# Analyzing the Cost of Graphs Relative to Tournaments

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**Abstract**

Let $G$ be an undirected graph and let $T$ be a tournament on the same vertex set as $G$. Define the cost of $G$ relative to $T$ to be $\sum_{u,v \in E(G)} (T(u,v) + T(v,u)) + |E|$, where $T(u,v)$ denotes the number of two-step paths from $u$ $v$, in $T$. In this paper, we determine for several classes of graphs which tournaments minimize the cost. Pelsmajer, et al. [5] conjecture that for each graph there is a transitive tournament that minimizes the graph's cost. We prove that a transitive tournament minimizes the cost for complete graphs, nearly complete graphs, paths, star graphs, and cycles.

## 1 Introduction

We first define some basic concepts of graph theory. A *graph*, $G$, consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each edge in $E(G)$ is an unordered pair of vertices. If $\{x, y\} \in E(G)$ we will denote the edge as $xy$ and say $x$ and $y$ are *adjacent* vertices. A *directed graph* (or *digraph*) $D$ consists of a vertex set $V(D)$ and an edge set $E(D)$, where each edge is represented by an ordered pair of vertices showing a direction for each edge. If $(u, v) \in E(D)$ we will denote it as $\overrightarrow{uv}$ and say $u$ *defeats* $v$ or $v$ is *defeated by* $u$. A *tournament* $T$ is a digraph where for all $u, v \in V(T)$ with $u \neq v$, exactly one of $\overrightarrow{uv}$ or $\overrightarrow{vu}$ is in $E(T)$. We refer the reader to [7] for other basic notation and terminology.

In this paper we make free use of the fact that in a tournament $d^+(v) + d^-(v) = n - 1$ for any $v \in V(T)$. We will call the set of vertices that defeat vertex $v$ the *in-neighborhood* of $v$, denoted $N^-(v)$, and call the set of vertices that are defeated by $v$ the *out-neighborhood* of $v$, denoted $N^+(v)$. We will denote the *out-degree* and *in-degree* of a vertex $v$ as $d^+(v)$ and

$d^-(v)$, respectively. There are $\binom{n}{2}$ edges in any tournament. Since each edge in a tournament can point in one of two directions, there are $2^{\binom{n}{2}}$ possible tournaments on a particular set of vertices.

We will now describe three common, named tournaments. For the following three types of tournaments, consider all the edges not explicitly shown to be oriented in the direction of the large arrow:

- Transitive ($TT_n$): Given any three vertices $a$, $b$ and $c$, if $\overrightarrow{ab}, \overrightarrow{bc} \in E(TT_n)$, then $\overrightarrow{ac} \in E(TT_n)$.

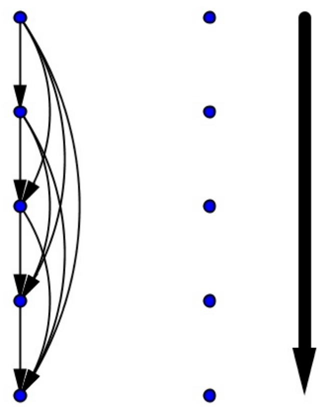

Figure 1: Two ways to display a $TT_5$

- One Major Upset ($OMUT_n$): Starting with a transitive tournament on $n$ vertices, we reverse the edge from the vertex of largest out-degree to the vertex of smallest out-degree. Thus, the $n$th vertex defeats the first vertex.

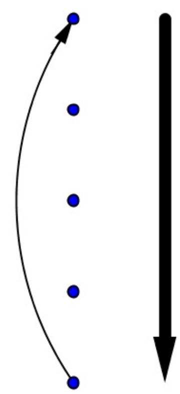

Figure 2: $OMUT_5$

- Path of Minor Upsets ($POmUT_n$): Starting with a transitive tournament on $n$ vertices, change the orientation on edges so that the $k$th vertex defeats the $(k-1)$th vertex, for all $k \in \{2, 3, \ldots, n\}$, where the $k$th vertex in the transitive tournament is the unique vertex of out-degree $n - k$.

