THE STRUCTURE OF UNITARY CAYLEY GRAPHS

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ABSTRACT. In this paper we explore structural properties of unitary Cayley graphs, including clique and chromatic number, vertex and edge connectivity, planarity, and crossing number.

1. INTRODUCTION

A graph is a pair $G = (V, E)$. $V(G)$ is the set of *vertices* of G, and $E(G)$ is the set of edges. An *edge* $\{u, v\}$ is an unordered pair of distinct vertices; the vertices u and v are the *endpoints* of the edge. The vertices of a graph can be pictured as dots, and the edges as line segments whose endpoints are vertices. An edge e is said to be *incident* at a vertex v if v is an endpoint of e. Two vertices are said to be *adjacent* if there is an edge between them. Adjacent vertices are sometimes referred to as *neighbors*

A graph is *simple* if there is at most one edge between any two vertices, and the endpoints of each edge are distinct. All graphs discussed in this paper are simple graphs. A graph H is a *subgraph* of a graph G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph H is an *induced subgraph* of G if $V(H) \subseteq V(G)$, and if $E(H)$ contains every edge of G whose endpoints are in H.

In this paper, we investigate the properties of unitary Cayley graphs. A *unitary Cayley graph*, G_R is the graph whose vertex set is a ring R, where $\{i, j\}$ is an edge of G_R if and only if $(i - j)$ is a unit in R. If R is the cyclic ring of order n, we denote its unitary Cayley graph G_n.

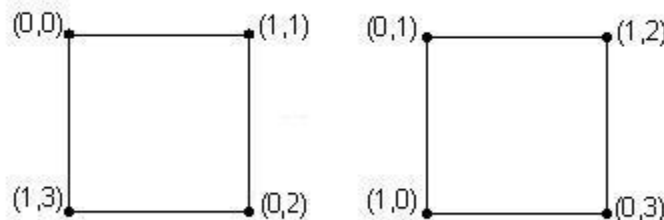Figures 1 and 2 show some examples of unitary Cayley graphs.

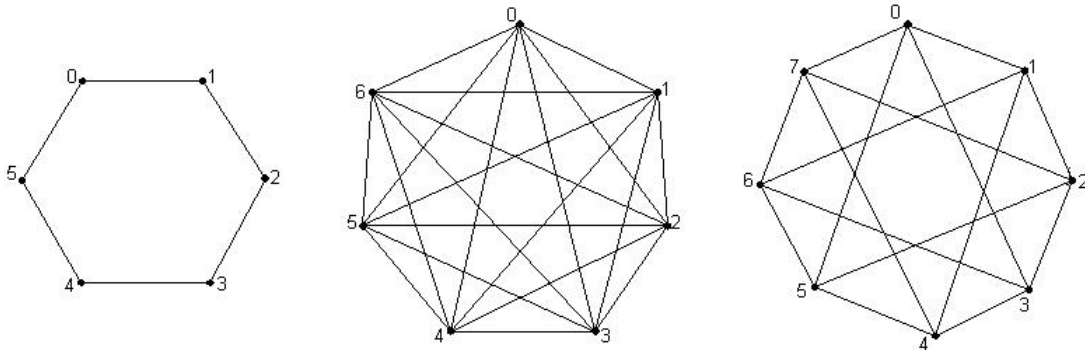FIGURE 1. The G_R graph for $R = \mathbb{Z}_2 \times \mathbb{Z}_4$.

FIGURE 2. The G_n graphs for $\mathbb{Z}_6$, $\mathbb{Z}_7$, and $\mathbb{Z}_8$

The motivation for this project is derived from the problem of finding representation numbers of general graphs. A *representative labeling* modulo n is a labeling of each vertex of the graph with an element of the cyclic ring of order n, such that two vertices are adjacent if and only if their difference is a unit in the ring. The *representation number* for an arbitrary graph is the smallest integer n for which we can create a representative labeling of the graph modulo n. Finding the representation number of an arbitrary graph remains an open problem [2].

If a graph is representable modulo n, then it must be an induced subgraph of the unitary Cayley Graph of of the cyclic ring $\mathbb{Z}_n$ [2].

Unitary Cayley graphs of rings which are the direct product of cyclic rings relate to the *generalized representation number* of a graph. The definition of a generalized representation number of a graph is similar to that of the (ordinary) representation number. Let R be a ring that is a direct product of cyclic rings. A representative labeling of a graph in R is a labeling which associates each vertex with an element in R, such that two vertices are adjacent if and only if the difference of their labels is a unit in R. The generalized representation number of an arbitrary graph is the order of the smallest ring in which the graph is representable. If a graph is representable in R, then it must be an induced subgraph of the unitary Cayley graph of R [2].

In this paper, we explore the properties of unitary Cayley graphs. The properties we examine include chromatic number, clique number, connectivity, and planarity.

2. DEFINITIONS

Let G be a graph and $v \in V(G)$. The *degree* of v, denoted $deg(v)$, is number of edges incident at v. The maximum degree of G, which we denote $\Delta(G)$, is degree of the vertex of largest degree. Similarly, the minimum degree $\delta(G)$ is the degree of the vertex of smallest degree. A graph is said to be *k-regular*, or simply *regular*, if every vertex of the graph has degree k.

A *walk* is a sequence of pairwise adjacent vertices of a graph. A walk is *closed* if its initial and terminal points are equal. A *path* is a walk in which no vertex is repeated. A closed path is called a *cycle* A *trail* is a walk in which no edge is repeated. An *Euler circuit* is a closed trail containing all the edges of a graph.

A *Hamiltonian cycle* is a cycle which contains every vertex of a graph.

A graph is *connected* if there exists a path between any two vertices. Otherwise, the graph is *disconnected*

The *distance* between two vertices of a graph is the number of edges of the shortest path between them. The *diameter* of a connected graph is the maximum distance between two vertices of the graph. A disconnected graph is said to have infinite diameter.

A graph is *complete* if every vertex is adjacent to every other vertex. The complete graph with n vertices is denoted K_n.

An *independent set* is a set of pairwise non-adjacent vertices.

A graph G is *multipartite* if its vertex set is a union of disjoint independent sets, which are known as *partite classes*. A *bipartite* graph is a multipartite graph with 2 partite classes. A multipartite graph is *complete multipartite* if each vertex in each partite class is connected to every vertex in every other paritite class. We denote the complete multipartite graph with m partite classes of size $n_1, n_2 \ldots, n_m$ $K_{n_1, n_2, \ldots, n_m}$. A bipartite graph is *complete bipartite* if every vertex is adjacent to every vertex outside its partite class.

An *automorphism* is a mapping from $\alpha : V(G) \rightarrow V(G)$, which preserves both adjacency and non-adjacency.

3. VERTEX DEGREE

Proposition 1. *Let $n = p_1^{e_1} \ldots p_s^{e_s}$ for distinct primes $p_1 \ldots p_s$, where $p_1 < p_2 \ldots < p_s$. The degree of a vertex $v \in V(G_n)$ is given by $deg(v) = \phi(n)$, where ϕ is Euler's totient function.*

Proof. For any $v \in V(G_n)$. Using the notation of Evans et. al., we assign each vertex v to a set $X(v_1, \ldots v_s)$, where $0 \leq v_i \leq p_i - 1$ and $v_i \equiv v \pmod{p_i}$ [2].

A vertex v in G_n is adjacent to a vertex $w \in X(w_1, \ldots w_s)$ if only if $v_i \neq w_i$ for all i with $1 \leq i \leq s$ [2]. For if $v_i = w_i$ for some i, then $v \equiv w \pmod{p_i}$, which implies that $v - w \equiv 0 \pmod{p_i}$. Hence, p_i divides $v - w$, and $v - w$ is not a unit of n. This implies v and w are non-adjacent.

Conversely, if $v_i \neq w_i$ for all i, then $v - w \not\equiv 0 \pmod{p_i}$ for all i. Hence, none of the prime divisors of n divide $v - w$. This implies that $v - w$ is a unit of $\mathbb{Z}_n$, and hence v and w are adjacent.

By construction, each set $X(v_1 \ldots v_s)$ has cardinality $\prod_{i=1}^{s} p_i^{e_i - 1}$. Finding the degree of v thus reduces to finding the number of sets whose vertices are adjacent to v, then multiplying by the number of elements per set.

In the collection of sets $X(v_1 \ldots v_s)$, each v_i ranges from 0 to p_i, and thus takes on one of p_i possible values. Suppose the vertices of $X(w_1, \ldots w_s)$ are adjacent to those of $X(v_1, \ldots v_s)$, the set containing v. Then each w_i must satisfy the conditions $w_i \neq v_i$, $0 \leq w_i \leq p_i - 1$. Each w_i thus takes on one of $p_i - 1$ possible values. The total number of sets $X(w_1, \ldots w_s)$ whose vertices are adjacent to those of any $X(v_1, \ldots v_s)$ is given by $\prod_{i=1}^{s}(p_i - 1)$. Hence, the degree of v is given by

$$
\begin{aligned}
deg(v) &= \left(\prod_{i=1}^{s}(p_i - 1) \right) \left(\prod_{i=1}^{s} p_i^{e_i - 1} \right) \\
&= \prod_{i=1}^{s}(p_i - 1)(p_i^{e_i - 1}) \\
&= \prod_{i=1}^{s} \phi(p_i^{e_i}) \\
&= \phi(n)
\end{aligned}
$$

□

This proposition has a number of useful consequences.

Corollary 1. G_n *is* $\phi(n)$-*regular for all* n.

Proof. This follows immediately from Proposition 1, which gives $\phi(n)$ as the degree of an arbitrary vertex of G_n.

□

Corollary 2. *The total number of edges of* G_n *is* $|E(G)| = \dfrac{1}{2} \displaystyle\prod_{i=1}^{s} p_i^{2e_i - 1}(p_i - 1) = n\left(\dfrac{\phi(n)}{2}\right)$.

