

Strong complete mappings for 3-groups

Reza Akhtar

Department of Mathematics

Miami University, Oxford, OH 45056, USA

akhtarr@miamioh.edu

Stephen M. Gagola III

Department of Mathematics

Miami University, Oxford, OH 45056, USA

gagola@mailbox.sc.edu

Abstract

A strong complete mapping of a group G is a bijection $\varphi : G \rightarrow G$ such that the maps $x \mapsto x\varphi(x)$ and $x \mapsto x^{-1}\varphi(x)$ are also bijections. It was shown by A.B. Evans that a finite abelian group admits a strong complete mapping if and only if its 2-Sylow subgroup and its 3-Sylow subgroup are either trivial or noncyclic. As a step towards the classification of nonabelian groups admitting a strong complete mapping, we prove that if a nontrivial 3-group G is neither cyclic nor isomorphic to $L_r = \langle a, b \mid a^{3^{r-1}} = b^3 = 1, bab^{-1} = a^{1+3^{r-2}} \rangle$ for some $r \geq 4$, then G admits a strong complete mapping.

1 Introduction

Let G be a group. A bijection $\theta : G \rightarrow G$ is called a *complete mapping* if $x \mapsto x\theta(x)$ is a bijection and an *orthomorphism* if $x \mapsto x^{-1}\theta(x)$ is a bijection. It is clear that $x \mapsto \theta(x)$ is a complete mapping if and only if $x \mapsto x\theta(x)$ is an orthomorphism. A *strong complete mapping* is a bijection which is both a complete mapping and an orthomorphism. We call G *admissible* if it admits a complete mapping and *strongly admissible* if it admits a strong complete mapping. Strong complete mappings are closely related to orthogonality problems for Latin squares [7] and have been used to study group sequencings [1], Knut Vic designs ([9], [10]), strong starters [11], solutions to the toroidal n -queens problem [13] and check digit systems [14].

The classification of admissible finite groups was begun in 1955 with the work of Hall and Paige [8] and completed in 2009 by Wilcox, Evans, and Bray et. al. (see [5], [15], [3]); it is now known that a finite group G is admissible if and only if the 2-Sylow subgroup of G is either trivial or noncyclic. The analogous question for infinite groups was settled by Bateman [2], who proved that every infinite group is admissible.

In contrast, the classification of strongly admissible finite groups is still open. It has been fully resolved for abelian groups: Evans [6] showed that a finite abelian group is strongly

admissible if and only if neither its 2-Sylow subgroup nor its 3-Sylow subgroup is nontrivial and cyclic. It is also known [7] that certain infinite families of dihedral and generalized quaternion groups are strongly admissible, but little is known in general in the nonabelian case. Part of the difficulty is that, aside from explicit constructions, the primary tools for proving the existence of such maps are inductive in nature. It is relatively easy to show (cf. [8, Cor. 2]) that if H is a normal subgroup of a group G such that both H and G/H are admissible, then G is admissible; however, one needs to assume further that H is contained in the center of G to deduce strong admissibility by a similar argument. That was precisely the strategy adopted by Evans in the case of finite abelian groups (see [6, Lemma 5] or [4, Theorem 3]), thus reducing the classification problem to a handful of explicit constructions for groups of a particular form. It seems appropriate to begin consideration of the nonabelian case with the class of nilpotent groups, since these at least have a nontrivial center. As every finite nilpotent group is isomorphic to the direct product of its Sylow subgroups and strong admissibility is preserved under products, we are thereby reduced to studying strong admissibility for p -groups. The map $x \mapsto x^2$ is a strong complete mapping for every finite group of order relatively prime to 6, and so the question is only of relevance when $p = 2$ or $p = 3$.

In this paper we study strong admissibility for 3-groups. Our main inductive tool is Proposition 2.4, which states that if such a 3-group G has a normal subgroup $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ such that G/H is strongly admissible, then G is strongly admissible. The proof of this statement relies on strong admissibility of H and a method to patch together a strong complete mapping for H and a strong complete mapping for G/H to produce one for G . To carry out the process of induction, we need a way to consider what happens when G/H is not strongly admissible, but in almost all cases it is possible to circumvent this problem. We were able to achieve near complete success in the classification of strongly admissible 3-groups, leaving the question open only for the special class of groups:

$$L_r = \langle a, b \mid a^{3^{r-1}} = b^3 = 1, bab^{-1} = a^{1+3^{r-2}} \rangle$$

when $r \geq 4$. Using **Mace4**, we were able to exhibit a strong complete mapping for L_3 and even obtain a formula for it with the aid of **GAP** (see Lemma 2.6); however, it was not clear to us how such a formula might be generalized. Our main result is:

Theorem.