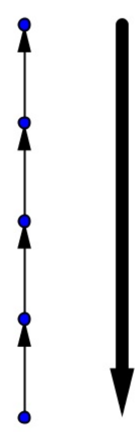

Figure 3: $POmUT_5$

We study a problem introduced in [5] and [6] in which the edges of an undirected graph $G$ are assigned costs based on a specific tournament $T$, where $G$ and $T$ have the same vertex sets. The cost of edge $uv$ of $G$ is $T(u, v) + T(v, u) + 1$, where $T(x, y)$ is the number of two-step paths in $T$ from $x$ to $y$. A *two-step path* from $u$ to $v$ in $T$ is a path $uwv$ for some

$w \in V(T) - \{u, v\}$. The cost of the entire graph $G$ with respect to a given tournament $T$ is then defined as the sum of the costs of the edges. We denote this sum as $Cost(G, T)$, which equals $|E| + \sum_{uv \in E(G)} (T(u, v) + T(v, u))$. We let $C(G)$ denote $min\{Cost(G, T)\}$ where the minimum cost of $G$ over all choices of associated tournaments $T$. We can represent the two-step paths of a given tournament as a matrix of the number of two-step paths between each of the vertices, which we call the *cost matrix*.

By comparing cost matrices, it can be shown easily that $OMUT_n$ will never result in a lower cost on any graph when compared with $TT_n$. This is because the $i, j$ entry of the cost matrix of $OMUT_n$ is at least as large as the $i, j$ entry of the cost matrix of $TT_n$.

# 2 Complete and Nearly Complete Graphs

Following the work of [5], we start by studying complete graphs. A graph is complete if for every pair of distinct vertices $u$ and $v$ in the graph there is an edge between $u$ and $v$. Removing a small number of edges from a complete graph results in what we refer to as a *nearly complete graph*.

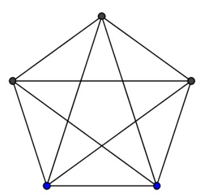

Figure 4: The Complete Graph on Five Vertices

## 2.1 The Complete Graph ($K_n$)

Note that $Cost(K_n, T)$ is simply $\binom{n}{2}$ plus the number of two-step paths in $T$. So, in a complete graph we consider what happens to the number of two-step paths in an underlying tournament when we reverse the direction of an edge. We begin our discussion of this in order to improve upon the original proof given in [5] that a transitive tournament minimizes the cost of $K_n$.

**Proposition 2.1.1** *Let $T'$ be obtained from tournament $T$ by reversing the edge $\overrightarrow{vu}$ to $\overrightarrow{uv}$ for some $\overrightarrow{vu} \in E(T)$. The decrease in the number of two-step paths resulting from reversing $\overrightarrow{vu}$ is $Cost(K_n, T) - Cost(K_n, T') = 2(d_T^+(u) - d_T^+(v) + 1)$.*

*Proof.* The number of two-step paths using $\overrightarrow{vu}$ equals $d_T^-(v) + d_T^+(u)$. Since the edge between $v$ and $u$ is reversed in $T'$, $v$ lost an out-edge, and $u$ lost an in-edge. Thus, in $T'$, $d_{T'}^+(v) = d_T^+(v) - 1$, and $d_{T'}^-(u) = d_T^-(u) - 1$.

The difference in number of two-step paths in $T$ versus in $T'$ is

$$\begin{aligned} & d_T^-(v) + d_T^+(u) - (d_{T'}^+(v) + d_{T'}^-(u)) \\ = \; & d_T^-(v) + d_T^+(u) - (d_T^+(v) - 1 + d_T^-(u) - 1) \\ = \; & 2(d_T^+(u) - d_T^+(v) + 1). \end{aligned}$$