Proof. The total number of edges of G_n is half the total degree of G_n. Since all vertices of G_n have the same degree, we simply multiply the degree of a vertex of G_n by $\dfrac{n}{2}$. Thus $|E(G)| = \dfrac{n}{2} \displaystyle\prod_{i=1}^{s} p_i^{e_i - 1}(p_i - 1)$.

Expanding n as $\displaystyle\prod_{i=1}^{s} p_i^{e_i}$ gives $|E(G)| = \dfrac{1}{2} \displaystyle\prod_{i=1}^{s} p_i^{2e_i - 1}(p_i - 1) = n\left(\dfrac{\phi(n)}{2}\right)$.

□

Corollary 3. G_n *is Eulerian for all* $n \geq 3$.

Proof. G_n is Eulerian if and only if every vertex of G_n is of even degree [3]. We will show that if $n \geq 3$, the degree of each vertex of G_n is even.

Expand n as $n = p_1^{e_1} \ldots p_s^{e_s}$, where the p_i are distinct primes.
If n is not a power of 2, then p_i is odd for some i, so $p_i - 1$ is even. For any

$v \in V(G)$, $deg(v) = \displaystyle\prod_{i=1}^{s} p_i^{e_i - 1}(p_i - 1)$ is even. Therefore, G_n is Eulerian.

If $n = 2^e$ and $n \geq 3$, then $e > 1$. For any $v \in V(G)$, we have $deg(v) = 2^{e-1}$, again by proposition 1. Since $k - 1 > 0$, $deg(v)$ is even. Hence G_n is Eulerian, and the proof is complete.

□

4. Vertex and Edge Coloring

Definition: The chromatic Nnumber, denoted $\chi(G_n)$, is the least number of colors that can be used to color vertices in such a way that adjacent vertices are not colored the same.

Definition: The Clique Number, denoted $\omega(G_n)$ is the size of largest complete subgraph that can be found in a graph.

It is known in Graph Theory that $\omega(G_n) \geq \chi(G_n$ because if there exist a clique subgraph the smallest the chromatic number can be is the clique number.

Theorem 1. G_n *is a Graph. Let* $n = p_1^{e_1} p_2^{e_2} ... p_s^{e_s}$, *where* $p_1 < p_2 < ... < p_s$ *and* $e_i \geq 1$ *for all* $i = 1, ..., s$. *Then* $\omega(G_n) = \chi(G_n) = p_1$.

Proof. Let m represent the vertices and k_m is the coloring. For each m, $0 \leq m \leq n - 1$, there is a unique k_m, such that $m \equiv k_m (mod p_1)$. The vertex m is assigned the color k_m. If two vertices m and m' receive the same color then $k_m \equiv k_{m'} (mod p_1)$, so $m \equiv m' (mod p_1)$. This allows m to not be adjacent to m'. Thus, this coloring is proven and therefore, $\chi(G_n \leq p_1)$

We observe that in G_n there exist a p_1-clique. The clique in G_n is $\{0, 1, ..., p_1 - 1\}$ which implies $p_1 \leq \omega(G_n)$.

Based upon the proof the following can be stated, $p_1 \leq \omega(G_n) \leq \chi(G_n) \leq p_1$ and therefore we conclude that $\omega(G_n) = \chi(G_n) = p_1$. □
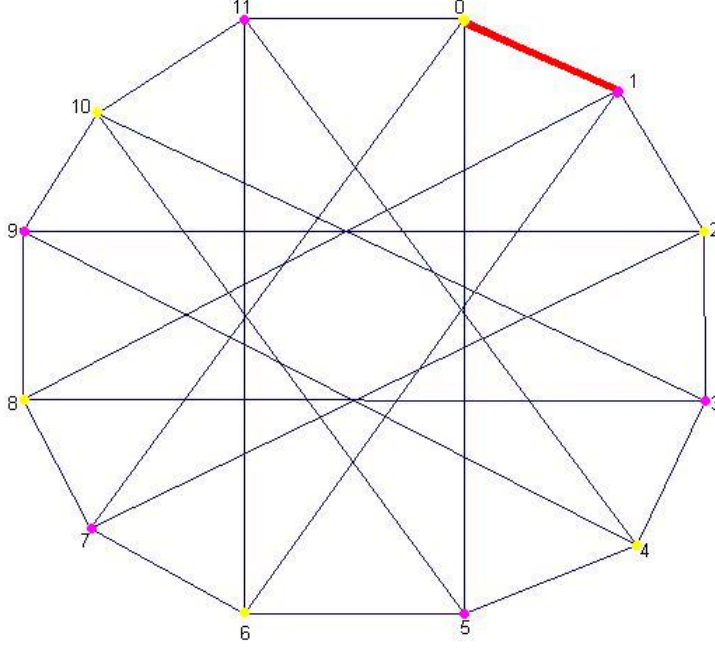

FIGURE 3. G_{12} where $\omega(G_{12}) = \chi(G_{12}) = p_1 = 2$.

Figure 3 shows that $\omega(G_{12}) = \chi(G_{12}) = p_1 = 2$. For the chromatic number, the vertices that are colored yellow are $0 \, modulo \, 2$ and the vertices that are colored pink are $1 \, modulo \, 2$. The clique number is represented by the vertices $\{0,1\}$.

Next, we consider the edge chromatic number of G_n.
A *proper edge coloring* of a graph G assigns a color to each edge of G such that no edges of like color are incident at the same vertex.

The *edge chromatic number* of G, denoted $\Lambda(G)$, is the smallest number of colors which can be used to generate a proper coloring of G.

To find the edge chromatic number of G_n, we use the following lemma.

Lemma 1. *Let $n \geq 3$ be given. Then G_n can be decomposed into $\dfrac{\phi(n)}{2}$ Hamiltonian cycles whose edge sets are disjoint.*

Proof. Let $u_1, -u_1, \ldots u_r, -u_r$ be the units of $\mathbb{Z}_n$. Clearly, for any u_i, the vertices labeled $0, u_i, 2u_i, \ldots nu_i$, where multiplication is modulo n, form a Hamiltonian cycle of G_n. We denote this cycle $C_n^{u_i}$. We will show that $C_n^{u_1}, C_n^{u_2}, \ldots C_n^{u_r}$ form a set of $\dfrac{\phi(n)}{2}$ Hamiltonian cycles in G_n whose edge sets are disjoint.

Let $\{a, b\}$ be an edge of $C_n^{u_i}$. Let $l(a)$ and $l(b)$ be the labels assigned to a and b, respectively. Without loss of generality, $l(a) > l(b)$. By definition of $C_n^{u_i}$, $l(a) - l(b) = u_i$. But this implies that $l(b) - l(a) = -u_i$, so $\{a, b\}$ is also an edge of $C_n^{-u_i}$. Thus, $C_n^{u_i}$ and $C_n^{-u_i}$ have the same edge set. By the definition of a Hamiltonian cycle, they also have the same vertex set. So $C_n^{u_i} = C_n^{-u_i}$. However, $\{a, b\}$ cannot be an edge of any cycle $C_n^{u_j}$, where $u_i \neq \pm u_j$. For if this were so, we would have $l(a) - l(b) = \pm u_j$, which cannot occur. Therefore, the edge set of $C_n^{u_i}$ is disjoint from that of every other cycle $C_n^{u_j}$, except for $C_n^{-u_i}$. $C_n^{u_1}, C_n^{u_2}, \ldots C_n^{u_r}$ thus form a set of $\dfrac{\phi(n)}{2}$ edge-disjoint Hamiltonian cycles in G_n.

Next, we will show that the union of these Hamiltonian cycles is the entire graph G_n. By definition of a Hamiltonian cycle, $\displaystyle\bigcup_{i=1}^{u_r} C_n^{u_i}$ has the same vertex set as G_n. Clearly, the edge set of $\displaystyle\bigcup_{i=1}^{u_r} C_n^{u_i}$ is a subset of $E(G_n)$. Each $C_n^{u_i}$ has exactly n edges, so the total number of edges in $\displaystyle\bigcup_{i=1}^{u_r} C_n^{u_i}$ is $n\left(\dfrac{\phi(n)}{2}\right)$. But this is precisely the number of edges in $E(G)$. Hence, the edge set of $\displaystyle\bigcup_{i=1}^{u_r} C_n^{u_i}$ is equal to that of G_n, and

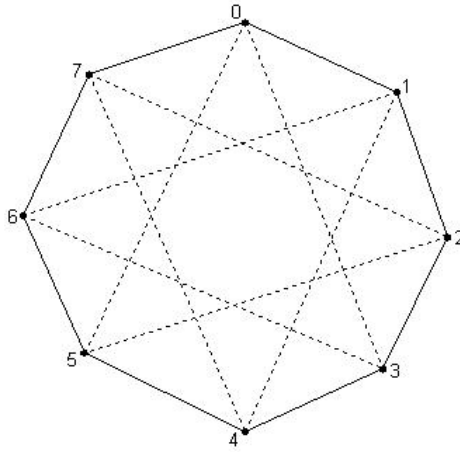$$\bigcup_{i=1}^{u_r} C_n^{u_i} = G_n.$$

$\square$

FIGURE 4. G_8 is a union of 2 disjoint Hamiltonian cycles.

Figure 4 shows G_8 as a union of 2 disjoint Hamiltonian cycles. Note that $\dfrac{\phi(8)}{2} = 2$. The solid lines form a cycle consisting of edges the difference of whose endpoints is 1; the dashed lines, a cycle consisting of edges the difference of whose endpoints is 3.

Our derivation of the edge chromatic number of G_n requires Vizing's theorem.

