SUBMATRICES OF THE CAYLEY ADDITION TABLE FOR $\mathbb{Z}_n$

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"Few mathematical objects could be considered more simple than the Cayley addition table of $\mathbb{Z}_n$, yet we show that even these simple objects have some interesting yet unproved properties"(Snevily, 1999). Hunter Snevily proposed the following conjecture:

> Let n be any positive odd integer. Then, for any $k \in \{1, 2, ..., n\}$, the $k \times k$ submatrix of the Cayley addition table of $\mathbb{Z}_n$ contains a latin transversal.

A latin transversal is defined as a collection of n distinct entries of an $n \times n$ matrix, no two of which are in the same row or column. We show that every 4×4 submatrix contains a latin transversal. Additionally, we prove that any $n-2 \times n-2$ submatrix contains a latin transversal when n is prime. These results, together with previous results (Coleman, Hall, Hutchings and Ruhnke, 1999), establish the existence of at least one latin transversal in any 4×4 or smaller submatrix as well as in many $n-2 \times n-2$ submatrices and any $n-1 \times n-1$ submatrix of the $\mathbb{Z}_n$ Cayley addition table where n is odd.

In practical terms, this enables us to solve a variety of logistical problems in manufacturing, experimental design and other areas. An illustration of possible use is presented following the results.

Definitions. A *rightward diagonal* consists of entries of the form $a_{(i+m) \bmod n, (j+m) \bmod n}$ for $m = 0, 1, .., n-1$ and $0 \le i, j \le n-1$.

Similarly, a *leftward diagonal* consists of entries of the form $a_{(i+m) \bmod n, (j-m) \bmod n}$ for $m = 0, 1, .., n-1$ and $0 \le i, j \le n-1$.

Example. The boxed entries below comprise a leftward diagonal.

$$\begin{bmatrix} 0 & \boxed{\mathbf{1}} & 2 & 3 & 4 & 5 & 6 \\ \boxed{\mathbf{1}} & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & \boxed{\mathbf{1}} \\ 3 & 4 & 5 & 6 & 0 & \boxed{\mathbf{1}} & 2 \\ 4 & 5 & 6 & 0 & \boxed{\mathbf{1}} & 2 & 3 \\ 5 & 6 & 0 & \boxed{\mathbf{1}} & 2 & 3 & 4 \\ 6 & 0 & \boxed{\mathbf{1}} & 2 & 3 & 4 & 5 \end{bmatrix}$$

Theorem 1. *In any $k \times k$ submatrix of an $n \times n$ Cayley table, the sum of the elements of any transversal will be congruent, mod n. Furthermore, this sum is equal to the sum of all elements a_{i1},a_{1j} (mod n), where $1 \le i, j \le k$ when the submatrix is translated to have $a_{11} = 0$.*

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Proof. Note that any submatrix can be translated such that $a_{11} = 0$ (Coleman, Hall, Hutchings, and Ruhnke, 1999). Now each element $a_{ij} = a_{i1} + a_{1j}$ by construction, where a_{ij} is the element in the i^{th} row and j^{th} column. A transversal picks one element from each row and each column. Thus the sum of the elements in any transversal will be equal to the sum of the first elements of each row and each column of the translated submatrix, (mod n). ☐

Example. A 4×4 submatrix would be translated as follows:

$$\begin{bmatrix} 0 & a & b & c \\ d & a+d & b+d & c+d \\ e & a+e & b+e & c+e \\ f & a+f & b+f & c+f \end{bmatrix}$$

Note that the elements of any transversal sum to $a+b+c+d+e+f$.

Corollary 1. *The sums of the elements of the main right and left diagonals of any submatrix will always be equal, (mod n).*

Proof. The diagonals are both transversals. Thus by the above theorem, the elements will sum to the same number, (mod n). ☐

Note. Corollary 1 verifies that every 2×2 submatrix has a latin transversal (Theorem 7, Coleman, Hall, Hutchings, and Ruhnke, 1999).

Theorem 2. *In an $n \times n$ Cayley table, the sum of the elements of a transversal in any submatrix is equal to zero minus the sum of the first elements of each row and each column that are deleted, (mod n).*

Proof. Consider that every Cayley table can be written as follows:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & \cdots & n-1 \\ 1 & 1+1 & 1+2 & 1+3 & \cdots & 0 \\ 2 & 2+1 & 2+2 & 2+3 & \ldots & 1 \\ 3 & 3+1 & 3+2 & 3+3 & \ldots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-1 & 0 & 1 & 2 & \ldots & n-2 \end{bmatrix}$$

By Theorem 1, the sum of any transversal of the $n \times n$ matrix is equal to 0 (mod n) since the sum of the first elements of each row and each column is equal to $2 * 0 + 2 * 1 + ... + 2 * (n-1) \equiv 0$ (mod n). Now the sum of any transversal of the submatrix formed will be missing the first elements of each row and each column deleted. Thus the sum of any transversal will be equal to zero minus the sum of the first elements of each row and each column deleted, (mod n). ☐

Theorem 3. *Given two distinct elements a_{ij} and a_{kl} such that $i \neq k, j \neq l$ in an $n \times n$ Cayley table, if $(|k-i|, n) = 1$, $(|l-j|, n) = 1$, and $(k-i) + (l-j) \not\equiv 0$ (mod n), then a latin transversal can be found containing those elements. If $(k-i) + (l-j) \equiv 0$ (mod n), then a latin transversal can be found containing a_{il} and a_{kj}.*

