The Homeless Traveling Salesman: Investigating Quadrilaterals with a Variable Home Point

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The traditional traveling salesman problem involves minimizing the total distance traveled for a salesman who needs to visit a given number of cities (visiting each city only once) and then return home. We examined an extension of the problem by allowing the position of the home point to vary.

We fixed four points, representing cities, on a Cartesian grid and allowed the coordinates of the home point to change. For each position, as the home point moves throughout the plane, the salesman has several possible paths from which to choose. The path with the shortest length is determined and called the best path for that home point. We assigned each possible best path a color and then colored each home point according to its best path, creating colored regions on the graph. A Matlab program (see appendix) was used to create colored graphs for different arrangements of the four cities.

We chose to examine cases that involved placing the four points on the grid to form certain quadrilaterals including a rhombus, a kite, and a trapezoid. Melissa Dejarlais proved in her unpublished 1999 SUMSRI paper that there are four, five, or six distinct best paths for any configuration of four distinct cities. For each of our quadrilaterals, we examined the conditions necessary for a fifth or sixth color to be present in the graph.

Rhombus

Consider a rhombus ABCD in which the vertices have the following coordinates: $(0, h)$, $(-t, 0)$, $(0, -h)$, and $(t, 0)$, where $t > 0$ and $h > 0$. When $t = 4$ and $h = 4$, the rhombus is a square and the colored regions are elementary, each occupying one quadrant of the plane and all meeting at the origin.

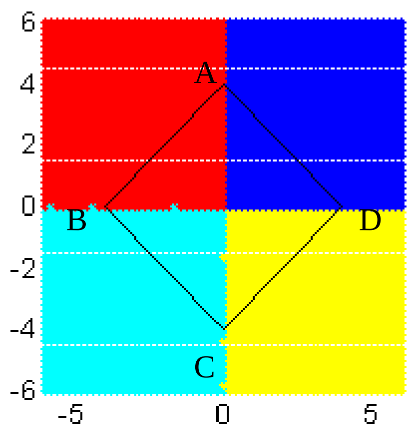


Figure 1: $t = 4$, $h = 4$

Varying h or t slightly while keeping the other constant also yields such four-color figures. However, once h is sufficiently large, a fifth color (green) comes in on both sides of the rhombus, as can be seen in Figure 2 when $h = 16$, four times the value of t. Similarly, when h is sufficiently small, a fifth color (black) comes in above and below the rhombus. In Figure 3 when $h = 1$, one-fourth the value of t, the black region is simply a 90° rotation and scaling of the green region.

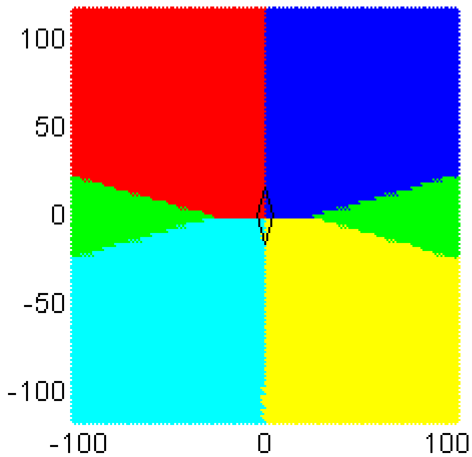


Figure 2: $t = 4$, $h = 16$

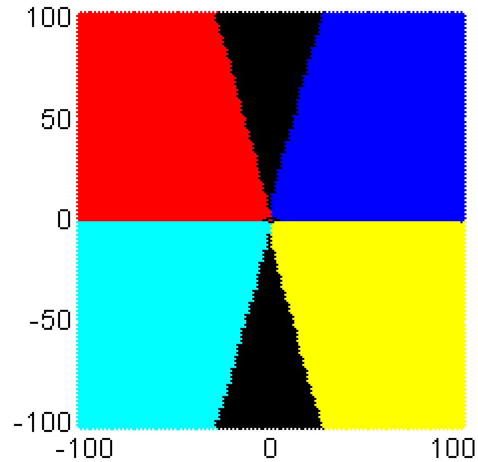


Figure 3: $t = 4$, $h = 1$

The graph below indicates the number of colors present for values of t and h ranging from 0.1 to 10. A red point indicates the presence of five colors for a given t and h, and a blue point indicates the presence of four colors.

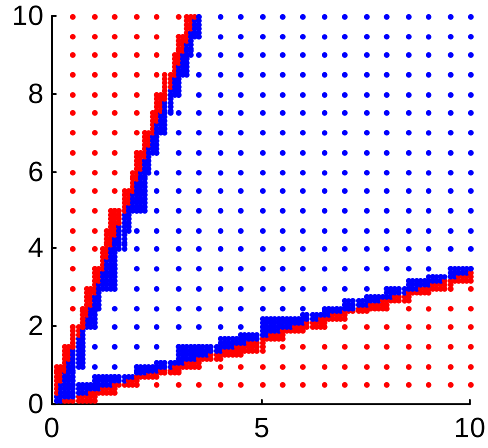


Figure 4: h versus t

As can be seen, for any given value of t, if h is sufficiently large or small, there will be a fifth color. The linearity and symmetry (about the line $h = t$) of the graph lead us to believe that there is a critical ratio of h to t (and vice versa) that determines whether or not a fifth color is present. Once we know this critical ratio for a specific case, the following lemma proves that the ratio is critical for all rhombi.

Lemma 1: Similar figures with n fixed points in the Cartesian plane will have similar colored regions for the traveling salesman problem.

Proof: Given n fixed points in the Cartesian plane and any given home city, scaling the entire figure by a factor of k will also scale the lengths of the possible best paths by a factor of k. Thus the path with the minimum length for the given home point prior to scaling will still be the best path after scaling. Therefore, scaling the figure by a factor of k will lead to a scaling of the colored regions by a factor of k as well, which means that similar figures have similar colored regions for the traveling salesman problem. QED

For our rhombus, scaling t and h by the same constant scales the entire figure. Thus if we find the critical value of h above which a fifth color is present for a given t, the ratio of h to t for that particular case will be the critical ratio for all rhombi. Before we find the critical ratio, let us examine some other aspects of this figure.

The six possible best paths for any given set of four fixed points are as follows.

p1 = H→A→D→C→B→H or H→A→C→D→B→H
p2 = H→A→B→D→C→H or H→A→D→B→C→H
p3 = H→A→B→C→D→H or H→A→C→B→D→H
p4 = H→B→A→D→C→H or H→B→D→A→C→H
p5 = H→B→A→C→D→H or H→B→C→A→D→H
p6 = H→C→B→A→D→H or H→C→A→B→D→H

Dejarlais proved that for any given figure, regardless of the position of the home point, either the two options listed above for each path will always be of equal length or one of the two options will always be shorter than the other.

For our rhombus, we chose $t = 4$ and $h \geq 4$. Thus the coordinates of the vertices of rhombus ABCD are $(0, h)$, $(-4,0)$, $(0, -h)$, and $(4,0)$, respectively. In this case, the first option for each path listed above will always be shorter than or equal to the second option listed. In p1, the first option is shorter because the second one crosses itself. In p2, the two options are the same length because each side of the rhombus is the same length. By similar reasoning, the first options for the other four paths are all shorter than or equal to the second options. Thus we can simply look at the first option for each path, ignoring the second.

The following chart lists the color that corresponds to each of the paths.

Color	Path
Red	p1 = H→A→D→C→B→H
Green	p2 = H→A→B→D→C→H
Blue	p3 = H→A→B→C→D→H
Cyan	p4 = H→B→A→D→C→H
Black	p5 = H→B→A→C→D→H
Yellow	p6 = H→C→B→A→D→H

We can find the equations for the boundaries between any two colored regions by setting the lengths for the two paths equal to each other and simplifying the expressions. We will let d(p*i*) denote the length of the *i*th path. Some of the equations for the boundaries are elementary. Looking at the boundary between red and blue, we set d(p1) = d(p3) and obtain HA + AD + DC + CB + BH = HA + AB + BC + CD + DH. Canceling terms, we obtain AD + BH = AB + DH. Since the figure is a rhombus, all the sides have the same length. Thus the equation for this boundary simplifies to BH = DH, which is simply the y-axis. The same is true for the boundary between cyan and yellow. Similar reasoning shows that the red and cyan regions and the blue and yellow regions meet along the x-axis.

The equations for the boundaries between blue and cyan and between red and yellow simplify to HA + HD = HB + HC and HA + HB = HC + HD, respectively. By inspection, one can see that the origin satisfies each of these equations. Thus the four colors, red, blue, cyan, and yellow, are each limited to one quadrant of the plane and all four meet in the center of the figure at the origin.

Of greater interest are the boundaries where the fifth color meets the other four. We hope to find the critical value of h greater than 4 above which the fifth color, green, is present on both sides of the rhombus. Thus we will look at the boundaries where green meets red, blue, cyan, and yellow, in other words the equations where d(p2) equals d(p1), d(p3), d(p4), and d(p6).

Theorem 1: If the ratio of h to t (or vice versa) for a rhombus with vertices at $(0, h)$, $(-t, 0)$, $(0, -h)$, and $(t, 0)$, where $t > 0$ and $h > 0$, is greater than 11.3137 / 4, then there will be a fifth color present in the traveling salesman graph. This is the critical ratio for a rhombus.

Proof: Setting d(p2) equal to d(p1), d(p3), d(p4), and d(p6), canceling terms as above, and rearranging yields four boundary equations for the green region.

	Colors	Boundary Equation
b1	Red/Green	HC – HB = BC – BD
b2	Blue/Green	HC – HD = BC – BD
b3	Cyan/Green	HA – HB = AD – BD
b4	Yellow/Green	HA – HD = AD – BD

Since BC = AD in a rhombus, these equations are all hyperbolas where the difference of the distances to two fixed points are equal to the same constant, BC – BD, which is denoted by 2a. Figure 5 shows the relevant branches of these hyperbolas for $t = 4$ and $h = 25$. The algebraic computations and Matlab program used to produce the hyperbolas in this figure are included in the appendix.

