A Variation of *Lights Out*, Modulo 3

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Abstract

In this project we investigate the properties of a mod 3 variation of the game "*Lights Out*" played on an arbitrary graph or digraph, following work done by Craft, Miller and Pritikin. [1] We prove that there are 2^{n-1} unwinnable configurations for trees with n vertices, and we show a procedure to derive and recognize these configurations. For each graph, we exhibit a nontrivial initial configuration X, which is unwinnable. We characterize which initial configurations are winnable on complete graphs, complete bipartite graphs, acyclic digraphs, certain tournaments, and some other graphs.

I. Introduction

The original game of *Lights Out* was developed in its simplest form by Tiger Electronics. It is played on a 5x5 grid of squares, each square being in a state of either "on" or "off". An initial state is specified for each square, perhaps generated at random. The object of the game is to turn all squares to the "off" state. "Clicking" on any square will change its status as well as the status of its four neighbors (to the right, left, top, and bottom). Popularity of this game was spread not only by kids playing the hand-held game and from the proliferation of websites offering a free interactive online version of the game, but it was also spread among mathematicians because the game can be used to illustrate linear algebraic and graph theoretic concepts.

For the linear algebraic connection, matrices can be used to represent which squares are affected when clicking on an initial square of the 5x5 grid. A linear system of equations with mod 2 variables can be set up for the solution set of winning sequences or combinations of moves for the game. The 5x5 grid can also be represented as a simple graph G whose vertices represent the squares of the grid. Vertices A and B are adjacent in G if and only if square A is affected by "clicking" on square B. Our research will utilize this graph theoretic approach of

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examining variations of *Lights Out*, so that we can analyze the related game played on any graph structure, not just a 5x5 grid.

Craft, Miller, and Pritikin [1] have studied a variation of *Lights Out* in which vertices begin in either an "on" or "off" state and are deleted as they are clicked on; only a vertex in an "on" state can be clicked on. The object here is to delete all vertices. The variation of the game we chose for our research is an extension of the research of this problem in [1].

The rules that we have been examining are the same as those in [1], except the "colorings" are modulo 3. We will denote the colors by 0, 1, and 2, and we can only click on vertices colored either 1 or 2. "Clicking" on either of these types of vertices will result in a deletion of that vertex *x*, as well as add 1 (mod 3) to the color of the vertices adjacent to *x*. The graph is won when all the vertices are deleted. Our interest is in determining which initial colorings in a given graph are winnable. (A variation of *Lights Out* in which clicking on a vertex results in deletion of that vertex has been marketed under the name "Entropy" by Smartgames, but their game allows clicking on an unlit vertex, and the object of their game is to turn out all of the lights in a minimum number of moves, not to remove all of the vertices.)

In this paper, we begin by studying trees. We introduce several theorems that apply to all trees such as the fact that there are 2^{n-1} unwinnable configurations, where n is the number of vertices, and procedures to derive these configurations. Then we present one theorem that applies only to paths, which are a special type of tree. Next we turn our attention to cycles, complete graphs, and complete bipartite graphs. We briefly examine some directed graphs, including acyclic digraphs, directed cycles, and tournaments. Finally, we present a nontrivial unwinnable configuration for any graph.

It is necessary to define some basic concepts about graph theory before we continue. All formal definitions are taken from West's Introduction to Graph Theory [2] unless otherwise denoted. Please refer to [2] for any other unclear terms regarding graph theory. A *graph G* is a set of exactly two vertices, a set of edges, and an assignment of a set at most two vertices as endpoints of each edge; this is what some is sometimes referred to as a simple graph. *Adjacent vertices* are "vertices that are endpoints of an edge." We call a vertex *g* a *neighbor* of a vertex *v* if *v* and *g* are adjacent.

We also define some terms that are outside the formal definitions of graph theory and are specific to our paper. *Clicking* is the act of deleting a vertex and increasing its adjacent vertices' color by 1 (mod 3). A 3-*coloring* (or *coloring* for short) of a graph G is a function that assigns one of the "colors" 0, 1, or 2 from Z_3 to each vertex in G. Given a coloring c of a graph, let $\boldsymbol{0}_c$ be the number of vertices of color 0, let $\boldsymbol{1}_c$ be the number of vertices of color 1, and let $\boldsymbol{2}_c$ be the number of vertices of color 2 in the coloring c. We define the *0-coloring* to be the case where each vertex of a graph has color 0, and we denote this by $\boldsymbol{0}$. Similarly, let the*2-coloring* be the case where each vertex of a graph has color 2, and we denote this by $\boldsymbol{2}$. A graph G with a coloring c is *winnable* if all the vertices can be deleted in some sequence of moves, and we denote this by $(G, c) \in W$, where W is the set of all winnable graphs. Otherwise, we say that G is *unwinnable* and denote it by $(G, c) \notin W$.

Now that we have some of the basic concepts, we will proceed with the discoveries we made in our research.

II. Trees

We began our research with a look at paths, one of the simplest types of trees. As we went further in our research, we found that many of the theorems we initially proved about paths could be generalized to the broader category of trees.

We begin this section by giving the definitions of some basic terms related to trees. A *path* is a "simple graph whose vertices can be listed so that vertices are adjacent if and only if they are consecutive in the list." The path on n vertices is denoted P_n. A graph G is *connected* if it has "a u,v-path for ever pair of vertices u, v." A *tree* is a connected graph that has $n+1$ edges, where n is the number of vertices in the tree. The *degree* of a vertex v in a graph G, written $deg(v)$ or $deg_G(v)$, is "the number times it appears in edges." A *leaf* of a tree is a "vertex of degree 1."

Theorem 2.5: *If (P_n, c) is not winnable, then $\boldsymbol{2}_c$ is even.*

Proof: We proceed by induction on the number of vertices in P_n.