□

**Theorem 2.1.2** [5] *The minimum tournament arrangement of $K_n$ has value $\binom{n+1}{3}$.*

*Proof.* Note that the minimum cost of $K_n$ equals the minimum total number of two-step paths among choices of underlying tournament $T$. Thus for an optimal choice of $T$, there will be no two vertices of $T$ with the same out-degree. Note that the out-degree of each vertex can only be $0, 1, 2, \ldots,$ or $n-1$. Thus, there will be exactly one of each out-degree in the tournament. It is not hard to verify from here that $T$ must be a transitive tournament. Notice that if we choose any three vertices in $T$, there is exactly one two-step path visiting all three vertices. Therefore the $Cost(K_n, TT_n) = \binom{n}{3} + \binom{n}{2}$. By "L-Law Identity" of Pascal's Triangle, this equals $\binom{n+1}{3}$. So, $Cost(K_n, T_n)$ can only be minimized by a transitive tournament. □

## 2.2 The Complete Graph Minus One Edge $(K_n - e)$

From [6], we know that the minimum cost of a tournament on $K_n$ is $\binom{n+1}{3}$. We will denote the minimum cost as $C(K_n)$. Between two vertices in $T_n$ there can be at most $n-2$ two-step paths, one per choice of intermediary vertex in such a path. Yet $Cost(K_n - uv, TT_n) = \binom{n+1}{3} - (n-1)$ when $u, v$ are the vertices of outdegree $n-1$ and $0$ in $TT_n$. Thus $C(K_n) \leq \binom{n+1}{3} - (n-1)$. So we get the following theorem.

**Theorem 2.2.1** *The cost of $K_n - e$ is minimized by $TT_n$, when $C(K_n) = \binom{n+1}{3} - (n-1)$*

*Proof.* Assume there is a non-transitive tournament, $T$, that minimizes the cost on $K_n - e$. Note that the upper bound for the minimum cost of $K_n - e$ is $\binom{n+1}{3} - (n-1)$. Assume for contradiction that $T$ results in a cost less than or equal to $\binom{n+1}{3} - (n-1) - 1$. Note that if we put $e$ back into the graph, $e$ can have at most a cost of $n-1$. Thus the $C(K_n) \leq \binom{n+1}{3} - (n-1) - 1 + (n-1)$ under $T$. But then $C(K_n) = \binom{n+1}{3}$ and $Cost(K_n, T) \leq \binom{n+1}{3} - 1$. Thus we have a contradiction that $T$ yields a cost that is less than the minimum cost of $K_n$.

□

## 2.3 The Complete Graph Minus Two Edges $(K_n - e - f)$

Now that we have proved $TT_n$ yields a minimum cost for the $K_n - e$ graph, we can also show the same holds for $K_n - e - f$.

**Theorem 2.3.1** *The cost of $K_n - e - f$ is minimized by a transitive tournament.*

*Proof.* Whether edges $e, f$ share a vertex or not, it is easy to have $TT_n$ underly $K_n$ in such a way that $n-1$ is the cost of $e$ and $n-2$ or $n-3$ is the cost of $f$. Thus $C(K_n - e - f \leq \binom{n+1}{3} - (2n-4)$. Assume for contradiction that there is a non-transitive tournament, $T_n$, yielding cost on $K_n - e - f$ lower still, with $C(K_n - e - f, T_n) \leq \binom{n+1}{3} - (2n-4) - 1$. Both removed edges have a cost of at most $n-1$ in the arbitrary tournament, so when replaced the maximum cost addition would be $2n-2$. So $Cost(K_n, T_n) \leq \binom{n+1}{3} - (2n-4) - 1 + (2n-2)$ or simply $\binom{n+1}{3} + 1$. This yields a contradiction because as proven earlier, the transitive tournament is the only tournament that minimizes the cost of $K_n$, with all others having cost $\binom{n+1}{3} + 2$ or larger. But the arbitrary $T_n$ is not transitive, a contradiction. □

## 2.4 The Complete Graph Minus Three Edges $(K_n - e - f - g)$

The cost of an edge between vertices $u$ and $v$ can be defined as the number of two-step paths between $u$ and $v$ plus the count of the edge: $\sum\limits_{u,v \in E(G)} (T(u,v)T(v,u)) + 1$. Given a transitive tournament, a nearly transitive tournament is one where the edge between $k$ and $k-2$ has its orientation reversed, for some $k$.

