How Many Ways Can You Divide Zero?

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Abstract

Let $\Sigma(R)$ be the graph whose vertices are the nonzero zero divisors of a ring R and whose edges are pairs $\{u, v\}$ where $u \neq v$ and $u + v$ is a zero divisor. We explore basic properties of sum graphs including degree, planarity, coloring, cycles, and isomorphisms by studying examples and finding various patterns.

1 Introduction

A graph is a mathematical structure consisting of points joined by lines. There are thousands of problems related to computer science that can be modeled by graphs, so the growth of computer use over the past 40 years has led to a corresponding development in graph theory. In this paper we focus on sum graphs. In essence, these are graphs whose vertices can be represented as a set S of positive integers, and have the property that vertex i and vertex j are adjacent if and only if $i + j$ is in S. Apart from giving rise to combinatorial problems, sum graphs are fascinating in their own right. They have potential applications in data compression, specifically compressed storage of graphs in memory.

2 Ring Theory Preliminaries

In order to understand our work with sum graphs, some ring definitions are helpful.

Definition 2.1. *A* **ring** *is a nonempty set R with two binary operations, $+$ and $*$ satisfying:*

- $(R, +)$ is an abelian group.
- $(R, *)$ is associative, commutative, and has an identity of 1.
- For all $a, b, c \in R$, the distributive law $a * (b + c) = (a * b) + (a * c)$ holds.

We work with the set $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$, which is a ring with respect to addition and multiplication modulo n. For convenience of notation we often omit the overhead bars.

Definition 2.2. *An* **integral domain** *is a ring R such that $a, b \in R$ and $ab = 0$ implies that either $a = 0$ or $b = 0$. Equivalently, $a \neq 0$ and $b \neq 0$ implies that $ab \neq 0$.*

Definition 2.3. *In a ring R, a* **zero divisor** *is an element $c \in R$ such that there exists $x \in R$, $x \neq 0$ where $cx = 0$. Clearly $0 \in R$ is always a zero divisor. We denote the set of zero divisors by $Z(R)$, and we let $Z(R)^* = Z(R) - \{0\}$ denote the set of nonzero zero divisors.*

Theorem 2.4. *[4, Theorem 19.3] In the ring $\mathbb{Z}_n$, the nonzero zero divisors are precisely those nonzero elements that are not relatively prime to n.*

Other than zero divisors, another important class of elements are units, which are helpful and key elements in our work.

Definition 2.5. *A* **unit** *is an element $u \in R$ that has a multiplicative inverse in R. That is, there exists $v \in R$ such that $uv = 1$.*

Theorem 2.6. *Let R be a finite ring. Then every element of R is either a zero divisor or a unit. In particular, every element of $\mathbb{Z}_n$ is either a zero divisor or a unit.*

Theorem 2.7. *(Chinese Remainder Theorem)[5, Corollary 2.26] If $m = p_1^{k_1} \cdots p_t^{k_t}$ ($k_i > 0; p_i$ distinct primes), then $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}$.*

3 Graph Theory Preliminaries

Our work is based on a graph-theoretic approach of studying zero divisors. Therefore, some basic definitions from graph theory are helpful.

Definition 3.1. *A* **graph** *G consists of a vertex set $V(G)$, an edge set $E(G)$, and an association to each edge, $e \in E(G)$, of two vertices called the* **endpoints** *of e.*

Definition 3.2. *The* **complement** *of a graph G, denoted $\bar{G}$, is the graph whose vertex set is $V(G)$ but for which $\{x, y\} \in E(\bar{G})$ if and only if $\{x, y\} \notin E(G)$*

An **adjacent vertex** of a vertex v in a graph is a vertex that is connected to v by an edge. The **neighborhood** of a vertex v, denoted $N(v)$, is the set of all vertices adjacent to v. An edge is **incident** at a vertex if that vertex is one of its endpoints. The **degree** of a vertex v is the number of edges incident at it, denoted $deg(v)$. A **dominating vertex** is a vertex in a graph that is adjacent to every other vertex of G.

Definition 3.3. *An* **independent set** *in a graph is a set of pairwise nonadjacent vertices.*

Definition 3.4. *A* **clique** *in a graph is a set of pairwise, adjacent vertices.*

Definition 3.5. *The* **complete graph** *on n vertices, denoted K_n, is a graph such that each pair of distinct vertices is adjacent.*

Definition 3.6. *A graph G is* **bipartite** *if $V(G)$ is the union of two disjoint independent sets called partite sets of G. The* **complete bipartite** *graph, denoted $K_{n,m}$, is a bipartite graph whose independent sets have size n and m, respectively, and has the maximum number of edges possible.*

3.1 Sum Graphs

We focus our research on sum graphs.

Definition 3.7. *Let R be a ring. Define the* **sum graph** *$\Sigma(R)$ as follows:*

$$V(\Sigma(R)) = Z(R)^*;$$

$$\{x, y\} \in E(\Sigma(R)) \Longleftrightarrow x + y \in Z(R).$$

4 Degrees

Based on examples of sum graphs we can see patterns that lead to some interesting propositions. In this section we focus on propositions that deal with the degrees of vertices in sum graphs.

Proposition 4.1. *Let $R = \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_r^{e_r}}$, where $\max\{e_1, \ldots, e_r\} \geq 2$ and $n = |V(\Sigma(R))|$. Then $\triangle(\Sigma(R)) = n - 1$.*

Proof. Let R and n be as above. Suppose that $e_1 > 1$. Consider $x = (p_1, 0 \ldots, 0) \in V(\Sigma(R))$. Now take another vertex $y = (b_1, b_2, \ldots, b_r)$. As $y \in V(\Sigma(R))$ at least one $b_j = cp_j, c \in \mathbb{Z}$.
If $j = 1$, then $x + y = (p_1(1 + c), b_2, \ldots, b_r)$ This implies that $x + y \in V(\Sigma(R))$. Now if $j > 1$, then $x + y = (p_1 + b_1, b_2, \ldots, b_{j-1}, cp_j, b_{j+1}, \ldots, b_r)$. This implies that $x + y \in V(\Sigma(R))$. Therefore x is adjacent to all other vertices in $\Sigma(R)$. This implies that $deg(x) = n - 1$. Therefore $\triangle(\Sigma(R)) = n - 1$

$\square$

Proposition 4.2. *Let $R = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$, where $p_i \le p_{i+1}$, p_i prime for $1 \le i \le r$ and $n = |V(\Sigma(R))|$. Then, $\triangle(\Sigma(R)) = n - (\prod_{i=1}^{r-1}(p_i - 1)) - 1$.*

Proof. Let p_i and R be as above. Let $w = (0, \ldots, 0, 1) \in V(\Sigma(R))$. Then the degree of w is n minus the number of vertices that are not adjacent to w, minus one. Now take $x \in V(\Sigma(R))$ a vertex that is not adjacent to w; this implies that $x = (u_1, ..., u_{r-1}, 0)$, where u_i is a unit in $\mathbb{Z}_{p_i}$, $\forall 1 \le i \le r-1$. Thus there are exactly $\prod_{i=1}^{r-1}(p_i - 1)$ vertices not adjacent to w. This implies that $deg(w) = n - (\prod_{i=1}^{r-1}(p_-1)) - 1$.

