The Structure of Zero-Divisor Graphs

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Abstract

Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph whose vertices are the nonzero zero-divisors of $\mathbb{Z}_n$, and such that two vertices u, v are adjacent if n divides uv. Here, the authors investigate the size of the maximum clique in $\Gamma(\mathbb{Z}_n)$. This leads to results concerning a conjecture posed by S. Hedetniemi, the core of $\Gamma(\mathbb{Z}_n)$, vertex colorings of $\Gamma(\mathbb{Z}_n)$ and $\overline{\Gamma(\mathbb{Z}_n)}$, and values of n for which $\overline{\Gamma(\mathbb{Z}_n)}$ is Hamiltonian. Additional work is done to determine the cases in which $\Gamma(\mathbb{Z}_n)$ is Eulerian.

1 Introduction

A *graph* G is a set of vertices $V(G)$ and a set of edges $E(G)$ consisting of unordered pairs of distinct vertices, such that there are no loops or multiple edges. We say that two vertices $u, v \in V(G)$ are *adjacent* if there exists an edge $\{u, v\} \in E(G)$. Also, we say that the edge $\{u, v\}$ is *incident* on the vertices u and v. Given a graph $H = (V(H), E(H))$, H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Our work here is on zero-divisor graphs, particularly, the zero-divisor graph of $\mathbb{Z}_n$, denoted $\Gamma(\mathbb{Z}_n)$. In order to understand the graph $\Gamma(\mathbb{Z}_n)$, we must first give a few definitions. The ring $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots \overline{n-1}\}$ is the set of all residue classes modulo n. The operations are addition mod n and multiplication mod n. From now on we will not use the bars in order to simplify notation. An example of such a ring is

$$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}.$$

An element x of a ring R is called a *zero-divisor* if there exists some nonzero element $y \in R$ such that $xy = 0$. The set of zero-divisors of $\mathbb{Z}_{12}$ is

$$Z(\mathbb{Z}_{12}) = \{0, 2, 3, 4, 6, 8, 9, 10\}.$$

For example, since $2 \times 6 = 12$ and $12|(2 \times 6)$, then 2 and 6 are both zero-divisors.

Now we can consider $\Gamma(\mathbb{Z}_n)$, the *zero-divisor* graph. The set of vertices of this graph is the set $V(\Gamma(\mathbb{Z}_n)) = Z(\mathbb{Z}_n) - \{0\}$. Also, $\{u, v\} \in E(\Gamma(\mathbb{Z}_n))$ if and only if $uv = 0$ in $\mathbb{Z}_n$ or $n|uv$ as ordinary integers. Here we provide the graph $\Gamma(\mathbb{Z}_{12})$ as an example in Figure 1. As we can see, there are only edges between those vertices whose product is zero.

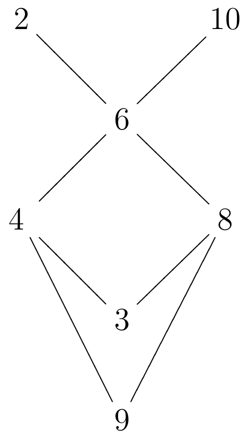

Figure 1: $\Gamma(\mathbb{Z}_{12})$

In the following section, we provide more definitions and notation which will be used in our work. Also, for the convenience of the reader, we have included a glossary of definitions at the end of this paper.

2 Preliminaries

Let G be a graph. The *subgraph induced* by S, where $S \subseteq V(G)$, is the graph H with vertex set $S = V(H)$ such that $\{x, y\} \in E(H)$ if and only if $\{x, y\} \in E(G)$ and $x, y \in V(H)$. A graph is called *complete* if for every $u, v \in V(G)$ there exists an edge $\{u, v\}$. For example, the graph in Figure 2 is complete. When the subgraph induced by a set of vertices $S \subseteq V(G)$ is complete, we call S a *clique*. The *clique number* $\omega(G)$ is the size of the largest clique in a graph G.

The number of edges incident on v, or the number of vertices adjacent to v, is the *degree* of v, denoted deg (v). The set of vertices adjacent to v is called the *neighborhood* of v. Each member of the neighborhood is called a *neighbor* of v. In particular, in a complete graph, each vertex is a neighbor of all other vertices.

A *walk* is a sequence of vertices and edges $v_0, e_0, v_1, e_1 \ldots v_{k-1}, e_{k-1}, v_k$ where $e_i = \{v_i, v_{i+1}\}$. Here, edges and vertices may be repeated any number of times. A walk is

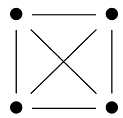

Figure 2: Complete Graph

called *closed* if the initial and terminal vertices coincide. A *path* is a walk in which no vertex is repeated, except possibly the initial and terminal vertices, and a closed path is called a *cycle*. A walk in which no edge is repeated is called a *trail*.

The *complement* of a graph G is the graph $\overline{G}$ whose vertex set is $V(G)$, but $\{x, y\} \in E(\overline{G})$ if and only if $\{x, y\} \notin E(G)$. For example, if the original graph G is complete, then $\overline{G}$ is a set of vertices with no edges between them, which we call an *independent set*. In Figure 3 we show $\overline{\Gamma(\mathbb{Z}_{12})}$ as an example.

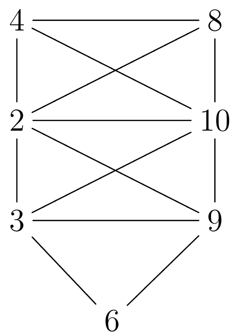

Figure 3: $\overline{\Gamma(\mathbb{Z}_{12})}$

In this paper we deal with the vertex coloring of zero-divisor graphs. A coloring of a graph G has the property that no two adjacent vertices have the same color. Moreover, we say that G is *k-colorable* if there exists a coloring of G using only k colors. We denote $\chi(G) = \min\{k : \text{G is k-colorable}\}$. For example, $\chi(\mathbb{Z}_{12}) = 2$.

Throughout this paper, we write $n = p_1^{e_1} \ldots p_r^{e_r}$, to mean a factorization of n, as a product of distinct primes $p_1 < \ldots < p_r$ with positive exponents e_i.

2.1 Associate Classes

Let R be a ring and x, y, and u be elements in R. An element u is called a *unit* if there exists an element y such that $uy = 1$. In this paper we only consider finite rings. Thus, every element is either a zero-divisor or a unit. If there exists a unit u such that $x = uy$, then we say that x and y are *associates*. As Lauve observed in [4], if two elements of the vertex set $V(\Gamma(\mathbb{Z}_n))$ are associates, then they have the same set of neighbors. Because of this, we can partition the elements of $V(\Gamma(\mathbb{Z}_n))$ in such a way that two vertices are in the same class if and only if they are associates. This partition divides the elements of $V(\Gamma(\mathbb{Z}_n))$ into *associate classes.* We denote by A_v the associate class represented by a vertex v. Note that the associate classes of $\Gamma(\mathbb{Z}_n)$ and $\overline{\Gamma(\mathbb{Z}_n)}$ are the same.