Proof. Since $(|k-i|, n) = 1$, $k-i$ is a generator of $\mathbb{Z}_n$. Then $i + (k-i)s$, $s \in \mathbb{Z}$, also generates $\mathbb{Z}_n$. Similarly, since $(|l-j|, n) = 1$, $l-j$ is a generator of $\mathbb{Z}_n$ and thus $j+(l-j)s$ also generates $\mathbb{Z}_n$. Now if $(k-i)+(l-j) \not\equiv 0$

(mod n) $\{a_{xy}$ such that $x \equiv i + (k-i)s$ and $y \equiv j + (l-j)s \pmod n$ where $s \in \mathbb{Z}, 0 \le s \le n-1\}$ is a latin transversal containing a_{ij} and a_{kl}. If $(k-i)+(l-j) \equiv 0 \pmod n$ then $(k-j)+(l-i) \not\equiv 0 \pmod n$ since n is odd. Also if $a_{il} = a_{kj}$ then $i+l \equiv k+j \pmod n$ and by substitution, $k \equiv i \pmod n$, which contradicts that the submatrix is 2×2. Thus a_{il} and a_{kj} are distinct and a latin transversal can be found containing these two elements by an argument similar to the one above. □

Theorem 4. *If n is prime, then any $n-2 \times n-2$ submatrix will have a latin transversal.*

Proof. Consider the 2×2 formed by the intersections of the rows and columns deleted. By Note 1 and Theorem 3, this 2×2 will contain two distinct elements a_{ij} and a_{kl} such that $i \ne k, j \ne l$, and $(k-i)+(l-j) \not\equiv 0$ (mod n) . Since n is prime, $(|k-i|, n) = 1$ and $(|l-j|, n) = 1$ $\forall i, j, k, l$. Thus a latin transversal can be found in the $n \times n$ Cayley table containing these two elements, along with $n-2$ other distinct elements. These $n-2$ elements will be a latin transversal for the $n-2 \times n-2$ submatrix. □

Theorem 5. *In any submatrix of the $\mathbb{Z}_n$ Cayley addition table where n is odd, an element $a_{ij} = x$ is from the upper portion of the leftward diagonal of x's and the element $a_{pq} = x$ is from the lower portion of that diagonal iff $i < p$ and $j < q$.*

Proof. Let $x \in \{0, 1, .., n-1\}$. Suppose $a_{ij} = x$. In a Cayley table, the leftward diagonal containing a_{ij} has entries all equal to x. Furthermore, no other entries of the table will equal x. Notice that leftward diagonals wrap at most once as shown below and the two portions of a leftward diagonal leave no gaps nor do they overlap across rows or columns.

$$\begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 4 & 5 & 6 \\ \boxed{1} & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & \boxed{1} \\ 3 & 4 & 5 & 6 & 0 & \boxed{1} & 2 \\ 4 & 5 & 6 & 0 & \boxed{1} & 2 & 3 \\ 5 & 6 & 0 & \boxed{1} & 2 & 3 & 4 \\ 6 & 0 & \boxed{1} & 2 & 3 & 4 & 5 \end{bmatrix}$$

If we choose a_{ij} from the lower portion of the leftward diagonal, we cannot find an $a_{pq} = x$ such that $i < p$ and $j < q$. So let us choose a_{ij} from the upper portion of the diagonal. Since any portion of a leftward diagonal runs uphill from left to right, we can see that any a_{pq} where $i < p$ and $j < q$ (ie: a_{pq} is below and to the right of a_{ij}) must be from the lower portion of the diagonal.

Conversely, if we have an a_{ij} from the upper portion of the diagonal and an a_{pq} from the lower portion of the diagonal, a_{pq} must be below and to the right of a_{ij} (ie: $i < p$ and $j < q$) since the portions of the diagonal do not overlap across rows or columns. □

Corollary 2. *In any submatrix, if an element x is the a_{ij} entry and the a_{pq} entry where $i < p$ and $j < q$, then x is not the a_{st} entry for any $s > p$ and $t > q$.*

Proof. Since $i < p$ and $j < q$, a_{pq} is from the lower section of the leftward diagonal of x's, we know $p < s$ and $q < t$. But because leftward diagonals wrap at most once, there cannot be another lower section of x's from which to choose a_{st}. Therefore $a_{st} \ne x$. □

Corollary 3. *Once an element x appears as the a_{ij} and a_{pq} entries in a submatrix where $i < p$ and $j < q$, if any other element y appears as the a_{rs} entry where $r \geq p$ and $s \geq q$, then y cannot be the a_{tu} entry where $t > r$ and $u > s$.*

Proof. In Cayley tables, the first leftward diagonal consists of zero, the second consists of ones, and so on until the main leftward diagonal which has entries all equal to $n-1$. The diagonals then wrap around and the lower portion of each leftward diagonal follows in the same order as before up through $n-2$. In this way, we see that the lower portion of the leftward diagonal consisting of x's cannot precede the upper portion of any other leftward diagonal. By Theorem 5, we see that if a submatrix contains a second x below and to the right of a previous x, that second x is necessarily from the lower section of the leftward diagonal of x's. Therefore the y diagonal must already have wrapped around. So when we have a y, either in the same row or below the second x, and either in the same column or to the right of the second x, it must be from the lower section of the diagonal of y's. Therefore, there cannot be another y below and to the right of that y. ☐

