Line Graphs of Zero Divisor Graphs

Camil I. Aponte, Patrice S. Johnson, and Nathan A. Mims

Abstract

Let $L(\Gamma(\mathbb{Z}_n))$ be the line graph of $\Gamma(\mathbb{Z}_n)$. The authors determine when $\overline{\Gamma(\mathbb{Z}_n)}$ and $L(\Gamma(\mathbb{Z}_n))$ are Eulerian. Moreover, studies are done on the diameter, girth, trees, planarity, center, eccentricity, clique, chromatic number, and the existence of Hamiltonian cycles for $L(\Gamma(\mathbb{Z}_n))$.

1 Introduction

Among the most interesting graphs are the zero divisor graphs, their complements and their line graphs, because these involve both ring theory and graph theory. By studying these graphs we can gain a broader insight into the concepts and properties that involve both graphs and rings. In order to discuss the zero divisor graphs we begin by defining several key terms.

A *simple graph* is a pair $G = (V, E)$, where V is the vertex set and E is the edge set. In fact, V can be any set and E consists of unordered pairs $\{u, v\}$ of distinct elements of V. When two vertices are connected by an edge they are said to be *adjacent.*Every time we say the word graph we mean a simple graph by the definition stated above.

Some graphs can be constructed with special elements of a ring R, where R is commutative and with identity. These graphs are the *zero divisor graphs*, denoted $\Gamma(R)$, where these special elements of R are called *zero divisors*. An element $z \in R$ is a zero divisor if there is an element $z' \in R$ such that $z' \neq 0$ and $z \cdot z' = 0$. The graph $\Gamma(\mathbb{Z}_n)$ is built by defining the vertex set $V(\Gamma)$ to be the nonzero zero divisors of R and the edge set $E(\Gamma)$ to to consists of all pairs $\{u, v\}$ where $u \neq v$ and $u \cdot v = 0$.

We study in more detail the zero divisor graph of the ring $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$ of residue classes modulo n. If we take two elements $\bar{a}$ and $\bar{b}$ from $\mathbb{Z}_n$, we define addition by $\bar{a}\bar{+}\bar{b} = \bar{r}$ where r is the remainder upon dividing $(a+b)$ by n, so $0 \leq r \leq n-1$. Multiplication is defined analogously. For example, look at $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$; where we will omit the bars for ease of notation. The addition of 5 and 8 yields 1, since $5 + 8 \equiv 13(mod 12) = 1$. Also, the product of 3 and 4 is 0 since $3 \cdot 4 \equiv 12(mod 12) = 0$. Note that $\mathbb{Z}_{12}$ has zero divisors $\{0, 2, 3, 4, 6, 8, 9, 10\}$. Hence, $V(\Gamma) = \{2, 3, 4, 6, 8, 9, 10\}$ and $E(\Gamma)$ is shown in Figure 1. Observe, for example, that $\{6, 10\}$ is an edge, because $6 \cdot 10 = 0$ in $\mathbb{Z}_{12}$.

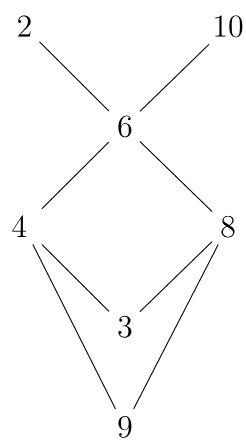

Figure 1: $\Gamma(\mathbb{Z}_{12})$

The *complement* $\overline{G}$ of a graph G, is a graph where vertices are the same as the vertices of G and an edge $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$. The complement of $\Gamma(R)$ is denoted $\overline{\Gamma(R)}$ and its edges $\{u, v\}$ are the pairs such that $u \cdot v \neq 0$.

The *line graph* $L(G)$ of a graph G, is the graph defined by $V(L(G)) = E(G)$ and $\{e_1, e_2\} \in E(L(G))$ if e_1 and e_2 are incident to a common vertex in G. If $\{x, y\} \in E(G)$ we will denote the corresponding vertex of $L(G)$ by $[x, y]$. We will denote the vertex set $V(L(G))$ and the edge set $E(L(G))$. In particular, $L(\Gamma(R))$ will have vertices of the form $[u, v]$ such that u and v are nonzero zero divisors where $u \cdot v = 0$.

In general, we study some properties such as diameter, girth, planarity, centers, and eccentricity of $L(\Gamma(\mathbb{Z}_n))$, as well as for what values of n this graph is Hamiltonian. Also, we determine precisely when $\overline{\Gamma(\mathbb{Z}_n)}$ and $L(\Gamma(\mathbb{Z}_n))$ are Eulerian. Moreover, it is instructive to observe that if n is prime, then zero is the only zero divisor of $\mathbb{Z}_n$.

Thus, $\Gamma(\mathbb{Z}_n)$ is an empty graph. Also, $\Gamma(\mathbb{Z}_4)$ is a single vertex because 2 is the only nonzero zero divisor. In these cases $L(\Gamma(\mathbb{Z}_n))$ is an empty graph which will not be studied in the sequel.

2 Preliminaries

Throughout this paper we make use of [7] that contains formulas for the degrees of $\Gamma(\mathbb{Z}_n)$ and $\overline{\Gamma(\mathbb{Z}_n)}$. Lemma 6.6 of [7] says that for any $v \in V(\Gamma(\mathbb{Z}_n))$:

$$\deg\ (v) = \begin{cases} \gcd(v,n) - 1, & \text{if } v^2 \neq 0; \\ \gcd(v,n) - 2, & \text{if } v^2 = 0. \end{cases} \tag{1}$$

Also in this lemma it is proved that for any $v \in V(\overline{\Gamma(\mathbb{Z}_n)})$:

$$\deg\ (v) = \begin{cases} n - \phi(n) - \gcd(v,n) - 1, & \text{if } v^2 \neq 0; \\ n - \phi(n) - \gcd(v,n), & \text{if } v^2 = 0. \end{cases} \tag{2}$$

Their formula differs from ours by one; they did not consider the fact that v is not its own neighbor. We are indebted to Leigh Cobbs for this observation. Below we generalize this to a formula for the degree of any vertex $[u,v] \in L(G)$ where G is a simple graph.

Lemma 2.1. *For any $[u,v] \in V(L(G))$:*

$$deg\ _{L(G)}[u,v] = deg\ _G(u) + deg\ _G(v) - 2.$$

Proof. We can see that $\{u,v\} \in E(G)$ is incident to the vertices of $L(G)$ representing edges of G incident at u and the edges incident at v, but we also have to consider that u and v are adjacent to each other. Hence, in $L(G)$ the degree of the vertex $[u,v]$ will be the degree of u in G plus the degree of v in G minus 2, because u is in the neighborhood of v and v is in the neighborhood of u.$\square$

In particular, we can specify the degree of any $[u,v] \in V(L(\Gamma(\mathbb{Z}_n)))$ by substituting, using the formula for degree in $\Gamma(\mathbb{Z}_n)$:

$$\deg_{L(\Gamma(\mathbb{Z}_n))}[u,v] = \begin{cases} \gcd(u,n) + \gcd(v,n) - 4, & \text{if } u^2 \neq 0 \text{ and } v^2 \neq 0; \\ \gcd(u,n) + \gcd(v,n) - 5, & \text{if either } u^2 = 0 \text{ or } v^2 = 0; \\ \gcd(u,n) + \gcd(v,n) - 6, & \text{if } u^2 = 0 \text{ and } v^2 = 0. \end{cases} \tag{3}$$

It is important to state some results and definitions about line graphs that are helpful to prove some of the properties we look at.

The proof of the next theorem uses the concepts of a walk and path. A *walk* of length k is a sequence $v_0, e_0, v_1, e_1, \ldots, v_{k-1}, e_{k-1}, v_k$ where for every i, $v_i \in V$, $e_i \in E$, and $e_i = \{ v_i, v_{i+1} \}$. A *path* is a walk in which no internal vertex is repeated. The initial and terminal vertices may coincide, in which case one has a cycle.

Theorem 2.2. *If a graph G is connected then $L(G)$ is connected.*

Proof. Let $[v,w]$, $[v_1,w_1] \in V(L(G))$ and $P = \{v = u_0, u_1, u_2, \ldots, u_n = v_1\}$ be a $(v - v_1)$ path in G. Also, let $e_0 = [v,w]$, $e_{n+1} = [v_1,w_1]$ and $e_i = [u_{i-1}, u_i]$ for $1 \leq i \leq n$. Then, the vertices $e_0, e_1, \ldots, e_{n+1}$ form an $(e_0 - e_{n+1})$ walk. Hence, it contains an $(e_0 - e_{n+1})$ path. Therefore $L(G)$ is connected.□

A *complete graph* K_m is a graph in which all the vertices are adjacent to each other, where m is the number of vertices. A subset $C \subseteq V(G)$ is called a *clique* if the subgraph of G induced by C is a complete graph. If the entire graph is a clique, it is called a complete graph K_m where m is the number of vertices. We illustrate an example of a star graph and its (complete) line graph, Figure 2 shows $\Gamma(\mathbb{Z}_{10})$ and Figure 3 shows $L(\Gamma(\mathbb{Z}_{10}))$.