Lemma 2.1. $|A_m| = \phi(n/gcd(n,m))$

Proof. Let m be a zero-divisor in $\mathbb{Z}_n$ and $m = uv$ where u is a unit and v divides n. Then, the ideal generated by m, (m), is a subset of $\mathbb{Z}_n$. Also, since $v = gcd(n,m)$, the ideal generated by v, $(v) = (m)$. Therefore, $|(m)| = |(v)| = \frac{n}{gcd(n,m)}$.

To find the number of associates of m, note that k is associate to $m \Leftrightarrow (k) = (m) \Leftrightarrow gcd(k, |(m)|) = 1$ and $k \neq 1$. Hence, the number of generators of (m) is equal to $\phi(|(m)|)$, which is equal to $\phi(n/gcd(m,n))$.

$\square$

Phillips, et al. [5], also worked with associate classes and proved that every associate class of zero divisors of $\mathbb{Z}_n$ is either a clique or an independent set in $\Gamma(\mathbb{Z}_n)$. We use this result in our paper.

Also, in order to facilitate our study of $\Gamma(\mathbb{Z}_n)$, we sometimes use the notion of structure graphs defined by Brickel in [1]. A *structure graph* is a graph whose vertices are associate classes of zero divisors in $\mathbb{Z}_n$. Here, two vertices X and Y are adjacent if and only if $xy = 0$ for every $x \in X$ and every $y \in Y$.

3 Eulerian Graphs

A graph is *Eulerian* if it contains a closed trail containing every edge. In this section, we determine for which values of n is $\Gamma(\mathbb{Z}_n)$ Eulerian, using a result from Euler and Hierholzer that states that a graph is Eulerian if and only if every vertex in $V(\Gamma(\mathbb{Z}_n))$ has even degree. An example of an Eulerian graph is shown below in Figure 4.

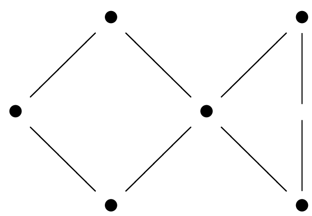

Figure 4: An Eulerian graph

Theorem 3.1. *Let $n = p_1^{e_1} \dots p_r^{e_r}$. The graph $\Gamma(\mathbb{Z}_n)$ is Eulerian if and only if:*
(i) n is odd and square-free or
(ii) $n = 4$.

Proof. Take an arbitrary zero-divisor of n, say $v = p_1^{f_1} p_2^{f_2} \dots p_r^{f_r}$ where $0 \leq f_i \leq e_i$. From [5], we know

$$\deg\,(v) = \begin{cases} gcd(n,v) - 1, & \text{if } v^2 \neq 0; \\ gcd(n,v) - 2, & \text{if } v^2 = 0. \end{cases} \tag{1}$$

To prove that a graph $\Gamma(\mathbb{Z}_n)$ is Eulerian, we need to show that there exists no vertex $v \in V(\mathbb{Z}_n)$ such that deg (v) is odd. With the preceding formula we have the tools we need to prove the theorem.

Case 1: When n is odd

Subcase 1a: Let $n = p_1^{e_1} \dots p_i^{e_i} \dots p_r^{e_r}$, such that $e_i \geq 2$. In other words, n is not square-free. Let $v = p_1^{e_1} \dots p_i^{e_i - 1} \dots p_r^{e_r}$. Then we know that $v^2 = 0$. This means that $deg(v) = gcd(n,v) - 2$. Since $gcd(n,v) = v$, and v is odd, then it follows that deg (v) is odd. Therefore, $\Gamma(\mathbb{Z}_n)$ is not Eulerian.

Subcase 1b: Let $n = p_1^{e_1} \dots p_i^{e_i} \dots p_r^{e_r}$, such that $e_1 = \dots = e_i = \dots = e_r = 1$, and let $v = p_1 \dots p_i^{e_i - 1} \dots p_r^{e_r}$. In this case, n is square-free, and $v^2 \neq 0$. This means $deg(v) = gcd(n,v) - 1$. Since $gcd(n,v) = v$, and v is odd, then it follows that deg (v) is even for every v of the form $v = p_1 \dots p_i^{e_i - 1} \dots p_r^{e_r}$. However, every vertex of $\Gamma(\mathbb{Z}_n)$ is associate to a vertex of this form. Hence, every vertex in $\Gamma(\mathbb{Z}_n)$ has even degree. So, if n is odd and square-free, then $\Gamma(\mathbb{Z}_n)$ is Eulerian.

Case 2: When n is even

Subcase 2a: $n \neq 4$
If $n = p_1^{e_1} \dots p_r^{e_r}$ then $p_1 = 2$. This means that 2 is a zero-divisor of $\mathbb{Z}_n$. If $n \neq 4$,

$\Gamma(\mathbb{Z}_n)$ is not an Eulerian graph because, the only neighbor of 2 is $n/2$.

Subcase 2b: $n = 4$
If $n = 4$, $\Gamma(\mathbb{Z}_n)$ consists of only one vertex. This is trivially Eulerian.

$\square$

4 S. Hedetniemi's Conjecture

The *direct-product* of two graphs G and G' is the graph in which $V(G \times G') = V(G) \times V(G')$. That is, the vertex set is the set of ordered pairs of vertices from G and G'. The edge set of $G \times G'$ consists of all $\{(u, u'), (v, v')\}$, such that, $u, v \in V(G)$ and $u', v' \in V(G')$, $\{u, v\} \in E(G)$ and $\{u', v'\} \in E(G')$.

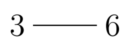

Figure 5: $\Gamma(\mathbb{Z}_9)$

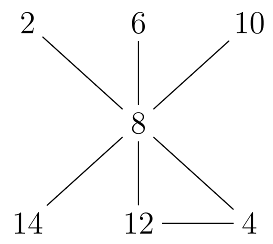

Figure 6: $\Gamma(\mathbb{Z}_{16})$

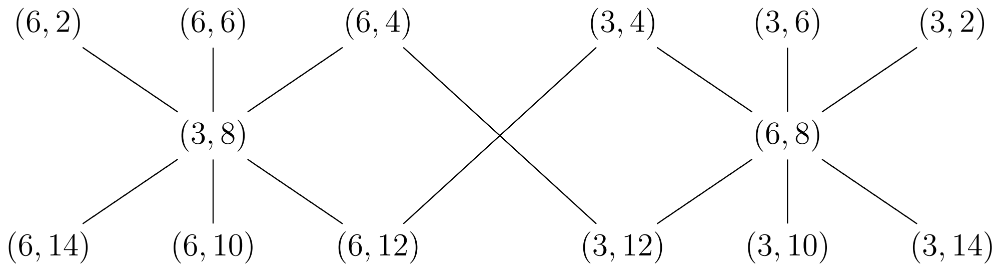

Figure 7: $\Gamma(\mathbb{Z}_9) \times \Gamma(\mathbb{Z}_{16})$

S. Hedetniemi conjectured that for all graphs G and G', $\chi(G \times G') = \min\{\chi(G), \chi(G')\}$ [3]. Here we prove Hedetniemi's conjecture for all zero-divisor graphs $\Gamma(R)$ that satisfy $\chi(\Gamma(R)) = \omega(\Gamma(R))$. In particular, we prove the conjecture for $\Gamma(\mathbb{Z}_n)$.