Now take $y = (y_1, \ldots, y_r) \in V(\Sigma(R))$. Let $J = \{i | y_i \ne 0\}$. Let $z = (z_1, \ldots, z_r) \in V(\Sigma(R))$ not adjacent to y. This implies that $z_i \ne 0$ for $i \notin J$ and $z_j \ne -y_j$ for $j \in J$. As $z \in V(\Sigma(R))$, $z_j = 0$ for some $j \in J$. Let $T = \{z | \text{such that z is not adjacent to y}\}$. Take $T_j \subset T$, $T_j = \{z \in T | z_j = 0\}$. Then, $|T| \ge |T_j| = \prod_{k \ne j}(p_k - 1)$. This implies that $deg(y) \le n - (\prod_{k \ne j}(p_k - 1)) - 1 \le deg(w)$.
Therefore $\triangle(\Sigma(R)) = n - (\prod_{s=1}^{r-1}(p_s - 1)) - 1$

□

Proposition 4.3. *$\Sigma(\mathbb{Z}_n)$ is a complete graph $\iff n = p^r$ where p is prime and $r \ge 2$*

Proof. ($\Leftarrow$) Suppose $n = p^r$ where p is prime. The vertices of $\Sigma(\mathbb{Z}_{p^r})$ are multiples of p so any vertex in the sum graph can be written in the form $v_i = kp$ for some $k \in \mathbb{N}$. Then $v_1 + v_2 = k_1 p + k_2 p = (k_1 + k_2)p$. Since any multiple of p is in $V(\Sigma(\mathbb{Z}_{p^r}))$, then $v_i, v_j \in V(\Sigma(\mathbb{Z}_{p^r}))\ \forall\ i, j \in \mathbb{N}$.
($\Rightarrow$) Let $\Sigma(\mathbb{Z}_n)$ be a complete graph and suppose $n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$,$p_i$ distinct primes and $r \ge 2$. Then by the Chinese Remainder Theorem, $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \times \ldots \times \mathbb{Z}_{p_r^{e_r}}$. Suppose $v_1, v_2 \in V(\Sigma(\mathbb{Z}_n))$, $v_1 = (1, 0, \ldots, 0)$ and $v_2 = (0, 1, \ldots, 1)$. Then $v_1 + v_2$ must have a zero divisor in one coordinate in order for there to be an edge between them, but $v_1 + v_2 = (1, \ldots, 1)$ and 1 is a unit in $\mathbb{Z}_{p_i} \forall 1 \le i \le r$. This means that there is no edge between v_1 and v_2, which is a contradiction.

□

Proposition 4.4. *In $\mathbb{Z}_n$, let $n = p_1 p_2$ where p_i is prime and $p_1 < p_2$. The vertices with maximum degree will be of the form $v = kp_1\ \forall\ k \in \mathbb{N}$.*

Proof. The number of vertices in the neighborhood of p_1 is equal to $\frac{n}{p_1} - 2 = p_2 - 2$ and the size of the neighborhood of p_2 is equal to $p_1 - 2$. Because p_i is only adjacent to the vertices of the form $v = kp_i\ \forall\ k \in \mathbb{N}$, $|N(p_i)| = |N(kp_i)|$ [3]. Since $p_1 < p_2$, $p_2 - 2 > p_1 - 2$ so $|N(p_1)| > |N(p_2)|$. This means that any vertex of the form $v = kp_1$ $\forall\ k \in \mathbb{N}$, will have maximum degree.

□

Proposition 4.5. *In $\mathbb{Z}_n$, let $n = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_r}^{e_r}$ where p_i is prime, $e_i > 1$ for some i, $r \geq 2$ and $p_1 < p_2 < \ldots < p_r$. The vertices with maximum degree will be of the form $v = kp_1p_2 \ldots p_r$ $\forall$ $k \in \mathbb{N}$.*

Proof. If we let $v = kp_1 \cdots p_r$, then v will be adjacent to every vertex in the sum graph because it is a multiple of all the p_i and $v \neq n$. If w is not of the form $kp_1 \cdots p_r$, then there exists p_i such that $p_i \nmid w$. Then it is not adjacent to vertices of the form $w = tp_i$ $\forall$ $t \in \mathbb{N}$. So v is a dominating vertex if and only if v is a multiple of all p_i. $\square$

5 Connectedness

Lemma 5.1. *Let $R_1, \ldots, R_r$ be rings and $R = R_1 \times \cdots \times R_r$. Then for any i, $1 \leq i \leq r$ the set of vertices of $\Sigma(R)$ with zero in the i^{th} coordinate forms a clique.*

Proof. Let $R = R_1 \times \cdots \times R_r$ be a ring with S the set of all vertices with 0 in the i^{th} coordinate. Let $a, b \in S$. Then, $a + b$ will also have a zero in the i^{th} coordinate and thus $a + b \in S$. Therefore, S is a clique. $\square$

Theorem 5.2. *Let $R = R_1 \times R_2$ where each R_i is an integral domain. Then $\Sigma(R)$ is disconnected and is the union of two cliques with respective size $|R_1| - 1$ and $|R_2| - 1$.*

Proof. Since R_1 and R_2 are integral domains, R_1 and R_2 have no nonzero zero divisors. Thus $Z(R)^* = S_1 \cup S_2$ where $S_1 = \{(a, 0) : a \neq 0\}$ and $S_2 = \{(0, b) : b \neq 0\}$. By Lemma 5.1 S_1 and S_2 are cliques.
To show that $\Sigma(R)$ is disconnected, suppose there exist adjacent vertices $(a, 0)$ and $(0, b)$. It follows that $(a, 0) + (0, b) = (a, b) \in Z(R)$. Since R_1 and R_2 are integral domains, nonzero zero divisors take the form $(a, 0)$ or $(0, b)$. Therefore $(a, b) \notin Z(R)$, and thus S_1 and S_2 are connected components of $\Sigma(R)$.
Now we need to show $|S_1| = |R_1| - 1$ and $|S_2| = |R_2| - 1$. Every vertex in S_1 is of the form $(a, 0)$. There are $|R_1|$ choices for a, but $0 \in R_1$, and $(0, 0) \notin S_1$. So there are $|R_1| - 1$ vertices in S_1. Similarly, there are $|R_2| - 1$ vertices in S_2.