**Lemma 2.4.1** *A nearly transitive tournament has only one edge of cost $n-1$.*

*Proof.* Assume there are two edges with cost $n-1$ in a nearly transitive tournament. Take two independent edges $xy$ and $uv$. For $xy$ and $uv$ to have a cost of $n-1$, $x$ must defeat either $u$ or $v$, then $y$ must defeat the vertex that $x$ does not defeat. This creates two two-step paths between $x$ and $y$ and two between $u$ and $v$. If $u$ defeats $v$ there is one 3-cycle, $uvyu$; however, no matter which direction the edge between $x$ and $y$ points, there will be another 3-cycle. Similarly, if $v$ defeats $u$, there will be two 3-cycles regardless of the orientation of the edge between $x$ and $y$. This yields a contradiction as a nearly transitive tournament has only one 3-cycle. If the two edges share a vertex, $xy$ and $xz$ consider an arbitrary vertex, $w$. For a cost of $n-1$, $w$ must be used as an intermediary point in creating two-step paths for $xy$ and $xz$. Using the edge $xy$ or $xz$ and an edge between $y$ and $z$ creates the other two-step paths. This leaves one edge left that no matter the orientation, creates two 3-cycles. This yields a contradiction. □

**Theorem 2.4.2** *Suppose edges $e, f, g$ of $K_n$ neither form a cycle nor are independent. Then the cost of $K_n - e - f - g$ is minimized by a transitive tournament.*

*Proof.* Assume for contradiction that there is a non-transitive tournament that yields a cost of $K_n - e - f - g$ better than the minimum transitive tournament. The highest cost of three edges on a transitive tournament is $3n-5$ created by a path, so in order to have a cost better than a transitive tournament, $C(K_n - e - f - g, T_n)$ must be strictly less than $\binom{n+1}{3} - (3n-5)$. Replacing three edges can add at most $3(n-1)$ to the cost. Now the cost of $K_n$ is less than $\binom{n+1}{3} - (3n-5) + (3n-3)$ or $\binom{n+1}{3} + 2$. Since this upper bound is too high there is another tournament that can be found by reversing an edge, which lowers the cost by two. This yields a contradiction. In the case where the three edges form a star, the maximum cost is $(3n-6)$, the new cost of $K_n$ with the edges replaced is less than $\binom{n+1}{3}$ +3. After flipping an edge, the cost is lowered by at least two. If this flipped edge creates a transitive tournament as proven earlier, it was not the case that there were three edges of cost $n-1$. Otherwise, we can flip another edge to lower the cost again. Reassessing the cost on this nearly transitive tournament would give us an upper bound of $\binom{n+1}{3}$ $-(3n-6)+(3n-5)$ or $\binom{n+1}{3}$ +1. Here we can flip an edge and lower the cost by two, yielding a contradiction. If the maximum cost of three edges removed is $(3n-7)$, created by one independent edge and two edges that share a vertex, so that after replacing the edges, we find $\binom{Cost(K_n,T_n)<n+1}{3}$ +4. By flipping an edge in the arbitrary tournament, we can lower the cost by at least two. If the cost savings in one flip is greater than or equal to 4, we are done. If not, then assuming the tournament is not nearly transitive, two flips can be made to lower the cost yielding a contradiction. If the tournament is nearly transitive, then $\binom{Cost(K_n,T_n)<n+1}{3}$ +2. Here one edge can be flipped to lower the cost to yield a contradiction. □

**Conjecture 2.4.3** *If the edges removed from a complete graph create a cycle or are independent, $K_n-e-f-g$ can be minimized on a transitive tournament.*

# 3 More Types of Graphs

Now that we know that the transitive tournament minimizes the cost on complete and nearly complete graphs, we will consider other types of graphs: path graphs, star graphs, and cycle graphs. We prove the Pelsmajer conjecture for other simple families of graphs, as suggested in [6]. Note, in this section we do not consider $|E|$ in the cost.

