Wallpaper: The Mathematics of Art

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July 2003

In this paper we rediscover the 17 wallpaper groups first classified by Fedorov and Schönflies. We explore all lattices and determine their possible point groups. The point groups then enable us to classify each distinct wallpaper group.

1. Introduction

Deep in the valleys of Granada, Spain is the majestic Alhambra. This Islamic palace was built after the Muslims conquered Spain in the 8th century [3]. The palace's interior design exhibits the Islamic necessity to use geometric shapes [2]. This mathematical labyrinth, as it has been referred to many people, also contains many mysteries and mathematical marvels. One features of the Alhambra that is both mathematically splendid and elusive is that its walls were decorated using all seventeen possible *wallpaper patterns* [4]. A wa*llpaper pattern* is a repeating pattern with two independent translations that fills the plane [1]. The formal definition of a wallpaper group is given is Section 2. A mystery lies in whether the architects had a mathematical knowledge of all the different wallpaper patterns or simply exhausted all the possibilities [4].

Interest in mathematical classification of patterns goes back many years. In the 19th century E.S. Fedorov and Schönflies started working on a classification of planar patterns. In 1891, they classified the 17 wallpaper groups [7]. Sources are undecided on whether Fedorov and A.M Schönflies independently came up with the same results, and should be equally credited with the discovery, or Fedorov made the classification himself, and Schönflies corrected minor errors in Fedorov's work [6, 9]. In either case, the classification of the 17 wallpaper groups was made popular by George Pólya in 1924 when he wrote a paper relating wallpaper patterns and crystal structure [7].

Among those who were intrigued by Pólya's paper was the acclaimed artist M.C. Escher. Escher became almost obsessed with the regular division of the plane when he visited the Alhambra in 1922 and saw the wallpaper patterns inside. He composed 137 regular division drawings in his lifetime. He strongly considered the Alhambra to be his greatest inspiration [5]. After his brother Berend noticed the connection between some of Escher's art and crystallography (the science of crystal structure), Berend showed Escher Pólya's paper, and Escher began studying the mathematics of art [8]. Even though Escher did not have an in depth understanding of some of the mathematics he read about, he was able to understand the key mathematical concepts involved with the 17 wallpaper groups. Although he spent countless hours sketching and appreciating wallpaper patterns in the Alhambra, he felt that his knowledge of certain mathematical concepts could help him create works of art better than those in the Alhambra [8].

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The architects of the Alhambra may not have provided a mathematical proof of the possible wallpaper groups, but in this paper we will do so. We will mathematically prove that there are only seventeen different wallpaper groups.

2. Preliminaries

In this section, we provide the mathematical definitions and notation to be used throughout the paper. Since wallpaper is a two-dimensional object we will be dealing with plane symmetries.

Definition 2.1 A *wallpaper* is a two dimensional repeating pattern with two independent translations that fill the plane [1].

Definition 2.2 A *plane symmetry* is an exact correspondence in position or form about a given line or point [4].

When dealing with two dimensional objects there are four types of symmetries: rotations, reflections, translations, and glide reflections. These symmetries are described below.

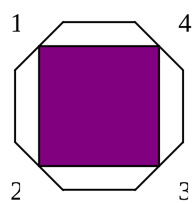


Figure 2.1a

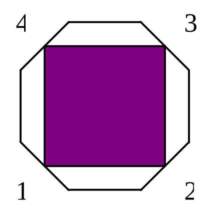


Figure 2.1b

Definition 2.3 A planar object has *rotational* symmetry about a point p if it is carried into itself by at least one nontrivial rotations around p [4].

In Figure 2.1a fix the origin to be at the center of the octagon and rotate the octagon counterclockwise 90^{o} about the origin. This gives you the octagon shown in Figure 2.1b.

Definition 2.4 An object is symmetric with respect to a line l if it is carried into itself by *reflection* in l. The axis of reflection, l, is the *mirror* of reflection [4].

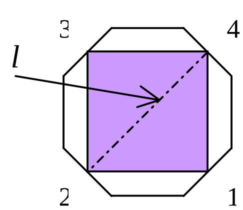


Figure 2.2

If we reflect the octagon in Figure 2.1a about the line l we get the octagon in Figure 2.2.

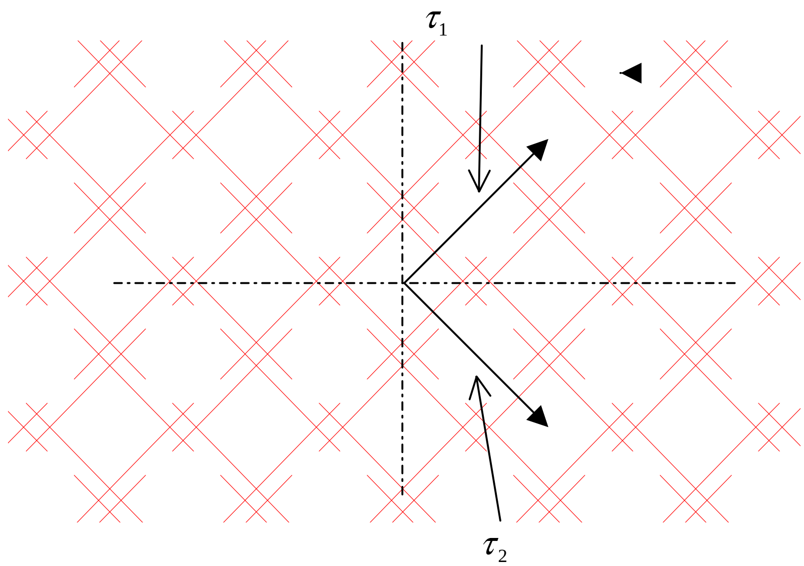


Figure 2.3

Definition 2.6 A wallpaper has *translational symmetry* if each point in the wallpaper can be moved along a fixed vector **v** to obtain the same wallpaper.

In Figure 2.3 translation by τ_2 is a symmetry of the wallpaper.

Definition 2.7 A wallpaper has a nontrivial *glide reflection* if each point in the wallpaper can be translated by a half-unit and then reflected to obtain the same wallpaper.

Definition 2.8 A function $f : R^n \to R^n$ is an *isometry* if it preserves distance.

Example 2.9
The function $f : R \to R$ defined by $f(x) = -x$ is an isometry because $|f(x) - f(y)| = |-x + y| = |x - y|$.

Before we define wallpaper groups it is necessary to introduce the Euclidean group. The *Euclidean group E_2* is the group of isometries of the plane under composition of functions.

A *direct isometry* is a rotation about the origin composed with a translation. An *opposite isometry* is a reflection about a line that passes through the origin composed with a translation. An element of E_2 is either a direct isometry or an opposite isometry refer to [1].

Theorem 2.10 *Every direct isometry is either a translation or a rotation. Every opposite isometry is either a reflection or a glide reflection*

For the proof refer to Theorem 24.1 of [1].

An isometry g can be written as an ordered pair, g = (**v**, M). This notation is shorthand for $g(x,y) = \mathbf{v} + f_M(x,y) = \mathbf{v} + M(x,y)^t$ where M is a 2×2 orthogonal matrix that represents g in the standard basis for R^2, and the vector **v** is the image of the origin under the action of a translation. When dealing with isometries we will use this ordered pair notation. Let $(\mathbf{v_1}, M_1)(\mathbf{v_2}, M_2) \in E_2$, then we define the composition of the two as follows:

$$(\mathbf{v_1}, M_1)(\mathbf{v_2}, M_2) = (\mathbf{v_1} + f_{M_1}(\mathbf{v_2}), M_1 M_2)$$

If M is the identity I, then g is a translation. We define A_θ to be the orthogonal matrix that represents a clockwise rotation of θ^o about the origin. Also, define B_θ to be the orthogonal matrix that represents a reflection whose mirror makes an angle of $\theta/2$ with the positive horizontal axis. In other words,

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } B_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

If $g = (\mathbf{v_1}, M_1)$ is a pure reflection with mirror m then $\mathbf{v_1}$ is perpendicular to m. On the other hand, if $\mathbf{v_1}$ is not perpendicular to m then g is a glide reflection.

Since E_2 is closed, the composition of isometries gives either direct or opposite isometries.

