq S_1$. Since $\mathcal{S}$ is complete, there exists some $B \in \mathcal{S}$ such that $x_p \in B$. Hence B contains all x_i, $1 \leq i \leq p$, so $B \supseteq S_1$. Since $A \subseteq S_1 \subseteq B$, and A and B are both members of the Sperner family $\mathcal{S}$, $A = B = S_1$ and so $S_1 \in \mathcal{S}$. A similar argument establishes that $S_2 \in \mathcal{S}$. It is now clear that $\mathcal{S}$ separates x from y if and only if one lies in S_1 and the other in S_2. It follows that the color classes associated to *any* linear coloring must be S_1 and S_2. $\square$

If $t = (\mathcal{P}, R, \mathcal{C}) \in M_2(S)$ is a triple with $\mathcal{P} = (S_1, S_2)$, $R = (r_1, r_2)$, we define the *opposite triple* $t^{op} = (\mathcal{P}^{op}, R^{op}, \mathcal{C}^{op})$ by $\mathcal{P}^{op} = (S_2, S_1)$, $R^{op} = (r_2, r_1)$ and $\mathcal{C}^{op} = \{(y, x) \in C_{|S_2|,|S_1|} : (x, y) \in \mathcal{C}\}$. In terms of the directed graph representation, if $\alpha(t) = (D, h)$, then $\alpha(t^{op}) = (D^R, h)$, where D^R is the reverse of D. It is clear from the definition that $\Phi(t^{op}) = \Phi(t)$. In light of this interpretation, the next result essentially asserts that the directed graph associated to a triple t is determined (up to reversal of arrows) by the Sperner family $\Phi(t)$.

Lemma 3.6. *Suppose $\mathcal{S}$ is a complete primitive Sperner family of linear chromatic number 2 on a set S. If $\mathcal{S} = \Phi(\mathcal{P}, R, \mathcal{C}) = \Phi(\mathcal{P}', R', \mathcal{C}')$, then either $\mathcal{P} = \mathcal{P}'$ and $\mathcal{C} = \mathcal{C}'$ or $\mathcal{P} = \mathcal{P}^{op}$ and $\mathcal{C}' = \mathcal{C}^{op}$.*

Proof.
That $\mathcal{P}'$ equals $\mathcal{P}$ or $\mathcal{P}^{op}$ follows from Lemma 3.5. Since $\Phi(\mathcal{P}', R', \mathcal{C}') = \Phi(\mathcal{P}'^{op}, R'^{op}, \mathcal{C}'^{op})$, we may assume that $\mathcal{P}' = \mathcal{P}$. First observe that $|\mathcal{C}| = |\mathcal{S}| = |\mathcal{C}'|$ by Proposition 2.2. We proceed to prove by induction on $k = |\mathcal{C}| = |\mathcal{C}'|$ that if $\Phi(\mathcal{P}, R, \mathcal{C}) = \Phi(\mathcal{P}, R', \mathcal{C}') = \mathcal{S}$, then $\mathcal{C} = \mathcal{C}'$. By Lemma 3.3, $(0, |S_2|)$ and $(|S_1|, 0)$ are in both $\mathcal{C}$ and $\mathcal{C}'$, so $k \geq 2$, and if $k = 2$, then $\mathcal{C} = \mathcal{C}'$. Now suppose $k > 2$. Let $c = (x, y) \in \mathcal{C}$ be the element for which x is positive and minimal (among all such choices); likewise, let $c' = (x', y') \in \mathcal{C}'$ be the element for which x' is positive and minimal. Then the elements of $Z_1 = \{z \in S_1 : 1 \leq r_1(z) \leq x\}$ are exactly those which appear in all but one element of $\mathcal{S}$; yet the same is true of $Z_1' = \{z \in S_1 : 1 \leq r_1'(z) \leq x'\}$, so it must be the case that $x = x'$. Similarly, the elements of $W = \{w \in S_2 : y \leq r_2(w) \leq |S_2|\}$

and $W' = \{w \in S_2 : y' \le r_2(w) \le |S_2|\}$ are exactly those which appear in a unique element of $\mathcal{S}$, so $y = y'$. Then $\mathcal{D} = \mathcal{C} \setminus \{(x, y)\}$ and $\mathcal{D}' = \mathcal{C}' \setminus \{(x, y)\}$ are antichains of size $k - 1$ with $\Phi(\mathcal{P}, R, \mathcal{D}) = \Phi(\mathcal{P}, R', \mathcal{D}') = \mathcal{S} \setminus \{A\}$, so by induction $\mathcal{D} = \mathcal{D}'$. Thus, $\mathcal{C} = \mathcal{C}'$, as desired. $\square$

We now come to our main result. For convenience, write $S \sqsubseteq [a, b]$ to mean a subset $S \subseteq [a, b]$ containing both a and b.