Let G be a nontrivial finite 3-group which is neither cyclic nor isomorphic to L_r for some $r \geq 4$. Then G is strongly admissible.

Nontrivial cyclic groups are known to be strongly inadmissible by [4, Theorem 2]. The principal difficulty in studying the groups L_r , $r \geq 4$, is that there is no normal subgroup

N such that both N and L_r/N are noncyclic. This renders an inductive strategy hopeless, although an explicit construction seems equally elusive.

It is natural to ask to what extent these techniques might extend to the case of 2-groups. Unfortunately, the program breaks down on several fronts. The success we achieved for 3-groups was driven in part by the fact that every noncyclic 3-group has a normal subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$; this allowed us to pursue an inductive strategy successfully. The analogous statement is no longer true for 2-groups, as witnessed by the family of dihedral groups, for example. Even worse, the key ingredient in the proof of Proposition 2.4 (the inductive step in the case of 3-groups) is the existence of an orthomorphism θ of $\mathbb{Z}_3 \times \mathbb{Z}_3$ such that the map $x \mapsto x\theta(x)$, after an appropriate “twisting”, becomes bijective. The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is too small to admit a similar orthomorphism that would allow this argument to be generalized. These difficulties suggest that a fresh approach is required to address the question of strong admissibility for 2-groups.

In Section 2 we review background results and prove the technical lemmas we need. The main result of the paper is proven in Section 3.

2 Preliminaries

We open with a well-known result from the literature (see [7, Theorem 7 and Cor. 2]). We provide a proof in the interest of completeness, but especially to emphasize that the construction used in our inductive step (Proposition 2.4) is based on this one.

Lemma 2.1.

- *Let H be a subgroup of the center of a finite group G . If H and G/H are strongly admissible, then G is strongly admissible.*
- *A direct product of strongly admissible groups is strongly admissible.*

Proof.

Suppose $H \leq Z(G)$, and let $\alpha : H \rightarrow H$ and $\beta : G/H \rightarrow G/H$ be strong complete mappings. Fix a choice of right transversal $T \subseteq G$ so that every $g \in G$ may be written uniquely as $g = h_g t_g$, with $h_g \in H$ and $t_g \in T$, and let $\Phi : G/H \rightarrow T$ denote the bijection $Hg \mapsto t_g$. We claim that $\gamma : G \rightarrow G$ defined by $\gamma(g) = \alpha(h_g)\Phi(\beta(Ht_g))$ is a strong complete mapping. Since G is finite, it suffices to show that the maps $g \mapsto \gamma(g)$, $g \mapsto g\gamma(g)$, and $g \mapsto g^{-1}\gamma(g)$ are injective. To this end, suppose $g, g' \in G$. If $\gamma(g) = \gamma(g')$, then

$$\alpha(h_g)\Phi(\beta(t_g)) = \alpha(h_{g'})\Phi(\beta(Ht_{g'})). \quad (1)$$

Reducing this equation modulo H , we have $\beta(Ht_g) = \beta(Ht_{g'})$ in G/H . Since β is a bijection from G/H to itself, we have $Ht_g = Ht_{g'}$, and because T is a transversal, $t_g = t_{g'}$. Substituting back into (1) and canceling yields $\alpha(h_g) = \alpha(h_{g'})$. Finally, the fact that α is a bijection from H to itself implies $h_g = h_{g'}$. Thus, $g = t_g h_g = t_{g'} h_{g'} = g'$ and so γ is injective.

Next, suppose $g\gamma(g) = g'\gamma(g')$, so $h_g t_g \alpha(h_g) \Phi(\beta(Ht_g)) = h_{g'} t_{g'} \alpha(h_{g'}) \Phi(\beta(Ht_{g'}))$. Because $H \leq Z(G)$, we have

$$h_g \alpha(h_g) t_g \Phi(\beta(Ht_g)) = h_{g'} \alpha(h_{g'}) t_{g'} \Phi(\beta(Ht_{g'})). \quad (2)$$

Arguing as in the previous case, reduction modulo H implies $(Ht_g)\beta(Ht_g) = (Ht_{g'})\beta(Ht_{g'})$ in G/H . The fact that β is a complete mapping of G/H establishes that $t_g = t_{g'}$. Substituting into (2) and canceling yields $h_g \alpha(h_g) = h_{g'} \alpha(h_{g'})$; finally, using the fact that α is a strong complete mapping of H , we conclude $h_g = h_{g'}$.