Theorem 2.11 *A reflection with mirror m followed by reflection with mirror* m' *is a translation when m is parallel to* m' *and a rotation otherwise.*

Proof:
Let f and g be reflections with axes of reflection m and m', respectively. Also, let $\theta/2$ be the angle of reflection for f and let $\phi/2$ be the angle of reflection for g. So, $f = (\mathbf{v}, B_\theta)$ and $g = (\mathbf{w}, B_\phi)$. Then,

$$fg = (\mathbf{v}, B_\theta)(\mathbf{w}, B_\phi) = (\mathbf{v} + f_{B_\theta}(\mathbf{w}), B_\theta B_\phi)$$

Suppose m and m' are parallel, then $\theta = \phi$ and let $\mathbf{u} = \mathbf{v} + f_{B_\theta}(\mathbf{w})$. Then,

$$B_\theta B_\phi = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

We find that, $fg = (\mathbf{u}, I)$, which is a translation.

Now, suppose m and m' are not parallel, then

$$B_\theta B_\phi = \begin{bmatrix} \cos(\theta-\phi) & -\sin(\theta-\phi) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix} = A_{\theta-\phi}$$

Therefore, $fg = (\mathbf{u}, A_{\theta-\phi})$, which is a rotation of $(\theta-\phi)$.

QED

Composition of isometries is not always a commutative operation; therefore, we examine some of the cases when it is.

Theorem 2.12 *Two reflections commute if and only if their mirrors either coincide or are perpendicular to one another.*

Proof:
Assume that two reflections f_1 and f_2 commute. Let $f_1 = (\mathbf{v_1}, B_\theta)$ and $f_2 = (\mathbf{v_2}, B_\phi)$ where B_θ and B_ϕ are reflection matrices, and assume f_1 and f_2 have angles of reflection ϕ and θ, respectively. Using the assumption that f_1 and f_2 commute gives

$$(\mathbf{v_1}, B_\theta)(\mathbf{v_2}, B_\phi) = (\mathbf{v_2}, B_\phi)(\mathbf{v_0}, B_\theta)$$

and, hence,

$$(\mathbf{v_1} + f_{B_\varpi}(\mathbf{v_2}), B_\theta B_\phi) = (\mathbf{v_2} + f_{B_\phi}(\mathbf{v_1}), B_\phi B_\theta)$$

We reach a conclusion by examining the matrices. Since

$$B_\theta B_\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} = B_\phi B_\phi$$

Matrix multiplication gives

$$\begin{bmatrix} \cos\phi\cos\theta + \sin\phi\sin\theta & \cos\phi\sin\theta - \sin\phi\cos\theta \\ \cos\phi\sin\theta - \sin\phi\cos\theta & \sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & \cos\theta\sin\phi - \sin\theta\cos\phi \\ \cos\theta\sin\phi - \sin\theta\cos\phi & \sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}.$$

Using matrix equality to examine the entries in the first row and second column yields

$$\cos\phi\sin\theta - \sin\phi\cos\theta = \cos\theta\sin\phi - \sin\theta\cos\phi.$$

After using a trigonometric identity this reduces to

$$\sin(\theta-\phi) = \sin(\phi-\theta).$$

If we let $\theta = c + \phi$, where c is a constant, we have

$$\sin(c + \phi - \phi) = \sin(\phi - \phi - c)$$

This reduces to $\sin(c) = \sin(-c)$. Which is true if and only if c is equal to 0° or 180°. When $c = 0^{o}$ then we have $\theta = 0 + \phi = \phi$. This implies that the mirrors of f_1 and f_2 coincide. If $c = 180^{0}$, then $\theta = 180^{o} + \phi$. This implies that $\theta/2 = 90^{o} + \phi/2$ and the mirrors of f_1 and f_2 are perpendicular. Therefore, if two reflections commute then their mirrors either coincide or are perpendicular to each other.

To prove the converse, we assume that the mirrors of f_1 and f_2 either coincide or are perpendicular, we show that the reflections commute in either case. The case when the mirrors coincide is trivial. Now, if the mirrors are perpendicular there is more to show.

If the mirrors of f_1 and f_2 are perpendicular then $\theta = 180 + \phi$. Multiplying matrices, we have:

$$B_\theta \ B_\phi = \begin{bmatrix} \cos\phi\cos\theta + \sin\phi\sin\theta & \cos\phi\sin\theta - \sin\phi\cos\theta \\ \cos\phi\sin\theta - \sin\phi\cos\theta & \sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\phi-\theta) & -\sin(\phi-\theta) \\ \sin(\phi-\theta) & \cos(\phi-\theta) \end{bmatrix}$$

Substituting $\theta = 180^{o} + \phi$ gives

$$\begin{bmatrix} \cos(-180) & -\sin(-180) \\ \sin(-180) & \cos(-180) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(180) & -\sin(180) \\ \sin(180) & \cos(180) \end{bmatrix}.$$

$$= \begin{bmatrix} \cos(\theta-\phi) & -\sin(\theta-\phi) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix}.$$

Using a trigonometric identity and matrix multiplication we have

$$\begin{bmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & \cos\theta\sin\phi - \sin\theta\cos\phi \\ \cos\theta\sin\phi - \sin\theta\cos\phi & \sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}$$

$$= B_\phi \ B_\theta .$$

We now show that the actions on the vectors commute. Since f_1 and f_2 are perpendicular, $\mathbf{v_1}$ lies on the mirror that passes through the origin and is parallel to the mirror of f_2. Also, $\mathbf{v_2}$ lies on the mirror that passes through the origin and is parallel to the mirror of f_1. This implies that $f_{M_1}(\mathbf{v_2}) = v_2$ and $f_{M_2}(\mathbf{v_1}) = \mathbf{v_1}$. Also, since vector addition is commutative $\mathbf{v_1} + \mathbf{v_2} = \mathbf{v_2} + \mathbf{v}_1$.

By substitution we get that $\mathbf{v_1} + M_1\mathbf{v_2^t} = \mathbf{v_2} + M_2\mathbf{v_1^t}$ and so the actions on the vectors commute. Therefore, since the vectors and the matrices commute, $f_1 f_2 = f_2 f_1$.
Thus, we conclude that if the mirrors of f_1 and f_2 either coincide or are perpendicular then the reflections commute.

QED

Let G be a subgroup of E_2 and let O_2 be the set of 2×2 orthogonal matrices. Define $\pi : E_2 \to O_2$ by $\pi(\mathbf{v}, M) = M$.

Definition 2.13 The intersection of G and the group of all translations, $T \subseteq E_2$, is called the *translation subgroup of G*. We denote the translation subgroup by *H*.

Definition 2.14 The *point group* of *G* is $\pi(G)$; we denote the point group by *J*.

The point group of *G* is the set of all orthogonal matrices corresponding to the isometries in *G*.

We are now able to provide the formal definition of a *wallpaper group*.

Definition 2.15 A subgroup $G \subseteq E_2$ is a *wallpaper group* if its translation subgroup *H* is generated by two independent vectors and its point group *J* is finite.

The translation subgroup of the wallpaper in Figure 2.3 is generated by τ_1 and τ_2, and its point group is $\{I, -I, B_0, B_\pi\}$. To show that two wallpaper groups are distinct we must show that they differ algebraically.

Theorem 2.16 *An isomorphism between wallpaper groups takes translations to translations, rotations to rotations, reflections to reflections, and glide reflections to glide reflections.*

For a proof of this refer to Theorem 25.5 of [3].

Theorem 2.17 *If the wallpaper groups G and G' are isomorphic, then the point groups $\pi(G)$ and $\pi(G')$ are isomorphic.*

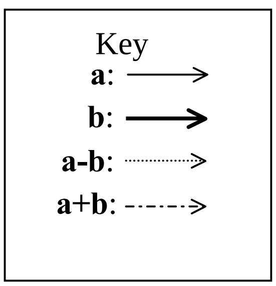
Proof:
Let G and G' be isomorphic wallpaper groups. Then, by Theorem 2.16, there is a bijection between G and G' that takes reflections to reflections, translations to translations, and rotations to rotations. This implies that for every reflection, translation, rotation, or glide reflection in *G* there is a corresponding reflection, translation, rotation, or glide reflection in G'. The reverse is true for G'. If the wallpaper group *G* has a given isometry, then the point group $\pi(G)$ has a 2×2 orthogonal matrix representation of that given isometry. Therefore, for every isometry represented by a matrix in $\pi(G)$, there is a corresponding isometry represented by a matrix in $\pi(G')$. This, also, is conversely true for G'. This implies that there is a bijection between $\pi(G)$ and $\pi(G')$. By Theorem 2.16 the operation is preserved, therefore, $\pi(G)$ *and* $\pi(G')$ are isomorphic.