Corollary 4. *If an element $a_{ij} = y$ appears to the right and below another element $a_{gh} = x$ and $a_{pq} = y$ appears to the right and below the first y, then there cannot be an x in any position a_{rs} where $r \geq p$ and $s \geq q$.*

Proof. Because $y = a_{ij}$ is to the right and below $x = a_{gh}$, we know that $y > x$ and the upper portion of the leftward diagonal of x's precedes the upper portion of the diagonal of y's. We can see this in the matrix below. As pointed out in the proof of Corollary 2, this order will not change when the diagonals wrap around. By Theorem 5, a_{pq} is from the lower portion of the diagonal of y's. There cannot be an x in any position a_{rs} where $r \geq p$ and $s \geq q$ since that would imply that $x > y$.

$$\begin{bmatrix} \boxed{\mathbf{0}} & 1 & \mathbf{2} & 3 & 4 & 5 & 6 \\ 1 & \boxed{\mathbf{2}} & 3 & 4 & 5 & 6 & 0 \\ \mathbf{2} & 3 & 4 & 5 & 6 & 0 & 1 \\ 3 & 4 & 5 & 6 & 0 & 1 & \mathbf{2} \\ 4 & 5 & 6 & 0 & 1 & \mathbf{2} & 3 \\ 5 & 6 & 0 & 1 & \boxed{\mathbf{2}} & 3 & 4 \\ 6 & 0 & 1 & \mathbf{2} & 3 & 4 & 5 \end{bmatrix}$$

Corollary 5:. *If x appears as the a_{ij} entry and as the a_{pq} entry where $i < p$ and $j < q$, then x can appear as neither the a_{rs} entry where $r > p$ and $s < j$ nor the a_{tu} entry where $t < i$ and $u > q$.*

Proof. By Theorem 5 we know that a_{ij} is from the upper portion of the leftward diagonal of x's and a_{pq} is from the lower portion of the leftward diagonal of x's.

Consider a_{rs}. If $r > p$, then a_{rs} is lower than a_{pq}. So a_{rs} must be from the lower portion of the diagonal. But $s < j$ so, by Theorem 5, a_{rs} cannot be from the lower portion of the diagonal.

Now consider a_{tu}. If $t < i$, then a_{tu} is above a_{ij} so a_{tu} must be from the upper portion of the diagonal. But $u > q$ so, by Theorem 5, a_{tu} cannot be from the upper portion of the diagonal. Therefore, $a_{rs} \neq x$ and $a_{tu} \neq x$. ☐

Theorem 6. *Every 4×4 submatrix of the $n \times n$ Cayley table has a latin transversal.*

Proof. By Corollary 2, the main diagonal of any 4×4 submatrix in the table contains no element more than twice. So the diagonal of a 4×4 will have four distinct elements,

$$\begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{bmatrix} \textbf{Case 1}$$

two distinct elements,

$$\begin{bmatrix} a & & & \\ & a & & \\ & & b & \\ & & & b \end{bmatrix} \textbf{Case 2} \quad \begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} \textbf{Case 3} \quad \begin{bmatrix} a & & & \\ & b & & \\ & & a & \\ & & & b \end{bmatrix} \textbf{Case 4}$$

or three distinct elements.

$$\begin{bmatrix} a & & & \\ & a & & \\ & & b & \\ & & & c \end{bmatrix} \textbf{Case 5} \quad \begin{bmatrix} b & & & \\ & c & & \\ & & a & \\ & & & a \end{bmatrix} \textbf{Case 6}$$

$$\begin{bmatrix} a & & & \\ & b & & \\ & & a & \\ & & & c \end{bmatrix} \textbf{Case 7} \quad \begin{bmatrix} b & & & \\ & a & & \\ & & c & \\ & & & a \end{bmatrix} \textbf{Case 8}$$

$$\begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & a \end{bmatrix} \textbf{Case 9} \quad \begin{bmatrix} b & & & \\ & a & & \\ & & a & \\ & & & c \end{bmatrix} \textbf{Case 10}$$

These ten cases completely classify all 4×4 submatrices of the *ntimesn* Cayley Table.

Case 1 has an obvious latin transversal in its main diagonal.

Case 2 does not exist because by Corollary 3, b cannot appear twice in the diagonal after a appears twice.

Case 3 cannot happen similarly since a cannot appear twice in the diagonal after b appears twice.

Case 4 must have elements a_{34} and a_{43}, namely c and d, such that $d \neq a, b$ and $c \neq a, b$, since that would require repetition in row or column. If c and d are distinct, there is a latin transversal $abdc$.

$$\begin{bmatrix} \boxed{a} & & & \\ & \boxed{b} & & \\ & & a & \boxed{c} \\ & & \boxed{d} & b \end{bmatrix}$$

If $c = d$, we have

$$\begin{bmatrix} a & & & \\ & b & & \\ & & a & c \\ & & c & b \end{bmatrix}$$

which requires new elements $a_{32} = f$ and $a_{23} = e$ with $f \neq a, b, c, e$ and $e \neq a, b, c, f$. By Note 1, f and e cannot be equal since $2c = a + b$, but $a + b = f + e$ as well; $f = e$ would force $2c = a + b = 2f = 2e \mod n$ since n and 2 are relatively prime. But this cannot happen since f and e cannot equal c. So there is a latin transversal $afeb$.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & \boxed{e} & \\ & \boxed{f} & a & c \\ & & c & \boxed{b} \end{bmatrix}$$

Note that **Case 4** works similarly for

$$\begin{bmatrix} a & d & & \\ c & b & & \\ & & a & \\ & & & b \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} a & c & & \\ c & b & & \\ & & a & \\ & & & b \end{bmatrix}$$

Case 5 and **Case 6** are proved similarly, so we will show only **Case 5**.