$\square$

Proposition 5.3. *Let $R = R_1 \times R_2 \cdots \times R_r$ where each R_i is a ring and $|R_1| = n_1, \ldots, |R_r| = n_r$. If $r \geq 3$, then $diam(\Sigma(R)) \leq 2$. In particular, $\Sigma(R)$ is connected.*

Proof. We give a proof in the case $r = 3$; the general case follows similarly.
Let $r = 3$, so $R = R_1 \times R_2 \times R_3$ and let $x, y \in Z(R)^*$ be distinct vertices where $x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$. If $x \leftrightarrow y$ then $d(x, y) = 1$. Suppose $x \not\leftrightarrow y$. In order for $d(x, y) \leq 2$, there must be a $z \in V(Z(R))$ such that $x \leftrightarrow z \leftrightarrow y$.

Choose $z = (-a_1, 0, -b_3)$. If $x = z$ or $y = z$, then $d(x, y) = 1$. Now assume $z \neq x$ and $z \neq y$. If $a_1 \neq 0$ or $b_3 \neq 0$, then $x \leftrightarrow z \leftrightarrow y$. If $a_1 = 0$ and $b_3 = 0$, $x = (0, a_2, a_3)$ and $y = (b_1, b_2, 0)$. Now we can choose $z' = (0, 1, 0)$ such that $x \leftrightarrow z' \leftrightarrow y$ and $d(x, y) = 2$. Therefore, when $r = 3$, $diam(R) \leq 2$.

$\square$

6 Isolated and Pendant Vertices

Proposition 6.1. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_p$, $p \neq 2$ prime, then $\Sigma(R)$ is the disjoint union of a clique and an isolated vertex.*

Proof. Since 2 and p are primes we know that $V(R) = \{(1, 0)\} \cup \{(0, b) : b \neq 0 \in \mathbb{Z}_p\}$. Also all the vertices of the form $(0, b)$ such that $b \in \mathbb{Z}_p$ are pairwise adjacent; therefore, they form a clique. The sum $(1, 0) + (0, b)$ gives us a unit $(1, b)$, therefore $(1, b) \notin V(Z(R))$. Hence $\Sigma(R)$ is the disjoint union of a clique and the isolated vertex, $(1, 0)$.

$\square$

Proposition 6.2. *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_p$, $p \neq 3$ prime, then $\Sigma(R)$ contains 2 pendant vertices, namely $(1, 0)$ and $(2, 0)$.*

Proof. Since 3 and p are primes we know that $V(R) = \{(1, 0), (2, 0)\} \cup \{(0, b) : b \neq 0 \in \mathbb{Z}_p\}$. Also all the vertices of the form $(0, b)$ are pairwise adjacent, therefore they form a clique. The sum $(a, 0) + (0, b)$ where $b \in \mathbb{Z}_p^*$ and $a \in \mathbb{Z}_3^*$ gives us units $(2, b)$ or $(1, b), b \in \mathbb{Z}_p$, therefore $(2, b)$ and $(1, b) \notin V(Z(R))$. But $(2, 0) + (1, 0) \in V(Z(R))$, so $(1, 0)$ and $(2, 0)$ are adjacent pendant vertices and are not adjacent to any of the vertices of the clique.

$\square$

Proposition 6.3. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_p$, where $p \neq 2$ is prime. Then $\overline{\Sigma(R)}$ is a star with $|V(R)| - 1$ pendant vertices.*

Proof. By Proposition 6.1 we know that the vertex $(1, 0)$ is an isolated vertex in the sum graph $\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ where $p \neq 2$ is a prime, and that the rest of the vertices are pairwise adjacent. Therefore, in the complement of the sum graph we have that the vertex $(1, 0)$ is adjacent to the rest of the vertices. Hence there are $|V(R)| - 1$ pendant vertices.

$\square$

Proposition 6.4. *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_p$, $p \neq 3$ prime, then $\overline{\Sigma(R)}$ contains 2 vertices of degree $|V(R)| - 2$ and the rest of the vertices are of degree 2.*

Proof. By Proposition 6.2 we know that the vertices $(1,0)$ and $(2,0)$ are pendant vertices in the sum graph $\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_p)$ where $p \neq 3$ is a prime, and that the rest of the vertices are pairwise adjacent. Therefore, in the complement of the sum graph we have that the vertex $(1,0)$ is adjacent to the rest of the vertices except $(2,0)$. The same situation holds for the vertex $(2,0)$. Hence there are $|V(R)|-2$ vertices of degree 2 and 2 vertices of degree $|V(R)|-2$.

□

Proposition 6.5. *Let $R = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$, where $r \geq 3$ and $p_i \geq 2$ is prime, $1 \leq i \leq r$. Then $\overline{\Sigma(R)}$ and $\Sigma(R)$ do not contain isolated vertices.*

Proof. Let $n = |V(\Sigma(R))|$. We will first examine $\Sigma(R)$. By Proposition 5.3 we know that $\Sigma(R)$ is connected therefore, it does not have an isolated vertex. Now, by Proposition 4.2, we know that the max degree is given by $n - (\prod_{s=1}^{r-1}(p_s - 1)) - 1$ but since $p_i \geq 2$ for all i, we know that the maximum degree of $\Sigma(R)$ is at most $n-2$ therefore none of the vertices are isolated in $\overline{\Sigma(R)}$.