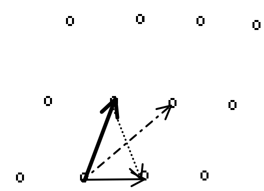
QED

3. Lattices

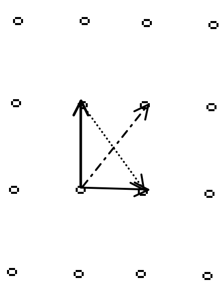
Definition 3.1 A *wallpaper lattice, L*, is the set of all the points that the origin gets mapped to under the action of *H, the translation subgroup of a wallpaper pattern.*

Recall that H is the translation subgroup and J is the point group of a wallpaper group *G*. We associate a *lattice* of points to every wallpaper pattern. Lattices are important to us because to understand the wallpaper groups it is necessary to understand the lattice associated with it. We see the lattices as the basic wallpapers. Hence, finding the point groups of the lattices will enable us to classify the wallpaper groups. If we select nonparallel, nonzero vectors **a** and **b** in *L*, where **a** is of minimum length and **b** is also as small as possible, then *L* is spanned by **a** and **b**. For a proof of this refer to Theorem 25.1 of []. In other words, all elements of *L* are of the form $m\mathbf{a} + n\mathbf{b}$, where *n* and *m* are integers. By examining all possible basic parallelograms determined by the vectors **a** and **b** we can conclude that there are only five different types of lattices. We categorize each lattice by the following inequalities:

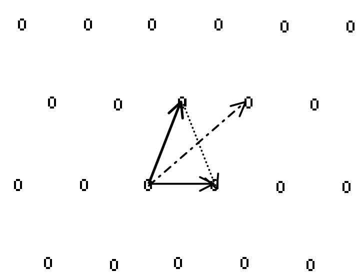




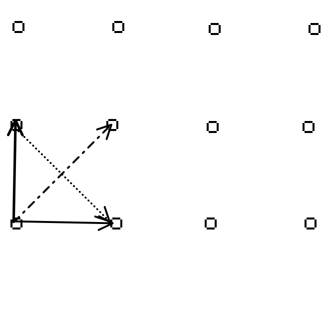
Oblique

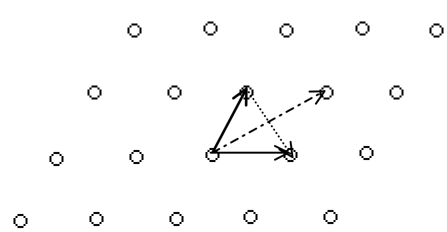


Rectangular



Centered Rectangular





Square **Hexagonal**

Oblique: $\|\mathbf{a}\| < \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$
Rectangular: $\|\mathbf{a}\| < \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} + \mathbf{b}\|$
Centered rectangular: $\|\mathbf{a}\| < \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$
Square: $\|\mathbf{a}\| = \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} + \mathbf{b}\|$
Hexagonal: $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$

It might be the case that a lattice spanned by the vectors **a** and **b** appears not to fall into any of the five categories. However, by manipulating the inequalities we can see that the lattice does actually fall into one of the categories. The next example illustrates this.

Example 3.2 Consider the lattice spanned by the vectors: $\mathbf{a} = \langle 1, -1 \rangle$, and $\mathbf{b} = \langle 3, -4 \rangle$.

If we calculate the magnitude of **a** and **b** we get, $\|\mathbf{a}\| = \sqrt{2}$ and $\|\mathbf{b}\| = 5$.
Now, $\mathbf{a} - \mathbf{b} = \langle 2, 3 \rangle$ and $\mathbf{a} + \mathbf{b} = \langle 4, -5 \rangle$. Thus, $\|\mathbf{a} - \mathbf{b}\| = \sqrt{13}$ and $\|\mathbf{a} + \mathbf{b}\| = \sqrt{41}$.
Hence, $\|\mathbf{a}\| < \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$.

Initially, this result might make us believe that the lattice spanned by these two vectors is not included in the five categories. However, by replacing **a – b** with **b** we can see that the lattice is actually oblique. This does not change the lattice since the vectors that span the lattice must of minimum length.

4. Point Groups

Recall from Section 2 that the point group J of a group G is the set of 2×2 orthogonal matrices that represent the isometries that preserve the lattice of the corresponding pattern of G. A thorough examination of all five lattices will provide all the possible point groups established by these lattices. Each lattice has either a dihedral group or a subgroup of a *dihedral group* associated to it. The dihedral group D_n of order $2n$ is the set of all symmetries of a regular n-gon. The subgroups of D_n are the identity, Z_n, and, sometimes when n is even, the Klein-4 group K_4. The group Z_n is isomorphic to the set of all rotations of the regular n-gon. The Klein 4 group contains two perpendicular reflections, and the 180^o rotation. When determining all of the possible point groups that are produced by each lattice we will consider the groups that are isomorphic to D_n or any of its subgroups.

When considering whether a possible point group is isomorphic to either of these groups we recall that the orders of the point groups have to be equal and that the conditions of Theorem 2.16 must be satisfied.

Theorem 4.1 *If G is a wallpaper group corresponding to a pattern with an oblique lattice, then the point group of G is* $\{I, -I\}$ *or* $\{I\}$.

Proof:
Let G be a wallpaper group corresponding to a pattern with an oblique lattice. Figure 4.1 illustrates an oblique lattice unit. We see that the only possible nontrivial rotations in G are of order two.

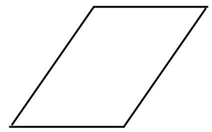


Figure 4.1

That is, if we rotate it 180^o we have the same figure. However, there are no possible reflections. Hence, the point group of G must be a subgroup of $\{I,-I\} \cong Z_2$. If we eliminate 180^o rotation, then $\{I\}$ is the point group of G. Since the only subgroup of Z_2 is $\{I\}$, we have established all possible point groups of G.

QED

Theorem 4.2 *If G is a wallpaper group corresponding to a pattern with a rectangular lattice, then the possible point groups of G are:* $\{I\}$, $\{I,-I\}$, $\{I,B_0\}$, $\{I,B_\pi\}$, or $\{I,-I,B_0,B_\pi\}$.

Proof:
Let G be a wallpaper group corresponding to a pattern with a rectangular lattice and having point group J. Figure 4.2 illustrates a lattice unit.

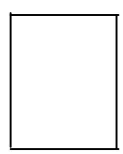


Figure 4.2

From Figure 4.2 we see that J can have 180^o rotation, vertical reflections, and horizontal reflections. Thus, J must be a subgroup of $\{I,-I,B_0,B_\pi\} \cong K_4$. The subgroups of K_4 are K_4, Z_2, and $\{I\}$. If J is isomorphic K_4, then $J = \{I,-I,B_0,B_\pi\}$. If J is isomorphic to Z_2 then we have three possibilities: $\{I,B_0\}$, $\{I,B_\pi\}$ and $\{I,-I\}$. Since $B_0B_0 = I$, $B_\pi B_\pi = I$, $-I(-I) = I$ these sets are closed and are groups. The only other possible point group left is $\{I\}$ we conclude that J must equal $\{I\}$, $\{I,-I\}$, $\{I,B_0\}$, $\{I,B_\pi\}$, or $\{I,-I,B_0,B_\pi\}$.

QED

Theorem 4.3 *If G is a wallpaper group corresponding to a pattern with a centered rectangular lattice, then the point groups of G is one of:* $\{I\}$, $\{I,B_\pi\}$, $\{I,B_0\}$, $\{I,-I\}$, *or* $\{I,-I,B_0,B_\pi\}$.

Proof:

Let G be a wallpaper group corresponding to a pattern with a centered rectangular lattice and with point group J. Figure 4.2 illustrates a lattice unit.

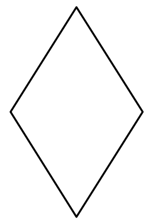


Figure 4.2

From the lattice unit we see that J may have rotations of order two, vertical reflections, and horizontal reflections. Therefore, as in the Theorem 4.2, J must be a subgroup of $\{I, -I, B_0, B_\pi\} \cong K_4$. Thus, the possible point groups are the same in Theorem 4.2.

QED

Theorem 4.4 *If G is a wallpaper group corresponding pattern with a square lattice then its point is one of:* $\{I\}$, $\{I, -I\}$, $\{I, B_0\}$, $\{I, B_\pi\}$, $\{I, B_{\pi/2}\}$, $\{I, B_{3\pi/2}\}$, $\{I, -I, B_0, B_\pi\}$, $\{I, -I, B_{\pi/2}, B_{3\pi/2}\}$, $\{I, -I, A_{\pi/2}, A_{3\pi/2}\}$, *or* $\{I, -I, A_{\pi/2}, A_{3\pi/2}, B_0, B_{\pi/2}, B_\pi, B_{3\pi/2}\}$.