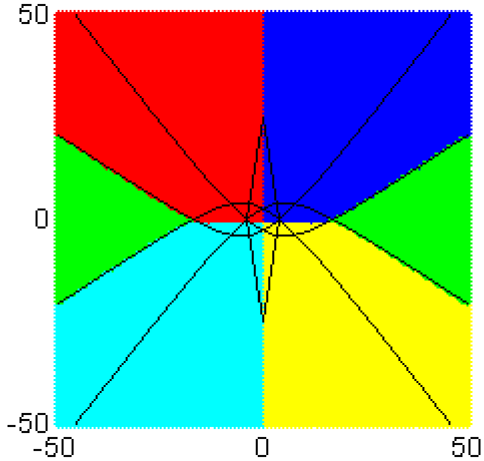
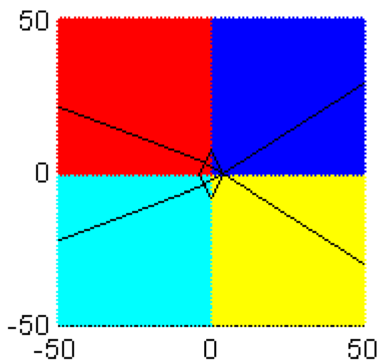


Figure 5: $t = 4$, $h = 25$

Due to the symmetry of the figure and the hyperbolas about the y-axis, we can concentrate on when the green is present on the left side, knowing that it will simultaneously be present on the right side as well. The following graphs show the relevant branches of b1 and b3, the hyperbolas governing the left side of the figure, for $h = 8$, 15, 30, and 50, when $t = 4$.



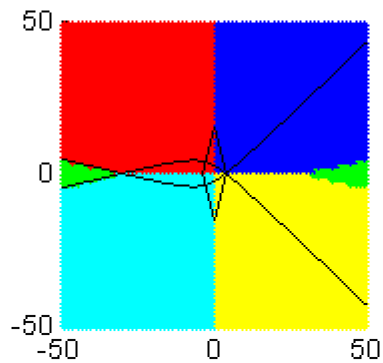


Figure 6: $t = 4$, $h = 8$

Figure 7: $t = 4$, $h = 15$

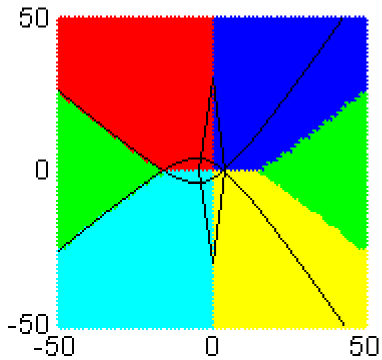


Figure 8: $t = 4$, $h = 30$

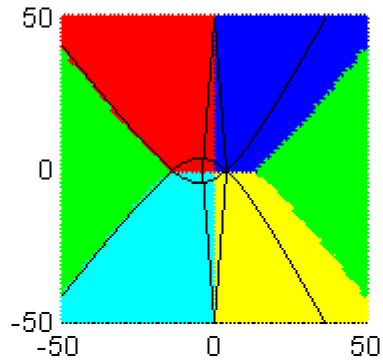


Figure 9: $t = 4$, $h = 50$

Point D satisfies the equations for both of these hyperbolas, which means that b1 and b3 will always intersect at D. The fifth color, green, is only present when b1 and b3 intersect a second time to the left of point B. Due to the symmetry of the hyperbolas about the x-axis, this intersection must occur on the x-axis. Thus the critical *h*-value is the maximum *h*-value at which these hyperbolas fail to intersect the second time along the x-axis. We know that hyperbolas intersect only when their asymptotes intersect; therefore, the critical value is the one at which the asymptotes extending to the left for each hyperbola are parallel. The asymptotes of the hyperbolas will be parallel to each other when they are both parallel to the x-axis, and when one asymptote is parallel to the x-axis the other will be as well. Thus, we will narrow our focus to finding the *h* at which the asymptote of b1 is parallel to the x-axis. Figure 10 shows the asymptote of b1.

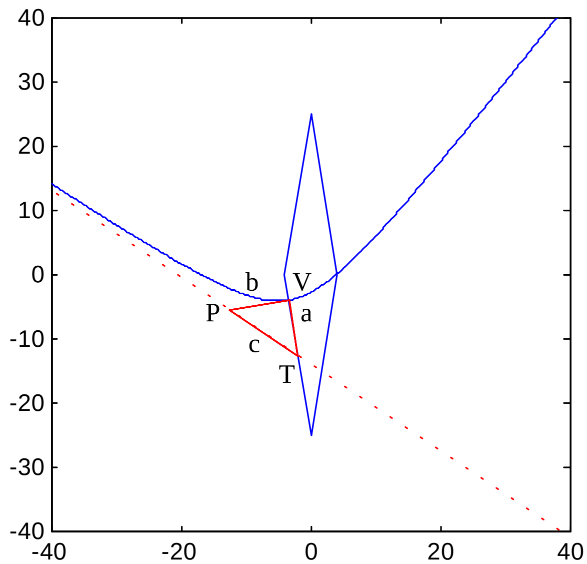


Figure 10

It suffices to show that two distinct points on the asymptote have the same y-coordinate. Point T on the asymptote is the midpoint of BC; thus its coordinates are (-2, -*h* / 2). The coordinates for P on the asymptote can be found by starting at point T, moving a length *a* along the unit vector from T to V, and then moving a length *b* along the unit vector from V to P, which is perpendicular to the first unit vector. Thus the coordinates for P are as follows:

$$P = (-2, -\tfrac{h}{2}) + \tfrac{(-4,h)}{\sqrt{16+h^2}}\,a + \tfrac{(-h,-4)}{\sqrt{16+h^2}}\,b.$$

The geometry of hyperbolas enables us to calculate the following values for *a* and *b*:

$$a = \tfrac{1}{2}(\mathrm{BC} - \mathrm{BD}) = \tfrac{1}{2}(\sqrt{16+h^2} - 8),$$
$$c = \tfrac{1}{2}\mathrm{BC} = \tfrac{1}{2}\sqrt{16+h^2}\text{, and}$$
$$b^2 = c^2 - a^2 = \tfrac{1}{4}(16+h^2) - \tfrac{1}{4}(\sqrt{16+h^2} - 8)^2,$$

which simplifies to

$$b = \sqrt{4\sqrt{16+h^2} - 16}.$$

Thus the y-coordinate for P is

$$-\tfrac{h}{2} + \tfrac{h}{\sqrt{16+h^2}}\tfrac{1}{2}(\sqrt{16+h^2} - 8) - \tfrac{4}{\sqrt{16+h^2}}\sqrt{4\sqrt{16+h^2} - 16}.$$

Setting the y-coordinates for T and P equal to each other, we obtain

$$-\tfrac{h}{2} = -\tfrac{h}{2} + \tfrac{h}{\sqrt{16+h^2}}\tfrac{1}{2}(\sqrt{16+h^2} - 8) - \tfrac{4}{\sqrt{16+h^2}}\sqrt{4\sqrt{16+h^2} - 16}.$$

Simplifying this equation yields

$$\tfrac{1}{2}h(\sqrt{16+h^2} - 8) - 4\sqrt{4\sqrt{16+h^2} - 16} = 0.$$

Using the fzero command in Matlab, which uses the bisection method to approximate the roots of an equation to any given tolerance level, we found that h ≈ 11.3137 with a tolerance of 0.0000001. Thus the critical *h*-value is 11.3137.

Therefore, when $t = 4$, if the ratio of *h* to *t* is greater than 11.3137 / 4, there will be a fifth color present in the traveling salesman graph. However, according to the previous lemma, we know that this is the critical ratio for all rhombi. Therefore, if the ratio of *h* to *t* (or vice versa) for a given rhombus is greater than 11.3137 / 4, then there will be a fifth color present. This is the critical ratio for all rhombi. QED

The following graph shows the hyperbolas when $h = 11.3137$ and $t = 4$, the critical values at which the hyperbolas fail to intersect a second time.

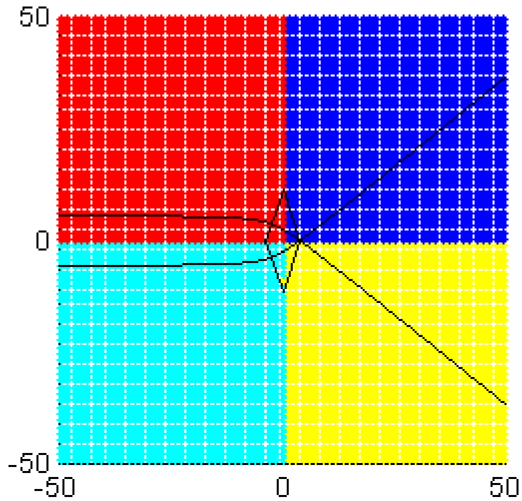


Figure 11: $t = 4$, $h = 11.3137$

Corollary 1: Let σ be the obtuse angle of a rhombus. If σ is greater than 141.0575°, then there will be a fifth color present in the traveling salesman graph. This is the critical angle for a rhombus.

Proof: The critical ratio of *h* to *t* (and vice versa) for a rhombus is 11.3137 / 4. Using this ratio, we can find the critical angle, σ, using $\tan(\sigma / 2) = 11.3137 / 4$. Solving this equation gives us $\sigma \approx 141.0575°$. QED

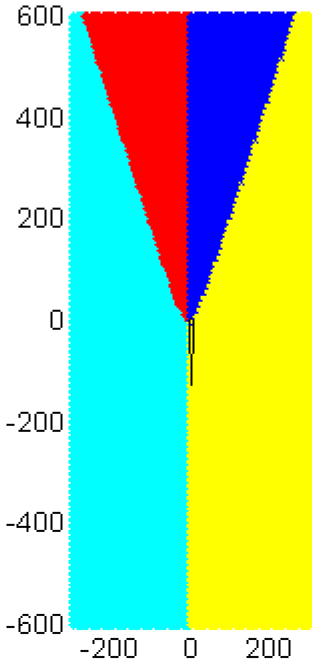
Right Kite

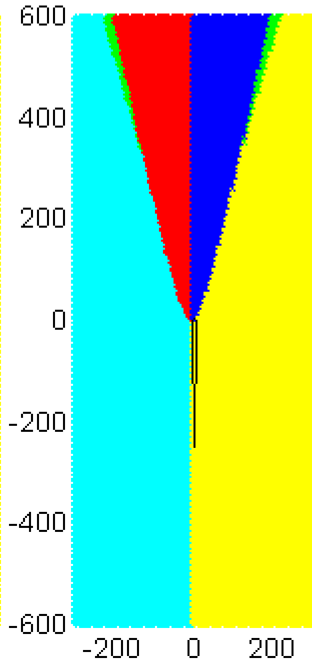
A figure that is a slight variation of the rhombus is a kite ABCD with vertices at $(0, t)$, $(-t, 0)$, $(0, -h)$, and $(t, 0)$, where $t > 0$ and $h > 0$. In this figure, angle A is always 90°; thus we will refer to this figure as a right kite.