BASE CASE: The case where $n = 1$ is easily shown.

INDUCTIVE STEP: Assume the statement is true for $n = 1,2,\ldots,k$ for some integer k. Consider the case where $n = k+1$, with $(P_{k+1}$, c) unwinnable.

Case 1: Let the first vertex v have color 0. The path on the remaining k vertices is unwinnable by Theorem 2.1. So there is an even number of vertices of color 2 by the induction hypothesis. Including v with color 0 does not change this number.

Case 2: Let the first vertex v have color 2. The second vertex w has to have color 1 or 2, because if it had color 0, clicking on v would give us a path with the first vertex of color 1, which is always winnable.

Suppose w has color 1. Click on v, in which case w becomes of color 2 and we obtain a path with k vertices. By the induction hypothesis and the 2,2-procedure, there is an even number of vertices of color 2 in P_k. This is exactly the same number of vertices of color 2 as in P_{k+1} because we deleted v, a vertex of color 2, but we added another vertex with color 2 when we changed the color of w after clicking on v.

Now, suppose w has color 2. We click on v, in which case the color of w will change to 0 and we have an unwinnable path with k vertices. By the induction hypothesis, there is an even number of vertices of color 2 in P_k. Since we deleted v, a vertex of color 2, in P_{k+1} and changed the color of w from 2 to 0 when we clicked on v, the original number of vertices of color 2 must have been even. □

III. Undirected Cycles

The next graph we will discuss is the undirected cycle. This follows easily from our observations about paths, since clicking on a vertex in a cycle results in a path. A *cycle* is a "simple graph whose vertices can be placed on a circle so that vertices are adjacent if an only if they appear consecutively on the circle." The cycle on n vertices is denoted C_n. We include two lemmas concerning paths that will be used in the proof of our theorem about the winnability of cycles.

Lemma 3.1: *Every path P_n with coloring 211…112 is unwinnable.*

Proof: We proceed by induction on the number of vertices.

BASE CASE: The case where $n = 2$ is easily shown.

INDUCTIVE STEP: Assume the statement is true for $n = 2,3,\ldots,k$ for some integer k. Consider the case when $n = k+1$.

Case 1: Click on one of the end vertices v. This removes v and changes the adjacent vertex's color to 2 (since its original color was 1). So we have a path of length k with the coloring 211…112, which is unwinnable by the induction hypothesis.

Case 2: Click on a vertex of color 1 that is adjacent to a vertex of color 2. This will turn the color of the end vertex to color 0 and isolate it, so this coloring is unwinnable.

Case 3: Click on any vertex v of color 1 that is not adjacent to a vertex of color 2. This removes v and changes the color of its two adjacent vertices to 2. We now have two components with coloring 211…112 that are paths with s and $k - s$ vertices, where $s \leq k$. If $s = k$, we are back in case 2. If not, these components are unwinnable by the induction hypothesis. □

Lemma 3.2: *Every path P_n with an even number of vertices and coloring 022…220 is unwinnable.*

Proof: We proceed by induction on the number of vertices.

BASE CASE: The case where $n = 2$ is easily shown.

INDUCTIVE STEP: Assume that the claim is true for $n = 2,3,\ldots,k$ for some integer k and consider the case where $n = k+1$.

Case 1: Click on a vertex v of color 2 adjacent to a vertex of color 0. Then we will have two components: a path with $k-2$ vertices and a singleton vertex of color 1. By clicking on v we deleted one vertex of color 2 and changed one vertex of color 2 to color 0. So we still have an even number of vertices of color 2 in the path of $k-2$ vertices. Thus the graph is unwinnable by the induction hypothesis.

Case 2: Click on a vertex v of color 2 that is not adjacent to a vertex of color 0. By doing this, the original path is divided into two components and the two neighbors of v now have color 0. Since we clicked on a vertex of color two and changed the colors of its neighbors from 2 to 0, we now have an odd number of vertices of color two. This means that at least one of the components of the new graph has an even number of vertices of color 2. Its total number of vertices is less than $k+1$, so we can apply the induction hypothesis and conclude that this component is unwinnable, making the original path on $k+1$ vertices also unwinnable. □

We are now ready to determine when a cycle is winnable.

Theorem 3.1: *(C_n, c) is unwinnable if and only if $c = \mathbf{0}$ or $c = \mathbf{1}$ or [$c = \mathbf{2}$ when $n \equiv 1 \pmod 2$].*

Proof: We first show that if the coloring is $\mathbf{0}$, $\mathbf{1}$, or [$\mathbf{2}$ where $n \equiv 1 \pmod 2$], then (C_n, c) is unwinnable. If $c = \mathbf{0}$, we have no starting vertex to click on, therefore (C_n, c) is unwinnable. If $c = \mathbf{1}$, then click on any vertex. We then get a path whose coloring is 211…112, which is unwinnable by Lemma 4.1. Finally, if $c = \mathbf{2}$ and $n \equiv 1 \pmod 2$, then we get a path with coloring 022…220, where there are an even number of vertices. This is unwinnable by Lemma 4.2.

Conversely, assume that the coloring c is not $\mathbf{0}$, $\mathbf{1}$, or [$\mathbf{2}$ where $n \equiv 1 \pmod 2$]. We show that (C_n, c) is winnable. Well, if the cycle has a vertex of color 0, then since not all the vertices are of color 0 we can click on a non-zero vertex that is adjacent to a vertex of color 0. We then get a path with an end vertex of color 1, which is winnable by Theorem 2.3.

If the cycle has an even number of vertices and $c = \mathbf{2}$, then when we click on a vertex v, we delete v, which is of color 2, and we change the two adjacent vertices of v to color 0. Therefore we now have an odd number of vertices of color 2 in the resulting path, which is winnable by Theorem 2.5.