## 3.1 Cost of Graphs with More Than One Component

To begin, we show that the costs of the components of the graph determine the cost of the graph.

**Lemma 3.1.1** Let $G_1, G_2, \ldots, G_k$ be the connected components of graph $G$, and let $T_1, T_2, \ldots, T_k$ be tournaments where $V(T_i) = V(G_i)$ for all $i$ in $\{1, 2, \ldots, k\}$. Then there exists a tournament $T$ on $V(G)$ so that $Cost(G,T) = \sum_{i=1}^{k} Cost(G_i, T_i)$.

*Proof.* Let $T$ be the tournament on vertex set $V(G)$ and $E(T) = E(T_1) \cup E(T_2) \cup \cdots \cup E(T_k) \cup \{u_i u_j | u \in V(G), i < j\}$. Then $Cost(G,T) = Cost(G_1,T_1) + Cost(G_2,T_2) + \cdots + Cost(G_k,T_k)$. □

**Corollary 3.1.2** The minimum cost of a graph is equal to the sum of the minimum costs of the components.

*Proof.* Let $G_1, G_2, \ldots, G_k$ be the connected components of graph $G$. Let $T_1, T_2, \ldots, T_k$ be underlying tournaments on $G_1, G_2, \ldots, G_k$ that minimize the cost on each respective component. Then $C(G) \leq C(G_1) + C(G_2) + \cdots + C(G_k) = \sum_{i=1}^{k} C(G_i)$.

Choose an arbitrary $T$. Consider the induced subgraphs on $T_1, T_2, \ldots, T_k$ from restricting $T$ to $V(G_1), V(G_2), \ldots, V(G_k)$. Then

$$C(G) \geq C(G_1,T_1) + C(G_2,T_2) + \cdots + C(G_k,T_k) \geq C(G_1) + C(G_2) + \cdots + C(G_k).$$

Therefore, $C(G) = \sum_{i=1}^{k} C(G_i)$. $\square$

## 3.2 Path Graphs

A *path* graph is a connected, acyclic graph in which each vertex has degree one or two.

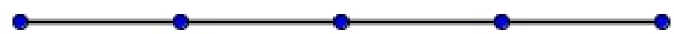

Figure 5: Path Graph on Five Vertices

We want to show that every graph with cost zero is a subgraph of a path. To prove this, it will suffice to show that if a graph does have zero cost, it must be acyclic and have only vertices with degree two or less.

**Lemma 3.2.1** *If $G$ is a graph for which $C(G) = 0$, then $G$ does not contain a vertex of degree greater than two.*

*Proof.* Consider a graph with $Cost(G, T_n) = 0$ for some $T_n$. Suppose for contradiction that vertex $v$ has degree at least three in a graph $G$. Let $x$, $y$, and $z$ be neighbors of $v$. Either the in-degree of $v$ is greater than one, or the out-degree of $v$ is greater than one. First suppose the in-degree of $v$ is at least two, where without loss of generality $y$ and $z$ are in-neighbors of $v$. If there is an edge from $y$ to $z$ in $T_n$, there is a two-step path from $y$ to $z$ to $v$, giving $vy$ a cost of at least one, which is a contradiction. Similarly, if $zy$ is an edge $T_n$, the two-step path from $y$ to $z$ to $v$ leads to a contradiction. So the in-degree of $v$ is at most one. But then the out-degree of $v$ is at least two. By symmetric argument, we similarly obtain a contradiction. $\square$