The six possible best paths for the right kite are the same as those for the rhombus, for the same reasoning. The color corresponding to each possible path is maintained as well for consistency.

Color	Path
Red	p1 = H→A→D→C→B→H
Green	p2 = H→A→B→D→C→H
Blue	p3 = H→A→B→C→D→H
Cyan	p4 = H→B→A→D→C→H
Black	p5 = H→B→A→C→D→H
Yellow	p6 = H→C→B→A→D→H

We will begin by examining cases where $h \geq t$. The following graphs are for $h = 125, 250$, and 500 when $t = 4$.





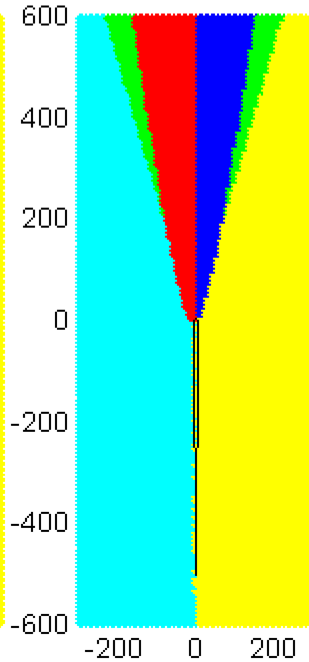


Figure 12: t = 4, h = 125   Figure 13: t = 4, h = 250   Figure 14: t = 4, h = 500

From these graphs, the fifth color, green, seems to appear only for h-values that are much larger than the critical h-value for the rhombus. The green regions appear on both sides of the figure well above the figure. While the green regions still appear symmetric about the y-axis, they are no longer symmetric about the x-axis. The symmetry of the colored regions corresponds to the symmetry of the kite.

Just as we did with the rhombus, we will examine the boundaries between colored regions by setting the lengths of two paths equal to each other and finding the boundary equation. The boundaries between red and blue and between cyan and yellow, b1 and b2, are both the y-axis, just as they were for the rhombus.

The equations for the boundaries between red and cyan and between blue and yellow simplify to b3: HC – HA = CB – AB and b4: HC – HA = CD – AD, respectively. In the right kite, CB = CD and AB = AD; therefore, b3 = b4. By inspection, B and D satisfy these equations; thus b3 (b4) is a hyperbola with foci at A and C, passing through points B and D. Note that the four-color point inside the figure is not at the origin, as it was for the rhombus. Instead, it occurs at the point where the hyperbola crosses the y-axis.

We wish to determine the critical ratio of h to t for $h \geq t$ above which a fifth color, in our case green, is present. Note that this ratio will not also be the critical ratio of t to h for $h \leq t$ that determines when a fifth color, in our case black, appears above or below the figure. For the rhombus, these ratios were the same, because the figures were similar, simply rotated 90°. For the right kite, allowing h to be less than t creates a new situation.

Theorem 2: If the ratio of h to t for a right kite with vertices at (0, t), (-t, 0), (0, -h), and

(t, 0), where $t > 0$ and $h > 0$, is greater than 147.1908 / 4, then there will be a fifth color present in the traveling salesman graph.

Proof: The boundary equations for the fifth color, where green meets red, blue, cyan, and yellow, respectively, simplify to the same boundary equations for green that governed the rhombus.

	Colors	Boundary Equation
b5	Red/Green	HC – HB = BC – BD
b5*	Blue/Green	HC – HD = BC – BD
b6	Cyan/Green	HA – HB = AD – BD
b6*	Yellow/Green	HA – HD = AD – BD

Clearly these boundary equations are symmetric about the y-axis; thus we need only examine the presence of green on one side of the figure. Therefore, we will focus on b5 and b6, knowing that b5* and b6* are simply reflections of b5 and b6 across the y-axis. (We use the b*i** notation to indicate a hyperbola that is simply a reflection of b*i*.) Note that the equations for b5 and b6 are both satisfied by point D. The following graphs show the relevant branches of b3, b5, and b6 for h = 75, 125, 250, and 500 when t = 4. The algebraic computations and Matlab programs used to plot these hyperbolas are similar to the ones shown in the appendix for the hyperbola in the case of the rhombus.

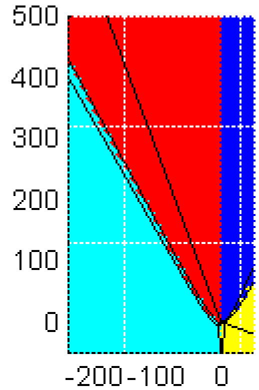


Figure 15: t = 4, h = 75

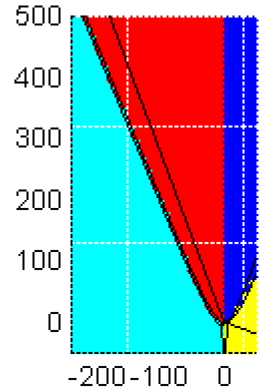
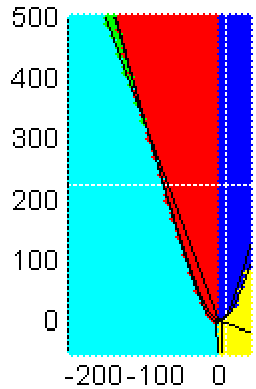


Figure 16: t = 4, h = 125



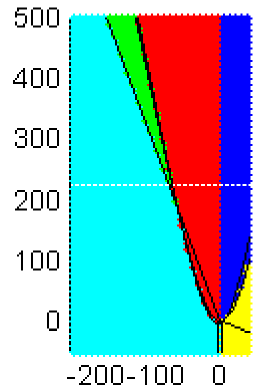


Figure 17: $t = 4$, $h = 250$ Figure 18: $t = 4$, $h = 500$

It is interesting to note that when b5 and b6 intersect, b3 also passes through the intersection point. This tricolor point occurs because green meets red along b5, and green meets cyan along b6; thus red and cyan must meet when b5 and b6 intersect. Therefore, between the figure and the tricolor intersection point, b3 is the governing boundary separating the red and cyan regions, and beyond the intersection point, b5 and b6 are the governing boundaries separating red from green and green from cyan.

Just as in the case of the rhombus, we need to find the critical h-value above which b5 and b6 intersect the second time. Once again, our task comes down to finding the h-value at which the asymptotes of the hyperbolas are parallel. Looking at b6, we see that for a fixed t, b6 will also be fixed. By our lemma, we know that the critical ratio for a fixed t will be the critical ratio for all t, since the right kites will be similar; thus we will let $t = 4$.

To find the h for which the asymptotes are parallel, we will find two points on each asymptote, find the slope of each asymptote, set the slopes equal to each other, and solve for h. Following the same methods used for the rhombus, we can find the coordinates for two points on each of the asymptotes. The asymptote of b6 contains the points

$$(-2, 2) \text{ and } (-2, 2) + \frac{(8-\sqrt{32})}{2\sqrt{32}}(4,4) + \frac{\sqrt{4\sqrt{32}-16}}{\sqrt{32}}(-4,4).$$

The slope of the asymptote of b6 is

$$\frac{2\text{-}\left(2+\frac{(8-\sqrt{32})}{2\sqrt{32}}4+\frac{\sqrt{4\sqrt{32}-16}}{\sqrt{32}}4\right)}{\text{-}2\text{-}\left(\text{-} 2+\frac{(8-\sqrt{32})}{2\sqrt{32}}4-\frac{\sqrt{4\sqrt{32}-16}}{\sqrt{32}}4\right)},$$

which simplifies to

$$\frac{4-\frac{\sqrt{32})}{2}+\sqrt{4\sqrt{32}-16}}{4-\frac{\sqrt{32})}{2}-\sqrt{4\sqrt{32}-16}} \approx -2.6703.$$

The asymptote of b5 contains the points

$$(-2, -\tfrac{h}{2}) \text{ and } (-2, -\tfrac{h}{2}) + \frac{\sqrt{16+h^2}-8}{2\sqrt{16+h^2}}(-4, h) + \frac{\sqrt{4\sqrt{16+h^2}-16}}{\sqrt{16+h^2}}(-h, -4).$$

Thus the slope of the asymptote of b5 is

$$\frac{-\frac{h}{2} - \left(-\frac{h}{2} + \frac{\sqrt{16+h^2}-8}{2\sqrt{16+h^2}}h - \frac{\sqrt{4\sqrt{16+h^2}-16}}{\sqrt{16+h^2}}4\right)}{-2 - \left(-2 - \frac{\sqrt{16+h^2}-8}{2\sqrt{16+h^2}}4 - \frac{\sqrt{4\sqrt{16+h^2}-16}}{\sqrt{16+h^2}}h\right)},$$

which simplifies to

$$\frac{-\frac{h}{2}(\sqrt{16+h^2}-8)+4\sqrt{4\sqrt{16+h^2}-16}}{2(\sqrt{16+h^2}-8)+h\sqrt{4\sqrt{16+h^2}-16}}.$$

Setting these two slopes equal to each other and using the fzero command in Matlab, as described previously, we find that $h \approx 147.1908$. Hence, for a right kite with $t = 4$, if the ratio of h to t is greater than 147.1908 / 4, there will be a fifth color present in the traveling salesman graph. According to our lemma, we know that this ratio is a critical ratio for all right kites. Therefore, if the ratio of h to t for a right kite is greater than 147.1908 / 4, then there will be a fifth color present. QED

The following graph shows b3, b5, and b6 when $h = 147.1908$ and $t = 4$, the critical values at which the hyperbolas fail to intersect a second time.

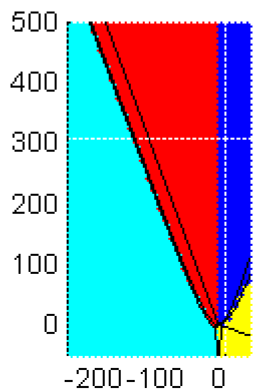


Figure 19: $t = 4$, $h = 147.1908$

In the right kite ABCD, angles B and D are congruent, and we will refer to them as the side angles. We will refer to ∠C as the bottom angle.

Corollary 2: Let σ be a side angle of a right kite ABCD. If σ is greater than 133.4433°, then there will be a fifth color present in the traveling salesman graph. This is a critical angle for the right kite.