Notation. For ease of notation, let $\chi(\Gamma(\mathbb{Z}_n) \times \Gamma(\mathbb{Z}_m)) = \chi_{n,m}$, $\chi(\Gamma(\mathbb{Z}_n)) = \chi_n$, and $\chi(\Gamma(\mathbb{Z}_m)) = \chi_m$. Also, let ω_n denote the clique number of $\Gamma(\mathbb{Z}_n)$. Here, the parity of the exponents of the primes in the prime factorization of n is relevant. Therefore, we now let $n = p_1^{e_1} \dots p_r^{e_r} q_1^{f_1} \dots q_s^{f_s}$, where p_i and q_j are distinct primes, such that for every $1 \leq i \leq r$, e_i is even, and for every $1 \leq j \leq s$, f_j is odd.

Theorem 4.1. $\chi_{n,m} = \min\{\chi_n, \chi_m\}$

In order to prove this theorem, we must prove $\chi_{n,m} \leq \min\{\chi_n, \chi_m\}$, and $\chi_{n,m} \geq \min\{\chi_n, \chi_m\}$. However, to prove $\chi_{n,m} \geq \min\{\chi_n, \chi_m\}$ we need Lemma 4.2.

Lemma 4.2. *Let $n = p_1^{e_1} \dots p_r^{e_r} q_1^{f_1} \dots q_s^{f_s}$, where p_i and q_j are distinct primes, such that for every $1 \leq i \leq r$, e_i is even, and for every $1 \leq j \leq s$, f_j is odd. In $\Gamma(\mathbb{Z}_n)$ there exists a clique of size*

$$|\bigcup_{v|n, n|v^2} A_v| + s = p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1-1}{2}} \dots q_s^{\frac{f_s-1}{2}} + s - 1.$$

Proof. Let $v_0 = p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1+1}{2}} \dots q_s^{\frac{f_s+1}{2}}$. Note that $v_0^2 = 0$ and $v_0 \in V(\Gamma(\mathbb{Z}_n))$. Then we have,

$$\bigcup_{v|n, n|v^2} A_v = \{m p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1+1}{2}} \dots q_s^{\frac{f_s+1}{2}} : 1 \leq m \leq p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1-1}{2}} \dots q_s^{\frac{f_s-1}{2}} - 1\}.$$

In other words, the above union is the set of all the nonzero multiples of v_0 that exist in $\mathbb{Z}_n$. Thus, we know,

$$\sum_{v|n, n|v^2} |A_v| = |\bigcup_{v|n, n|v^2} A_v| = p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1-1}{2}} \dots q_s^{\frac{f_s-1}{2}} - 1.$$

Hence, for each vertex u, w in the union of associate classes, v_0 divides both u and w, and v_0^2 divides uw. Because $v_0^2 = 0$, we know n also divides uw. Therefore, $uw = 0$.

We claim that there exists s more multiples $x_i \in V(\Gamma(\mathbb{Z}_n))$ of v_0 such that $x_i v = 0$, for all

$$v \in \bigcup_{v|n, n|v^2} A_v$$

where s is as in the statement of Lemma 4.2. To see this, recall $v_0 = p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1+1}{2}} \dots q_i^{\frac{f_i+1}{2}} \dots q_s^{\frac{f_s+1}{2}}$. For $1 \leq i \leq s$, let $x_i = p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1+1}{2}} \dots q_i^{\frac{f_i-1}{2}} \dots q_s^{\frac{f_s+1}{2}}$, so that $x_i = v_0/q_i$. Clearly, x_i is not in the above union. Then $v_0 x_i = p_1^{e_1} \dots p_r^{e_r} q_1^{f_1+1} \dots q_i^{f_i} \dots q_s^{f_s+1}$, which is a multiple of n. Therefore, we also know that $v_0 x_i = 0$. Hence, $mv_0 x_i = 0$ for any m. So x_i is adjacent to all elements in

$$\bigcup_{v|n, n|v_2} A_v.$$

Also, choose j such that $i \neq j$. Then $x_i x_j = p_1^{e_1} \dots p_r^{e_r} q_1^{f_1+1} \dots q_i^{f_i} \dots q_j^{f_j} \dots q_s^{f_s+1}$. Since n clearly divides $x_i x_j$, then $x_i x_j = 0$. So every x_i is adjacent to every x_j, as well as all elements in

$$\bigcup_{v|n, n|v^2} A_v.$$

The total number of x_i's is s. Therefore, in $\Gamma(\mathbb{Z}_n)$, there exists a clique of size

$$|\bigcup_{v|n, n|v^2} A_v| + s = p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1-1}{2}} \dots q_s^{\frac{f_s-1}{2}} + s - 1.$$

$\square$

¿From Beck's results (cited in [4]) we know $\chi(\Gamma_0) = N' + s$, where $N' = p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1-1}{2}} \dots q_s^{\frac{f_s-1}{2}}$, and s denotes the number of primes raised to an odd power in the prime factorization of n. We also know that $\chi(G - \{v\}) = \chi(G) - 1$ if v is adjacent to every other vertex $w \in G$. Thus, it follows that

$$\chi_n = N' + s - 1.$$

Therefore, $\chi_n = p_1^{\frac{e_1}{2}} \dots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1-1}{2}} \dots q_s^{\frac{f_s-1}{2}} + s - 1$, which, from Lemma 4.2, is the size of a clique in $\Gamma(\mathbb{Z}_n)$.

Corollary 4.3. $\chi_n = \omega_n$

We now return to the proof of the theorem.

Proof. Assume without loss of generality that $\chi_n \leq \chi_m$.
I. $\chi_{n,m} \leq \min\{\chi_n, \chi_m\}$
Fix $f : V(\Gamma(\mathbb{Z}_n)) \to \{1, 2, \dots, \chi_n\}$. Define $\hat{f} : V(\Gamma(\mathbb{Z}_n) \times \Gamma(\mathbb{Z}_m)) \to \{1, 2, \dots, \chi_n\}$ by $\hat{f}(v, u) = f(v)$. Claim: $\hat{f}$ is a coloring, i.e., if (v, u) and $(\hat{v}, \hat{u})$ are adjacent in $\Gamma(\mathbb{Z}_n) \times \Gamma(\mathbb{Z}_m)$, then $\hat{f}(v, u) \neq \hat{f}(\hat{v}, \hat{u})$.

If (v, u) and $(\hat{v}, \hat{u})$ are adjacent in $\Gamma(\mathbb{Z}_n) \times \Gamma(\mathbb{Z}_m)$, then v and $\hat{v}$ are adjacent in $\Gamma(\mathbb{Z}_n)$. It follows that, since f is a coloring, $f(v) \neq f(v)$. Hence, $\hat{f}(v, u) \neq \hat{f}(\hat{v}, \hat{u})$.