□

Proposition 6.6. *Let $R = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r} \times \mathbb{Z}_{p^n}$, where $m_i, n \geq 2, r \in \mathbb{N}$ and p is a prime. Then $\overline{\Sigma(R)}$ has at least one isolated vertex given by $v_0 = (0_1, 0_2, \cdots, 0_r, p^{n-1})$.*

Proof. We prove this by contradiction. Let R and v_0 be as above. Now suppose that v_0 is not a dominating vertex in $\Sigma(R)$, which implies there exists $v_2 \in V(\Sigma(R))$ such that $v_0 + v_2 = v_1$ where $v_1 = (u_1, u_2, \cdots, u_r, u_{r+1})$ with u_j is a unit for $1 \leq j \leq r+1$. Hence $v_2 = (u_1, u_2, \cdots, u_r, u_{r+1} - p^{n-1})$. Since u_{r+1} is a unit in $\mathbb{Z}_{p^n}$, $p \nmid u_{r+1}$ and $p^{n-1} | p^n$, therefore we have that $p \nmid u_{r+1} - p^{n-1}$. This implies that $u_{r+1} - p^{n-1}$ is a unit in $\mathbb{Z}_{p^n}$ and therefore $v_2 \notin V(\Sigma(R))$ which is a contradiction. Hence v_0 is a dominating vertex of $\Sigma(R)$ and an isolated vertex in the complement $\Sigma(R)$.

□

7 Coloring

Definition 7.1. *Let G be a graph. A* **k − coloring** *of G is a map $f : V(G) \to \{1, 2, \cdots, k\} = [k]$. A k-coloring is called* **proper** *if adjacent vertices are assigned different colors. G is called* **k − colorable** *if there exists a proper k-coloring. The chromatic number of G is denoted by $\chi(G)$ and is given by*

$$\chi(G) = \min\{k : G \text{ is } k\text{-colorable}\}.$$

Lemma 7.2. *For any graph G, $\chi(G) \geq \omega(G)$.*

Proposition 7.3. *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_p$, $p \neq 2$ prime, then $\chi(\Sigma(R)) = p - 1$.*

Proof. By Proposition 6.2 we know that the $\Sigma(R)$ is a disjoint union of a clique with $p-1$ vertices and a single edge. Therefore $\chi(\Sigma(R)) \geq p - 1$. In a proper coloring of $\Sigma(R)$, $p-1$ distinct colors must be used on the clique, but any two distinct colors used there can be used to color the remaining vertices. Hence $\chi(\Sigma(R)) = p - 1$. □

Proposition 7.4. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_p$, p prime, then $\chi(\Sigma(R)) = p - 1$.*

Proof. By Proposition 6.1 we know that the $\Sigma(R)$ is a disjoint union of a clique with $p-1$ vertices and a vertex. Therefore $\chi(\Sigma(R)) \geq p-1$. In a proper coloring of $\Sigma(R)$, $p-1$ distinct colors must be used on the clique, but any color can be used to color the remaining vertex. Hence $\chi(\Sigma(R)) = p - 1$. □

Proposition 7.5. *Let $R = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \neq 2$ prime, then $\chi(\Sigma(R)) = p - 1$.*

Proof. We know $\Sigma(R)$ is a graph composed of two components that are cliques, since p is prime each clique contains $p-1$ vertices and therefore we need at least $p-1$ colors for a proper coloring. Since the two cliques are disjoint we have that $\chi(\Sigma(R)) = p-1$. □

By studying the complement graphs of the results just mentioned, we are able to see some patterns that led us to the following general proposition.

Proposition 7.6. *Let G be a graph which is the disjoint union of two cliques. Then $\chi(\overline{G}) = 2$.*

Proof. Let G be a graph which is the disjoint union of two cliques. Then $\overline{G}$ is complete bipartite since nonedges become edges and vice versa. Now since $V(\overline{G})$ consists of two independent sets, S_1 and S_2, we can assign each vertex in S_i the same color. Now since all the vertices in S_2 are adjacent to those in S_1, in order to obtain a proper coloring we need at least two colors. Hence, $\chi(\overline{G}) = 2$. □

8 Planarity

Definition 8.1. *A graph G is* **planar** *if G can be drawn in the plane with no edge crossings.*

Definition 8.2. *An elementary* **subdivision** *of a graph G is a graph obtained by splitting an edge of G. Two graphs are* **homeomorphic** *if one can be obtained from the other by elementary subdivisions.*

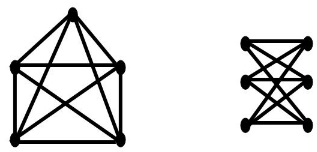

Figure 1: K_5 and $K_{3,3}$

Theorem 8.3. *(Kuratowski's Theorem) A graph G is planar if and only if G contains no subgraph homeomorphic to K_5 or $K_{3,3}$ See Figure 1. [2]*

Proposition 8.4. *Let $R = \mathbb{Z}_n \times \mathbb{Z}_m$. If $\max\{n, m\} \geq 6$, $\Sigma(R)$ is nonplanar.*

Proof. The set of nonzero zero divisors includes $S_1 = \{(a, 0) : a \neq 0\}$ and $S_2 = \{(0, b) : b \neq 0\}$. The order of S_1 is $n - 1$. Similarly, there are $m - 1$ vertices in S_2. Without loss of generality, assume $n \geq m$. Therefore, S_1 has more vertices than S_2. By Lemma 5.1, S_1 is a clique. Since $n \geq 6$, $|V(S_1)| \geq 6-1 = 5$. Therefore, S_1 contains a 5-clique and thus S_1 contains a subgraph isomorphic to K_5. By Kuratowski's Theorem, $\Sigma(R)$ is nonplanar.

□

Proposition 8.5. *Let $R = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}$. If $\max\{n_1, n_2, n_3\} \geq 3$, then $\Sigma(R)$ is nonplanar.*

Proof. Let R be a ring such that $R = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}$ and assume without loss of generality $n_1 \leq n_2 \leq n_3$. The set of nonzero zero divisors will include $S = \{(0, a_2, a_3) | a_2 \in \mathbb{Z}_{n_2}, a_3 \in \mathbb{Z}_{n_3}\} - \{(0, 0, 0)\}$. By Lemma 5.1, S is a clique. Since $n_3 \geq 3$, there are at least three options for a_3 and two options for a_2. Thus, any vertex in S is adjacent to at least $3 * 2 - 1 = 5$ other vertices in S. Therefore S contains a 5-clique and thus S contains a subgraph isomorphic to K_5. By Kuratowski's Theorem, since S is nonplanar, then $\Sigma(R)$ is nonplanar.