Vizing's Theorem. *Let G be a simple graph, and let $\Delta(G)$ be the maximum degree of G. Then either $\Lambda(G) = \Delta(G)$ or $\Lambda(G) = \Delta(G) + 1$ [5].*

Proposition 2. *If n is even, $\Lambda(G_n) = \phi(n)$. If n is odd, $\Lambda(G_n) = \phi(n) + 1$*

Proof. Case 1: Suppose n is even. By lemma 1, G_n can be decomposed into $\dfrac{\phi(n)}{2}$ edge-disjoint Hamiltonian cycles. Since n is even, each of these cycles has even length. Each cycle can thus be colored with two colors, simply by assigning the two colors to alternate edges of the cycle. We color each Hamiltonian cycle of G_n in this manner, such that no two cycles have a color in common. The result is a proper coloring of G_n using $\phi(n)$ colors. This implies that $\Lambda(G_n) \leq \phi(n)$ Since $\phi(n)$ is the maximum degree of G_n, Vizing's theorem shows that $\Lambda(G_n) \geq \phi(n)$. Combining these results gives $\Lambda(G_n) = \phi(n)$, and the proof is complete in this case.

Case 2: Suppose n is odd. We will show that G_n cannot be colored by only $\phi(n)$ colors, and thus, by Vizing's theorem, a proper coloring of n requires $\phi(n) + 1$ colors.

In a proper coloring, no two edges of like color can share an endpoint. Thus, for odd n, a proper coloring of G_n contains no more than $\dfrac{n-1}{2}$ edges of a given color. Using $\phi(n)$ colors, we can color at most $\phi(n)\left(\dfrac{n-1}{2}\right)$ edges of G_n. But G_n has a total of $\phi(n)\left(\dfrac{n}{2}\right)$ edges. So it is impossible to color all the edges of G_n using only $\phi(n)$ colors. Thus $\Lambda(G_n)$ is greater than $\phi(n)$, the maximum degree of G_n. By Vizing's theorem, $\Lambda(G)_n = \phi(n) + 1$.

$\square$

5. Partite Classes of G_n

Proposition 3. *: G_n is multipartite with p_1 partite sets, where $n = p_1^{e_1} p_2^{e_2} ... p_s^{e_s}$ and $p_1 < p_2 < \ldots < p_s$ are primes.*

Proof. : For $0 \leq i \leq p_1$ we define each partite set as $A_i = \{x \in V(G_n) : n \equiv i (mod p_1)\}$. The $A_1 \cup ... \cup A_{p_1 - 1} = V(G_n)$ and A_i forms an independent set because they are equivalence classes *modulo p_1*. Therefore the number of partite sets is less than or equal to p_1.

The set $\{0, 1, \ldots, p_1 - 1\}$ shows that there exist a clique with p_1 elements. The clique shows that the number of partite sets is greater than or equal to p_1. This concludes that p_1 is equal to the number of partite sets.

$\square$

6. Connectivity

We now consider the edge connectivity of G_n.

Let G be a connected graph. A *disconnecting set* of edges in G is a subset $D \subseteq E(G)$, such that removing the edges in D from G yields a disconnected graph.

The *edge connectivity*, denoted κ', of a connected graph G, is the order of the smallest disconnecting set of edges in G.

Let x and y be distinct vertices of a graph G. An x, y-disconnecting set of edges in G is a subset C of $E(G)$ such that removing the edges of C from G yield a graph with no paths from x to y.

To find the edge connectivity of G_n, we make use of Menger's theorem for edge connectivity.

Menger's Theorem *Suppose x and y are distinct vertices of a graph G. Then the minimim size of an x, y-disconnecting set of edges equals the maximum number of pairwise edge-disjoint x, y-paths in G.*

Proposition 4. *The edge connectivity $\kappa'(G_n)$ of G_n is equal to $\phi(n)$.*

Proof. For $n < 3$, the proposition is trivial. Let $n \geq 3$ be given. First, we show $\kappa'(G_n) \geq \phi(n)$.

Let v and w be vertices of G_n. By lemma 1, G_n can be decomposed into $\dfrac{\phi(n)}{2}$ disjoint Hamiltonian cycles. Hence, there are at least $\phi(n)$ paths from v to w whose edge sets are disjoint. By Menger's theorem, to disconnect v and w, we must remove at least $\phi(n)$ edges. Since v and w are arbitrary vertices of G_n, $\kappa'(G_n) \geq \phi(n)$.

To see that $\kappa'(G_n) = \phi(n)$, note that we can isolate a vertex v of G_n, simply by removing all edges incident at v. Since the degree of any vertex of G_n is $\phi(n)$, we can isolate a vertex by removing $\phi(n)$ edges. Hence, $\kappa'(G_n) \leq \phi(n)$. Since we have already shown the other inequality, $\kappa'(G_n) = \phi(n)$, and the proof is complete. □

Now we consider the vertex connectivity of G_n. $\kappa(G_n) = \min\{|S|: G - S$ is disconnected or only has one vertex$\}$.

We use Whitney's theorem as an upperbound for vertex connectivity, since we have proven that $\kappa'(G) = \phi(n)$. We use Menger's Theorem as the basis for the vertex connectivity proof.

Whitney's Theorem[4]. $\kappa(G) \leq \kappa'(G) \leq \phi(n)$.

Menger's Theorem[1]. *If two vertices $x, y \in V(G)$ are nonadjacent, the maximum number of internally disjoint xy-paths in G is equal to the minimum number of vertices whose deletion destroys all xy-paths.*

Proposition 5. $\kappa(G_n) = \kappa'(G_n) = \phi(G_n)$.

Proof. We know from Proposition 4 that $\kappa'(G_n) = \phi(n)$. In this proof, we will show that G_n has $\phi(n)$ internally disjoint paths from any two nonadjacent vertices. Thus the minimum number of vertices whose deletion destroys all xy-paths will be $\phi(n)$, by Menger's Theorem.

Consider G_n where $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$ and $p_1, \ldots, p_n$ are distinct primes. Let there be two nonadjacent vertices labeled $(0, 0, \ldots, 0)$ and $x = (p_1 a_1, p_2 a_2, \ldots, p_t a_t, b_{t+1}, \ldots, b_m)$, where $p_i \nmid b_i$. Then there exist common neighbors of the form $(c_1, \ldots, c_t, c_{t+1}, \ldots, c_m)$, where $p_i \nmid c_i$ and $p_i \nmid (b_i - c_i)$ for all i. This connects $(0, 0, \ldots, 0)$ and x by internally disjoint paths of length two.

However, there may be vertices adjacent to $(0, 0, \ldots, 0)$ which are not adjacent to x. Choose a vertex that is adjacent to $(0, 0, \ldots, 0)$ but not to x. Let this vertex be u, where $u = (u_1, \ldots, u_m)$, $p_i \nmid b_i$ for all i and $p_i | (u_i - b_i)$ for some $t + 1 \leq i \leq m$. Thus, u is adjacent to $(0, 0, \ldots, 0)$ and nonadjacent to x.

Case 1: All p_i are odd or $p_i = 2$ for some $t + 1 \leq i \leq m$.
Now we choose a vertex, $v = (v_1, \ldots, v_m)$, that is adjacent to both u and x. We define this vertex as follows. If $p_i | (u_i - b_i)$, then $v_i = u_i - b_i$. For $1 \leq i \leq t$, let $v_i = \begin{cases} u_i + 2 & \text{if } u_i \equiv (p_i - 1) \pmod{p_i} \\ u_i + 1 & \text{otherwise} \end{cases}$

If $p_i \nmid (u_i - b_i)$ and $t + 1 \leq i \leq m$, then

$$v_i = \begin{cases} u_i + 2 & \text{if } (u_i - b_i) \equiv (p_i - 1) \pmod{p_i} \\ u_i + 1 & \text{otherwise} \end{cases}$$

So v is adjacent to x. Thus $(0, 0, \ldots, 0)$ is connected to x by internally disjoint paths of length 3. We know $(0, 0, \ldots, 0)$ has $\phi(n)$ neighbors. We have now shown there are also $\phi(n)$ internally disjoint paths from $(0, 0, \ldots, 0)$ to x for the case where all primes are odd or $p_i = 2$ for some $t + 1 \leq i \leq m$.

Case 2: Let $p_i = 2$ for some $1 \leq i \leq t$. Without loss of generality, let $p_1 = 2$. Let there be two nonadjacent vertices labeled $(0, 0, \ldots, 0)$ and x. Preserving the notation of the previous case, choose a vertex u which is adjacent to x.

Notice that $p_1 a_1 = 2a_1$ is even and u_1 is odd, so there is no vertex $v = (v_1, \ldots, v_m)$ that is adjacent to both x and u.

We create a path from u to x as follows:

Let $w = (w_1, \ldots, w_m)$ be defined by $w_i = u_i + 1$ for all $1 \le i \le t$ and when $p_i \nmid (u_i - b_i)$ for $t + 1 \le i \le m$. If $p_i | (u_i - b_i)$ for $t + 1 \le i \le m$, then let
$$w_i = \begin{cases} b_i + 2 & \text{if } b_i \equiv (p_i - 1) \pmod{p_i} \\ b_i + 1 & \text{otherwise} \end{cases}$$
Notice that for $t + 1 \le i \le m$, $p_i \ne 2$.

Let $v = (v_1, \ldots, v_m)$ by defined by $v_i = u_i$ for all $1 \le i \le t$ and when $p_i \nmid (u_i - b_i)$ for $t + 1 \le i \le m$. If $p_i | (u_i - b_i)$ for $t + 1 \le i \le m$ then $v_i = u_i - b_i$.

Then u, w, v, x is a path from u to x and all such paths are vertex disjoint.

Again, we have shown that there are still $\phi(n)$ internally disjoint paths from $(0, \ldots, 0)$ to $(p_1 a_1, p_2 a_2, \ldots, p_t a_t, b_{t+1}, \ldots, b_m)$, even if $p_i = 2$ for some $1 \le i \le t$. Thus, we have proven that there are $\phi(n)$ internally disjoint paths between any nonadjacent vertices of the graph G_n.

$\kappa(G_n) = \kappa'(G_n) = \phi(n)$. $\qquad\qquad\square$

7. Connectedness and Diameter

Proposition 6. *Let $R = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$. Then G_R is connected if and only if n_i is even for at most one value of i.*

Proof. ($\Rightarrow$) Suppose there exists $i \ne j$ such that n_i and n_j are both even. Let z be the vertex $(0, \ldots, 0)$ of G_R. We will show that G_R has a vertex w such that there is no path in G_R from z to w.