Finally, if the cycle has no vertices of color 0, but the vertices are not all of color 1 and not all of color 2, then we will have a sequence of consecutive vertices v, x, y, where $c(x) = 2$ and $c(y) = 1$. We click on v (which is non-zero) and we get a path where the first 2 vertices are of color 0 and 1 respectively. This path is winnable by Theorem 2.1 and Theorem 2.3. □

IV. Complete Graphs

We now turn our attention to complete graphs. A *complete graph* is a "simple graph in which each two vertices are adjacent." The complete graph with n vertices is denoted K_n. These graphs have the nice inductive quality that if you delete any vertex you get another complete graph.

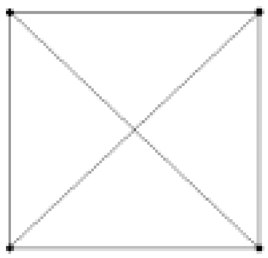


Figure 4.1: an example of a complete graph on 4 vertices, K_4

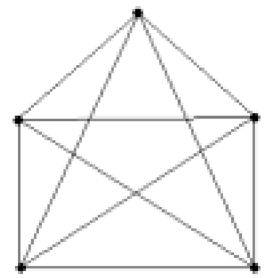


Figure 4.2: an example of a complete graph on 5 vertices, K_5

We present the following theorem about the winnability of this type of graph.

Theorem: *For $n \equiv 0 \pmod 3$: $(K_n, c) \notin W$ if and only if $\mathbf{0}_c$, $\mathbf{1}_c$, or $\mathbf{2}_c \geq \frac{2n+3}{3}$.*

For $n \equiv 1 \pmod 3$: $(K_n, c) \notin W$ if and only if $\mathbf{0}_c \geq \frac{2n+1}{3}$ or $\mathbf{1}_c$ or $\mathbf{2}_c > \frac{2n+1}{3}$.

For $n \equiv 2 \pmod 3$: $(K_n, c) \notin W$ if and only if $\mathbf{0}_c$ or $\mathbf{2}_c \geq \frac{2n+2}{3}$ or $\mathbf{1}_c > \frac{2n+2}{3}$.

Proof: We proceed by induction on the number of vertices.

BASE CASES: The cases where $n = 1, 2$, and 3 are easily shown.

INDUCTIVE STEP:

Case 1: $n \equiv 0 \pmod 3$.

Assume the statement is true for $n = 3, 6, \ldots, 3k$ for some integer k. Consider any (K_{3k+3}, c). We first suppose that $\mathbf{0}_c$, $\mathbf{1}_c$, or $\mathbf{2}_c \geq \frac{2(3k+3)+3}{3}$ and show that $(K_{3k+3}, c) \notin$ W. Let A = $\mathbf{0}_c$ and assume that A $\geq \frac{2(3k+3)+3}{3}$. It is not hard to show that unless $n \leq 3$ or one of $\mathbf{0}_c$, $\mathbf{1}_c$, or $\mathbf{2}_c$ equals n, then we can make a sequence of three moves. If A = $3k+3$, then we are done when $k > 1$ since A = $\mathbf{0}$. If not, we make three arbitrary moves and show that the remaining graph and coloring are unwinnable. Let A′ be the new number of vertices of color 0 after making three moves. The minimum value that A′ can have is obtained when two of our moves were made inside the A-many vertices. Note that we cannot move three times in the A-many vertices because the first move cannot be among them. (Note later, when A = $\boldsymbol{1}_c$ or $\boldsymbol{2}_c$ we still will have color 0 at some point in making the three moves.) So,

$$A' \geq A - 2 \geq \frac{2(3k+3)+3}{3} - 2 = \frac{2(3k)+3}{3}.$$

Since $A' \geq \frac{2(3k)+3}{3}$, we can apply the induction hypothesis, and thus, $(K_{3k+3}, c) \notin W$. The proof follows similarly if $\boldsymbol{1}_c$ or $\boldsymbol{2}_c$ is greater than or equal to $\frac{2(3k+3)+3}{3}$.

To prove the converse of the statement, we prove the contrapositive. Let Z be the maximum number of vertices of the same color in (K_{3k+3}, c) and let Z' be the maximum number of vertices of the same color in (K_{3k}, c'), the resulting complete graph after making three moves in a manner specified later. Consider (K_{3k+3}, c) such that $Z < \frac{2(3k+3)+3}{3}$.

Subcase 1: Suppose that there is the same number Z of vertices of each color in (K_{3k+3}, c). We click on a vertex of color 1 in each of our three moves. We have then removed one vertex from each of the original color sets. So Z' is equal to $Z - 1$. Since the sizes of each of the original sets were equal, they were all of size $Z = \frac{3k+3}{3}$. Therefore,

$$Z' = Z - 1 = \frac{3k+3}{3} - 1 = \frac{3k}{3} < \frac{2(3k)+3}{3}.$$

Since $Z' < \frac{2(3k)+3}{3}$, we can apply the induction hypothesis and obtain $(K_{3k+3}, c) \in W$.

Subcase 2: Suppose there is the same number Z of vertices of two colors in (K_{3k+3}, c) with less than Z of the third color. Then $Z \leq \frac{3k+3}{2}$. We make three moves, moving twice in one set of Z-many and once in the other set of Z-many so as to drop Z by one. Therefore,

$$Z' = Z - 1 \leq \frac{3k+3}{2} - 1 < \frac{2(3k)+3}{3}.$$

Since $Z' < \frac{2(3k)+3}{3}$, we can apply the induction hypothesis. Therefore $(K_{3k+3}, c) \in W$.