Proof: Since the side angles are congruent, we can let σ = ∠B without any loss of generality. Let O represent the origin. Thus ∠B = ∠ABO + ∠OBC. Since Δ ABO is a right, isosceles triangle, ∠ABO must be 45°. Using the critical ratio of h to t for a right kite, we can find the critical measure of ∠OBC, using tan (∠OBC) = 147.1908 / 4. Solving this equation gives us ∠OBC ≈ 88.4433°. Combining these steps gives us σ = ∠B = ∠ABO + ∠OBC ≈ 45° + 88.4433° = 133.4433°, which means σ ≈ 133.4433°. QED

We will now look at cases of the right kite where $0 < h \leq t$. In the following graphs, $t = 4$ and $h = 0.25$, 0.5, and 1.

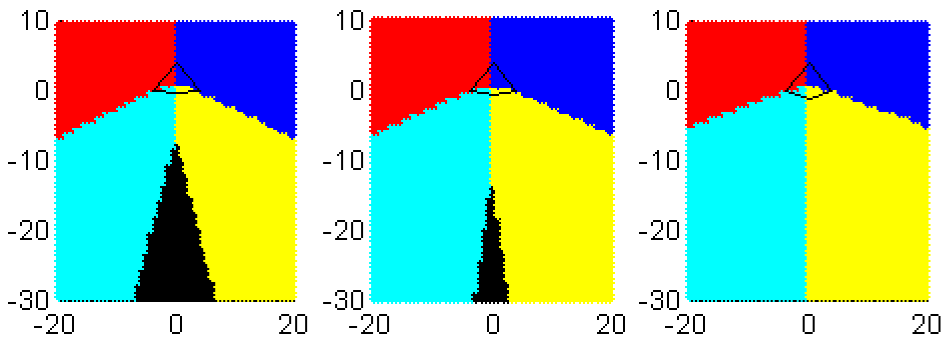


Figure 20: $t = 4$, $h = 0.25$ Figure 21: $t = 4$, $h = 0.5$ Figure 22: $t = 4$, $h = 1$

From these graphs, it appears that we can find a critical value for h below which a fifth color, black, appears below the right kite.

Theorem 3: If the ratio of h to t for a right kite with vertices at $(0, t)$, $(-t, 0)$, $(0, -h)$, and $(t, 0)$, where $t > 0$ and $h > 0$, is less than 0.8302 / 4, then there will be a fifth color present beyond the bottom angle in the traveling salesman graph.

Proof: The boundary equations for the fifth color appearing below the right kite are the equations where black meets cyan and yellow, or d(p5) = d(p4) and d(p5) = d(p6).

	Colors	Boundary Equation
b7	Cyan/Black	HD – HC = AD – AC
b7*	Yellow/Black	HB – HC = AD – AC

These boundaries are hyperbolas with foci at C and D and at C and B with the same value for 2a. The two hyperbolas intersect once at A. As in the previous cases, the fifth color is present beyond the second intersection of these two hyperbolas. We wish to find the critical value for h below which these hyperbolas intersect a second time. Following the argument used for the rhombus and because the boundaries are symmetric about the y-axis, we can simply find the h at which the asymptote of one of the hyperbolas is parallel to the y-axis. As before, it will be sufficient to find the h at which two distinct points on the asymptote have the same x-coordinate.

Once again, we can find the coordinates for two points on one of the asymptotes. The asymptote of b7 contains the points

$$(2,-\tfrac{h}{2}) \text{ and } (2,-\tfrac{h}{2}) + \frac{\frac{1}{2}\left|4+h-\sqrt{32}\right|}{\sqrt{16+h^2}}(-4,-h) + \frac{\frac{1}{2}\sqrt{16+h^2-\left|4+h-\sqrt{32}\right|^2}}{\sqrt{16+h^2}}(h,-4).$$

Setting the x-coordinates for these two points equal to each other, we obtain

$$2 = 2 - \frac{\frac{1}{2}\left|4+h-\sqrt{32}\right|}{\sqrt{16+h^2}}4 + \frac{\frac{1}{2}\sqrt{16+h^2-\left|4+h-\sqrt{32}\right|^2}}{\sqrt{16+h^2}}h.$$

Simplifying this expression gives us

$$-2\left|4+h-\sqrt{32}\right| + \tfrac{h}{2}\sqrt{16+h^2-\left|4+h-\sqrt{32}\right|^2} = 0.$$

Using Matlab's fzero command, we find that $h \approx 0.8302$. Thus when the $t = 4$, the critical h-value below which a fifth color will appear for the right kite is 0.8302. Once more, we can use our lemma to conclude that this ratio holds for all right kites. Therefore, if the ratio of h to t for a right kite is less than 0.8302 / 4, there will be a fifth color present in the traveling salesman graph. QED

Corollary 3: Let σ be the bottom angle of a right kite ABCD. If σ is greater than 156.5495°, then there will be a fifth color present in the traveling salesman graph. This is another critical angle for the right kite.

Proof: If the ratio of h to t for a right kite is less than 0.8302 / 4, then there will be a fifth color present in the traveling salesman graph. Using this ratio, we can find the critical bottom angle, using $\tan(\sigma / 2) = 4 / 0.8302$. Solving this equation gives us $\sigma \approx$ 156.5495°. Since the fifth color occurs when the ratio of h to t is less than 0.8302 / 4, the fifth color will occur when the bottom angle is greater than this critical angle. QED

In all of the traveling salesman graphs for the right kite, we failed to see any black appear above the right angle of the figure. We can explain this fact by examining the boundaries separating black from red and blue.

Theorem 4: For a right kite with vertices at $(0, t)$, $(-t, 0)$, $(0, -h)$, and $(t, 0)$, where $t > 0$ and $h > 0$, a fifth color can never appear directly beyond the right angle in a traveling salesman graph.

Proof: The boundary equations for the fifth color appearing above the right kite are the equations where black meets red and blue, in other words d(p5) = d(p1) and d(p5) = d(p3).

	Colors	Boundary Equation
b8	Red/Black	HA – HD = AC – BC
b8*	Blue/Black	HA – HB = AC – BC

These boundaries are hyperbolas with foci at A and D and at A and B that both pass through C and are symmetric about the y-axis. As before, the fifth color, black, will be present only if these hyperbolas intersect a second time, somewhere above point A. We know that the hyperbolas will intersect when their asymptotes intersect. The following figure shows an asymptote of b8 and demonstrates that there are cases when the asymptotes will not intersect beyond point A.

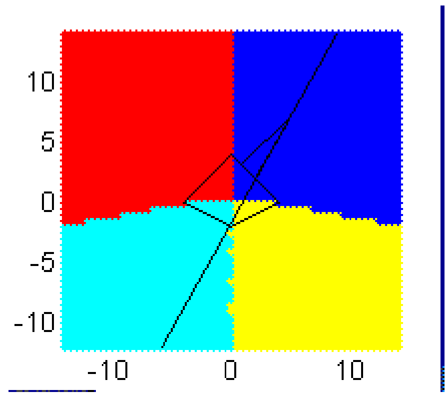


Figure 23: $t = 4$, $h = 2$

Before the asymptotes intersect, they must reach a point when they are parallel. Thus, we can show that the hyperbolas will not intersect beyond point A by proving that there are no values for h at which the asymptotes become parallel. Due to the symmetry of the hyperbolas about the y-axis, the problem is reduced to proving that either one of the asymptotes can never be parallel to the y-axis.

As before, we will let $t = 4$ and then use our lemma to generalize the results. Using the method discussed earlier, the coordinates for two points on b8 are

$$(2,2) \text{ and } (2,2)+\frac{\frac{1}{2}(4+h-\sqrt{16+h^2})}{\sqrt{32}}(-4,4)+\frac{\sqrt{8-\frac{1}{4}(4+h-\sqrt{16+h^2})^2}}{\sqrt{32}}(4,4).$$

Setting the x-coordinates of these points equal to each other gives us

$$2=2-\frac{\frac{1}{2}(4+h-\sqrt{16+h^2})}{\sqrt{32}}4+\frac{\sqrt{8-\frac{1}{4}(4+h-\sqrt{16+h^2})^2}}{\sqrt{32}}4,$$

which reduces to

$$-\tfrac{1}{2}(4+h-\sqrt{16+h^2})+\sqrt{8-\tfrac{1}{4}(4+h-\sqrt{16+h^2})^2}=0.$$

We now need to prove that this equation has no positive solutions. First we will make the following substitution:

$$a=4+h-\sqrt{16+h^2}.$$

Thus our equation becomes

$$-\tfrac{1}{2}a+\sqrt{8-\tfrac{1}{4}a^2}=0.$$

Solving for *a*, we find $a = \pm 4$. Thus we have two cases to examine.

Case1: a = 4

$$a=4+h-\sqrt{16+h^2}$$
$$4=4+h-\sqrt{16+h^2}$$
$$h=\sqrt{16+h^2}$$
$$h^2=16+h^2$$
$$0=16$$

Thus there are no solutions for Case 1.

Case 2: a = -4

$$a=4+h-\sqrt{16+h^2}$$
$$-4=4+h-\sqrt{16+h^2}$$
$$h+8=\sqrt{16+h^2}$$
$$h^2+16h+64=16+h^2$$
$$16h=-48$$
$$h=-3$$

Thus there are no positive solutions for Case 2.

Therefore when $t = 4$, there are no positive values for *h* at which the asymptotes will be parallel to the y-axis, which means that the hyperbolas will never cross beyond point A. According to our lemma for similar figures, we can generalize this statement to say that the hyperbolas governing the presence of a fifth color beyond the right angle of a right kite will never intersect. Therefore, for all positive values of *t* and *h*, a fifth color can never appear directly beyond the right angle of a right kite in the traveling salesman graph. QED

Trapezoid

Consider an isosceles trapezoid ABCD in which the vertices have the following coordinates: (0,0), (4-*t*,*h*), (4+*t*,*h*), (8,0), where $t \neq 0$ or $h \neq 0$. If either *t* or *h* is equal to zero, then the figure ceases to be a trapezoid. We will label AD as b_1 and BC as b_2 and we will keep h constant at $h = 2$ while we vary t. Since $b_1 = 8$ and $b_2 = 2t$, $b_1/h = 4$ and $b_2/h = 2t/2 = t$.

Let us first examine values of t where $0 < t < 4$ and hence, $b_1 > b_2$. If we begin with $t = .5$, then there are four colors present in the graph. A value of $t = 1$, however, will produce a graph with six colored regions. Similarly, a value of $t = 2.5$ will produce a six color graph while a value of $t = 3.5$ will produce a graph with only four colors.