Let $v = (v_1, \ldots, v_r)$ be any vertex of G_R not equal to z, and suppose there exists a path of length l between v and z. Then v_i is the sum of $l - 1$ units in $\mathbb{Z}_{n_i}$, and v_j is the sum of $l - 1$ units in $\mathbb{Z}_{n_j}$. Since n_1 and n_2 are even, each unit of $\mathbb{Z}_{n_1}$ or $\mathbb{Z}_{n_2}$ must be odd. Thus, v_i and v_j are each the sum of $l - 1$ odd numbers, which need not be distinct. This implies that either v_i and v_j are both even (if $l - 1$ is even), or they are both odd (if $l - 1$ is odd). Consider a vertex $w = (w_1, w_2, \ldots, w_r)$, where w_i is even and w_j is odd. By the above argument, there is no path from this vertex to $(0, 0, \ldots, 0)$. This implies that G_R is not a connected graph.

($\Leftarrow$) Next, suppose n_i is even for at most 1 value of i. We will show that there exists a path between any two vertices of G_n. By translation, it suffices to show that there exists a path between the vertex $z = (0, 0, \ldots, 0)$ and each other vertex of G_n.

Assume there exists a non-empty subset D of $V(G_R)$ such for any $d \in D$, no path exists between z and d.

Consider the following partial ordering of R. For any $a, b \in R$, we say $a < b$ if $a_i \le b_i$ for all i. Choose a vertex $d = (d_1 \ldots, d_r) \in D$ such that d is minimal with respect to the partial ordering described above.

Case 1: Suppose n_i is odd for all i. By construction, $d_j \ne 0$ for some $1 \le j \le s$. If $d = (d_1, d_2, \ldots, d_r)$, where $d_i = \begin{cases} 0 & \text{if } i \ne j \\ 1 & \text{if } i = j \end{cases}$. Then $(0, 0, \ldots, 1, \ldots, 0), (1, 1, \ldots, 2, \ldots, 1), (0, 0, \ldots, 0)$ is a path from z to d. This path is valid because n_j is odd, and hence 1 and 2 are units of $\mathbb{Z}_{n_j}$.

If d has some other value, we define $f = d - (0, 0, \ldots 1, 0)$. Because d is a minimal element of d, $f \notin D$, and there exists a path from z to f. Hence, it suffices to show there exists a path from f to d. By translation, this reduces to finding a path from z to $(0, 0, \ldots 1, 0)$. Such a path is given above. Hence, we can find a path between f and d

Concatenating this path with a path from z to f gives a path from z to d. This contradicts the claim that no path exists between z and d.

Therefore, our initial assumption was false. For any vertex v of G_R, there exists a path between z and v. Thus, if n_i is odd for all i, G_R is a connected graph.

Case 2: Suppose n_i is even for exactly one value of i. Choose d a minimal element of D. If d_i is 0, then $d_j \neq 0$ for some $j \neq i$. Then n_j is odd, and we employ the argument used in the odd case.

If $d_i \neq 0$, we must modify our argument slightly. If $d = (d_i, d_2, \ldots, d_r)$, where $d_k = \begin{cases} 0 & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases}$.

Then $(0, 0, \ldots, 1, \ldots, 0), (1, 1, \ldots, 0, \ldots, 1), (2, 2, \ldots, 1, \ldots 2), (0, 0, \ldots, 0)$ is a path from z to d.

Otherwise, we define $f = d - (0, 0, \ldots, 1, \ldots, 0)$. As in the odd case, there exists a path between z and f, and it suffices to show that there exists a path between f and d. But this reduces to finding a path between $(0, 0, \ldots, 0)$ and $(0, 0, \ldots, 1, \ldots, 0)$, which we have already accomplished. Hence, there exists a path between f and d.

Concatenating this path with the path from z to f gives a path from f to d. This contradictions our assumption that no path exists between d and z. Hence, our initial assumption is false, and G_R is a connected graph. $\qquad\square$

Note that the above result implies that G_n is connected for all n. Expanding n as $p_1^{e_1} \ldots p_s^{e_s}$, where the p_i are distinct primes, we see that $\mathbb{Z}_n$ is isomorphic to R, where $R = \mathbb{Z}_{p_1^{e_1}} \times \ldots \times \mathbb{Z}_{p_s^{e_s}}$. But $p_i^{e_i}$ is even for at most one value of i, because 2 is the only even prime. Hence, the above result shows that G_R is a connected graph. Since $\mathbb{Z}_n$ is isomorphic to R, G_n is isomorphic to G_R. This implies that G_n is connected.

Using an argument similar to that used in in the proof of proposition 6, we now derive the diameter of G_n for all n.

Proposition 7. *The diameter of G_n is 1 if n is prime, 2 if n is an odd composite number, and 3 if n is an even composite number.*

Proof. Case 1: Suppose n is prime. Claim: $G_n = K_n$, and the diameter of G_n is 1. [2]

Let n be prime, and let v and w be vertices of G_n. Then $v - w \in \mathbb{Z}_n$, and $v - w < n$. Because n is prime, this implies that $v - w$ is relatively prime to n. Hence $v - w$ is a unit of the ring $\mathbb{Z}_n$, and v and w are adjacent. Since v and w are arbitrary, each vertex of G_n is adjacent to every other vertex, and $G_n = K_n$. *Case 2:* Next, suppose n is an odd composite number, where $n = p_1^{e_1} \ldots p_s^{e_s}$. As in Proposition 1, we assign each vertex v of G_n to a class $X(h_1 \ldots h_s)$, where each h_i ranges from 0 to $p_i - 1$. Let $v, w \in V(G)$ be given, where $v \in X(v_1, \ldots, v_s)$ and $w \in X(w_1, \ldots, w_s)$. If $v_i \neq w_i$ for all $1 \leq i \leq s$, then v and w are adjacent.

Suppose $v_i = w_i$ for at least one value of i. Then v and w are non-adjacent, which implies that the diameter of G_n is at least 2. If n is odd, $p_i \geq 3$ for all i. Hence, each h_i has at least three possible values. Thus, we can construct a class $X(u_1, \ldots u_n)$ such that $u_i \neq v_i$ and $u_i \neq w_i$ for all $0 \leq i \leq n$. Let $u \in X(u_1, \ldots, u_n)$ be given. Then u is adjacent to both v and w, so v, u, w is a path of length two between v and w. Since v and w are arbitrary nonadjacent vertices of G_n, the diameter of G_n is 2.

Case 3: Suppose n is even, and $n \geq 2$. We again let $v, w \in V(G)$ be given, where $v \in X(v_1, \ldots, v_s)$ and $w \in X(w_1, \ldots, w_s)$. If $v_i \neq w_i$ for all $1 \leq i \leq s$, then v and w are adjacent. If $v_i = w_i$ for some i, the situation is more complicated. We have $n = p_1^{e_1} \ldots p_s^{e_s}$, with $p_1 = 2$. Thus, for each $X(h_1 \ldots h_s)$, h_1 has only two possible values. If $v_1 = w_1$ for v and w, we can again construct a class $X(u_1, \ldots u_n)$ such that $u_i \neq v_i$ and $u_i \neq w_i$ for all $0 \leq i \leq n$. Thus, any vertex in l is adjacent to both v and w, and the minimum length for a path between v and w is 2.

If, $v_1 \neq w_1$, but $v_i = w_i$ for some $i \geq 2$, the above argument fails. Clearly, v and w are non-adjacent. However, we cannot construct a class whose vertices are adjacent to both v and w. The first coordinate of any class will be equal either to v_1 or w_1. Thus, there is no path of length two between v and w. Since

there exist pairs of vertices in G_n which are not connected by any path of length 1 or 2, the diameter of G_n is at least 3.

We now show that the diameter of G_n is exactly 3. We construct a class of vertices $X(u_1, \ldots, u_s)$, as follows. If $v_i \neq w_i$, let $u_i = w_i$. If $v_i = w_i$, let u_i be any integer such that $u_i \neq w_i$, $0 < u_i < p_i - 1$. Let $u \in X(u_1, \ldots u_s)$ be given. Then u is adjacent to v. Also, u and w agree in their first coordinate, so there exists a vertex y adjacent to both u and w. Hence v, u, y, w is a path of length three between v and w. So the minimun length for such a path is three, so the diameter of G_n is at most 3. Combining this result with the lower bound for diameter given above, this implies that the diameter of G_n is exactly 3.

$\square$

8. Planarity

A graph is *planar* if it can be drawn in the plane with no edge crossings.

A planar graph is to be *embedded in the plane* if it is drawn in the plane with no edge crossings. Such a drawing is called a *planar embedding* of the graph or, alternatively, a *plane graph*.

We use several well-known theorems to find the values of n for which G_n is planar. The first is a consequence of Euler's formula.

A *face* of a plane graph is a region of the plane enclosed by edges of the graph. The space surrounding the graph is considered a face.

The *face length* of a face of a plane graph is the number of edges that form the boundary of that face.

Euler's formula. [5] *Let G be a plane graph with v vertices, e edges, and f faces, then $v - e + f = 2$.*

Corollary 4. [5] *Let G be a plane graph with v vertices, e edges, and f faces, where $v \geq 3$. Then $e \leq 3v - 6$.*

Proof. It suffices to show that the result holds when G is connected. If G is disconnected, we simply add edges to G to form a connected graph, G'. Then G' has e' edges, where $e' > e$. If we can prove that $e' \leq 3v - 6$, it follows that $e \leq 3v - 6$.