Theorem 3.7. *Suppose $m \ge 2$. Then*

$$ls_p(m; 2) = \frac{m!}{2} \sum_{1 \le k \le m-1} \sum_{I \sqsubseteq [0,k]} \sum_{\substack{J \sqsubseteq [0,m-k] \\ |J|=|I|}} \frac{1}{e(I)e(J)}$$

$$= m!\Big(\sum_{1 \le k \le \lfloor m/2 \rfloor} \sum_{I \sqsubseteq [0,k]} \sum_{\substack{J \sqsubseteq [0,m-k] \\ |J|=|I|, J \ne I}} \frac{1}{e(I)e(J)} + \frac{1}{2} \sum_{I \sqsubseteq [m/2]} \frac{1}{e(I)^2} \Big)$$

Proof.
We prove the first formula, as the second then follows immediately. Given any complete primitive Sperner family $\mathcal{S}$ on S of linear chromatic number 2, Lemma 3.6 implies that if we fix a triple $t = (\mathcal{P}, R, \mathcal{C}) \in M_2(S)$ with $\Phi(t) = \mathcal{S}$, then

$$|\Phi^{-1}(\mathcal{S})| = 2|\{R' : \Phi(\mathcal{P}, R', \mathcal{C}) = \mathcal{S}\}|$$

Now if we let $\alpha(\mathcal{P}, R, \mathcal{C}) = (D, h)$ then $\alpha(\mathcal{P}, R', \mathcal{C}) = (D, h')$ for some other labeling $h' : V'(D) \to S$, so we have:

$$|\Phi^{-1}(\mathcal{S})| = 2|\{h' : V'(D) \to S,\ h' \text{ is bijective },\ \Phi(\beta(D, h')) = \mathcal{S}\}|$$

To compute the term on the right, let $I = \{0 = a_0, a_1, \ldots, a_k = |S_1|\} \subseteq [0, |S_1|]$ be the set of initial vertices of edges in D and $J = \{0 = b_0, b_1, \ldots, b_k = |S_2|\} \subseteq [0, |S_2|]$ the set of terminal vertices, where the notation is chosen so that the sequences $\{a_i\}$ and $\{b_i\}$ are increasing. By construction, if $1 \le i \le k$ and $a_{i-1} < x \le a_i$, the element of S labeled $h(x)$ appears in exactly i members of the Sperner family $\mathcal{S}$; similarly, if $b_{i-1} < y \le b_i$, then $h(y)$ appears in exactly i members of $\mathcal{S}$. If one obtains the same Sperner family upon replacing h by h', it must be the case that for all i, $1 \le i \le k$, h' must be obtained from h by permutation of the labels on the various sets of vertices $A_i = \{x \in X : a_{i-1} < x \le a_i\}$ and $B_i = \{y \in Y : b_{i-1} < y \le b_i\}$, $1 \le i \le k$. Conversely, it is clear that any labeling obtained from h by permutation of these sets of labels will yield the same Sperner family. Hence if we define the *elasticity* of a

Table 1:

m	$ls_p(m;2)$	$ls(m;2)$
2	1	1
3	3	6
4	19	37
5	135	270
6	1351	2521
7	11823	24906

subset $C = \{c_0, \ldots, c_k\} \subseteq [0, n]$ (where $c_0 < \ldots < c_k$) by $e(C) = \prod_{i=1}^{k}(c_i - c_{i-1})!$, the number of such labelings h' is precisely $e(I)e(J)$, so $|\Phi^{-1}(\mathcal{S})| = 2e(I)e(J)$.