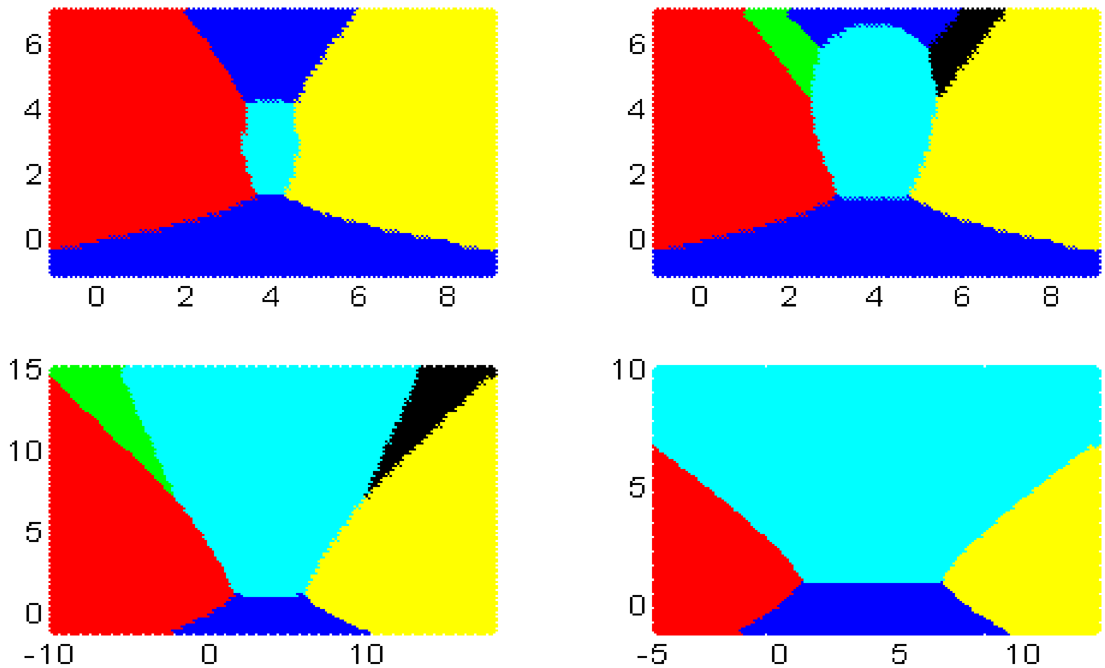


Figure 24: (top) left $t = .5$, $h = 2$, right $t = 1$, $h = 2$; (bottom) left $t = 2.5$, $h = 2$, right $t = 3.5$, $h = 2$

When $t = 4$, $b_1 = b_2$ and ABCD is a rectangle, which produces a graph with four colors.

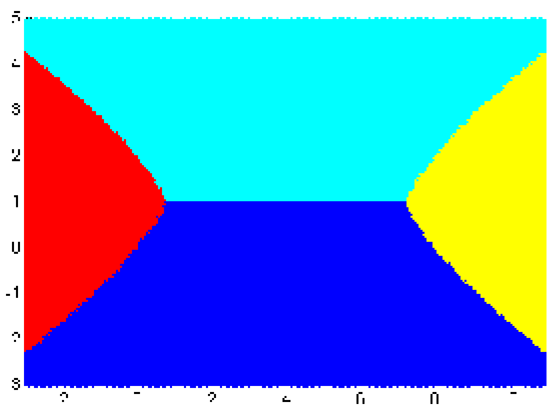


Figure 25: $t = 4$, $h = 2$

When $t > 4$, $b_1 < b_2$. If we choose $t = 4.5$, then there are four colors present in the graph, while a value of $t = 5.5$ will produce a graph with six colors.

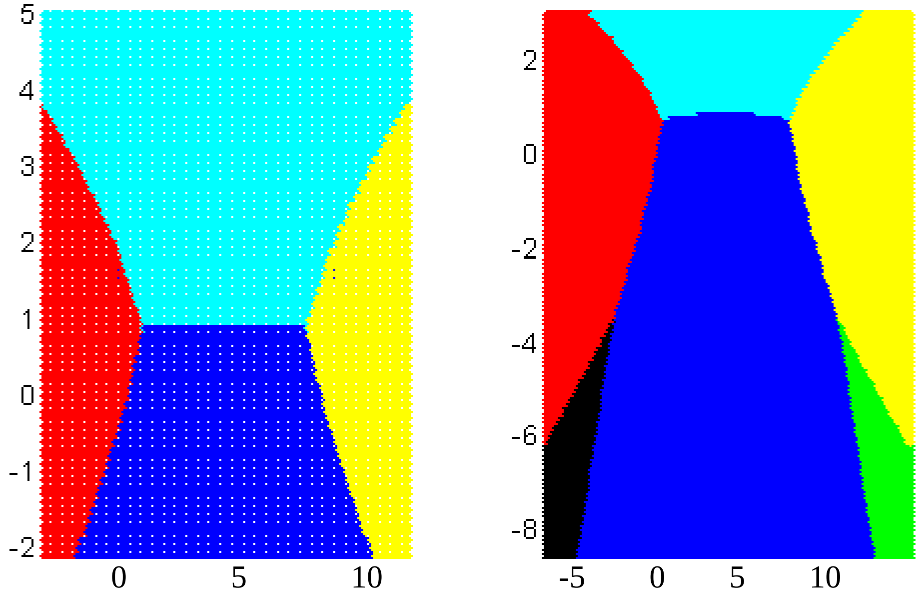


Figure 26: left $t = 4.5$, $h = 2$, right $t = 5.5$, $h = 2$

The graph below indicates the number of colors present for values of t and h ranging from 0 to 10. A blue point represents the presence of four colors for a given t and h, and a green point indicates the presence of six colors.

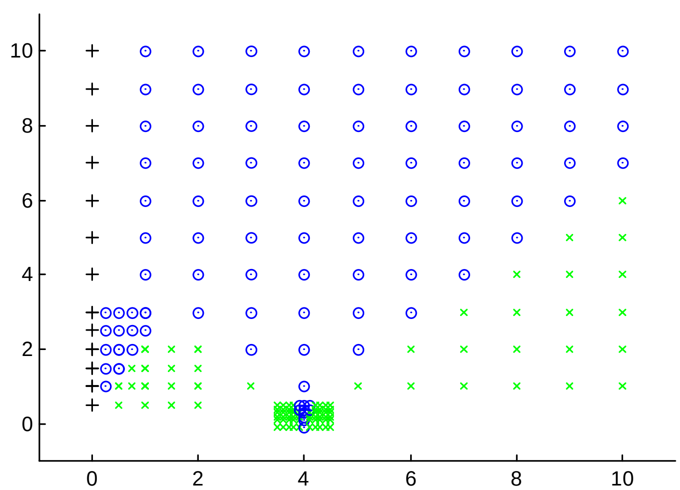


Figure 27: *h versus t*

As can be seen, for $h = 2$, there are two points where the number of colors present shifts from four to six when $0 < t < 4$. In the case of $t = 4$, it appears that there will always be four colors in the graph regardless of the height. For values of t when $t > 4$, there is only one critical value where the number of colors present shifts from four to six. The linearity of the graph leads us to believe that there are critical ratios of *t* to *h* that will determine whether or not six colors are present.

Red, yellow, blue and cyan are the four colors that are always present in our graph. The following table shows the boundary equations for green and black, the fifth and sixth colors, with the other four colors.

Colors	Boundary Equation
Red/Green	HC-HB = BC+AD-AB-BD
Blue/Green	HD-HC = BD-BC
Cyan/Green	HA-HB = AD-BD
Black/Green	HD+HB-HA-HC = BD-AC
Yellow/Green	HA-HD = -BD-CD+BC+AD

Boundary Equation	Colors
HC-HB = -BC-AD+CD+AC	Yellow/Black
HA-HB = AC-BC	Blue/Black
HD-HC = AD-AC	Cyan/Black
HD+HB-HA-HC = BD-AC	Green/Black
HA-HD = AC+AB-BC-AD	Red/Black

Lemma 2: Isosceles trapezoids will generate either four or six color regions.

Proof: Since ABCD is a quadrilateral, there is the possibility of four, five, or six colors for the traveling salesman graph. Because ABCD is an isosceles trapezoid, AB = CD (the legs of an isosceles trapezoid are congruent) and AC = BD (the diagonals of an isosceles trapezoid are congruent). By substituting these values into the boundary equations above, the following equations are obtained.

Colors	Boundary Equation
Red/Green	HC-HB = BC+AD-AB-BD
Blue/Green	HD-HC = BD-BC
Cyan/Green	HA-HB = AD-BD
Black/Green	HD+HB-HA-HC = BD-AC
Yellow/Green	HA-HD = -AC-AB+BC+AD

Boundary Equation	Colors
HC-HB = -BC-AD+AB+BD	Yellow/Black
HA-HB = BD-BC	Blue/Black
HD-HC = AD-BD	Cyan/Black
HD+HB-HA-HC = BD-AC	Green/Black
HA-HD = -BC-AD+AC+AB	Red/Black

The Red/Green and Yellow/Black boundary equations are the equations of opposite branches of the same hyperbola. Since the foci are symmetric about the line x = 4, the hyperbola branches will also be symmetric about the line x = 4. By the same argument, this is also true of the Yellow/Green and the Red/Black equations. The Black/Green and the Green/Black equations are, of course, equal as well as symmetric about x = 4. Since AB = CD, the Blue/Green and the Blue/Black equations as well as the Cyan/Green and the Cyan/Black equations are equations of congruent hyperbolas. Also, since the foci points A and B of the Blue/Black and the Cyan/Green hyperbolas are symmetric to the foci points D and C of the Blue/Green and the Cyan/Black hyperbolas about the line x = 4, respectively, the congruent hyperbolas are also symmetric about the line x = 4. Since each of the boundary equations that govern the green and black regions are the same, equal, or congruent hyperbolas that are all symmetric about the line x = 4, the green and black regions will only appear under identical conditions. Thus five colors will never be present, only four or six. Q.E.D.

