DISCRETE MORSE THEORY AND FUNDAMENTAL GROUPS IN MODULI SPACES OF PLANAR LINKAGES

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1. Abstract

Following R. Forman[2], G. Panina and A. Zhukova[4] determined a discrete gradient vector field for the moduli spaces on planar linkages. Using this discrete gradient vector field and applying the rewriting system develop by D. Farley and L. Sabalka[1], we were able to compose an easier method for computing the fundamental groups of these moduli spaces.

2. Introduction

We consider the moduli spaces of planar polygonal n-linkages in which cells are labeled by cyclically ordered partitions of the set $[n] = \{1, \ldots, n\}$[4]. In their paper, G. Panina and A. Zhukova introduce a method of applying a version of R. Forman's perfect discrete Morse function on these spaces. We, however, did not develop techniques following the *perfect* discrete Morse function, as our research does not utilize the *path reversing technique* developed originally by R. Forman. The purpose of understanding this discrete Morse function was to assist us in efficiently computing the fundamental groups of the moduli spaces.

In order to compute the fundamental groups we adopted D. Farley and L. Sabalka's rewrite techniques. We used **Theorem 3.21** to tie together the ideas of the discrete Morse function and the fundamental groups of the moduli spaces. This theorem allowed us to represent the fundamental groups as the relations of *critical cells* in the configuration spaces.

In our research we developed algorithms which simplify the rewriting of the fundamental groups and provide us the means to easily observe which *redundant 1-cells* are *collapsible* and which are *critical*.

More specifically, Section 3 of our paper provides the background information regarding moduli spaces, cell structures, discrete Morse theory and the rewriting system. The final section of our paper presents our research and the procedures we found which shorten the act of computing fundamental groups considerably.

3. Background

3.1. Moduli Spaces of Planar Polygons.

The definitions used in this section are those used by G. Panina and A. Zhukova in their work to explain the attributions of moduli spaces.

Definition 3.1. [4] A *polygonal n-linkage* is a sequence of positive numbers $L = (l_1, ..., l_n)$. It should be interpreted as a collection of rigid bars of lengths l_i joined consecutively in a

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chain by revolving joints. We always assume that the triangle inequality holds, that is,

$$\forall j, l_j < \frac{1}{2}\sum_{i=1}^{n} l_i \tag{1}$$

which guarantees that the chain of bars can close. A *planar configuration* of L is a sequence of points

$$P = (p_1, ..., p_n), p_i \in \mathbb{R}^2 \tag{2}$$

With $l_i = |p_i, p_{i+1}|$, and $l_n = |p_n, p_1|$. We also call P a *polygon*.

Definition 3.2. [4]The *moduli space of the linkage L*, or the *configuration space* M(L) is the set of all configurations modulo orientation preserving isometries of $\mathbb{R}^2$.

Equivalently, we can define $M(L)$ as

$$M(L) = \{(u_1, ..., u_n) \in (S^1)^n : \sum_{i=1}^{n} l_i u_i = 0\}/SO(2). \tag{3}$$

Here SO(2) acts diagonally on $(S^1)^n$: if $A \in$ SO(2), then $A(u_1, u_2, , u_n) = (Au_1, , Au_n)$. The latter definition shows that M(L) does not depend on the ordering of $\{l_i, ..., l_n\}$; however, it does depend on the values of l_1. We also assume that $l_1 \leq l_2 \leq \ldots \leq l_n$.

Definition 3.3. A subset I of $[n] = \{1, 2..., n\}$ is said to be *short* if

$$\sum_I l_i < \frac{1}{2}\sum_I^{n} l_i \tag{4}$$

Definition 3.4. A subset I of $[n] = \{1, 2..., n\}$ is said to be *long* if

$$\sum_I l_i > \frac{1}{2}\sum_I^{n} l_i \tag{5}$$

Definition 3.5. A partition of $[n] = \{1, 2, ...n\}$ is said to be *admissible* if all parts are short.

Definition 3.6. The set containing the entry "n" is call the $n - set$. By convention, the $n - set$ is written at the end.

3.2. **Cell Structure.**

Definition 3.7. In [], states that a *cell* is labeled by cyclically ordered admissible partitions of the set $[n]$ into $(n - k)$ non-empty parts,where the $n - set$ strand is always at the end. Here k represents the dimension of the cell,and therefore a cyclically ordered partition of $[n]$ into $(n - k)$ pieces determines the $k - dimensional$ cell in $M(L)$.

Example 3.1. Suppose $n = 5$.

(1) A 0-cell is determined by a partition with 5 elements. So, $(\{1\}, \{2\}, \{3\}, \{4\}, \{5\})$ and $(\{2\}, \{1\}, \{3\}, \{4\}, \{5\})$ are 0-cells.
(2) A 1-cell is determined by a partition with 4 elements. Thus, $(\{1, 2\}, \{3\}, \{4\}, \{5\})$ and $(\{2\}, \{1, 3\}, \{4\}, \{5\})$ are 1-cells.
(3) A 2-cell is determined by a partition with 3 elements.Thus, we have two different examples of a 2-cell.