Now if we wish to sum over *all* complete primitive Sperner families, we need to consider all such possible bipartite directed graphs and all possible labelings of such graphs. The bipartition is determined uniquely by an integer $1 \leq k \leq m-1$; having fixed this, the edges are determined by specifying subsets $I \subseteq [0, k]$ and $J \subseteq [0, m-k]$ of equal size, and a labeling is simply a permutation of $\{1, \ldots, m\}$.

Finally, for convenience we write $Sp_2'(S)$ for the set of complete primitive Sperner families of linear chromatic number 2, $M_2'(S) = \Phi^{-1}(Sp_2'(S))$, and $\mathcal{B}' = \alpha(M_2'(S))$. Then

$$ls_p(m;2) = \sum_{\mathcal{S} \in Sp_2'(S)} 1 = \sum_{t \in M_2'(S)} \frac{1}{|\Phi^{-1}(\Phi(t))|} = \sum_{(D,h) \in \mathcal{B}'} \frac{1}{|\Phi^{-1}(\Phi(\beta(D,h)))|}$$

$$= \sum_{k=1}^{m-1} \sum_{I \subseteq [0,k]} \sum_{J \subseteq [0,m-k] : |J| = |I|} \frac{m!}{2e(I)e(J)} = \frac{m!}{2} \sum_{I \subseteq [0,k]} \sum_{J \subseteq [0,m-k] : |J| = |I|} \frac{1}{e(I)e(J)}$$

□

When making computations, the second formula of Theorem 3.7 is usually more convenient. Table 1 displays the values of $ls_p(m;2)$ and $ls(m;2)$ for small values of m, confirming the calculations in [1, Section 2].

4 On the size of Sperner families of specified chromatic number

A well-known result of Sperner asserts that the size of a maximum antichain in the Boolean lattice B_m (or equivalently the maximum size of a Sperner family on $S = [m]$) is $\binom{m}{\lfloor m/2 \rfloor}$; this is met by considering the family of subsets of S of size $\lfloor m/2 \rfloor$. A trivial lower bound of 1 is met by the Sperner family $\{S\}$. In this section we address the analogous question, raised in [1], of finding upper and lower bounds for the size of a Sperner family with specified linear chromatic number. For convenience, we introduce some notation. Given $m \geq n$, write $m = qn + r$ for some integers q, r, $0 \leq r < n$, and define:

$$s(m,n) = |\{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i = \lfloor m/2 \rfloor,\ 0 \leq x_i \leq \lceil m/n \rceil \text{ for } 1 \leq i \leq r$$
$$\text{and } 0 \leq x_i \leq \lfloor m/n \rfloor \text{ for } r+1 \leq i \leq n\}|$$

and

$$t(n) = \min\{t : \binom{t}{\lfloor t/2 \rfloor} \geq n\}.$$

Theorem 4.1. *Let S be a set of cardinality m and $\mathcal{S}$ a complete Sperner family on S of linear chromatic number n. Then $t(n) \leq |\mathcal{S}| \leq s(m,n)$.*

Proof.
For the lower bound, let S be a set with m elements and suppose $\mathcal{S}$ is a Sperner family of cardinality t, where $\ell = \binom{t}{\lfloor t/2 \rfloor} < n$. After fixing a bijection between $\mathcal{S}$ and $[t]$, we may identify, for each $x \in S$, $\mathcal{S}_x$ with a subset $T_x \subseteq [t]$. By a now well-known result [3], there exists a partition of the poset $2^{[t]}$ into disjoint chains $C_1, \dots C_\ell$; thus, we may construct an ℓ-linear coloring $f : S \to [\ell]$ of $\mathcal{S}$ by declaring $f(x)$ to be the unique index i such that T_x is a member of C_i. Hence $\lambda(\mathcal{S}) \leq \ell < n$.