Finally, suppose $g^{-1}\gamma(g) = g'^{-1}\gamma(g')$, so $h_g^{-1}\alpha(h_g)t_g^{-1}\beta(t_g) = h_{g'}^{-1}\alpha(h_{g'})t_{g'}^{-1}\beta(t_{g'})$. Upon reduction modulo H we apply normality of H in G to deduce that $(Ht_g)^{-1}\beta(Ht_g) = (Ht_{g'})^{-1}\beta(Ht_{g'})$ in G/H , and then proceed as before. Note that we used centrality of H in arguing that γ is a complete mapping but only normality in arguing that γ is an orthomorphism.

For the second statement, suppose $\{G_i\}_{i \in I}$ is a family of strongly admissible groups. For each i , fix a choice of strong complete mapping $\varphi_i : G_i \rightarrow G_i$. Then a coordinate-wise argument easily establishes that the map $(x_i)_{i \in I} \mapsto (\varphi_i(x_i))_{i \in I}$ is a strong complete mapping of $\prod_{i \in I} G_i$. \square

We also recall a key negative result on strong admissibility.

Theorem 2.2. (*Evans, [4, Theorem 2]*) *If a finite group G has a nontrivial, cyclic 3-Sylow subgroup S , and H is a normal subgroup of G for which $G/H \cong S$, then G does not admit strong complete mappings.*

Finally, we will need the classification of strongly admissible finite abelian groups.

Theorem 2.3. (*Evans, [5]*) *A finite abelian group admits strong complete mappings if and only if neither its 2-Sylow subgroup nor its 3-Sylow subgroup is nontrivial and cyclic.*

Our own work begins with an appropriate generalization of Lemma 2.1 to 3-groups. Once again, we work with a normal subgroup $H \leq G$, but this time we do not assume that H is contained in the center $Z(G)$, so we need to keep track of the conjugation action of G on H .

Proposition 2.4. *Let G be a 3-group and $N \triangleleft G$, with $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. If G/N has a strong complete mapping, then G has a strong complete mapping.*

Proof.

Since a normal subgroup of G must intersect $Z(G)$ nontrivially, we may assume that N is generated by elements $a, b \in G$, each of order 3, where $a \in Z(G)$. Direct computation shows that $\alpha : N \rightarrow N$ defined by $\alpha(a^i b^j) = a^{2j} b^i$ is a strong complete mapping of N . For $\varepsilon \in \{0, 1\}$, define $\alpha_\varepsilon : N \rightarrow N$ by $\alpha_\varepsilon(a^i b^j) = a^{2j+\varepsilon i} b^i$. Then $\alpha_0 = \alpha$, and it is easy to check that α_1 is an orthomorphism.

Fix a choice of right transversal $T \subseteq G$ so that every $g \in G$ may be written uniquely as $g = n_g t_g$, with $n_g \in N$ and $t_g \in T$, and let $\Phi : G/N \rightarrow T$ denote the bijection $Ng \mapsto t_g$. Let $\beta : G/N \rightarrow G/N$ be a strong complete mapping. Observe that each $g \in G$ may be written uniquely as $g = n_g t_g$, where $n_g \in N$ and $t_g \in T$. Now G acts on N by conjugation, and since G acts trivially on the subgroup $M = \langle a \rangle$, G therefore acts on $N/M \cong \mathbb{Z}_3$. Since G is a 3-group and $\text{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2$, the conjugation action of G on N/M must be trivial, so for every $g \in G$, $t_g b = a^{k(t_g)} b t_g$ for some $k(t_g)$, $0 \leq k(t_g) \leq 2$.