We will now investigate the critical values of *t* that cause the shift between four and six colors. Even though there are five equations that govern each the green and black regions, the entire region can be discovered by only examining three of the boundary equations for each of the two colors. In the graphs above where $0 < t < 4$, the green and yellow and the green and black regions do not intersect. The boundary equations still exist but they fall within the region of another color. In this region, the other color will always be the best path and hence the boundary colors do not intersect in the graph. Since red, blue and cyan all intersect green in these graphs, the intersection of the green sides of the Red/Green, Blue/Green, and Cyan/Green boundary equations will determine the green region in the graph. When $t > 4$, the green and red regions as well as the green and black regions do not intersect. By the same reasoning, the green region will be determined by the intersection of the green sides of the Yellow/Green, Blue/Green, and the Cyan/Green boundary. Since $t = b_2/h$, we will find critical values of t and use them with the ratio b_2/h.

Theorem 5: For any isosceles trapezoid ABCD where $b_1/h = 4$:

i) if $.7466 < b_2/h < 3.0803$, there are six colors present in the traveling salesman graph;

ii) if $b_2/h > 4.7882$, there are six colors present in the traveling salesman graph;

iii) for all other positive values of b_2/h, the traveling salesman graph will have four colors.

Proof: i) The lower critical t-value occurs when the three relevant boundary equations have a common solution.

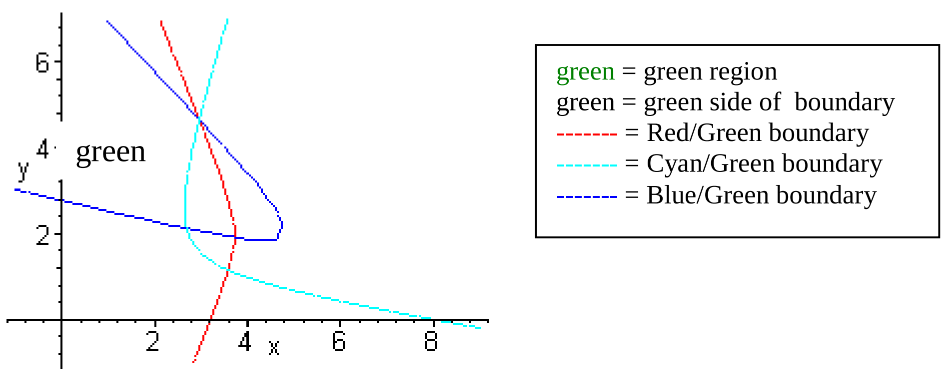


Figure 28: color boundaries when t is equal to the critical value

Using a Matlab program (see appendix) that uses Newton's Method, we are able to find the intersection of the Blue/Green and Cyan/Green boundary equations as well as the intersection of the Red/Green and the Cyan/Green boundary equations. The program then minimizes the distance between the two within a given tolerance. Using the Matlab program, we find that $t = .7466$ is a critical value. When $t < .7466$, there is no common solution to the three equations and hence no green region.

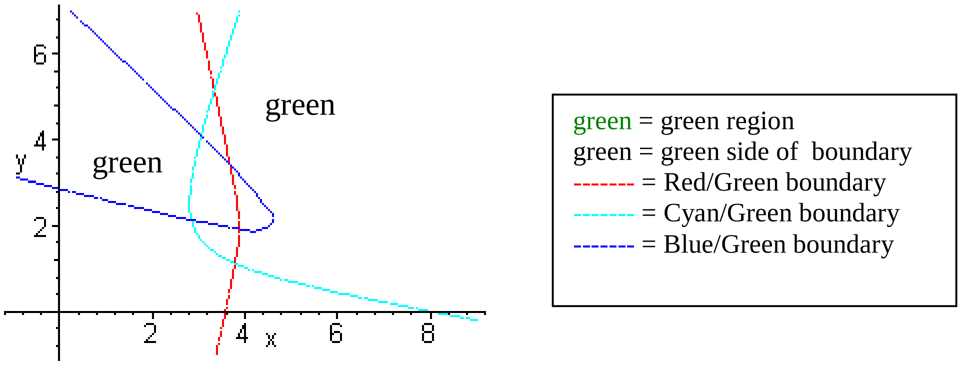


Figure 29: color boundaries when *t* is less than the critical value

When $t > .7466$, the equations bound a region of the graph which produces the green color. Thus, when $b_2/h > .7466$, the color green will appear on the graph.

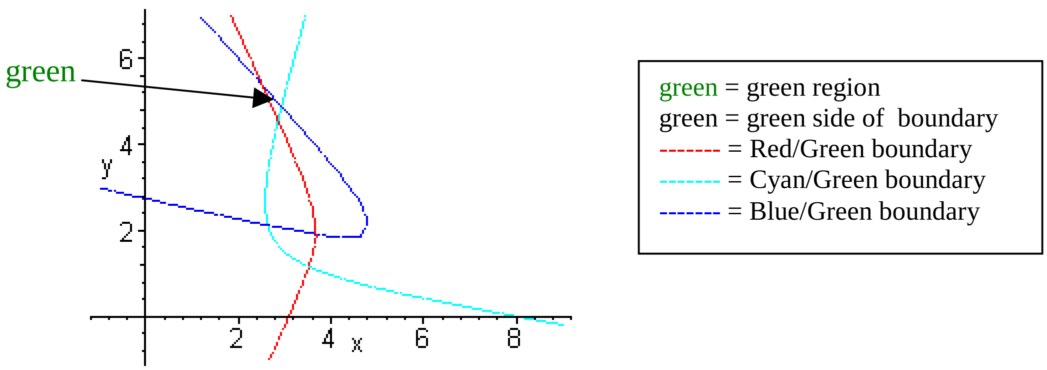


Figure 30: color boundaries when *t* is greater than the critical value

Using Lemma 2, the ratio $b_2/h > .7466$ will also be when the black region will appear on the graph.

The upper *t*-value for the critical ratio occurs when a common solution stops existing for the three equations. More specifically, the hyperbolas formed from the Red/Green and the Cyan/Green boundary equations have no point of intersection within the green side of the Blue/Green boundary equation and thus their asymptotes are parallel in this region.

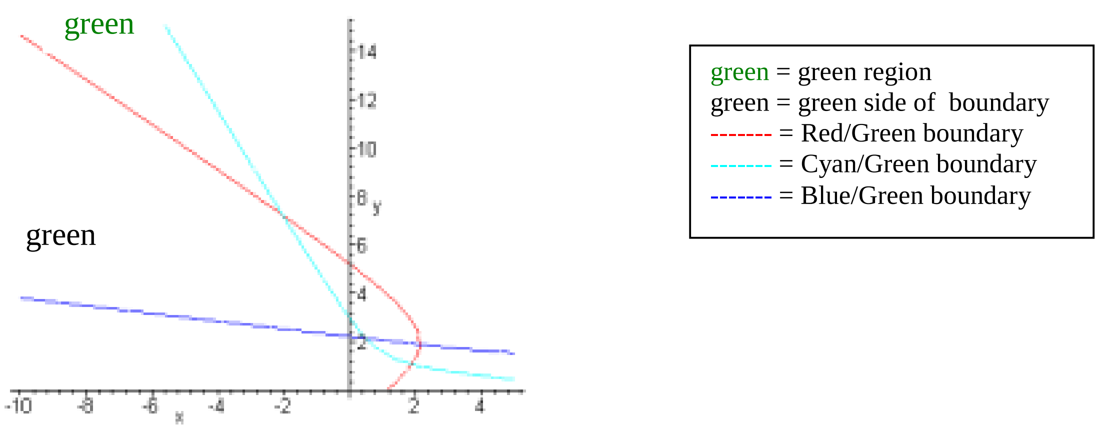


Figure 31: color boundaries when *t* is less than the critical value

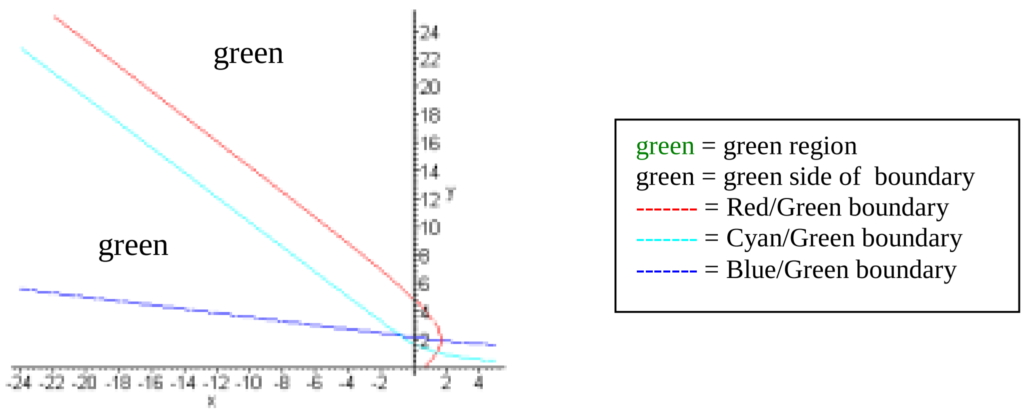


Figure 32: color boundaries when *t* is equal to the critical value

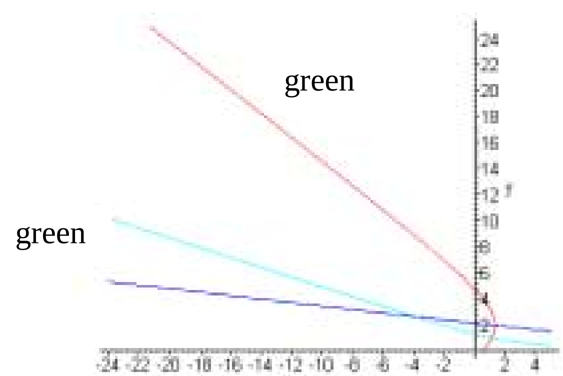


Figure 33: color boundaries when *t* is greater than the critical value

To find the *t* for which the asymptotes are parallel, as in the case of the kite, we will find two points on each asymptote, find the slope of each asymptote, set them equal to each other, and solve for t. Following the same methods used for the rhombus and the kite, we can find the coordinates for two points on each of the asymptotes. The asymptote of the Red/Green boundary equation contains the points

$$(4,2) \text{ and } (4,2)+\frac{1}{2}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)(-1,0)$$
$$+\left(\sqrt{t^2-\frac{1}{4}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)^2}\right)(0,1), \text{ and has slope}$$
$$\frac{-\sqrt{t^2-\frac{1}{4}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)^2}}{\frac{1}{2}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)}.$$