(a) The first type would have two sets that contain two entries, while the other set is a singleton. An example of this is $(\{1,2\},\{3,4\},\{5\})$.
(b) The second type would have one set containing three entries, while the other two sets are singletons. Thus, $(\{1,2,3\},\{4\},\{5\})$ is an example.

In general, the first type of two-cell would have two sets that contain two entries, while the other sets are singletons, and the second type would have one set containing three entries, while the other sets are singletons.

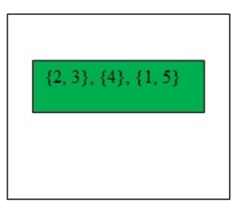
Now, we can begin to draw some 2-cells. So, given a cell c, we obtain the faces by splitting one of the partite sets c into two nonempty parts[4]. In our drawing of the cell, the faces of the particular cell are represented by the sides of the figure and the vertices represent the transition from one face to the next. Also, it should be noted that the order in a partite set does not matter, but our convention is to write the numbers in increasing order. Therefore, all of our examples will be done using this convention.

Example 3.2. Steps to draw the first type of 2-cell.
First, let's list the faces and vertices of ({2,3},{4},{1,5}).

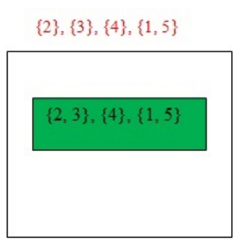
(1) ({2,3},{4},{1},{5})
  - ({2},{3},{4},{1},{5})
  - ({3},{2},{4},{1},{5})
(2) ({1},{2,3},{4},{5})
  - ({1},{2},{3},{4},{5})
  - ({1},{3},{2},{4},{5})
(3) ({2},{3},{4},{1,5})
  - ({2},{3},{4},{1},{5})
  - ({1},{2},{3},{4},{5})
(4) ({3},{2},{4},{1,5})
  - ({3},{2},{4},{1},{5})
  - ({1},{3},{2},{4},{5})

Thus, we have 4 faces and 4 vertices, and our figure is a square.

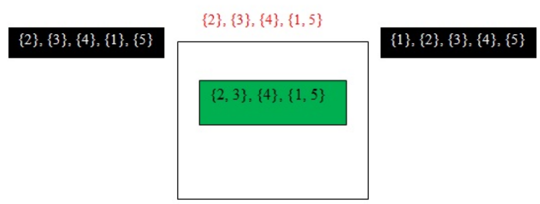
(1) Draw a square, and label the square by writing the 2-cell we are going to draw.



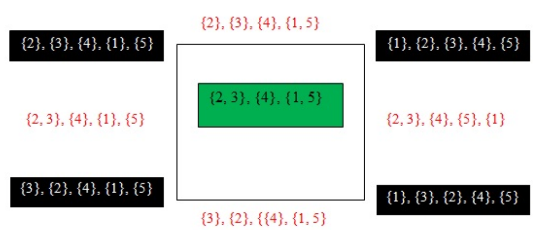
(2) Start with the top edge, and write a 1-cell that would be a face of the 2-cell.



(3) Next, fill in the adjacent vertices, by referring to our list above.
  (a) The vertex that is always on the left of a particular face is all singletons, and in the same order as the 1-cell on that face. The vertex that always on the right is all singletons, in the same order as the 1-cell on the face, but the singletons that are paired will split in the opposite order. Thus, we get the top edge of the square, and the first face of the 2-cell.



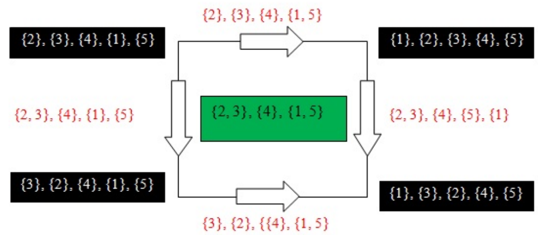
(4) Getting the other faces of the 2-cell
  (a) By referring to our list above, we can look to see the vertices that the faces have in common,fill in the next face,then vertex, and eventually fill in the whole square. For example, as we can see from our list,({2},{3},{4},{1,5}) and ({1},{2,3},{4},{5}) have a vertex in common, so they should be next to each other, and the only vertex left for that face is ({1},{3},{2},{4},{5}), so that should be the next vertex.