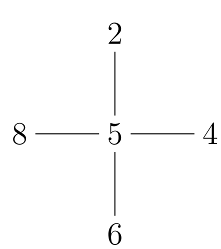

Figure 2: $\Gamma(\mathbb{Z}_{10})$

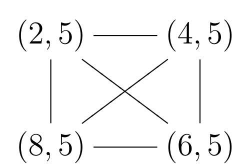

Figure 3: $L(\Gamma(\mathbb{Z}_{10}))$

Theorem 2.3. *If a graph contains a star subgraph, then its line graph will contain a clique.*

Proof. If a graph G contains a star subgraph H with central vertex v, then all the edges of H will have v as an endpoint. Hence, every vertex of $V(L(H))$ will have the form $[v, w]$ where w is a vertex that is adjacent in H to v, so all these vertices will be adjacent in $L(H)$ and form a clique.$\square$

As we proceed to study the graphs $L(\Gamma(\mathbb{Z}_n))$ and $\overline{\Gamma(\mathbb{Z})}$ in the remainder of the paper, we write $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ to represent the prime factorization of n. Precisely, $p_1, p_2, \dots, p_r$ are distinct primes and $p_1 < p_2 < \dots < p_r$ and $e_1, e_2, \dots, e_r$ are positive integers.

3 Diameter

Let G be a graph, where the *distance* between vertices u and v in G, denoted $d(u, v)$, is the minimum length of a (u, v)-path. If no (u, v)-path exists, i.e. if u and v lie in different components, then $d(u, v) = \infty$. Evidently, u and v are adjacent exactly when $d(u, v) = 1$. The *diameter* of G denoted $diam(G) = \max d(u, v)$, where $u \neq v$. Notice, that $L(\Gamma(\mathbb{Z}_9))$ is a single vertex, $[3, 6]$, and thus, the $diam(L(\Gamma(\mathbb{Z}_9)) = 0$. We will now find the diameter of $L(\Gamma(\mathbb{Z}_n))$ where $n \neq 9$.

Theorem 3.1. *If $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, $r \leq 2$ and $n \geq 10$, then $diam(L(\Gamma(\mathbb{Z}_n))) = 2$. When $n = 2p$ (p is an odd prime) or $n = 8$, then $diam(L(\Gamma(\mathbb{Z}_n))) = 1$.*

Proof.

Case 1: $r = 1$, $e_1, e_2 \in \Gamma(\mathbb{Z}_n)$ so $n = p^e$. We may assume $e \geq 2$. Write $e = e_1 + e_2$ where $e/2 \leq e_1, e_2 \leq (e + 1)/2$.

$\lozenge$ *Subcase 1.1:* e is even.

We can see that $e_1 = m$, $e_2 = m$, and $e = 2m$. Let $\{u_1, v_1\}, \{u_2, v_2\} \in E(\Gamma(\mathbb{Z}_n))$, where $u_1 \cdot v_1 = 0$ and $u_1 \neq v_1$, and the same for $u_2 \cdot v_2 = 0$ and $u_2 \neq v_2$. Then, p^{e_1} divides at least one of u_1 or v_1, otherwise p^{e_1} does not divide u_1v_1. Assume without loss of generality that $p^{e_1} \mid u_1$ and $p^{e_2} \mid u_2$; then $p^{e_1} \cdot p^{e_2} \mid u_1u_2$ gives us $p^e \mid u_1u_2$, which means $n \mid u_1u_2$. Therefore, $u_1 \cdot u_2 = 0$. So, then $[v_1, u_1]$, $[u_1, v_2]$, $[v_2, u_2]$ is a path from $[u_1, v_1]$ to $[u_2, v_2]$ in $L(\Gamma(\mathbb{Z}_n))$. Then the graph has $diam([u_1, v_1], [u_2, v_2]) \leq 2$.

We want $\{u_1, v_1\}, \{u_2, v_2\} \in E(\Gamma(\mathbb{Z}_n))$ and not be adjacent, to show that there is a pair of nonadjacent vertices, $[u_1, v_1], [u_2, v_2] \in L(\Gamma(\mathbb{Z}_n))$.

- If $p = 2$ then $e \geq 4$ and $n \geq 16$. Let, $[2, 2^{e-1}], [4, 3 \cdot 2^{e-2}] \in V(L(\Gamma(\mathbb{Z}_n)))$, then $[2, 2^{e-1}]$ is not adjacent to $[4, 3 \cdot 2^{e-2}]$.
- If $p = 3$ then $e \geq 3$, and $n \geq 27$. Let, $[3, 3^{e-1}], [6, 2 \cdot 3^{e-1}] \in V(L(\Gamma(\mathbb{Z}_n)))$, then $[3, 3^{e-1}]$ is not adjacent to $[6, 2 \cdot 3^{e-1}]$.
- If $p \geq 5$, then $e \geq 2$ and $n \geq 25$. Let, $[p, 2p^{e-1}], [3p, 4p^{e-1}] \in V(L(\Gamma((Z)_n)))$, then $[p, 2p^{e-1}]$ is not adjacent to $[3p, 4p^{e-1}]$.

Thus, we have a pair of nonadjacent vertices when $n = p^e$, and therefore $diam([u_1, v_1], [u_2, v_2]) = 2$.

◊ *Subcase 1.2:* e is odd.

We can see that $e_1 = m + 1, e_2 = m$, and $e = 2m + 1$. The same reasoning and conclusion from Subcase 1.1 arise in this case.

Case 2: $r = 2$, $n = 2p$, and p is an odd prime.

In this case, $\Gamma(\mathbb{Z}_n)$ is a star graph in which there is a common vertex that is adjacent to all other vertices. Let $\{u_1, v_1\}$, $\{u_2, v_2\} \in E(\Gamma(\mathbb{Z}_n))$. Assume without loss of generality that, $v_1 = v_2$, $2 \mid v_1, v_2$, $p \mid u_1$, and $p \mid u_2$. Then $2p \mid u_1v_2$ enforces that $\{u_1, v_2\}$ is an edge of $\Gamma(\mathbb{Z}_n)$ likewise $\{u_2, v_1\}$ is an edge. Then $[u_1, v_2], [u_2, v_2]$ is a path in $L(\Gamma(\mathbb{Z}_n))$ from $[u_1, v_1]$ to $[u_2, v_2]$ and the graph has $diam([u_1, v_1], [u_2, v_2]) = 1$.

Case 3: $r = 2$, $n \neq 2p$ so $n = p_1^{e_1} p_2^{e_2}$.

Let $\{u_1, v_1\}$, $\{u_2, v_2\} \in E(\Gamma(\mathbb{Z}_n))$. In this case $\Gamma(\mathbb{Z}_n)$ is a bipartite graph. Thus, assume without loss of generality that, $p_1 \mid u_1$, $p_1 \mid u_2$, $p_2 \mid v_1$, and $p_2 \mid v_2$, then $p_1p_2 \mid v_1u_2$ shows that $\{v_1, u_2\}$ is an edge. Then $[u_1, v_1], [v_1, u_2], [u_2, v_2]$ is a path in $L(\Gamma(\mathbb{Z}_n))$ from $[u_1, v_1]$ to $[u_2, v_2]$ and the graph has $diam([u_1, v_1], [u_2, v_2]) \leq 2$.

We want $\{u_1, v_1\}, \{u_2, v_2\} \in E(\Gamma(\mathbb{Z}_n))$ and not be adjacent, to show that there is a pair of nonadjacent vertices, $[u_1, v_1], [u_2, v_2] \in L(\Gamma(\mathbb{Z}_n))$.

- $\Diamond$ *Subcase 3.1:* $p_1 = 2$ and $p_2 \geq 3$.

  Then $e_1 \geq 2, e_2 \geq 1$ and $n \geq 12$. Let, $[p_1^{e_1-1}, p_1^{e_1-1}p_2], [p_1^{e_1}, p_2^{e_2}] \in V(L(\Gamma(\mathbb{Z}_n)))$, then $[p_1^{e_1-1}, p_1^{e_1-1}p_2]$ is not adjacent to $[p_1^{e_1}, p_2^{e_2}]$.

- $\Diamond$ *Subcase 3.2:* $p_1 = 2$ and $p_2 \geq 3$.

  Then $e_1 = 1, e_2 \geq 2$ and $n \geq 18$. Let, $[p_1, p_2^{e_2}], [p_1p_2, p_2^{e_2-1}] \in V(L(\Gamma(\mathbb{Z}_n)))$, then $[p_1, p_2^{e_2}]$ is not adjacent to $[p_1p_2, p_2^{e_2-1}]$.

- $\Diamond$ *Subcase 3.3:* $p_1 \geq 3$ and $p_2 \geq 5$.