By Corollary 2, **Case 5** cannot have either a_{43} or a_{34} be a. Also, these elements cannot be b or c since that would force repetition in row or column. So let $a_{43} = d$ and $a_{34} = w$ with $d \neq a, b, c$ and $w \neq a, b, c$.

$$\begin{bmatrix} a & & & \\ & a & & \\ & & b & w \\ & & d & c \end{bmatrix}$$

Let $a_{21} = x$ and $a_{12} = y$. By Note 1, $x \neq y$. If x, y, d and w are distinct, there is a latin transversal $xydw$.

$$\begin{bmatrix} a & \boxed{y} & & \\ \boxed{x} & a & & \\ & & b & \boxed{w} \\ & & \boxed{d} & c \end{bmatrix}$$

We will check $d = w$ and $d \neq w$.

Case 5.1: $d = w$

If $x \neq b, c$ and $y \neq b, c$, then $xybc$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{y} & & \\ \boxed{x} & a & & \\ & & \boxed{b} & d \\ & & d & \boxed{c} \end{bmatrix}$$

Note that $x = b$ and $y = c$ or $y = b$ and $x = c$ cannot happen because by Note 1, $2a = b + c = 2d$ implying that $a = d$ mod n, which cannot happen by Corollary 2.

$$\begin{bmatrix} a & c & & \\ b & a & & \\ & & b & d \\ & & d & c \end{bmatrix}$$

Also $x \neq y$, so $x = y = d$ cannot happen. So we will check the cases when x is b, c or d.

Case 5.1.1: $x = b$

$$\begin{bmatrix} a & y & & \\ b & a & & \\ & & b & d \\ & & d & c \end{bmatrix}$$

First, assume y is a new element. Introduce elements $m, z \neq a, b, d, y$. We know $z \neq m$ because $2a = b + y = m + z$, which would force $a = z = m$ if $z = m$. If $c \neq z, m$ there is a latin transversal $bmzc$.

$$\begin{bmatrix} a & y & \boxed{z} & \\ \boxed{b} & a & & \\ & \boxed{m} & b & d \\ & & d & \boxed{c} \end{bmatrix}$$

By Corollary 3 $m = c$ cannot happen, but we must check $z = c$.

$$\begin{bmatrix} a & y & z & \\ b & a & & \\ & c & b & d \\ & & d & c \end{bmatrix} m = c \qquad \begin{bmatrix} a & y & c & \\ b & a & & \\ & m & b & d \\ & & d & c \end{bmatrix} z = c$$

If $z = c$, introduce element $r \neq a, b, c, d$. If $r \neq m$, $amrc$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & y & c & \\ b & a & \boxed{r} & \\ & \boxed{m} & b & d \\ & & d & \boxed{c} \end{bmatrix}$$

If $r = m$, introduce $a_{24} = s \neq a, b, c, d, r$. Then $ards$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & y & c & \\ b & a & r & \boxed{s} \\ & \boxed{r} & b & d \\ & & \boxed{d} & c \end{bmatrix}$$

Now we will check the case that $x = b$ but y equals an existing element. By Note 1 $y \neq b, c$, so check $y = d$.

$$\begin{bmatrix} a & d & & \\ b & a & & \\ & & b & d \\ & & d & c \end{bmatrix}$$

Introduce elements $a_{41} = m \neq a, b, c, d, n$ and $a_{24} = n \neq a, b, c, d, m$. We have $m \neq n$ by Note 1 because $m + n = b + c = 2d$. Then $mdbn$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{d} & & \\ b & a & & \boxed{n} \\ & & \boxed{b} & d \\ \boxed{m} & & d & c \end{bmatrix}$$

Case 5.1.2. $x = c$

$$\begin{bmatrix} a & y & & \\ c & a & & \\ & & b & d \\ & & d & c \end{bmatrix}$$

First, assume that y is a new element. Introduce elements $a_{31} = k \neq a, b, c, d, r$ and $a_{23} = r \neq a, b, c, d, k$. We know $r \neq k$ because $2d = b + c = k + r$ by Note 1, but $k, r \neq d$. If $y \neq k, r$, then there is a latin transversal $kyrc$.

$$\begin{bmatrix} a & \boxed{y} & & \\ c & a & \boxed{r} & \\ \boxed{k} & & b & d \\ & & d & \boxed{c} \end{bmatrix}$$

Case 5.1.2.1. $y = k$

If $y = k$, introduce $a_{14} = i \neq k, a, d$. Then $kadi$ is a latin transversal.

$$\begin{bmatrix} a & k & & \boxed{i} \\ c & \boxed{a} & r & \\ \boxed{k} & & b & d \\ & & \boxed{d} & c \end{bmatrix}$$

Case 5.1.2.2. $y = r$

If $y = r$, introduce $a_{42} = j \neq a, y, d$. Then $ajyd$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & y & & \\ c & a & \boxed{y} & \\ k & & b & \boxed{d} \\ & \boxed{j} & d & c \end{bmatrix}$$

