Representations of Graphs by Rings

Megan Bernstein

July 17, 2008

Abstract

A graph is representable by a ring if its vertices can be labeled with distinct ring elements so the difference of the labels is a unit in the ring if and only if the vertices are adjacent. We explore representation by rings composed of the direct products of cyclic rings, including an upper bound on the representation number for a graph with a fixed number of vertices and representation numbers for various families of graphs.

1 Introduction

A graph, G, is an ordered pair such that $G = (V,\, E)$ where V is a set of vertices and E is a set of unordered pairs of distinct vertices of G. An edge, e, between vertices $v, w \in V(G)$ is denoted $e = \{v, w\}$. For a given ring R, the graph representing it, G_R, is a graph with $|R|$ vertices, each labeled with a distinct ring element, whose edges are $\{(v, w) : v - w \text{ is a unit in R}\}$.

We say G' is an induced subgraph of G if $V(G') \subseteq V(G)$ and if $(v, w) \in E(G)$ with $v, w \in V(G')$, then $(v, w) \in E(G')$.

A graph G is represented by a ring R if G is an induced subgraph of G_R. The representation number of a graph, G is the smallest natural number r such that $\mathbb{Z}_r$ represents G. This is denoted $rep(G) = r$. In this case, G is also said to be representable modulo r.

The size of largest set of unconnected vertices is the independence number, denoted $\alpha(G)$. The clique number $\omega(G)$ of a graph is the largest complete graph as an induced subgraph, and the chromatic number $\chi(G)$ is the fewest number of colors needed to color each vertex such that no vertices of the same color are connected by an edge.

The original work in representation followed Evans and Erdös proof that finite graph has a finite representation number [0]. Narayan established a sharp upper bound for the representation number of a graph with r vertices [4]. Considerable work has been done on establishing the representation number for various families of graphs.

A natural extension this previous work in representation of graphs modulo n, comes from the realization that the original work for representable modulo n is in essence representing the graph of the ring $\mathbb{Z}_n$. We call this collection of rings $\mathcal{C}$. This opens the question of how to represent graphs by arbitrary rings, for example rings of the form $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$, which we call $\mathcal{D}$. Let the representation number of a graph G in $\mathcal{D}$, denoted $rep_{\mathcal{D}}(G)$ be the order of the smallest ring in $\mathcal{D}$ that represents G. Since $\mathcal{D}$ contains $\mathcal{C}$, all the existing upper bounds on representation numbers are maintained, but some simple examples show that significant improvements in lower bounds or actual representation numbers can be made. For example, take the graph C_5, which has a representation number for $\mathcal{C}$ of 105, while it can be represented in $\mathcal{D}$ by $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, This happens to be optimal, so the representation number in $\mathcal{D}$ is 27.

It is also possible to consider representation number $rep_{\mathcal{F}}(G)$ for any collection $\mathcal{F}$ of rings; however, in this paper, we will focus on the case $\mathcal{F} = \mathcal{D}$.

2 Notation

Following Evans et al [1], we will establish a coordinate representation for G and array representation for G_R.

Let $R \in \mathcal{D}$, so $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$. By the Chinese Remainder Theorem, we get an equivalent representation, $R = \mathbb{Z}_{p_1^{l_{1,1}}} \times \cdots \times \mathbb{Z}_{p_1^{l_{1,r_1}}} \times \cdots \times \mathbb{Z}_{p_s^{l_{s,1}}} \times \cdots \times \mathbb{Z}_{p_s^{l_{s,r_s}}}$ where each p_i is a distinct prime, each $l_{i,j} > 0$, and $\prod_{i=1}^{t} n_i = \prod_{i=1}^{s} \prod_{j=1}^{r_s} p_i^j$. We will use this factorization as the basis for our notation. We call r_i the *dimension* of the i^{th} prime, and the sum of the r_i's is the *total dimension* of the representation.

2.1 Coordinate Representation

A vertex label is of the form $(i_{p_1,1}, ..., i_{p_1,r_1}, i_{p_2,1}, ..., i_{p_s,r_s})$ where each $0 \leq i_{p_j,k} < p_j^{l_{j,r_j}}$. Two vertices are adjacent if and only if their coordinates differ mod p_i, for each respective coordinate. We can see this definition of adjacency is equivalent to the difference of the labels being units in R.

2.2 Array Representation

The graph $G_{\mathbb{Z}_{p_1^{l_{1,1}}} \times \cdots \times \mathbb{Z}_{p_s^{l_{s,r_s}}}}$ can be viewed as an $\sum_{i=0}^{s} r_i$-dimensional array of cells of independent vertices with coordinates $(a_{(1,1)}, ..., a_{(s,r_s)})$ such that:

- each cell has cardinality $\prod_{j=1}^{s} \prod_{i=1}^{r_s} p_i^{l_{j,i}-1}$
- each cell contains the vertices with labels congruent mod p_i for each i
- a vertex in the cell $(a_{(p_1,1)}, ..., a_{(p_s,r_s)})$ is adjacent to a vertex in the cell $(b_{(p_1,1)}, ..., b_{(p_s,r_s)})$ if and only if $a_{(i,j)} \neq b_{(i,j)}$ for all $1 \leq i < s$ and $1 \leq j \leq r_s$.

As an example, we display $G_{\mathbb{Z}_4 \times \mathbb{Z}_4}$ as follows.