(5) Lastly, we write the orientation on the faces of the 2-cell. To determine orientation, we look at the face of the 2-cell and the vertices that are adjacent to that face.
  (a) The arrow should be leaving the adjacent vertex to a particular edge that is all singletons, and is in the same order as the 1-cell on that face.
  (b) The arrow should be pointing towards the vertex that is all singletons, in the same order as the 1-cell on the face, but the singletons that are paired will split in the opposite order.
  (c) Note: This orientation, where the arrows on the left and right faces both point downwards, and the arrows on the top and bottom faces both point right will always be the same for 2-cells of this type.

Next, we will draw the second type of 2-cell, where we have a set with three entries and the other sets are just singletons.

The 2-cell we will be drawing is ({1,2,3},{4},{5}), and we will still adhere to the rules about the orientation of the arrows in relation to the vertices and faces, as we did with the squares.



Example 3.3. Drawing the second type of 2-cell.

Below, there is a list of all of the faces and their associated vertices. Again, we determined the faces of this 2-cell by the face relation,which is that we obtain faces of a cell by splitting one of its partite sets into 2 nonempty parts []. In this case, a face would be a cell where 2 out of the 3 entries in the 3 entry set are paired and the 3^{rd} entry is a singleton placed either on the right or left of this 2 entries that are paired. Thus, there are 6 possible faces and 6 vertices.

(1) ({1,2},{3},{4},{5})
- ({1},{2},{3},{4},{5})
- ({2},{1},{3},{4},{5})

(2) ({3},{1,2},{4},{5})
- ({3},{1},{2},{4},{5})
- ({3},{2},{1},{4},{5})

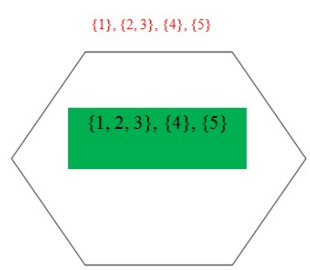
(3) ({2,3},{1},{4},{5})
- ({2},{3},{1},{4},{5})
- ({3},{2},{1},{4},{5})

(4) ({1},{2,3},{4},{5})
- ({1},{2},{3},{4},{5})
- ({1},{3},{2},{4},{5})

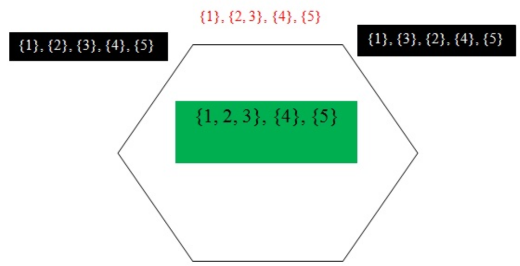
(5) ({1,3},{2},{4},{5})
- ({1},{3},{2},{4},{5})
- ({3},{1},{2},{4},{5})

(6) ({2},{1,3},{4},{5})
- ({2},{1},{3},{4},{5})
- ({2},{3},{1},{4},{5})

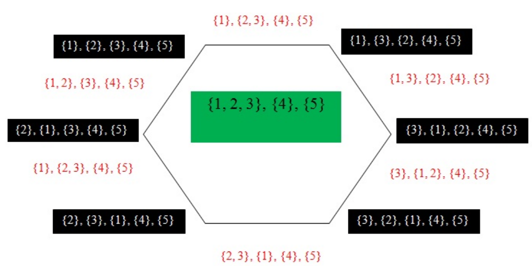
(1) Draw a hexagon, start at the top edge with a face of the 2-cell.



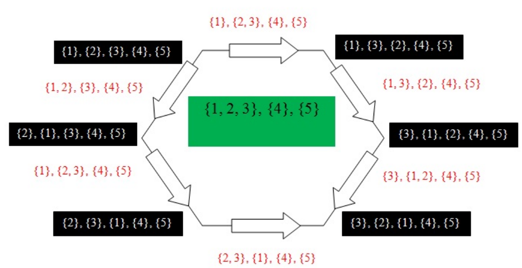
(2) Next, write in the adjacent vertices on the top edge, using our rule from drawing the squares and referring to the list above.



(3) Finishing the hexagon:
Start from the top right edge and go around the hexagon. By referring to our list above, we can look to see the vertices that the faces have in common,fill in the next face,then vertex, and eventually fill in the whole hexagon. For example, as we can see from our list,({1},{2,3},{4},{5}) and ({1,3},{2},{4},{5}) have a vertex in common, so they should be next to each other, and the only vertex left for that face is ({3},{1},{2},{4},{5}), so that should be the next vertex.



(4) Lastly, we would place arrows on the edges of the hexagon based on orientation, as we did for the squares.