For $g \in G$, define

$$\varepsilon(t_g) = \begin{cases} 1 & \text{if } k(t_g) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then define $\Gamma : G \rightarrow G$ by

$$\Gamma(n_g t_g) = \alpha_{\varepsilon(t_g)}(n_g) \Phi(\beta(Nt_g)).$$

We claim that Γ is a strong complete mapping of G . To show that Γ is bijective, suppose $g, g' \in G$ and $\Gamma(g) = \Gamma(g')$, i.e.

$$\alpha_{\varepsilon(t_g)}(n_g) \Phi(\beta(Nt_g)) = \alpha_{\varepsilon(t_{g'})}(n_{g'}) \Phi(\beta(Nt_{g'})). \quad (3)$$

Reducing this equation modulo N , we must have $\beta(Nt_g) = \beta(Nt_{g'})$. Because β is a strong complete mapping, we have $Nt_g = Nt_{g'}$, and because T is a transversal, $t_g = t_{g'}$. Substituting into (3) and cancelling on the right, we have $\alpha_{\varepsilon(t_g)}(n_g) = \alpha_{\varepsilon(t_{g'})}(n_{g'})$. Finally, $\alpha_{\varepsilon(t_g)}$ is a bijection, so $n_g = n_{g'}$.

Next, we check that Γ is a complete mapping. With the same notation as above, suppose $n_g t_g \Gamma(n_g t_g) = n_{g'} t_{g'} \Gamma(n_{g'} t_{g'})$, i.e.

$$n_g t_g \alpha_{\varepsilon(t_g)}(n_g) \Phi(\beta(t_g N)) = n_{g'} t_{g'} \alpha_{\varepsilon(t_{g'})}(n_{g'}) \Phi(\beta(t_{g'} N)). \quad (4)$$

Reducing this equation modulo N , we have $(Nt_g)\beta(Nt_g) = (Nt_{g'})\beta(Nt_{g'})$. Since β is a complete mapping of G/N , we must have $Nt_g = Nt_{g'}$. Because T is a transversal, this forces $t_g = t_{g'}$. Thus, (4) becomes

$$n_g t_g \alpha_{\varepsilon(t_g)}(n_g) = n_{g'} t_g \alpha_{\varepsilon(t_g)}(n_{g'}). \quad (5)$$

Writing $n_g = a^i b^j$ and $n_{g'} = a^{i'} b^{j'}$, where $i, j, i', j' \in \{0, 1, 2\}$ and recalling that $a \in Z(G)$, (5) reads:

$$a^{2j+(1+\varepsilon(t_g))i} b^j t_g b^i = a^{2j'+(1+\varepsilon(t_g))i'} b^{j'} t_g b^{i'}. \quad (6)$$

Because $t_g b = a^{k(t_g)} b t_g$, (6) becomes

$$a^{2j+(1+\varepsilon(t_g)+k(t_g))i} b^{i+j} t_g = a^{2j'+(1+\varepsilon(t_g)+k(t_g))i'} b^{i'+j'} t_g. \quad (7)$$

Canceling t_g from both sides and comparing the exponents on a and b , we conclude

$$2(j - j') + (1 + \varepsilon(t_g) + k(t_g))(i - i') \equiv 0 \pmod{3}$$

$$(j - j') + (i - i') \equiv 0 \pmod{3}.$$

Now from the definition of $\varepsilon(t_g)$, we always have $1 + \varepsilon(t_g) + k(t_g) \not\equiv 2 \pmod{3}$, so the only solution to the above system is $i - i' = 0$, $j - j' = 0$, i.e. $n_g = n_{g'}$. Thus, Γ is a complete mapping.

Finally, we verify that Γ is an orthomorphism. To this end, suppose $(n_g t_g)^{-1} \Gamma(n_g t_g) = (n_{g'} t_{g'})^{-1} \Gamma(n_{g'} t_{g'})$, i.e.

$$t_g^{-1} n_g^{-1} \alpha_{\varepsilon(t_g)}(n_g) \Phi(\beta(Nt_g)) = t_{g'}^{-1} n_{g'}^{-1} \alpha_{\varepsilon(t_{g'})}(n_{g'}) \Phi(\beta(Nt_{g'})) \quad (8)$$

Arguing as above, we reduce (8) modulo N and invoke the fact that β is an orthomorphism to show that $t_g = t_{g'}$. Substituting and canceling, we obtain

$$n_g^{-1} \alpha_{\varepsilon(t_g)}(n_g) = n_{g'}^{-1} \alpha_{\varepsilon(t_g)}(n_{g'})$$

Because $\alpha_{\varepsilon(t_g)}$ is an orthomorphism, we conclude $n_g = n_{g'}$ and hence that Γ is an orthomorphism. \square

We conclude this section with a few results on the structure of 3-groups.

Lemma 2.5. *Every noncyclic 3-group contains a normal subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof.