The asymptote of the Cyan/Green boundary equation contains the points

$$\left(\frac{4-t}{2},1\right) \text{ and } \left(\frac{4-t}{2},1\right)+\frac{1}{2}\left(\frac{8-\sqrt{t^2+8t+20}}{\sqrt{t^2-8t+20}}\right)(4-t,2)$$
$$+\frac{\sqrt{t^2-8t+20-\left(8-\sqrt{t^2+8t+20}\right)^2}}{2\sqrt{t^2-8t+20}}(-2,4-t), \text{ and has slope}$$
$$\frac{\left(8-\sqrt{t^2+8t+20}\right)+(4-t)\frac{1}{2}\sqrt{t^2-8t+20-\left(8-\sqrt{t^2+8t+20}\right)^2}}{(4-t)\frac{1}{2}\left(8-\sqrt{t^2+8t+20}\right)-\sqrt{t^2-8t+20-\left(8-\sqrt{t^2+8t+20}\right)^2}}.$$

By setting the slopes equal to each other, we produce the following equation:

$$\frac{\left(8-\sqrt{t^2+8t+20}\right)+(4-t)\frac{1}{2}\sqrt{t^2-8t+20-\left(8-\sqrt{t^2+8t+20}\right)^2}}{(4-t)\frac{1}{2}\left(8-\sqrt{t^2+8t+20}\right)-\sqrt{t^2-8t+20-\left(8-\sqrt{t^2+8t+20}\right)^2}}$$
$$+\frac{\sqrt{t^2-\frac{1}{4}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)^2}}{\frac{1}{2}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)}=0.$$

By using the fzero command in Matlab to solve for t, we obtain a critical t-value of 3.0803. When $t < 3.0803$ the green region is present in the graph. Since green and black occur under the same conditions, the black region is also present. Thus, when $b_2/h < 3.0803$, there are six colors present in the graph. Combining the two results with Lemma 1 gives us the open interval (.7466,3.0803) for b_2/h where six colors will be seen in the traveling salesman graph for all isosceles trapezoids with $b_1/h = 4$.

ii) For values of t, where $t > 4$, the Yellow/Green, Blue/Green, and Cyan/Green boundary equations determine the green region. Below the critical t value, the Yellow/Green and Blue/Green boundary equations have no common solution on the green side of the Cyan/Green boundary causing there to be no green region.

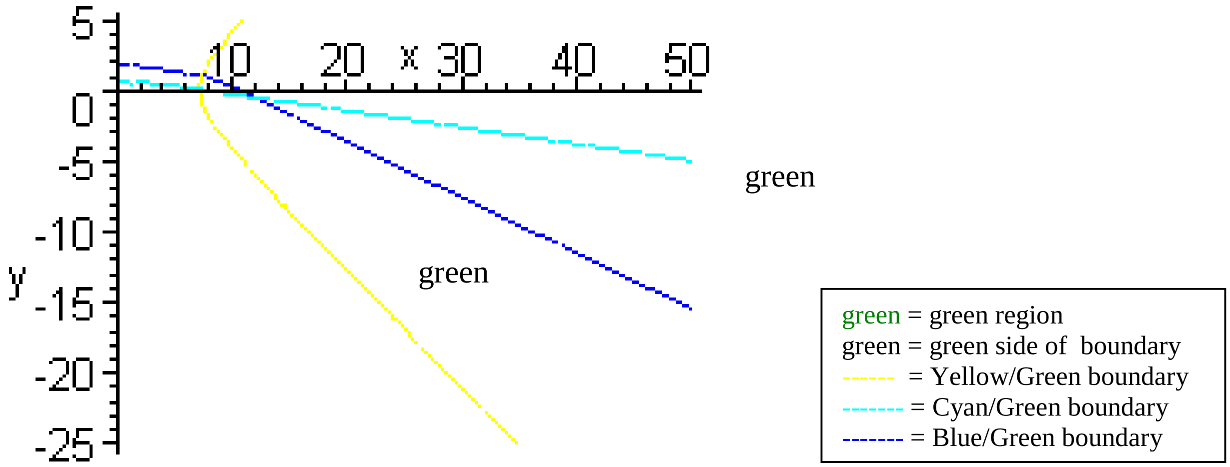


Figure 34: Yellow/Green, Cyan/Green, and Blue/Green boundaries when t is less than the critical value

At the critical t value, the asymptotes of the hyperbolas formed by the Yellow/Green and Blue/Green boundary equations are parallel.

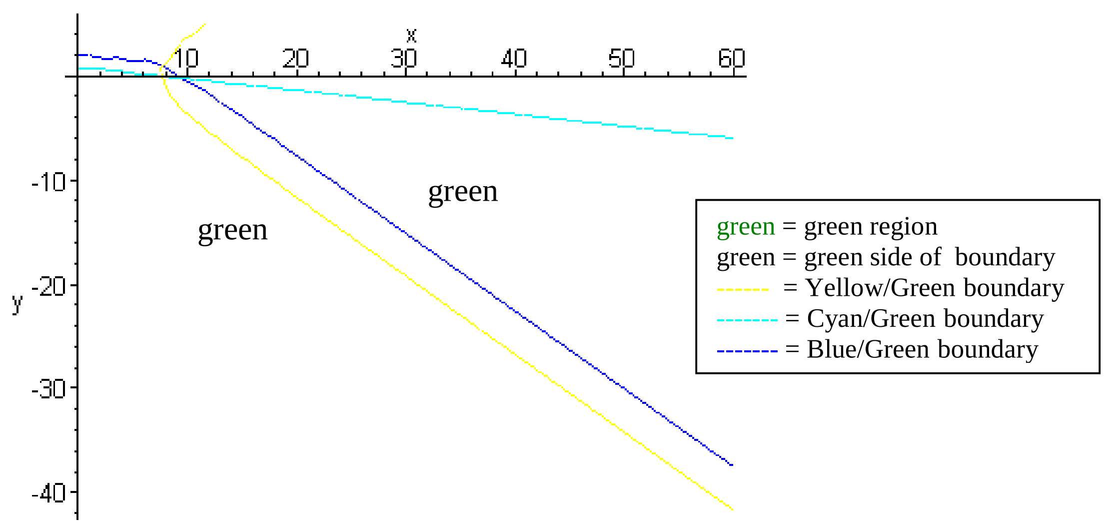


Figure 35: Yellow/Green, Cyan/Green, and Blue/Green boundaries when t is equal to the critical value

For t values larger than the critical value, the two hyperbolas intersect and the green region is created.

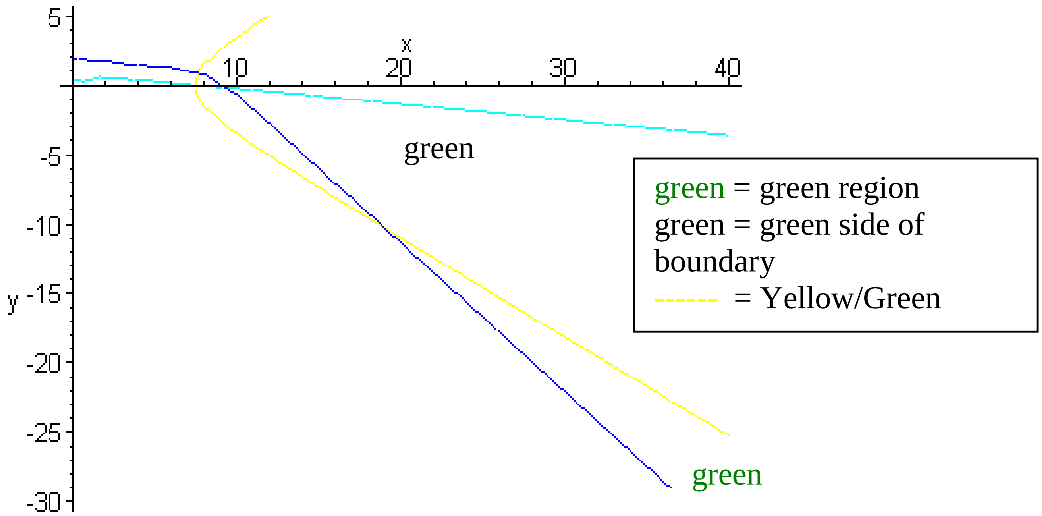


Figure 34: Yellow/Green, Cyan/Green, and Blue/Green boundaries when t is greater than the critical value

The asymptote of the hyperbola formed by the Yellow/Green boundary equation contains the points

$$(4,0) \text{ and } (4,0)+\left(\frac{2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}}{2}\right)(1,0)$$
$$+\left(\sqrt{16-\frac{1}{4}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)^2}\right)(0,-1), \text{ and has slope}$$
$$-\frac{\sqrt{16-\frac{1}{4}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)^2}}{\frac{1}{2}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)}.$$

The asymptote of the hyperbola formed by the Blue/Green boundary equation contains the points

$$\left(\frac{6+t}{2},1\right) \text{and} \left(\frac{\sqrt{t^2+8t+20}-2t}{2\sqrt{t^2-8t+20}}\right)(-4+t,2)$$
$$+\left(\frac{\sqrt{t^2-8t+20-\left(\sqrt{t^2+8t+20}-2t\right)^2}}{2\sqrt{t^2-8t+20}}\right)(2,4-t), \text{ and has slope}$$
$$\frac{\sqrt{t^2+8t+20}-2t+(4-t)\frac{1}{2}\sqrt{t^2-8t+20-\left(\sqrt{t^2+8t+20}-2t\right)^2}}{(-4+t)\frac{1}{2}\left(\sqrt{t^2+8t+20}-2t\right)+\sqrt{t^2-8t+20-\left(\sqrt{t^2+8t+20}-2t\right)^2}}.$$

Since the critical value of t is found when the asymptotes are parallel, we will set the two slopes equal to each other to obtain the following equation.

$$\frac{\sqrt{t^2+8t+20}-2t+(4-t)\frac{1}{2}\sqrt{t^2-8t+20-\left(\sqrt{t^2+8t+20}-2t\right)^2}}{(-4+t)\frac{1}{2}\left(\sqrt{t^2+8t+20}-2t\right)+\sqrt{t^2-8t+20-\left(\sqrt{t^2+8t+20}-2t\right)^2}}$$
$$+\frac{\sqrt{16-\frac{1}{4}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)^2}}{\frac{1}{2}\left(2t+8-\sqrt{t^2+8t+20}-\sqrt{t^2-8t+20}\right)}=0$$

As before, we use the fzero command in Matlab to find that 4.7882 is a root of the equation and hence the critical value of t. When t and hence $b_2/h > 4.7882$, the green region, as well as the black by Lemma 2, is present in the graph. Using Lemma 1, we can now state that for all isosceles trapezoids with $b_1/h = 4$, there will be six colors present if $b_2/h > 4.7882$.

iii) We showed earlier in Lemma 2 that there can only be four or six colors present for an isosceles trapezoid in a traveling salesman graph. Since we have found the positive b_2/h values where six colors are present in the graph, all other positive values will have four colors in their respective traveling salesman graphs. Q.E.D.