Subcase 3: Suppose there are Z-many of only one color in (K_{3k+3}, c). The sizes for the other two color sets are one set of at most $Z - 1$ and the remaining set being of at most $Z - 2$ (since $n \equiv$

0(mod 3)). Therefore $Z \leq k+2$. We click twice in the Z-many and once in a color set with the next highest size. So,

$$Z' \leq Z-2 \leq k+2-2 = k < \frac{2(3k)+3}{3}.$$

Since $Z' < \frac{2(3k)+3}{3}$, we apply the induction hypothesis, and therefore $(K_{3k+3}, c) \in W$.

Case 2: $n \equiv 1 \pmod 3$.

Assume the statement is true for $n = 1,4,\ldots,3k+1$ for some integer k. Consider (K_{3k+4}, c). We first suppose that $\boldsymbol{0}_c \geq \frac{2(3k+4)+1}{3}$ and show that $(K_{3k+4}, c) \notin W$. If $\boldsymbol{0}_c = 3k+4$, we are done since $c = \mathbf{0}$. If $\boldsymbol{0}_c < 3k+4$, we make three arbitrary moves and show that the remaining graph and coloring are unwinnable. Because the maximum number of moves we can make inside the $\boldsymbol{0}_c$ many vertices is two, $\boldsymbol{0}_{c'} \geq \boldsymbol{0}_c - 2$, where $\boldsymbol{0}_{c'}$ is the number of vertices of color 0 in K_{3k+1}. So,

$$\boldsymbol{0}_{c'} \geq \boldsymbol{0}_c - 2 \geq \frac{2(3k+4)+1}{3} - 2 = \frac{2(3k+1)+1}{3}.$$

Therefore, since $\boldsymbol{0}_{c'} \geq \frac{2(3k+1)+1}{3}$, we can apply the induction hypothesis and obtain that $(K_{3k+4}, c) \notin W$.

Now suppose that $\boldsymbol{1}_c$ or $\boldsymbol{2}_c > \frac{2(3k+4)+1}{3}$. Let $\mathrm{A} = \boldsymbol{1}_c$ and assume that $\mathrm{A} > \frac{2(3k+4)+1}{3}$. If $\mathrm{A} = 3k+4$, we are done since after making two moves, all the remaining vertices have color 0. If $\mathrm{A} < 3k+4$, we make three moves. Because the maximum number of moves we can make inside the A-many vertices is two, $\mathrm{A}' \geq \mathrm{A} - 2$. Therefore,

$$\mathrm{A}' \geq \mathrm{A} - 2 > \frac{2(3k+4)+1}{3} - 2 = \frac{2(3k+1)+1}{3}.$$

Thus, since $\mathrm{A}' > \frac{2(3k+1)+1}{3}$, we can apply the induction hypothesis and get that $(K_{3k+4}, c) \notin W$.

The proof follows similarly for $\boldsymbol{2}_c > \frac{2(3k+4)+1}{3}$.

Conversely, we assume that $\boldsymbol{0}_c < \frac{2(3k+4)+1}{3}$ and $\boldsymbol{1}_c$ and $\boldsymbol{2}_c \leq \frac{2(3k+4)+1}{3}$ and show that $(K_{3k+4}, c) \in W$. There are three cases.

Subcase 1: Suppose $\boldsymbol{0}_c$, $\boldsymbol{1}_c < \frac{2(3k+4)+1}{3}$ and $\boldsymbol{2}_c = \frac{2(3k+4)+1}{3}$. Since $\boldsymbol{2}_c = \frac{2(3k+4)+1}{3}$, either $\boldsymbol{0}_c$ or $\boldsymbol{1}_c \leq \frac{(3k+4)-1}{3}$. Note that both $\boldsymbol{0}_c$ and $\boldsymbol{1}_c$ cannot equal $\frac{(3k+4)-1}{3}$. So assume that $\boldsymbol{0}_c \leq \frac{(3k+4)-1}{3}$ and $\boldsymbol{1}_c < \frac{(3k+4)-1}{3}$. We click three times: twice in the $\boldsymbol{2}_c$-many and once in the $\boldsymbol{0}_c$-many. So, in the remaining colored graph we will have

$$\boldsymbol{0}_{c'}, \leq \frac{(3k+4)-1}{3} - 1 < \frac{2(3k+1)+1}{3}.$$

Also,

$$\boldsymbol{1}_{c'}, < \frac{(3k+4)-1}{3} < \frac{2(3k+1)+1}{3}.$$

Finally,

$$\boldsymbol{2}_{c'} = \frac{2(3k+4)+1}{3} - 2 = \frac{6k+3}{3} = \frac{2(3k+1)+1}{3}.$$

Since $\boldsymbol{0}_{c'}$, $\boldsymbol{1}_{c'} < \frac{2(3k+1)+1}{3}$ and $\boldsymbol{2}_{c'} \leq \frac{2(3k+1)+1}{3}$, we apply the induction hypothesis and obtain that $(K_{3k+4}, c) \in W$. The proof follows similarly if $\boldsymbol{1}_c \leq \frac{(3k+4)-1}{3}$ and $\boldsymbol{0}_c < \frac{(3k+4)-1}{3}$.

Subcase 2: Let $\boldsymbol{0}_c$, $\boldsymbol{2}_c < \frac{2(3k+4)+1}{3}$ and $\boldsymbol{1}_c = \frac{2(3k+4)+1}{3}$. This case follows similar to subcase 1.

Subcase 3: Let $\boldsymbol{0}_c$, $\boldsymbol{1}_c$, $\boldsymbol{2}_c < \frac{2(3k+4)+1}{3}$. Since n ≡ 1 (mod 3) we have two cases.