Let G be a noncyclic 3-group. If $Z(G)$ is noncyclic, then the structure theorem assures the existence of a normal subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, so we assume henceforth that $Z(G)$ is cyclic of order 3^s , $s \geq 1$. Since G is not itself cyclic, $Z(G)$ is a proper, nontrivial, normal subgroup of G , so there exists a subgroup $Y \subseteq Z(G)$ of order 3.

Now let $C = \langle a \rangle$ be a cyclic normal subgroup of G of maximum order; define r by $|a| = 3^{r-1}$. Because C intersects $Z(G)$ nontrivially, $C \supseteq Y$, so in fact $Y = \langle a^{3^{r-2}} \rangle$. Let N be a normal subgroup of G containing C such that $[N : C] = 3$. Then N is not itself cyclic but has a maximal cyclic subgroup, so by the classification of such groups (see, for example, [12, 5.3.4]), $N = \langle a, b \rangle$ where $|b| = 3$ and either $bab^{-1} = a$ or $bab^{-1} = a^{1+3^{r-2}}$. If we define $M = \langle a^{3^{r-2}}, b \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $Y \subseteq M \subseteq N$. By normality of N in G , for every $g \in G$ gMg^{-1} is a subgroup of N containing Y . However, in either case above, M is the unique subgroup of N which is both isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ and contains Y , so we must have $gMg^{-1} = M$ for all $g \in G$; hence, $M \triangleleft G$. \square

We will need the following calculation as the base step for the inductive argument in the next section.

Lemma 2.6. *Every noncyclic group of order 9 or 27 is strongly admissible.*

Proof.

The noncyclic group of order 9 and the abelian noncyclic groups of order 27 are strongly admissible by Theorem 2.3, so the only remaining groups to consider are the Heisenberg group $H_3 = \langle x, y, z \mid x^3 = y^3 = z^3 = 1, xz = zx, yz = zy, xy = zyx \rangle$ and $L_3 = \langle a, b \mid a^9 = b^3 = 1, bab^{-1} = a^4 \rangle$. Every element of H_3 may be written (uniquely) as $x^i y^j z^k$, where $0 \leq i, j, k \leq 2$; a strong complete mapping φ of H_3 found by Mace4 is exhibited in Table 1 along with the maps $s \mapsto s\varphi(s)$ and $s \mapsto s^{-1}\varphi(s)$ in Table 2. Using GAP, we found a formula for a strong complete mapping of L_3 . First, we identify L_3 with the subgroup $K \leq SL_2(\mathbb{F}_3)$ defined by:

$$K = \left\{ \begin{bmatrix} 1 & c & c - c^2 & a \\ 0 & 1 & c & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{F}_3 \right\}$$

A strong complete mapping is then defined by:

$$\begin{bmatrix} 1 & c & c - c^2 & a \\ 0 & 1 & c & c - c^2 + b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & b & b - b^2 & -ba + c \\ 0 & 1 & b & -(b + b^2) - a + (1 - b^2)c \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Table 1: Strong complete mapping for H_3

s	$\varphi(s)$	s	$\varphi(s)$	s	$\varphi(s)$
1	1	x	y	x^2	z^2
z	yz^2	xz	y^2z^2	x^2z	yz
z^2	xz^2	xz^2	x^2y^2	x^2z^2	$x^2y^2z^2$
y	xy^2z	xy	x^2y	x^2y	y^2
yz	x^2yz	xyz	z	x^2yz	xy
yz^2	x^2y^2z	xyz^2	x	x^2yz^2	x^2z
y^2	y^2z	xy^2	xy^2z^2	x^2y^2	x^2
y^2z	x^2z^2	xy^2z	x^2yz^2	x^2y^2z	xy^2
y^2z^2	xyz^2	xy^2z^2	xz	$x^2y^2z^2$	xyz

Table 2: Images for maps $s \mapsto s\varphi(s)$ and $s \mapsto s^{-1}\varphi(s)$