Proof:
Recall that the subgroups of D_4 are isomorphic to D_4, K_4, Z_4, Z_2, or $< e >$. Let G be a wallpaper group with a square lattice and with point group J. Figure 4.4 illustrates a lattice unit of the square lattice. We see that J may have rotations of order four, vertical and horizontal reflections, and two diagonal reflections.

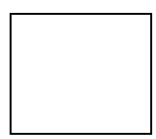


Figure 4.4

Thus, J must be a subgroup of
$J_1 = \{I, -I, A_{\pi/2}, A_{3\pi/2}, B_0, B_\pi, B_{\pi/2}, B_{3\pi/2}\} \cong D_4$. The only point group isomorphic to D_4 is J_1. If the point group of G is isomorphic to K_4 then it contains orthogonal matrices that represent perpendicular reflections, the 180^o rotation and the identity. Thus, a point group containing two reflections such as B_0 and $B_{\pi/2}$, whose mirrors are not perpendicular, will not be isomorphic to K_4. Thus, the only possible groups are $\{I, -I, B_0, B_\pi\}$ and $\{I, -I, B_{\pi/2}, B_{3\pi/2}\}$. We have to check that these sets are groups. Let us start with $\{I, -I, B_0, B_\pi\}$. It contains the identity and each element is the inverse of itself. Also, the set is closed because $B_\pi B_\pi = I$, and $B_0 B_\pi = -I$. Finally, matrix multiplication is always associative; therefore, it is a group.

Next, we look at $\{I,-I,B_{\pi/2},B_{3\pi/2}\}$. It also contains the identity and each element is an inverse of itself. Again, the set is closed because $B_{3\pi/2}B_{3\pi/2}=I$, $B_{\pi/2}B_{\pi/2}=I$, $B_{\pi/2}B_{3\pi/2}=-I$. Thus, $\{I,-I,B_{\pi/2},B_{3\pi/2}\}$ is a group.

Next, we will examine the possible point groups that are isomorphic to Z_4. In this case the point group must have rotations of order four and no reflections. This is possible only when the point group is $\{I,-I,A_{\pi/2},A_{3\pi/2}\}$.

Now, we will examine the possible point groups of G that are isomorphic to Z_2. These point groups have order two and include the identity. Therefore, the only possible point groups are:

$\{I,-I\}$, $\{I,B_0\}$, $\{I,B_\pi\}$, $\{I,B_{\pi/2},\}$, and $\{I,B_{3\pi/2}\}$.

Given the trivial $\{I\}$ we conclude that the only possible point groups of G are: $\{I\}$, $\{I,-I\}$, $\{I,B_0\}$, $\{I,B_\pi\}$, $\{I,B_{\pi/2},\}$, $\{I,B_{3\pi/2}\}$, $\{I,-I,B_0,B_\pi\}$ $\{I,-I,B_{\pi/2},B_{3\pi/2}\}$ $\{I,-I,A_{\pi/2},A_{3\pi/2}\}$, *or* $\{I,-I,A_{\pi/2},A_{3\pi/2},B_0,B_{\pi/2},B_\pi,B_{3\pi/2}\}$.

QED

Theorem 4.5 *If G is a wallpaper group corresponding to a patter with a hexagonal lattice, then its point group must be one of the following:* $\{I\}$, $\{I,-I\}$, $\{I,B_{k\pi/2}\}$, *for* $0\le k\le 5$, $\{I,-I,B_0,B_\pi\}$, $\{I,-I,B_{\pi/3},B_{4\pi/3}\}$, $\{I,-I,B_{2\pi/3},B_{5\pi/3}\}$, $\{I,A_{2\pi/3},A_{4\pi/3}\}$, $\{I,A_{2\pi/3},A_{4\pi/3},B_{\pi/3},B_\pi,B_{5\pi/3}\}$, $\{I,A_{2\pi/3},A_{4\pi/3},B_0,B_{2\pi/3},B_{4\pi/3}\}$, $\{I,-I,A_{\pi/3},A_{2\pi/3},A_{4\pi/3},A_{5\pi/3}\}$, *or* $\{I,-I,A_{\pi/3},A_{2\pi/3},A_{4\pi/3},A_{5\pi/3},B_0,B_{\pi/3},B_\pi,B_{2\pi/3},B_{4\pi/3},B_{5\pi/3}\}$

Proof:
The subgroups of D_6 are isomorphic to Z_6, K_4, D_3, Z_3, Z_2, and $\{I\}$. Let G be a wallpaper group corresponding to a pattern with a hexagonal lattice and point group J. Figure 4.5 illustrates a lattice unit of the hexagonal lattice.

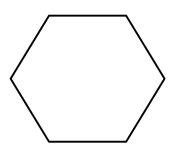


Figure 4.5

We see that J may have rotations of order six, vertical and horizontal reflections, and reflections whose mirrors make 30^o, 60^o, 120^o, and 150^o angles with the positive horizontal axis. Hence, J must be a subgroup of
$J_1=\{I,-I,A_{\pi/3},A_{2\pi/3},A_{4\pi/3},A_{5\pi/3},B_0,B_{\pi/3},B_\pi,B_{2\pi/3},B_{4\pi/3},B_{5\pi/3}\}\cong D_6$. We first take a look at all possible point groups isomorphic to Z_6. In this case J must contain orthogonal matrices that represent rotations of order six. That is,

$J = \{I, -I, A_{\pi/3}, A_{2\pi/3}, A_{4\pi/3}, A_{5\pi/3}\}$. This is closed since products of rotations are rotations. Therefore, *J* is a possible point group.

Now, if *J* is isomorphic to K_4, then $\{I, -I, B_0, B_\pi\}$, $\{I, -I, B_{\pi/3}, B_{4\pi/3}\}$, and $\{I, -I, B_{2\pi/3}, B_{5\pi/3}\}$ are the only possible point groups (refer to proof of Theorem 4.4). To verify that these sets are groups we must check for closure. Recall that two reflections commute if their mirrors are perpendicular; see Theorem 2.12.

$\{I, -I, B_0, B_\pi\}$*:* In Theorem 4.4 we proved that $\{I, -I, B_0, B_\pi\}$ is a group.
$\{I, -I, B_{\pi/3}, B_{4\pi/3}\}$*:* $B_{\pi/3}\ B_{\pi/3} = I$, $B_{4\pi/3}\ B_{4\pi/3} = I$, $B_{\pi/3}\ B_{4\pi/3} = -I = B_{4\pi/3}\ B_{\pi/3}$
$\{I, -I, B_{2\pi/3}, B_{5\pi/3}\}$*:* $B_{2\pi/3}\ B_{2\pi/3} = I$, $B_{5\pi/3}\ B_{5\pi/3} = I$, $B_{2\pi/3}\ B_{5\pi/3} = -I = B_{5\pi/3}\ B_{2\pi/3}$

Since all three sets are closed they are groups and are possible point groups of *G*.

If *J* is isomorphic to D_3, then *J* can have matrices that represent 120° rotation, vertical reflections, and reflections of 30° and 150°. This is the case when
$J = \{I, A_{2\pi/3}, A_{2\pi/3}, B_{\pi/3}, B_\pi, B_{5\pi/3}\}$. The group *G* can also have rotations of 120°, horizontal reflections, and reflections of 60° and 120°. That is
$J = \{I, A_{2\pi/3}, A_{4\pi/3}, B_0, B_{2\pi/3}, B_{4\pi/3}\}$,

If *J* is isomorphic to Z_3 then it must have rotations of order three. This is the case when
J $= \{I, A_{2\pi/3}, A_{4\pi/3}\}$. If *J* is isomorphic to Z_2 then it must be one of the following $\{I, B_{k\pi/2}\}$ for $0 \le k \le 5$. These sets are groups since a reflection composed with itself is the identity.