3.3. **Discrete Morse Theory.**

Definition 3.8. [2] A *discrete vector field* is a map

$$W : K \to K \cup \{0\} \tag{6}$$

satisfying

(1) for each p, $W(K_p) \subseteq K_{(p+1)} \cup \{0\}$
(2) for each $\sigma^p \in K_p$, either $W(\sigma) = 0$ or σ is a regular face of $W(\sigma)$

(3) if $\sigma \in \text{Image}(W)$ then $W(\sigma) = 0$
(4) for each $\sigma^p \in K_p$

$$\#\{v^{p-1} \in K_{p-1} | W(v) = \sigma\} \leq 1 \tag{7}$$

Definition 3.9. [2] Let W be a combinatorial vector field. A *W-path of dimension p* is a sequence of p-cells

$$\gamma = \sigma_1, \sigma_1, ..., \sigma_r$$

such that

(1) if $W(\sigma_i) = 0$ then $\sigma_{i+1} = \sigma_i$
(2) if $W(\sigma_i) \neq 0$ then $\sigma_{i+1} \neq \sigma_i$ and $\sigma_{1+1} < W(\sigma_i)$

say γ is a *closed path* if $\sigma_r = \sigma_0$ and γ is *non-stationary* if $\sigma_1 \neq \sigma_0$.

Theorem 3.10. *Let W be a discrete vector field. There is a discrete Morse function f with $W = V_f$ if and only if W has no non-stationary closed paths. Moreover, for every such W, f can be chosen to have the property that if σ^p is critical, then*

$$f(\sigma) = p$$

Such a Morse function is said to be self-indexing.

Remark 3.11. Note that W determines the critical points of f. Namely, if $W = V_f$, then, by the lemma 6.3, $\sigma \in K_p$ is critical for f if and only if $W(\sigma) = 0$ and $\sigma \notin Image(W)$.

Definition 3.12. Let W be a discrete vector field on X. A *W-path of dimension p* is a sequence of p-cells $\sigma_0, \sigma_1, ..., \sigma_r$ such that if $W(\sigma_i)$ is undefined, then $\sigma_{i+1} = \sigma_i$, and otherwise $\sigma_{i+1} \neq \sigma_i$ and $\sigma_{i+1} < W(\sigma_i)$. The W-path is closed if $\sigma_r = \sigma_0$, and *non-stationary* if $\sigma_1 \neq \sigma_0$. A discrete vector field W is a *discrete gradient vector field* if W has no non-stationary closed paths.

Definition 3.13. Given any discrete gradient vector field W on X, there is an associated classification of cells in X into 3 types: redundant, collapsible, and critical (this terminology is partially borrowed from [12] as well as from Ken Brown, [7]). A cell $\sigma \in K$ is *redundant* if $\sigma \in$ domW, *collapsible* if $\sigma \in$ imW, and critical otherwise. The *rank* of a cell c with respect to a discrete gradient vector field W is the length of the longest W-path $c = c_1, ..., c_r$ having the property that $c_i \neq c_j$ if $i \neq j$. Critical and collapsible cells all have rank 1, and redundant cells are of rank at least 2. If $c' < W(c)$ and $c \neq c'$, then clearly $rank(c') < rank(c)$.

Definition 3.14. An *alphabet* is simply a set $\sum$. The *free monoid* on $\sum$, denoted $\sum^*$, is the collection of all positive words in the generators $\sum$, together with the empty word, endowed with the operation of concatenation.

Definition 3.15. A *monoid presentation* denoted $\langle \sum |R\rangle$, consists of an alphabet $\sum$ together with a collection R of ordered pairs of elements in $\sum^*$. An element of R should be regarded as an equality between words in $\sum^*$, but, in what follows, the order will matter.

Definition 3.16. A *rewrite system* Γ is an oriented graph. The vertices of Γ are called *objects* and the positive edges are called *moves*. If v_1 is the initial vertex of some positive edge in Γ and v_2 is the terminal vertex, then write $v_1 \rightarrow_\Gamma v_2$ or $v_1 \rightarrow v_2$ if the name of the specific rewrite system is clear from the context. An object is called *reduced* if it is not the initial vertex of any positive edge (move). The reflexive, transitive closure of $\rightarrow$ is denoted $\dot{\rightarrow}$.

Definition 3.17. Every monoid presentation $\langle \sum | R\rangle$ has a natural rewrite system, called a *string rewriting system*, associated to it. The set of objects of this string rewriting system is the free monoid $\sum^*$. There is a move from $w_1 \in \sum^*$ to $w_2 \in \sum^*$ if $w_1 = ur_1v$ and $w_2 = ur_2v$ in $\sum^*$, where u, v $\in \sum^*, (r_1, r_2) \in R$.