II. $\chi_{n,m} \geq \min\{\chi_n, \chi_m\}$
We know by Corollary 4.3 that in $\mathbb{Z}_n$ there exists a clique C_n of size $k = \chi_n$, and in

$\mathbb{Z}_m$ there exists a clique C_m of size $j = \chi_m$. Let $V(C_n) = \{v_1, ..., v_k\}$, and $V(C_m) = \{u_1, ..., u_j\}$. Consider the subgraph of $\Gamma(\mathbb{Z}_n) \times \Gamma(\mathbb{Z}_m)$ induced by $\{(v_i, u_i) : 1 \leq i \leq \chi_n\}$. Since every v_i is adjacent to every other vertex in $V(C_n)$ and every u_i is adjacent to every other vertex in $V(C_m)$, the subgraph induced by $\{(v_i, u_i) : 1 \leq i \leq \chi_n\}$ forms a clique in $\Gamma(\mathbb{Z}_n) \times \Gamma(\mathbb{Z}_m)$ of size χ_n. This means that we need at least χ_n colors to color $\Gamma(\mathbb{Z}_n) \times \Gamma(\mathbb{Z}_m)$.

By I. and II., $\chi_{n,m} = \min\{\chi_n, \chi_m\}$.

□

Theorem 4.4. *Let R and R' be any rings such that $\chi(\Gamma(R)) = \omega(\Gamma(R))$ and $\chi(\Gamma(R')) = \omega(\Gamma(R'))$. Then, $\chi(\Gamma(R) \times \Gamma(R')) = \min\{\chi(\Gamma(R)), \chi(\Gamma(R'))\}$.*

Proof. We prove this theorem using the same reasoning as in the proof of Theorem 4.1.

As in Theorem 4.1, we assume without loss of generality that $\chi(\Gamma(R)) \leq \chi(\Gamma(R'))$. Then, we prove that $\chi(\Gamma(R)\times\Gamma(R')) \leq \min\{\chi(\Gamma(R)), \chi(\Gamma(R'))\}$ by defining a coloring for $\Gamma(R) \times \Gamma(R')$ using only the colors necessary to color $\Gamma(R)$.

On the other hand, we prove that $\chi(\Gamma(R) \times \Gamma(R')) \geq \min\{\chi(\Gamma(R)), \chi(\Gamma(R'))\}$ using the fact that $\chi(\Gamma(R)) = \omega(\Gamma(R))$ and $\chi(\Gamma(R')) = \omega(\Gamma(R'))$. Let C_R and $C_{R'}$ be the maximal cliques in $\Gamma(R)$ and $\Gamma(R')$ respectively. Also, let $V(C_R) = \{v_1, \ldots, v_u\}$ and $V(C_{R'}) = \{w_1, \ldots\}$. Consider the subgraph of $\chi(\Gamma(R) \times \Gamma(R'))$ induced by $\{(v_i, u_i) : 1 \leq i \leq \chi(\Gamma(R))\}$. By the same reasoning as in Theorem 4.2, the subgraph forms a clique in $\Gamma(R) \times \Gamma(R')$ of size $\chi(\Gamma(R))$. This means that we need at least $\chi(\Gamma(R))$ colors to color $\Gamma(R) \times \Gamma(R')$. Hence, $\chi(\Gamma(R) \times \Gamma(R')) = \min\{\chi(\Gamma(R)), \chi(\Gamma(R'))\}$.

□

5 The Core of a Graph

In this section, we determine the core of the graph $\Gamma(\mathbb{Z}_n)$. To understand what a core is, we must first define homomorphism and automorphism. A *homomorphism* is a mapping f from $V(G)$ to $V(H)$, such that for any $x, y \in V(G)$, where $\{x, y\} \in E(G)$, $\{f(x), f(y)\} \in E(H)$. An *automorphism* is a bijective homomorphism f from a graph G to itself. Two graphs G and H are called *isomorphic* if and only if there exists a one-to-one and onto mapping of vertices $f : V(G) \longrightarrow V(H)$ which preserves adjacency.

By definition, a graph G is a *core* if any homomorphism from G to itself is an automorphism. Also, a subgraph H of G is called a core of G if H is a core itself, and there is a homomorphism from G to H. Every graph has at least one core, and in cases where a graph has more than one core, all of the cores are isomorphic [3].

Theorem 5.1. *The core of the graph $\Gamma(\mathbb{Z}_n)$ is the maximal clique in $\Gamma(\mathbb{Z}_n)$.*

Proof. By Corollary 4.3 we know that $\chi(\Gamma(\mathbb{Z}_n)) = \omega(\Gamma(\mathbb{Z}_n))$. Let $\chi(\Gamma(\mathbb{Z}_n)) = m$, and C_m be a clique of size m. Also, let $V(C_m) = \{v_1, \ldots, v_m\}$ where v_i is colored i .

Let $f : V(\Gamma(\mathbb{Z}_n)) \to V(C_m)$ be given by $f(w) = v_i$, where $\chi(w) = \chi(v_i)$. Note that $f(v_i) = v_i$ for every $i = 1, \ldots, m$, and hence, f is onto.

Let $\{w, w'\} \in E(\Gamma(\mathbb{Z}_n))$. It follows that the color of w is not equal to the color of w', which implies that $f(w) \neq f(w')$. Hence, $\{f(w), f(w')\} \in E(\Gamma(\mathbb{Z}_n))$. Therefore, f is a homomorphism. By definition, C_m is the core of $\Gamma(\mathbb{Z}_n)$.

In particular, since the maximal clique of a complete graph is the graph itself, then the core of a complete graph is itself a core.

$\square$

Corollary 5.2. *$\Gamma(\mathbb{Z}_n)$ is a core if and only if $n = p^2$.*

Proof. The proof of this is rather simple. We know that $\Gamma(\mathbb{Z}_n)$ is complete if and only if $n = p^2$. By Theorem 5.1, the core of $\Gamma(\mathbb{Z}_n)$ is the largest clique. Since $\Gamma(\mathbb{Z}_{p^2})$ is complete, it is its own largest clique, and therefore also the core. Hence, $\Gamma(\mathbb{Z}_n)$ is a core if and only if $n = p^2$.

$\square$

Theorem 5.3. *Let R be a ring such that $\chi(\Gamma(R)) = \omega(\Gamma(R))$. Then, the core of $\Gamma(R)$ is the maximal clique in $\Gamma(R)$.*

Proof. We prove this theorem using the same reasoning we used for the proof of Theorem 5.1.

Since, $\chi(\Gamma(R)) = \omega(\Gamma(R))$ we can let $\chi(\Gamma(R) = m$, and C_m be a clique of size m. Also let, $V(C_m) = \{v_1, \ldots, v_m\}$, and $\chi(v_i) = i$.