In a simple graph, the length of each face is at least 3. When we sum the lengths of the faces of G, each edge is counted twice. Let $\{l_i\}$ be a list of lengths of the faces of G. Then $2e = \sum_{i=1}^{f} l_i \geq 3f$. Substituting into Euler's formula, $v - e + f = 2$, we obtain $e \leq 3v - 6$. $\square$

Corollary 5. *Any planar graph has a vertex of degree less than 6 [3].*

Proof. Let G be a plane graph with v vertices, e edges, and f faces. If $v \leq 6$, the result follows. If $v > 6$, we note that by Corollary 4, $e \leq 3v - 6$, which implies $\dfrac{2e}{v} \leq 6 - \dfrac{12}{v}$, and thus $\dfrac{2e}{v} < 6$. However, $2e$ is the sum of the degrees of the vertices of G, because each edge has exactly two endpoints. So the inequality $\dfrac{2e}{v} < 6$ is equivalent to the statement that the average degree of G is less than 6. This implies that G has at least one vertex of degree less than 6, and the proof is complete.

$\square$

The corollaries to Euler's formula give a simple way to show that certain graphs are non-planar. However, a graph which meets the conditions for planarity given by Corollies 4 and 5 may still be non-planar. Kuratowski's theorem, stated below, provides a criterion that can be used to determine the planarity of any graph.

Kuratowski's Theorem. *A graph is planar if and only if it contains no subgraph which is a subdivision of K_5, the complete graph on 5 vertices, or $K_{3,3}$, the complete bipartite graph with partite sets of 3 vertices each [5].*

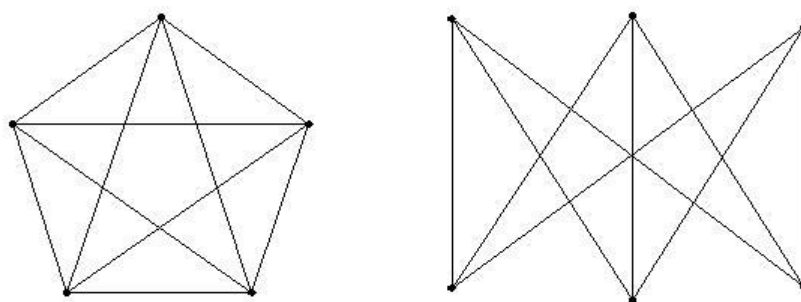Note that both K_5 and $K_{3,3}$ are themselves non-planar.

FIGURE 5. The non-planar graphs K_5 (left) and $K_{3,3}$ (right)

Given a graph G, we obtain a *subdivision* of G by inserting an arbitrary number of vertices which subdivide the edges in a subset of $E(G)$
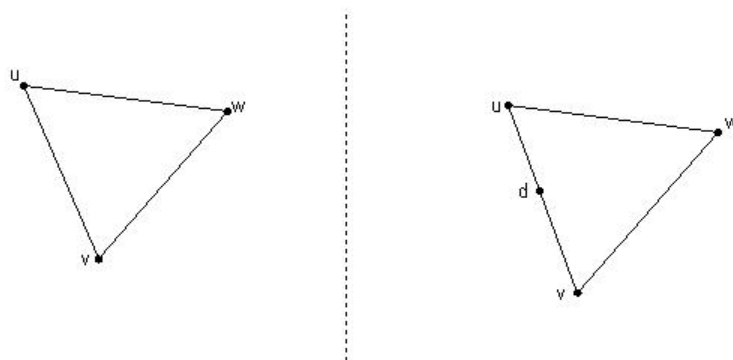

FIGURE 6. An example of a subdivision.

In figure 6, the graph on the right is a subdivision of the graph on the left. The edge $\{u, v\}$ is subdivided by the addition of the vertex d; the edge $\{u, v\}$ is replaced by the path u, d, v.

Proposition 8. *G_n is planar if and only if $n \in \{1, 2, 3, 4, 6\}$.*

Proof. For any of the given values of n, we can easily produce a planar drawing of G_n, as shown in figure 7.

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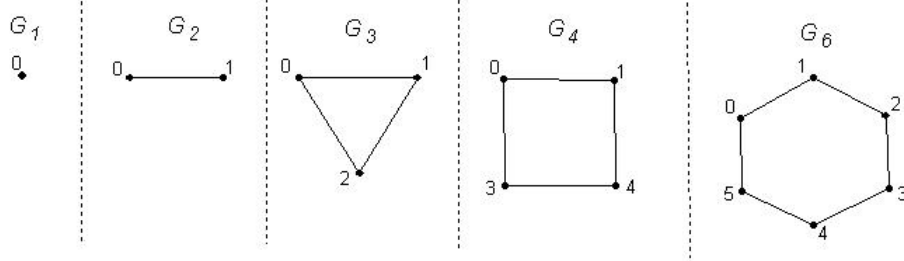
FIGURE 7. The planar G_n graphs.

We will now show that G_n is not planar for any other value of n. By the corollary to Euler's formula, a planar graph must have a vertex with degree less than six. Earlier, we showed that G_n is a $\phi(n)$-regular. Thus, if $\phi(n) \geq 6$, G_n cannot be planar. This implies that G_n is nonplanar for $n = 7$, $n = 9$, and $n \geq 13$. It remains to be shown that G_5, G_8, G_{10} and G_{12} are nonplanar.

G_5 is equal to K_5, the complete graph on five vertices, and is thus nonplanar. G_8, G_{10}, and G_{12} each contain subgraphs that are subdivisions of $K_{3,3}$, as shown in figure 8.
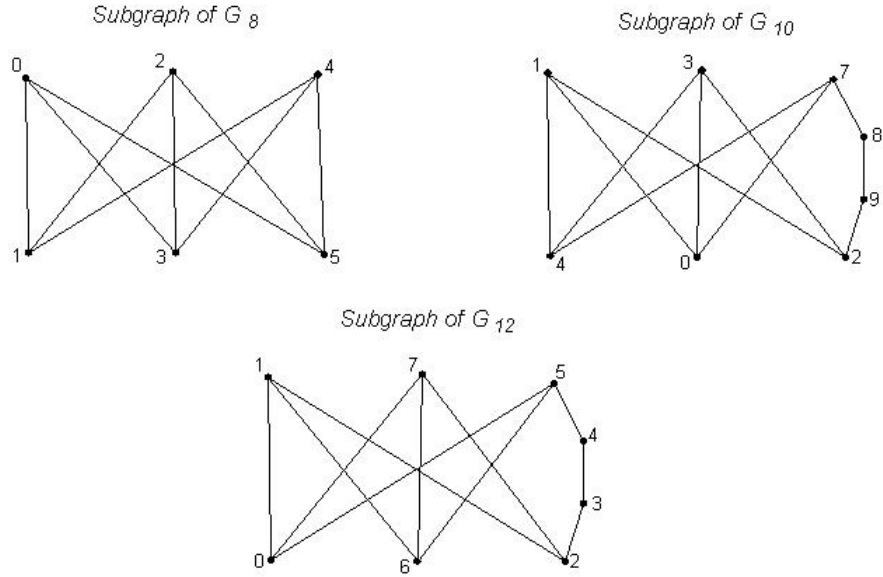
FIGURE 8. Subgraphs of G_8, G_{10} and G_{12} which are subdivisions of $K_{3,3}$.

By Kuratowski's theorem, G_8, G_{10}, and G_{12} are non-planar. Therefore, the only values of n for which G_n is planar are $1, 2, 3, 4$ and 6.

□

We now seek to extend this result to G_R graphs, where R is an arbitrary direct product of cyclic rings.

Lemma 2. *Let $R = \mathbb{Z}_{n_1} \times \ldots \mathbb{Z}_{n_r}$. Then G_R is k-regular, and $deg(v) = \prod_{i=1}^{r} \phi(n_i)$ for all $v \in V(G)$.*

Proof. Let v be a vertex of G_R, and suppose w is a vertex of G_R adjacent to v. Let $l(v) = (v_1, v_2, \ldots v_r)$ be the label assigned to v, and let $l(w) = (w_1, w_2, \ldots w_r)$ be the label assigned to w. Then w_i must be the label of a vertex adjacent to the vertex labeled v_i in the graph G_{n_i}. Since the degree of any vertex of G_{n_i} is $\phi(n_i)$, w_i has $\phi(n_i)$ possible values. Since v is an arbitrary vertex of G_R, this argument implies that G_R is k-regular, with degree $\prod_{i=1}^{r} \phi(n_i)$.

$\square$

Proposition 9. *Let R be a ring of the form $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$. Then G_R is planar if and only if R has the form $\mathbb{Z}_2 \times \ldots \mathbb{Z}_2 \times \mathbb{Z}_n$, where $n \in \{1, 2, 3, 4, 6\}$.*

Proof. ($\Rightarrow$) Let $R = \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \mathbb{Z}_n$, where $n \in \{1, 2, 3, 4, 6\}$. We will show that G_R is a planar graph. When $n = 1$, $R = \mathbb{Z}_2 \times \ldots \mathbb{Z}_2$. We note that the degree of any vertex of G_R is one. This follows from the lemma, and the fact that $\phi(2) = 1$. G_R is thus a disjoint union of K_2 graphs, which can easily be drawn in the plane without edge crossings.

The cases $R = \mathbb{Z}_2 \times \ldots \mathbb{Z}_2 \times \mathbb{Z}_4$ and $R = \mathbb{Z}_2 \times \ldots \mathbb{Z}_2 \times \mathbb{Z}_6$ require a little more work. We begin by observing that both G_4 and G_6 are even cycles.

Choose n such that G_n is an even cycle, and suppose $m \in \mathbb{N}$. Let R be the ring obtained from $\mathbb{Z}_n$ by taking the direct product with $\mathbb{Z}_2$ $m-1$ times. We will show that G_R is a disjoint union of even cycles, which implies that G_R is a planar graph.