(*i*). Suppose there is the same number Z of vertices of two colors in (K_{3k+4}, c). (Note that there is less than Z of the third color since $n \equiv 1 \pmod 3$). Then $Z \leq \frac{3k+4}{2}$. We move twice in one set of Z-many and once in the other set of Z-many. Then

$$Z' \leq Z - 1 \leq \frac{3k+4}{2} - 1 < \frac{2(3k+1)+1}{3}.$$

Since $Z' < \frac{2(3k+1)+1}{3}$, we apply the induction hypothesis and obtain that $(K_{3k+4}, c) \in W$.

(ii). Suppose there are Z-many of only one color in (K_{3k+4}, c). The sizes for the other two color sets are at most $Z - 1$. So $Z \leq \frac{3k+6}{3}$. We click on a vertex of color 1 each time so as to reduce Z by one. Therefore,

$$Z' \leq Z - 1 \leq \frac{3k+6}{3} - 1 < \frac{2(3k+1)+1}{3}.$$

Since $Z' < \frac{2(3k+1)+1}{3}$, we can apply the induction hypothesis. Thus $(K_{3k+4}, c) \in W$.

Case 3: $n \equiv 2 \pmod 3$.

Assume the statement is true for $n = 2, 5, \ldots, 3k+2$ for some integer k. Consider (K_{3k+5}, c). We first suppose that $\boldsymbol{0}_c$ or $\boldsymbol{2}_c \geq \frac{2(3k+5)+2}{3}$ and show that $(K_{3k+5}, c) \notin W$. Let $A = \boldsymbol{0}_c$ and assume $A \geq \frac{2(3k+5)+2}{3}$. If $A = 3k+5$, we are done. If not, we make three arbitrary moves and show that the remaining graph and coloring are unwinnable. Let A' be the number of vertices of color 0 after making the three moves. The minimum value that A' can be is $A - 2$, which is obtained when two of our three moves were made inside the A-many vertices. So,

$$A' \geq A - 2 \geq \frac{2(3k+5)+2}{3} - 2 = \frac{2(3k+2)+2}{3}.$$

Since $A' \geq \frac{2(3k+2)+2}{3}$, we can apply the induction hypothesis. Therefore $(K_{3k+5}, c) \notin W$. The proof follows similarly if $\boldsymbol{2}_c \geq \frac{2(3k+5)+2}{3}$.

Now suppose $\boldsymbol{1}_c > \frac{2(3k+5)+2}{3}$. We show that $(K_{3k+5}, c) \notin W$. If $\boldsymbol{1}_c = 3k+5$, we are done. If not, we make three arbitrary moves and show the remaining graph and coloring are unwinnable. Since we can move at most two times inside the $\boldsymbol{1}_c$-many vertices,

$$\boldsymbol{1}_{c'} \geq \boldsymbol{1}_c - 2 > \frac{2(3k+5)+2}{3} - 2 = \frac{2(3k+2)+2}{3}.$$

We apply the induction hypothesis, since $\boldsymbol{1}_{c'} > \frac{2(3k+2)+2}{3}$. Thus, $(K_{3k+5}, c) \notin W$.

Conversely, we assume that $\boldsymbol{0}_c$ and $\boldsymbol{2}_c < \frac{2(3k+5)+2}{3}$ and $\boldsymbol{1}_c \leq \frac{2(3k+5)+2}{3}$ and show that $(K_{3k+5}, c) \in W$. There are two cases.

Subcase 1: Let $\boldsymbol{0}_c, \boldsymbol{2}_c < \frac{2(3k+5)+2}{3}$ and $\boldsymbol{1}_c = \frac{2(3k+5)+2}{3}$. Since $\boldsymbol{1}_c = \frac{2(3k+5)+2}{3}$, either $\boldsymbol{0}_c$ or $\boldsymbol{2}_c \leq \frac{(3k+5)-2}{3}$. Note that both $\boldsymbol{0}_c$ and $\boldsymbol{2}_c$ cannot equal $\frac{(3k+5)-2}{3}$. So assume that $\boldsymbol{0}_c \leq \frac{(3k+5)-2}{3}$ and $\boldsymbol{2}_c < \frac{(3k+5)-2}{3}$. We click three times: twice in the $\boldsymbol{1}_c$-many and once in the $\boldsymbol{0}_c$-many. So, in the remaining colored graph we will have

$$\boldsymbol{0}_{c'} \leq \frac{(3k+5)-2}{3} - 1 < \frac{2(3k+2)+2}{3}.$$

Also,

$$\boldsymbol{2}_{c'} < \frac{(3k+5)-2}{3} < \frac{2(3k+2)+2}{3}.$$

$$\boldsymbol{1}_{c'} = \boldsymbol{1}_c - 2 = \frac{2(3k+5)+2}{3} - 2 = \frac{2(3k+2)+2}{3}.$$

Since $\boldsymbol{0}_{c'}, \boldsymbol{2}_{c'} < \frac{2(3k+2)+2}{3}$ and $\boldsymbol{1}_{c'} \leq \frac{2(3k+2)+2}{3}$, we can apply the induction hypothesis and obtain that $(K_{3k+5}, c) \in W$. The proof follows similarly if $\boldsymbol{2}_c \leq \frac{(3k+5)-2}{3}$ and $\boldsymbol{0}_c < \frac{(3k+5)-2}{3}$.

Subcase 2: Let $\boldsymbol{0}_c, \boldsymbol{1}_c, \boldsymbol{2}_c < \frac{2(3k+5)+2}{3}$. Since $n \equiv 2 \pmod 3$, we have two cases.

(i). Suppose there is the same number Z of vertices of two colors in (K_{3k+5}, c). (Note that there is less than Z of the third color since $n \equiv 2 \pmod 3$). Then $Z \leq \frac{3k+5}{2}$. We move twice in one set of Z-many and once in the other set of Z-many. So,

$$Z' \leq Z - 1 \leq \frac{3k+5}{2} - 1 < \frac{2(3k+2)+2}{3}.$$

Since $Z' < \dfrac{2(3k+2)+2}{3}$, we apply the induction hypothesis. Therefore, $(K_{3k+5}, c) \in W$.