**Lemma 3.2.2** *If $G$ is a graph for which $C(G) = 0$, then $G$ does not contain a cycle.*

*Proof.* Consider a graph with $Cost(G, T_n) = 0$ for some $T_n$. Suppose for contradiction that $G$ contains an undirected $k$-cycle, $C = v_0, v_1, ..., v_{k-1}$. Suppose without loss of generality that $v_0$ defeats $v_1$. Since $v_i$ and $v_{i+1}$ are adjacent in $G$, in $T_n$ there cannot be a two-step path between $v_i$ and $v_{i+1}$. So if $v_0$ and $v_1$ both defeat $v_i$ for some $i$, $1 < i < k-2$, then $v_0$ and $v_1$ both defeat $v_{i+1}$. If $v_0$ and $v_1$ defeat all other vertices, there is a two-step path from either $v_0$ to $v_1$ to $v_{k-1}$. If $v_0$ and $v_1$ are defeated by all other vertices, there is a two-step path from $v_2$ to $v_0$ to $v_1$. We reach a contradiction either way, showing that the undirected graph will have a cost greater than zero. Therefore, in order to have cost zero, there cannot be a cycle within $G$. $\square$

Using these two lemmas, the following theorem follows easily.

**Theorem 3.2.3** *Every component of a graph $G$ for which $C(G) = 0$ is a path.*

## 3.3 Star Graphs

The next simple graph family we consider is the star graph. These graphs have a central vertex that is connected to all other vertices, as shown. These graphs are acyclic, and at most one vertex has degree greater than one. Instead of the radial pattern, we will discuss the star graph as the graph on the right in the figure.

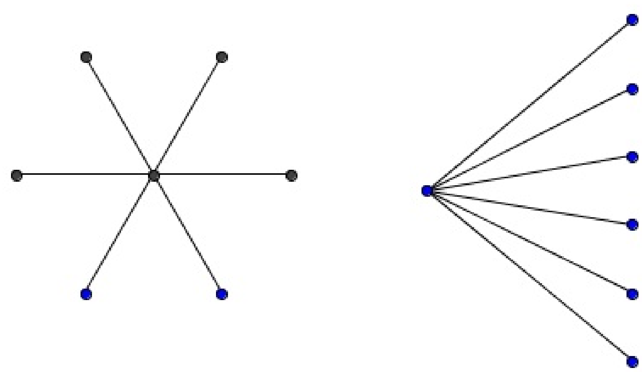

Figure 6: Two Representations of a Star Graph on Seven Vertices, $S_7$

The Pelsmajer conjecture is once again confirmed for this type of graph.

**Theorem 3.3.1** *The minimum cost of a tournament arrangement of a star graph is* $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor$.

*Proof.* Let $v_0$ be the central vertex of a star graph $S_n$, with underlying arbitrary tournament $T_n$ of least cost. Since each edge in the star graph has an endpoint at $v_0$, the only costs that will be assessed are the two-step paths in $T_n$ that start or end at $v_0$. If some vertex $x$ of $N^+(v_0)$ defeats a vertex $y$ of $N^-(v_0)$, replacing $xy$ by $yx$ will not increase the number of two-step paths that use $v_0$ as an endpoint. So, we may assume that every vertex of $N^-(v_0)$ defeats every vertex of $N^+(v_0)$. So, the only two-step paths that contribute to the total cost are the paths from $v_0$ to a vertex in $N^+(v_0)$ to another vertex in $N^+(v_0)$, and paths from a vertex in $N^-(v_0)$ to another vertex in $N^-(v_0)$ to $v_0$.

Thus the cost of $S_n$ relative to $T_n$ is the number of edges in $N^+(v_0)$ plus the number of edges in $N^-(v_0)$. Let $k$ be the number of vertices in $N^+(v_0)$, so $n-k-1$ is the number of vertices in $N^-(v_0)$. Thus,

$$Cost(S_n, T_n) = \binom{k}{2} + \binom{n-k-1}{2} = \frac{1}{2}\left[n^2 + 2k^2 - 3n - 2kn + 2k + 2\right].$$