Let $f : V(\Gamma(R)) \to V(C_m)$ be given by $f(w) = v_i$, where $\chi(w) = \chi(v_i)$. Note that $f(v_i) = v_i$ for every $i = 1, \ldots, m$, and hence, f is onto.

Let $\{w, w'\} \in E(\Gamma(R)$. It follows that $\chi(w) \neq \chi(w')$, which implies that $f(w) \neq f(w')$. Hence, $\{f(w), f(w')\} \in E(\Gamma(R)$. Therefore, f is a homomorphism. By definition, C_m is the core of $\Gamma(R)$.

$\square$

6 Coloring of $\overline{\Gamma(\mathbb{Z}_n)}$

Melody Brickel addressed the coloring of $\overline{\Gamma(\mathbb{Z}_n)}$ in [1]. She did not determine a formula for the coloring of $\overline{\Gamma(\mathbb{Z}_n)}$; however, she did determine that the chromatic number of $\overline{\Gamma(\mathbb{Z}_n)}$ equals the clique number of $\overline{\Gamma(\mathbb{Z}_n)}$ for the cases $n = p^e$, and $n = p_1p_2p_3$, where $p_1 < p_2 < p_3$. Here we include the proofs of these cases, and extend to proofs of some additional cases. From these results, we make the following conjecture:

Conjecture 6.1. *$\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$ for all $n \geq 2$.*

Lemma 6.2. *$\chi(G) \geq \omega(G)$ for any graph G.*

Proof.

Consider a clique in G. For any vertices v, w in this clique, v and w must be colored differently. Thus, we need at least $\omega(G)$ to color G.

$\square$

Proposition 6.3. *Let $n = p^e$. Then $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.*

Proof. Suppose e is even and let A_{p^i} be the associate class of p^i, where $1 \leq i \leq e$. The largest clique in $\overline{\Gamma(\mathbb{Z}_n)}$ is

$$(\bigcup_{i=1}^{\frac{e}{2}-1} A_{p^i})\bigcup\{x\}$$

for some $x \in A_{p^{\frac{e}{2}}}$. All of these vertices have a different color. Any vertex v not in the clique is included in

$$\bigcup_{i=\frac{e}{2}}^{e} A_{p^i},$$

and thus $xv = up^q$ where $q \geq e$, and u is a unit. Also v_iv_j is a multiple of p^e for any v_i, v_j not in the maximal clique. Therefore, all v can be colored with the color of x. Hence, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.

Now suppose e is odd. The largest clique is now

$$\bigcup_{i=1}^{\frac{e-1}{2}} A_{p^i}.$$

All of the vertices not in this clique form an independent set, since the product of these elements is divisible by p^e. Also, for the same reason any v not in the maximal clique cannot be adjacent to any y in $A_{p^{\frac{e-1}{2}}}$. Therefore, all v can be colored with the color of any y. Thus, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.

$\square$

Proposition 6.4. *Let $n = p_1p_2p_3$. Then $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.*

Proof. We know from Lemma 2.1 that $|A_{p_1}| > |A_{p_2}| > |A_{p_3}|$, and $|A_{p_1p_2}| > |A_{p_1p_3}| > |A_{p_2p_3}|$.

Case 1: The largest clique is $A_{p_1} \cup A_{p_2} \cup A_{p_3}$. Since A_{p_1} is not adjacent to $A_{p_2p_3}$ in the structure graph, the colors of A_{p_1} can be used to color $A_{p_2p_3}$. The same strategy can be used for the coloring of $A_{p_1p_2}$ and $A_{p_1p_3}$. Thus, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) \leq \omega(\overline{\Gamma(\mathbb{Z}_n)})$. So, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.

Case 2: Suppose $|A_{p_3}| < |A_{p_1p_2}|$. The largest clique is now $A_{p_1} \cup A_{p_2} \cup A_{p_1p_2}$. Color each of the vertices of the clique. The vertices of A_{p_3} can be colored with the colors of $A_{p_1p_2}$. Similarly, $A_{p_2p_3}$ can be colored with the colors of A_{p_1}, and $A_{p_1p_3}$ can be colored with the colors of A_{p_2}. So, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) \leq \omega(\overline{\Gamma(\mathbb{Z}_n)})$, and $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$. Thus, the proposition is proved.

$\square$

Proposition 6.5. *Let $n = p_1p_2$, where $p_1 < p_2$. Then $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.*

Proof. Consider the associate classes of zero-divisors, and note that there are only two: A_{p_1} and A_{p_2}. In $\overline{\Gamma(\mathbb{Z}_n)}$, each class forms a disjoint clique. From Lemma 2.1, and since $p_1 < p_2$, we know $p_2 - 1 = |A_{p_1}| > |A_{p_2}| = p_1 - 1$. Hence A_{p_1} forms the larger clique. All of the vertices of A_{p_1} must have a different color. These colors can be used to color all of the vertices of A_{p_2}. Thus, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) \leq \omega(\overline{\Gamma(\mathbb{Z}_n)})$. So, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.

$\square$

Proposition 6.6. *Let $n = p_1^2 p_2$, where $p_1 < p_2$. Then $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.*

Proof. Again consider the associate classes of zero-divisors. In $\overline{\Gamma(\mathbb{Z}_n)}$, $A_{p_1^2}$ is adjacent to A_{p_1}, which is also adjacent to A_{p_2}, which is also adjacent to $A_{p_1p_2}$ in the structure graph. The sizes of the classes determined using Lemma 2.1 are as follows:
$|A_{p_1}| = p_1p_2 - p_1 - p_2 + 1$
$|A_{p_2}| = p_1^2 - p_1$
$|A_{p_1^2}| = p_2 - 1$
$|A_{p_1p_2}| = p_1 - 1$.
Case 1: $|A_{p_1^2}| < |A_{p_2}|$ Then we have, $|A_{p_1p_2}| < |A_{p_1^2}| < |A_{p_2}| < |A_{p_1}|$. Thus, the largest clique is $A_{p_1} \cup A_{p_2}$. Color each vertex of the largest clique a different color. Then color all of $A_{p_1p_2}$ with one color of A_{p_1}, and use colors of A_{p_2} to color $A_{p_1^2}$. Therefore, we have $\chi(\overline{\Gamma(\mathbb{Z}_n)}) \leq \omega(\overline{\Gamma(\mathbb{Z}_n)})$. Hence, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.

Case 2: $|A_{p_1^2}| > |A_{p_2}|$ Then we have, $|A_{p_1p_2}| < |A_{p_2}| < |A_{p_1^2}| < |A_{p_1}|$. So the largest clique is now $A_{p_1} \cup A_{p_1^2}$. Color each vertex of the largest clique a different color. Then color all of $A_{p_1p_2}$ with one color of A_{p_1}, and use colors of $A_{p_1^2}$ to color A_{p_2}. Therefore, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.