(*ii*). Suppose there are Z-many of only one color in (K_{3k+5}, c). The sizes for the other two color sets are either at most $Z - 2$, or one of size $Z - 1$ and the other of size at most $Z - 3$ (since $n \equiv 2 \pmod 3$). So $Z \le \dfrac{3k+9}{3}$. We click twice in the Z-many vertices and once in a set of the next highest size. Therefore,

$$Z' \le Z - 2 \le \frac{3k+9}{3} - 2 < \frac{2(3k+2)+2}{3}.$$

Since $Z' < \dfrac{2(3k+2)+2}{3}$, we apply the induction hypothesis and get that $(K_{3k+5}, c) \in W$. ☐

V. *X*-Coloring Modulo 3

Perhaps one of our most exciting finds, the discovery of the ***X***-coloring modulo 3, helps us to give an example of an unwinnable coloring for any given graph. A graph G has the ***X***-*coloring* if for each vertex v,

$$\boldsymbol{X}(v) = \begin{cases} 0, & \text{if } \deg(v) \equiv 0 \pmod 3 \\ 1, & \text{if } \deg(v) \equiv 2 \pmod 3 \\ 2, & \text{if } \deg(v) \equiv 1 \pmod 3 \end{cases}.$$

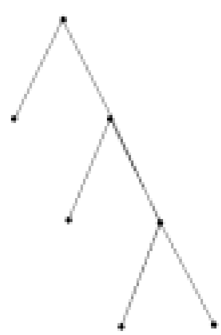


Figure 2.1: an example of a tree with 7 vertices, H_7

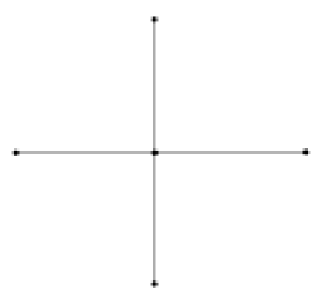


Figure 2.2: an example of a tree with 5 vertices, H_5

Figure 3.1: an example of a path with 6 vertices, P_6

We begin by exploring ways to generate unwinnable colorings for a given tree H_n. We find two procedures that once applied to any 3-colored tree do not change its winnability.

Theorem 2.1: *When we attach a leaf of color 0 to any colored tree (H_n, c) to produce a colored tree (H_{n+1}, c'), $(H_n, c) \in W$ if and only if $(H_{n+1}, c') \in W$. We call the process of adding a zero leaf to a 3-colored tree the* <u>*0-procedure*</u>.

Proof: Suppose $(H_n, c) \in W$. Consider (H_{n+1}, c'). We show that (H_{n+1}, c') is also winnable. Apply the winning sequence to H_n. This changes the color of v once, since v is a leaf; so, the color of v is now 1. Thus, after "winning" on the subgraph H_n, we can click on v, and therefore (H_{n+1}, c') is winnable.

Suppose for contradiction that (H_n, c) is not winnable and (H_{n+1}, c) is winnable. Then we can assume that there is a winning sequence of moves for (H_{n+1}, c). Now the vertex adjacent to v must be clicked on before v; however, once this happens, v is isolated and no longer affects the rest of the graph, so it does not matter when v is clicked on. Then without loss of generality, assume v is the last vertex to be clicked on in the winning sequence of moves. Then the graph without v, which is (H_n, c) would have to be winnable, which is a contradiction. So, (H_{n+1}, c) is not winnable. ☐

Theorem 2.2: *When we add 2 (mod 3) to any vertex v of a colored tree (H_n, c) and attach a leaf v_1 of color 2 to v, the resulting colored tree (H_{n+1}, c') is unwinnable if and only if (H_n, c) is unwinnable. We call "this" process the* <u>2,2-procedure</u>.

Proof: Consider any $(H_n, c) \notin W$, and let (H_{n+1}, c') be the result of applying the 2,2-procedure. Assume for contradiction that $(H_{n+1}, c') \in W$. Then there is some winning strategy for (H_{n+1}, c'). Since clicking on v before clicking on v_1 results in isolating v_1 with a color of 0, that strategy must click on v_1 before v. Also, clicking on v_1 only affects v. So, there exists a winning strategy in which we click on v_1 first. The resulting colored graph is (H_n, c). (H_n, c) has a winning strategy (following the remainder of our strategy for (H_n+1, c)). However, this is a contradiction because H_n is unwinnable. Thus, our assumption is false, and $(H_{n+1},c') \notin W$.

Now consider any $(H_n, c) \in W$ and its associated (H_{n+1},c') via the 2,2 procedure. Click on v_1. The resulting graph is (H_n, c), which is winnable. So, $(H_{n+1},c') \in W$. □

We make the following basic observation while examining characteristics of winnable trees.

Theorem 2.3: *If H is a tree and (H_n, c) has a leaf of color 1, then H is winnable.*

Proof: We proceed by induction on the number of vertices.

BASE CASE: The case where $n = 2$ is easily verified.

INDUCTIVE STEP: Assume the statement is true for $n = 2,3,\ldots,k$ for some integer $k \geq 2$. Consider any tree H_{k+1} on $k+1$ vertices that has at least one leaf x of color 1. H_{k+1} has at least one other leaf, and all leaves other than x are not adjacent with x.

Case 1: Suppose there are at least two non-zero leaves. Click on any such leaf other than x, so that x is a leaf of color 1 in the colored tree H_k remaining. By the induction hypothesis, what remains is winnable, so the original graph is winnable.