Choose a vertex v of G_n. We give v a new label $l(v)$, an ordered m-tuple, as follows. Set the last coordinate of $l(v)$ equal to v, and let the remaining coordinates be an arbitrary list L of ones and zeroes. We then proceed around the cycle, assigning an m-tuple $l(v_i)$ to every vertex v_i, according to the following rule: we set the last coordinate $l(v_i)$ equal to v_i. If the distance from v to v_i is even, we fill in the remaining coordinates with the ordered list L. If the distance from v to v_i is odd, we fill in the first $m-1$ coordinates with a list of values obtained from L by interchanging the $1's$ and $0's$

The resulting labeling shows that G_n is a subgraph of G_R, where R is the ring obtained from $\mathbb{Z}_n$ by taking the direct product with $\mathbb{Z}_2$ $m-1$ times. We repeat this process a total of 2^{m-1} times, constructing 2^{m-1} copies of G_n and labeling the vertices of each in a manner analogous to that described above. For each copy of G_n, we let the first $m-1$ coordinates of the label assigned to v be a distinct list of ones and zeroes. We will show that the resulting union of disjoint cycles is equal to G_R, with R as above.

The scheme described assigns each element of R to a unique vertex of the resulting graph. By construction, this graph is isomorphic to G_R. Since this graph is a disjoint union of cycles, it is clearly planar. Hence, for $R = \mathbb{Z}_2 \times \ldots \mathbb{Z}_2 \times \mathbb{Z}_4$ and $R = \mathbb{Z}_2 \times \ldots \mathbb{Z}_2 \times \mathbb{Z}_6$, G_R is a planar graph.

Figure 9 provides an example of how this process works.
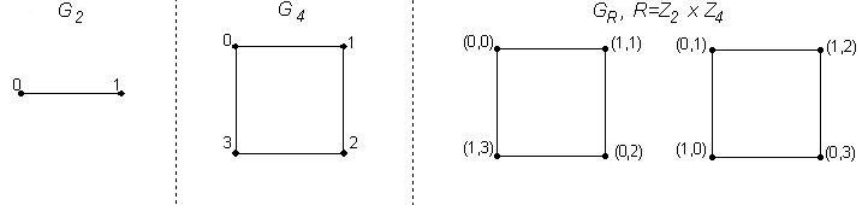
FIGURE 9. G_R, with $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ is isomorphic to the disjoint union of two G_4 graphs.

For the remaining case, $n = 3$, we simply note that $\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$ by the Chinese Remainder Theorem. Hence, for $R = \mathbb{Z}_2 \times \mathbb{Z}_3$, G_R is isomorphic to G_6. The result for G_6 thus implies that G_R, $R = \mathbb{Z}_2 \times \ldots \mathbb{Z}_2 \times \mathbb{Z}_3$ is a planar graph.

($\Longleftarrow$) Next, we show that no other direct product of cyclic rings yields a planar G_R graph. Again, we can eliminate any ring whose graph has degree greater than 5. However, this is not sufficient to yield a finite list. Taking the direct product of a ring R with $\mathbb{Z}_2$ gives a graph with different structure, but the same degree. Thus, we may obtain rings whose graphs have sufficiently small degree by multiplying any of the following rings by $\mathbb{Z}_2$ an arbitrary number of times:

$$\mathbb{Z}_5, \ \mathbb{Z}_7, \ \mathbb{Z}_8, \ \mathbb{Z}_{10}, \ \mathbb{Z}_{12}, \ \mathbb{Z}_3 \times \mathbb{Z}_3, \ \mathbb{Z}_3 \times \mathbb{Z}_6, \ \mathbb{Z}_4 \times \mathbb{Z}_4, \ \mathbb{Z}_4 \times \mathbb{Z}_6, \ \text{and} \ \mathbb{Z}_6 \times \mathbb{Z}_6.$$

We have already shown that the cyclic rings listed above are non-planar. Consider the non-cyclic rings in the above list. In Figure 11, we demonstrate that each of their respective G_R graphs contains a subgraph which is a subdivision of $K_{3,3}$, and is thus non-planar. More specifically, the G_R graphs of all except for $\mathbb{Z}_3 \times \mathbb{Z}_3$ contain a subgraph of the form shown in figure 10. This graph is a subdivision of $K_{3,3}$, where any path between vertices in different partite classes has odd length. In addition, G_8, G_{10} and G_{12} have subgraphs of this form, as shown in the planarity proof for G_n graphs. G_7 is the complete graph on 7 vertices, and therefore has $K_{3,3}$ as a subgraph.
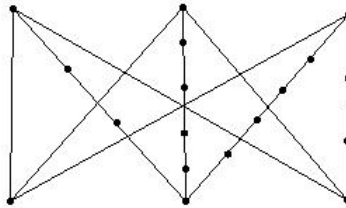

FIGURE 10. This graph is a subdivision of $K_{3,3}$. Note that each edge in the underlying $K_{3,3}$ has been replaced by a path of odd length.
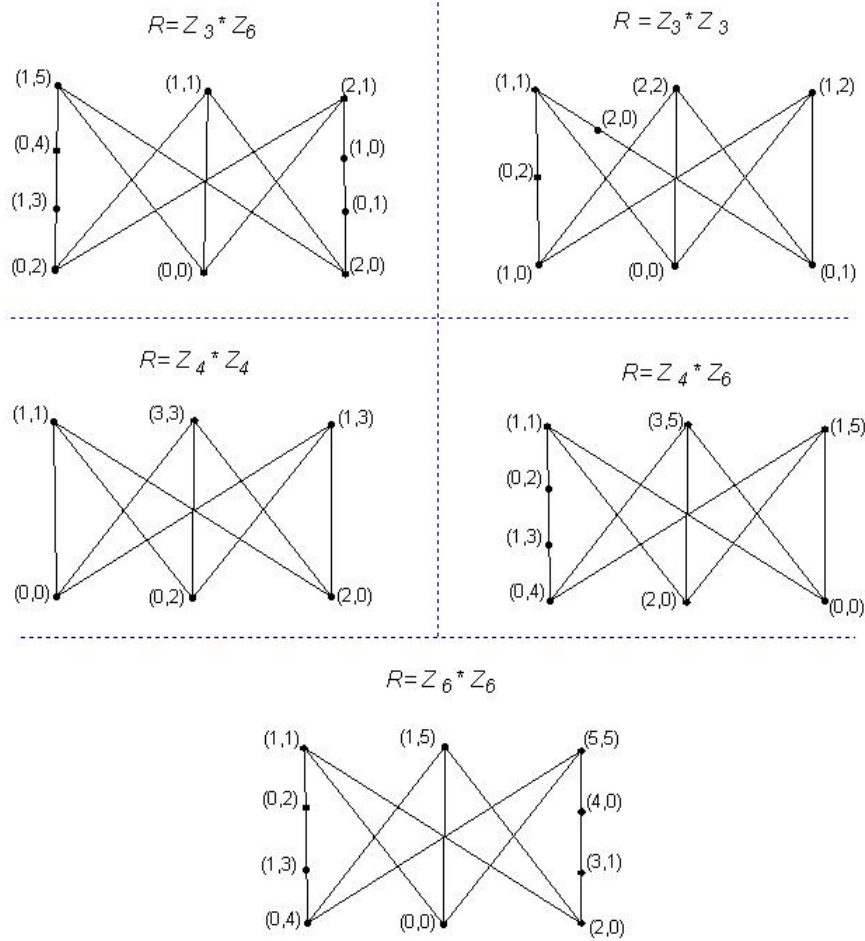
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FIGURE 11. subgraphs of non-cyclic rings which are subdivisions of $K_{3,3}$.

Consider a G_R graph which has a subgraph of the form shown in Figure 10. Suppose we take direct product of R with $\mathbb{Z}_2$ an arbitrary number of times. We will prove that the resulting ring has a non-planar G_R graph.

Let R be a ring, and let H be a subgraph of G_R such that H is a subdivision of $K_{3,3}$, and each path corresponding to an edge of the underlying $K_{3,3}$ has odd length. We label the vertices of this subgraph, as follows. Let H_0 denote one of the partite sets in H, and let H_1 denote the other. We append an additional coordinate, with a value of zero, to the label of each vertex in H_0. We then proceed along each path which corresponds to an edge in $K_{3,3}$, appending a coordinate to the label of each. If the distance between a given vertex and the nearest vertex of H_0 is odd, let this additional coordinate have a value of 1; if the distance is even, let it have a value of 0. The resulting labeling shows that H is a subgraph of $R \times \mathbb{Z}_2$. Thus, the G_R graph coresponding to $R \times \mathbb{Z}_2$ is nonplanar. By repeating this process an arbitrary number of times, we can show that the ring obtained by taking the direct product of R with $\mathbb{Z}_2$ an arbitrary number of times has a G_R graph with H as a subgraph. Thus, the G_R graph corresponding to such a ring is nonplanar.

Finally, we examine the case of $\mathbb{Z}_3 \times \mathbb{Z}_3$, which has no subgraph of the form H. We note that $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_3$, which does have such a subgraph. The above argument thus implies that $\mathbb{Z}_3 \times \mathbb{Z}_3$ with $\mathbb{Z}_2$ an arbitrary number of times will yield a ring whose graph is nonplanar.

We have shown that the only non-cyclic rings yielding planar G_R graphs are those obtained by taking the direct product of $\mathbb{Z}_2$ with a ring $\mathbb{Z}_n$ such that G_n is planar.

$\square$

9. Crossing Number

Proposition 10. : *For any $n \geq 5$, the complete graph K_n is non-planar.*

Proof. Assume for contradiction that K_n for some $n \geq 5$ is a planar graph. Since K_n is a complete graph, then the number of vertices is n and $e = \dfrac{kn}{2}$, where k represents the degree of each vertex. In this case $k = n - 1$. By replacing k by $n - 1$ we get: $e = \dfrac{(n-1)n}{2}$. Now by using *Euler's Formula* we get that $e \leq 3(n) - 6$ which implies that $\dfrac{(n-1)n}{2} \leq 3(n) - 6$. By simple algebra this reduces to $n^2 - 7n + 12 \leq 0$. Thus $n \leq 4$, which is a contradiction of our assumption that $n \geq 5$.

$\square$

The minimum number of edge crossings required to represent a graph G in the plane is said to be the *crossing number* denoted as $\gamma(G)$.