	0	1
0	$(0,0),(0,2),(2,0),(2,2)$	$(1,0),(1,2),(3,0),(3,2)$
1	$(0,1),(0,3),(2,1),(2,3)$	$(1,1),(1,3),(3,1),(3,3)$

Table 1: Array Representation of $\mathbb{Z}_4 \times \mathbb{Z}_4$

3 Rings with the Same Representative Graph

For rings in $\mathcal{C}$, each ring is determined uniquely by its order, so each representative graph is also uniquely determined by the number of vertices. So we get that $G_R \cong G_{R'} \Leftrightarrow R \cong R'$. However, in $\mathcal{D}$ there can be multiple rings of the same order, and we find more than one ring can sometimes generate the same graph. So $G_R = G_{R'}$ does not necessarily imply $R \cong R'$ for $R, R' \in \mathcal{D}$ An example of this is $G_{\mathbb{Z}_2 \times \mathbb{Z}_8} = G_{\mathbb{Z}_4 \times \mathbb{Z}_4}$. See Figure 1 and Figure 2.

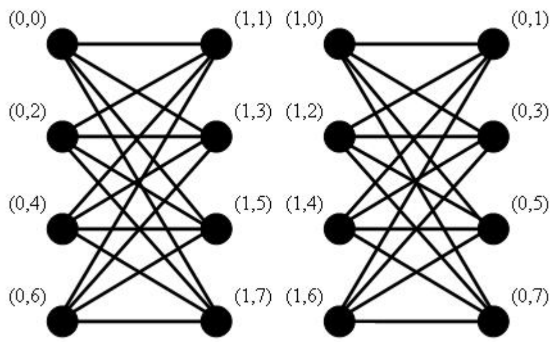

Figure 1: $2K_{4,4}$ represented by $\mathbb{Z}_2 \times \mathbb{Z}_8$

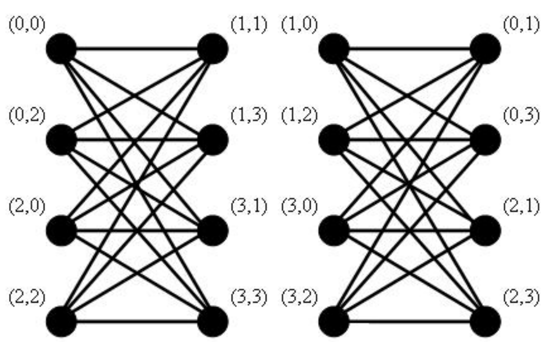

Figure 2: $2K_{4,4}$ represented by $\mathbb{Z}_4 \times \mathbb{Z}_4$

In general, this is a consequence of the array representation of the rings: the dimensions of the array are based on the sum of product dimensions for each prime ($\sum_{i=1}^{s} r_i$ in our notation). Therefore, the number of ring elements in each cell is based on the product $\prod_{i=1}^{s}\prod_{j=1}^{r_s} p_i^{l_{i,j}-1}$. So for any $l_{i,j} > 1$ and $l_{i,k}$, substituting $l'_{i,j} = l_{i,j} - 1$ and $l'_{i,k} = l_{i,k} + 1$ yields the same array representation since $p^{l_{i,j}}p^{l_{i,k}} = p^{l'_{i,j}}p^{l'_{i,k}}$. This means that $R = \mathbb{Z}_{p_1^{l_{1,1}}} \times \cdots \times \mathbb{Z}_{p_s^{l_{s,r_s}}}$, has the same graph as $R' = (\mathbb{Z}_{p_1})^{e_1} \times \mathbb{Z}_{p_1^{f_1}} \times \cdots \times (\mathbb{Z}_{p_s})^{e_s} \times \mathbb{Z}_{p_s^{f_s}}$ where $e_i \geq 0, f_i \geq 1$ for each distinct prime p_i. And $e_i + 1$ is the dimension of the i^{th} prime and the total number of vertices per cell is $\prod_{i=1}^{s} p_i^{f_i-1}$.

It immediately follows that:

Proposition 3.1. *For each $R = \mathbb{Z}_{p_1^{l_{1,1}}} \times \cdots \times \mathbb{Z}_{p_s^{l_{s,r_s}}}$ with distinct primes p_i, there exists a $R' = (\mathbb{Z}_{p_1})^{e_1} \times \mathbb{Z}_{(p_1)^{f_1}} \times \cdots \times \mathbb{Z}_{p_s})^{e_s} \times \mathbb{Z}_{(p_s)^{f_s}}$ with $e_i \geq 0, f_i \geq 1$, such that $G_R \cong G_{R'}$.*

4 An Upper Bound for the $Rep_{\mathcal{D}}$ of Graphs with Fixed Order

Narayan found the strict upper bound for the representation number of a graph with r vertices to be $p_s \cdots p_{s+r-2}$ where p_s is the smallest prime greater than or equal to $r-1$ and p_{s+i} is the i^{th} prime after it [4].

We can adapt his proof to representations with rings in $\mathcal{D}$.

First we need to establish some notion: the family of complete graphs, K_n, is comprised of the graphs with n vertices and an edge between each pair of vertices. A disjoint union of graphs, $G \cup H$, is the graph whose vertex set is $V(G) \cup V(H)$, and in which $\{v, w\}$ is an edge iff $[v, w \in V(G)$ and $\{v, w\} \in E(G)]$ or $[v, w \in V(H)$ and $\{v, w\} \in E(G)]$.