s	$s\varphi(s)$	$s^{-1}\varphi(s)$	s	$s\varphi(s)$	$s^{-1}\varphi(s)$	s	$s\varphi(s)$	$s^{-1}\varphi(s)$
1	1	1	x	xy	x^2y	x^2	x^2z^2	xz^2
z	y	yz	xz	xy^2	x^2y^2z	x^2z	x^2yz^2	xy
z^2	xz	x	xz^2	y^2z^2	xy^2z	x^2z^2	xy^2z	y^2
y	x	xyz^2	xy	y^2z	xz	x^2y	x^2	xyz
yz	x^2y^2	x^2z^2	xyz	xyz^2	$x^2y^2z^2$	x^2yz	y^2	x^2z
yz^2	x^2z	x^2yz	xyz^2	x^2yz	y^2z	x^2yz^2	xyz	y^2z^2
y^2	yz	z	xy^2	x^2y	z^2	x^2y^2	xy^2z^2	y
y^2z	$x^2y^2z^2$	x^2yz^2	xy^2z	z^2	xy^2	x^2y^2z	yz^2	x^2
y^2z^2	xz^2	xy^2z^2	xy^2z^2	x^2y^2z	yz^2	$x^2y^2z^2$	z	x^2y^2

□

Finally, we record a result which will allow us to bypass the problem of a cyclic quotient group in the induction process.

Lemma 2.7. *Let G be a noncyclic 3-group of order 3^r , $r \geq 4$. Let $N = \langle z, b \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ be a normal subgroup of G , where $z \in Z(G)$ and $b \notin Z(G)$. If G/N is cyclic, then either $G \cong \mathbb{Z}_{3^{r-1}} \times \mathbb{Z}_3$ or $G \cong L_r$, or else there exists $N' \triangleleft Z(G)$, $N' \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ such that G/N' is not cyclic.*

Proof.

Since G/N is cyclic, there exists $x \in G$ which maps to a generator $\bar{x} = xN \in G/N$

under the quotient map, so in fact $G = \langle x, z, b \rangle$. Because G itself is not cyclic, either $|x| = 3^{r-1}$ or $|x| = 3^{r-2}$. If $|x| = 3^{r-1}$, then G has a maximal cyclic subgroup, so by [12, 5.3.4], $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^{r-1}}$ or $G \cong L_r$. If $|x| = 3^{r-2}$, then $\langle z, b \rangle \cap \langle x \rangle = \{1\}$, so let $N' = \langle z, x^{3^{r-3}} \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. We claim that $N' \subseteq Z(G)$. It suffices to show that $x^{3^{r-3}}$ commutes with b . Because $N \triangleleft G$, we have $xbx^{-1} = z^i b$ for some $i \in \{1, 2\}$, i.e. $bx b^{-1} = z^{-i} x$; thus, $bx^{3^{r-3}} b^{-1} = (bx b^{-1})^{3^{r-3}} = z^{-3^{r-3}i} x^{3^{r-3}} = x^{3^{r-3}}$. Finally, the quotient group G/N' contains elements $\bar{x}^{3^{r-4}}$ and \bar{b} , both of order 3, each of which generates a distinct subgroup of G/N' . Therefore, G/N' is not cyclic. \square

3 Main result

Theorem 3.1. *Let G be a nontrivial 3-group which is neither cyclic nor isomorphic to L_r , $r \geq 4$. Then G admits a strong complete mapping.*

Proof.

Suppose G be as in the statement and let $|G| = 3^r$. By Lemma 2.6 we may assume $r \geq 4$. We proceed by induction on r . By Lemma 2.5, G has a normal subgroup $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. If G/N is noncyclic and $G/N \not\cong L_{r-2}$, then G is strongly admissible by the induction hypothesis and Proposition 2.4. If G/N is cyclic and $N \subseteq Z(G)$, then G is abelian, so G has a strong complete mapping by Theorem 2.3. If G/N is cyclic and $N \not\subseteq Z(G)$, then G and N satisfy the hypotheses of Lemma 2.7, so either $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^{r-1}}$ (and hence G is strongly admissible by Theorem 2.3) or we may replace N with another subgroup $N' \triangleleft G$, $N' \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ such that G/N' is not cyclic. If $G/N' \not\cong L_{r-2}$, then G/N' is strongly admissible by induction, and G is strongly admissible by Proposition 2.4.

