Uncharted Territory of Zero Divisor Graphs and Their Complements

Amanda Phillips, Julie Rogers, Kevin Tolliver, and Frances Worek

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Abstract

Let $\Gamma(\mathbb{Z}_n)$ be a zero divisor graph whose vertices are nonzero zero divisors of $\mathbb{Z}_n$ and whose edges connect two vertices whose product is zero modulo n. Then $\overline{\Gamma(\mathbb{Z}_n)}$ represents the complement of $\Gamma(\mathbb{Z}_n)$. The authors explore the center of $\Gamma(\mathbb{Z}_n)$ and $\overline{\Gamma(\mathbb{Z}_n)}$. Further study is done on planarity, independent sets and cliques, vertices of minimum degree, and connectivity of $\overline{\Gamma(\mathbb{Z}_n)}$.

1 Introduction

The set of zero divisors in a ring are not easily studied using algebraic means because of their unique structure. Therefore, zero divisor graphs are studied to learn more about zero divisors. One relatively unexplored area in graph theory is that of zero divisor graphs and their complements. Our intention in this paper is to introduce and discuss the concept of zero divisor graphs with a focus on their complements by expanding upon the work done by Melody Brickel. An element a in a ring R is a *zero divisor* if there exists a nonzero element b in R such that $ab = 0$. We examine the ring $\mathbb{Z}_n = \{\overline{0}, \overline{1}, ..., \overline{n-1}\}$, defined as the set of all residue classes modulo n. For ease of notation, we will omit the bars. For example, let us consider $\mathbb{Z}_8$.

$$\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}.$$

The set of zero divisors of $\mathbb{Z}_8$ is

$$Z(\mathbb{Z}_8) = \{0, 2, 4, 6\}$$

Since 2×4 and 6×4 are 0 modulo 8, the elements 2,4, and 6 along with 0 are zero divisors.

A *simple graph* G is a pair (V, E) where V is a set of vertices and E is a set of edges–unordered pairs $\{v, w\}$ of distinct elements of V. In order to construct the graph of zero divisors of $\mathbb{Z}_n$, denoted $\Gamma(\mathbb{Z}_n)$, we need to define the set of vertices and edges. The vertices of $\Gamma(\mathbb{Z}_n)$, denoted $V(\Gamma(\mathbb{Z}_n))$, are the nonzero zero divisors. The edges of $\Gamma(\mathbb{Z}_n)$ are pairs $\{x, y\}$, written xy for ease of notation, in $V(\Gamma(\mathbb{Z}_n))$ such that $x \neq y$ and $xy = 0$ under multiplication modulo n.

The *complement* of a graph G, denoted $\overline{G}$, is a simple graph in which all the vertices of $\overline{G}$ are the same as the vertices of G, but the edges $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. Consider the zero divisor graph $G = \Gamma(\mathbb{Z}_n)$. The edges of the complement, $\overline{\Gamma(\mathbb{Z}_n)}$, are the pairs xy in $V(\overline{\Gamma(\mathbb{Z}_n)})$ such that $x \neq y$ and $xy \neq 0$ under multiplication modulo n.

We provide definitions and notation in a preliminary section, and a glossary of definitions is provided at the end of the paper. We will explore the centers of both zero divisor graphs and their complements. The remainder of our research concentrates on the complements of zero divisor graphs. We determine when these graphs are planar and when an associate class is an independent set. We find a vertex of minimum degree, and this result leads us to the connectivity of $\overline{\Gamma(\mathbb{Z}_n)}$ for n of a specific type.

2 Preliminaries

Let R be a ring and a, b, and u be elements in R. An element a is called a *zero divisor* if there exists an element $b \neq 0$ such that $ab = 0$. The set of zero divisors in R is denoted $Z(R)$. Denote by $Z(R)^*$ the set of nonzero zero divisors in R. An element a is called a *unit* if there exists an element b such that $ab = 1$. For our purposes in this paper we look at $\mathbb{Z}_n$ which is a finite ring, so we can say that every element of $\mathbb{Z}_n$ is either a zero divisor or a unit. Also, we say that two elements a and b are *associates* if there exists a unit u such that $a = ub$. We define $a \parallel b$ to mean that a and b are associates. For every a, the *annihilator* of a, denoted $\text{ann}(a)$, is the set of all elements x in R such that $ax = 0$. Note, $\text{ann}(a)$ is an ideal of R.

We also make use of the *Chinese Remainder Theorem*, which states that $\mathbb{Z}_{m_1 \cdots m_r} \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_r}$ when $m_1, \ldots, m_r$ are pairwise relatively prime integers.

Let $S \subseteq V(G)$. The *subgraph induced by* S is the graph H such that $V(H) = S$ and $E(H) = \{v_1v_2 \in E(G){:}v_1, v_2 \in S\}$.

A *path* is a sequence of vertices and edges in G such that no vertex is repeated. G is *connected* if for every u and v in $V(G)$, with $u \neq v$, there exists a u, v-path.

Throughout this paper, we write $n = p_1^{e_1} \cdots p_r^{e_r}$, where $p_1 < p_2 < \ldots < p_r$ are the prime factors of n. Let v be a vertex in the graph $\Gamma(\mathbb{Z}_n)$. Let us define $v_i = \frac{n}{p_i} = p_1^{e_1} \cdots p_i^{e_i - 1} \cdots p_r^{e_r}$, $1 \leq i \leq r$. Let N_v represent the neighborhood set of v. In other words, N_v consists of all vertices adjacent to v. Let A_v represent the associate class of v; that is, A_v consists of all the associates of v.

Since we are considering $\mathbb{Z}_n$ in this paper we state the following definition. Let $n \geq 2$ be an integer with $x \in \mathbb{Z}_n$. Pick $\tilde{x} \in \mathbb{Z}$, $0 \leq \tilde{x} \leq n-1$, such that the class of $\tilde{x}$ (mod n) is x. We then say $m \mid x$ if $m \mid \tilde{x}$ as integers where $m \in \mathbb{Z}$.

3 Centers

In this section we will look at the centers of zero divisor graphs and their complements. To be able to understand what a center is we must first define distance and eccentricity. Let G be a graph and $u, v \in V(G)$ where $u \neq v$. Then the *distance* from u to v, denoted $d(u, v)$, is the length of the shortest $u, v-$path. If there is no $u, v-$path then $d(u, v) = \infty$. For example, in $\Gamma(\mathbb{Z}_{12})$, we can see from Figure **??** that $d(2, 3) = 3$ and $d(4, 8) = 2$.

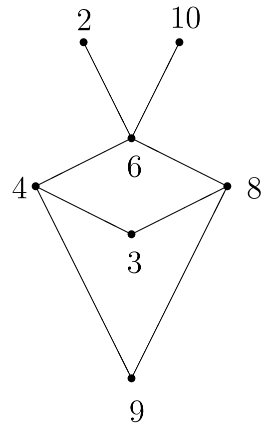

Figure 1: $\Gamma(\mathbb{Z}_{12})$

The *eccentricity* of $u \in V(G)$, denoted $\varepsilon(u)$, is the maximum distance from u to any other vertex. Looking back at $\Gamma(\mathbb{Z}_{12})$, we see that $\varepsilon(10) = 3$ because $d(10, v)$ is at most 3 for any vertex v. Finally we say that the *center* of G is the subgraph induced by the set of vertices of minimum eccentricity. Note that the center of $\Gamma(\mathbb{Z}_{12})$ consists of the vertices 4, 6, and 8 together with the edges from 4 to 6 and 6 to 8 because $\varepsilon(4) = \varepsilon(6) = \varepsilon(8) = 2$, which is the minimum eccentricity of $\Gamma(\mathbb{Z}_{12})$. Now let us explore the centers of zero divisor graphs in general.