Again, given the group $\{I\}$ we conclude that the only possible point group of *G* must be $\{I\}$, $\{I, -I\}$, $\{I, B_{k\pi/2}\}$ *for* $0 \le k \le 5$, $\{I, -I, B_0, B_\pi\}$, $\{I, -I, B_{\pi/3}, B_{4\pi/3}\}$, $\{I, -I, B_{2\pi/3}, B_{5\pi/3}\}$, $\{I, A_{2\pi/3}, A_{4\pi/3}\}$, $\{I, A_{2\pi/3}, A_{2\pi/3}, B_{\pi/3}, B_\pi, B_{5\pi/3}\}$,
$\{I, A_{2\pi/3}, A_{4\pi/3}, B_0, B_{2\pi/3}, B_{4\pi/3}\}$, $\{I, -I, A_{\pi/3}, A_{2\pi/3}, A_{4\pi/3}, A_{5\pi/3}\}$, or
$\{I, -I, A_{\pi/3}, A_{2\pi/3}, A_{4\pi/3}, A_{5\pi/3}, B_0, B_{\pi/3}, B_\pi, B_{2\pi/3}, B_{4\pi/3}, B_{5\pi/3}\}$.

QED

5. Wallpaper Groups

Having found all the possible point groups we now move on to classifying all possible wallpaper groups. To do this, we must look at each lattice again, and using our knowledge of the point groups define each wallpaper group associated with that lattice. After defining the wallpaper groups we will determine which ones are distinct. Each wallpaper group has a name made up of the letters *c*, *g*, *p*, *m* and the integers 1,2,3,4, and 6. We denote a wallpaper group that has a primitive lattice with a *p* and one that has a centered lattice includes a *c*. The name of a wallpaper group that has non-trivial glides is most often includes a *g*. In this paper we will not consider trivial glides. A wallpaper that has reflections will be denoted with an *m*. Finally, if a wallpaper group has rotations of order 1,2,3,4 or 6 that number will be present in its name.

Theorem 5.1 *The order of rotation in a wallpaper group can be 1,2,3,4, or 6.*

For a proof refer to Theorem 25.3 of [1]. Let us begin with the oblique lattice.

Theorem 5.2 *An oblique lattice produces two distinct wallpaper groups:* ***p1*** *and* ***p2****.*

Proof:
Let G be a wallpaper group corresponding to a pattern with an oblique lattice and having point group J. Then we know from Theorem 4.1 that J is $\{I,-I\}$ or $\{I\}$. If $J = \{I\}$ then we define G to be the group **p1**. That is **p1** contains only translations. If $J = \{I,-I\}$ then we define G to be the group **p2**, which only has translations and rotations of 180°. Since $\pi(\mathbf{p1})$ and $\pi(\mathbf{p2})$ are not isomorphic, by the contrapositive of Theorem 2.17 we have that **p1** and **p2** are not isomorphic. Since $-I$ can only be realized by a rotation of order two; each possible point group has just one corresponding wallpaper group. Hence, there are no more groups produced by this lattice.

QED

Now, we move on to look at the rectangular lattice.

Theorem 5.3 *A rectangular lattice produces five distinct wallpaper groups that are different from those produced by an oblique lattice:* **pm**, **pg**, **pmm**, **pmg**, and **pgg.**

Proof:
Let G be a wallpaper group a corresponding pattern with a rectangular lattice and having point group J. From Theorem 4.2 we know that J is $\{I\}, \{I, B_\pi\}$, $\{I, B_0\}$, $\{I,-I\}$, or $\{I,-I, B_0, B_\pi\}$.

Case 1: $J = \{I\}$
In this case G only has translation and thus is isomorphic to **p1.** Therefore, this case does not provide us with any new groups.

Case 2: $J = \{I, B_0\}$
Theorem 2.1 states that every opposite isometry is a reflection or a glide reflection. Therefore, B_0 can either be realized by a reflection or by a glide reflection. Suppose, B_0 is realized by a reflection in a horizontal mirror. This gives us the group defined as **pm**, which contains translations and horizontal reflections. Recall from Theorem 2.16 that an isomorphism between wallpaper groups takes a reflection to a reflection. Thus, the group **pm** is not isomorphic to **p1** or **p2** because neither **p1** nor **p2** contain reflections. Now, suppose that B_0 is realized by a horizontal glide reflection. Then we define G to be the group **pg**, which contains translations and horizontal glide reflections. Again, by the statement of Theorem 2.16 **pg** cannot be isomorphic to **p1**, **p2**, or **pm** since **pg** contains glide reflections.

Case 3: $J = \{I, B_\pi\}$

If B_π is realized by a vertical reflection then by changing our perspective, G is isomorphic to **pm**. If B_π is realized by a vertical glide reflection then G is isomorphic to **pg**. Therefore, this case does not give us any new groups.

Case 4: $J = \{I, -I, B_0, B_\pi\}$

Recall that B_0 and B_π can be realized by either reflections or glide reflections. Suppose B_0 and B_π are realized by reflections. Then we define G to be the group **pmm**, which contains translations, horizontal reflections, and vertical reflections. The group **pmm** is not isomorphic to **p1**, **p2**, **pm**, or **pg** because their point groups are not isomorphic. Next, suppose B_0 and B_π are realized by glide reflections. We define G to be the group **pgg**, which contains translations, vertical glide reflections, and horizontal glide reflections. Since the point group of **pgg** is not isomorphic to the point groups of **p1**, **p2**, **pm**, or **pg** it cannot be isomorphic to any of these groups. Also, **pgg** and **pmm** are not isomorphic because by definition **pmm** does not contain glide reflections. Now, suppose B_0 is realized by a reflection and B_π by a glide reflection. We define G to be the group **pmg**, which contains translations, horizontal reflections, and vertical glide reflections. Again, the group **pmg** is not isomorphic to **p1**, **p2**, **pm**, or **pg** because the point group of **pmg** has a different order. Moreover, **pmg** is not isomorphic to **pmm** or **pgg** due to the fact that **pmm** does not contain any glide reflections and **pgg** does not contain any reflections. Interchanging the roles of B_0 and B_π yields a group isomorphic to **pmg**.

Having examined all possible point groups of G we conclude that there are five distinct wallpaper groups produced by a rectangular lattice that differ from those produced by an oblique lattice.

QED

Next, we take a look at the point groups of a centered rectangular lattice.

Theorem 5.4 *The centered rectangular lattice produces two distinct wallpaper groups that are different from those produced by the oblique and rectangular lattice:* ***cm*** *and* ***cmm***.

Proof:

Let G be a wallpaper group corresponding to a pattern with a centered rectangular lattice and having point group J. Then, from Theorem4.3 we know the point group of G is either $\{I\}$, $\{I, B_\pi\}$, $\{I, B_0\}$, $\{I, -I\}$, *or* $\{I, -I, B_0, B_\pi\}$. We will not consider the cases where

$J = \{I\}$ or $J = \{I, -I\}$ since these have been previously explored.

Case 1: $J = \{I, B_0\}$

It is unnecessary to consider the cases when B_0 is realized by a reflection only or by a glide reflection only since those groups are either isomorphic to **pm** or **pg**, respectively. Suppose B_0 is realized by either a horizontal reflection or a horizontal glide reflection. In this case we define G to be the group **cm.** The group **cm** is not isomorphic to **pm** or **p2** due to the fact that neither *pm* nor *p2* contains any glide reflections. Furthermore, G is not isomorphic to **pg** since **pg** does not contain any reflections. Therefore, **cm** is a new group.

Case 2: $J = \{I, B_\pi\}$

If we interchange the roles of B_π with B_0 then we get a group that is isomorphic to **cm**.

Case 3: $J = \{I, -I, B_0, B_\pi\}$

Suppose B_0 and B_π can be realized in G by either a reflection or a glide reflection. In this case we define G to be the group **cmm**. The group **cmm** is not isomorphic to **pgg** because **pgg** does not contain any reflections. Also, **cmm** has both vertical and horizontal reflections and **pmg** has horizontal reflections or vertical reflections, but not both. Now, the composition of two reflections in **pmg** would only give us a translation. On the other hand, if we take a vertical reflection and compose it with a horizontal reflection in **cmm** we would get a half turn. Thus, **cmm** is not isomorphic to **pmg**. Moreover, since **pmm** does not contain any glide reflections it cannot be isomorphic to **cmm**. Hence, **cmm** is a distinct wallpaper group. If B_0 is realized by a glide reflection then B_π must be realized by a glide reflection. Consider the horizontal glide (n**a** + m**b**, B_0), where $n, m \in Z$ and $n, m \neq 0$, in G. The composition of (n**a** + m**b**, B_0) and (**0**, $-I$) gives us a vertical glide reflection: (n**a** + m**b**, B_0) (**0**, $-I$) = (n**a** + m**b**, B_π). By the same argument if B_π is realized by glide reflection then B_0 must be realized by a glide reflection also. Hence, this point group does not give us any new groups.