For the upper bound, note that Proposition 2.2 allows us to reduce our question to that of finding the size of a maximum antichain in a chain product $C_{m_1,\dots,m_n}$, where $m_1, \dots, m_n$ are positive integers whose sum is m. By Proposition 2.1, the size of the largest antichain is equal to $|\{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i = \lfloor m/2 \rfloor, 0 \leq x_i \leq m_i\}|$. Hence, the problem is reduced to finding the values of $m_1, \dots, m_n$ satisfying $\sum_{i=1}^{n} m_i = m$ which maximize this quantity. We claim that this is achieved by choosing each m_i

to be either $\lfloor m/n \rfloor$ or $\lceil m/n \rceil$; clearly there is only one way of achieving this, up to permutation of the m_i. Suppose instead that $m_1 \le \lfloor m/n \rfloor - 1$ and $m_2 \ge \lceil m/n \rceil + 1$; we will show that any such choice of $m_1, m_2, \ldots, m_n$ does not yield a maximum. If $n = 2$, $s(m_1, m_2) = m_1 + 1 < m_1 + 2 = s(m_1 + 1, m_2 - 1)$. If $n > 2$, then

$$s(m_1 + 1, m_2 - 1, m_3, \ldots, m_n) - s(m_1, m_2, m_3, \ldots, m_n)$$

$$\begin{aligned}
= |\{(x_2, \ldots, x_n) : \sum_{i=2}^{n} x_i = \lfloor m/2 \rfloor - m_1 - 1 : 0 \le x_2 \le m_2 - 1, 0 \le x_i \le m_i \text{ for } i \ge 3\}| \\
-|\{(x_1, x_3, \ldots, x_n) : x_1 + \sum_{i=3}^{n} x_i = \lfloor m/2 \rfloor - m_2 : 0 \le x_i \le m_i \text{ for all } i\}| \\
= \sum_{j=0}^{m_2-1} |\{(x_3, \ldots, x_n) : \sum_{i=3}^{n} x_i = \lfloor m/2 \rfloor - m_1 - 1 - j\}| \\
- \sum_{j=0}^{m_1} |\{(x_3, \ldots, x_n) : \sum_{i=3}^{n} x_i = \lfloor m/2 \rfloor - m_2 - j\}| \\
= \sum_{j=m_1+1}^{m_2-1} |\{(x_3, \ldots, x_n) : \sum_{i=3}^{n} x_i = \lfloor m/2 \rfloor - j\}|
\end{aligned}$$

Since $m_1 \le m_2 - 2$, this sum has at least one term in it and hence is positive. $\square$

Remarks.

1. If $\ell = \binom{t(n)}{\lfloor t(n)/2 \rfloor} \le m$, we can show that the lower bound is tight. Consider the family $\mathcal{A} = \{A \subseteq [t(n)] : |A| = \lfloor t(n)/2 \rfloor\}$ of subsets of $[t(n)]$. For each $x \in [t(n)]$, let $\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\}$. Then $\mathcal{T} = \{\mathcal{A}_x : x \in [t(n)]\}$ is a Sperner family on $\mathcal{A}$. Since any two elements of $\mathcal{A}$ are separated by $\mathcal{T}$, the linear chromatic number of $\mathcal{T}$ must be exactly ℓ. Finally, if S is any set with m elements, fix a bijection $h : [\ell] \to L$ where $L \subseteq S$ is any subset of cardinality ℓ. By Lemma 3.1, the complete Sperner family $\mathcal{S} = \{h(X) \cup (S - L) : X \in \mathcal{T}\}$ of subsets of S has linear chromatic number ℓ. To show tightness of the upper bound, write $m = qn + r$ for $q, r \in \mathbb{Z}$ and $0 \le r < n$, and partition S into n subsets $S_1, \ldots, S_n$ such that $|S_i| = \lceil m/n \rceil$ for $1 \le i \le r$ and $|S_i| = \lfloor m/n \rfloor$ for $r + 1 \le i \le n$. Then $\mathcal{S} = \{\cup_{i=1}^{n} T_i : T_i \subseteq S_i \text{ for all } i,\ \sum_{i=1}^{n} |T_i| = \lfloor m/2 \rfloor\}$ is a Sperner family on S of size $s(m, n)$.

2. Note that when $n = m$, the upper bound of Theorem 4.1 is merely $\binom{m}{\lfloor m/2 \rfloor}$. This is not surprising in view of the fact the Sperner family on S consisting of all subsets of size $\lfloor m/2 \rfloor$ has linear chromatic number m.

5 Sperner families of maximum linear chromatic number

In this last section we address a question raised in [1] concerning unimodality of the numbers $ls(m;n)$: does there exist $k \leq m$ such that $ls(m;1) \leq \ldots \leq ls(m;k) \geq ls(m;k+1) \geq \ldots \geq ls(m;m)$? We show that for sufficiently large m, $ls(m;m)$ is much bigger than $\sum_{n<m} ls(m;n)$; thus the only hope for unimodality of the numbers $ls(m;n)$ would lie in a proof that they are increasing as a function of n.