  Then $e_1 \geq 1, e_2 \geq 1$ and $n \geq 15$. Let, $[p_1^{e_1}, p_2^{e_2}], [2p_1^{e_1}, 2p_2^{e_2}] \in V(L(\Gamma(\mathbb{Z}_n)))$, then $[p_1^{e_1}, p_2^{e_2}]$ is not adjacent to $[2p_1^{e_1}, 2p_2^{e_2}]$.

Thus, we have a pair of nonadjacent vertices when $n \neq 2p$ so $n = p_1^{e_1}p_2^{e_2}$, and therefore $diam([u_1, v_1], [u_2, v_2]) = 2$. $\square$

Theorem 3.2. *If $n = p_1^{e_1}p_2^{e_2} \ldots p_r^{e_r}$ and $r \geq 3$, then $diam(L(\Gamma(\mathbb{Z}_n))) \leq 3$.*

Proof. If $r \geq 3$, $n = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_r}^{e_r}$.

We want to find a path between $\{u_1, v_1\}$, $\{u_2, v_2\} \in E(\Gamma(\mathbb{Z}_n)$. There exists i such that $p_i \mid u_1$. Also, assume without loss of generality $p_i \mid u_2$. Moreover p_i is adjacent in $\Gamma(Z_n)$ to w_i, where $w_i = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_i}^{e_i-1} \ldots {p_r}^{e_r}$, because $p_1 \cdot w_1 = p_i \cdot {p_1}^{e_1}{p_2}^{e_2} \ldots {p_i}^{e_i-1} \ldots {p_r}^{e_r} = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_i}^{e_i} \ldots {p_r}^{e_r} = n = 0$. We know u_1 is a multiple of p_i, thus it follows that $u_1 \cdot w_i$ is a multiple of $p_i \cdot w_i = 0$. Therefore, $u_1 \cdot w_i = 0$ implies that u_1 is adjacent to w_i. Similarly, u_2 is adjacent to w_i. We now have the path $[v_1, u_1], [u_1, w_i], [w_i, u_2], [u_2, v_2]$ is path in $L(\Gamma(\mathbb{Z}_n))$ from $[u_1, v_1]$ to $[u_2, v_2]$. Thus, $diam(L(\Gamma(\mathbb{Z}_n))) \leq 3$. $\square$

4 Girth

The *girth* of a graph G, denoted $gr(G)$, is the length of the shortest cycle. If no cycle exists, $gr(G) = \infty$. The shortest possible cycle consists of three pairwise adjacent vertices.

Theorem 4.1. *$gr(L(\Gamma(\mathbb{Z}_n))) = 3$ if and only if $n \geq 10$. If $n = 6, 8$ or 9, then $gr(L(\Gamma(\mathbb{Z}_n))) = \infty$.*

Proof. We know from Theorem 2.3 that if for some n there exists $v \in V(\Gamma(\mathbb{Z}_n))$ such that deg $(v) \geq 3$, then its line graph will have a clique of three vertices as a subgraph. This clique is the smallest cycle, thus $gr(L(\Gamma(\mathbb{Z}_n))) = 3$.

For this proof we use formula (1) for the degree of a vertex in $V(\Gamma(\mathbb{Z}_n))$ given in the preliminaries and assume $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ as usual. In the proof, we focus on the value of r rather than the value of n.

Case 1: $r \geq 3$.

Consider the vertex p_1p_2 in $\Gamma(\mathbb{Z}_n)$. Since, $p_1p_2 \geq 6$ and $(p_1p_2)^2 \neq 0$, then, deg $(p_1p_2) = \gcd(p_1p_2, n) - 1 = p_1p_2 - 1 \geq 6 - 1 = 5$. Thus, $gr(L(\Gamma(\mathbb{Z}_n))) = 3$, by the comments above.

Case 2: $r = 2$ so $n = p_1^{e_1} p_2^{e_2}$.

◊ *Subcase 2.1:* $p_2 \geq 5$.

Note that vertex $p_2^{e_2} \geq 5$ and $(p_2^{e_2})^2 \neq 0$. Then, deg $(p_2^{e_2}) = \gcd(p_2^{e_2}, n) - 1 = p_2^{e_2} - 1 \geq 5 - 1 = 4$. Thus, $gr(L(\Gamma(\mathbb{Z}_n))) = 3$.

◊ *Subcase 2.2:* $p_1 = 2, p_2 = 3$ so $n = 2^{e_1} 3^{e_2}$.

- If $e_1 \geq 2$ the vertex $2^{e_1} \geq 4$ and $(2^{e_1})^2 \neq 0$. Then deg $(2^{e_1}) = \gcd(2^{e_1}, n) - 1 = 2^{e_1} - 1 \geq 4 - 1 = 3$ and $gr(L(\Gamma(\mathbb{Z}_n))) = 3$.
- If $e_1 = 1$, $n = 2 \cdot 3^{e_2}$ and $e_2 \geq 2$ then the vertex $3^{e_2} \geq 9$ and $(3^{e_2})^2 \neq 0$. Then deg $(3^{e_2}) = \gcd(3^{e_2}, n) - 1 = 3^{e_2} - 1 \geq 9 - 1 = 8$.
- If $e_1 = 1$ and $e_2 = 1$, then $n = 2 \cdot 3 = 6$ and $L(\Gamma(\mathbb{Z}_6))$ consists of two adjacent vertices. Thus $gr(L(\Gamma(\mathbb{Z}_6)) = \infty$.

Case 3: $r = 1$ so $n = p_1^{e_1}$.

◊ *Subcase 3.1:* $p_1 \geq 3$.

- If $e_1 \geq 3$ then $p_1^{e_1} \geq 27$. The vertex $p_1^{e_1-1} \geq 9$ and $(p_1^{e_1-1})^2 = 0$. So, deg $(p_1^{e_1-1}) = \gcd(p_1^{e_1-1}, n) - 2 = p_1^{e_1-1} - 2 \geq 9 - 2 = 7$.
- If $e_1 = 2$ and $p_1 \geq 5$. In this case $(p_1)^2 = 0$ and deg $(p_1) = \gcd(p_1, n) - 2 = p_1 - 2 \geq 5 - 2 = 3$.
- If $e_1 = 2$, $p_1 = 3$, then $n = 3^2 = 9$ and $L(\Gamma(\mathbb{Z}_9))$ consists of only one vertex. Therefore $gr(L(\Gamma(\mathbb{Z}_9)) = \infty$.

◊ *Subcase 3.2:* $p_1 = 2$.

- If $e_1 \geq 4$ then $n = 2^{e_1}$. The vertex $2^{e_1-1} \geq 8$ and $(2^{e_1-1})^2 = 0$, so deg $(2^{e_1-1}) = \gcd(2^{e_1-1}, n) - 2 = 2^{e_1-1} - 2 \geq 8 - 2 = 6$.
- If $e_1 = 3$ then $n = 2^3 = 8$ and $L(\Gamma(\mathbb{Z}_8))$ consists of two adjacent vertices and $gr(L(\Gamma(\mathbb{Z}_8)) = \infty$.
- If $e_1 = 2$ then $n = 2^2 = 4$ and $L(\Gamma(\mathbb{Z}_4))$ is an empty graph, therefore $gr(L(\Gamma(\mathbb{Z}_4)) = \infty$.□

Now that we know which $L(\Gamma(\mathbb{Z}_n$ contain at least one cycle then it is possible to determine which $L(\Gamma(\mathbb{Z}_n$ does not contain a cycle. A connected acyclic graph is known as a *tree*.

Corollary 4.2. *$L(\Gamma(\mathbb{Z}_n))$ is a tree if and only if $n = 6, 8,$ or 9.*

Proof. We showed in Theorem 2.2 that $L(\Gamma(\mathbb{Z}_n)$ is connected, so we need to determine which n it does not contain a cycle. By Theorem 4.1 for any $n \neq 6, 8$ or 9 we know $gr(L(\Gamma(\mathbb{Z}_n)) = 3$. So in this case $L(\Gamma(\mathbb{Z}_n))$ contains at least one cycle of three vertices, and therefore, it is not a tree. When $n = 6$ or 8, the line graph consists of two connected vertices, and when $n = 9$ the line graph is one vertex. Therefore, these line graphs are the only trees in $L(\Gamma(\mathbb{Z}_n))$.□

5 Planarity

In this section, we determine when $L(\Gamma(\mathbb{Z}_n))$ is planar. A graph is *planar* if it can be drawn in the plane without any edge crossings. *Kuratowski's Theorem* can be used to prove or disprove planarity. The weak version of this theorem states that a graph is not planar if it contains a K_5 or $K_{3,3}$ subgraph. K_5, is the complete graph on five vertices and $K_{3,3}$ is the complete bipartite graph with three vertices for each partition. If some $v \in V(\Gamma(\mathbb{Z}_n))$ has deg $(v) \geq 5$, then $L(\Gamma(\mathbb{Z}_n))$ is not planar because by Theorem 2.3 it will have a subgraph isomorphic to K_5.