To adapt the proof, we need to establish a relationship between the product representation of a graph and the coordinate representation of a graph in $\mathcal{D}$ and prove that the $rep_{\mathcal{D}}(K_{r-1} \cup K_1) = p_s^{r-1}$. The crucial information from product representation still holds: that the maximum product dimension of a r-vertex graph is $r-1$ and the only graph with that product dimension is $K_{r-1} \cup K_1$ [4].

4.1 Product and Coordinate Representation

A product representation of a graph of length t assigns distinct vectors of length t to each vertex so that vertices u, v are adjacent if and only if their vectors differ in every position. The product dimension of a graph is the minimum length of such a representation of G.

To go from a coordinate representation of a graph by $R \in \mathcal{D}$ to a product representation: If $|R| = p_1^{k_1} \cdots p_s^{k_s}$ and $R = \mathbb{Z}_{p_1^{l_{1,1}}} \cdots \mathbb{Z}_{p_s^{l_{s,r_s}}}$, let t the length of the vector be $\sum_{i=1}^{s} r_i$. Then assign each vertex's coordinate label to be the vector. This is not guaranteed to be a minimal product representation.

And then to go from product representation for each position of the vector, choose the smallest prime for each coordinate such that the prime is larger than the number of values used in that position. That gives a representation by some element of $\mathcal{D}$.

$Rep_{\mathcal{D}}(K_{r-1} \cup K_1)$

First some examples. Note that $rep_{\mathcal{D}}(K_0 \cup K_1) = rep_{\mathcal{D}}(K_1) = 1$ and that $rep_{\mathcal{D}}(K_1 \cup K_1) = rep_{\mathcal{D}}(2K_1) = 4$. Finally, $rep_{\mathcal{D}}(K_2 \cup K_1) = 4$ as represented by $\mathbb{Z}_2 \times \mathbb{Z}_2$; however, $rep(K_2 \cup K_1) = 6$.

Proposition 4.1. *For $r \geq 3$, $rep_{\mathcal{D}}(K_{r-1} + K_1) = p^{r-1}$ where p is the smallest prime greater than or equal to $r - 1$.*

Proof.

Construct a representation for $K_{r-1} + K_1$ by the ring $(\mathbb{Z}_p)^{r-1}$ by letting the vertices of K_{r-1} be labeled $(0, 0, ..., 0)$ through $(r-2, r-2, ..., r-2)$ and the vertex of K_1 be labeled $(0, 1, ..., r-2)$.

We know from the product dimension of $K_{r-1} + K_1$ [4], that the total dimension of its representation must be at least $r-1$. If the smallest prime dividing $rep_{\mathcal{D}}(K_{r-1} + K_1)$ was less than p, we could not represent K_{r-1}, since two vertices would have labels with the same value in that coordinate. So $rep_{\mathcal{D}}(K_{r-1} + K_1) = p^{r-1}$ and its representation type is $(\mathbb{Z}_p)^{r-1}$. ■

Theorem 4.2. *For $r \geq 3$, the maximum of $rep_{\mathcal{D}}(G)$ over graphs of order r is p^{r-1}, where p is the smallest prime strictly greater than $r - 1$.*

Proof.

We can see this bound is sharp when $r-1$ is not prime since this is the representation number of the graph $K_{r-1} + K_1$.

Suppose we have a graph G of order r. Then it has a product representation of length $r-1$ [4]. For each dimension of the product representation i, from 1 to $r-1$, we may assume the values used are in the set $\{0, 1, ..., c_i - 1\}$ for some c_i a positive integer. Note that since there are r vertices, each $c_i \leq r$.

Case 1: G is a complete graph

If G is a complete graph, then its representation number is p if $r-1$ is not prime or $r-1$ if it is prime. In either case, $r - 1 < p < p^{r-1}$.

Case 2: $r-1$ is not prime and G is not complete

Assume G is not complete, and $r-1$ is not prime, then we know $p \geq r$. And we also know that for each i, $c_i \leq r$; so to form a coordinate representation from the product representation, we choose the smallest prime greater than each c_i which will be less than or equal to p. Thus, $rep_{\mathcal{D}}(G) \leq p^{r-1}$ in this case.

Case 3: $r-1$ is prime, G is not complete

We have already shown that the representation number of $K_{r-1} + K_1 = (r-1)^{r-1} < p^{r-1}$ in the case that $r-1$ is prime. Since this is the only graph with r vertices and product dimension $r-1$, so all other graphs with r vertices have product dimension $\leq r-2$. So for the $r-2$ dimensions, each has a maximum of r values, so to convert to coordinate representation we use the smallest prime greater than or equal to r, namely p. So the maximum representation number is $p^{r-2} < p^{r-1}$. ■

5 Graphs Representable by $\mathbb{Z}_p \times \mathbb{Z}_p$:

The only graphs representable with $\mathbb{Z}_p$ are complete graphs, and the only graphs representable by $\mathbb{Z}_{p^e}$ are graphs that reduce to complete graphs, which the complete multipartite graphs, discussed later. We will now explore graphs representable with a product dimension of 2.

From Evans et al [1] we have the following theorem:

Theorem 5.1. *A graph, G is representable modulo pq if and only if it does not contain any of the following as an induced subgraph: $K_2 + 2K_1$, $K_3 + K_1$, or $K_n - C_n$ for n odd and $n \geq 5$.*

They also demonstrated the last condition is equivalent to the statement $\omega(G) = \chi(G)$.

Lemma 5.2. *If a graph G is representable by $\mathbb{Z}_p \times \mathbb{Z}_q$, p, q distinct primes, $p < q$ then G is representable by $\mathbb{Z}_q \times \mathbb{Z}_q$*

Proof.

