The Cayley Addition Table of $\mathbf{Z}_n$

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In the June-July 1999 American Mathematical Monthly, Hunter Snevily stated, "Few mathematical objects could be considered more simple than the Cayley addition table of $\mathbf{Z}_n$ but we show that even these simple objects have some interesting yet unproved properties." He then proposed the following conjecture:

> Let n be any positive odd integer. Then, for any $k \in \{1, \ldots, n\}$, the $k \times k$ submatrix of the Cayley addition table of $\mathbf{Z}_n$ contains a latin transversal.

This paper examines this conjecture. We begin with preliminary definitions and examples.

Lemma 1 is a basic result found in most abstract algebra textbooks.

Definition 1: A transversal of a square matrix is a collection of n entries, no two of which are in the same row or column.

Definition 2: A transversal is called a latin transversal if no two of its entries are the same.

Example 1: In the figure below, the shaded region is a transversal. However, it is not a latin transversal since the elements 0 and 2 each appear twice.

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Note that the above conjecture is not true if n is even because there exists an element x in $\mathbf{Z}_n$ that has order two. When considering the 2×2 submatrix formed by 0 and x, there does not exist a latin transversal. Hence, for the remainder of the paper, unless stated otherwise, n will be an odd positive integer. "Submatrix" refers to any submatrix of the Cayley addition table and for any matrix A, a_{ij} refers to the element in the i^{th} row and j^{th} column.

Definition 3: A rightward diagonal is a collection of entries of the form $a_{i+m(\mathbf{mod}\ n),\, j+m(\mathbf{mod}\ n)}$ for $m = 0, 1, \ldots, n-1$ and $0 \le i \le j \le n-1$.

Example 2: The shaded region in the figure below is a rightward diagonal.

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Definition 4: Any collection of latin transversals is called non-concurrent if there is no element common to every latin transversal in the collection.

Definition 5: Let S be the set of $k \times k$ matrices with integer entries. Then, $T: S \to S$ is a translation iff $\exists\, b \in \mathbf{R}$ such that $T(A) = A + b\mathbf{1}$ where $\mathbf{1}$ is the $k \times k$ matrix with all entries equal to 1.

Definition 6: A submatrix of the Cayley addition table is equispaced if the distance between successive rows is equal to the distance between successive columns.

Example 3: The shaded region in the figure below is an equispaced submatrix.

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Lemma 1: Let $a, b, c \in \mathbf{Z}$. Then $\gcd(a,b) = 1$ and $a \mid bc \Rightarrow a \mid c$.

Proof:

$\mathbf{gcd}(a,b) = 1 \Rightarrow sa + tb = 1$ for some $s,t \in \mathbf{Z}$. Thus $sac + tbc = c$. So $a \mid (sac + tbc)$ since $a \mid a$ and $a \mid bc$. $\therefore a|c$.

Theorem 2: The Cayley addition table of $\mathbf{Z}_n$ contains a latin transversal.

Proof:

In the case where $n = 1$, the theorem is trivially true because there is only one element of the Cayley table.

Consider $n > 1$.

The main diagonal of the Cayley table for $\mathbf{Z}_n$ is as follows:

$+_{\text{mod } n}$	**0**	**1**	**2**	**…**	$\boldsymbol{n-1}$
0	$2\cdot 0$				
1		$2\cdot 1$			
2			$2\cdot 2$		
…				…	
$\boldsymbol{n-1}$					$2\cdot(n-1)$

Clearly, the entries of the main diagonal are a transversal since no two entries are in the same row or column. Note that elements on the main diagonal are of the form $2a$ **mod** n for $a = 0, 1, 2, \ldots, n-1$.

CLAIM: The main diagonal is a latin transversal.

Suppose $2a \equiv 2b$ **mod** n for distinct $a, b \in \mathbf{Z}_n$. By definition, $n \mid 2a - 2b$, i.e., $n \mid 2(a - b)$.

gcd$(2,n) = 1$ because n is odd. Thus, by Lemma 1, $n \mid (a - b) \Rightarrow a \equiv b$ **mod** n.

But this is a contradiction because a and b are distinct elements from $\mathbf{Z}_n$.