Proposition 3.1. *Let $n = p_1^{e_1} \cdots p_r^{e_r}$, where $r > 2$ or if $r = 2$ then $n \neq 2p_2$. Also let $B = \{x \in \mathbb{Z}_n : p_i | x$ for all $i\}$. The center of $\Gamma(\mathbb{Z}_n)$ is the subgraph induced by $(\bigcup_{i=1}^{r} A_{v_i}) \bigcup B$.*

Proof.
Let $v \in V(G)$ be any vertex. Then, by Lemma **??**, $v = cp_1^{f_1} \dots p_r^{f_r}$, where c is a unit, $p_1 \leq v < n$, and $0 \leq f_i \leq e_i$ for all i. Because $v \neq 0$, there exists k such that $f_k \leq e_k - 1$. Fix $j \neq k$. If $v \neq p_j$, then $p_j \notin N_v$. If $v = p_j$, then $2p_j \notin N_v$. Therefore $d(p_j, v) > 1$, which implies that $\varepsilon(v) > 1$. Thus, no vertex has eccentricity 1. Let $1 \leq i \leq r$. Choose $v \in A_{v_i}$ and $t \in V(G)$, $t \neq v$. If t is adjacent to v, then $d(v, t) = 1$. If not, then $p_i \nmid t$ and there exists $j \neq i$ such that $t = lp_j$ for some $l \in \mathbb{Z}$. Note that t is adjacent to v_j. We also know that all the vertices in A_{v_i} are adjacent to all the vertices in A_{v_j}. Therefore, if any vertex t is not adjacent to v, there still exists a $v, t-$path through v_j of length 2. Thus $\varepsilon(v) = 2$, which implies that $\varepsilon(u) = 2$ for all $u \in A_{v_i}$ and for all i. Similarly, for any vertex $u = mp_1 \cdots p_r$ for some $m \in \mathbb{Z}$, if $t \in V(G)$ and t is adjacent to u, then the distance is 1. Otherwise, there is some j such that there exists a $u, t-$path through v_j. Therefore, $\varepsilon(u) = 2$.

Now we will show that every other vertex in the graph is not in the center because they have eccentricity greater than 2. Choose u such that u is not associate to p_i, for any i, and $p_1 \cdots p_r \nmid u$. Then there exists j such that $p_j \nmid u$ implies that u is not adjacent to any vertex in A_{v_j}. Note, p_j is only adjacent to v_j and its associates, but no other vertex. So p_j is not adjacent to u. Thus, $d(u, p_j) > 2$ because any u, p_j-path has to start at u and end at p_j, and must contain an associate of v_j as an internal vertex; since u is adjacent to neither p_j nor A_{v_j}, this path must contain at least one more internal vertex. Hence $d(u, p_j) \geq 3$ and from [**?**] we know that $diam(\Gamma(\mathbb{Z}_n)) \leq 3$, so we can conclude that $d(u, p_j) = 3$. Therefore $\varepsilon(u) = 3$ implies that u is not in the center. ■

Proposition 3.2. *If $n = 2p$, then the center of $\Gamma(\mathbb{Z}_n)$ is p.*

Proof.
The vertex set of G is defined as $V(G) = \{2, 4, \dots, (p-1)2, p\}$. Since p is adjacent to all multiples of 2, $\varepsilon(p) = 1$. Also note that any multiple of 2 is adjacent only to p, which means that $\varepsilon = 2$ for all multiples of 2. Therefore, p is the only element in the center since it has the minimum eccentricity. ■

Proposition 3.3. *If $n = p^e$, then the center of $\Gamma(\mathbb{Z}_n)$ is $A_{p^{e-1}}$.*

Proof.
Every vertex in $A_{p^{e-1}}$ has $\varepsilon = 1$ because $V(G) = \{p, 2p, \ldots, p^2, 2p^2, \ldots, (p-1)p^2, \ldots, (p-1)p^{e-1}\}$ and all the vertices are multiples of p; therefore, every vertex in $A_{p^{e-1}}$ is adjacent to all vertices in the graph. Every other vertex in the graph has some non-neighbor and therefore has $\varepsilon \geq 2$ and is not in the center. ■

We now look at the center of $\overline{\Gamma(\mathbb{Z}_n)}$.

Lemma 3.4. *Let $n = p_1^{e_1} \cdots p_r^{e_r}, r \geq 2$, and let $v \in V(\overline{\Gamma(\mathbb{Z}_n)})$. For all $i \neq j$, either $p_i \in N_v$ or $p_j \in N_v$.*

Proof.
Explore two cases, either $v \in A_{v_i}$ or $v \notin A_{v_i}$. First case $v \in A_{v_i}$. The vertex p_i is not adjacent to $v = up_1^{e_1} \cdots p_i^{e_i - 1} \cdots p_r^{e_r}$ and $v \cdot p_j = up_1^{e_1} \cdots p_i^{e_i-1} \cdots p_j^{e_j+1} \cdots p_r^{e_r}$, which is not divisible by n. So v and p_j are adjacent. Second case, $v \notin A_{v_i}$. Then $v = up_1^{f_1} \cdots p_r^{f_r}$, where u is a unit and $0 \leq f_i \leq e_i$ for all $1 \leq i \leq r$. There exists $f_i < e_i - 1$ or there exists k such that $f_k \leq e_k - 1$. Therefore, $v \cdot p_i$ is not divisible by n, since $f_i < e_i - 1$ or $f_k \leq e_k - 1$. This implies that p_j is adjacent to v. ■

Proposition 3.5. *The center of $\overline{\Gamma(\mathbb{Z}_n)}$, denoted $\overline{G}$, for $n = p_1^{e_1} \cdots p_r^{e_r}$ is $\overline{\Gamma(\mathbb{Z}_n)}$ as long as $r \geq 3$.*

Proof.
Let $x \in V(\overline{G})$. Consider three cases, either there exists an i such that $x \in A_{v_i}$, there exists an i such that $x \in N_{v_i}$, or $x \in E = V(\overline{G}) - (\bigcup_{i=1}^{r} A_{v_i}) - (\bigcup_{i=1}^{r} N_{v_i})$. Suppose $x \in A_{v_i}$, x is adjacent to every vertex in N_x. Note $p_j, p_k \in N_x$, such that j, k are distinct and not equal to i. If $w \notin N_x$, either w is adjacent to p_j or p_k, by Lemma **??**. Then there is a x, w-path with length 2.

Suppose $x \in N_{v_i}$. Note $p_j, p_k \in N_x$, such that j, k are distinct and not equal to i. If $w \notin N_x$, either w is adjacent to p_j or p_k, by Lemma **??**. Then there is a x, w-path with length 2.

Suppose $x \in E$. Note $p_j, p_k \in N_x$, such that j, k are distinct and not equal to i. If $w \notin N_x$, either w is adjacent to p_j or p_k, by Lemma **??**. Then there is a x, w-path with length 2.

Proposition 3.6. *The center of $\overline{\Gamma(\mathbb{Z}_n)}$, denoted $\overline{G}$, for $n = p_1^{e_1} p_2^{e_2}$, where $e_1 > 1$ or $e_2 > 1$, is the subgraph induced by $V(\overline{G}) - (A_{v_1} \cup A_{v_2})$.*

Proof.
Consider the five cases, either x is in A_{v_1}, A_{v_2}, N_{v_1}, N_{v_2}, or $E = V(G) - A_{v_1} - A_{v_2} - N_{v_1} - N_{v_2}$. Suppose $x \in A_{v_1}$. The vertex x is adjacent to every vertex in N_{v_1}. Say $w \in V(\overline{G}) - A_{v_2}$. Note $p_2 \in N_x$, which is adjacent to every vertex w. Then there is a xw-path with length two. Say $w \in V(\overline{G}) - N_{v_2} - E$. Now N_{v_1} and N_{v_2} are disjoint sets because there is no vertex in $\overline{G}$ that is not divisible by p_1 and not divisible by p_2. Also, p_2 is adjacent to p_1. Note $p_1 \in N_{v_2}$. Therefore, there is a x, w-path with length 3, and every vertex in A_{v_1} has eccentricity 3. Similarly, every vertex in A_{v_2} has eccentricity 3.