Theorem 2. $\gamma(K_6) = 3$.

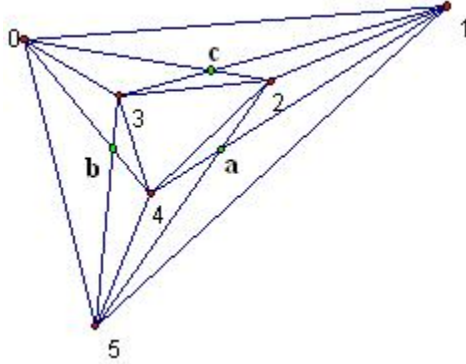*Proof.* Figure 12 demonstrates that $\gamma(K_6) \leq 3$.

Figure 12.

We prove $\gamma(K_6) \geq 3$.

Case 1: There exists a vertex v such that adding v to $K_6 - v$ note $K_6 - v \cong K_5$ yields no more edge crossings.

This implies that if we start with a K_5 drawing and add another vertex to get K_6, the edges incident at the vertex do not increase the number of edge crossings. This vertex must reside inside a face of K_5 where it can connect to all the other vertices without any more edge crossings. Without loss of generality, assume this is the inside face, then the other vertices would have to use the outer face to connect. Start by using the vertex labeled 0 and connect it to the vertex labeled 2. Then start use the vertex labeled 1 and connect it to the vertex labeled 3. No matter what side one chooses, this will create one edge crossing. Repeat this

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procedure to connect the vertex labeled 2 to the vertex labeled 4 and this will create at least one more crossing. Then use the same method to connect the vertices 3 and 0.This yields at least 5 crossings.

Case 2: For every vertex v, adding v to $K_7 - v$ yields at least one more edge crossing.

*Subcase 1:*If there exists a v such that K_6-v has at least two edge crossings, then we are done because adding v to the drawing has to add at least 1 more edge crossing (If not, we would be back in the first case).

Subcase 2: If not, then for every v K_6-v has exactly 1 edge crossing. The resulting drawing is an isomorphism of K_5. Also we know that this drawing has one form, which uses a subgraph of K_4. Thus, there is only a unique way of drawing it. By doing an exhaustive check, we can verify that regardless where we put the vertex, it creates at least two more edge crossings. Therefore, $\gamma(K_6) \geq 3$. Combining the two proofs we get that the $\gamma(K_6) = 3$. □

Proposition 11. $: 5 \leq \gamma(G_7) \leq 9$.

Proof. We prove that $\gamma(G_7) \geq 5$. Note: $G_7 \cong K_7$.

Case 1: There exists a vertex v, such that adding v to K_7 - v yields no more edge crossings.

This implies that if we start with a K_6 drawing and add another vertex to get K_7, this vertex does not increase the number of edge crossings. Insert the vertex inside a face of K_6 where the vertex can connect to all the other vertices without any edge crossing. Without loss of generality, assume this is the inside face, then the other vertices would have to use the outer face to connect. Start by using the vertex labeled 0 and connect it to the vertex labeled 2. Then start with the vertex labeled 1 and connect it to the vertex labeled 3. No matter what side you choose, this will create one edge crossing. Repeat this procedure to connect the vertex labeled 2 to the vertex labeled 4 and this will create at least one other crossing. Then use the same method to connect the vertices 3 and 5 and 4 and 6 and 5 and 0. This yields six edge crossings.

*Case 2:*For every vertex v, adding v to $K_7 - v$ yields at least one more edge crossing.

*Subcase 1:*If there exists a v such that K_7-v has at least 4 edge crossings, then we are done because v has to add at least one more edge crossing (If not, we would be in the first case).

*Subcase 2:*If not, then $K_7 - v \cong K_6$ has exactly 3 edge crossings. By our result that K_6 has a crossing number of 3 and doing an exhaustive check to verify that regardless of where we insert the vertex v, we have two extra edge crossings. This yields five edge crossings. Therefore, $\gamma(G_7) \geq 5$.

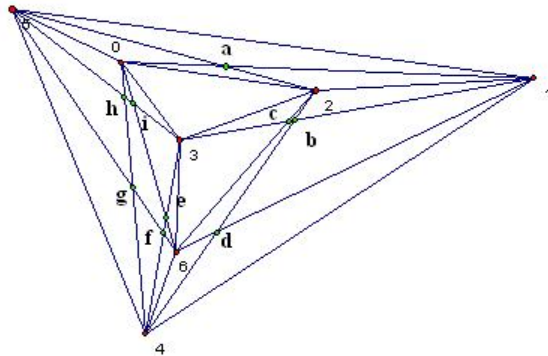We prove that $\gamma(G_7) \leq 9$ by looking at Figure 13.

FIGURE 13.

□

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Theorem 3. $\gamma(G_8) = 4$.

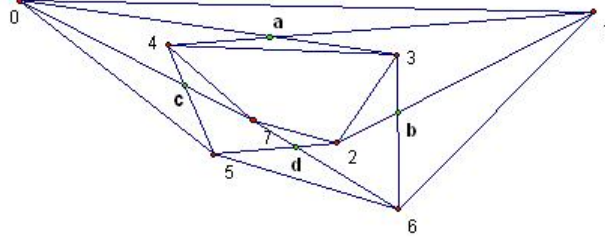*Proof.* We prove $\gamma(G_8) \leq 4$ by using Figure 14

FIGURE 14.

We prove $\gamma(G_8) \geq 4$ using the following Lemma.

Lemma 3. [5]. *If G is a triangle-free planar graph with n vertices and e edges, then $e \leq 2n - 4$.*

Note: $G_8 = K_{4,4}$ is bipartite, and therefore any subgraph of it is triangle free. Now suppose that G_8 can be drawn in the plane with 3 or fewer crossings. This means that by removing 3 edges (the edges that create the crossings) from G (but keeping all vertices), we may produce a planar subgraph G'. However, G' is then triangle-free with 8 vertices and 13 edges, so this contradicts Lemma 3 because we have $13 \leq 12$. Therefore, $\gamma(G_8) \geq 4$. Combining the two statements above we get that $\gamma(G_8) = 4$. □

Proposition 12. $\gamma(G_9) \leq 16$.
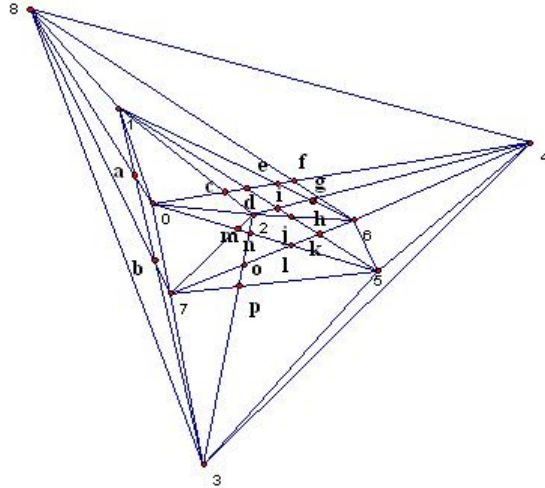
Proof. Figure 15 yields this result.

FIGURE 15.

□

Theorem 4. $\gamma(G_{10}) = 4$.

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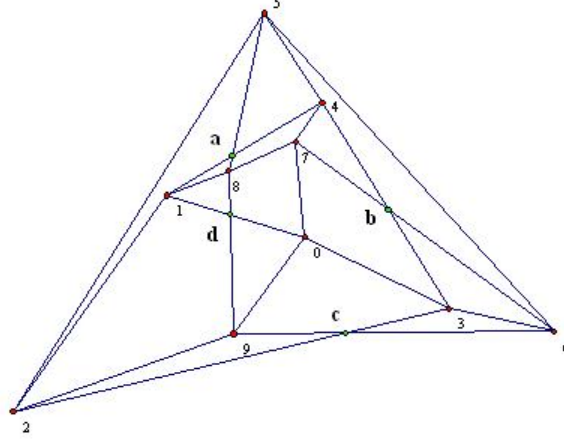*Proof.* First, we prove $\gamma(G_{10}) \leq 4$ by using Figure 16.

FIGURE 16.

Next, we prove $\gamma(G_{10}) \geq 4$.

Note: G_{10} is bipartite. Using the same technique, we used in Theorem 3 G', is triangle-free with 10 vertices and 17 edges, which leads us to the contradiction of Lemma 3 because we have $17 \leq 16$. Therefore, $\gamma(G_{10}) \geq 4$. Combining the two statements above we get that $\gamma(G_{10}) = 4$. □

Proposition 13. $: 4 \leq \gamma(G_{12}) \leq 6$.

Proof. First, we prove $\gamma(G_{12}) \leq 6$ by observing Figure 17
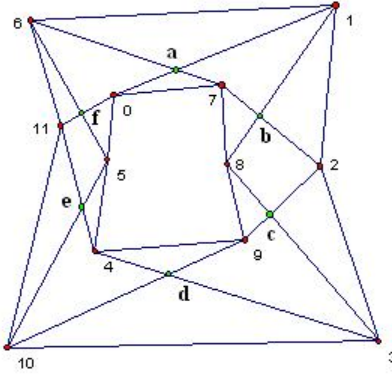

FIGURE 17.

We prove $\gamma(G_{12}) \geq 4$.

Note: G_{12} is bipartite. Using the same technique we used in Theorem 3, G' is triangle-free with 12 vertices and 21 edges, which leads us to the contradiction of Lemma 3 because we have $21 \leq 20$. Therefore, $\gamma(G_{12}) \geq 4$.

□

10. Automorphisms

Finally, we look at the automorphisms of G_n. An *automorphism* of a graph is a bijective mapping from the vertex set of the graph to itself which preserves adjacency and non-adjacency. The automorphisms of a given graph form a group under the operation composition of functions.