□

Proposition 6.7. *Let $n = p_1 p_2^2$, where $p_1 < p_2$. Then $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.*

Proof. This proof follows the same logic as the previous one, so we will be brief. Again, the largest clique is formed by $A_{p_1} \cup A_{p_2}$. All of $A_{p_1p_2}$ can again be colored by one color, this time from A_{p_2}, and $A_{p_2^2}$ can be colored with colors of A_{p_1}. Thus, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.

□

Proposition 6.8. *Let $n = p_1^2 p_2^2$. Then $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$.*

Proof. Consider the associate classes of zero-divisors. Using the formula in Lemma 2.1, the sizes of the classes are as follows:
$|A_{p_1}| = p_1p_2^2 - p_1p_2 - p_2^2 + p_2$
$|A_{p_2}| = p_1^2p_2 - p_1p_2 - p_1^2 + p_1$
$|A_{p_1p_2}| = p_1p_2 - p_1 - p_2 + 1$
$|A_{p_1^2}| = p_2^2 - p_2$
$|A_{p_2^2}| = p_1^2 - p_1$

$|A_{p_1p_2^2}| = p_1 - 1$
$|A_{p_1^2p_2}| = p_2 - 1.$
Thus, we have $|A_{p_1p_2^2}| < |A_{p_1^2p_2}| < |A_{p_2^2}| < |A_{p_1p_2}| < |A_{p_1^2}| < |A_{p_2}| < |A_{p_1}|$. The classes that form cliques in $\overline{\Gamma(\mathbb{Z}_n)}$ are A_{p_1}, A_{p_2}, $A_{p_1^2}$, and $A_{p_2^2}$, however $A_{p_1^2}$ and $A_{p_2^2}$ are not adjacent in the complement. Also, all of the members of $A_{p_1p_2}$ are adjacent to all of the classes that form cliques. Thus, the largest clique in $\overline{\Gamma(\mathbb{Z}_n)}$ is $A_{p_1} \cup A_{p_2} \cup A_{p_1^2} \cup \{v\}$, where $v \in A_{p_1p_2}$.

Color each member of the clique a different color. All of the members of $A_{p_1p_2}$ can be the same color since the members of $A_{p_1p_2}$ form an independent set. Since $A_{p_1^2}$ and $A_{p_2^2}$ are not adjacent in $\overline{\Gamma(\mathbb{Z}_n)}$, $A_{p_2^2}$ can be colored with colors of $A_{p_1^2}$. Also, $A_{p_1p_2^2}$, which forms an independent set, is not adjacent to A_{p_1} and thus, can be colored with one color of A_{p_1}. By symmetry $A_{p_1^2p_2}$ can be colored with one color of of A_{p_2}. Therefore, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) \leq \omega(\overline{\Gamma(\mathbb{Z}_n)})$. So, $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$

□

The proof of the following result is immediate:

Corollary 6.9. *If* $\chi(\overline{\Gamma(\mathbb{Z}_n)}) = \omega(\overline{\Gamma(\mathbb{Z}_n)})$, *and* $\chi(\overline{\Gamma(\mathbb{Z}_m)}) = \omega(\overline{\Gamma(\mathbb{Z}_m)})$, *then* $\chi(\overline{\Gamma(\mathbb{Z}_n)} \times \overline{\Gamma(\mathbb{Z}_m)}) = \min\{\chi(\overline{\Gamma(\mathbb{Z}_n)}), \chi(\overline{\Gamma(\mathbb{Z}_m)})\}$.

7 Hamiltonian Graphs

A graph is *Hamiltonian* if it has a cycle which contains every vertex. Figure 3 is an example of a Hamiltonian graph. In this section we ask the question: For which values of n is $\overline{\Gamma(\mathbb{Z}_n)}$ Hamiltonian? M. Brickel [1] proved by induction that $\overline{\Gamma(\mathbb{Z}_n)}$ is Hamiltonian when $n = p_1 \ldots p_r$, where $r \geq 3$. In this section, we extend this result to the cases where $n = p_1^{e_1}p_2^{e_2}$, and $n = p_1^2p_2 \ldots p_r$.

A subset $S \subseteq V(G)$ is an *independent set* in a graph G if the subgraph induced by S has no edges. Moreover, the *independence number* $\alpha(G)$ is the number of vertices in the largest independent set of G. On the other hand, the *connectivity* $\kappa(G)$ of a graph G is the size of the smallest subset $S \subseteq V(G)$ such that $G - S$ is disconnected.

In this section we also refer to the vertex of *minimum degree* $\delta(\Gamma(\mathbb{Z}_n))$. We know from Phillips, et al. [5] that in $\overline{\Gamma(\mathbb{Z}_n)}$ the vertex of minimum degree has degree
$\delta(G) = p_1^{e_1-1}(p_1-1)p_2^{e_2} \ldots p_r^{e_r} - p_1^{e_1-1}(p_1-1) \ldots p_r^{e_r-1}(p_r-1)$. Also, since the largest independent set in $\overline{\Gamma(\mathbb{Z}_n)}$ is equal to the largest clique in $\Gamma(\mathbb{Z}_n)$, we know from Lemma 4.2 that $\alpha(\overline{\Gamma(\mathbb{Z}_n)}) = p_1^{\frac{e_1}{2}} \ldots p_r^{\frac{e_r}{2}} q_1^{\frac{f_1-1}{2}} \ldots q_s^{\frac{f_s-1}{2}} + s - 1$.

Theorem 7.1. *If $n = p_1^{e_1} p_2^{e_2}$, then $\overline{\Gamma(\mathbb{Z}_n)}$ is Hamiltonian.*

Proof. To prove this theorem we will use a result from Diestel [2] that states that, for any graph G, if $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian, for any graph G.

Case 1: $e_1 > 1$
We know from Phillips, et al. [5] that when $n = p_1^{e_1} p_2^{e_2}$ and $e_1 > 1$, then $\kappa(\overline{\Gamma(\mathbb{Z}_n)}) = \delta(\overline{\Gamma(\mathbb{Z}_n)})$. Hence, for this case, we only need to show that $\delta(\overline{\Gamma(\mathbb{Z}_n)}) \geq \alpha(\overline{\Gamma(\mathbb{Z}_n)})$.

Subcase 1: e_1, e_2 even
Here we need to show that
$p_1^{e_1-1}(p_1-1)p_2^{e_2} - p_1^{e_1-1}(p_1-1)p_2^{e_2-1}(p_2-1) \geq p_1^{\frac{e_1}{2}} p_2^{\frac{e_2}{2}} - 1$.
Simplifying the left hand side we obtain,
$(p_1^{e_1-1})(p_2^{e_2-1})(p_1-1)$.
But,
$(p_1^{e_1-1})(p_2^{e_2-1})(p_1-1) \geq (p_1^{e_1-1})(p_2^{e_2-1}) \geq p_1^{\frac{e_1}{2}} p_2^{\frac{e_2}{2}} \geq p_1^{\frac{e_1}{2}} p_2^{\frac{e_2}{2}} - 1$.