Proposition 3.18. (1) *The inclusion of X'_n into X induces an isomorphism from $\pi_{n-1}(X'_n)$ to $\pi_{n-1}(X)$.*
(2) *If X has no critical cells of dimension greater than k, then $X \searrow X'_k$.*
(3) *X is homotopy equivalent to a CW complex with m_n cells of dimension n, where m_n is the number of critical n-cells in X.*
(4) *The subcomplex of X generated by the collapsible and critical edges is connected.*
(5) *The subcomplex of X generated by the collapsible edges and the 0- skeleton of X is a maximal forest.*
(6) *If there is only one critical 0-cell, then the graph consisting of (the closures of) the collapsible edges is a maximal tree in X.*

3.3.1. *Monoid Presentation.* Choose a maximal tree T of X consisting of all of the collapsible edges in X, and additional critical edges, as needed.
Define a monoid presentation $MP_{W,T}$ as follows: Generators are oriented edges in X, both positive and negative, so that there are two oriented edges for each geometric edge in X. If e denotes a particular oriented edge, let $\overline{e}$ denote the edge with the opposite orientation. If w denotes a sequence of oriented edges $e_1...e_m$, let $\overline{w}$ denote sequence of oriented edges $\overline{e_m}...\overline{e_1}$.

Definition 3.19. A *boundary word* of a 2-cell c is simply one of the possible relations determined by an attaching map for c (cf. [19], page 139); if w_1 and w_2 are two boundary words for a cell c, then w_1 can be obtained from w_2 by the operations of inverting and taking cyclic shifts.

There are several types of relations.
(1) For a given oriented edge e in T , introduce the relations $(e, 1)$ and $(\overline{e}, 1)$.
(2) For any oriented edge e, introduce relations $(e\overline{e}, 1)$ and $(\overline{e}e, 1)$.
(3) For a collapsible 2-cell c, consider the (unique) geometric 1-cell e such that $e = W^{-1}(c)$. Suppose that a boundary word of c is ew. In this case, the word w contains no occurrence of e or $\overline{e}$, since the geometric edge corresponding to e is a regular face of c. Introduce the relations $(e, \overline{w})$ and $(\overline{e}, w)$.

Proposition 3.20. *The rewrite system associated to the monoid presentation $MP_{W,T}$ is complete.*

In view of Proposition [], there is a unique reduced word in each equivalence class modulo the presentation $MP_{W,T}$. If w is any word in the generators of $MP_{W,T}$, let $r(w)$ denote the unique reduced word that is equivalent to w.

Theorem 3.21. *Let X be a finite connected CW complex with a discrete gradient vector field W. Then:*

$$\pi_1(X) \cong \langle \Sigma | R\rangle, \tag{8}$$

where $\sum$ is the set of positive critical 1-cells that are not contained in T , and $R = \{r(w) | w$ is the boundary word of a critical 2-cell$\}$.

In case there is just one critical 0-cell, the discrete gradient vector field W completely determines the maximal tree T , and we denote the presentation $\langle \Sigma | R\rangle$ from the previous theorem P_W , where the oriented CW complex is understood. The presentation P_W depends only on the choice of the boundary words for the critical 2-cells, since the string rewriting system associated to the monoid presentation $MP_{W,T}$ is complete and by the previous theorem.

3.4. **Discrete Gradient Vector Field on M(L).**

Definition 3.22. [4] A *discrete vector field* is a set of pairs (α^p, β^{p+1}) such that: each cell of the complex participates in at most one pair, and
in each pair, the cell α^p, is a facet of β^{p+1}.

Below we describe a discrete gradient vector field. According to the definition, we introduce some pairings of the cells.
Step 1. We pair together

$$\alpha = (...\ \{1\}\ I\ ...) \text{ and } \beta = (...\ \{1\} \cup I...)$$

iff the following holds:

(1) the set I does not contain n, and
(2) the set $\{1\} \cup I$ is short.

Before we pass to step 2, observe that the non-paired cells are labeled by one of the following types of labels:

$$(...\ \{1,*,n\})$$
$$(...\ \{1\}\ \{*,n\})$$
$$(...\ \{1\}\ (\text{a 1-prelong set})...)$$

Step 2. We pair together

$$\alpha = (...\ \{2\}\ I...) \text{ and } \beta = (...\ \{2\} \cup I...)$$

iff the following holds:

(1) The set I contains neither n, nor 1.
(2) The set $\{2\} \cup I$ is short.
(3) α and β were not paired at the previous step.

After this step, the non-paired cells are labeled by one of the following types of labels:

$$(...,1,2,*,\{n\})$$
$$(...,\{1\}\{2,*,n\})$$
$$(...,\{2\}\{1,*,n\})$$
$$(...,\{2\}\{1\}\{*,n\})$$
$$(...,\{2\}\ \{1\}(\text{a 1-prelong set})...)$$
$$(...,\{1\}\ (\text{a 1-prelong set})...\ \{2\}\{*,n\})$$
$$(...,\{2\}\ (\text{a 2-prelong set not containing 1})...\ \{1\}\{*,n\})$$
$$(...,\{1\}\ (\text{a 1-prelong set not containing 2})...\ \{2,*,n\})$$
$$(...,\{2\}\ (\text{a 2-prelong set not containing 1})...\ \{1,*,n\})$$

We proceed this way for all k $<n$, assuming that the step number k looks as follows:
Step k. We pair together

$$\alpha = (...\ \{k\}\ I...) \text{ and } \beta = (...\{k\} \cup I...)$$

iff the following holds:

(1) The set I contains none of n, 1, 2, ..., k-1.
(2) The set {k} $\cup$ I is short.
(3) α and β were not paired at the previous steps.