□

Proposition 8.6. *Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ be a ring. If $r \geq 4$, then $\Sigma(R)$ is nonplanar.*

Proof. Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ be a ring with $r \geq 4$.
When $r = 4$, the set of nonzero zero divisors will include the vertices $(0,0,0,1)$, $(0,0,1,0)$, $(0,1,0,0)$, $(1,0,0,0)$, $(1,1,0,0)$. These vertices form a 5-clique. Thus $\Sigma(R)$ contains a subgraph isomorphic to K_5 and therefore, by Kuratowski's Theorem, $\Sigma(R)$ is nonplanar. A similar argument holds for $r \geq 4$ by simply adding zeros to the additional coordinates.

□

This leaves only a finite number of rings of the form $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$ that could give rise to planar sum graphs.
Let $R_1 = \mathbb{Z}_3 \times \mathbb{Z}_4$, $R_2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, and $R_3 = \mathbb{Z}_5 \times \mathbb{Z}_5$ be rings. The sum graphs of these rings contain subgraphs isomorphic to K_5 and therefore $\Sigma(R_1)$, $\Sigma(R_2)$, and $\Sigma(R_3)$ are nonplanar.

Proposition 8.7. *The only sum graphs of rings of the form $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$ where $r \geq 2$ that are planar are the following: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_5 \times \mathbb{Z}_2$, $\mathbb{Z}_5 \times \mathbb{Z}_3$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. See Figures 2 - 5. □

To check for the planarity of $\Sigma(\mathbb{Z}_n)$ where $n = p_1^{e_1} \cdots p_r^{e_r}$ and p_i are distinct primes, by the Chinese Remainder Theorem and our work above, we can reduce to the case where $n = p^e$ for some prime p. By the definition for sum graphs given in [3], any nonplanar graph will still be nonplanar by our definition because the graph will only have more edges. Hence by [3, Corollary 5.16], we only have to check when $n = 4, 8, 9, 25, 49$. For $n = p^e$, $\Sigma(\mathbb{Z}_n)$ is a complete graph by Proposition 4.3. When $n = 49$, $\Sigma(Z_n)$ contains a subgraph isomorphic to K_5 and therefore is nonplanar. When $n = 4, 8, 9, 25$, $\Sigma(\mathbb{Z}_n)$ has $1, 3, 2, 4$ vertices, respectively, and therefore does not contain a subgraph isomorphic to K_5. Therefore, when $n = 4, 8, 9, 25$, $\Sigma(\mathbb{Z}_n)$ is planar.
The same arguments show that for a finite direct product of any finite rings, if the number of factors is greater than or equal to 4, if there are three factors, R_1, R_2, R_3 where $\max\{|R_1|, |R_2|, |R_3|\} \geq 3$, and if there are two factors with one having size of at least 6, then the sum graphs of these rings are nonplanar.

Proposition 8.8. *Let $R \cong R_1 \times \cdots \times R_r$ where R_i is a finite local ring. Let m_i be the unique maximal ideal of R_i. If some m_i has six elements or more, $\Sigma(R)$ is nonplanar.*

Proof. Let $R \cong R_1 \times \cdots \times R_r$ where R_i is a finite local ring. Let m_i be the unique maximal ideal of R_i such that $|m_i| \geq 6$; m_i consists of all the zero divisors of R_i.

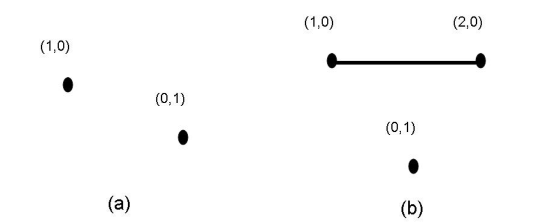

Figure 2: (a) $\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and (b) $\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_2)$

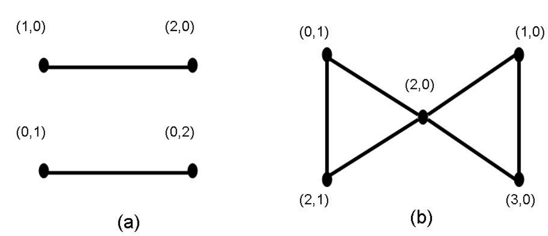

Figure 3: (a) $\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and (b) $\Sigma(\mathbb{Z}_4 \times \mathbb{Z}_2)$

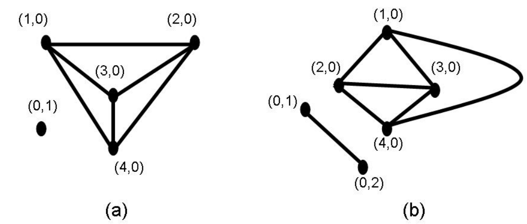

Figure 4: (a) $\Sigma(\mathbb{Z}_5 \times \mathbb{Z}_2)$ and (b) $\Sigma(\mathbb{Z}_5 \times \mathbb{Z}_3)$

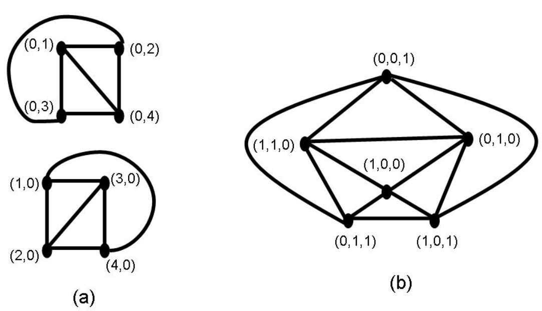

Figure 5: (a) $\Sigma(\mathbb{Z}_5 \times \mathbb{Z}_5)$ and (b) $\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Therefore, $V(R_i) = m_i - \{0\}$. Since every ideal is an Abelian group under addition, m_i is closed under addition. Therefore, for every $a, b \in V(R_i)$, $a + b \in V(R_i)$. Since there are at least six vertices in m_i, five of them are nonzero. Therefore $\Sigma(R_i)$ contains a 5-clique and thus contains a subgraph isomorphic to K_5 which implies that $\Sigma(R)$ is nonplanar.