Now assume y is an existing element. We know $y = d$ because $y \neq b, c$ by Note 1. So we now check $x = c, y = d$.

$$\begin{bmatrix} a & d & & \\ c & a & & \\ & & b & d \\ & & d & c \end{bmatrix}$$

Introduce $a_{31} = m \neq a, b, c, d, n$ and $a_{23} = n \neq a, b, c, d, n$. Then $mdnc$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{d} & & \\ c & a & \boxed{n} & \\ \boxed{m} & & b & d \\ & & d & \boxed{c} \end{bmatrix}$$

Case 5.1.3. $x = d$

$$\begin{bmatrix} a & y & & \\ d & a & & \\ & & b & d \\ & & d & c \end{bmatrix}$$

We know $y \neq d$ by Note 1. If $y \neq b, c$, then $dybc$ is a latin transversal. If $y = b$, introduce $a_{42} = m \neq a, b, c, d, n$ and $a_{14} = n \neq a, b, c, d, m$. We know $m \neq n$ because $m + n = b + c = 2d$. So $dmbn$ is a latin transversal.

$$\begin{bmatrix} a & b & & \boxed{n} \\ \boxed{d} & a & & \\ & & \boxed{b} & d \\ & \boxed{m} & d & c \end{bmatrix}$$

If $y = c$, introduce $a_{32} = m \neq a, b, c, d, n$ and $a_{13} = n \neq a, b, c, d, m$. We know $m \neq n$ because $m + n = b + c = 2d$. So $dmnc$ is a latin transversal.

$$\begin{bmatrix} a & c & \boxed{n} & \\ \boxed{d} & a & & \\ & \boxed{m} & b & d \\ & & d & \boxed{c} \end{bmatrix}$$

Case 5.2. $d \neq w$

$$\begin{bmatrix} a & & & \\ & a & & \\ & & b & w \\ & & d & c \end{bmatrix}$$

Introduce $a_{32} = x \neq a, b, w, d, c$ and $a_{23} = y \neq a, b, d, w, c$. Corollary 3 gives us $x \neq c, d$ and $y \neq w, c$. If $x \neq y$, then $axyc$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & a & \boxed{y} & \\ & \boxed{x} & b & w \\ & & d & \boxed{c} \end{bmatrix}$$

If $x = y$, introduce $a_{24} = e \neq a, c, w, x$. If $e \neq d$, then $axde$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & a & x & \boxed{e} \\ & \boxed{x} & b & w \\ & & \boxed{d} & c \end{bmatrix}$$

If $d = e$, then introduce $a_{42} = z \neq a, c, d, x, w$. We know $z \neq w$ because $z + w = x + c = 2d$. So $azxw$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & a & \boxed{x} & d \\ & x & b & \boxed{w} \\ & \boxed{z} & d & c \end{bmatrix}$$

This concludes **Case 5**.

For **Case 7**, we introduce $a_{43} = d \neq a, c$ and $a_{34} = r \neq a, c$. If d, b and r are pairwise distinct, there is a latin transversal $abdr$.

$$\begin{bmatrix} \boxed{a} & & & \\ & \boxed{b} & & \\ & & a & \boxed{r} \\ & & \boxed{d} & c \end{bmatrix}$$

We must check the cases for $r = d = b$, $r = b$, $d = b$, and $d = r$.

Case 7.1. $r = d = b$

$$\begin{bmatrix} a & & & \\ & b & & \\ & & a & b \\ & & b & c \end{bmatrix}$$

Introduce $a_{41} = h \neq a, b, c, g$ and $a_{14} = g \neq a, b, c, h$. We know $h \neq g$ because $h + g = a + c = 2b$ and $b \neq h, g$. So $hbag$ is a latin transversal.

$$\begin{bmatrix} a & & & \boxed{g} \\ & \boxed{b} & & \\ & & \boxed{a} & b \\ \boxed{h} & & b & c \end{bmatrix}$$

Case 7.2. $r = b$

$$\begin{bmatrix} a & & & \\ & b & & \\ & & a & b \\ & & d & c \end{bmatrix}$$

Introduce $a_{14} = g \neq a, b, c$ and $a_{41} = h \neq a, d, c, b$. We know $h \neq b$ by Corollary 5. If $h \neq g$, then there is a latin transversal $hbag$.

$$\begin{bmatrix} a & & & \boxed{g} \\ & \boxed{b} & & \\ & & \boxed{a} & b \\ \boxed{h} & & d & c \end{bmatrix}$$

If $h = g$, introduce $a_{31} = m \neq a, b, g$. If $m \neq d$, then $mbdg$ is a latin transversal.

$$\begin{bmatrix} a & & & \boxed{g} \\ & \boxed{b} & & \\ \boxed{m} & & a & b \\ g & & \boxed{d} & c \end{bmatrix}$$

If $m = d$, introduce $a_{13} = n \neq a, d, g, b$. We know $n \neq b$ because $n + b = a + g = 2d$. If $n \neq c$, then $dbnc$ is a latin transversal.

$$\begin{bmatrix} a & & \boxed{n} & g \\ & \boxed{b} & & \\ \boxed{d} & & a & b \\ g & & d & \boxed{c} \end{bmatrix}$$

If $n = c$, then introduce $a_{42} = z \neq a, b, c, d, g$. We have $z \neq a$ because $2a = c + d$ already, so $z = a$ would force $a_{32} = c$. That cannot happen since that would force $2c = a + b$ and $a_{12} = b$, which is impossible.