We proceed pairing for all k = 1, 2, ..., n-1.

Definition 3.23. [4] An entry k is *forward-movable* (with respect to the cell α), if it forms a singleton, which followed by a set I, $n \notin I$ such that

(1) $k < i$ for every $i \in I$, and
(2) $\{k\} \cup I$ is short.

Example 3.24. Suppose n=7

(1) Say we have a 0-cell: ({2}, {1}, {3}, {6}, {5}, {4},{7}) where L= {1,1,1,2,4,5,5}. First we will join {1} with I. In this particular case I is {3}. Since {1,3} is short ({2}, {1}, {3}, {6}, {5}, {4},{7}) is forward movable since it is in the domain.
(2) **Non-example:** The 1-cell: ({6}, {4}, {3}, {2}, {1}, {5,7}) is not forward movable because {1} cannot join with I. Since {1} cannot join with I we move onto {2}. Because {2} cannot join with I we move onto $\{k\}$. Therefore, ({6}, {4}, {3}, {2}, {1}, {5,7}) is not forward-movable.

Definition 3.25. [4] An entry k is *backward-movable* if the following holds:

(1) entry k lies in a non-singleton set J, $n \notin$ J;
(2) k=min(J);
(3) one of the following conditions hold:
    (a) the set J is preceded by a non-singleton set;
    (b) the set J is preceded by a singleton $\{m\}$ with $m >$ k;
    (c) the set J is preceded by the n-set.

Example 3.26. Suppose n=7

(1) Say we have a 1-cell: ({5}, {2}, {1,3}, {4}, {6}, {7}) where L= {1,1,1,2,4,5,5}. Note that we have an entry k that lies in a non-singleton set J, $n \notin$ J. Since k = min(J) then 1 is backward-movable.
    Note that this 1-cell came from ({5}, {2}, {1}, {3}, {4}, {6}, {7}).
(2) **Non-Example:** In the 1-cell ({2}, {4}, {3}, {5}, {6}, {1,7}) K does not lie in a non-singleton set J, $n \notin$ J. Therefore, it is not backward-movable.
(3) **Non-Example:** In the 1-cell ({3}, {2,4}, {1}, {6}, {5}, {7}) k lies in a non-singleton set J, $n \notin J$. In this case k=2 and k = min J. But J is not preceded by a non-singleton set. Nor does it satisfy that the set J is preceded by a singleton {m} with $m > k$; or that the set J is preceded by the n-set. Therefore k is not backward-movable.
    Note that the 1-cell is forward-movable since {1} can join with {6} making it ({3}, {2,4}, {1,6}, {5}, {7}).

Definition 3.27. [4]*Critical cells* are the cells that are non-paired. They are exactly those with empty set of movable entries.

Notation:

(1) By ". . . " we denote any ordered admissible collection of subsets of [n],which as well can be the empty set.
(2) By "*"we denote any subset of [n], which as well can be the empty set.
(3) A set I $\subset$ [n] is *k-prelong*, if I is short, and I $\cup$ {k} is long.
(4) For a set $I \cup [n]$ and an entry $k \in [n]$, we write $k < I$ whenever $\forall i \in I$, $k < i$.

(5) Analogously, we write $k = min(I)$ whenever k is the minimal entry of the set I.
(6) Unlike "...", by "♥" and "◆" we denote a (possibly empty) string of singletons going in the decreasing order. For instance, "◆" can be ({7}, {4}, {2}), but can be neither ({7,4,2} nor ({4}, {2}, {7}).

Theorem 3.28. [4] *The critical cells of the introduced above discrete Morse function are exactly all cells of the two following types.*

Type 1.

$$(\heartsuit\ \{*, n\}).$$

Example:

(1) $\{\{6\}, \{5\}, \{4\}, \{2\}, \{1\}, \{3, 7\}\}$ is an example of a critical 1-cell.
(2) $\{\{6\}, \{5\}, \{4\}, \{1\}, \{2, 3, 7\}\}$ is an example of a critical 2-cell.

Type 2.

(♥ { k} I ◆ { n,*}), if the following conditions hold:

(1) I is a k-prelong set not containing n.
(2) $k < I$
(3) $k <$ ♥
(In other words, (♥ $\{k\}$) is an ordered string of singletons.)

Example: Suppose n=7 where L= {1,1,1,2,4,5,5}.
Say we have the 2-cell $\{\{3\}, \{5, 6\}, \{4\}, \{2\}, \{1, 7\}\}$. In this case, k={3}. k ∪ I is k-prelong and $k < I$ therefore $\{\{3\}, \{5, 6\}, \{4\}, \{2\}, \{1, 7\}\}$ is an example of a critical 2-cell.