Next, let $x \in N_{v_1}$. Note $p_2 \in N_x$. If $w \notin N_x$, w is adjacent to p_2. Then there is a x, w-path with length 2. So N_{v_1} has eccentricity 2. Similarly, N_{v_2} has eccentricity 2.

Suppose $x \in E$. Note $p_2 \in N_x$. If $w \notin N_x$, w is adjacent to p_2. Then there is a x, w-path with length 2. So E has eccentricity 2.

Let $v \in V(G)$ be any vertex. Then $v = cp_1^{f_1}p_2^{f_2}$, where c is a unit and $0 \leq f_i \leq e_i$ for all $1 \leq i \leq 2$. There exists k such that $f_k \leq e_k - 1$. The vertex $w = p_1^{e_1 - f_1}p_2^{e_2 - f_2} \notin N_v$. Therefore, $d(v, w) > 1$ implies $\varepsilon(v) > 1$. Thus, no vertex has eccentricity 1. It follows that every vertex in $G - A_{v_1} - A_{v_2}$ has eccentricity of two. The subgraph induced by these vertices is the center. ■

4 Planarity

In this section, we determine when $\overline{\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r})}$ is planar where $n_1, \ldots, n_r \in \mathbb{Z}$. A graph is called *planar* if it can be drawn in the plane with none of its edges overlapping. Using *Kuratowski's Theorem*, if there exists a subgraph of G which is isomorphic to K_5, the complete graph on 5 vertices, or $K_{3,3}$, the complete bipartite graph with 3 elements in each subset of the partition, then G is not planar. A *complete graph*, denoted K_n, is the graph on n vertices with all possible edges. G is *bipartite* if its vertices can be partitioned into two nonempty independent subsets.

Theorem 4.1. $\overline{\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r})}$ *is planar if and only if* $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r} \cong \mathbb{Z}_{p^2}$ *for some prime* p *or* $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ *is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_4 \times \mathbb{Z}_3$, *or* $\mathbb{Z}_3 \times \mathbb{Z}_5$.

Proof.
Consider the four cases: $r \geq 4$, $r = 3$, $r = 1$, and $r = 2$.

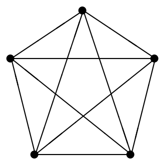

Figure 2: K_5

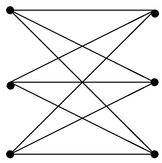

Figure 3: $K_{3,3}$

Case 1: Consider $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ with $r \geq 4$. There is a subgraph isomorphic to $K_{3,3}$ induced by the bipartition $\{(1,0, 0, 0, \ldots), (0, 1, 0, 1, \ldots), (1, 1, 0, 0, \ldots)\} \cup \{(1, 1, 1, 0, \ldots), (1, 1, 0, 1, \ldots), (1, 0, 0, 1, \ldots)\}$. Thus $\overline{\Gamma(\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r})}$ is not planar.

Case 2: Consider $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}$. If $n_i \geq 3$ for some i, then a $K_{3,3}$ subgraph may be induced. Without loss of generality, assume $n_3 \geq 3$. Then a $K_{3,3}$ subgraph may be induced by the bipartition $\{(1, 0, 2), (0, 0, 2), (0, 1, 2)\} \cup \{(0, 0, 1), (0, 1, 1), (1, 0, 1)\}$. Therefore $\overline{\Gamma(\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r})}$ is not planar. If $n_1 = n_2 = n_3 = 2$, then $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}$ is planar, as shown in Figure **??**.

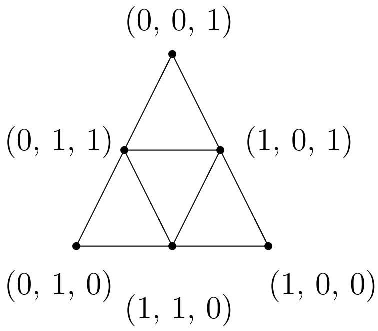

Figure 4: $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}$

Case 3: Let $r = 1$. That is, consider $\mathbb{Z}_n$, where $n = p_1^{e_1} \cdots p_s^{e_s}$. We shall examine three subcases, a) $s \geq 3$, b) $s = 2$, c) $s = 1$.

a) If $s \geq 3$, then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \times \cdots \times \mathbb{Z}_{p_s^{e_s}}$, and we may construct a subgraph isomorphic to $K_{3,3}$ using the bipartition $\{(1, 0, \ldots, 0), (1, 1, 1, \ldots, 0), (1, 2, 2, \ldots)\} \cup \{(1, 1, 0, \ldots, 0), (1, 2, 1, \ldots, 0), (1, 1, 2, \ldots, 0)\}$. Therefore $\overline{\Gamma(\mathbb{Z}_n)}$ is not planar.

b) Consider $n = p_1^{e_1} p_2^{e_2}$. First let us look at $p_2 \geq 7$. In this case, there is a subgraph isomorphic to K_5 induced by the vertices $\{p, 2p, \ldots, 5p\}$.

Next, consider the case $n = 2^{e_1}3^{e_2}$. When $e_1 = 1$ and $e_2 = 1$, $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)}$ is planar as shown in Figure **??**.

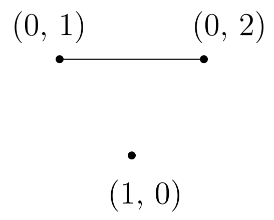

Figure 5: $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)}$

When $e_2 \geq 2$, the graph contains a K_5 subgraph induced by $\{2, 4, 6, 8, 10\}$. Therefore $\overline{\Gamma(\mathbb{Z}_n)}$ is not planar. When $e_1 = 2$ and $e_2 = 1$, $\overline{\Gamma(\mathbb{Z}_n)}$ is planar, as shown in Figure **??**.

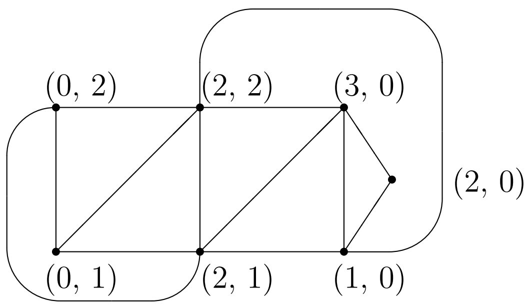

Figure 6: $\overline{\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_3)}$

When $e_1 \geq 3$, a subgraph isomorphic to K_5 is induced by the vertices $\{2, 4, 8, 10, 14\}$.

Now consider the case where $n = 2^{e_1}5^{e_2}$. If $e_1 = 1$ and $e_2 = 1$, the graph is planar, as shown in Figure **??**.

When $e_2 \geq 2$, the vertices $\{5, 15, 25, 35, 45\}$ induce a subgraph isomorphic to K_5. This may also be done when $e_1 \geq 2$, using the vertices $\{2, 4, 6, 8, 12\}$.

Consider the case $n = 3^{e_1}5^{e_2}$. For the case $e_1 = 1$ and $e_2 = 1$, the graph is planar, as shown in Figure **??**.

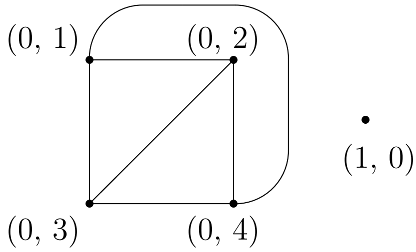

Figure 7: $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)}$

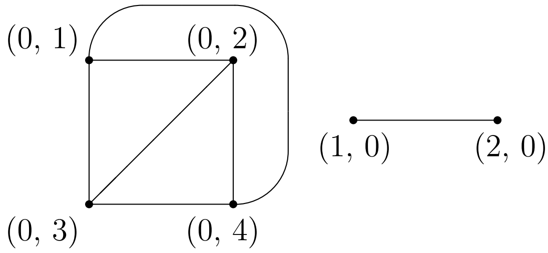

Figure 8: $\overline{\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5)}$

When $e_1 \geq 2$ or $e_2 \geq 2$, $\overline{\Gamma(\mathbb{Z}_n)}$ is not planar since a subgraph isomorphic to K_5 may be constructed from the vertices $\{5, 10, 20, 25, 30\}$.

c) For $n = p^e$ when $e = 1$, $\overline{\Gamma(\mathbb{Z}_p)}$ is the empty graph since $Z(\mathbb{Z}_p)^* = \emptyset$. For $n = p^e$ when $p \geq 7$ and $e \geq 3$ the graph is not planar since we can form a K_5 subgraph from the vertices $p, 2p, \ldots, 5p$.