Theorem 5.1. *$L(\Gamma(\mathbb{Z}_n))$ is planar if and only if $n = 6, 8, 9, 12,$ or 25.*

Proof. Assume $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$.

Case 1: $r \geq 3$.

Then $p_2p_3 \geq 15$ and $(p_2p_3)^2 \neq 0$. Thus, deg $(p_2p_3) = \gcd(p_2p_3, n) - 1 = p_2p_3 - 1 \geq 15 - 1 = 14$.

Case 2: $r = 2$ so $n = p_1^{e_1} p_2^{e_2}$.

◊ *Subcase 2.1:* $p_2 \geq 5$.

Then, $p_1p_2 \geq 10$. Then, deg $(p_1p_2) \geq \gcd(p_1p_2, n) - 2 \geq 10 - 2 = 8$.

◊ *Subcase 2.2:* $p_2 = 3$ so $n = 2^{e_1}3^{e_2}$.

- If $e_2 \geq 2$, the vertex $3^{e_2} \geq 9$ and $(3^{e_2})^2 \neq 0$. Thus, deg $(3^{e_2}) = \gcd(3^{e_2}, n) - 1 = 3^{e_2} - 1 \geq 9 - 1 = 8$.
- If $e_1 \geq 3$ and $e_2 = 1$, then, $n = 2^{e_1}3$. Then, vertex $2^{e_1-1}3 \geq 12$ and $(2^{e_1-1}3)^2 = 0$. Then, deg $(2^{e_1-1}3) = \gcd(2^{e_1-1}3) - 2 = 2^{e_1-1}3 - 2 \geq 10$.
- If $e_1 = 2$ and $e_2 = 1$, then, $n = 2^2 \cdot 3 = 12$. $L(\Gamma(\mathbb{Z}_{12}))$ is planar.
- If $e_1 = 1$, $e_2 = 1$, then, $n = 2 \cdot 3 = 6$. $L(\Gamma(\mathbb{Z}_6))$ is planar.

Case 3: $r = 1$ so $n = p_1^{e_1}$.

◊ *Subcase 3.1:* $e_1 \geq 3$ and $p_1 \geq 3$.

Then, $p_1^{e_1} \geq 27$, so $p_1^{e_1-1} \geq 9$ and $(p_1^{e_1-1})^2 = 0$. Hence, deg $(p_1^{e_1-1}) = \gcd(p_1^{e_1-1}, n) - 2 = p_1^{e_1-1} - 2 \geq 9 - 2 = 7$.

◊ *Subcase 3.2:* $e_1 \geq 4$ and $p_1 = 2$.

Then, $n = 2^{e_1}$, the vertex $2^{e_1-1} \geq 8$ and $(2^{e_1-1})^2 = 0$. Hence, deg $(2^{e_1-1}) = \gcd(2^{e_1-1}, n) - 2 = 2^{e_1-1} - 2 \geq 8 - 2 = 6$.

◊ *Subcase 3.3:* $e_1 = 3$ and $p_1 = 2$.

Then, $n = 2^3 = 8$, which consists of two adjacent vertices; thus it is planar.

◊ *Subcase 3.4:* $e_1 = 2$, $n = p_1^2$.

- If $p_1 \geq 7$, $p_1^2 = 0$, thus deg $(p_1) = \gcd(p_1, n) - 2 = p_1 - 2 \geq 7 - 2 = 5$.
- If $p_1 = 5$, $n = 5^2 = 25$. $L(\Gamma(\mathbb{Z}_{25}))$ is planar as we can see in Figure 4.

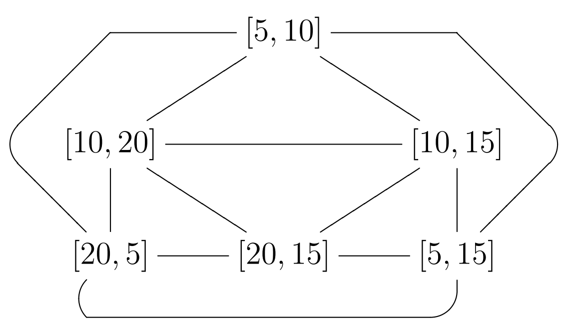

Figure 4: $L(\Gamma(\mathbb{Z}_{25}))$

- If $p_1 = 3$, $n = 3^2 = 9$. $L(\Gamma(\mathbb{Z}_9))$ consists of a single vertex, $[3, 6]$; therefore it is planar.$\square$

6 Center and Eccentricity

To find the center of a line graph, we must know the concepts of; distance and eccentricity. The first vertex of a path is called the initial vertex and the last vertex is called the terminal vertex. If the initial and terminal vertices coincide the path is a closed path. The other vertices in the path are internal vertices, which can not be repeated. The length of the path is the number of edges traversed. We can see in Figure 5 that the path $(a - c - d - b)$ has length three.

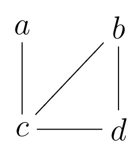

Figure 5: The path $(a - c - d - b)$ has length 3

Let $L(G)$ be a line graph of G and $u, v \in L(V(G))$ where $u \neq v$. The distance, denoted $d(u, v)$, is the $\min\{length(P) \colon P$ is a (u, v)- path$\}$. If no (u, v)- path exists then $d(u, v) = \infty$. For example $L(\Gamma(\mathbb{Z}_{12}))$, we can see in Figure 7 that $d([2, 6], [3, 8]) = 2$.

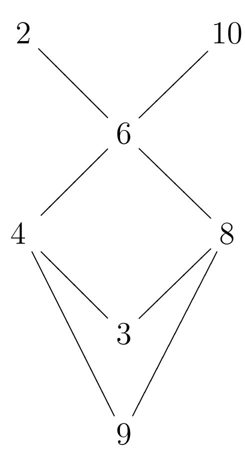

Figure 6: $\Gamma(\mathbb{Z}_{12})$

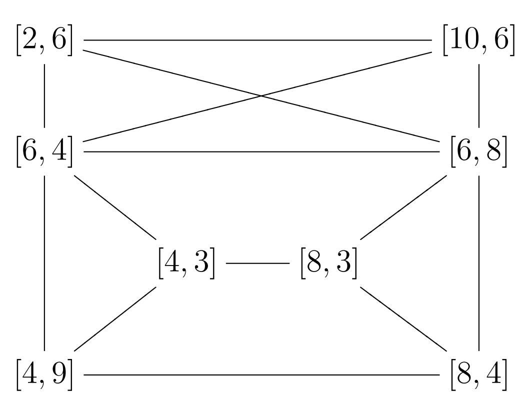

Figure 7: $L(\Gamma(\mathbb{Z}_{12}))$

The *eccentricity* of $u \in V(L(G))$, denoted $\varepsilon(u)$, is the maximum distance from u to any other vertex. In Figure 7, we see $\varepsilon([2,6]) = 2$, because $d[(2,6)]$ is at most 2 from any other vertex v.

The *center* of $L(G)$ is the subgraph induced by the set of vertices of minimum eccentricity. The center of $L(\Gamma(\mathbb{Z}_{12}))$ is $L(\Gamma(\mathbb{Z}_{12}))$ since $\varepsilon(u) = 2$ for every $u \in L(\Gamma(\mathbb{Z}_{12}))$. Now we can find the eccentricity and the center of $L(\Gamma(\mathbb{Z}_n))$ for every n.

Lemma 6.1. *For $n \neq 27, 8$, or $2p$, p is an odd prime, $2 \leq \varepsilon(v) \leq 3$ for every $v \in V(L(\Gamma(\mathbb{Z}_n)))$*

Proof. In general for any graph G, $\varepsilon(v) \leq diam(G)$, for every $v \in G$. Then, $\varepsilon(v) \leq diam(L(\Gamma(\mathbb{Z}_n)))$ for every $v \in V(L(\Gamma(\mathbb{Z}_n)))$.

Case 1: $n = p^e$.

In this case we know from Theorem 3.2 that $diam(L(\Gamma(\mathbb{Z}_n))) = 2$ and $\varepsilon(v) > 1$.

◊ *Subcase 1.1:*$p \leq 5$.

Let $[v_1, v_2]$ for every $V(L(\Gamma(\mathbb{Z}_n)))$, where $v_1 \cdot v_2 = 0$. We can choose $v_3 \in \{2v_1 3v_1 4v_1\} - \{v_2\}$ and $v_4 \in \{2v_2 3v_2 4v_2\} - \{v_1, v_3\}$. Let $v_3 = u_1 \cdot v_1$ where u_1 is a unit and $v_4 = u_2 \cdot v_2$ where u_2 is a unit. Therefore, $v_3 \cdot v_4 = 0$. So, $d([v_1, v_2], [v_3, v_4]) > 1$ so the $\varepsilon(v) > 1$.

◊ *Subcase 1.2:*$p = 2$ and $e \geq 5$.