Consider the $p \times q$ array from the array representation of the graph of $\mathbb{Z}_p \times \mathbb{Z}_q$. We can increase the length length of the p dimension of the array to another prime without changing any of the original induced subgraphs. Therefore we can increase p to q and obtain the graph of $\mathbb{Z}_q \times \mathbb{Z}_q$. ■

This immediately yields:

Lemma 5.3. *If a graph G is representable by $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p, then for any prime q greater than p, G is representable by $\mathbb{Z}_p \times \mathbb{Z}_q$.*

Theorem 5.4. *A graph G is representable by $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p, if and only if it does not contain any of the following as an induced subgraph: $K_2 + 2K_1$, $K_3 + K_1$, or $K_n - C_n$ for n odd and $n \geq 5$.*

Proof.

If a graph G is representable by $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p, then it is representable for some prime q by $\mathbb{Z}_p \times \mathbb{Z}_q$, and by 5.1 it does not contain any of the necessary induced subgraphs.

If a graph G does not contain any of the necessary induced subgraphs, then it is representable by $\mathbb{Z}_p \times \mathbb{Z}_q$ for some primes p, q such that $q > p$, so it is also representable by $\mathbb{Z}_q \times \mathbb{Z}_q$. ■

6 Reductions of Graphs

We will show reduction in $\mathcal{D}$ works similarly as in $\mathcal{C}$, as originally shown by Evans et al [1].

First we define reduction. Let two vertices be equivalent if their open neighborhoods (the set of vertices they are adjacent to) are equal. We can see this is an equivalence relation on the vertices of the graph since a vertex has the same open neighborhood as itself, and equality of open neighborhoods is transitive and symmetric. Moreover, the subgraph induced by two equivalence classes is either an edgeless graph if no vertex from one is in the open neighborhood of the other, or else, a complete bipartite graph since if one vertex from the first equivalence class is in the open neighborhood of a vertex from the second, all the vertices of the first are in all of the open neighborhoods of the second. This means we can represent each equivalence class as just one vertex. The reduction of a graph G is the graph, H, whose vertices are the equivalence classes of G, and two vertices are adjacent if and only if two vertices in their respective equivalence classes (and therefore all the vertices in the equivalence classes) are adjacent.

A complete bipartite graph is a graph whose reduction is K_2. Similarly, a complete multipartite graph is any graph whose reduction is K_n for some n.

Proposition 6.1. *A graph is representable by* $R = \mathbb{Z}_{p_1^{l_{1,1}}} \times \cdots \times \mathbb{Z}_{p_s^{l_{s,r_s}}}$ *if and only if its reduction is representable by* $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}$ *where the dimension of each* p_i *is preserved.*

Proof.

From Proposition 3.1, we know there exists R' such that $G_R \cong G_{R'}$ and $R' = (\mathbb{Z}_{p_1})^{e_1} \times \mathbb{Z}_{(p_1)^{f_1}} \times \cdots \times (\mathbb{Z}_{p_s})^{e_s} \times \mathbb{Z}_{(p_s)^{f_s}}$ with $e_i \geq 0, f_i \geq 1$. We will use R' in this proof.

($\Leftarrow$) If the reduction of G is representable by $(\mathbb{Z}_{p_1})^{e_1+1} \times \cdots \times (\mathbb{Z}_{p_s})^{e_1+1}$, it is possible to choose f_i sufficiently large so that in array representation we may fill the cells of the equivalence classes with the vertices in the equivalence classes.

($\Rightarrow$) If all the vertices with the same open neighborhood in G are already in the same cell in array representation, the equivalence classes of the cells serve as the basis for the reduction. Suppose there exist two vertices with the same open neighborhood but in different cells. We can relabel with the ring formed by raising each f_i by one, so that those vertices are in the same cell. Repeating this, for some values of h_i, G can be represented by $R' = (\mathbb{Z}_{p_1})^{e_1} \times \mathbb{Z}_{(p_1)^{h_1}} \times \cdots \times \mathbb{Z}_{p_s})^{e_s} \times \mathbb{Z}_{(p_s)^{h_s}}$ with $e_i \geq 0, h_i \geq 1$ so that all the vertices with the same open neighborhood are in the same cell. We can then reduce to $(\mathbb{Z}_{p_1})^{e_1+1} \times \cdots \times (\mathbb{Z}_{p_s})^{e_1+1}$. ■

We finish our study of $\mathbb{Z}_p \times \mathbb{Z}_p$ with:

Corollary 6.2. *If a graph* G *is representable by* $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^q}$ *for* $r, q > 0$ *if and only if the reduction of* G *is representable by* $\mathbb{Z}_p \times \mathbb{Z}_p$.

Proof.

This immediately follows from 6.1. ■

7 Representation Numbers of Families of Graphs

7.1 K_n

K_n is the family of complete graphs. From Evans et al, we get that $rep(K_n) = p$ where p where p is the first prime greater than or equal to n [1].

Proposition 7.1. *$rep(K_n) = rep_{\mathcal{D}}(K_n) = p$ where p is the smallest prime greater than or equal to n.*

Proof.