4. Our Research

Definition 4.1. Let e be a 1-cell in M(L). We define $p(e)$ as follows:

(1) If m is neither forward nor backward-movable $\forall m \in \{1, \ldots, n\}$, then $p(e) = e$.
(2) If m is the smallest movable entry and is backward-movable, then $p(e) = 1$.
(3) If m is the smallest movable entry and is forward-movable, then $e = (\ldots, \{m\}, I, \ldots)$ and we define $p(e) = (\ldots, I, \{m\}, \ldots)$.

Remark 4.2. We define $p(\overline{e}) = \overline{p(e)}$.

Example 4.3. Continuing to use the equilateral heptagon from the previous examples, the following statements hold:

(1) Let $e = (6, 5, 4, 2, 3, \{1, 7\})$, then $p(e) = (6, 5, 4, 3, 2, \{1, 7\})$.
(2) Let $e = (\{4, 6\}, 5, 3, 2, 1, 7)$, then $p(e) = 1$.
(3) Let $e = (2, 4, 6, 5, 1, \{3, 7\}$, then $p(e) = (4, 2, 6, 5, 1, \{3, 7\})$.

Definition 4.4. Given a redundant 1-cell $e = (\ldots, \{k\}, \{l\}, \ldots, \{m, n\})$, where k is the smallest forward-movable integer, we define $q(e)$ such that $q(e) = (\{m\}, \ldots, \{k, l\}, \ldots, \{n\})$.

Example 4.5. Given $e = (4, 6, 2, 3, 5, \{1, 7\})$, $q(e) = (1, 4, 6, \{2, 3\}, 5, 7)$.

Definition 4.6. For a 1-cell e, there is $m \in \mathbb{N}$ such that $p^m(e) = 1$ or $p^m(e) = p^{m+1}(e)$. We define $p^\infty(e) = p^m(e)$.

Example 4.7. In reference to the same heptagon that we have been using in the previous examples, we can conclude the following:

(1) If $e = (1, 2, \{3, 4\}, 5, 6, 7)$, then $p^\infty(e) = 1$.
(2) If $e = (3, 2, 4, 1, 6, \{5, 7\})$, then $p^\infty(e) = (6, 4, 3, 2, 1, \{5, 7\})$.

Theorem 4.8. *If e is a 1-cell such that n is the only element in the n-set, then $r(e) = p^{\infty}(e)$.*

Proof. $r(e)$ follows the discrete gradient vector field, i.e. $p(e)$ uses the boundary map of $v_1(e)$ to rewrite e.

(1) Case I: (I is a singleton) In this case e flows into a 2-cell, (...,{m,k},...,{i,j},...,n), which is in the shape of a square (as demonstrated **Section 2.2**). The edge e can be rewritten using the boundary of the rest of the square: $e_l e_b \overline{e_r}$ (l=left, b=bottom, r=right). For e_l and e_r the smallest unstuck integer, m, is in a pair because that is what flowing across the 2-cell represents. However, m is a collapser and so both $e_l = 1$ and $e_r = 1$. Therefore $r(e) = e_b = p(e)$ which is either the identity (if i is a collapser) or $e = p(e_b)$.
(2) Case II: (I is a pair) In this case e flows into a 2-cell, (...,{m,i,j},...,n), which is in the shape of a hexagon. So e can be rewritten as $e_1 e_2 e_3 \overline{e_4 e_5}$ (sides $1 \ldots 5$ are obtained by tracing the hexagon counterclockwise). The sides $e_1, e_2, \overline{e_4}$ and $\overline{e_5}$ all include the pairs {m,i} or {m,j}, making all those sides equal the identity (since m is a collapser). Therefore $r(e) = e_3 = p(e)$ and either $e_3 = 1$ if i is a collapser, or $e_3 = p(e_3)$.

Since v_1 is an injective function, r is vacuously confluent. Also because r follows a discrete gradient vector field, it cannot flow to non-stationary closed paths, and terminates. Therefore this rewriting system is complete and $p^{\infty}(e)$ represents a finite flow over the discrete gradient vector field. □

Theorem 4.9. *If $e = (\ldots, \{m, n\})$, where the n-set has exactly 2 elements, then*

(1) *$e \dashrightarrow p(e)$, if e is critical or the smallest forward-movable integer is less than m,*
(2) *$e \dashrightarrow p(e)\overline{q(e)}$ otherwise.*

$r(e)$ can be computed by continuously applying this procedure.

Proof.