□

The finite local rings whose maximal ideal has size less than six as follows:

- Rings with $|\mathfrak{m}| = 2$: $\mathbb{Z}_4$ and $\dfrac{\mathbb{Z}_2[x]}{(x^2)}$.
- Rings with $|\mathfrak{m}| = 3$: $\mathbb{Z}_9$ and $\dfrac{\mathbb{Z}_3[x]}{(x^2)}$.
- Rings with $|\mathfrak{m}| = 4$: $\mathbb{Z}_8$, $\dfrac{\mathbb{Z}_4[x]}{(2x, x^2-2)}$, $\dfrac{\mathbb{Z}_2[x,y]}{(x^2, y^2, xy)}$, $\dfrac{\mathbb{Z}_2[x]}{(x^3)}$, $\dfrac{\mathbb{Z}_4[x]}{(x^2)}$, $\dfrac{\mathbb{Z}_2[x,y]}{(x^2, y^2+y+1)}$.
- Rings with $|\mathfrak{m}| = 5$: $\mathbb{Z}_{25}$ and $\dfrac{\mathbb{Z}_5[x]}{(x^2)}$.

Definition 8.9. *A graph G is* **outerplanar** *when G is planar and in some planar drawing every vertex of G is on the boundary of the unbounded face.*

Corollary 8.10. *(To Kuratowski's Theorem) A graph G is outerplanar if and only if G contains no subgraph homeomorphic to K_4 or $K_{2,3}$ See Figure 6.*

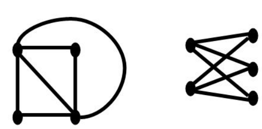

Figure 6: K_4 and $K_{2,3}$

Since a graph must be planar to be outerplanar, there are only a finite number of rings of the form $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$ where $r \geq 2$ that could have outerplanar sum graphs. As seen in Figures 4 and 5, of the eight planar sum graphs we have found, $\Sigma(\mathbb{Z}_5 \times \mathbb{Z}_2)$, $\Sigma(\mathbb{Z}_5 \times \mathbb{Z}_3)$, $\Sigma(\mathbb{Z}_5 \times \mathbb{Z}_5)$, and $\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ all have subgraphs isomorphic to K_4. Therefore, these graphs are not outerplanar. This leaves $\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_2)$, $\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)$, and $\Sigma(\mathbb{Z}_4 \times \mathbb{Z}_2)$ as the only sum graphs of this form that are outerplanar, which can be seen in Figures 2 and 3. For the case where $r = 1$, $\Sigma(\mathbb{Z}_4)$, $\Sigma(\mathbb{Z}_8)$, and $\Sigma(\mathbb{Z}_9)$ are outerplanar.

9 Cycles and Girth

Definition 9.1. *A* **path** *is a sequence of vertices and edges with no repeated internal vertices. A path is called* **closed** *if the initial and final vertices conicide. A* **cycle** *is a closed path. The* **length** *of a cycle is the number of edges in the cycle. The smallest possible length of a cycle is* 3.

Lemma 9.2. *Let* $R = \mathbb{Z}_n \times \mathbb{Z}_m$. *Then* $\Sigma(R)$ *has a cycle if and only if* $\max\{n, m\} \geq 4$.

Proof. ($\Leftarrow$) Let $R = \mathbb{Z}_n \times \mathbb{Z}_m$ be a ring with $\max\{n, m\} \geq 4$. The nonzero zero divisors of $\Sigma(R)$ will include $S_1 = \{(a, 0) : a \neq 0\}$ and $S_2 = \{(0, b) : b \neq 0\}$. The order of S_1 is $n - 1$ and the order of S_2 is $m - 1$. Without loss of generality, assume $n \geq m$. By Lemma 5.1, S_1 is a clique and has 3 or more vertices. Thus, there are 3 pairwise adjacent vertices in $\Sigma(R)$ which shows that $\Sigma(R)$ contains a 3-cycle.
($\Rightarrow$) As can be seen by Figures 2 and 3 (a), $\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_2)$, and $\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ do not have cycles. □

Definition 9.3. *The* **girth** *of a graph* G *is the length of the shortest cycle in* G *or* ∞ *if there are no cycles in* G. *The girth of* G *is denoted* $gr(G)$.

Proposition 9.4. *Let* $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$. *If* $\Sigma(R)$ *is nonplanar, then* $gr(\Sigma(R)) = 3$.

Proof. Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ with $\Sigma(R)$ a nonplanar graph. The results from Section 8 show if $\Sigma(R)$ is nonplanar, then $\Sigma(R)$ contains a subgraph isomorphic to K_5. Since K_5 is a 5-clique, it contains a 3-cycle. Therefore, $gr(\Sigma(R)) = 3$.

□

Proposition 9.5. *Let* $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$. *Then* $gr(\Sigma(R))$ *is either* 3 *or* ∞.

Proof. By Proposition 9.4, if $\Sigma(R)$ is nonplanar, then $gr(\Sigma(R)) = 3$. This leaves finitely many planar sum graphs to check. By Lemma 9.2, $\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_2)$, and

$\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ do not have any cycles and therefore $gr(\Sigma(R)) = \infty$. Also, $gr(\Sigma(\mathbb{Z}_4)) = \infty$ and $gr(\Sigma(\mathbb{Z}_9)) = \infty$ because they are isomorphic to K_1 and K_2, respectively. Since $\Sigma(\mathbb{Z}_8)$ is isomorphic to K_3 and $\Sigma(\mathbb{Z}_{25})$ is isomorphic to K_4, these graphs have girth of 3. The rest of the planar graphs, namely $\Sigma(\mathbb{Z}_4 \times \mathbb{Z}_2)$, $\Sigma(\mathbb{Z}_5 \times \Sigma\mathbb{Z}_2)$, $\Sigma(\mathbb{Z}_5 \times \mathbb{Z}_3)$, $\Sigma(\mathbb{Z}_5 \times \mathbb{Z}_5)$, and $\Sigma\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, have girth of 3 as can be seen by Figures 3 (b), 4, and 5. $\square$

Lemma 9.6. *[7] Let G be a graph. Then $gr(G) \leq 2 * diam(R) + 1$*

Proposition 9.7. *Let $R = R_1 \times R_2 \times \cdots \times R_r$ where each R_i is a ring and $r \geq 3$. Then $gr(\Sigma(R)) \leq 5$.*

Proof. By Proposition 5.3, $diam(\Sigma(R)) \leq 2$. Now by Lemma 9.6, $gr(\Sigma(R)) \leq 2*2+1 = 5$ $\square$

By considering different examples, we find $gr(\overline{\Sigma(R)})$ is not as easily determined as $gr(\Sigma(R))$.