If $rep_{\mathcal{D}}(K_n) < rep(K_n)$ then there would be a ring, R, of the form $Z_{m_1} \times \cdots \times Z_{m_s}$ for some $s > 1$ that represents K_n with the order of the ring, $\prod_{i=1}^{s} m_i < p$, where p is the smallest prime greater than or equal to n. Without loss of generality we can assume one of the vertices is labeled all zeros. This means each Z_{m_i} must have at least $n-1$ units, since all vertices are adjacent, so $m_i \geq n$, so the order of R is at least n^2, but we are guaranteed a prime between n and n^2 for $n \geq 2$. ■

7.2 mK_1

mK_1 is the family of edgeless graphs. It has a representation number of $2m$ [1]. The complement of a graph, written $\bar{G}$, is the graph with the same vertices as G, with edges and non-edges switched. So $mK_1 = \bar{K_m}$.

Proposition 7.2. *$rep(mK_1) = rep_{\mathcal{D}}(mK_1) = 2m$ and all rings in $\mathcal{D}$ of order $2m$ represent mK_1.*

Proof.

For a ring R to represent mK_1, the graph G_R must contain m independent vertices, where a set of independent vertices is a set of vertices with no edge connecting any pair of vertices.

Claim: For a ring R, let p_1 be the smallest prime factor of the order of the ring n, then $\alpha(G_R) = n/p_1$

Two vertices are adjacent if and only if their coordinates differ mod p_i for each coordinate i. Looking at the set of vertices in G_R whose labels have first coordinates equivalent to $0(\mod p_1)$, all such vertices in this set are independent since they all have the same value mod p_1 in the first coordinate. There are $\prod_{j=1}^{s}\prod_{i=1}^{r_s} p_i^{l_{j_i}-1}$ vertices per cell and $p_1^{r_1-1}\cdot p_2^{r_2}\cdot\ldots\cdot p_s^{r_s}$ cells with first coordinate 0, so there are a total of n/p_1 independent vertices in this set.

For the lower bound, write $R \cong \mathbb{Z}_{p_1^{e_1}} \times R'$. We know that $\alpha(G_R) = \omega(\bar{G_R})$. We also know that $\omega(\bar{G_R}) \leq \chi(\bar{G_R})$, so we will show $\chi(\bar{G_R}) \leq n/p_1$. Note that in $\bar{G_R}$ the rules of adjacency are reversed, so that two vertices are adjacent if and only if their values are the same at any point. We can then construct an independent set of size p_1 by looking at the diagonal $\{(a_1p_1, a_2, ..., a_s), (a_1p_1+1, a_2+1, ..., a_s+1), ..., (a_1p_1+p_1-1, a_2+p_1-1, ..., a_s+p_1-1)\}$ for any of the $n/(p_1^{e_1})$ possible choices for $a_2, ..., a_s$ and the p^{e_1-1} choices for a_1. By assigning one color to the vertices in each such independent set, we obtain a proper coloring of $\bar{G_R}$ with n/p_1 colors, so we can conclude $\alpha(G_R) \leq n/p_1$, and $\alpha(G_R) = n/p_1$.

So a ring of order p_1n where $p_1|n$ and $n \geq m$ can represent mK_1. Since the smallest prime is 2, this means the smallest ring that can represent mK_1 is a ring of order $2m$; moreover, all rings in $\mathcal{D}$ of order $2m$ represent mK_1, since all will have 2 as their smallest prime factor, and so will contain an independent set of vertices of size m. ■

Although this result might seem to indicate a trend that all rings of order $2m$ will represent mK_1, this is not the case. Take for example the ring $\mathbb{F}_8 \times \mathbb{Z}_3$, where $\mathbb{F}_8$ is the field with eight elements; this does not represent $12K_1$ although it is of order 24.

7.3 Stars

A star, $K_{1,n}$ is a complete bipartite graph with one vertex adjacent to all the other vertices n, but no additional edges. This means it has an independence number of n, and a maximum degree of n. We showed in Proposition 7.2, $\alpha(G_R) = n/p_1$ where n is the order of the ring and p_1 is the smallest prime factor of n. So the independence number is based only on the order of a ring, not its algebraic structure. If $R \in \mathcal{D}$ and $p_1, ..., p_s$ are the prime divisors of R with respective dimensions $r_1, .., r_s$, then the *degree* of R is defined to be $\prod_{i=1}^{s}(p_i-1)^{r_i}/p_i^{r_i-1}$.

Therefore, as for complete graphs, $rep_{\mathcal{D}}(K_{1,m}) = rep(K_{1,m})$.

Proposition 7.3. *$rep_{\mathcal{D}}(K_{1,m}) = rep(K_{1,m})$ and if $|R| = rep(K_{1,m})$ and R represents $K_{1,m}$, then the dimension of the prime 2 is 1.*

Proof.

Assume a ring $R \in \mathcal{D}$ of order r with at least one prime p_i of dimension 2 represents $K_{1,m}$. That means $\alpha(G_R) = r/p_1$ and the degree of each vertex is at most $\phi(r)(p_i - 1)/p_i$. In the case $R = \mathbb{Z}_r$, $\alpha(G_{\mathbb{Z}_r}) = r/p_1$ and the degree of each vertex is $\phi(r)$, and $\phi(r) > \phi(r)(p_i - 1)/p_i$, so $\mathbb{Z}_r$ also represents $K_{1,m}$. Therefore, $rep_{\mathcal{D}}(K_{1,m}) = rep(K_{1,m})$.