Theorem 5. *Let $n = \prod_{i=1}^{s} p_i^{e_i}$, where the p_i are distinct primes, and $p_1, p_2, ...p_s$. Then a bijective mapping $\alpha : V(G_n) \to V(G_n)$ induces an automorphism of G_n if and only if α preserves congruence modulo p_i for all i.*

Proof. ($\Longleftarrow$) Suppose α preserves congruence modulo p_i for all i. Let v and w be vertices of G_n. If v and w are non-adjacent, then we have $v \equiv w \pmod{p_i}$ for some i. Because α preserves congruence modulo p_i, $\alpha(v) \equiv \alpha(w) \pmod{p_i}$. Hence, by the rules of modular representation, $\alpha(v)$ is adjacent to $\alpha(w)$. So α maps nonadjacent vertices of G_n to nonadjacent vertices of G_n.

Next, suppose that v and w are adjacent. We will show that $\alpha(v)$ and $\alpha(w)$ are adjacent. Consider a prime divisor p_i of n. Suppose α maps a vertex u into a residue class r mod p_i. Since α preserves congruence modulo p_i, every vertex whose label is congruent to u modulo p_i maps into r. G_n contains $\dfrac{n}{p_i}$ vertices whose labels are congruent to u modulo p_i, which is exactly the number of vertices with labels in r. By the pigeonhole principle, no two vertices which are not congruent modulo p_i can map to vertices whose labels are in the same residue class modulo p_i. Because v and w are adjacent, we have $v \not\equiv w \pmod{p_i}$ for all i. By the above argument, this implies that $\alpha(v) \not\equiv \alpha(w) \pmod{p_i}$ for all i. Hence, $\alpha(v)$ is adjacent to $\alpha(w)$. So α maps adjacent vertices of G_n to adjacent vertices of G_n. Hence, α is an automorphism of G_n.

($\Longleftarrow$) Next, suppose that α is an automorphism of G_n. We will show that G_n preserves congruence modulo p_i for all i.

We first show that α preserves congruence modulo p_1, the smallest prime divisor of n. Consider the set A of vertices of G_n whose labels belong to a residue class a modulo p_1. A is an independent subset of $V(G_n)$. Since α is an automorphism of G_n, α maps A to an independent subset S of $V(G_n)$. We will show that the elements of S are congruent modulo p_1.

Let t and u be vertices in S. Then clearly, $|t - u| > p_1$. For otherwise, we would have $t \not\equiv u \pmod{p_i}$ for all i, so t and u would be adjacent, contradicting the definition of S. Hence, S contains $\dfrac{n}{p_1}$ elements between 0 and n, with the difference between any two elements greater than or equal to p_1. We arrange the elements of S in a list from smallest to largest, and consider consecutive elements of S. Therefore, the difference between consecutive elements of S must be exactly p_1. This implies that the elements of S are congruent modulo p_1.

Next, suppose α preserves congruence modulo $p_1, p_2, \ldots p_m$. We will show that α preserves congruence modulo p_{m+1}.

Case 1: Suppose n is odd. Let v and w vertices of G_n such that $v \equiv w \pmod{p_{m+1}}$. Assume $|\alpha(v) - \alpha(w)| < p_{m+1}$. Then $\alpha(v)$ and $\alpha(w)$ are in different residue classes modulo p_i for all $i \geq m$.

The number of common neighbors of v and w in G_n is given by $\prod_{i=1}^{s} p_i^{e_i-1} d_i$ where $d_i = p_i - 1$ if $v \equiv w$ $\pmod{p_i}$ and $d_i = p_i - 2$ if $v \not\equiv w \pmod{p_i}$. To see this, we assign each vertex v to a set $X(v_1, \ldots, v_s)$, where $0 \leq v_i \leq p - 1$ and $v_i \equiv l(v) \pmod{p_i}$.

Let u be a vertex adjacent to both v and w, and let $X(u_1, \ldots, u_s)$ be the class containing u. Then each u_i must differ from both v_i and w_i. Hence, u_i can take on either $p_i - 1$ possible values, if $u_i = v_i$, or $p_i - 2$

possible values, if $u_i \neq v_i$. Hence, u belongs to one of $\prod_{i=1}^{s} d_i$ sets, each of which contain $\prod_{i=1}^{s} p_i^{e_i-1}$ elements.

Hence, the number of common neighbors of v and w is $\prod_{i=1}^{s} p_i^{e_i-1} d_i$. Because n is even, $p_i \geq 3$ for all i, and hence $d_i \geq 1$ for all i. Therefore, any two vertices have at least one common neighbor.

Since α is an automorphism of G_n, the number of common neighbors of $\alpha(v)$ and $\alpha(w)$ must be equal to that of v and w. Since α preserves congruence modulo $p_1, p_2, \ldots, p_m$, we know that $\prod_{i=1}^{m} p_i^{m_i-1} d_i$ is identical for the two pairs v, w and $\alpha(v), \alpha(w)$. For $\alpha(v)$ and $\alpha(w)$, $d_i = p_i - 2$ for all $i \geq m + 1$. Since $v \equiv w$ (mod q), we have $d_m + 1 = p_{m+1} - 1$ for v and w. The value of $\prod_{i=1}^{s} p_i^{k_i-1} d_i$ is thus greater for v and w than for $\alpha(v)$ and $\alpha(w)$. This implies that v and w have more common neighbors than do $\alpha(v)$ and $\alpha(w)$, which is a contradiction. Hence, $|\alpha(v) - \alpha(w)| \geq p_{m+1}$.

Consider the subset Q of $V(G_n)$, consisting of all vertices of G_n whose labels are congruent to v modulo p_{m+1}. Q is an independent set. Since α is an automorphism, α maps Q to an independent set, which we denote R. R contains exactly $\dfrac{n}{p_{m+1}}$ elements between 0 and $n - 1$. By the above argument, any two elements of R must differ by at least p_{m+1}. Arrange the elements of R in a list from smallest to largest. By the pigeonhole principle, successive elements on this list must differ by exactly p_{m+1}. But this implies that the elements of R are congruent modulo p_{m+1}. Hence, α preserves congruence modulo p_{m+1}.

Case 2: Suppose n is even. In this case, the proof is more complicated. Again, we consider two vertices labeled v and w, where $v \equiv w$ (mod p_{m+1}). If, in addition, $v \equiv w$ (mod 2), we can apply the common neighbors argument used above, and show that $|\alpha(v) - \alpha(w)| \geq p_{m+1}$. If, $v \not\equiv w$ (mod 2), this argument fails, because v and w have no common neighbors. We therefore use a variation on this argument. Instead of considering common neighbors of v and w, we examine paths of length 3 between v and w.

— Claim: the number of paths of length 3 between two vertices of G_n which are not congruent modulo 2 is given by $\prod_{i=1}^{s} p_i^{2(e_i-1)} c_i$, where the value of c_i is given by the following:

If $v \equiv w$ (mod p_i), then $c_i = p_i^2 - 3p_i + 2$.

If $v \not\equiv w$ (mod p_i), then $c_i = p_i^2 - 3p_i + 3$.

To prove this, we again assign each vertex of G_n to a set X, as described above. Suppose v, t, u, w is a path in G_n. Let $X(v_1, \ldots, v_s)$ be the set containing v, $X(t_1, \ldots, t_s)$ the set containing t, $X(u_1, \ldots, u_s)$ the set containing u, and $X(w_1, \ldots, w_s)$ the set containing w.

Suppose $v_i = w_i$ for a given value of i. Then t_i can take on any value between 0 and $p_i - 1$ not equal to v_i, or one of $p_i - 1$ different values. Similarly, u_i can take on any value between 0 and $p_i - 1$ not equal to w_i or u_i, or one of $p_i - 2$ values. Hence, the number of possible combinations of values of t_i and u_i is given by $c_i = (p_i - 1)(p_i - 2) = p_i^2 - 3p_i + 2$.

On the other hand, suppose $v_i \neq w_i$. Then t_i can again take on one of $p_i - 1$ possible values. If $t_i = w_i$, then u_i can take on any value not equal to t_i, or one of $p_i - 1$ possible values. If $t_i \neq w_i$, then we again have $p_i - 2$ possible values for u_i. Hence, the number of possible combinations of values of t_i and u_i is given by $c_i = 1(p_i - 1) + (p_i - 2)(p_i - 2) = p_i^2 - 3p_i + 3$.

Multiplying the values of c_i for $1 \leq i \leq s$ gives the number of possible combination of coordinates for the sets $X(t_1, \ldots, t_s)$ and $X(u_1, \ldots, u_s)$. To obtain the total number of paths of length 3 between v and w, we

multiply this expression by the square of the number of elements in each class, and obtain a total of $\prod_{i=1}^{s} p_i^{2(e_i-1)} c_i$ such paths between any two vertices.

Since α is an automorphism of G_n, the number of paths of length 3 between $\alpha(v)$ and $\alpha(w)$ is equal to the number of such paths between v and w. Since α preserves congruence modulo $p_1, \ldots, p_m$, $\prod_{i=1}^{m} p_i^{2(e_i-1)} c_i$ is identical for the pair $\alpha(v)$ and $\alpha(w)$ and the pair v and w. Suppose $|\alpha(v) - \alpha(w)| < p_{m+1}$. Then $\alpha(v) \not\equiv \alpha(w) \pmod{p_i}$ for all $p_{m+1} \leq i \leq s$. Hence, $c_i = p_i^2 - 3p_i + 3$ for all $p_{m+1} \leq i \leq s$. Since $v \equiv w$ $\pmod{p_{m+1}}$, we have $d_m + 1 = p_i^2 - 3p_i + 2$. Thus, there are fewer paths of length 3 between v and w than between $\alpha(v)$ and $\alpha(w)$. This gives a contradiction. Hence, $|\alpha(v) - \alpha(w)| \geq p_{m+1}$.

By the argument used in the odd case, this implies that α preserves congruence modulo $p_m + 1$. Again, repeating this argument for each prime p_i shows that α preserves congruence modulo p_i for all i. This completes the proof.

$\square$

11. Acknowledgements

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