Subcase 2: e_1, e_2 odd
For this case we need to show that
$p_1^{e_1-1}(p_1-1)p_2^{e_2} - p_1^{e_1-1}(p_1-1)p_2^{e_2-1}(p_2-1) \geq p_1^{\frac{e_1-1}{2}} p_2^{\frac{e_2-1}{2}} + 1$.
As in Subcase 1, we have,
$p_1^{e_1-1}(p_1-1)p_2^{e_2} - p_1^{e_1-1}(p_1-1)p_2^{e_2-1}(p_2-1) = (p_1^{e_1-1})(p_2^{e_2-1})(p_1-1) \geq (p_1^{e_1-1})(p_2^{e_2-1})$.
Claim: $(p_1^{e_1-1})(p_2^{e_2-1}) \geq p_1^{\frac{e_1-1}{2}} p_2^{\frac{e_2-1}{2}} + 1$.
This is equivalent to, $(p_1^{e_1-1})(p_2^{e_2-1}) - p_1^{\frac{e_1-1}{2}} p_2^{\frac{e_2-1}{2}} \geq 1$.
Factoring we obtain, $p_1^{\frac{e_1}{2}} p_2^{\frac{e_2}{2}} (p_1^{\frac{e_1}{2}} p_2^{\frac{e_2}{2}} - 1) \geq 1$, which is a true statement since $e_1 \geq 3$ and $p_1 \geq 2$.

Subcase 3: e_1 odd, and e_2 even
Here we use the same reasoning as in Subcase 1 to show that
$p_1^{e_1-1}(p_1-1)p_2^{e_2} - p_1^{e_1-1}(p-1)p_2^{e_2-1}(p_2-1) \geq p_1^{\frac{e_1}{2}} p_2^{\frac{e_2-1}{2}}$.
Note that if e_1 is even, and e_2 is odd we can also apply the same reasoning.

Case 2: $e_1 = 1, e_2 \geq 1$

¿From Theorem 7.3 [5], we know that $\kappa(G) = \delta(G) - |A_{v_1}| + 1$, where v_1 is a vertex of minimum degree. Here, $v_1 = p_2^{e_2}$. Therefore, $|A_{v_1}| = |A_{p_2^{e_2}}| = p_1 - 1$.

Subcase 1: e_2 even
In this case, $\alpha(\overline{\Gamma(\mathbb{Z}_n)}) = p_2^{\frac{e_2}{2}}$. Therefore, we need to show that
$\kappa(\overline{\Gamma(\mathbb{Z}_n)}) = (p_1-1)p_2^{e_2} - (p_1-1)p_2^{e_2-1}(p_2-1) - (p_1-1) + 1 \geq p_2^{\frac{e_2}{2}}$.
Factoring the left hand side we obtain,
$(p_1-1)[p_2^{e_2} - p_2^{e_2-1}(p_2-1) - 1] + 1 = (p_1-1)(p_2^{e_2-1} - 1) + 1 \geq p_2^{e_2-1} \geq p_2^{\frac{e_2}{2}}$.

Subcase 2: e_2 odd

Here, $\alpha(\overline{\Gamma(\mathbb{Z}_n)}) = p_2^{\frac{e_2-1}{2}} + 1$. So, we have to show that
$p_1^{e_1}(p_1-1)p_2^{e_2} - p_1^{e_1-1}(p_1-1)p_2^{e_2-1}(p_2-1) + 1 \geq p_2^{\frac{e_2-1}{2}} + 1$.
As in Subcase 1, we manipulate the left hand side and obtain,
$(p_1-1)[p_2^{e_2} - (p_1-1)] + 1 = (p_1-1)[p_2^{e_2-1}(p_1-1)] + 1 = (p_1-1)(p_2^{e_2-1}-1) \geq p_2^{e_2-1} - 1 = (p_2^{\frac{e_2-1}{2}} - 1)(p_2^{\frac{e_2-1}{2}} + 1) \geq p_2^{\frac{e_2-1}{2}} + 1 \geq p_2^{\frac{e_2-1}{2}}$.

□

Theorem 7.2. *If n=$p_1^2p_2\dots p_r$, then $\overline{\Gamma(\mathbb{Z}_n)}$ is Hamiltonian.*

Basecase : *Let* $n = p_1^2p_2$ This is a special case of $n = p_1^{e_1}p_2^{e_2}$ which is Hamiltonian in the previous theorem.

Proof. We will use induction on r For the induction hypothesis, assume $\overline{\Gamma(\mathbb{Z}_n)}$ is hamiltonian when $n = p_1^2p_2\dots p_r$ and $r \leq k$. Now suppose n=$p_1^2p_2\dots p_{k+1}$. Then we can partition the vertices of $\overline{\Gamma(\mathbb{Z}_n)}$ into two subsets $V_1 = \{v \in \overline{\Gamma\mathbb{Z}_n} : p_{k+1} \mid v\}$ and $V_2 = \{v \in \overline{\Gamma\mathbb{Z}_n} : p_{k+1} \nmid v\}$. Let $\mathbf{R}$ be the subgraph induced by V_1 and $\mathbf{H}$ be the subgraph induced by V_2. Note that $V(\mathbf{H})$ is a clique, and thus $\mathbf{H}$ is Hamiltonian. Similarly we will show that $\mathbf{R}$ is Hamiltonian.

Let $K = \overline{\Gamma(\mathbb{Z}_{p_1^2p_2\dots p_k})}$. To show $\mathbf{R}$ to be Hamiltonian we will first show that $\mathbf{R}$ has a subgraph isomorphic to K.

Let $\mathbf{B}$=$\mathbf{R} - A_{p_{k+1}}$ and let $v \in V(\mathbf{B})$. Then $v = m \cdot p_{k+1}$ where m is a zero-divisor. Let $A_m{}'$ be the associate class of m in K. Using the Euler ϕ function we have $\mid A_v \mid$= $\phi(\frac{n}{v})$= $\phi\frac{n}{mp_{k+1}}$= $\phi(\frac{p_1^2\dots p_{k+1}}{mp_{k+1}})$= $\phi(\frac{p_1^2\dots p_k}{m})$. In the subgraph K, $\mid A_m{}' \mid$= $\phi(\frac{p_1^2\dots p_k}{m})$. Therefore $\mid A_v \mid$=$\mid A_m{}' \mid$.

We claim that $A_m{}'$ is a clique iff A_v is a clique and $A_m{}'$ is an independent set if A_v is an independent set. If A_v is a clique, then $v^2 \neq 0$ in $\mathbb{Z}_{p_1^2p_2\dots p_k}$. Therefore, there exists a $p_i, 1 \leq i \leq k+1$ such that p_i does not divide v^2. Thus there exists a $p_i, 1 \leq i \leq k$ such that p_i does not divide $A_m{}'$. From this we know that $A_m{}'$ is a clique. Similarly if A_v is an independent set, then $v^2 = 0$ in $\mathbb{Z}_{p_1^2p_2\dots p_k}$. Thus $p_1^2\dots p_{k+1} \mid v^2$ and so every p_i, $1 \leq i \leq k+1$ divides v^2. Therefore every p_i divides v/p_{k+1}, hence $(v/p_{k+1})^2 = 0$ in $\mathbb{Z}_{p_1^2p_2\dots p_k}$. This provides sufficient evidence that $A_m{}'$ is an independent set.