$$\begin{bmatrix} a & b & c & g \\ & b & & \\ d & c & a & b \\ g & a & d & c \end{bmatrix} IMPOSSIBLE$$

So we have $z \neq a$. Now introduce $a_{23} = f \neq a, b, c, d$. If $z \neq f$, then $azfb$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & c & g \\ & b & \boxed{f} & \\ d & & a & \boxed{b} \\ g & \boxed{z} & d & c \end{bmatrix}$$

If $z = f$, introduce $a_{12} = q \neq a, b, c, f, g$. So $gqzb$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{q} & c & g \\ & b & \boxed{z} & \\ d & & a & \boxed{b} \\ \boxed{g} & z & d & c \end{bmatrix}$$

Case 7.3. $d = b$

$$\begin{bmatrix} a & & & \\ & b & & \\ & & a & r \\ & & b & c \end{bmatrix}$$

Introduce $a_{42} = m \neq b, c, n$ and $a_{23} = n \neq a, b, m$. If $m \neq a$ and $r \neq m, n$, we have a latin transversal $amnr$.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & \boxed{n} & \\ & & a & \boxed{r} \\ & \boxed{m} & b & c \end{bmatrix}$$

We must check $m = a$, $m = r$, and $n = r$.

Case 7.3.1. $m = a$

$$\begin{bmatrix} a & & & \\ & b & n & \\ & & a & r \\ & a & b & c \end{bmatrix}$$

Introduce $a_{32} = s \neq a, b, r$ and $a_{24} = o \neq b, n, r, c$. If $o \neq a, s$, then $asbo$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & n & \boxed{o} \\ & \boxed{s} & a & r \\ & a & \boxed{b} & c \end{bmatrix}$$

If $o = a$ or $o = s$, we must find latin transversals.

Case 7.3.1.1. $o = a$

$$\begin{bmatrix} a & & & \\ & b & n & a \\ & s & a & r \\ & a & b & c \end{bmatrix}$$

Introduce $a_{21} = t \neq a, b, n$ and $a_{12} = z \neq a, b, c$. We have $s = c$ because $s + n = b + a = c + n$. If $t \neq z$, then $tzac$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{z} & & \\ \boxed{t} & b & n & a \\ & c & \boxed{a} & r \\ & a & b & \boxed{c} \end{bmatrix}$$

Now $t \neq c$ by Corollary 2. Introduce $a_{13} = p \neq a, b, n, t, y$ and $a_{41} = y \neq a, b, c, t, p$ to examine the case when $t = z$. We have $y \neq p$ since $y = p$ would imply that $y + p = 2y = a + b = 2t$ and $t \neq y$. Also, $a_{14} = y$ because $y + b = t + a$. If $p \neq c$, then $ycpa$ is a latin transversal.

$$\begin{bmatrix} a & t & \boxed{p} & y \\ t & b & n & \boxed{a} \\ & \boxed{c} & a & r \\ \boxed{y} & a & b & c \end{bmatrix}$$

If $p = c$, then $n = y$ because $2y = c + a = p + a = n + y$. Then $tcby$ is a latin transversal (it's also great yogurt).

$$\begin{bmatrix} a & t & c & \boxed{y} \\ \boxed{t} & b & y & a \\ & \boxed{c} & a & r \\ y & a & \boxed{b} & c \end{bmatrix}$$

Case 7.3.1.2. $o = s$

$$\begin{bmatrix} a & & & \\ & b & n & s \\ & s & a & r \\ & a & b & c \end{bmatrix}$$

When $o = s$, $n \neq c$ because $n + c = 2c = b + s = 2a$ and $a \neq c$, so $asnc$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & \boxed{n} & s \\ & \boxed{s} & a & r \\ & a & b & \boxed{c} \end{bmatrix}$$

Case 7.3.2.: $m = r$

$$\begin{bmatrix} a & & & \\ & b & n & \\ & & a & r \\ & r & b & c \end{bmatrix}$$

Now $m = r \neq a, b, c, n$. Remember that we are assuming that $r \neq n$, which is a later case. Introduce $a_{24} = s \neq b, c, n, r, a$ and $a_{32} = q \neq a, r, b$. We know $s \neq a$ since $s = a$ implies $s + a = 2a = n + r = 2b$ and $a \neq b$. If $s \neq q$, then $aqbs$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & n & \boxed{s} \\ & \boxed{q} & a & r \\ & r & \boxed{b} & c \end{bmatrix}$$

If $s = q$ and $n \neq c$, then $asnc$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & \boxed{n} & s \\ & \boxed{s} & a & r \\ & r & b & \boxed{c} \end{bmatrix}$$

We must check $n = c$.

$$\begin{bmatrix} a & & & \\ & b & c & s \\ & s & a & r \\ & r & b & c \end{bmatrix}$$

Introduce $a_{41} = k \neq a, b, c, r, u$ and $a_{14} = u \neq a, c, r, s, k, b$. We have $u \neq k$ because $2u = k + u = a + c$, but $a + c$ already equals $b + r$ which in turn equals $2s$. So $u = k$ would imply $u = k = s$, which is impossible.