Thus we are reduced to the case in which $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ is a normal subgroup of G such that $G/N \cong L_{r-2}$. If $r = 4$, then $L_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ is strongly admissible and G is strongly admissible by Proposition 2.4. If $r = 5$, then L_3 is strongly admissible by Lemma 2.6, so G is strongly admissible by Proposition 2.4. We assume henceforth $r \geq 6$. Fix generators z, b of order 3 for N , where $z \in Z(G)$, and let $\pi : G \rightarrow G/N$ denote the quotient map. Choose generators $\bar{x}, \bar{y} \in L_{r-2}$ such that $\bar{x}^{3^{r-3}} = \bar{y}^3 = 1$ and $\bar{y}\bar{x}\bar{y}^{-1} = \bar{x}^{1+3^{r-4}}$ and then select $x, y \in G$ such that $\pi(x) = \bar{x}$ and $\pi(y) = \bar{y}$; evidently, $G = \langle x, y, z, b \rangle$. From normality of N in G we have $xbx^{-1} = z^{-e_x}b$, $yby^{-1} = z^{-e_y}b$, i.e.

$$bx b^{-1} = z^{e_x} x \text{ and } by b^{-1} = z^{e_y} y \text{ for some } e_x, e_y, 0 \leq e_x, e_y \leq 2. \quad (9)$$

Since $G/N = \langle \bar{x}, \bar{y} \rangle = \langle \bar{y}\bar{x}, \bar{y} \rangle = \langle \bar{y}^2\bar{x}, \bar{y} \rangle$, we may replace x by yx or y^2x without disturbing anything claimed henceforth. In particular, if $e_y \neq 0$, then replacing x by $y^{-e_x}x$, we may

assume $e_x = 0$. Replacing z with z^2 , we may assume without loss of generality that at one of e_x, e_y is 0 and the other is either 0 or 1. We claim furthermore that $x^3 \in Z(G)$; to prove this, it suffices to show that x^3 commutes with y and b . From normality of N in G we have $xbx^{-1} = z^i b$ for some i , $0 \leq i \leq 2$, i.e. b and x commute modulo $\langle z \rangle$. Then $bx^3b^{-1} = (bxb^{-1})^3 = (z^{-i}x)^3 = x^3$. From the relation $\bar{y}\bar{x}\bar{y}^{-1} = \bar{x}^{1+3^{r-4}}$ in G/N we have $yxy^{-1} = z^i b^j x^{1+3^{r-4}}$ for some i, j , $0 \leq i, j \leq 2$. Again, since b and x commute modulo $\langle z \rangle$, it follows that $yx^3y^{-1} = (yxy^{-1})^3 = x^3$.

Now let $c = 3^{r-4}$ and define $M = \langle z, c \rangle \subseteq Z(G)$. Observe that $M \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and also $|c| = 3$ or $|c| = 9$. If $|c| = 3$, then $|x| = 3^{r-3}$. If G/M is cyclic, then G is abelian and therefore strongly admissible by Theorem 2.3. Since $r \geq 6$, the residues of $x^{3^{r-5}}$ and b are (respectively) central elements of order 3 in G/M , but they generate different subgroups. Since $Z(L_{r-2}) \cong \mathbb{Z}_3$, G/M cannot be isomorphic to L_{r-2} . By induction, G/M is strongly admissible, and thus G is strongly admissible by Lemma 2.1. If $|c| = 9$, then $c^3 \in \text{Ker } \pi = N$; hence, $c^3 = z^i b^j$ for some i, j , $0 \leq i, j \leq 2$. If $j \neq 0$, then $b \in M \cong \mathbb{Z}_3 \times \mathbb{Z}_9$, and $G/M \cong (G/N)/\langle \pi(c) \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^{r-4}}$ is admissible by Theorem 2.3. Thus, G is strongly admissible by Lemma 2.1. If $j = 0$ then we must have $i \neq 0$, and so $z \in M' = \langle c, b \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_9$. Note that $M' \triangleleft G$ and $G/M' \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^{r-4}}$ is strongly admissible. If $e_x = e_y = 0$, then $M' \subseteq Z(G)$, so we may conclude as before by Lemma 2.1. Thus we assume henceforth that either $e_x = 1$ and $e_y = 0$, or else $e_x = 0$ and $e_y = 1$.

We will use a technique similar to that used in Proposition 2.4 to construct a strong complete mapping for G . Observe that every member of M' may be written uniquely as $z^i b^j c^k$, where $0 \leq i, j, k \leq 2$. Now consider the transversal $T = \{x^l y^m : 0 \leq l \leq 3^{r-4} - 1, 0 \leq m \leq 2\}$ for M' in G , and let $\Phi : G/M' \rightarrow T$ be the bijection which assigns to each (right) coset $M'g$ the unique element $t_g \in T$ such that $M'g = M't_g$. By Theorem 2.3, G/M' admits a strong complete mapping $\beta : G/M' \rightarrow G/M'$, and M' likewise admits a strong complete mapping $\alpha_0 : M' \rightarrow M'$. Define maps $\alpha_s : M' \rightarrow M'$, $s = 1, 2$ as in Table 3; for convenience, we use the string ijk to represent the element $z^i b^j c^k \in M'$.