We can use our methods to find critical b_2/h values when $b_1/h \neq 4$ by using $b_1/h = n$, where n is any real number except zero because b_1 can never be zero. Then, given any isosceles trapezoid, we can calculate n and write the coordinates of the vertices of a similar isosceles trapezoid as (0,0), (4-t, 8/n), (4+t, 8/n), (8,0). First, we substitute these new coordinates into our Matlab program that uses Newton's method to find the first critical t value. We then find the critical values when the asymptotes are parallel, as we did before. Ideally some conditions could be found for all isosceles trapezoids, using a step function.

Acknowledgements

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Appendix

Matlab Program for Coloring the Traveling Salesman Graphs

%This program produces the regions corresponding to the six possible %paths for the traveling salesman problem with four cities.

```
%The following need to be defined before running the
program:
%P: which contains the coordinates of the four points
%h,k: amount grid goes beyond the graph of P(h sides,k
vertical)
%step: distance between grid points

d=0*P';        %create the distance array the same size as
P'
for i=1:4      %calculate all distances (some not needed)
   for j=1:4
      d(i,j)=norm(P(i,:)-P(j,:));
   end
end

xleft = min(P(:,1))-h; xright = max(P(:,1))+h;
ybottom = min(P(:,2))-k; ytop = max(P(:,2))+k;

axis([xleft, xright,ybottom, ytop])
hold on

P=[P;P(1,:)];          %change P in order to plot graph
plot(P(:,1)',P(:,2)')
P=P(1:4,:);     %change P back to original form for further
use

x=xleft:step:xright;       %determine the size of the mesh
y=ybottom:step:ytop;

clear f
for i = 1:length(x)
   for j = 1:length(y);
      X=[x(i),y(j)];    %calculate the six paths' lengths

      a=d(1,3)+d(3,4)+d(4,2); b=d(1,4)+d(4,3)+d(3,2);
      f(1) = norm(X-P(1,:))+min(a,b)+norm(X-P(2,:));

      a=d(1,2)+d(2,4)+d(4,3); b=d(1,4)+d(4,2)+d(2,3);
      f(2) = norm(X-P(1,:))+min(a,b)+norm(X-P(3,:));
```

```
        a=d(1,2)+d(2,3)+d(3,4); b=d(1,3)+d(3,2)+d(2,4);
        f(3) = norm(X-P(1,:))+min(a,b)+norm(X-P(4,:));

        a=d(2,1)+d(1,4)+d(4,3); b=d(2,4)+d(4,1)+d(1,3);
        f(4) = norm(X-P(2,:))+min(a,b)+norm(X-P(3,:));

        a=d(2,3)+d(3,1)+d(1,4); b=d(2,1)+d(1,3)+d(3,4);
        f(5) = norm(X-P(2,:))+min(a,b)+norm(X-P(4,:));

        a=d(3,1)+d(1,2)+d(2,4); b=d(3,2)+d(2,1)+d(1,4);
        f(6) = norm(X-P(3,:))+min(a,b)+norm(X-P(4,:));

        [m,q]=min(f);
        if q==1
           plot(x(i),y(j),'.r')
        end
        if q==2
           plot(x(i),y(j),'.g')
        end
        if q==3
           plot(x(i),y(j),'.b')
        end
       if q==4
           plot(x(i),y(j),'.c')
        end
       if q==5
           plot(x(i),y(j),'.k')
        end
       if q==6
           plot(x(i),y(j),'.y')
        end

   end
end
```

Algebraic Computations and Matlab Program for the Hyperbolas Governing the Fifth Color in a Rhombus

Due to the symmetry of the figure, we need only find the equation for any one of the hyperbolas and then we can reflect that hyperbola across the x-axis and y-axis. We will find the equation for the first of the hyperbolas listed, HC – HB = BC – BD. Letting t=4, the coordinates of B, C, and D are (-4, 0), (0, -h), and (4,0) respectively. Let the coordinates of the movable home point be (x, y). Thus the equation becomes

$$\sqrt{x^2+(y+h)^2}-\sqrt{(x+4)^2+y^2}=\sqrt{16^2+h^2}-8.$$

Rearranging and then squaring both sides yields

$$x^2+(y+h)^2=(\sqrt{16^2+h^2}-8)^2+2(\sqrt{16^2+h^2}-8)\sqrt{(x+4)^2+y^2}+(x+4)^2+y^2,$$

which simplifies to

$$2hy+16\sqrt{16^2+h^2}-8x-96=2(\sqrt{16^2+h^2}-8)\sqrt{(x+4)^2+y^2}.$$

Squaring again, we obtain

$$\begin{aligned}4h^2y^2+4hy(16\sqrt{16^2+h^2}-8x-96)+(16\sqrt{16^2+h^2}-8x-96)^2\\ =4(\sqrt{16^2+h^2}-8)^2((x+4)^2+y^2).\end{aligned}$$

Distributing and regrouping, we arrive at a quadratic equation,

$$\begin{aligned}(4(\sqrt{16^2+h^2}-8)^2-4h^2)y^2-(4h(16\sqrt{16^2+h^2}-8x-96))y\\ +(4(\sqrt{16^2+h^2}-8)^2(x+4)^2)-(16\sqrt{16^2+h^2}-8x-96)^2=0.\end{aligned}$$

We use this equation and the quadratic formula in the following Matlab program to graph the relevant branches of the hyperbolas.

```
%This program draws the relevant branches of the hyperbolas
for the %rhombus case of the traveling salesman problem,
where the width of the %rhombus is 8 and the height, h, is
variable and must be entered prior %to running the program.
X=[0 4 0 -4 0];
Y=[h 0 -h 0 h];
plot(X,Y)
hold on
x=-40:0.1:40;
L=length(x);
S=sqrt(16+h^2);
a=4*(S-8)^2-4*h^2;
for i=1:L
   b=-4*h*(16*S-8*x(i)-96);
   c=4*((S-8)^2)*(x(i)+4)^2-(16*S-8*x(i)-96)^2;
   y1(i)=(-b-sqrt(b^2-4*a*c))/(2*a);
   y2(i)=-y1(i);
end
plot(x,y1)
plot(x,y2)
plot(-x,y1)
plot(-x,y2)
axis('equal')
```

Twohypinta Program

```
%twohypinta
%This program uses Newton's Method to
%find the intersection of two hyperbolas

x=origx;
y=origy;
dAB=norm(P(2,:)-P(1,:));
dAC=norm(P(1,:)-P(3,:));
dAD=norm(P(1,:)-P(4,:));
dBC=norm(P(2,:)-P(3,:));
dBD=norm(P(2,:)-P(4,:));
dCD=norm(P(3,:)-P(4,:));

test=1;
while test>.00001
   q=[x,y];
   dA=norm(q-P(1,:));
   dB=norm(q-P(2,:));
   dC=norm(q-P(3,:));
   dD=norm(q-P(4,:));

   f=dC-dB+dBD+dAB-dBC-dAD;%Red/Green boundary
   g=dA-dB+dBD-dAD;%Cyan/Green boundary
   fx=(x-P(3,1))/dC-(x-P(2,1))/dB;
   fy=(y-P(3,2))/dC-(y-P(2,2))/dB;
   gx=(x-P(1,1))/dA-(x-P(2,1))/dB;
   gy=(y-P(1,2))/dA-(y-P(2,2))/dB;
   H=[fx,fy;gx,gy];
   b=[f;g];
   del=H\b;
   test=norm(del);
   x=x-del(1);
   y=y-del(2);
end
```

Twohypintb Program

```
%twohypintb
%This program uses Newton's Method to
```

```
%find the intersection of two hyperbolas

x=origx;
y=origy;
dAB=norm(P(2,:)-P(1,:));
dAC=norm(P(1,:)-P(3,:));
dAD=norm(P(1,:)-P(4,:));
dBC=norm(P(2,:)-P(3,:));
dBD=norm(P(2,:)-P(4,:));
dCD=norm(P(3,:)-P(4,:));

test=1;
while test>.00001
   r=[x,y];
   dA=norm(r-P(1,:));
   dB=norm(r-P(2,:));
   dC=norm(r-P(3,:));
   dD=norm(r-P(4,:));

   f=dA-dB+dBD-dAD;%Cyan/Green boundary
   g=dD-dC+dBC-dBD;%Blue/Green boundary
   fx=(x-P(1,1))/dA-(x-P(2,1))/dB;
   fy=(y-P(1,2))/dA-(y-P(2,2))/dB;
   gx=(x-P(4,1))/dD-(x-P(3,1))/dC;
   gy=(y-P(4,2))/dD-(y-P(3,2))/dC;
   H=[fx,fy;gx,gy];
   b=[f;g];
   del=H\b;
   test=norm(del);
   x=x-del(1);
   y=y-del(2);
end
```

Newmeth Program

```
%Newmeth Program
%Minimizes the distance between two points
%Calls twohypinta and twohypintb

dAB=norm(P(2,:)-P(1,:));
dAC=norm(P(1,:)-P(3,:));
dAD=norm(P(1,:)-P(4,:));
dBC=norm(P(2,:)-P(3,:));
dBD=norm(P(2,:)-P(4,:));
```

```
dCD=norm(P(3,:)-P(4,:));

test=1;
while test>.00001
   z=[x,y];
   dA=norm(z-P(1,:));
   dB=norm(z-P(2,:));
   dC=norm(z-P(3,:));
   dD=norm(z-P(4,:));

   f=dC-dB+dBD+dAB-dBC-dAD;
   g=dA-dB+dBD-dAD;
   h=dC+dB-dA-dD+dAD-dBC
   fx=(x-P(1,1))/d1-(x-P(2,1))/d2;
   fy=(y-P(1,2))/d1-(y-P(2,2))/d2;
   gx=(x-P(1,2))/d1-(x-P(3,1))/d2;
   gy=(y-P(1,2))/d1-(y-P(3,2))/d2;
   H=[fx,fy;gx,gy];
   b=[f;g];
   del=H\b;
   test=norm(del);
   x=x-del(1);
   y=y-del(2);
end
```