The vertex set of $\overline{\Gamma(\mathbb{Z}_{p^2})}$ is $\{p, 2p, \ldots, (p-1)p\}$. Thus the product of any two vertices is congruent to zero modulo p^2, resulting in $\overline{\Gamma(\mathbb{Z}_{p^2})}$ being completely disconnected. Thus our graph is planar.

When $p = 2$ and $e \geq 4$, we can form a K_5 subgraph from the vertices $\{2, 4, 6, 10, 14\}$. Therefore, $\overline{\Gamma(\mathbb{Z}_n)}$ is not planar.
When $p = 2$ and $e = 3$, $n = 8$. $\overline{\Gamma(\mathbb{Z}_8)}$ is shown to be planar in Figure **??**.

When $p = 3$ and $e \geq 3$, we can form a K_5 subgraph from the vertices $\{3, 6, 12, 15, 21\}$. Therefore $\overline{\Gamma(\mathbb{Z}_n)}$ is not planar.
When $p = 5$ and $e \geq 3$, we can form a K_5 subgraph from the vertices $\{5, 10, 15, 20, 30\}$. Therefore the graph is not planar.

Case 4: Consider $r = 2$. $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} = \mathbb{Z}_{{p_1}^{e_1} \ldots {p_r}^{e_r}} \times \mathbb{Z}_{{q_1}^{f_1} \ldots {q_s}^{f_s}} \cong \mathbb{Z}_{p_1}^{e_1} \times \ldots \times \mathbb{Z}_{p_r}^{f_r} \times \mathbb{Z}_{q_1}^{f_1} \times \ldots \times \mathbb{Z}_{q_s}^{f_s}$, with $p_1, \ldots, p_r$ distinct primes and $q_1, \ldots, q_s$ distinct primes.

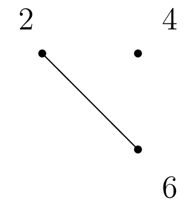

Figure 9: $\overline{\Gamma(\mathbb{Z}_8)}$

When $r \geq 2$ or $f \geq 2$, $\mathbb{Z}_n$ is a direct product of three or more factors, thus reducing the problem to cases 1 and 2.

When $r = 1$, $f = 1$, and $p_1 \neq q_1$, $\mathbb{Z}_{p_1}^{e_1} \times \mathbb{Z}_{q_1}^{f_1} \cong \mathbb{Z}_{{p_1}^{e_1}{q_1}^{f_1}}$, reducing the problem to case 3.

When $r = 1$, $f = 1$, $p_1 = q_1$, and $p_1^{e_1}, {p_1}^{f_1}$ are both greater than or equal to 4, a $K_{3,3}$ subgraph is induced by the bipartition $\{(0, 1), (0, 2), (0, 3)\} \cup \{(1, 0), (2, 0), (3, 0)\}$. Thus $\mathbb{Z}_{p_1}^{e_1} \times \mathbb{Z}_{p_1}^{f_1}$ is not planar.

The previous case does not cover when ${p_1}^{e_1} < 4$ or ${p_1}^{f_1} < 4$.

When ${p_1}^{e_1} < 4$ and ${p_1}^{f_1} < 4$, the only cases to consider are $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)}$ and $\overline{\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)}$. Both graphs are planar as shown in Figures **??** and **??**.

Figure 10: $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)}$

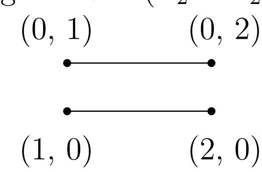

Figure 11: $\overline{\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)}$

When ${p_1}^{f_1} > 4$ and ${p_1}^{e_1} < 4$, a K_5 subgraph may be induced from the vertices $\{(1, 0), (0, 2), (0, 3), (0, 4), (0, 5)\}$. A similar argument holds for ${p_1}^{e_1} > 4$ and ${p_1}^{f_1} < 4$.

If ${p_1}^{e_1} < 4$ and ${p_1}^{f_1} = 4$, it is the case $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)}$. This graph is planar, as shown in Figure **??**.

■

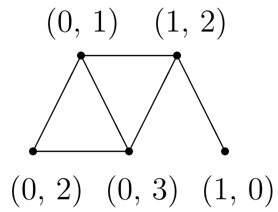

Figure 12: $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)}$

5 Independent Sets and Cliques

A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* in a graph is a set of pairwise nonadjacent vertices. Our goal in this section is to show that the associate class of a zero divisor in $\mathbb{Z}_n$ is either a clique or an independent set in the graph $\overline{\Gamma(\mathbb{Z}_n)}$.

For example let us examine $n = p_1^2 p_2$ for $\overline{\Gamma(\mathbb{Z}_n)}$, where p_1, p_2 are distinct primes.

There are four associate classes:

$A_{p_1^2} = \{x \in \mathbb{Z}_n : p_1^2 \| x\}, A_{p_1} = \{x \in \mathbb{Z}_n : p_1 \| x\}$,
$A_{p_2} = \{x \in \mathbb{Z}_n : p_2 \| x\}, A_{p_1 p_2} = \{x \in \mathbb{Z}_n : p_1 p_2 \| x\}$

$$A_{p_1^2} \text{ —— } A_{p_1} \text{ —— } A_{p_2} \text{ —— } A_{p_1 p_2}$$

We can say that A_{p_1} is a clique. For all $x_1, x_2 \in A_{p_1}$, x_1 is adjacent to x_2 since $x_1 \cdot x_2 = up_1^2$, where u is a unit, which is not divisible by n.

Also, $A_{p_1 p_2}$ is an independent set. For all $x_1, x_2 \in A_{p_1 p_2}$, x_1 is not adjacent to x_2 since $x_1 \cdot x_2 = up_1^2 p_2^2$, where u is a unit, which is divisible by n.

Proposition 5.1. *$A_{p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}}$ is an independent set if and only if $a_1 \geq \frac{e_1}{2}$, $a_2 \geq \frac{e_2}{2}$, $\ldots$, $a_r \geq \frac{e_r}{2}$.*

Proof.
($\Rightarrow$): Let $A_{p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \cdots p_r^{a_r}}$ be an associate class of the graph, where $1 \leq k \leq r$. Choose $x_1, x_2 \in A_{p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \cdots p_r^{a_r}}$. Assume that the associate class is an independent set and there exists some number $a_k < \frac{e_k}{2}$, where $a_1 \leq a_k \leq a_r$. Now $x_1 \cdot x_2 = up_1^{2a_1} p_2^{2a_2} \cdots p_k^{2a_k} \cdots p_r^{2a_r}$, where u is a unit. Since $a_k < \frac{e_k}{2}$, then $2a_k < e_k$, which means that $x_1 \cdot x_2$ it is not divisible by n. This means that x_1 and x_2 are adjacent which is a contradiction to the definition of an independent set.

($\Leftarrow$): Assume that $a_i \geq \frac{e_i}{2}$ for all i. Choose $x_1, x_2 \in A_{p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \cdots p_r^{a_r}}$. Now $x_1 \cdot x_2 = up_1^{2a_1} p_2^{2a_2} \cdots p_k^{2a_k} \cdots p_r^{2a_r}$, where u is a unit. Since $a_i \geq \frac{e_i}{2}$ for all i, then $2a_i \geq e_i$, which means the product is divisible by n. This means that x_1 and x_2 are not adjacent and the associate class is an independent set.