$2a \neq 2b$ **mod** n, for all distinct $a, b \in \mathbf{Z}_n$. Then, by definition, the main diagonal is a latin transversal. Thus, the $n \times n$ matrix of the Cayley addition table of $\mathbf{Z}_n$ contains a latin transversal.

Corollary 3: Any rightward diagonal of the Cayley addition table is a latin transversal.

Proof: Let us inspect a rightward diagonal of $\mathbf{Z}_5$ (the shaded region):

$+_{\text{mod } n}$	**0**	**1**	**2**	**3**	**4**
0				$2\cdot 3+2$	
1					$2\cdot 4+2$
2	$2\cdot 0+2$				
3		$2\cdot 1+2$			
4			$2\cdot 2+2$		

It is clear that the shaded diagonal forms a transversal. Additionally, each entry has the form $2l + 2$ for $l = 0, 1, \ldots, 4$. To test whether this is a latin transversal, we may subtract 2 from each entry, since it is common to all entries. But this leaves a collection of entries exactly of the form $2\cdot 0, 2\cdot 1, \ldots, 2\cdot 4$, which is a latin transversal by Theorem 2.

In general, one may select a random integer k from the first column, and then transverse the matrix in a rightward diagonal fashion. The elements will have the form $2l + k$ for
$l = 0, 1, \ldots, n - 1$. Subtracting the common k from each entry gives a collection of entries exactly of the form of the main diagonal, which we know is a latin transversal. Therefore, any rightward diagonal of the Cayley addition table is a latin transversal.

Corollary 4: Any $(n - 1) \times (n - 1)$ submatrix will contain a latin transversal.

Proof: The $(n - 1) \times (n - 1)$ submatrix is essentially formed by removing one row and one column from the Cayley addition table. The stricken row and column will intersect at one common point. Consider the rightward diagonal latin transversal of the $n \times n$ matrix that passes through this point (which we know exists by Corollary 3). By removing one row and one column, we have removed one element from a latin transversal. Since removing an element from a latin transversal implies that it is still a latin transversal, we know that there exists at least one latin transversal in the $(n - 1) \times (n - 1)$ submatrix.

Theorem 5: Let S be the set of $k \times k$ submatrices. Then for all $A \in S$ there exists a translation $T: S \to S$ such that $T(A) = B$ and $b_{11} = 0$. Moreover, A has a latin transversal iff B has a latin transversal.

Proof:

Let $A \in S$ such that the first entry is $a + b$, where $a, b \in \mathbf{Z}_n$. Then A has the following form.

$a + b$	$a + b + a_1$	$a + b + a_2$	. . .	$a + b + a_{(k-1)}$
$a + b + b_1$	$a + b+ a_1 + b_1$	$a + b + a_2 + b_1$	. . .	$a + b + a_{(k-1)} + b_1$
$a + b + b_2$	$a + b + a_1 + b_2$	$a + b + a_1 + b_2$	. . .	$a + b + a_{(k-1)} + b_2$
. . .	. . .	. . .	. . .	. . .
$a + b + b_{(k-1)}$	$a + b + a_1 + b_{(k-1)}$	$a + b + a_1 + b_{(k-1)}$	. . .	$a + b + a_{(k-1)} + b_{(k-1)}$

Subtracting $(a + b)$ from each entry, we have the following form:

0	a_1	a_2	. . .	$a_{(k-1)}$
b_1	$a_1 + b_1$	$a_2 + b_1$	. . .	$a_{(k-1)} + b_1$
b_2	$a_1 + b_2$	$a_2 + b_2$	. . .	$a_{(k-1)} + b_2$
. . .	. . .	. . .	. . .	. . .
$b_{(k-1)}$	$a_1 + b_{(k-1)}$	$a_2 + b_{(k-1)}$	. . .	$a_{(k-1)} + b_{(k-1)}$

So, $T(A) = A - (a + b)\mathbf{1} = B$, where $\mathbf{1}$ is the $k \times k$ matrix with all entries equal to 1.

Note that $(x_1, x_2, \ldots, x_k)$ is a latin transversal iff $(x_1 + c, x_2 + c, \ldots, x_k + c)$ is a latin transversal for $x_i, c \in \mathbf{Z}_n$ and $i = 0, 1, \ldots, k$. In this case, let $c = a + b$. Then, A has a latin transversal iff B has a latin transversal.