Suppose $n = rep_{\mathcal{D}}(K_{1,r}) = rep(K_{1,r})$ and R represents $K_{1,r}$, $|R| = n$. Suppose further that the dimension of the prime 2 is greater than or equal to 2, so $4|n$. Then we know, there is an independent set of size r since $n/p_1 = n/2 \geq r$ and there exists a vertex is of degree r, so $r \leq \phi(n)(2-1)/2 \leq n/4$, so $r \leq n/4$. Looking at the ring $\mathbb{Z}_{n/2}$, it has an independent set of vertices of size $n/4 \geq r$. Moreover, each vertex has degree $\phi(n/2)$ and since $2|(n/2)$, $\phi(n/2) = 1/2\phi(n) \geq r$. So $\mathbb{Z}_{n/2}$ represents $K_{1,r}$ and $rep(K_{1,n}) \leq n/2$.

And $2|rep(K_{1,m})$ [5], so the dimension of the prime 2 is exactly 1. ■

8 Cycles and Paths

A cycle of length n, denoted C_n is a graph with n vertices and n edges so that starting from one vertex each edge can be traversed, returning to the original vertex. The graph P_n called a path is a cycle with one edge removed.

The representation numbers of cycles and paths are high due to the need for a large number distinct primes because of their relatively high product dimension. The representation number in $\mathcal{D}$ will benefit from not needing distinct primes. The representation number for C_n where $n = 2^k + 1$ is not generally known when k is even, so that case will be excluded.

Theorem 8.1. *For $m \geq 3$, if C_m can be represented by R a ring in $\mathcal{D}$, $R \times \mathbb{Z}_3$ represents C_{2m-2}.*

Proof.

Let $v_1, ..., v_n$ be the ordered s-tuples of the coordinate representation of R where s is the total dimension of R. And let v_i0, v_i1, v_i2 represent the ordered $(s+1)$-tuples of $R \times \mathbb{Z}_3$. Then a representation of C_{2m-2} is:

if m is odd: $v_12, v_20, v_31, ..., v_{m-1}0, v_m2, v_{m-1}1, v_{m-2}0, ..., v_30, v_21$

if m is even: $v_12, v_20, v_31, ..., v_{m-1}1, v_m2, v_{m-1}0, v_{m-2}1, ..., v_30, v_21$ ■

Corollary 8.2. *If $m = 2^s + 2$ for some $s \geq 2$ C_m can be represented by $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1}$*

Proof.

$\mathbb{Z}_2 \times \mathbb{Z}_3$ represents C_6 and $6 = 2^2 + 2$. And by induction on s if $k = 2^s + 2$ and C_k is represented by $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1}$, then Theorem 8.1 tell us $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1} \times \mathbb{Z}_3$ represents the cycle of size $2k - 2 = 2 * 2^s + 2 * 2 - 2 = 2^{s+1} + 2$. ■

Corollary 8.3. *For $m \leq 2^s + 1$ for $s \geq 2$, P_m can be represented by $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1}$.*

Proof.

The above corollary tells us for a given s, we can represent the cycle of order $2^s + 2$ by $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1}$. But a path of smaller length than a given cycle is just an induced subgraph of the cycle, so for $m < 2^s + 2$, we can represent the path of length m by $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1}$.■

Corollary 8.4. *For m even and $m \leq 2^s + 2$ for some $s \geq 2$, C_m can be represented by $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1}$*

Proof.

If $m = 2^s + 2$ for some s we have already shown that C_m is representable by $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1}$. Since m is even, we may assume $2^{s-1} + 4 \leq m \leq 2^s$ for some s. Substituting $m = 2n - 2$ for some n, and reducing we get $2^{s-2} + 3 \leq n \leq 2^{s-1} + 1$. We also know the path P_n is representable by $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-2}$, using some labels $v_1, ..., v_n$. And now we can construct a labeling for C_m as follows:

If n is odd: $v_12, v_20, ..., v_{n-1}1, v_n2, v_{n-1}0, ..., v_21$

If n is even: $v_12, v_20, ..., v_{n-1}0, v_n2, v_{n-1}1, ..., v_21$ ■

Theorem 8.5. *If n is even and $2^{s-1}+3 \leq n \leq 2^s+2$ for some s, then $rep_{\mathcal{D}}(C_n) = 2(3)^{s-1}$.*

Proof.

We have already shown that $\mathbb{Z}_2 \times (\mathbb{Z}_3)^{s-1}$ represents C_n. No ring of the form $\mathbb{Z}_2 \times \mathbb{Z}_2 \times R$ can represent C_n if $\mathbb{Z}_2 \times R$ does not represent C_n since $G_{\mathbb{Z}_2 \times \mathbb{Z}_2 \times R} = 2G_{\mathbb{Z}_2 \times R}$. So the representing ring can only have $p_1 = 2$ with a dimension of 1. Also, Lovasz et al proved that the product dimension of C_n is s [3]. So $p_2 = 3$ with a dimension of $s-1$ is the minimal representation. ■

Theorem 8.6. *If m odd, $2^{s-1}+1 < m \leq 2^s - 1$, and $s \geq 4$, then $rep_{\mathcal{D}}(C_m) = 3^s$.*

Proof.

Since the chromatic number of C_n for n odd is 3, the smallest prime that can divide the representation number is 3 [1]. Lovasz et al proved that the product dimension is s in this case too, so $rep_{\mathcal{D}}(C_m) = 3^s$ [3]. ■

9 Wheels and Lifting

We can *lift* a wheel out of a cycle by raising each prime factor of a ring that represents a cycle, adding one to each of the original labels in each coordinate, and then adding a vertex labeled (0,...,0). The original idea for lifting is from Evans et al [3].