The maps α_1 and α_2 were in fact found by Mace4, which was asked to search for orthomorphisms of $\mathbb{Z}_3 \times \mathbb{Z}_9$ satisfying the following additional condition: for $\ell \in \{1, 2\}$, the map

$$z^i b^j c^k \mapsto z^{i+\ell j} b^j c^k \alpha_\ell(z^i b^j c^k) \quad (10)$$

is bijective.

Every $g \in G$ may be rewritten uniquely as $g = m'_g t_g$, with $m'_g = z^i b^j c^k \in M'$ and $t_g = x^s y^t \in T$, where $0 \leq i, j, k, t \leq 2$ and $0 \leq s \leq 3^{r-3} - 1$. Now define $\Gamma : G \rightarrow G$ by

$$\Gamma(g) = \alpha_u(m_g) \Phi(\beta(M't_g)).$$

Table 3:

ijk	$\alpha_1(ijk)$	$\alpha_2(ijk)$	ijk	$\alpha_1(ijk)$	$\alpha_2(ijk)$	ijk	$\alpha_1(ijk)$	$\alpha_2(ijk)$
000	000	000	100	210	022	200	222	011
001	002	002	101	120	012	201	110	122
002	220	220	102	001	001	202	112	112
010	100	100	110	211	210	210	122	202
011	202	021	111	101	101	211	221	120
012	011	212	112	012	010	212	020	020
020	022	221	120	111	111	220	121	222
021	102	211	121	010	200	221	021	110
022	212	102	122	201	121	222	200	201

where

$$u = \begin{cases} s \pmod{3} & \text{if } e_y = 0 \\ t \pmod{3} & \text{if } e_x = 0 \end{cases}$$

The argument of Proposition 2.4, *mutatis mutandis*, shows that Γ is an orthomorphism. To show that Γ is a complete mapping, suppose $g\Gamma(g) = g'\Gamma(g')$, i.e.

$(z^i b^j c^k)(x^s y^t)\Gamma((z^i b^j c^k)(x^s y^t)) = (z^{i'} b^{j'} c^{k'})(x^{s'} y^{t'})\Gamma((z^{i'} b^{j'} c^{k'})(x^{s'} y^{t'}))$. Then

$$(z^i b^j c^k)(x^s y^t)\alpha_u(z^i b^j c^k)\Phi(\beta(M'x^s y^t)) = (z^{i'} b^{j'} c^{k'})(x^{s'} y^{t'})\alpha_u(z^{i'} b^{j'} c^{k'})\Phi(\beta(M'x^{s'} y^{t'})). \quad (11)$$

Reducing modulo M' , we have $x^s y^t \beta(M'x^s y^t) = x^{s'} y^{t'} \beta(M'x^{s'} y^{t'})$. Since β is a complete mapping, $M'x^s y^t = M'x^{s'} y^{t'}$, and because T is a transversal, $s = s'$ and $t = t'$. Substituting into (11) and canceling, we have

$$(z^i b^j c^k)(x^s y^t)\alpha_u(z^i b^j c^k) = (z^{i'} b^{j'} c^{k'})(x^s y^t)\alpha_u(z^{i'} b^{j'} c^{k'}). \quad (12)$$

If $e_y = 0$, then $e_x = 1$ and $u = s$, so (12) may be rewritten:

$$z^{i+sj} b^j c^k \alpha_s(z^i b^j c^k) = z^{i'+sj'} b^{j'} c^{k'} \alpha_s(z^{i'} b^{j'} c^{k'}).$$

Using the bijectivity of the map (10), it follows that $i = i'$, $j = j'$, $k = k'$. Likewise, if $e_x = 0$, then $e_y = 1$ and $u = t$, so (12) becomes:

$$z^{i+tj} b^j c^k \alpha_t(z^i b^j c^k) = z^{i'+tj'} b^{j'} c^{k'} \alpha_t(z^{i'} b^{j'} c^{k'}).$$

As before, we conclude $i = i'$, $j = j'$, and $k = k'$ and thus that Γ is a strong complete mapping. \square

Corollary 3.2. *Let G be a nilpotent group of odd order whose 3-Sylow subgroup is neither cyclic nor isomorphic to L_r , $r \geq 4$. Then G is strongly admissible.*

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