We know $n = 2^e$, there exists $i, j = 1 \ldots e-1$ and units u_1, u_2 such that $v_1 = u_1 \cdot 2^i$ and $v_2 = u_2 \cdot 2^j$ and $i + j \geq e$. We ca choose $k, l = 1 \ldots e-1$, where $k \neq i, j$ and $l \neq i, j, k$ such that $k + l \geq e$. Let $v_3 = 2^k$ and $v_4 = 2^l$, so $v_3 \cdot v_4 = 0$. So, $d([v_1, v_2], [v_3, v_4]) > 1$ so the $\varepsilon(v) > 1$.

◊ *Subcase 1.3:* $p = 3$ $e \geq 4$.

- If $v_1 \neq 2v_2$ and $v_2 \neq 2v_1$, then let $v_3 = 2v_1$, so $v_3 \neq v_1$ or v_2. Let $v_4 = 2v_2$, so $v_4 \neq v_1, v_2$ or v_3. So $v_3 \cdot v_4 = 0$. Thus, $d([v_1, v_2], [v_3, v_4]) > 1$ so the $\varepsilon(v) > 1$.

- If $v_1 \sim v_2$ and $v_3 \sim v_4$. Let $v_2 = 2v_1$. Fix i such that $v_1 = u_1 \cdot 3^i$ where u is a unit. Choose $k \neq i$ such that $2k \geq e$. Let $v_3 = 3^k$ and $v_4 = 2v_3 = 2(3^k)$. Therefore $v_3 \cdot v_4 = 2(3^{2k}) = 0$, so, $d([v_1, v_2], [v_3, v_4]) > 1$ so the $\varepsilon(v) > 1$.
- Similar argument works for $v_1 = 2v_2$.$\square$

Case 2: $n = p_1p_2$ and $p_1 \geq 3$.

Since $\Gamma(\mathbb{Z}_n)$ is a complete bipartite graph, any vertex $v \in V(L(\Gamma(\mathbb{Z}_n)))$ has the form $[v_1, v_2]$ where $p_1 \mid v_1$, and $p_2 \mid v_2$. Let $[v_1, v_2]$ for every $V(L(\Gamma(\mathbb{Z}_n)))$. Since $p_1 > 2$ there exists $v_3 = 2v$ such that $p_1 \mid v_3$. Similarly there exists $v_4 \neq v_2$ such that $p_2 \mid v_4$. Thus $d([v_1, v_2], [v_3, v_4]) > 1$, so the $\varepsilon([v_1, v_2]) \geq 2$. Therefore $\varepsilon(v) \geq 2$ for every $v \in L(\Gamma(\mathbb{Z}_n))$. Since this is a complete bipartite graph we can take another edge that is adjacent to both edges, $[v_1, v_4]$ and connect the edges such that $\varepsilon(v) = 2$.

Case 3: $n = p_1^{e_1}p_2^{e_2}\dots p_r^{e_r}$ and $r \geq 2$.

Note that $diam(L(\Gamma(\mathbb{Z}_n))) \leq 3$, so $d(u, v) \leq 3$. We claim that no $v \in V(L(\Gamma(\mathbb{Z}_n)))$ has $\varepsilon(v) = 1$.

Let $[v_1, v_2] \in V(L(\Gamma(\mathbb{Z}_n)))$. There exists i such that $p_i^{e_i} \nmid v_1$, if $p_i^{e_i} \nmid v_2$, then let $v_3 = p_i^{e_i}p_{i+1} \neq v_1$ or v_2 and $v_4 = p_1^{e_1}\cdots p_i^{e_i}p_{i+1}^{e_{i+1}-1}\cdots p_r^{e_r} \neq v_2$ or v_1. Again, $v_3 \cdot v_4 = 0$. If there exists $j \neq i$ such that $p_j^{e_j} \nmid v_2$, let $v_3 = p_i^{e_i} \neq v_1$, $v_4 = p_1^{e_1}p_2^{e_2}\cdots p_{i-1}^{e_{i-1}}p_{i+1}^{e_{i+1}}\cdots p_r^{e_r} \neq v_2$. Note: $v_3 \cdot v_4 = n = 0$. If $p_j^{e_j} \mid v_2$ for every $j \neq i$, For either case, $d([v_1, v_2], [v_3, v_4]) > 1$, and hence $\varepsilon(v) \geq 2$. Since $diam(L(\Gamma(\mathbb{Z}_n))) \leq 3$ the center is everything that has $\varepsilon(v) = 2$.$\square$

Theorem 6.2. *For $L(\Gamma(\mathbb{Z}_n))$:*

i) when $n = 27$ the center is the vertex $[9, 18]$.

ii) when $n = 8$ the center is $L(\Gamma(\mathbb{Z}_8))$ and the $\varepsilon(v) = 1$ for every $v \in L(\Gamma(\mathbb{Z}_8))$

iii) when $n = 2p$ the center is $L(\Gamma(\mathbb{Z}_{2p}))$

iv) when $n = 16$ the center is $[4, 8], [8, 12]$

v)Otherwise the center is the graph induced by the vertices with eccentricity equal to 2.

Proof.

Case 1: $n = 27$.

We can see in Figure 8 that $\varepsilon([9, 18]) = 1$ while $\varepsilon(v) = 2$ for every $v \neq [9, 18]$. Therefore the center is $\{[9, 18]\}$.

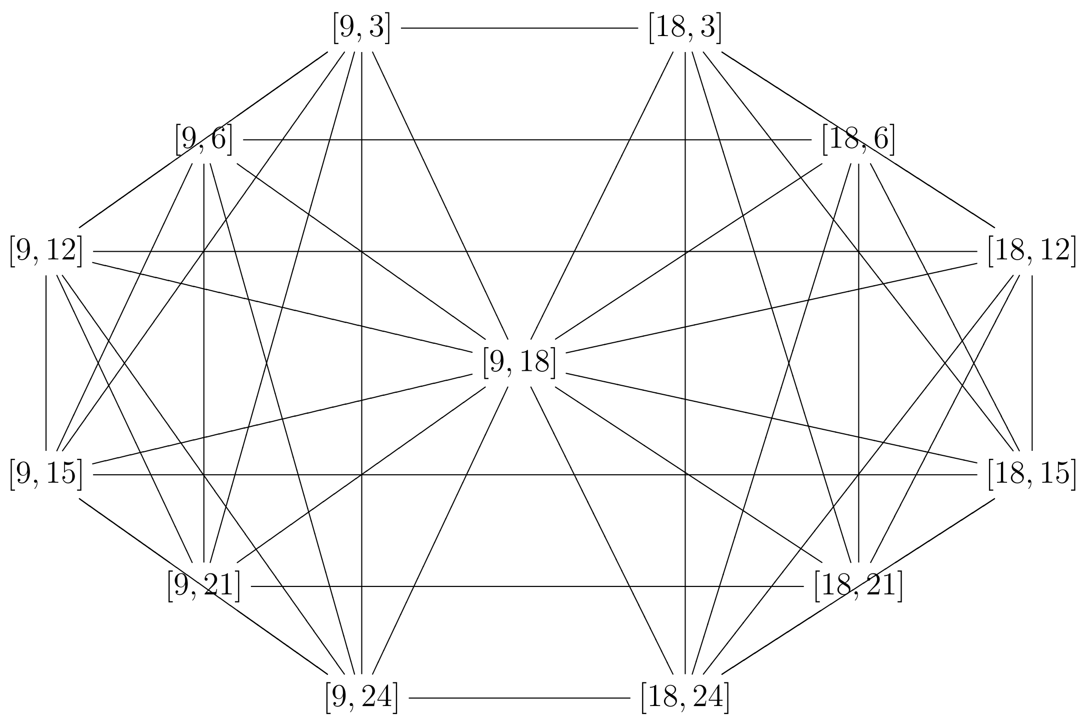

Figure 8: $L(\Gamma(\mathbb{Z}_{27}))$

Case 2: $n = 8$.

The center is $L(\Gamma(\mathbb{Z}_8))$ since $\varepsilon(v) = 1$, for each $v \in V(L(\Gamma(\mathbb{Z}_8)))$.

$$[2,4] \text{ —— } [4,6]$$

Figure 9: $L(\Gamma(\mathbb{Z}_8))$

Case 3: $n = 2p$.

Since $n = 2p$, $\Gamma(\mathbb{Z}_n)$ is a star graph, so $L(\Gamma(\mathbb{Z}_n))$ is a complete graph, and hence $\varepsilon(v) = 1$ for every $v \in V(L(\Gamma(\mathbb{Z}_n)))$. Therefore, the center is $L(\Gamma(\mathbb{Z}_n))$.

Case 4: $n = 16$.

The center is $[4,8]$ and $[8,12]$ as we can observe in Figure 10.