Having looked at all possible point groups we conclude that the centered rectangular lattice produces three distinct wallpaper groups.

QED

We now move on to look at the point groups of the square lattice.

Theorem 5.5 *A square lattice produces three distinct wallpaper groups that are different from those produced by an oblique, rectangular, and centered rectangular lattice:* ***p4****,* ***p4mm****, and* ***p4gm****.*

Proof:

Let G be a wallpaper group corresponding to a pattern with a square lattice and point group J. From Theorem 4.4, J must be one of the following: $\{I\}$, $\{I,-I\}$, $\{I,B_0\}$, $\{I,B_\pi\}$, $\{I,B_{\pi/2},\}$, $\{I,B_{3\pi/2}\}$, $\{I,-I,B_0,B_\pi\}$, $\{I,-I,B_{\pi/2},B_{3\pi/2}\}$, $\{I,-I,A_{\pi/2},A_{3\pi/2}\}$, or $\{I,-I,A_{\pi/2},A_{3\pi/2},B_0,B_{\pi/2},B_\pi,B_{3\pi/2}\}$. We will not consider the cases $\{I\}$, $\{I,-I\}$, $\{I,B_0\}$, $\{I,B_\pi\}$, or $\{I,-I,B_0,B_\pi\}$ since they have been explored above.

Case 1: $J=\{I,B_{\pi/2}\}$

Since G corresponds to a pattern with a square lattice, $B_{\pi/2}$ can either be realized by a reflection, a glide reflection, or both. Suppose that $B_{\pi/2}$ is realized in G by a reflection, then if we change our perspective G is isomorphic to **pm**. Next, suppose that $B_{\pi/2}$ is realized by a glide reflection, then G is isomorphic to **pg**. Finally, suppose that $B_{\pi/2}$ is realized by both a reflection and a glide reflection, then G is isomorphic to **cm**. Hence, this case does not give us any new wallpaper groups.

Case 2: $J=\{I,B_{3\pi/2}\}$

This case follows exactly as Case 1 because $\{I,B_{\pi/2}\}\cong\{I,B_{3\pi/2}\}$.

Case 3: $J=\{I,-I,B_{\pi/2},B_{3\pi/2}\}$

In this case J is isomorphic to the already explored $\{I,-I,B_0,B_\pi\}$ and thus does not provide us with any new groups.

Case 4: $J=\{I,-I,A_{\pi/2},A_{3\pi/2}\}$

We define G to be the group **p4**, which has translations and rotations of 90°. The group **p4** is the only group of rotations with a point group of order four. Thus, it is not isomorphic to **pmm**, **pmg**, **pgg**, or **cmm**.

Case 5: $J=\{I,-I,A_{\pi/2},A_{3\pi/2},B_0,B_{\pi/2},B_\pi,B_{3\pi/2}\}$

Suppose $B_0, B_{\pi/2}, B_\pi$, and $B_{3\pi/2}$ can be realized by reflections and that $B_{\pi/2}$, and $B_{3\pi/2}$ can be realized by glide reflections , then we define G to be the group **p4m**. This is a new group since it has a point group with order eight. Now, suppose that B_0 and B_π can be realized by reflections and $B_{\pi/2}$, and $B_{3\pi/2}$ are realized by glide reflections, then we define G to be the group **p4g**. The groups **p4m** and **p4g** are not isomorphic because the rotation $(\mathbf{a},A_{\pi/2})$ can be written as the product of two reflections in **p4m**:

$$(\mathbf{a},B_\pi)(\mathbf{0},B_{\pi/2})=(\mathbf{a},A_{\pi/2})$$

On the other hand, the rotation $(\mathbf{a},A_{\pi/2})$ cannot be written as the product of two reflections in **p4g**.

Claim: The rotation $(\mathbf{a},A_{\pi/2})$ cannot be factorized as the product of two reflections in **p4g**.

For the proof of this claim refer to Appendix A. The following shows that there are only two new groups coming from this *J*. Recall that a $f=(\mathbf{v},B_\theta)$ pure reflection if $\mathbf{v}$ is perpendicular to the mirror of f and a glide reflection otherwise. If B_0 and B_π are realized by reflections then $B_{\pi/2}$ and $B_{3\pi/2}$ must be realized as glide reflections also: $(\mathbf{a},B_\pi)\ (\mathbf{0},A_{\pi/2})\ =\ (\mathbf{a},B_{\pi/2})$ and $(\mathbf{b},B_0)\ (\mathbf{b},A_{\pi/2})\ =\ (\mathbf{b},B_{3\pi/2})$. Hence, $B_0,B_{\pi/2},B_\pi$ and $B_{3\pi/2}$ cannot just be realized by reflections. Also, if $B_{\pi/2}$ and $B_{3\pi/2}$ are realized by glide reflections then they must also be realized by reflections: $(\mathbf{a},B_\pi)\ (\mathbf{a},A_{\pi/2})\ =\ (\mathbf{0},B_{\pi/2})$ and $(\mathbf{a},A_{\pi/2})\ (\mathbf{a},B_\pi)\ =\ (\mathbf{a}+\mathbf{b},B_{3\pi/2})$. Therefore, we cannot just have two orthogonal matrices realized by reflections and two by glide reflections.

Suppose B_0 is realized by a glide reflection and $B_{\pi/2},B_\pi$, and, $B_{3\pi/2}$ are realized by reflections. If this is the case, then B_0 must also be realized by a reflection: $(\mathbf{0},B_{3\pi/2})\ (\mathbf{0},A_{3\pi/2})=(\mathbf{b},B_0)$. Thus, this group is **p4m**. By a similar argument, if any three of the four matrices are realized by reflections the other must be realized by a reflection also.

Finally, suppose that all four matrices are realized by both reflections and glide reflections then, the vertical and horizontal glides are trivial.

$$(m\mathbf{a}+n\mathbf{b},B_0)\ =\ (m\mathbf{a}+n\mathbf{b},B_{3\pi/2})\ (\mathbf{0},A_{3\pi/2})$$

A similar argument can be made to show that a vertical glide is trivial. Hence, this group is **p4m**.

We have examined all point groups and have arrived at three new groups: **p4**, **p4m**, and **p4g**.

QED

Finally, we must take a look at the hexagonal lattice.

Theorem 5.6 *A hexagonal lattice produces five distinct wallpaper groups different from those produced by an oblique, rectangular, centered rectangular, and square lattice* ***p3, p3m1, p31m, p6, p6mm.***

Proof:
Let *G* be a wallpaper group corresponding to a pattern with a hexagonal lattice and having point group *J*. Recall from Theorem 4.5 that *J* is either: $\{I\}$, $\{I,-I\}$, $\{I,B_{k\pi/2}\}$ *for* $0\le k\le 5$, $\{I,-I,B_0,B_\pi\}$, $\{I,-I,B_{\pi/3},B_{4\pi/3}\}$, $\{I,-I,B_{2\pi/3},B_{5\pi/3}\}$, $\{I,A_{2\pi/3},A_{4\pi/3}\}$, $\{I,A_{2\pi/3},A_{2\pi/3},B_{\pi/3},B_\pi,B_{5\pi/3}\}$, $\{I,A_{2\pi/3},A_{4\pi/3},B_0,B_{2\pi/3},B_{4\pi/3}\}$,
$\{I,-I,A_{\pi/3},A_{2\pi/3},A_{4\pi/3},A_{5\pi/3}\}$, or
$\{I,-I,A_{\pi/3},A_{2\pi/3},A_{4\pi/3},A_{5\pi/3},B_0,B_{\pi/3},B_\pi,B_{2\pi/3},B_{4\pi/3},B_{5\pi/3}\}$
We will not examine the cases where *J* is $\{I\}$, $\{I,-I\}$, $\{I,B_{k\pi/2}\}$ *for* $0\le k\le 5$, or $\{I,-I,B_0,B_\pi\}$ since these have been previously dealt with.

Case 1: $J = \{I, A_{2\pi/3}, A_{4\pi/3}\}$
In this case we define G to be the group **p3**, which only has rotations of order three. The wallpaper group **p3** is not isomorphic to any of the wallpaper groups discussed above since it is the only wallpaper group with a point group of order three.

Case 2: $J = \{I, -I, B_{\pi/3}, B_{4\pi/3}\}$, $J = \{I, -I, B_{2\pi/3}, B_{5\pi/3}\}$
These point groups are isomorphic to the aforementioned $\{I, -I, B_0, B_\pi\}$. Hence, they do not provide us with any new wallpaper groups.