Case 2: Suppose x is the only non-zero leaf. We look at a subtree H_k by disregarding one of the leaves of color 0. This subtree has at least one leaf of color 1, and thus H_k is winnable by the induction hypothesis. Since the colored tree H_{k+1} can be obtained from the colored H_k by the 0 procedure, by Theorem 2.1, the original graph is winnable. □

Using the procedures described in the first two theorems of this section, we are able to determine the exact number of unwinnable initial configurations for any given tree.

Theorem 2.4: *There are exactly 2^{n-1} unwinnable initial colorings for a given vertex labeled tree H_n on n vertices.*

Proof: We proceed by induction on the number of vertices.

BASE CASE: The case where $n = 1$ is easily shown.

INDUCTIVE STEP: Assume the statement is true for $n = 1,2,\ldots,k$ for some integer k. Consider the case where $n = k+1$. We choose a particular vertex v in a tree H_{k+1} with $k+1$ vertices and count the number of unwinnable initial colorings. The color of v cannot be 1 in an unwinnable initial coloring, by Theorem 2.3. Therefore, we consider the cases where the color of v is 0 or 2.

First we count the number of unwinnable initial colorings of H_{k+1} in which v has color 0. By Theorem 2.1, any such coloring c of H_{k+1} is unwinnable if and only if the colored graph formed by deleting v is unwinnable. By the induction hypothesis, there are exactly 2^{k-1} unwinnable colorings for that subgraph, each corresponding by the 0-procedure to just one unwinnable coloring of H_{k+1}. Therefore, the number of unwinnable colorings c in which $c(v) = 0$ is 2^{k-1}.

Next we count the number of unwinnable initial colorings of H_{k+1} in which v has color 2. By Theorem 2.2, any such coloring c of H_{k+1} is unwinnable if and only if the colored graph formed by performing the 2,2-procedure "in reverse" is unwinnable. By the induction hypothesis, there are exactly 2^{k-1} unwinnable colorings for that subgraph, each corresponding by the 2,2-procedure to just one unwinnable coloring of H_{k+1}. Therefore, the number of unwinnable colorings c in which $c(v) = 2$ is 2^{k-1}. Thus the total number of unwinnable colorings c for H_{k+1} is $2^{k-1} + 2^{k-1} = 2^k$, as desired. □

All of the above theorems for trees hold for paths as well, because paths are a specific type of tree. In addition to those theorems, we found one result that was unique to paths and could not be generalized to trees.

Theorem 2.5: *If (P_n, c) is not winnable, then $\mathbf{2}_c$ is even.*

Proof: We proceed by induction on the number of vertices in P_n.

BASE CASE: The case where $n = 1$ is easily shown.

INDUCTIVE STEP: Assume the statement is true for $n = 1,2,\ldots,k$ for some integer k. Consider the case where $n = k+1$, with (P_{k+1}, c) unwinnable.

Case 1: Let the first vertex v have color 0. The path on the remaining k vertices is unwinnable by Theorem 2.1. So there is an even number of vertices of color 2 by the induction hypothesis. Including v with color 0 does not change this number.

Case 2: Let the first vertex v have color 2. The second vertex w has to have color 1 or 2, because if it had color 0, clicking on v would give us a path with the first vertex of color 1, which is always winnable.

Suppose w has color 1. Click on v, in which case w becomes of color 2 and we obtain a path with k vertices. By the induction hypothesis and the 2,2-procedure, there is an even number of vertices of color 2 in P_k. This is exactly the same number of vertices of color 2 as in P_{k+1} because we deleted v, a vertex of color 2, but we added another vertex with color 2 when we changed the color of w after clicking on v.

Now, suppose w has color 2. We click on v, in which case the color of w will change to 0 and we have an unwinnable path with k vertices. By the induction hypothesis, there is an even number of vertices of color 2 in P_k. Since we deleted v, a vertex of color 2, in P_{k+1} and changed the color of w from 2 to 0 when we clicked on v, the original number of vertices of color 2 must have been even. □

III. Undirected Cycles

The next graph we will discuss is the undirected cycle. This follows easily from our observations about paths, since clicking on a vertex in a cycle results in a path. A *cycle* is a "simple graph whose vertices can be placed on a circle so that vertices are adjacent if an only if they appear consecutively on the circle." The cycle on n vertices is denoted C_n. We include two lemmas concerning paths that will be used in the proof of our theorem about the winnability of cycles.

Lemma 3.1: *Every path P_n with coloring 211…112 is unwinnable.*

Proof: We proceed by induction on the number of vertices.

BASE CASE: The case where $n = 2$ is easily shown.

INDUCTIVE STEP: Assume the statement is true for $n = 2,3,\ldots,k$ for some integer k. Consider the case when $n = k+1$.

Case 1: Click on one of the end vertices v. This removes v and changes the adjacent vertex's color to 2 (since its original color was 1). So we have a path of length k with the coloring 211…112, which is unwinnable by the induction hypothesis.
Case 2: Click on a vertex of color 1 that is adjacent to a vertex of color 2. This will turn the color of the end vertex to color 0 and isolate it, so this coloring is unwinnable.
Case 3: Click on any vertex v of color 1 that is not adjacent to a vertex of color 2. This removes v and changes the color of its two adjacent vertices to 2. We now have two components with coloring 211…112 that are paths with s and $k - s$ vertices, where $s \leq k$. If $s = k$, we are back in case 2. If not, these components are unwinnable by the induction hypothesis. □

Lemma 3.2: *Every path P_n with an even number of vertices and coloring 022…220 is unwinnable.*