¿From what we have just written we can conclude that the subgraph $\mathbf{B}$=$\mathbf{R} - A_{p_{k+1}}$ is isomorphic to $\overline{\Gamma\mathbb{Z}_{p_1^2p_2\dots p_k}}$. Therefore, by induction $\mathbf{B}$ contains a Hamiltonian path P.

Let u and x be the terminal vertices of P respectively. Without loss of generality, assume $u \notin A_{p_1 p_2 \dots p_{k+1}}$. Also, $A_{p_{k+1}}$ is adjacent to every vertex in B and is a clique. So **R** has a Hamiltonian path P_r given as follows: Follow P from u to x; then follow an edge from x to w where w is in A_{p_k+1}; then follow a Hamiltonian path of A_{p_k+1} from w to y.

Finally construct a Hamiltonian cycle in $\overline{\Gamma\mathbb{Z}_n}$ as follows: Recall $u \notin A_{p_1 p_2 \dots p_{k+1}}$. Thus $\exists\ 1 \leq i \leq k$ such that $p_i \nmid u$. Choose $a \in V(H)$ such that $p_i \nmid a$ and let $b = p_i$. Then $a \neq b$, a is adjacent to u, and b is adjacent to y. Let P_H be a Hamiltonian path in H from a to b. We take a Hamiltonian cycle by first taking P_H from a to b. Next we take an edge from b to y. Then we can follow P_R from y to u. Finally we take an edge from u to a. Thus the claim holds by induction.

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A Appendix: Glossary of Terms

In all the following definitions, let R be a ring, and G be a graph with vertex set $V(G)$ and edge set $E(G)$.

adjacent: Two vertices $u, v \in V(G)$ are said to be *adjacent* if $\{u, v\} \in E(G)$.

associates: Two ring elements x and y are *associates* if there exists a unit u such that $x = uy$.

associate class: The *associate class* of a vertex v, denoted A_v, consists of all the associates of v.

automorphism: An *automorphism* of a graph is a bijective homomorphism from the graph to itself.

clique: A subset $C \subseteq V(G)$ is called a *clique* if the subgraph of G induced by C is a complete graph.

clique number: The *clique number* of G, denoted $\omega(G)$, is the size of the maximal clique.

closed walk: A walk is called *closed* if the initial and terminal vertices coincide, and the closed path is called a *cycle*.

complete graph: A graph G is called *complete* if for every $u, v \in V(G)$, $u \neq v$, there exists an edge $\{u, v\}$.

complement: The *complement* of a graph G is the graph $\overline{G}$ whose vertex set is $V(G)$, but $\{x, y\} \in E(\overline{G})$ if and only if $\{x, y\} \notin E(G)$.

connectivity: The *connectivity* $\kappa(G)$ of a graph G is the least number k of vertices $\{v_1, v_2, \ldots, v_k\}$ such that $G - \{v_1, v_2, \ldots, v_k\}$ is disconnected.

core: A graph G is a *core* if any homomorphism from G to itself is an automorphism.

degree: For each $v \in V(G)$, the *degree* of v is the number of edges incident on v.

direct-product graph: A *direct-product graph* of two graphs G and G' is the graph where $V(G \times G') = V(G) \times V(G')$. Also, for any $u, v \in V(G)$ and $u', v' \in V(G')$, $\{(u, u'), (v, v')\}$ is an edge if and only if $\{u, v\} \in E(G)$ and $\{u', v'\} \in E(G')$.

Eulerian: A graph is called *Eulerian* if there exists a closed trail containing every edge.

Euler phi function: The *Euler phi function* $\phi(n)$, defined for $n \geq 1$, counts the number of $a's$, $1 \leq a \leq n - 1$ satisfying $gcd(a, n) = 1$.

graph: A *graph* is a pair $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The edge set consists of unordered pairs of distinct elements of $V(G)$.

Hamiltonian: A graph is called *Hamiltonian* if there exists a cycle containing every vertex.

homomorphism: Let G and H be graphs. A mapping f from $V(G)$ to $V(H)$ is a *homomorphism* if $f(x)$ and $f(y)$ are adjacent in H whenever x and y are adjacent in G.

incident: An edge $\{u, v\} \in E(G)$ is said to be *incident* to the vertices $u, v \in V(G)$.

independent set: A subset $S \subseteq V(G)$ is called an *independent set* if the subgraph induced by S has no edges. The *independence number* of a graph G is the number of elements in the largest independent set of G. The independence number is denoted $\alpha(G)$.

isomorphic Two graphs G and H are called *isomorphic* if there exists a one-to-one and onto mapping $f : V(G) \longrightarrow V(H)$ which preserves adjacency. If G and H are isomorphic, we write $G \cong H$ and say there is an isomorphism between G and H.

k-colorable: A graph G is called *k-colorable* if there exists a vertex coloring of G using only k colors. Note: $\chi(G) = \min\{k : \text{G is k-colorable}\}$.

minimal degree: The *minimal degree* $\delta(G)$ of G, is defined as $\min\{\deg\,(v) : v \in V\}$.

neighbors: We say two vertices $u, v \in V(G)$ are *neighbors* in a graph G if there exists an edge $\{u, v\} \in E(G)$ that connects u and v.

path: A *path* is a walk in which no vertex is repeated, except possibly the initial and terminal vertices.

subgraph: A *subgraph* of G is a graph $H = (V(H), E(H))$, where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

subgraph induced: For a graph G and a subset $S \subseteq V(G)$, the *subgraph induced* by S is the graph H with vertex set $S = V(H)$ such that $\{x, y\} \in E(H)$ if and only if $\{x, y\} \in E(G)$ and $x, y \in V(H)$.

structure graph: A *structure graph* is a graph whose vertices are associate classes of zero divisors in $\mathbb{Z}_n$ and where two vertices X and Y are adjacent if and only if $xy = 0$ for some $x \in X$ and some $y \in Y$.

trail: A *trail* is a walk in which no edge is repeated.

unit: An element $x \in R$ is a *unit* if there exists $y \in R$ such that $x \cdot y = 1$.

vertex coloring: A *vertex coloring* of G is a map $f : V(G) \longrightarrow \mathbb{N}$ such that $f(v) \neq f(w)$ if $\{v, w\} \in E$.

walk: A *walk* is a sequence of vertices and edges $v_0, e_0, v_1, e_1 \ldots v_{k-1}, e_{k-1}, v_k$ where $e_i = \{v_i, v_{i+1}\}$.

zero-divisor: An element $x \in R$ is a *zero-divisor* if there exists a nonzero $y \in R$ such that $xy = 0$.

zero-divisor graph: The *zero-divisor graph* is the graph denoted $\Gamma(R)$, where the vertex set consists of the zero-divisors of a ring R, and $\{x, y\}$ is an edge if and only if $xy = 0$ in R.

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