Proposition 5.2. *Every non-independent associate class is a clique.*

Proof.
Assume $A_{p_1^{f_1} \cdots p_k^{f_k} \cdots p_r^{f_r}}$ is not an independent set. This implies $f_k < \frac{e_k}{2}$. Let $x, y \in A_{p_1^{f_1} \cdots p_k^{f_k} \cdots p_r^{f_r}}$. We know that $x \cdot y = up_1^{2f_1} \cdots p_k^{2f_k} \cdots p_r^{2f_r}$, where u is a unit. Since $2f_k < e_k$, the product will not be divisible by n and all vertices will be adjacent. Therefore, the associate class will be a clique.

6 Minimum Degree

We now look at vertices of minimum degree in $\overline{\Gamma(\mathbb{Z}_n)}$. First we will discuss some definitions and give some examples referring back to Figure **??**. Let G be a finite graph. The *degree* of a vertex v in $V(G)$, denoted deg v, is the number of edges containing v as an endpoint. For example in $\Gamma(\mathbb{Z}_{12})$, deg $(4) = 3$ and deg $(9) = 2$. The minimum degree of G is defined as $\delta(G) = \min\{deg(v) : v \in V(G)\}$, and the vertices which have the minimum degree are called *vertices of minimum degree.* In $\Gamma(\mathbb{Z}_{12})$, 2 and 10 are both vertices of minimum degree because deg $(2) =$ deg $(10) = 1$.

We will prove in Theorem **??** that a vertex of minimum degree in $\overline{\Gamma(\mathbb{Z}_n)}$ is $p_1^{e_1-1} p_2^{e_2} \cdots p_r^{e_r}$, but first we must establish the following lemmas.

Lemma 6.1. *Every nonzero zero divisor in $\mathbb{Z}_n$ is associate to a zero divisor of the form $p_1^{f_1} \cdots p_r^{f_r}$ where $0 \leq f_i \leq e_i$.*

Proof.
Let x be a nonzero zero divisor. Then there exists some p_{i_1} such that $x = c_1 p_{i_1}$. If c_1 is a unit in $\mathbb{Z}_n$, then x and p_{i_1} are associates. If not, then there exists some p_{i_2} such that $x = c_2 p_{i_1} p_{1_2}$. If c_2 is a unit in $\mathbb{Z}_n$, then x and $p_{i_1} p_{i_2}$ are associates. If not, we continue this process inductively until c_k is a unit in $\mathbb{Z}_n$. Then we can say that x is associate to a zero divisor of the form $p_1^{f_1} \cdots p_r^{f_r}$ where $0 \leq f_i \leq e_i$. ■

Lemma 6.2. *If u and v are associates, then $N_u - \{u, v\} = N_v - \{u, v\}$.*

Proof.
($\subseteq$): Let $x \in N_u - \{u, v\}$. We can say that $n \nmid xu$. By the previous lemma and because u and v are associates, this implies that $n \nmid xv$ which implies $x \in N_v - \{u, v\}$. Also, we must exclude $\{u, v\}$ because u could be in N_v and vice versa but neither u nor v can be in their own neighborhood sets.
($\supseteq$): A similar argument can be made. ■

Lemma 6.3. *Let $u = {p_1}^{n_1}{p_2}^{n_2} \cdots {p_r}^{n_r}$ and $x_i = {p_1}^{n_1} \cdots {p_i}^{n_i+1} \cdots {p_r}^{n_r}$ be vertices in G. Then $N_{x_i} \subsetneq N_u$.*

Proof.
Suppose $n_1, \ldots, n_r$ are integers such that for all k we have $n_k \leq e_k$, and such that $n_i < e_i$ and $n_j < e_j$ for some $1 \leq i < j \leq r$. Let $u = {p_1}^{n_1}{p_2}^{n_2} \cdots {p_r}^{n_r}$ and $x_i = {p_1}^{n_1} \cdots {p_i}^{n_i+1} \cdots {p_r}^{n_r}$. Since $x_i \cdot p_1^{e_1-n_1} p_2^{e_2-n_2} \cdots p_i^{e_i-n_i-1} \cdots p_r^{e_r-n_r} = 0$,

$$N_{x_i} = \{w \in V(G) : p_i^{-1} \prod_{j=1}^{r} p_j^{e_j-n_j} \nmid w\}.$$

Similarly,

$$N_u = \{w \in V(G) : \prod_{j=1}^{r} p_j^{e_j-n_j} \nmid w\}.$$

If $y \in N_{x_i}$ then $p_i^{-1} \prod_{j=1}^{r} p_j^{e_j-n_j} \nmid y$ which implies that $\prod_{j=1}^{r} p_j^{e_j-n_j} \nmid y$ meaning $y \in N_u$. However, $z = {p_1}^{e_1-n_1} \cdots {p_i}^{e_i-n_i-1} \cdots {p_r}^{e_r-n_r} \in N_u$ and $z \notin N_{x_i}$. Therefore, $N_{x_i} \subsetneq N_u$. ■

Lemma 6.4. *For all $u \in V(G)$ where $G = \overline{\Gamma(\mathbb{Z}_n)}$, there exists some k such that $N_{v_k} - \{v_k, u\} \subseteq N_u - \{v_k, u\}$.*

Proof.
Choose an arbitrary vertex u. By Lemma **??** we know that u is associate to a vertex of the form $w = p_1^{f_1} \cdots p_r^{f_r}$. By Lemma **??** we can say that $N_u - \{u, w\} = N_w - \{u, w\}$. If there exists a k such that $w = v_k$ then $N_w = N_{v_k}$ which implies $N_{v_k} - \{v_k, u\} = N_u - \{v_k, u\}$. If no such k exists, then there exists an $i_1 \in \{1, \ldots, r\}$ such that $x_1 = p_{i_1} w \neq 0$. Then $N_{x_1} - \{x_1, w\} \subseteq N_w - \{x_1, w\} \subseteq N_w - \{w\}$ by Lemma **??**. If there exists a k such that $x_1 = v_k$ then stop. If no such k exists, then there exists an $i_2 \in \{1, \ldots, r\}$ such that $x_2 = p_{i_2} x_1$. Then $N_{x_2} - \{x_2, x_1, w\} \subseteq N_{x_1} - \{x_2, x_1, w\} \subseteq N_w - \{x_2, w\} \subseteq N_w - \{w\}$ by Lemma **??**. If there exists a k such that $x_2 = v_k$ then stop. If no such k exists, then continue this process inductively until $x_m = v_k$ for some k. Let

$B = \{v_k, w, x_1, x_2, \ldots, x_{m-1}\}$. By construction, $N_{v_k} - B = N_{x_m} - B \subseteq N_w - \{v_k, w\}$. It is easy to see that if $x_k w = 0$ then $x_k v_k = 0$. It follows that if $x_k \in N_{v_k}$ i.e. $x_k v_k \neq 0$ then $x_k \in N_w$ i.e. $x_k w \neq 0$. Therefore, $N_{v_k} - \{v_k, w\} \subseteq N_w - \{v_k, w\}$ which implies $N_{v_k} - \{v_k, u, w\} \subseteq N_w - \{v_k, u, w\} = N_u - \{v_k, u, w\}$ by Lemma **??**. Using a similar argument if $w \in N_{v_k}$ then $w \in N_u$. Therefore, $N_{v_k} - \{v_k, u\} \subseteq N_u - \{v_k, u\}$. ■

Lemma 6.5. *Let $n \geq 2$ be an integer. Then $|Z(\mathbb{Z}_n)^*| = n - \phi(n) - 1$ is the cardinality of the set of nonzero zero divisors of $\mathbb{Z}_n$, i.e the number of vertices in $\Gamma(\mathbb{Z}_n)$ and $\overline{\Gamma(\mathbb{Z}_n)}$.*

Proof.
There are n elements in $\mathbb{Z}_n$, and the number of units of $\mathbb{Z}_n$ is given by $\phi(n)$. Since every element of a finite ring is either a zero divisor or a unit, $n - \phi(n)$ will tell us the number of zero divisors in $\mathbb{Z}_n$. Since $Z(\mathbb{Z}_n)^*$ denotes nonzero zero divisors we have the formula $|Z(\mathbb{Z}_n)^*| = n - \phi(n) - 1$. ■

Lemma 6.6. *Let $v_i = {p_1}^{e_1} \cdots {p_i}^{e_i - 1} \cdots {p_r}^{e_r}$ be an element of $V(\overline{\Gamma(\mathbb{Z}_n)})$. Then*
$|N_{v_i}| = \begin{cases} n - \phi(n) - |ann(v_i)| & \text{if } v_i^2 \neq 0 \\ n - \phi(n) - |ann(v_i)| + 1 & \text{if } v_i^2 = 0 \end{cases}$.