Also, $u \neq b$ because $u + r = b + r = a + c$, which would force $a_{12} = a$. That, of course, is impossible. So $kbau$ is a latin transversal.

$$\begin{bmatrix} a & & & \boxed{u} \\ & \boxed{b} & c & s \\ & s & \boxed{a} & r \\ \boxed{k} & r & b & c \end{bmatrix}$$

Case 7.3.3. $n = r$

$$\begin{bmatrix} a & & & \\ & b & r & \\ & & a & r \\ & m & b & c \end{bmatrix}$$

Here $m \neq r$ since $r + m = 2b$. In fact, $r \neq a, b, c, m$. Introduce $a_{32} = i \neq a, b, r, m, c$. We know $i \neq c$ because $i + c = 2c = r + m = 2b$. So $airc$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & \boxed{r} & \\ & \boxed{i} & a & r \\ & m & b & \boxed{c} \end{bmatrix}$$

Case 7.4. $r = d$

$$\begin{bmatrix} a & & & \\ & b & & \\ & & a & d \\ & & d & c \end{bmatrix}$$

Introduce $a_{21} = x \neq a, b$ and $a_{12} = y \neq a, b$. If $x \neq y$ and $c \neq x, y$, then $xyac$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{y} & & \\ \boxed{x} & b & & \\ & & \boxed{a} & d \\ & & d & \boxed{c} \end{bmatrix}$$

We must examine $x = c$, $y = c$, $x = y$, and $x = y = c$.

Case 7.4.1. $x = c$

$$\begin{bmatrix} a & y & & \\ c & b & & \\ & & a & d \\ & & d & c \end{bmatrix}$$

Introduce $a_{32} = m \neq a, b, c, d, y$ and $a_{23} = n \neq a, b, c, d$. We know $m \neq c$ by Corollary 2. If $m \neq n$, then $amnc$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & y & & \\ c & b & \boxed{n} & \\ & \boxed{m} & a & d \\ & & d & \boxed{c} \end{bmatrix}$$

We must check $m = n$.

$$\begin{bmatrix} a & y & & \\ c & b & m & \\ & m & a & d \\ & & d & c \end{bmatrix}$$

Introduce $a_{42} = i \neq b, c, d, m, y$. We have $a_{24} = i$ as well because $m + d = a + i$. If $i \neq a$, then $amdi$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & y & & \\ c & b & m & \boxed{i} \\ & \boxed{m} & a & d \\ & i & \boxed{d} & c \end{bmatrix}$$

If $i = a$, introduce $a_{14} = h \neq a, c, d, y$. If $h \neq m$ then $cmdh$ is a latin transversal.

$$\begin{bmatrix} a & y & & \boxed{h} \\ \boxed{c} & b & m & a \\ & \boxed{m} & a & d \\ & a & \boxed{d} & c \end{bmatrix}$$

If $h = m$, then $a_{13} = b$ because $2m = b + a$.

$$\begin{bmatrix} a & y & b & m \\ c & b & m & a \\ & m & a & d \\ & a & d & c \end{bmatrix}$$

Then let $a_{31} = e \neq a, c, d, m, b$. We have $e \neq b$ since $2a = b + e$. So $ebdm$ is a latin transversal.

$$\begin{bmatrix} a & y & b & \boxed{m} \\ c & \boxed{b} & m & a \\ \boxed{e} & m & a & d \\ & a & \boxed{d} & c \end{bmatrix}$$

Case 7.4.2. $y = c$

$$\begin{bmatrix} a & c & & \\ x & b & & \\ & & a & d \\ & & d & c \end{bmatrix}$$

This argument is similar to 7.4.1, so the proof is omitted here.

Case 7.4.3, 7.4.4. $x = y$

$$\begin{bmatrix} a & x & & \\ x & b & & \\ & & a & d \\ & & d & c \end{bmatrix}$$

Introduce $a_{31} = m \neq a, d, x$ and $a_{24} = n \neq b, d, x$. If $m \neq n$, then $mxdn$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{x} & & \\ x & b & & \boxed{n} \\ \boxed{m} & & a & d \\ & & \boxed{d} & c \end{bmatrix}$$

If $m = n$, introduce $a_{23} = i \neq a, b, d, m, x$. If $x = y \neq c$ (Case 7.4.3), $mxic$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{x} & & \\ x & b & \boxed{i} & m \\ \boxed{m} & & a & d \\ & & d & \boxed{c} \end{bmatrix}$$

Otherwise (Case 7.4.4), $x = y = c$.

$$\begin{bmatrix} a & c & & \\ c & b & i & m \\ & & a & d \\ & & d & c \end{bmatrix}$$

Now introduce $a_{41} = e \neq a, c, d, m$. So $ecam$ is a latin transversal.

$$\begin{bmatrix} a & \boxed{c} & & \\ c & b & i & \boxed{m} \\ & & \boxed{a} & d \\ \boxed{e} & & d & c \end{bmatrix}$$

Note. Case 7 was for

$$\begin{bmatrix} a & & & \\ & b & & \\ & & a & \\ & & & c \end{bmatrix}$$

and we worked with $a_{43} = d$ and $a_{34} = r$. Note that using a_{21} and a_{12} instead would allow for the same arguments to be used to prove **Case 8**. Thus the proof of **Case 8** is omitted in this paper.