(1) The same proof applies here as that of **Theorem 3.11**.
(2) The 2-cell that e flows into is (...,{k,l},...,{m,n}). So if e were to be rewritten using the boundary of the 2-cell $e = e_l e_b \overline{e_r}$. We can see that $e_l = 1$ because after the repeated application of $p^{\infty}(e_l)$ the integer k becomes a collapser. Also observe that $e_b = p(e)$ and $\overline{e_r} = \overline{q(e)}$. Thus $r(e) = p(e)\overline{q(e)} = s(e)$ which, follows the discrete gradient vector field so, is again confluent and terminates.

□

Theorem 4.10. *If a critical 2-cell is of the form (◆,{a,b,n}),then the generators (◆,{a,n}) and (◆,{b,n}) commute.*

Proof. Starting with the 2-cell of the form $(\blacklozenge, \{a, b, n\})$, the boundary gives the relation:

$$(\blacklozenge, a, bn)(\blacklozenge, an, b)(\blacklozenge, n, ab)\overline{(\blacklozenge, bn, a)(\blacklozenge, b, an)(\blacklozenge, ab, n)} = 1.$$

This simplifies to:

$$(\blacklozenge, bn)(\blacklozenge, an)(\blacklozenge, n, ab)\overline{(\blacklozenge, bn)(\blacklozenge, an)(\blacklozenge, ab, n)} = 1.$$

Since a is a backward-movable, $(\blacklozenge, n, ab)$ and $\overline{\blacklozenge, ab, n}$ both go to the identity, and we are left with the relation:

$$(\blacklozenge, bn)(\blacklozenge, an)\overline{(\blacklozenge, bn)(\blacklozenge, an)} = 1.$$

□

Theorem 4.11. *If a critical 2-cell is of the form (♥,{r,q},◆,{a,n}), then the critical 1-cells (♥,{r,q},◆,n) and (◆,{a,n}) commute.*

Proof. Starting with the 2-cell of the form (♥,{r,q},◆,{a,n}), the boundary give the relation:

$$(\heartsuit, r, q, \blacklozenge, \{a,n\})(a, \heartsuit, \{r,q\}, \blacklozenge, n)\overline{(\heartsuit, q, r, \blacklozenge, \{a,n\})(\heartsuit, \{r,q\}, \blacklozenge, a, n)} = 1.$$

Applying $p^{\infty}(x)$ to each edge:

$$(\blacklozenge, \{a,n\})(\heartsuit, \{r,q\}, \blacklozenge, n)\overline{(\blacklozenge, \{a,n\})(\heartsuit, \{r,q\}, \blacklozenge, n)} = 1.$$

□

Corollary 4.12. *Assume that* $\{n-3, n-2, n-1\}$ *is short. Let* $I = \{i \in \{1, \dots, n-1\} \mid \{i,n\}$ *is short*$\}$*; let* $J = \{\{j,k\} \subseteq \{1, \dots, n-1\} \mid \{j,k,n\}$ *is short*$\}$*.*

$$\pi_1(X) \cong \langle x_i \, (i \in I) : [x_j, x_k] \, (\{j,k\} \in J) \rangle.$$

Proof. Let X be a moduli space with a regular cell complex. By assumption, since {n-3,n-2,n-1} is short, there are no long 3-element sets (not including the n-sets). Therefore critical 2-cells of Type II cannot exist, leaving critical 2-cells of Type I. By **Theorem 3.13** there are $\binom{n-1}{2}$ Type I critical 2-cells withis $\binom{n-1}{n-2} = n-1$ critical 1-cells. This means that there are $n-1$ generators and they commute.

□

Corollary 4.13. *If {n-2,n-1} is long, then* $\pi_1(X) \cong \mathbb{Z}^{n-3}$.

Proof. Let X be a moduli space with a regular cell complex. Since {n-2,n-1} is long, X is disconnected and any 2-cell of type (…,{l,k},…,{m,n}) is always collapsible. This is because there will always be a collapser in the pair that is not the n-set. Therefore there are only Type II critical 2-cells of the form (◆,{a,b,n}) or (n-2,n-1,◆,{a,b,n}). Since {n-2,n-1} is long, so are {n-1,n} and {n-2,n}. Thus the only critical 1-cells are of the form (◆,{v,n}) such that $v \leq n-3$. So there are $n-3$ generators, and as a result of **Theorem 3.13** they all commute.

□

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Bibliography

[1] Daniel Farley and Lucas Sabalka. *Discrete Morse Theory and graph braid groups*. Algebraic and Geometric Topology, 5(3):1079-1085, 2005.

[2] Robin Forman. *Morse theory for cell complexes*. Advances in Mathematics, 134(1):90-145, 1998.

[3] Michael Kapovich and John Millson. *On the Moduli Space of Polygons in the Euclidean plane*. Journal of Differential Geometry, 42(1):133-164, 1995.

[4] Gaiane Panina and Alena Zhukova. *Discrete Morse theory for Moduli Spaces of Flexible Polygons, or Solitaire Game on the Circle*. ArXiv e-prints, 2015.