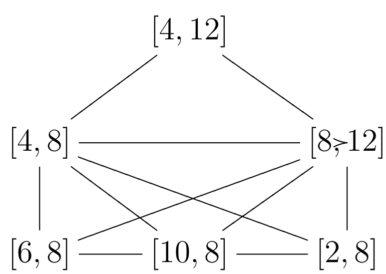

Figure 10: $L(\Gamma(\mathbb{Z}_{16}))$

Case 5:

We know by Lemma 6.1 that when $n \neq 27, 8$, or $2p$, $2 \leq \varepsilon(v) \leq 3$ for every $v \in V(L(\Gamma(\mathbb{Z}_n)))$. We need to show that there is at least one vertex with eccentricity equal to 2. Consider the vertex $[p_1^{e_1-1} \dots p_r^{e_r}, p_1]$. Since, $\varepsilon([p_1^{e_1-1} \dots p_r^{e_r}, p_1]) \geq 2$ we can take a vertex $[u, v]$, such that it is not adjacent to $[p_1^{e_1-1} \dots p_r^{e_r}, p_1]$. Assume without loss of generality that $p_1 | u$. Hence, $[p_1^{e_1-1} \dots p_r^{e_r}, p_1]$, $[u, p_1^{e_1-1} \dots p_r^{e_r}]$, $[u, v,]$ is a path of length two. Therefore, the center is the graph induced by the vertices with eccentricity 2.$\square$

7 Chromatic Number

Several research papers will give us the background needed to find the chromatic number of $L(\Gamma(\mathbb{Z}_n))$. The *chromatic number* (see [4]) is:

$$\chi(\Gamma(\mathbb{Z}_n)) = \min\{\, k \in \mathbb{N} \mid \Gamma(\mathbb{Z}_n) \text{ is } k\text{-colorable}\}.$$

A graph $\Gamma(\mathbb{Z}_n)$ is *k-colorable* if there is a proper k-coloring. A *proper k-coloring* of a graph is a coloring of its vertices such that no adjacent vertices are colored with the same color. [3]

A *k-edge coloring* of a graph G is an assignment of k colors $\{\, 1, \dots, k \,\}$ to the edges of G such that no two adjacent edges are colored the same color. The edge chromatic number is denoted $\chi'(G)$. A graph G is said to be *critical* if G is connected and $\chi'(G) = \Delta(G) + 1$ and for any edge e of G, we have $\chi'(\frac{G}{\{e\}}) < \chi'(G)$. [1]

Theorem 7.1. *In $\Gamma(\mathbb{Z}_n)$, the edge coloring leads to the vertex coloring in $L(\Gamma(\mathbb{Z}_n))$. Let $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$. Moreover, $\chi(L(\Gamma(\mathbb{Z}_n))) = \omega(L(\Gamma(\mathbb{Z}_n))) = \Delta(\Gamma(\mathbb{Z}_n))$.*

Proof. It is easy to see that $\chi(L(\Gamma(\mathbb{Z}_n))) = \chi'(\Gamma(\mathbb{Z}_n))$, because we know that the edges of $\Gamma(\mathbb{Z}_n)$ are the vertices in $L(\Gamma(\mathbb{Z}_n))$. We use Vizing's Theorem, which states that if G is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.

In [1], with the same reasoning it is proved that if R is a finite ring, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$, unless $\Gamma(R)$ is a complete graph of odd order. Notice that $\Gamma(\mathbb{Z}_n)$ is complete only when $n = p^2$ for some p, however $|V(\Gamma(\mathbb{Z}_n))| = p(p-1)$ which is even. Thus, $\chi'(\Gamma(\mathbb{Z}_n)) = \Delta(\Gamma(\mathbb{Z}_n))$ for all n. Finally, if v is a vertex of $\Gamma(\mathbb{Z}_n)$ of maximal degree Δ, the edges incident to v form a clique of size Δ in $L(\Gamma(\mathbb{Z}_n))$. Thus, $\omega(L(\Gamma(\mathbb{Z}_n))) \geq \Delta(\Gamma(\mathbb{Z}_n))$. Now, putting it together we have

$$\Delta(\Gamma(\mathbb{Z}_n)) \leq \omega(L(\Gamma(\mathbb{Z}_n))) \leq \chi(L(\Gamma(\mathbb{Z}_n))) = \chi'(\Gamma(\mathbb{Z}_n)) = \Delta(\Gamma(\mathbb{Z}_n)).$$

So, all the inequalities are equalities, which condenses the formula!□

8 Eulerian Graphs

A graph is *Eulerian* if there exists a closed trail containing every edge. A graph is Eulerian if and only if all its vertex degrees are even (Euler, 1736)[5]. Now, in order to find when $L(\Gamma(\mathbb{Z}_n))$ and $\overline{\Gamma(\mathbb{Z}_n)}$ are Eulerian we only need to look at their degree formulas and see for which values of n are the degrees always even.

Theorem 8.1. *$L(\Gamma(\mathbb{Z}_n))$ is Eulerian if and only if n is odd and square-free.*

Proof. $\Rightarrow$ If $n = p_1p_2 \dots p_r$, (square-free) and $p_1 \geq 3$.

In this case, any vertex $[u, v]$ has $u^2 \neq 0$ and $v^2 \neq 0$. Therefore, by Lemma 2.1, deg $[u, v] = \gcd(u, n) + \gcd(v, n) - 4$, which is always even. Hence, $L(\Gamma(\mathbb{Z}_n))$ is Eulerian.

$\Leftarrow$ If $n = p_1p_2 \dots p_r$ and $p_1 = 2$.

Assume $u = p_1 = 2$ and $v = p_2p_3 \dots p_r$, $u^2 \neq 0$ and $v^2 \neq 0$. Therefore, deg $[2, v] = 2 + p_2p_3 \dots p_r - 4 = p_2p_3 \dots p_r - 2$, which is odd. Hence, it is not Eulerian.

If $n = p_1^{e_1} \dots p_i^{e_i} \dots p_r^{e_r}$ and $e_i \geq 2$ for some i, $1 \leq i \leq r$ we have the following cases:

- If $p_1 \geq 3$, let $u = p_1^{e_1} \dots p_i^{e_i-1} \dots p_r^{e_r}$ then, $v = p_i$, $u^2 = 0$ and $v^2 \neq 0$. So, deg $[u, v] = \gcd(u, n) + \gcd(v, n) - 5 = p_1^{e_1} \dots p_i^{e_i-1} \dots p_r^{e_r} - p_1 - 5$, which is odd. Therefore, is not Eulerian.

- If $p_1 = 2$ and $e_1 \geq 2$, we take $u = p_1 = 2$ and $v = p_1^{e_1-1} \dots p_i^{e_i} \dots p_r^{e_r}$. Since, $u^2 \neq 0$ and $v^2 = 0$, deg $[2, v] = \gcd(2, n) + \gcd(v, n) - 5 = \gcd(v, n) - 3$, which is odd. Therefore, is not Eulerian.

- If $p_1 = 2$ and $e_1 = 1$, assume $r \geq 2$ (else the graph is empty). Then, we take $u = p_1 = 2$ and $v = p_2^{e_2} \dots p_r^{e_r}$. Since $u^2 \neq 0$ and $v^2 \neq 0$, we have $\deg\,[u, v] = \gcd(u, n) + \gcd(v, n) - 4 = 2 + p_2^{e_2} \dots p_r^{e_r} - 4 = p_2^{e_2} \dots p_r^{e_r} - 2$, which is odd. Therefore, is not Eulerian.□

Theorem 8.2. *$\overline{\Gamma(\mathbb{Z}_n)}$ is Eulerian if and only if $n = p^2$ for some prime p.*

Proof. Recall the formula (2) for the degree of $v \in V(\overline{\Gamma(\mathbb{Z}_n)})$ from the preliminaries. Note that $\phi(2) = 1$, which is odd, but if $n > 1$, $\phi(n)$ is even because $\phi(p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}) = p_1^{e_1-1}(p_1 - 1)p_2^{e_2-1}(p_2 - 1) \dots p_r^{e_r-1}(p_r - 1)$ and for any odd prime p, $p - 1$ is even.

If $n = p^2$ and the only zero divisors are multiples of p and $\overline{\Gamma(\mathbb{Z}_n)}$ is a completely disconnected graph, thus all the vertices has degree zero. Therefore, the graph is trivially Eulerian.

If $n \neq p^2$, then consider the following cases.

Case 1: $p_1 = 2$.

This means that $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ is even. In particular, 2 is a vertex of $\overline{\Gamma(\mathbb{Z}_n)}$ and $2^2 \neq 0$ so $\gcd(2, n) = 2$ which is even so $\deg\,(2) = n - \phi(n) - 3$ which is odd. Therefore, the graph is not Eulerian.

Case 2: $p_1 \geq 3$.

This means that $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ is odd. In this case $\gcd(v, n)$ of any vertex is odd because it will be a product of odd primes. So, $\deg\,(v)$ is odd if $v^2 \neq 0$ or even if $v^2 = 0$.

◊ *Subcase 2.1:* $r \geq 2$.

There will always be a vertex v such that $v^2 \neq 0$ so, $\deg\,(v) = n - \phi(n) - \gcd(v, n) - 1$. This vertex will have odd degree. In particular, $p_1^2 \neq 0$ because $r \geq 2$. Therefore, the graph is not Eulerian.