Theorem 9.1. *Given a representation for C_n by a ring $R \in F$ with $|R| = rep(C_n)$ and $R = (\mathbb{Z}_{p_1})^{e_1} \times ... \times (\mathbb{Z}_{p_s})^{e_s}$, if q_i is the next prime larger than p_i, then $S = (\mathbb{Z}_{q_1})^{e_1} \times ... \times (\mathbb{Z}_{q_s})^{e_s}$ represents W_n.*

Proof.

Note, that since cycles of length longer than 4 are reduced graphs, we may replace R with R' as in Proposition 3.1. Given a representation in R for C_n, with labels $c_1, ..., c_n$, then C_n is also representable by S. Let $w_1, ..., w_n$ be a new cycle represented by S in which each coordinate value has been increased by 1. This is still a cycle. Moreover, no vertex label includes any 0s. Let $w_0 = (0, ..., 0)$ be the center of the wheel. It is therefore adjacent to every vertex in the cycle. ■

This also gives the representation number for W_n in $\mathcal{D}$ where the representation number for C_n is known.

Theorem 9.2. *For n even, $rep_{\mathcal{D}}(W_n) = 3 * 5^{s-1}$ where $2^{s-1} + 2 < n \leq 2^s + 2$. And for n odd, $rep_{\mathcal{D}}(W_n) = 5^k$ where $2^{k-1} + 1 < n < 2^k + 1$.*

Proof.

The cases $n = 3, 4$ do not fit the usual pattern for C_n. We know $rep_{\mathcal{D}}(C_3) = 3$ and $W_3 = K_4$, so its representation number is 5, the next prime greater than 3. Also, $rep_{\mathcal{D}}(C_4) = 4$, and $rep_{\mathcal{D}}(W_4) = 9$, although it is not represented by $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Since C_n is an induced subgraph of W_n, any representation of W_n must induce a representation of C_n. That means it must have a total dimension greater than or equal to the total dimension of C_n. If n is odd, the chromatic number of W_n is 4, so each prime factor must be at least 5. If n is even, the chromatic number is 3, so each prime must be at least 3. Therefore, since we know for n even, $rep_{\mathcal{D}}(C_n) = 2 * 3^{s-1}$, we can conclude $rep_{\mathcal{D}}(W_n) = 3 * 5^{s-1}$. And for n odd, $2^{k-1} + 1 < n < 2^k + 1$, $rep_{\mathcal{D}}(C_n) = 3^k$, so the minimal lift is $rep_{\mathcal{D}}(W_n) = 5^k$.■

10 Split Graphs and Disjoint Unions of Complete Graphs

We have shown above the independence number is a function of ring size, while small degree is a primary strength of rings in $\mathcal{D}$ but not $\mathcal{C}$. Split graphs are graphs of the form $K_m \cup tK_1$. We will also consider graphs of the more general disjoint unions of complete graphs $K_{m_1} \cup ... \cup K_{m_s}$. Some that are particularly tractable in $\mathcal{D}$ are mK_2 and $K_2 \cup tK_1$.

10.1 mK_2

Proposition 10.1. *$rep_{\mathcal{D}}(mK_2) = 2m$ when $m = 2^k$ for some k.*

Proof.

The graph of $(\mathbb{Z}_2)^k$ is mK_2 since the degree of each vertex is $(2-2/2)^k = 1$ and there are exactly $2^{k+1} = 2m$ vertices. ■

Theorem 10.2. $rep_{\mathcal{D}}(mK_2) = 2^{\lceil \log_2(2m) \rceil}$

Proof.

Let $k = \lceil \log_2(2m) \rceil$. From Lovasz et al, we get the product dimension of mK_2 is k [3]. The chromatic number of mK_2 is 2, so a lower bound on the representation number is 2^k. Finally $(\mathbb{Z}_2)^k$ represents mK_2 since it is an induced subgraph of $2^{k-1}K_2$. ■

10.2 $K_2 \cup tK_1$

This is a case in which the ordinary representation number is given by $min\{2^k n : n$ is odd, $t < 2^{k-1}(n - \phi(n))\}$ [5]. Our goal is to minimize the representation number. We find that k is independent of this optimization, so let $k = 1$. We also need to minimize the fraction of n lost to $\phi(n)$, in other words, maximize $(n - \phi(n))/n$ or just minimize $\phi(n)/n$, which is done by maximizing the number of prime factors of n. Note that even as $\phi(n)/n$ approaches 0, the formula approaches $t < 2^{k-1}n$,and the representation number approaches $2(t+1)$. In general, since $\alpha(K_2 \cup tK_1) = t+1$, the representation number is $> 2(t+1)$.

$K_2 \cup tK_1$ is an induced subgraph of mK_2. By example we can show $rep_{\mathcal{D}}(K_2 \cup tK_1) < rep(K_2 \cup tK_1)$ for small values of $t > 0$ considering representations by $(\mathbb{Z}_2)^{\lceil \log_2(2t+2) \rceil}$. Also note that the representation number for any t, is $\geq 2(t+1)$. Let $\mathcal{M}$ be the collection of rings of the form $(\mathbb{Z}_2)^s$ for $s \geq 1$. Note that $\mathcal{M}$ is contained in $\mathcal{D}$ so $rep_{\mathcal{M}}(G) \geq rep_{\mathcal{D}}(G)$ for all graphs G.