Corollary 6: Any submatrix that is equispaced will contain a latin transversal.

Proof:

Consider the following matrix A, where $0 \le x_i < n$, and $x_i \neq x_j$ whenever $i \neq j$.

$a + b$	$a + b + x_1$	$a + b + x_2$	. . .	$a + b + x_{(k-1)}$
$a + b + x_1$	$a + b + x_1 + x_1$	$a + b + x_2 + x_1$	. . .	$a + b + x_{(k-1)} + x_1$
$a + b + x_2$	$a + b + x_1 + x_2$	$a + b + x_2 + x_2$	. . .	$a + b + x_{(k-1)} + x_2$
. . .	. . .	. . .	. . .	. . .
$a + b + x_{(k-1)}$	$a + b + x_1 + x_{(k-1)}$	$a + b + x_1 + x_{(k-1)}$	. . .	$a + b + x_{(k-1)} + x_{(k-1)}$

By Theorem 5, A can be translated to the following submatrix.

0	x_1	x_2	. . .	$x_{(k-1)}$
x_1	$2x_1$	$x_2 + x_1$	. . .	$x_{(k-1)} + x_1$
x_2	$x_1 + x_2$	$2x_2$	. . .	$x_{(k-1)}+x_2$
. . .	. . .	. . .	. . .	. . .
$x_{(k-1)}$	$x_1 + x_{(k-1)}$	$x_1 + x_{(k-1)}$	. . .	$2x_{(k-1)}$

By Theorem 2, $2a \neq 2b$ **mod** n for all distinct a,b in Z_n. Since $x_i \neq x_j$ whenever $i \neq j$, the main diagonal is clearly a latin transversal.

By Theorem 5, then, A has a latin transversal.

Note (♣): Each row and each column of a Cayley group table contain distinct elements.

Theorem 7: Every 2×2 submatrix contains a latin transversal.

Proof:

Each such 2×2 submatrix is equivalent to the following form by Theorem 5.

0	a
b	$a + b$

If $a \neq b$, then there exists a latin transversal, so suppose $a = b$.

Then, this implies that $a + b = 2a \neq 0$ **mod** n since n is odd and $a \neq 0$ by ♣.

Thus $(0, a + b)$ is a latin transversal.

Theorem 8: Every 3×3 submatrix contains a latin transversal.

Proof: Let a, b, c and d be distinct group elements of $(\mathbf{Z}_n, +)$, and let X be a 3×3 submatrix.

Case I: Suppose X only contains three elements, as in the figure below.

a	b	c
b	c	a
c	a	b

It is clear that this is the only possible configuration of a 3×3 submatrix with only three elements (with row and column interchanges excluded). It is also clear that there exists a latin transversal, namely (a, c, b), on the main diagonal (shaded).

Note that if there exists a 2×2 submatrix Y of X such that Y contains two non-concurrent latin transversals, then X contains a latin transversal. This can easily be seen in the illustration below by entering any group element in the shaded region.

a	b	
d	c	

Case II: We will now consider an arbitrary 2×2 submatrix Y of X, such that Y has non-distinct elements on its main diagonal. Please inspect the figure below.

	a	
		a

Let us now consider the 2×2 submatrix (call it W) indicated in the figure below. By ♣, x_{13} cannot equal a, since $x_{33} = a$. The off diagonal of W is hence a latin transversal by Theorem 7. Therefore, the only way to construct W such that it does not contain two non-concurrent latin transversals is to require the main diagonal to contain non-distinct elements.

	b	
	a	b
		a

Finally, let us now consider the 2×2 submatrix formed by (x_{12}, x_{13}, x_{32}, x_{33}). (x_{12}, x_{33}) forms a latin transversal. Neither x_{13} nor x_{32} equal a or b by ♣. Additionally, if both x_{13} and x_{32} are the same element, then X is the same as in Case I, so X contains a latin transversal (see fig. A). If x_{13} and x_{32} are distinct elements, then we have two non-concurrent latin transversals. Therefore, X must contain a latin transversal (see fig. B).

	b	c
	a	b
	c	a

fig. A

	b	d
	a	b
	c	a

fig. B

Of course, a similar argument follows if the original arbitrarily chosen 2×2 submatrix Y contains a non-latin transversal on its off diagonal.

<u>References</u>:

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