◊ *Subcase 2.2:*

If $r = 1$ then $n = p_1^{e_1}$. Then, p_1 has odd degree because $p_1^2 \neq 0$, $\deg\,(p_1) = n - \phi(n) - \gcd(p_1, n) - 1$. Therefore, the graph is not Eulerian.□

9 Hamiltonian Cycles

A graph is *Hamiltonian* if there exist, a cycle containing every vertex. Here we state some specific results concerning the values n for which $L(\Gamma(\mathbb{Z}_n))$ is Hamiltonian.

Lemma 9.1. *If $n = 2p$ for a prime $p \geq 3$, then $L(\Gamma(\mathbb{Z}_{2p}))$ is Hamiltonian.*

Proof. Since by Theorem 2.3, $\Gamma(\mathbb{Z}_{2p})$ is a star graph, $L(\Gamma(\mathbb{Z}_{2p}))$ is a complete graph. Thus, $\Gamma(\mathbb{Z}_{2p})$ is a Hamiltonian graph.$\square$

Lemma 9.2. *If $n = p_1 p_2 \cdots p_r$ and $p_1 \geq 3$, then $L(\Gamma(\mathbb{Z}_n))$ is Eulerian and Hamiltonian.*

Proof. Theorem 3.1 of [4] states that the graph $\Gamma(\mathbb{Z}_n)$ is Eulerian if $n = p_1 p_2 \cdots p_r$ and $p_1 \geq 3$ [4]. The line graph of an Eulerian graph is both Eulerian and Hamiltonian [8]. Therefore, in this case $L(\Gamma(\mathbb{Z}_n))$ is Eulerian and Hamiltonian.$\square$

10 Generalization of Girth and Chromatic Number

We have proven some results about $L(\Gamma(\mathbb{Z}_n))$. We will use similar ideas to generalize some of the concepts for arbitrary rings R. To be *local* a ring has exactly one maximal ideal, M. For example, $\mathbb{Z}_{p^m}$ is local, because $M = (p)$ is the only maximal ideal. However, $\mathbb{Z}_6$ is not local, because $M_1 = (2)$ and $M_2 = (3)$ are both maximal ideals. If R is a finite local ring with maximal ideal M, then $Z(R)^* = M$. A ring R is said to be *reduced* if R has no non-zero nilpotent element.

10.1 Girth

Lemma 10.1. *If a graph G has a triangle subgraph, then, $L(G)$ will contain a triangle subgraph.*

Proof. If G has a triangle, that means that there are three vertices, x, y, and z that are pairwise adjacent. Thus, $\{x,y\},\{x,z\},\{y,z\} \in E(G)$. Then in $L(G)$, $[x,y],[x,z],[y,z] \in V(L(G))$ and $\{[x,y],[x,z]\},\{[x,y],[y,z]\},\{[x,z],[y,z]\} \in E(L(G))$. Therefore, these vertices in the line graph form a triangle subgraph.□

Theorem 10.2. *If R is a local finite ring with nonzero maximal ideal M, then $gr(L(\Gamma(R))) = 3$.*

Proof. If R is a local finite ring with nonzero maximal ideal M, then $gr(\Gamma(R)) = 3$ by Theorem 2.4 from [2]. This means $\Gamma(R)$ contains a triangle. By Lemma 10.1, $L(\Gamma(R))$ has a triangle, thus, $gr(L(\Gamma(R))) = 3$.□

Theorem 10.3. *Let a ring $R = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a local finite ring and $1 \leq i \leq t$.*

(i) If $R \cong \mathbb{Z}_6, or \mathbb{Z}_2 \times \mathbb{Z}_2$, then $gr(L(R)) = \infty$.

(ii) If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $gr(L(R)) = 4$.

(iii) Otherwise, $gr(L(R)) = 3$

Proof. Let $R = R_1 \times R_2 \times \cdots \times R_t$.

Case 1: $|R_i| \geq 4$ for some i.

Let a_1, a_2 be distinct elements of R_1 both distinct from 0 and 1. Without loss of generality take $|R_1| \geq 4$. In this case we can construct a triangle in $L(\Gamma(R))$ using the vertices, $[(0,1,0,\ldots,0),\ (1,0,\ldots,0)]$, $[(0,1,0,\ldots,0),\ (a_1,0,\ldots,0)]$, and $[(0,1,0,\ldots,0),\ (a_2,0,\ldots,0)]$. Therefore, in this case $gr(L(\Gamma(R))) = 3$.

Case 2: $|R_i| < 4$ for all i.

◊ *Subcase 2.1:* $t \geq 3$.

Without loss of generality take $2 \leq |R_1| \leq 3$. We can find the vertices, $[(0,1,0,\ldots,0),(1,0,\ldots,0)]$, $[(0,1,0,\ldots,0),(0,0,1,0,\ldots,0)]$, and $[(0,1,0,\ldots,0),\ (1,0,1,0,\ldots,0)]$, which form a triangle. Therefore, in this case $gr(L(\Gamma(R))) = 3$.

$\Diamond$ *Subcase 2.2:* $t = 2$.

In this case $2 \leq |R_1| \leq 3$.

- If $|R_1| = 3$ and $|R_2| = 3$. Then $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Note that, $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a square as in Figure 11. In this case $L(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))$ is also a square, thus, $gr(L(\Gamma(R))) = 4$.

$$\begin{tikzcd} (0,1) \arrow[r, dash] \arrow[d, dash] & (1,0) \arrow[d, dash] \\ (2,0) \arrow[r, dash] & (0,2) \end{tikzcd}$$

Figure 11: $(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))$

$$\begin{tikzcd} {[(0,1),(1,0)]} \arrow[r, dash] \arrow[d, dash] & {[(1,0),(0,2)]} \arrow[d, dash] \\ {[(0,1),(2,0)]} \arrow[r, dash] & {[(2,0),(0,2)]} \end{tikzcd}$$

Figure 12: $L(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))$

- If $|R_1| = 2$ and $|R_2| = 3$, it follows that $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \mathbb{Z}_3$. Recall that the *Chinese Remainder Theorem* states that if $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \times \cdots \mathbb{Z}_{p_r^{e_r}}$. Thus, $R \cong \mathbb{Z}_6$. Since $L(\Gamma(\mathbb{Z}_6))$ consists of two adjacent vertices, $gr(L(R)) = \infty$. The same argument holds if $|R_1| = 3$ and $|R_2| = 2$.
- If $|R_1| = 2$ and $|R_2| = 2$, then $R_1 \cong R_2 \cong \mathbb{Z}_2$. The graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ only contains the two adjacent vertices ($(0,1)$ and $(1,0)$) contains two vertices, so $L(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))$ is just one vertex $[(0,1),(1,0)]$.$\Box$

10.2 Chromatic Number

Proposition 10.4. *Let R be a local finite ring with maximal ideal M. Then $M = Z(R)$.*

Proof. Let R be a local ring with maximal ideal M. If $M = 0$ (i.e. R is a field), the assertion is clear. If not, there is some $r > 0$ such that $M^r = 0$, $M^{r-1} \neq 0$. Let $x \neq 0$, $x \in M$. Then $x^r \in M^r$, thus $x^r = 0$. Since, $x \cdot x^{r-1} = 0$, there exist a $k \leq r$ such that $x^k = 0$ and $x^{k-1} \neq 0$. So, x^k where $x \cdot x^{k-1} = 0$. Therefore, x is a zero divisor which implies M consists of zero-divisors. Where then $M \subseteq Z(R)$ and $R^* = R - M$, where R^* are units. Thus $Z(R) \subseteq M$. Therefore, $M = Z(R)$.$\square$

Proposition 10.5. *$\Gamma(R)$ is complete if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local and has an index of nilpotency two; that is, $M^2 = 0$, where M is the maximal ideal.*

Proof. $(\Leftarrow)$: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is complete as seen in Figure 12. So, assume R is local with maximal ideal M. Since $M = V(\Gamma(R))$ and $M^2 = 0$, any two elements of $V(\Gamma(R))$ are adjacent. Therefore, $\Gamma(R)$ is complete.
$(\Rightarrow)$: Observe, that if R is local and $M^2 \neq 0$, then $\Gamma(R)$ is not complete; we may simply observe $x, y \in M = Z(R)$ such that $x \cdot y \neq 0$. Also, if R is not local and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R)$ is not complete.

Case 1: $R \cong R_1 \times R_2$.

Assume without loss of generality $|R_2| \geq 3$. Then we can find a pair of nonadjacent vertices, such as $(0, 1)$ and $(0, z)$ where z is any element. Thus, we have a pair of nonadjacent vertices when $R \cong R_1 \times R_2$, and therefore $\Gamma(R)$ is not complete .

Case 2: $R \cong R_1 \times R_2 \times \ldots R_t$ and $t \geq 3$.