For **Case 9**, introduce $a_{43} = x \neq a, c, b$ and $a_{34} = y \neq a, c, b$. We know $b \neq x, y$ by Corollary 3. If $x \neq y$, then $abxy$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & \boxed{b} & & \\ & & c & \boxed{y} \\ & & \boxed{x} & a \end{bmatrix}$$

If $x = y$, introduce $a_{42} = m \neq b, a, x$ and $a_{23} = n \neq b, c, x, a$. We know $n \neq a$ by Corollary 2. If $n \neq m$, then $amnx$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & \boxed{n} & \\ & & c & \boxed{x} \\ & \boxed{m} & x & a \end{bmatrix}$$

We must check $n = m$.

$$\begin{bmatrix} a & & & \\ & b & m & \\ & & c & x \\ & m & x & a \end{bmatrix}$$

Note that $x \neq b$ now because $2m = b + x$. Introduce $a_{24} = r \neq a, b, m, x$. If $r \neq c$, then $amcr$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & m & \boxed{r} \\ & & \boxed{c} & x \\ & \boxed{m} & x & a \end{bmatrix}$$

For $r = c$, let $a_{32} = e \neq b, c, x, a$. We know $e \neq a$ by Corollary 2. So $aexc$ is a latin transversal.

$$\begin{bmatrix} \boxed{a} & & & \\ & b & m & \boxed{c} \\ & \boxed{e} & c & x \\ & m & \boxed{x} & a \end{bmatrix}$$

For **Case 10**, introduce $a_{32} = x \neq a, y$ and $a_{23} = y \neq a, x$. We know $x \neq y$ since $x + y = 2a$. If $x, y \neq b, c$, then $bxyc$ is a latin transversal.

$$\begin{bmatrix} \boxed{b} & & & \\ & a & \boxed{y} & \\ & \boxed{x} & a & \\ & & & \boxed{c} \end{bmatrix}$$

We will check $x = b$ and $x = c$ (which are similar arguments to $y = c$ and $y = b$, whose proofs are omitted).

Case 10.1. $x = b$

$$\begin{bmatrix} b & & & \\ & a & y & \\ & b & a & \\ & & & c \end{bmatrix}$$

Introduce $a_{43} = s \neq a, c, x, b$ and $a_{34} = z \neq a, b, c$. We know by Corollary 2 that $s \neq b$. If $s \neq z$, then $basz$ is a latin transversal.

$$\begin{bmatrix} \boxed{b} & & & \\ & \boxed{a} & y & \\ & b & a & \boxed{z} \\ & & \boxed{s} & c \end{bmatrix}$$

For $s = z$, introduce $a_{42} = m \neq a, b, c, z$. If $m \neq y$, then $bmyz$ is a latin transversal.

$$\begin{bmatrix} \boxed{b} & & & \\ & a & \boxed{y} & \\ & b & a & \boxed{z} \\ & \boxed{m} & z & c \end{bmatrix}$$

If $m = y$, let $a_{24} = r \neq a, m, z, c, b$. We have $r \neq b$ since $r + m = a + c$ but $b + m = 2a$. So $bmar$ is a latin transversal.

$$\begin{bmatrix} \boxed{b} & & & \\ & a & m & \boxed{r} \\ & b & \boxed{a} & z \\ & \boxed{m} & z & c \end{bmatrix}$$

Case 10.2. $x = c$

$$\begin{bmatrix} b & & & \\ & a & y & \\ & c & a & \\ & & & c \end{bmatrix}$$

Introduce $a_{21} = s \neq a, b, y, c$ and $a_{12} = z \neq a, b, c$. We know $s \neq c$ by Corollary 2. If $s \neq z$, then $szac$ is a latin transversal.

$$\begin{bmatrix} b & \boxed{z} & & \\ \boxed{s} & a & x & \\ & c & \boxed{a} & \\ & & & \boxed{c} \end{bmatrix}$$

If $s = z$, then introduce $a_{31} = m \neq a, b, c, s$. If $m \neq y$ then $msyc$ is a latin transversal.

$$\begin{bmatrix} b & \boxed{s} & & \\ s & a & \boxed{y} & \\ \boxed{m} & c & a & \\ & & & \boxed{c} \end{bmatrix}$$

If $m = y$, then introduce $a_{13} = r \neq a, m, c$. We have $r \neq c$ because $r + c = 2c = y + a = 2m$. So, *marc* is a latin transversal.

$$\begin{bmatrix} b & s & \boxed{r} & \\ s & \boxed{a} & m & \\ \boxed{m} & c & a & \\ & & & \boxed{c} \end{bmatrix}$$

To make mathematics more than recreational, one must examine the practical applications of any mathematical finding. In the case of Cayley tables, one useful application is for factory worker assignments. Letting the rows of the table represent individuals and the columns represent times, the body of the table may be filled in with machine assignments. Assuming only one person can work a machine at once, we maximize production by letting all machines be assigned at any given time. To avoid repetition, there is rotation of workers and machinery. Then for an odd number of workers, n, and an equivalent number of machines, the Cayley table gives an optimal assignment sheet.

Worker	2:00	3:00	4:00
Bob	a	b	c
Sue	b	c	a
Tom	c	a	b

A latin transversal of this table is also useful. Suppose Pat, the supervisor, wants to check on every worker with every piece of equipment. Then Pat can do this in n days for the aforementioned Cayley table, choosing a different latin transversal every day.

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References:

Coleman, Hall, Hutchings and Ruhnke. 1999. The Cayley Addition Table of $\mathbb{Z}_n$.

Roberts, F.S. 1984. *Applied Combinatorics.* Englewood Cliffs, N.J.: Prentice-Hall, Inc.

Snevily, H. 1999. Problems and Solutions. *The American Mathematical Monthly*, vol. 106, no.6, pp. 584-585.