Using methods of calculus, we find the cost is minimized when $k = \frac{n-1}{2}$ if $n$ is odd and when $k = \frac{n}{2}$ if $n$ is even. So, when $n$ is odd, the minimum is

$$\tfrac{1}{2}\left[n^2 + 2(\tfrac{n-1}{2}) - 3n - 2n(\tfrac{n-1}{2}) + 2(\tfrac{n-1}{2}) + 2\right] = \tfrac{1}{4}\left[n^2 - 4n + 3\right].$$

And when $n$ is even the minimum is

$$\tfrac{1}{2}\left[n^2 + 2(\tfrac{n}{2}) - 3n - 2n(\tfrac{n}{2}) + 2(\tfrac{n}{2}) + 2\right] = \tfrac{1}{4}\left[n^2 - 4n + 4\right].$$

Therefore, the minimum cost is $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor$ among underlying tournament structures. $\square$

**Corollary 3.3.2** *The minimum cost of a star graph can be obtained using a transitive tournament.*

*Proof.* Let $S_n$ be a star graph on $n$ vertices. Let $T_n$ be an underlying transitive tournament for $S_n$. If $n$ is even, choose $v_0$ so that $d^+(v_0) = \frac{n}{2}$. If $n$ is odd, choose $v_0$ so that $d^+(v_0) = \frac{n-1}{2}$. Note that every vertex in $N^-(v_0)$ defeats every vertex in $N^+(v_0)$ within $T_n$. So, by the proof of the previous theorem, the cost of $S_n$ under this particular transitive tournament is minimized. $\square$

## 3.4 Cycle Graphs

Sometimes we will refer to the transitive tournament as a linear ordering of the vertices since we can create a function, $\phi : V \to \{1, 2, \ldots, n\}$, that will label the vertices from the vertex with the highest number of wins to the lowest. Hence, by permuting $\phi$ we can obtain as close to a transitive tournament as possible.

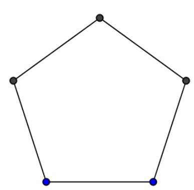

Figure 7: Cyclic Graph on Five Vertices

In order to prove the conjecture for the cycle graph, we first prove a statement about the cost of cycles relative to transitive tournaments.

**Theorem 3.4.1** *The cost of a transitive tournament underlying a cycle $C_n$ is at least $n-2$, including $|E|$ in our count.*

*Proof.* Let $C_n$ be an undirected cycle in a graph $G$. Let $T_n$ be a transitive tournament for $C_n$. Let $x_0$ and $n$ be distinct vertices in $C_n$. There are two paths that travel around the cycle from $x_0$ to $n$. Let one path consist of the vertices $x_0, x_1, x_2, ..., x_k, n$ for some $k \leq n-2$ and let the second path consist of $x_0, y_1, y_2, ..., y_{n-k-2}, n$. The cost of one path from $x_0$ to

$n$ is $\sum_{i=0}^{k} |x_i - x_{i+1}|$. Using the Triangle Inequality, this value is greater than or equal to $|\sum_{i=0}^{k} x_i - x_{i+1}|$. Since this sum is telescoping,

$$\sum_{i=0}^{k} |x_i - x_{i+1}| \geq |x_0 - x_{k+1}| = n - 1.$$

By symmetric argument, the second path from $x_0$ to $n$ also has total cost at least $n - 1$. Therefore the cost of $C_n$ is at least $2(n - 1)$. $\square$

Note that if the vertices are not ranked in increasing order, the cost will be strictly greater than $2(n - 1)$. So, for a transitive tournament, ranking the vertices in increasing order from $x_0$ to $n$ will give $G$ the lowest cost.