Proof.
The annihilator of an element a in $\mathbb{Z}_n$ is denoted ann(a) and consists of all elements r in $\mathbb{Z}_n$ such that $ra = 0$. Therefore $r = 0$ is contained in every ann(a). Also, a is contained in ann(a) if and only if $a^2 = 0$. In $\Gamma(\mathbb{Z}_n)$ since v_i is adjacent to all other vertices w such that $aw = 0$, and no vertex can be adjacent to itself or 0,
$\deg(v_i) = \begin{cases} |\text{ann}(v_i)| - 1 & \text{if } v_i^2 \neq 0 \\ |\text{ann}(v_i)| - 2 & \text{if } v_i^2 = 0 \end{cases}$.
Now looking at $\overline{\Gamma(\mathbb{Z}_n)}$, we know that edges are replaced with non-edges and vice versa. Therefore, $|N_{v_i}|$ will be equal to the total number of vertices in the graph minus the degree of v_i in the original graph. Using the previous lemma,
$|N_{v_i}| = |Z(\mathbb{Z}_n)^*| - \deg(v_i) = \begin{cases} n - \phi(n) - |\text{ann}(v_i)| & \text{if } v_i^2 \neq 0 \\ n - \phi(n) - |\text{ann}(v_i)| + 1 & \text{if } v_i^2 = 0 \end{cases}$. ■

Lemma 6.7. *Let $n \geq 2$ be an integer then $|\mathbb{Z}_n| = |ann(a)| \cdot |(a)|$.*

Proof.
Fix a in a ring R and a mapping $m_a : R \to R$ where $m_a(x) = ax$. Then m_a is a group homomorphism and Ker $(m_a) = \{r \in R : m_a(r) = 0\}$. Then we can say $|R| = |ker(m_a)| \cdot |im(m_a)| = |ann(a)||(a)|$. ■

Lemma 6.8. *Let $n \geq 2$ be an integer then $|(a)| = \frac{n}{\gcd(a,n)}$.*

Proof.
We can say that $|(a)|$ is equal to the order of a in $\mathbb{Z}_n$. Let $j = |a|$ and let $d = gcd(a, n)$. We can say

1. $ja = rn$ for some $r \in \mathbb{Z}$ and
2. $\frac{n}{d} \cdot a = n \cdot \frac{a}{d} \equiv 0 (\text{mod } n)$.

We also know that $j \mid \frac{n}{d}$ which implies that for some $k \in \mathbb{Z}$, then $jk = \frac{n}{d}$. This implies that $jkd = n$, which gives us $kd \mid n$. We can also say that $jakd = na$ which leads to $rnkd = na$. By cancellation, $rkd = a$ which implies that $kd \mid a$. Now we have $kd \mid a$ and $kd \mid n$ and since $gcd(a, n) = d$ this implies that $k = 1$. Since $jk = \frac{n}{d}$, this implies that $j = \frac{n}{d}$. This leads us to $\frac{n}{d} = |a|$ where d was defined as $gcd(a, n)$. Therefore, $|a| = \frac{n}{\gcd(a,n)}$. ■

Theorem 6.9. *Let $G = \overline{\Gamma(\mathbb{Z}_n)}$ where $n = {p_1}^{e_1} \cdots {p_r}^{e_r}$ and $e_i \geq 1$. Then a vertex of minimum degree is $v_1 = {p_1}^{e_1 - 1}{p_2}^{e_2} \cdots {p_r}^{e_r}$.*

Proof.
For each i, $1 \leq i \leq r$, and each integer f_i, $0 \leq f_i \leq e_i$, define $A_{p_1^{f_1} \cdots p_r^{f_r}} = \{x \in \mathbb{Z}_n : {p_1}^{f_1} \cdots {p_r}^{f_r} \| x\}$. Then the classes $A_{p_1^{f_1} \cdots p_r^{f_r}}$ partition $V(G)$ into equivalence classes as a consequence of Lemma **??**.

From Lemma **??** we can say that for all u in $V(G)$, $N_{v_i} - \{v_i, u\} \subseteq N_u - \{v_i, u\}$ so there exists some i such that N_{v_i} has the smallest cardinality of all the vertices in G. The cardinality of $N_{v_i} = \begin{cases} n - \phi(n) - |\text{ann}(v_i)| & \text{if } v_i^2 \neq 0 \\ n - \phi(n) - |\text{ann}(v_i)| + 1 & \text{if } v_i^2 = 0 \end{cases}$
as shown in Lemma **??** and Lemma **??**.

Therefore, since we are looking at the vertices of the form v_i and if we choose i such that $|\text{ann}(v_i)|$ is maximized, then $|N_{v_i}|$ will be minimized. We know from Lemma **??** that $|\mathbb{Z}_n| = |\text{ann}(a)| \cdot |(a)|$ for all $a \in V(G)$ and by Lemma **??** $|(a)| = \frac{n}{\gcd(a,n)}$. Therefore, $n = |\text{ann}(v_i)| \cdot \frac{n}{\gcd(v_i,n)} \Rightarrow |\text{ann}(v_i)| = \gcd(v_i, n)$. Since $n = p_i v_i$, this implies that $|\text{ann}(v_i)| = v_i$.
We claim that $v_i \geq v_{i+1} + 1$ for all $1 \leq i \leq r - 1$. Since $v_i = {p_1}^{e_1} \cdots {p_i}^{e_i - 1} \cdots {p_r}^{e_r}$ and $v_{i+1} = {p_1}^{e_1} \cdots {p_i}^{e_i} {p_{i+1}}^{e_{i+1} - 1} \cdots {p_r}^{e_r}$, we can write $v_{i+1} = \frac{p_i}{p_{i+1}} v_i$. Because $p_i < p_{i+1}$ for all i, we can say $v_{i+1} < v_i$ which leads to $v_{i+1} + 1 \leq v_i$. Now,

$$|N_{v_i}| \leq n - \phi(n) - v_i + 1$$

$$= n - \phi(n) - (v_i - 1)$$

$$\leq n - \phi(n) - v_{i+1}$$
$$\leq |N_{v_{i+1}}|$$

and therefore $|N_{v_i}| \leq |N_{v_{i+1}}|$.
Now we can say $|N_{v_1}|$ has the minimum cardinality of all the vertices in G and $v_1 = {p_1}^{e_1-1}{p_2}^{e_2} \cdots {p_r}^{e_r}$ is a vertex of minimum degree. ■

7 Connectivity

Using the vertex of minimum degree that we found in the previous section, we will now look at the connectivity of $\overline{\Gamma(\mathbb{Z}_n)}$ when $n = p_1^{e_1}p_2^{e_2}$. A *vertex cut* of a graph G is a subset $S \subseteq V(G)$ such that $G - S$ has more than one component or is a single vertex. The *connectivity* of G, denoted $\kappa(G)$, is the cardinality of the smallest set S such that S is a vertex cut.

Let us establish the following lemma to use in our proofs.