We can also find a pair of nonadjacent vertices, such as $(0, 1, 0, \ldots)$ and $(0, 1, 1, \ldots)$. Thus, we have a pair of nonadjacent vertices when $R \cong R_1 \times R_2 \times \ldots R_t$, and therefore $\Gamma(R)$ is not complete. $\square$

In [1], it states that Beck proved that if R is a direct product of finitely many reduced rings and principal ideal rings, then $\chi(\Gamma(R)) = \omega(\Gamma(R))$.

Theorem 10.6. *Let R be a finite ring and $G = L(\Gamma(R))$. Then $\chi(G) = \omega(G)$, unless $\Gamma(R)$ is complete and of odd order.*

Proof. We see that $\chi(L(\Gamma(H))) = \chi'(\Gamma(H))$, for any graph H. So, we can write $\chi(G) = \chi'(\Gamma(R))$ and we know since $\chi'(\Gamma(G)) = \Delta(\Gamma(R))$ then $\chi(G) = \Delta(\Gamma(R))$. It is useful to know that $\Delta(\Gamma(R)) \leq \omega(G)$, because combined with the inequalities $\omega(G) \leq \chi(G)$ this leads to $\chi(G) = \omega(G)$. $\square$

Suppose, R is local and $M^2 = 0$. When is $|V(\Gamma(R))|$ odd? Well, we know that $\Gamma(R) = M$, which is the set of zero divisors, and $|M| - 1$ is odd which means that

$|M|$ is even, thus $|R| = 2^k$. Then $\Gamma(R)$ is not complete which means R is not local. Hence, $|\Gamma(R)|$ is critical and $\chi'(\Gamma(R)) = \Delta(\Gamma(R)) + 1$.

11 Conclusion

In closing, we want to pose several suggestions for further research on line graphs. There are other areas that were not discussed in this paper, such as total coloring, minimum degree, connectivity, and Hamiltonian circuits. Also, we have discussed the vertex and edge coloring of the line graphs that will aid in total coloring. Furthermore, we have a formula for the degree of a vertex of any line graph, and we also have a formula for the degree of a vertex in $L(\Gamma(\mathbb{Z}_n))$, which is useful for generalizing results with respect to minimum degree, connectivity, and Hamiltonian cycles.

12 Acknowledgements

We would like to give a personal thanks to our educator, Dr. Reza Akhtar of Miami University, for his intuitiveness, devotion, and of course, his jokes to get us though the seminar, in our development throughout this research experience. Also, we can't go without mentioning our graduate assistant, Leigh Cobbs of Rutgers University, for her time well spent on helping us to improve and bring out our true capabilities. We thank Miami University for their faculty in teaching us different areas of graduate mathematics. Also, for Dr. Thomas Farmer in his willingness to help construct our paper. Additionally, we like to thank the Summer Undergraduate Mathematics Science Research Institute program for giving undergraduate students an opportunity to succeed in the mathematical sciences. Again we Thank-you!

13 Glossary

Below is a list of terms used in this paper. For all of these definitions, R is a ring and G is a graph with vertex set V and edge set E.

adjacent: Two distinct vertices $u, v \in V$ are said to be *adjacent* if there exists $e \in E$ such that $e = \{\, u, v \,\}$.

bipartite: A graph is called *bipartite* if V may be partitioned into two disjoint sets U_1 and U_2 such that no two vertices of U_i are adjacent, i.e. $\{a, b\} \in E \Rightarrow a$ and b are not from the same U_i. The complete bipartite graph on n vertices in U_1 and m vertices in U_2 is denoted $K_{n,m}$.

center: The *center* of G is the subgraph induced by the set of central vertices of G.

central vertex: A *central vertex* of G is vertex of minimum eccentricity.

chromatic number: The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum k such that G is k-colorable.

clique: A subset $C \subseteq V(G)$ is called a *clique* if the subgraph of G induced by C is a complete graph.

complement: The *complement* of G is the graph $\overline{G}$ whose vertex set is V, but $\{\, x, y \,\} \in E(\overline{G}) \Leftrightarrow \{\, x, y \,\} \notin E(G)$.

component: A *component* of G is a maximal connected subgraph of G.

connected: G is called *connected* if for every $u, v \in V(G)$, there exists a path from u to v.

connectivity: The *connectivity* of G, denoted $\kappa(G)$ is $\max\{\, k \geq 0 : G$ is k-connected$\,\}$ or is $\min\{\, l \geq 0 :$ there exist a vertex cut $S \subseteq V$ with $|S| = l \,\}$.

cycle: A closed path is called a *cycle*, where the initial and terminal vertices coincide.

degree: The *degree* of $v \in V(G)$, denoted $\delta(v)$, is the number of edges incident at v.

diameter: The *diameter* of G is $diam(G) = \max d(u, v)$.

distance: The *distance* between u and v is $u, v \in V(G)$denoted $d(u, v) = \min\{\text{ length } (P)\colon\ P \text{ is a } (u, v)\text{-path}\}$.

eccentricity: The *eccentricity* of u is $\varepsilon_G(u) = \varepsilon(u) = \max d(u, v)$.

Eulerian: A graph is called *Eulerian* if there exists a closed trail containing every edge, and the vertices have an even degree.

girth: The *girth* of G is the length of the shortest cycle in G.

Hamiltonian: A graph is called *Hamiltonian* if there exists a cycle containing every vertex.

incident: An edge $e \in E$ is said to be *incident* at a vertex $v \in V$ if v is an endpoint of e.

independent set: A subset $S \subseteq V$ is called an *independent set* if the subgraph of G induced by S has no edges.

***k*-edge colorable:** G is called *k-edge colorable* if there exists an edge coloring of G using only k colors.

***k*-vertex colorable:** G is called *k-vertex colorable* if there exists a vertex coloring of G using only k colors.

line graph: The *line graph* of G, denoted $L(G)$, is defined by $V(L(G)) = E$ and $\{ e_1, e_2 \} \in E(L(G)) \Leftrightarrow e_1$ and e_2 share a common vertex in G.

local: A ring is called *local* if there is exactly one maximal ideal.

nilpotent: An element in R is called *nilpotent* if $x^r = 0$ for some $r > 0$.

path: A *path* is a walk in which no internal vertex is repeated. (The initial and terminal vertices may coincide, in which case one has a cycle).

planar: A graph is called *planar* if it can be drawn in the plane without any edge crossings.

reduced: A ring R is said to be *reduced* if R has no non-zero nilpotent element.

subgraph: Let $G = (V(G), E(G))$ be a graph. A *subgraph* of G is a graph $H = (V(H), E(H))$ where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

subgraph induced by $\boldsymbol{S}$: Let G be a graph and $S \subseteq V$ be any subset. The *subgraph induced by* S is the graph $H = (V(H), E(H))$, where $V(H) = S$ and $\{ x, y \} \in E(H) \Leftrightarrow \{ x, y \} \in E(G)$ and $x, y \in S$.

trail: A *trail* is a walk in which no edge is repeated.

vertex coloring: A *vertex coloring* of G is a map $f : V \to \mathbb{N}$ such that $f(v) \neq f(w)$ if $\{ v, w \} \in E$, i.e. no two adjacent vertices have the same color.

vertex cut: A *vertex cut* of a graph, G, is a subset $S \subseteq V$ such that $G - S$ has more than one component or is a single vertex.

walk: A *walk* of length k is a sequence $v_0, e_0, v_1, e_1, \ldots, v_{k-1}, e_{k-1}, v_k$, where for every i, $v_i \in V$, $e_i \in E$, and $e_i = \{ v_i, v_{i+1} \}$.

References

[1] S. Akbari and A. Mohammadian. *On the zero-divisor graph of a commutative ring*. J. Algebra. **274**, p. 847-855, (2004).

[2] D. F. Anderson and P. Livingston. *The Zero-Divisor Graph of a Commutative Ring*. J. Algebra. **217**, p. 434-447, (1999).

[3] M. Brickel. *Complements of Zero Divisor Graphs*. Plan B Project, Miami University, 2004.

[4] N. Córdova, C. Gholston, and H. Hauser. *The Structure of Zero-Divisor Graphs*. SUMSRI, Miami University, 2005.

[5] R. Diestel. *Graduate Texts in Mathematics: Graph Theory*. Springer-Verlag New York, Inc., p. 18, (1997).

[6] A. Lauve. *Zero-Divisor Graphs of Finite Commutative Rings*. Senior Thesis, University of Oklahoma, 1999.

[7] A. Phillips, J. Rogers, K. Tolliver, and F. Worek. *Uncharted Territory of Zero Divisor Graphs and Their Complements*. SUMSRI, Miami University, 2004.

[8] Eric W. Weisstein. "Line Graph." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LineGraph.html, © 1999 CRC Press LLC, © 1999-2005.

Camil I. Aponte
University of Puerto Rico, Río Piedras
San Juan, PR
camili@gmail.com

Patrice S. Johnson
Oakwood College
Huntsville, AL
Patrice.Johnson@oakwood.edu

Nathan A. Mims
Savannah State University
Savannah, GA
nathanmims@hotmail.com