Given a poset P, we denote by $a(P)$ the number of antichains in P. If P and Q are posets with the same underlying set, we say that Q is a *relaxation* of P if whenever y covers x in Q, y also covers x in P. It is moreover clear that $a(Q) \geq a(P)$. The main tool we will need is

Theorem 5.1. *([5, Theorem 1.3]) Let P be a graded poset with levels $P_1, \ldots, P_k$, with $|P_k| \leq M$. Suppose $d_+(x) \geq n$ for all $x \in P_1 \cup \ldots P_{k-1}$ and $d_-(x) \leq n$ for all $x \in P_2 \cup \ldots P_k$. Then $a(P) \leq (k2^n - (k-1))^{M/n}$.*

For our purposes, the weaker inequality $a(P) \leq k^{M/n}2^M = 2^{M(1+\frac{\log_2 k}{n})}$ suffices.

Proposition 5.2. *There are at least $2^{\binom{m}{\lfloor m/2 \rfloor}} - m2^{\binom{m-1}{\lfloor m/2 \rfloor}}$ complete Sperner families on a set of cardinality m.*

Proof.
Suppose $|S| = m$, let $\mathcal{B} = \{B \subseteq S : |B| = \lfloor m/2 \rfloor\}$. Observe that each of the $2^{\mathcal{B}} = 2^{\binom{m}{\lfloor m/2 \rfloor}}$ subsets $\mathcal{S} \subseteq \mathcal{B}$ is a Sperner family; it suffices to count the primitive families among these. If $\mathcal{S}$ is imprimitive, some $s \in S$ must be a base point of $\mathcal{S}$; that is, $\mathcal{S} \subseteq \mathcal{B}_s = \{B \in \mathcal{B} : s \in B\}$. Since $|\mathcal{B}_s| = \binom{m-1}{\lfloor m/2 \rfloor - 1}$ and there are m choices for s, it follows that there are at least $2^{\binom{m}{\lfloor m/2 \rfloor}} - m2^{\binom{m-1}{\lfloor m/2 \rfloor - 1}}$ complete Sperner families on S. $\square$

Remark.
The result of Proposition 5.2 indicates that – as one might expect – most Sperner families are complete. Indeed, $\binom{m-1}{\lfloor m/2 \rfloor - 1} = \frac{\lfloor m/2 \rfloor}{m}\binom{m}{\lfloor m/2 \rfloor} \leq \frac{1}{2}\binom{m}{\lfloor m/2 \rfloor}$, so for large values of m, $m2^{\binom{m-1}{\lfloor m/2 \rfloor}} \leq 2^{\log_2 m + \frac{1}{2}\binom{m}{\lfloor m/2 \rfloor}}$ is much smaller than $2^{\binom{m}{\lfloor m/2 \rfloor}}$.

To study complete linear Sperner families $\mathcal{S}$ of chromatic number at most $m-1$ on a set S with m elements, first note that any such family, being $(m-1)$-linearly colorable, is determined by a partition of S into $m-2$ singletons and one doubleton, an ordering of the two elements within that doubleton, and an antichain in the poset $P = C_{2,\underbrace{1,\ldots,1}_{m-2}}$. Clearly there are $\dfrac{m(m-1)}{2}$ such doubletons and two orderings for each choice, so to obtain an upper bound on the number of such $\mathcal{S}$, it remains to bound the number of antichains in P. To this end, we use Theorem 5.1, following a construction similar to that found in [5, Prop. 5.1]. Note that P is a ranked poset with levels L_i, $i = 0, \ldots, m$, where $\ell_i = |L_i| = \binom{m-2}{i} + \binom{m-2}{i-1} + \binom{m-2}{i-2} = \binom{m-1}{i} + \binom{m-2}{i-2}$. By Proposition 2.1, level $m/2$ is the largest level when m is even and levels $\lfloor m/2 \rfloor$ and $\lceil m/2 \rceil$ are the largest when m is odd.