Case 3: $J = \{I, -I, A_{\pi/3}, A_{2\pi/3}, A_{4\pi/3}, A_{5\pi/3}\}$
In this case we define G to be the group **p6**, which contains rotations of order six. Since **p6** is the only wallpaper group with point group of order six it is not isomorphic to any of the groups discussed thus far.

Case 4: $J = \{I, A_{2\pi/3}, A_{4\pi/3}, B_0, B_{2\pi/3}, B_{4\pi/3}\}$
Suppose $B_0, B_{2\pi/3}, B_{4\pi/3}$ are realized by glide reflections. In this case we define G to be the group **p3m1**. The wallpaper group **p3m1** has a point group of order six with reflections. Hence, it is not isomorphic to any of the groups mentioned above.

The following shows that there is only one new wallpaper group coming from J. Recall that $(\mathbf{v}, M)$ is a reflection if $\mathbf{v}$ is perpendicular to M and a glide reflection otherwise. Assume B_0 is realized by a reflection then $B_{2\pi/3}, B_{4\pi/3}$ must be realized as glides. To see this, we take the product of a horizontal reflection and a 120^{o} rotation, $(2\mathbf{b} - \mathbf{a}, B_0)(\mathbf{0}, A_{2\pi/3}) = (2\mathbf{b} - \mathbf{a}, B_{4\pi/3})$. This product is a glide reflection. Now, the product of a horizontal rotation and 240^{o} rotation is also a glide reflection: $(2\mathbf{b} - \mathbf{a}, B_0)(\mathbf{0}, A_{4\pi/3}) = (2\mathbf{b} - \mathbf{a}, B_{2\pi/3})$. Let us now take a look at the following products: $(\mathbf{0}, A_{2\pi/3})(2\mathbf{b} - \mathbf{a}, B_{4\pi/3}) = (-(\mathbf{a} + \mathbf{b}), B_0))$, $(\mathbf{0}, A_{4\pi/3})(2\mathbf{b} - \mathbf{a}, B_{2\pi/3}) = (2\mathbf{a} - \mathbf{b}, B_0)$. These products are glide reflections and since the constituent parts are in G then so is the product. Hence, B_0 must also be realized as a glide reflection. Next, we take a horizontal glide reflection and compose it with a 120^{o} rotation: $(\mathbf{0}, A_{2\pi/3})(-(\mathbf{a} + \mathbf{b}), B_0) = (2\mathbf{a} - \mathbf{b}, B_{4\pi/3})$. This is a reflection in G. Also, $(-(\mathbf{a} + \mathbf{b}), B_0)(\mathbf{0}, A_{2\pi/3}) = (-(\mathbf{a} + \mathbf{b}), B_{4\pi/3})$ is a reflection. Hence, $B_{2\pi/3}, B_{4\pi/3}$ must also be realized as reflections in G. Assuming B_0 is realized by a reflection creates a chain reaction. That is having one isometry leads to the next. Hence, $B_0, B_{2\pi/3}, B_{4\pi/3}$ must be realized by both reflections and glide reflections. Thus, we conclude that if G has point group J it must be G = **p3m1**.

Case 5: $J = \{I, A_{2\pi/3}, A_{4\pi/3}, B_{\pi/3}, B_\pi, B_{5\pi/3}\}$

Suppose $B_{\pi/3}, B_{\pi}$ and $B_{5\pi/3}$ are realized by both reflections and glide reflections, then we define G to be the group **p31m**.

The following shows that there is only one new wallpaper group coming from J.
Assume that B_{π} is realized by a reflection, then $B_{\pi/3}, B_{\pi}$ and $B_{5\pi/3}$ must be realized by glides. The following glides are in G: $(\mathbf{a}, B_{\pi})(\mathbf{0}, A_{2\pi/3}) = (\mathbf{a}, B_{\pi/3})$, $(\mathbf{a}, B_{\pi})(\mathbf{0}, A_{4\pi/3}) = (\mathbf{a}, B_{5\pi/3})$, $(\mathbf{0}, A_{4\pi/3})(\mathbf{a}, B_{\pi/3}) = (\mathbf{b} - \mathbf{a}, B_{\pi})$.
In addition, $B_{\pi/3}$ and $B_{5\pi/3}$ must also be realized by reflections in *G:*
$(\mathbf{b} - \mathbf{a}, B_{\pi})(\mathbf{0}, A_{2\pi/3}) = (\mathbf{b} - \mathbf{a}, B_{\pi/3})$, $(\mathbf{b} - \mathbf{a}, B_{\pi/3})(-\mathbf{b}, A_{2\pi/3}) = (-\mathbf{b}, B_{5\pi/3})$ are reflections in G. Hence, $B_{\pi/3}, B_{\pi}$ and $B_{5\pi/3}$ are realized by reflections and glide reflections. We conclude that if G has point group J then it must be **p31m**. Also, **p31m** is not isomorphic to **p3m1**. Each rotation of order three can be written as the product of two reflections in **p3m1:** $(\mathbf{a}, A_{2\pi/3}) = (\mathbf{a}, B_{\pi})(\mathbf{0}, B_{\pi/3})$ and $(\mathbf{a}, A_{4\pi/3}) = (\mathbf{a}, B_{\pi})(\mathbf{0}, B_{5\pi/3})$. On the other hand, according to this is not the case in **p31m**.

Claim: The rotation $(\mathbf{a}, A_{2\pi/3})$ *cannot be factorized as the product of two reflections in* ***p3m1***.

For the proof of this claim refer to Appendix B.

Thus, **p31m** is not isomorphic to **p3m1**.

Case 6: $J = \{I, -I, A_{\pi/3}, A_{2\pi/3}, A_{4\pi/3}, A_{5\pi/3}, B_0, B_{\pi/3}, B_{\pi}, B_{2\pi/3}, B_{4\pi/3}, B_{5\pi/3}\}$
In this case we define G to be the wallpaper group **p6mm**. If B_0 is realized by a reflection then B_{π} is realized by a glide reflection: $(2\mathbf{b} - \mathbf{a}, B_0)(\mathbf{0}, A_{\pi}) = (\mathbf{b} - \mathbf{a}, B_{\pi})$. With this information and from Cases 4 and 5 we know that $B_0, B_{\pi/3}, B_{\pi}, B_{2\pi/3}, B_{4\pi/3}, B_{5\pi/3}$ are realized by both glide reflections and reflections. Thus, if G has point group J it must be **p6mm**. The group **p6mm** is the only wallpaper group with a point group of order twelve and thus is a distinct group.

Having examined all the possible point groups in the hexagonal lattice we conclude that it produces five distinct wallpaper groups: **p3**, **p3m1**, **p31m**, **p6**, and **p6mm**

QED

6. Conclusion

After a strategic mathematical analysis of the makeup of a wallpaper pattern, we were able to rediscover a way to classify the wallpaper patterns into 17 distinct wallpaper groups. Our proof that there are exactly 17 distinct wallpaper groups is a little different than that of E.S. Fedorov. Fedorov, being a crystallographer, took a more scientific approach to the problem. On the other hand we, as mathematicians, used a more mathematical approach.

After fully exploring two dimensional crystallographic groups, the next logical step is to look at three dimensions.

E.S. Fedorov moved on to classify all the 320 three dimensional crystallographic groups [6]. However, Escher did not desire to move beyond the realm of two dimensions, and relished in the splendor of wallpaper patterns. Escher's "obsession" with the wallpaper patterns in the Alhambra is just another example of the power of the mathematics of art.

Acknowledgement

We would first like to thank the National Security Agency and Miami University for supporting SUMSRI, and giving us the opportunity to participate in this enlightening program. Furthermore, we would like to thank Bonita Porter for coordinating the program, and Dr. Dennis Davenport, Dr. Vasant Waiker, and Dr. Paddy Dowling for directing the program. We would also like to give a special thanks to Dr. Chawne Kimber for directing our seminar group, and for assisting us in our research. We would like to thank our graduate assistant Christian Roldán-Santos for assisting us with our research. Finally, we would like to thank the Miami University Mathematics and Statistics department for its technical support and advisory. In addition, we would like to thank any other Faculty and Staff member who contributed to our research experience.

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Appendix A: Proof of claim for Theorem 5.5.