Lemma 7.1. *Let $G = \overline{\Gamma(\mathbb{Z}_n)}$ where $n = {p_1}^{e_1} \cdots p_r^{e_r}$ and $e_1 > 1$. Then $|N_{v_1}| = \delta(G) = p_1^{e_1-1}(p_1-1)p_2^{e_2} \cdots p_r^{e_r} - p_1^{e_1-1}(p_1-1)p_2^{e_2-1}(p_2-1) \cdots p_r^{e_r-1}(p_r-1)$.*

Proof.
Recall $v_1 = p_1^{e_1-1}p_2^{e_2} \cdots p_r^{e_r}$. Identifying $\mathbb{Z}_n$ with $\mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_r^{e_r}}$ using the Chinese Remainder Theorem, we can write $v_1 = (p_1^{e_1-1}, 0, \ldots, 0)$. Then N_{v_1} contains all vertices w such that the first component contains a unit in $\mathbb{Z}_{p_1^{e_1}}$ and the other components can be anything as long as one of the components is a zero divisor. The number of choices for the first component can be found using the Euler phi function to give us the number of units in $\mathbb{Z}_{p_1^{e_1}}$. This will give us $p_1^{e_1-1}(p_1-1)$ choices for the first component. Since the other components can be anything, there are $p_i^{e_i}$ choices for each of the other components $2 \leq i \leq r$. We must also subtract off the cases where all the components are units which will be $p_1^{e_1-1}(p_1-1) \cdots p_r^{e_r-1}(p_r-1)$. Therefore, there are $p_1^{e_1-1}(p_1-1)p_2^{e_2} \cdots p_r^{e_r} - p_1^{e_1-1}(p_1-1)p_2^{e_2-1}(p_2-1) \cdots p_r^{e_r-1}(p_r-1)$ neighbors of v_1, and since v_1 is a vertex of minimum degree it is also the minimum degree of G, denoted $\delta(G)$. ■

Theorem 7.2. *Let $G = \overline{\Gamma(\mathbb{Z}_n)}$, where $n = {p_1}^{e_1}{p_2}^{e_2}$ and $e_1 > 1$. Then the connectivity $\kappa(G)$ is equal to the minimum degree $\delta(G)$.*

Proof.
It is well-known that $\kappa(G) \leq \delta(G)$. We will show that $\kappa(G) \geq \delta(G)$, leading to $\kappa(G) = \delta(G)$.
Let $n = p_1^{e_1} p_2^{e_2}$. By the Chinese Remainder Theorem, $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}}$. By Theorem **??** a vertex of minimum degree in $\mathbb{Z}_n$ is $v_1 = {p_1}^{e_1-1} {p_2}^{e_2}$. Since we are using the Chinese Remainder Theorem, this vertex is identified with $(p_1^{e_1-1}, 0)$. The neighborhood set of v_1 contains all vertices w such that $w = (u_1, a)$ where u_1 is a unit in $\mathbb{Z}_{p_1^{e_1}}$ and a is not a unit in $\mathbb{Z}_{p_2^{e_2}}$. Therefore, we have two cases.
Case 1: $w = (u_1, 0)$
Case 2: $w = (u_1, k_1 p_2^{n_2})$ where $k_1 \in \mathbb{Z}$, $k_1 \neq 0$, and $1 \leq n_2 < e_2$
In Case 1, w is not adjacent to $z = (0, b)$ where $b \neq 0$.
In Case 2, w is not adjacent to $z = (0, k_2 p_2^{e_2 - n_2})$ where $k_2 \in \mathbb{Z}$, $k_2 \neq 0$.
We would like to find a vertex y such that y is adjacent to w and z in both of the cases above. We choose $y = (x_1, u_2)$ where x_1 is a nonzero zero divisor in $\mathbb{Z}_{p_1^{e_1}}$ and u_2 is a unit in $\mathbb{Z}_{p_2^{e_2}}$.
We now look at G', a subgraph of G where $|V(G')| = |V(G)| - |N_{v_1}| + 1$. By construction, we know that at least one vertex of the form w will remain in G'. We want to prove that for every vertex u in G' there exists a u,w-path. If u is adjacent to w, then we are done. If u is not adjacent to w, then it is of the form z described above. If one vertex of the form y exists in G', then a u,w-path exists through y. We know $|N_{v_1}| = p_1^{e_1-1}(p_1-1)p_2^{e_2} - p_1^{e_1-1}(p_1-1)p_2^{e_2-1}(p_2-1)$ by Lemma **??**. If we have more than $|N_{v_1}|-1$ vertices like y, then we will have $\kappa(G) \geq \delta(G)$. To find the number of vertices of the form y, we multiply the number of nonzero zero divisors in $\mathbb{Z}_{p_1^{e_1}}$ by the number of units in $\mathbb{Z}_{p_2^{e_2}}$. The number of nonzero zero divisors in $\mathbb{Z}_{p_1^{e_1}}$ is $p_1^{e_1} - p_1^{e_1-1}(p_1-1) - 1$ by Lemma **??**, and the number of units in $\mathbb{Z}_{p_2^{e_2}}$ is $p_2^{e_2-1}(p_2-1)$. Therefore, the total number of possibilities for y is $(p_1^{e_1} - p_1^{e_1-1}(p_1-1) - 1)(p_2^{e_2-1}(p_2-1))$.
We now want to prove that the following inequality is true

$$(p_1^{e_1} - p_1^{e_1-1}(p_1-1) - 1)(p_2^{e_2-1}(p_2-1)) > p_1^{e_1-1}(p_1-1)p_2^{e_2} - p_1^{e_1-1}(p_1-1)p_2^{e_2-1}(p_2-1) - 1$$

which can be reduced to

$$p_2^{e_2-1}(p_1^{e_1-1}p_2 - p_2 - p_1^{e_1} + 1) > -1.$$

We know that $p_2^{e_2-1}$ will always be positive, so if we prove $p_1^{e_1-1}p_2 - p_2 - p_1^{e_1} + 1 \geq 0$ the inequality will hold. Now

$$p_2(p_1^{e_1-1} - 1) \geq p_1^{e_1} - 1$$

$$\Leftrightarrow p_2 \geq \frac{p_1^{e_1} - 1}{p_1^{e_1-1} - 1}$$

$$\Leftrightarrow p_2 \geq \frac{p_1^{e_1} - p_1 + p_1 - 1}{p_1^{e_1-1} - 1}$$
$$\Leftrightarrow p_2 \geq p_1 + \frac{p_1 - 1}{p_1^{e_1-1} - 1}$$

Since $\frac{p_1-1}{p_1^{e_1-1}-1} \leq 1$ when $e_1 > 1$, the inequality holds when $p_2 \geq p_1 + 1$. Since $p_1 < p_2$ by construction, $p_2 \geq p_1 + 1$ will always hold. This proves that $\kappa(G) \geq \delta(G)$, and we can conclude that $\kappa(G) = \delta(G)$. ■

Theorem 7.3. *Let $G = \overline{\Gamma(\mathbb{Z}_n)}$, where $n = p_1 {p_2}^{e_2}$ and $e_2 > 1$. Then $\kappa(G) = \delta(G) - |A_{v_1}| + 1$.*

Proof.
We will once again use the Chinese Remainder Theorem to say $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2^{e_2}}$ and a minimum degree vertex is $v_1 = (1, 0)$. Knowing that $\mathbb{Z}_{p_1}$ is a field, we can partition the vertices of G into four sets as follows.

$$X_1 = \{(a, 0) : a \neq 0\}$$
$$X_2 = \{(a, x) : a \neq 0, x \in Z(\mathbb{Z}_{p_2^{e_2}})^*\}$$
$$X_3 = \{(0, x) : x \in Z(\mathbb{Z}_{p_2^{e_2}})^*\}$$
$$X_4 = \{(0, u) : u \, \text{is a unit in} \, \mathbb{Z}_{p_2^{e_2}}\}$$