	$\mathcal{D}$	$\mathcal{C}$
t= 0	2	2
t= 1	4	6
t= 2,3	8	30
t=4,5,6	16	30

Table 2: Representation number of $K_2 \cup tK_1$ for $\mathcal{D}$ and $\mathcal{C}$ for $t \leq 6$

However, it is not the case that $rep_{\mathcal{M}}(K_2 \cup tK_1) \leq rep(K_2 \cup tK_1)$ for all t, or even $rep_{\mathcal{M}}(K_2 \cup tK_1) \geq rep(K_2 \cup tK_1)$ for $t > r$ for some r. We can demonstrate the former with $t = 9254$, which is represented by $\mathbb{Z}_{2*3*5*7*11*13}$ and $2*3*5*7*11*13 =$

$30,030 < 2^{\lceil \log_2(2t+2) \rceil} = 32,768$. This generates an infinite set of values for which $rep_{\mathcal{M}}(K_2 \cup tK_1) > rep(K_2 \cup tK_1)$ since $t = 9252 * 2^k$ is represented by $2^{k+1} * 3 * 5 * 7 * 11 * 14 < 2^{\lceil \log_2(2t+2) \rceil}$. And $rep_{\mathcal{M}}(K_2 \cup tK_1) < rep(K_2 \cup tK_1)$ when $t = 2^k - 1$, since the representation number will always be 2^{k+1}, exactly $2(t+1)$.

So the representation number for $K_2 \cup tK_1 \leq min\{2^k n, 2^{\lceil \log_2(2t+2) \rceil} : n \text{ is odd}, t < 2^{k-1}(n - \phi(n))\}$

10.3 Split Graphs and Total Dimension

We ask, does a lower representation number ever come from increasing the dimension of a prime compared to lifting a prime?

Let us look at $K_m \cup tK_1$, in particular $K_2 \cup tK_1$. For $t \leq (m-1)!$ the representation number is p^m where $p \geq m$ is prime. For $t > (m-1)!$ the product dimension is $m + 1$ [5]. For the case of $m = 2$, there are two choices on how to represent for $t > 1$, maintaining the total dimension as 3. These are $(\mathbb{Z}_2)^2 \times \mathbb{Z}_q$ for some prime q or $\mathbb{Z}_2 \times \mathbb{Z}_q \times \mathbb{Z}_p$ for primes p, q. Alternatively we can increase the total dimension. The simplest, (but not the only way) to do it is to add additional copies of $\mathbb{Z}_2$. There may be more effective ways to do this by adding $\mathbb{Z}_p$s with $p \neq 2$. So our third case is $(\mathbb{Z}_2)^k$. We find the first can represent up to $q + 1$, the second $p + q + 1$ and the last up to $2^{k-1} - 1$. This second case is not going to be optimal (since $4q/(q+1) < 4pq/(p+q+1)$) so we can ignore it.

We find that generally increasing the total dimension gives the representation number, except in cases where $q = 2^k - 1$ is prime. In that case, the representation of $K_2 \cup 2^k K_1$ will be $(\mathbb{Z}_2)^2 \times \mathbb{Z}_q$ rather than $(\mathbb{Z}_2)^k + 2$. Generally if $2^{k-1} + 1 \leq p < 2^k - 1$, the graph $K_2 \cup (p+1)K_1$ is represented by $(\mathbb{Z}_2)^{k+1}$ since $2^{k+1} < 2^{k+1} + 4 \leq 4p$.

11 Multipartite Graphs

We conjecture that for any complete multipartite graph G, $rep(G) = rep_{\mathcal{D}}(G)$. We have already seen that this is true when G is of the form K_n, mK_1, or $K_{1,n}$.

However, one thing of interest is that there are multiple ways to represent (with rings of equal order) $2K_{4,4}$ and other graphs of the form $2^s K_{2^r,2^r}$, for $s \geq 1, r \geq 2$ by 3.1.

In this case, we have the rings $(\mathbb{Z}_2)^s \times \mathbb{Z}_{2^r}$ and $(\mathbb{Z}_2)^{s-1} \times \mathbb{Z}_4 \times \mathbb{Z}_{2^{r-1}}$.

12 Acknowledgments

We would like to express our deepest gratitude to Dr. Reza Akhtar for his instruction and guidance throughout our research. We also thank for her advice and help Amanda Phillips, our graduate assistant from Purdue University. Many thanks to Dr. Davenport and Dr. Waikar, directors of SUMSRI, and all others who made this experience in SUMSRI possible for us. We would also like to thank the National Security Agency, National Science Foundation, and Miami University for their support of this program.

References

[0] P. Erdos, A.B. Evans. *Representations of graphs and orthogonal latin square graphs*, J. Graph Theory 13 (1989), 593-595.

[1] A.B. Evans, G.H. Fricke, C.C. Maneri, T.A. McKee, and M. Perkel. *Representations of Graphs Modulo n*, Discrete Math **233** (2000), 109-123.

[2] A.B. Evans, D. Pritikin. *Some Thoughts on Representation Numbers*, preprint.

[3] L. Lovasz, J. Nesetril, A. Pultr. *On a product dimension of graphs*, J. Combin. Theory B 29 (1980), 47-67.

[4] D.A. Narayan. *An Upper Bound for the Representation Number of Graphs with Fixed Order*, Integers **3** (2003) #A12.

[5] D.A. Narayan, J. Urick. *Representation of Split Graphs, their Complements, Stars, and Hypercubes*, Integers **7** (2007) #A07.

Megan Bernstein
University of California, Berkeley
Berkeley, California
mbernstein@berkeley.edu