Viewing P as a directed graph whose arcs originate at vertices in L_i and terminate at vertices in L_{i+1}, observe that for $x \in L_i$, $i-1 \le d_-(x) \le i$ and $m-i \le d_+(x) \le m-i+1$. Now suppose $i > \lceil m/2 \rceil$ and $x \in L_i$ with $d_-(x) > \lceil m/2 \rceil$; write $x = (a_1, \ldots, a_{m-1})$, where $0 \le a_1 \le 2$ and $0 \le a_j \le 1$ for $2 \le j \le m-1$. Then let $J_x = \{j : a_j \ne 0\}$ and let $K_x \subseteq J_x$ consist of the $i - \lceil m/2 \rceil$ largest elements of J_x. For each $k \in K_x$, let $x(k)$ denote the vertex $(b_1, \ldots, b_{m-1})$ in level $i-1$, where $b_j = a_j$ for $j \ne k$ and $b_k = 0$, and let $\mathcal{A}_x$ denote the set of arcs connecting $x(k)$ to x for all $k \in K_x$. Define P' to be the poset obtained from P by deleting the set of arcs $\bigcup_{i > \lceil m/2 \rceil} \bigcup_{x \in L_i} \mathcal{A}_x$. Observe that P' is a relaxation of P and that every vertex in P' has in-degree at most $\lceil m/2 \rceil$. Moreover, if $i > \lceil m/2 \rceil$, then every vertex in L'_i, the ith level of P', has in-degree exactly $\lceil m/2 \rceil$.

We now describe the construction of a poset Q of which P' is a subposet. For each $i > \lceil m/2 \rceil$, add $\ell_{\lceil m/2 \rceil} - \ell_i$ new vertices to L'_i, and add new arcs originating at vertices in L'_{i-1} and terminating at these new vertices in such a way that all vertices in the (new) $i-1$st level have out-degree $\lceil m/2 \rceil$ and all vertices in the (new) ith level have in-degree $\lceil m/2 \rceil$. Denote by Q the resulting poset. Now Q satisfies the hypotheses of Theorem 5.1 with $k = m+1$, $n = \lceil m/2 \rceil$, and

$$M = \ell_{\lceil m/2 \rceil} = \binom{m-1}{\lceil m/2 \rceil} + \binom{m-2}{\lceil m/2 \rceil - 2} \le \frac{3}{4}\left(1 + \frac{5}{4(m-2)}\right)\binom{m}{\lfloor m/2 \rfloor},$$

so $a(P) \le a(P') \le a(Q) \le 2^{\frac{3}{4}(1+\frac{5}{4(m-2)})\binom{m}{\lfloor m/2 \rfloor}(1+2\frac{\log_2 m+1}{m})}$.

Thus we have proven the following:

Proposition 5.3. *Let S be a set of size m. The number of $(m-1)$-colorable complete Sperner families on S of linear chromatic number is bounded above by:*

$$2^{\log_2(m)+\log_2(m-1)+\frac{3}{4}(1+\frac{5}{4(m-2)})\binom{m}{\lfloor m/2\rfloor}(1+2\frac{\log_2 m+1}{m})}$$

In particular, for any $\varepsilon > 0$, the number of such families is at most $2^{(\frac{3}{4}+\varepsilon)\binom{m}{\lfloor m/2\rfloor}}$ for sufficiently large m.

Combining this with the estimate of Proposition 5.2 and the succeeding remark, we have:

Corollary 5.4. *On a set of cardinality m, there are at least*

$$2^{\binom{m}{\lfloor m/2\rfloor}} - 2^{\log_2 m+\frac{1}{2}\binom{m}{\lfloor m/2\rfloor}} - 2^{\log_2(m)+\log_2(m-1)+\frac{3}{4}(1+\frac{5}{4(m-2)})\binom{m}{\lfloor m/2\rfloor}(1+2\frac{\log_2 m+1}{m})}$$

complete Sperner families of linear chromatic number m. In particular, for any $\varepsilon > 0$, the number of such families is at least

$$2^{\binom{m}{\lfloor m/2\rfloor}} - 2^{(3/4+\varepsilon)\binom{m}{\lfloor m/2\rfloor}} - 2^{(1/2+\varepsilon)\binom{m}{\lfloor m/2\rfloor}}$$

for sufficiently large m.

Therefore, the number of Sperner families of maximum linear chromatic number grows much faster (as a function of m) than the number of those of lesser chromatic number.

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