**Theorem 3.4.2** *The minimum cost of a cycle $C_n$ is $n - 2$.*

*Proof.* This proof is by induction on $n$; the base case $n = 3$ is easily verified. Let $e$ be an edge on $C_n$ with positive cost. Denote the vertices around the cycle as $1, 2, ..., n$. And let $e$ be the edge between 1 and $n$. Let $A$ denote the set of vertices in the middle of 2-step paths between 1 and $n$. If $|A| = n - 2$, then there is nothing left to prove. So we can suppose that there exist consecutive vertices $x$ and $y$ in $C_n$ with $x \in A$ and $y \notin A$ so there is a 2-step path between $x$ and $y$ through either $n$ or 1. Delete $n$ from $C_n$ and from $T_n$ and add an edge between 1 and $n - 1$ on $C_n - \{n\}$ to obtain $C_{n-1}$ and $T_{n-1}$. By the inductive hypothesis, $Cost(C_{n-1}, T_{n-1}) \geq n - 1 - 2 = n - 3$. Let $k$ denote the total cost of all edges in $C_n \cap C_{n-1}$ and let $l$ denote the cost on the edge between 1 and $n - 1$. Let $W$ denote the set of vertices in the middle of 2-step paths between 1 and $n - 1$. For each $w \in W$, if $w \in A$ then we have a 2-step path between 1 and $n$ through $w$. Otherwise we have a 2-step path between $n$ and $n - 1$ through $w$. Thus we add $|W|$ to the cost of $C_n$ relative to $T_n$. So $Cost(C_n, T_n) \geq |W| + k + 1 = l + k + 1 = Cost(C_{n-1}, T_{n-1}) + 1 \geq n - 2$. This completes the proof by induction. $\square$

# 4 Maximizing Cost

**Theorem 4.1.1:** *The maximum cost of a complete graph is $\frac{n(n-1)^2}{4}$ when $n$ is odd and is obtained by a regular tournament, and the maximum cost is $\frac{n(n^2-2n)}{4}$ when $n$ is even and is obtained by a nearly regular tournament.*

*Proof.* We will start in the same manner as [5]. Let $A$ denote the number of two-step paths in $T$. So

$$A = \sum_{uv \in E(T)} T(u, v) = \sum_{w \in V(T)} d^+(w)d^-(w). \tag{1}$$

We will denote the imbalance of $w$ as $i(w) = d^+(w) - d^-(w)$. Then

$$B = \sum_{w \in V(T)} i(w)^2 = \left(\sum_w d^+(w)^2\right) + \left(\sum_w d^-(w)^2\right) - 2A. \tag{2}$$

So

$$n(n-1)^2 = \sum_w (d^+(w) + d^-(w))^2 = \left(\sum_w d^+(w)^2\right) + \left(\sum_w d^-(w)^2\right) + 2A. \tag{3}$$

Using (1.2) and (1.3), we obtain $4A = n(n-1)^2 - B$. In [5], we are to maximize $B$ to get the lowest cost. Hence, we will minimize $B$ to obtain the highest cost.

If $B = i(w) = 0$, then $d^+(w) = d^-(w)$ for all $w \in V(T)$. Thus we have a regular tournament on an odd number of vertices, and we can see that $A = \frac{n(n-1)^2}{4}$.

If $i(w) = 1$ for all $w \in V(T)$, then $B = n$ for an even number of vertices. Thus we have that $A = \frac{n(n^2-2n)}{4}$. $\square$

# 5 Concluding Remarks

The conjecture of Pelsmajer et al. [5] has been proved for several families of simple graphs, but a general proof that would hold for all graphs has not been found. If the conjecture holds, the transitive tournament could then be used to analyze and find the cost of arbitrary graphs, so that one could look only at linear arrangements without having to consider non-transitive underlying tournaments. A counter example may be found by building a graph from the low cost edges relative to a cost matrix.

One possible direction for this topic would be to look at different cost systems. Instead of using the typical cost matrix in this case, one could square each cost entry to see how it would affect overall graph costs. Again it may be helpful to first know what tournaments minimize the cost for any graph.

Another research extension would be to look at similar problems with multigraphs or edge-weighted graphs. How is the cost on a multigraph minimized? Does the conjecture still hold? Using multigraphs would open this particular problem up to many new questions about even more families of graphs.

A suggestion for more research in this field could be to continue with the question of maximizing costs. It could be helpful to find and prove maximum cost values. If this cannot be solved in the general case, one could look into specific families of graphs.

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