Proposition 9.8. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_p$. Then $gr(\overline{\Sigma(R)}) = \infty$.*

Proof. By Proposition 6.3 we know that $\overline{\Sigma(R)}$ is a star, therefore it does not have cycles. Hence $gr(\overline{\Sigma(R)}) = \infty$. $\square$

Proposition 9.9. *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_p$, with $p \neq 3$ is prime. Then $gr(\overline{\Sigma(R)}) = 4$.*

Proof. By Proposition 6.4 we know that $\overline{\Sigma(R)}$ is bipartite. From this, it follows that $gr(\overline{\Sigma(R)}) \neq 3$. There is a 4-cycle $(1,0),(0,1),(2,0),(0,2),(1,0)$ in $\overline{\Sigma(R)})$. This shows that $gr(\overline{\Sigma(R)}) = 4$. $\square$

We find $\overline{\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_6)}$ is an interesting case because $gr(\overline{\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_6)} = 6$ as can be seen by Figure 7.

10 Isomorphisms

Definition 10.1. *Let G and H be graphs. An* **isomorphism** *is a bijective function*

$$f : V(G) \to V(H)$$

such that

$$xy \in E(G) \Leftrightarrow f(x)f(y) \in E(H).$$

Definition 10.2. *The* **clique number**, *denoted $\omega(G)$, is the* $\max\{|S| : S \subseteq V(G), \textit{where } S \textit{ is a clique}\}$.

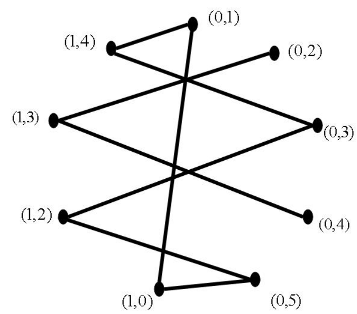

Figure 7: $\overline{\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_6)}$

Proposition 10.3. *Let $p_1, p_2, p_3,$ and p_4 be distinct primes. Then $\Sigma(\mathbb{Z}_{p_1p_2})$ is not isomorphic to $\Sigma(\mathbb{Z}_{p_3p_4})$.*

Proof. Assume $\Sigma(\mathbb{Z}_{p_1p_2})$ is isomorphic to $\Sigma(\mathbb{Z}_{p_3p_4})$ and each p_i is distinct. This means that each sum graph has the same number of vertices and edges. Using the Euler Phi Function, the number of vertices of $\Sigma(\mathbb{Z}_{p_1p_2})$ is $p_1 + p_2 - 2$. Using the same method to find the number of vertices of $\Sigma(\mathbb{Z}_{p_3p_4})$ and the fact that the number of vertices in each sum graph is equal,

$$p_1 + p_2 = p_3 + p_4. \tag{1}$$

Also, each sum graph has the same number of edges. Since $\mathbb{Z}_{p_1p_2} \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, the sum graph will have two disjoint cliques (5.2) one with $p_1 - 1$ vertices and the other with $p_2 - 1$ vertices. Since each is complete, the first component has $\frac{(p_1-1)(p_1-2)}{2}$ edges and the second has $\frac{(p_2-1)(p_2-2)}{2}$ for a total of

$$\frac{(p_1 - 1)(p_1 - 2)}{2} + \frac{(p_2 - 1)(p_2 - 2)}{2} = \frac{p_1(p_1 - 3) + p_2(p_2 - 3) + 4}{2} \tag{2}$$

edges in the graph. Since the graphs are isomorphic,

$$p_1(p_1 - 3) + p_2(p_2 - 3) + 4 = p_3(p_3 - 3) + p_4(p_4 - 3) + 4.[2]$$

Using (1), $p_1 = p_3 + p_4 - p_2$. Substituting this into the second equation:

$$(p_3 + p_4 - p_2)(p_3 + p_4 - p_2 - 3) + p_2(p_2 - 3) + 4 = p_3(p_3 - 3) + p_4(p_4 - 3) + 4$$

$$\Rightarrow p_3^2 + 2p_3p_4 - 2p_2p_3 - 3p_3 + p_4^2 - 2p_2p_4 - 3p_4 + 2p_2^2 = p_3^2 - 3p_3 + p_4^2 - 3p_4$$

$$\Rightarrow 2p_3p_4 - 2p_2p_3 - 2p_2p_4 + 2p_2^2 = 0$$
$$\Rightarrow p_3p_4 + p_2^2 = p_2(p_3 + p_4)$$

Therefore, p_2 divides $p_3p_4 + p_2^2$. Since p_2 divides p_2^2, p_2 must also divide p_3p_4, but since p_2, p_3, p_4 are primes, either $p_2 = p_3$ or $p_2 = p_4$, so the primes are not distinct and this is a contradiction. Therefore the sum graphs of $\mathbb{Z}_{p_1p_2}$ and $\mathbb{Z}_{p_3p_4}$ cannot be isomorphic.

□

Proposition 10.4. *Let p_1, p_2 be distinct primes and r, s integers. Then $\Sigma(\mathbb{Z}_{p_1^r})$ is not isomorphic to $\Sigma(\mathbb{Z}_{p_2^s})$.*

Proof. Assume that $\Sigma(\mathbb{Z}_{p_1^r}) \cong \Sigma(\mathbb{Z}_{p_2^s})$ and p_1 and p_2 are distinct. Applying the Euler Phi Function,

$$|Z(\mathbb{Z}_{p_1^r})^*| = p_1^r - p_1^{r-1}(p_1 - 1) - 1$$

and

$$|Z(\mathbb{Z}_{p_2^s})^*| = p_2^s - p_2^{s-1}(p_2 - 1) - 1.$$

Since the sum graphs are isomorphic, they have the same number of vertices, so

$$p_1^r - p_1^{r-1}(p_1 - 1) - 1 = p_2^s - p_2^{s-1}(p_2 - 1) - 1$$
$$\Rightarrow p_1^{r-1} = p_2^{s-1}$$
$$\Rightarrow p_1p_1^{r-2} = p_2^{s-1}$$

This implies that p_1 divides p_2^{s-1}. Since p_1, p_2 are primes, p_1 must divide p_2, so $p_1 = p_2$. This is a contradiction since we assumed the two primes were distinct. Therefore $\Sigma(\mathbb{Z}_{p_1^r}) \ncong \Sigma(\mathbb{Z}_{p_2^s})$.