Proof:
We will prove this theorem by assuming that $(\mathbf{a}, A_{\pi/2})$ can be factorized as the product of two reflections in **p4g** and then arrive at a contradiction. Let $g = (\mathbf{a}, A_{\pi/2})$ and suppose g can be written as a product of f_1 and f_2, where $f_1 f_2$ are reflections in **p4g**. This proof has a few cases. When the point group of **p4g** is $\{I, -I, B_0, B_\pi\}$ the cases are as follows: either f_1 and f_2 are both horizontal reflections, or f_1 and f_2 are both vertical reflects, or if f_1 and f_2 are horizontal and vertical reflections, respectively. Taking the point group of *p4g* to be $\{I, -I, B_{\pi/2}, B_{3\pi/2}\}$, gives the same result because the two point groups are isomorphic.

Case 1: f_1 and f_2 are both horizontal reflections.
Let f_1 and f_2 be horizontal reflections where $f_1 = (\mathbf{v_1}, B_0)$ and $f_2 = (\mathbf{v_2}, B_0)$. Recall that g is the rotation of ninety degrees where $g_1 = (\mathbf{a}\ \ , A_{\pi/2})$. Assuming that g can be factored into reflections f_1 and f_2 gives

$$(\mathbf{a}\ \ , A_{\pi/2}) = (\mathbf{v_1}, B_0)(\mathbf{v_2}, B_0) = (\mathbf{v_1} + \mathbf{v_2} B_{.0}^2, B_0^2)\,.$$

Examination of the matrix product B_0^2 gives

$$B_0^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Taking $A_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ we have a contradiction because this is not equal to *I*.

Therefore $A_{\pi/2}$ cannot be factored into these two reflections.

Case 2: f_1 and f_2 are both vertical reflections.
This case is similar to Case 1 where the product of the two horizontal reflections produces a translation. Therefore, $(\mathbf{a}, A_{\pi/2})$ cannot be factored into two reflections in this case either.

Case 3: f_1 and f_2 are horizontal reflections and vertical reflections respectively.
In this case $f_1 = (\mathbf{v_1}, B_0)$ and $f_2 = (\mathbf{v_2}, B_\pi)$. Again we assume that g can be factored into these two reflections.

Therefore we have

$$(\mathbf{a}\ , A_{\pi/2}) = (\mathbf{v_1}, B_0)(\mathbf{v_2}, B_\pi) = (\mathbf{v_1} + \mathbf{v_2} B_{.0}, B_0 B_\pi)$$

Examining the matrixes again gives

$$B_0 B_\pi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Recall that $A_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq -I$. This is a contradiction, therefore, $(\mathbf{a}, A_{\pi/2})$ cannot be factorized into these two reflections. All possible cases have been exhausted; hence, we conclude that the rotation $(\mathbf{a}, A_{\pi/2})$ cannot be factorized into a product of two reflections in **p4g**.

QED

Appendix B: Proof of claim for Theorem 5.6.

Proof:
The point group of p3m1 is generated by $A_{3\pi/2}$ and B_0 . As a result, the only reflections in p3m1 have matrices B_0 , $B_{2\pi/3}$, or $B_{4\pi/3}$. Let $g = (\mathbf{a}, A_{2\pi/3})$, where **a** is the horizontal vector that spans the lattice of the pattern corresponding **p3m1**, and suppose g can be factorized as a product of two reflections of **p3m1**. That is, $g = f_1 f_2$, where $f_1, f_2 \in$ **p3m1** and $f_1 = (\mathbf{v}, M_1)$ and $f_2 = (\mathbf{w}, M_2)$. Since $M_1 M_2 = A_{2\pi/3}$ we have three cases: $M_1 = B_{2\pi/3}$ and $M_2 = B_0$, $M_1 = B_0$ and $M_2 = B_{4\pi/3}$, or $M_1 = B_{4\pi/3}$ and $M_2 = B_{2\pi/3}$. These are the only possible products that will give us $A_{2\pi/3}$.

Case 1: $M_1 = B_{2\pi/3}$ and $M_2 = B_0$
Here, $g = (\mathbf{v} + \mathbf{w} B_{2\pi/3}, A_{2\pi/3}) = (\mathbf{a}, A_{2\pi/3})$.

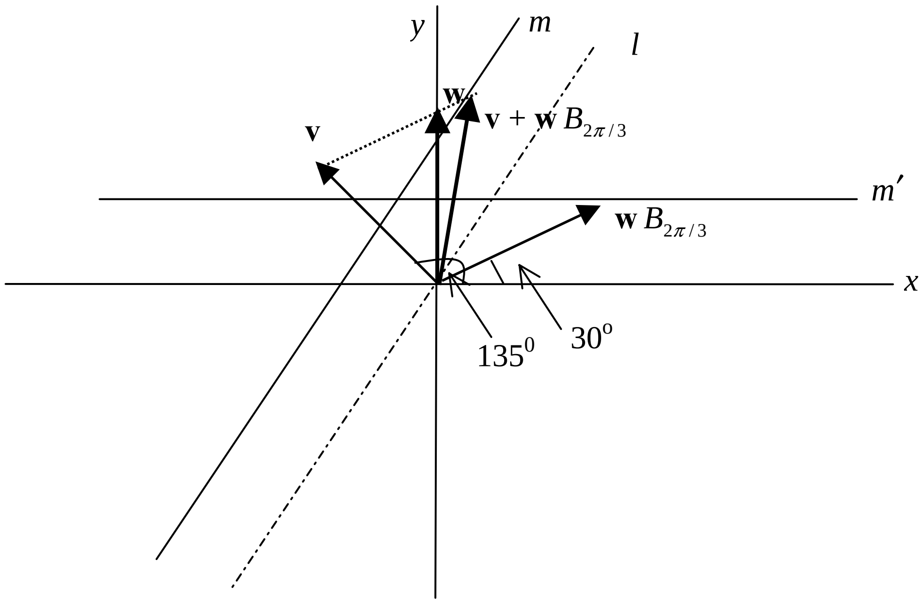


Figure 5.1

In Figure 5.1 let m and m' be the mirrors of reflection of f_1 and f_2, respectively. The angle between the mirror and the positive horizontal axis is 60^o. Also, let l be the line parallel to m that passes though the origin. If we multiply **w** and $B_{2\pi/3}$ we are actually reflecting **w** across the line l. Since l makes a 60^o angle with the positive horizontal axis, then $\mathbf{w}B_{2\pi/3}$ makes a 30^o angle with the positive horizontal axis. In addition, the angle between **v** and the positive horizontal axis is 135^o. We can see that $\mathbf{v} + \mathbf{w}B_{2\pi/3}$ will not make a 0^o angle with the positive horizontal axes. However, **a** is a horizontal vector hence $\mathbf{v} + \mathbf{w}B_{2\pi/3} \neq \mathbf{a}$. This gives us a contradiction.

Case 2 and Case 3: $M_1 = B_0$ and $M_2 = B_{4\pi/3}$, $M_1 = B_{4\pi/3}$ and $M_2 = B_{2\pi/3}$

In these cases the reflections are skewed as well, therefore, the argument is the same as that of Case 1. Hence, it follows that $\mathbf{v} + \mathbf{w}B_0 \neq \mathbf{a}$ and $\mathbf{v} + \mathbf{w}B_{4\pi/3} \neq \mathbf{a}$, we conclude that $(\mathbf{a}, A_{2\pi/3})$ cannot be factorized as the product of two reflection in **p3m1**.

QED

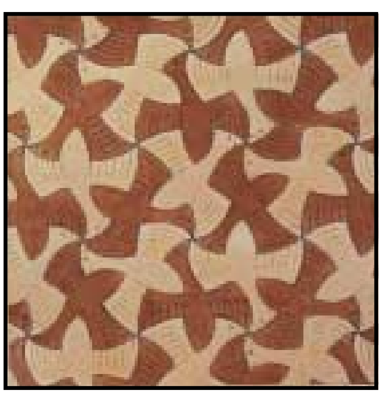
Appendix C: Sample Escher Wallpaper

p1

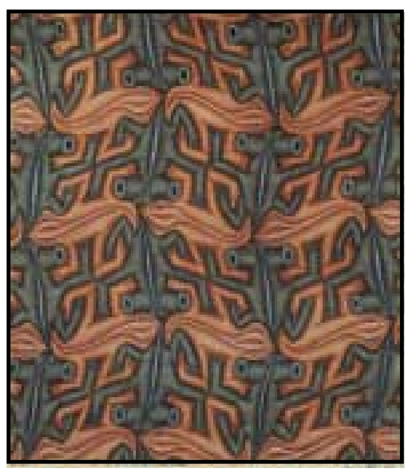
p31m

pg (without color)

p4



p3



p2