Note that the sets above will be nonempty as long as $e_2 > 1$. We know that all vertices in X_1 are adjacent to all vertices in X_2. Also, all vertices in X_2 and X_3 are adjacent to all vertices in X_4. The vertices in X_3 and X_4 will not be adjacent to the vertices in X_1. Note that $v_1 \in X_1$ and there are $p_1 - 1$ vertices in X_1. Also, there are $(p_1 - 1)(p_2^{e_2} - p_2^{e_2-1}(p_2 - 1) - 1)$ vertices in X_2, derived similarly to the number of vertices of the form y in Theorem **??**. Since $X_1 = A_{v_1}$ is a clique and v_1 is not adjacent to itself this allows us to conclude that deg $v_1 = |X_2| + |X_1| - 1 = p_1 - 1 + (p_1 - 1)(p_2^{e_2} - p_2^{e_2-1}(p_2 - 1) - 1) - 1$. In addition, there are $p_2^{e_2-1}(p_2 - 1)$ vertices in X_4.
Now we will show that $\kappa(G) \leq \delta(G) - |A_{v_1}| + 1$. Examine $G - X_2$. If we remove X_2 from the graph, then X_1 will be isolated from the rest of the graph, i.e. the graph will be disconnected. The number of vertices in X_2 is $(p_1 - 1)(p_2^{e_2} - p_2^{e_2-1}(p_2 - 1) - 1)$ which is equivalent to $\delta(G) - |A_{v_1}| + 1$. Therefore, we can conclude that $\kappa(G) \leq \delta(G) - |A_{v_1}| + 1$. Since all vertices in X_2 and X_3 are adjacent to all vertices in X_4 if we prove $|X_4| \geq$ deg $v_1 - |A_{v_1}| + 1$ we can conclude that $\kappa(G) \geq \delta(G) - |A_{v_1}| + 1$. Therefore, we want to prove the following inequality

$$p_2^{e_2-1}(p_2 - 1) \geq (p_1 - 1)(p_2^{e_2} - p_2^{e_2-1}(p_2 - 1) - 1)$$

which can be reduced to

$$p_2^{e_2-1}(p_2 - p_1) \geq 1 - p_1.$$

We can say that $1 - p_1 \leq -1$ and $p_2^{e_2-1}(p_2 - p_1) \geq 1$. Therefore, the inequality is always true and we can conclude that $|X_4| \geq \deg v_1 - |A_{v_1}| + 1$. The vertices in X_4 correspond to our vertices of the form y used in the proof of Theorem **??** and the vertices in X_2 correspond to the vertices of the form w. By a similar argument, we can conclude $\kappa(G) \geq \delta(G) - |A_{v_1}| + 1$ leading us to $\kappa(G) = \delta(G) - |A_{v_1}| + 1$. ■

8 Conclusion

We conclude this paper with a suggestion for further research. We have looked at connectivity for $\overline{\Gamma(\mathbb{Z}_n)}$ when $n = p_1^{e_1}p_2^{e_2}$, but these findings could be generalized to the case where n is the product of three or more powers of primes. Our conjecture for this case is presented below.

Conjecture 8.1. *Let $G = \overline{\Gamma(\mathbb{Z}_n)}$ where $n = {p_1}^{e_1}{p_2}^{e_2} \cdots p_r^{e_r}$ for primes $p_1 < \ldots < p_r$. Then $\kappa(G) = \delta(G) - |A_{v_1}| + 1$ when $e_1 = 1$ and $\kappa(G) = \delta(G)$ when $e_1 > 1$.*

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10 Glossary

adjacent Two vertices u and v are *adjacent* if there is an edge connecting u and v in the graph.

annihilator Let R be a ring and $a \in R$. Define the *annihilator* of a to be the set $\text{ann}(a) = \{x \in R : ax = 0\}$.

associates Let R be a ring. Two elements $a, b \in R$ are called *associates* if there exists a unit $u \in R$ such that $a = ub$.

bipartite A graph is *bipartite* if its vertex set can be partitioned into two non-empty independent subsets.

center Let G be a graph and $u, v \in V(G), u \neq v$. The *center* of G is the subgraph induced by the set of vertices of minimum eccentricity.

clique A *clique* is a subset $C \subseteq V(G)$ such that the subgraph induced by C contains all the possible edges.

complement The *complement* of G, denoted $\bar{G}$ is the graph such that $V(\bar{G}) = V(G)$ and $xy \in E(\bar{G}) \Leftrightarrow xy \notin E(G)$.

complete graph The *complete graph* on n-vertices, denoted K_n, is the graph with n vertices and all possible edges.

connected Let G be a graph. G is *connected* if for every $u, v \in V(G), u \neq v$, there exists a $u, v - path$.

connectivity The *connectivity* of G is $\kappa(G) = min\{k :$ there exists a vertex cut S, $|S| = k\}$.

degree The *degree* of a vertex $v \in V(G)$ is the number of edges containing v as an endpoint.

distance Let G be a graph and $u, v \in V(G), u \neq v$. The *distance* from u to v is $d(u, v) = min\{length(P) :$ P is a $u, v - path\}$. If there is no $u, v - path$ then $d(u, v) = \infty$.

eccentricity Let G be a graph. The *eccentricity* of $u \in V(G)$ is $\varepsilon_G(u) = \varepsilon(u) = \max d(u, v)$ where $v \in V(G)$ and $v \neq u$.

independent set An *independent set* is a subset $S \subseteq V(G)$ such that the subgraph induced by S has no edges.

path A *path* is a sequence of vertices and edges in G such that no vertex is repeated, unless this path is a cycle where the first and last vertices are the same.

planar A graph is *planar* if it can be drawn in the plane without any two edges crossing.

simple graph A *simple graph* is a pair (V, E) where $V = V(G)$, a vertex set, and $E = E(G)$, an edge set, and each $e \in E$ is an unordered pair $\{u, v\}$ of distinct vertices $\{u, v\} \in V$.

subgraph Let G be a graph. A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and for every $e \in E(H)$, the endpoints of e are in $V(H)$.

subgraph induced by S Let G be a graph, $S \subseteq V(G)$. The *subgraph induced by* S is the graph H such that $V(H) = S$ and $E(H) = \{v_1v_2 \in E(G) : v_1, v_2 \in S\}$.

unit Let R be a ring. An element $a \in R$ is called a *unit* if there exists a $b \in R$ such that $ab = 1$.

vertex cut A *vertex cut* of graph G is a subset $S \subseteq V(G)$ such that $G - S$ has more than one component or is a single vertex.

vertices of minimum degree Let G be a finite graph. The *vertices of minimum degree* are defined as $\delta(G) = min\{deg(v) : v \in V(G)\}$.

zero divisor Let R be a ring. An element $z \in R$ is called a *zero divisor* if there exists a $b \in R$, $b \neq 0$ such that $zb = 0$.

References

[1] R. Akhtar, L. Lee. *Zero-divisor graphs via ideals*, preprint.

[2] D. F. Anderson, A. Frazier, A. Lauve, P. Livingston. *The zero-divisor graph of a commutative ring. II* Lect. Notes Pure and Appl. Math. **220** (2001), 61-72.

[3] D. F. Anderson and P. Livingston. *The zero-divisor graph of a commutative ring.* J. Algebra. **217** (1999), no. 2, 434–447.

[4] M. Brickel. *Complements of Zero Divisor Graphs*. Plan B Project, Miami University, 2004.

[5] A. Lauve. *Zero-divisor graphs of finite commutative rings*. Senior Thesis, University of Oklahoma, 1999.

Amanda Phillips
Franklin College
Franklin, IN
`aphillips2@franklincollege.edu`

Julie Rogers
Marymount University
Arlington, VA
`jfr12301@marymount.edu`

Kevin Tolliver
Morehouse College
Atlanta, GA
`kptco06@hotmail.com`

Frances Worek
The Pennsylvania State University
University Park, PA
`few112@psu.edu`