□

Theorem 10.5. *Let $p < q < r$ be primes and $R = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$. Then $\omega(\Sigma(R)) = qr - 1$.*

Proof. First, we note that $V(\Sigma(R))$ is partitioned into three subsets as follows:

$$S_1 = \{(0, y, z) : y, z \text{ not both zero }\}$$
$$S_2 = \{(x, 0, z) : x \neq 0\}$$
$$S_3 = \{(x, y, 0) : x \neq 0, y \neq 0\}$$

By Lemma 5.1, each S_i is a clique. By an elementary counting argument, $|S_1| = qr-1$, $|S_2| = (p-1)r$, $|S_3| = (p-1)(q-1)$. Clearly $|S_1| > |S_2| > |S_3|$.

It remains to show that every clique has size at most $|S_1| = qr - 1$. Let C be an arbitrary clique in $\Sigma(R)$. Suppose C contains $v = (x, 0, z) \in S_2$ with $z \neq 0$. Any of the other $(p-1)r - 1$ elements of S_2 is adjacent to v; moreover, if $w = (0, y', z') \in S_1$ is adjacent to v, then either $y' = 0$ and $z' \neq 0$ or $z' = -z$ and y' is arbitrary. This gives at most $r + q - 2$ possibilities for w. Finally, if $u = (x'', y'', 0) \in S_3$ is adjacent to v, then $x'' = -x$. This gives at most $q - 1$ possibilities for u. Thus, $|C| \leq (p-1)r + r + q - 1 + q - 1 = pr + 2q - 2$. We claim that $pr + 2q - 2 \leq qr - 1$. If not, then $pr + 2q - 2 > qr - 1$; that is

$$q < \frac{pr-1}{r-2} = \frac{pr - 2p + 2p - 1}{r-2} = \frac{p(r-2) + 2p - 1}{r-2} = p + \frac{2p-1}{r-2} \leq p + \frac{2p-1}{q} < p + 2.$$

Thus, $q < p + 2$, so $q \leq p + 1$, which is a contradiction unless $p = 2$, $q = 3$. In the case, the given upper bound for $|C|$ becomes $pr + 2q - 2 = 2r + 4 \leq qr - 1 = 3r - 1$. Now suppose C contains a vertex $v = (x, 0, 0) \in S_2$ but no vertices of type $(x, 0, z)$ with $z \neq 0$. Thus, C contains at most $p-2$ other vertices in S_2. If $w = (0, y', z') \in S_1$ is adjacent to v, then either $y' = 0$ or $z' = 0$, which accounts for $q + r - 1$ possibilities. Now any vertex of S_3 is adjacent to v; thus in this case

$$|C| \leq p - 1 + q + r - 1 + (p-1)(q-1) = pq + r - 1 \leq (q-1)r + r - 1 = qr - 1.$$

Finally, suppose C contains no vertex of S_2 but contains some vertex $v = (x'', y'', 0) \in S_3$. Evidently any of the other $(p-1)(q-1) - 1$ vertices of S_3 is adjacent to v. In order for $w = (0, y', z') \in S_1$ to be adjacent to v, either $y' = -y''$ or $z' = 0$, which gives $r + q - 1$ possibilities for w. Thus,

$$|C| \leq (p-1)(q-1) + r + q - 1 = pq + r - p - 1 \leq pq + r - 1 \leq qr - 1$$

as in the previous paragraph.
Therefore, any clique in $\Sigma(R)$ containing a vertex of S_2 or a vertex of S_3 has size at most $|S_1| = qr - 1$, and hence S_1 is a clique of maximum size.

□

Theorem 10.6. *Let $R_1 = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3}$ and $R_2 = \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \mathbb{Z}_{q_3}$ where R_1 and R_2 are rings and p_i, q_j primes where $p_i \neq q_j$ for some i and j. Then $\Sigma(R_1) \ncong \Sigma(R_2)$.*

Proof. Assume $\Sigma(R_1) \cong \Sigma(R_2)$, so the clique number must be equal for each graph. By Theorem 10.5,

$$\omega(\Sigma(R_1)) = p_2 \cdot p_3 - 1$$

and

$$\omega(\Sigma(R_2)) = q_2 \cdot q_3 - 1$$

.

$$\Rightarrow p_2 \cdot p_3 = q_2 \cdot q_3$$

For any q_i, q_i divides $p_2 \cdot p_3$, so $q_i = p_j$ for some i and j. This hold for each q and p. Therefore the sum graphs cannot be isomorphic unless the rings are equal.

□

11 Open Problems and Conjectures

Conjecture 11.1. *In $\mathbb{Z}_n$, let $n = p_1 p_2 \ldots p_r$ where p is a distinct prime $p_1 < p_2 < \ldots < p_r$, then*
$|N(p_1)| > |N(p_2)| \geq |N(p_3)| \geq \ldots \geq |N(p_r)|$.

The motivation for our conjecture is the following: the neighborhood of any vertex has to have size at least $n/p_i - 2$ since it has to be adjacent to all the vertices of the form $v = kp_i$ where $k \in \mathbb{N}$ that do not include zero and itself (hence the subtraction of two). This is only a lower bound because it is possible that the vertex could be adjacent to a vertex of the form $w = kp_j$ where $k \in \mathbb{N}$ and $i \neq j$.

Conjecture 11.2. *Let $R = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$ where p_i is prime and $p_1 < p_2 < \ldots < p_r$. The clique number is $p_2 \cdot p_3 \cdots p_n - 1$.*

Proof. This proof should be similar to Lemma 10.5. The problem arises when showing that the only maximal cliques are the S_i. □

Corollary 11.3. *Let $R_1 = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$ and $R_2 = \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k}$ where R_1 and R_2 are rings and p_i, q_j primes, where $p_i \neq q_j$ for some i and j. Then $\Sigma(R_1) \ncong \Sigma(R_2)$.*

Proof. This proof follows from the previous lemma and uses the same technique as the proof of Theorem 10.6. □

12 Conclusions

We have explored various aspects of sum graphs, but there is still room for further research. We suggest looking into the edge coloring, more investigation into the complements, and continued work on isomorphisms of sum graphs.

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