Proof: We proceed by induction on the number of vertices.
BASE CASE: The case where $n = 2$ is easily shown.
INDUCTIVE STEP: Assume that the claim is true for $n = 2,3,\ldots,k$ for some integer k and consider the case where $n = k+1$.
Case 1: Click on a vertex v of color 2 adjacent to a vertex of color 0. Then we will have two components: a path with $k-2$ vertices and a singleton vertex of color 1. By clicking on v we deleted one vertex of color 2 and changed one vertex of color 2 to color 0. So we still have an even number of vertices of color 2 in the path of $k-2$ vertices. Thus the graph is unwinnable by the induction hypothesis.
Case 2: Click on a vertex v of color 2 that is not adjacent to a vertex of color 0. By doing this, the original path is divided into two components and the two neighbors of v now have color 0. Since we clicked on a vertex of color two and changed the colors of its neighbors from 2 to 0, we now have an odd number of vertices of color two. This means that at least one of the components of the new graph has an even number of vertices of color 2. Its total number of vertices is less than $k+1$, so we can apply the induction hypothesis and conclude that this component is unwinnable, making the original path on $k+1$ vertices also unwinnable. □

We are now ready to determine when a cycle is winnable.

Theorem 3.1: *(C_n, c) is unwinnable if and only if $c = \mathbf{0}$ or $c = \mathbf{1}$ or [$c = \mathbf{2}$ when $n \equiv 1 \pmod 2$].*

Proof: We first show that if the coloring is $\mathbf{0}$, $\mathbf{1}$, or [$\mathbf{2}$ where $n \equiv 1 \pmod 2$], then (C_n, c) is unwinnable. If $c = \mathbf{0}$, we have no starting vertex to click on, therefore (C_n, c) is unwinnable. If $c = \mathbf{1}$, then click on any vertex. We then get a path whose coloring is 211…112, which is unwinnable by Lemma 4.1. Finally, if $c = \mathbf{2}$ and $n \equiv 1 \pmod 2$, then we get a path with coloring 022…220, where there are an even number of vertices. This is unwinnable by Lemma 4.2.

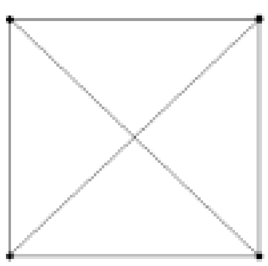
Conversely, assume that the coloring c is not $\mathbf{0}$, $\mathbf{1}$, or [$\mathbf{2}$ where $n \equiv 1 \pmod 2$]. We show that (C_n, c) is winnable. Well, if the cycle has a vertex of color 0, then since not all the vertices are of color 0 we can click on a non-zero vertex that is adjacent to a vertex of color 0. We then get a path with an end vertex of color 1, which is winnable by Theorem 2.3.

If the cycle has an even number of vertices and $c = \mathbf{2}$, then when we click on a vertex v, we delete v, which is of color 2, and we change the two adjacent vertices of v to color 0. Therefore we now have an odd number of vertices of color 2 in the resulting path, which is winnable by Theorem 2.5.

Finally, if the cycle has no vertices of color 0, but the vertices are not all of color 1 and not all of color 2, then we will have a sequence of consecutive vertices v, x, y, where $c(x) = 2$ and $c(y) = 1$. We click on v (which is non-zero) and we get a path where the first 2 vertices are of color 0 and 1 respectively. This path is winnable by Theorem 2.1 and Theorem 2.3. □

IV. Complete Graphs

We now turn our attention to complete graphs. A *complete graph* is a "simple graph in which each two vertices are adjacent." The complete graph with n vertices is denoted K_n. These graphs have the nice inductive quality that if you delete any vertex you get another complete graph.



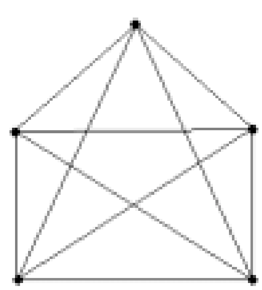


Figure 4.1: an example of a complete graph on 4 vertices, K_4

Figure 4.2: an example of a complete graph on 5 vertices, K_5

We present the following theorem about the winnability of this type of graph.

Theorem: *For $n \equiv 0 \pmod 3$: $(K_n, c) \notin W$ if and only if $\boldsymbol{0}_c$, $\boldsymbol{1}_c$, or $\boldsymbol{2}_c \geq \frac{2n+3}{3}$.*

For $n \equiv 1 \pmod 3$: $(K_n, c) \notin W$ if and only if $\boldsymbol{0}_c \geq \frac{2n+1}{3}$ or $\boldsymbol{1}_c$ or $\boldsymbol{2}_c > \frac{2n+1}{3}$.

For $n \equiv 2 \pmod 3$: $(K_n, c) \notin W$ if and only if $\boldsymbol{0}_c$ or $\boldsymbol{2}_c \geq \frac{2n+2}{3}$ or $\boldsymbol{1}_c > \frac{2n+2}{3}$.

Proof: We proceed by induction on the number of vertices.

BASE CASES: The cases where $n = 1$, 2, and 3 are easily shown.

INDUCTIVE STEP:

Case 1: $n \equiv 0 \pmod 3$.

Assume the statement is true for $n = 3, 6, \ldots, 3k$ for some integer k. Consider any (K_{3k+3}, c). We first suppose that $\boldsymbol{0}_c$, $\boldsymbol{1}_c$, or $\boldsymbol{2}_c \geq \frac{2(3k+3)+3}{3}$ and show that $(K_{3k+3}, c) \notin$ W. Let A = $\boldsymbol{0}_c$ and assume that A $\geq \frac{2(3k+3)+3}{3}$. It is not hard to show that unless $n \leq 3$ or one of $\boldsymbol{0}_c$, $\boldsymbol{1}_c$, or $\boldsymbol{2}_c$ equals n, then we can make a sequence of three moves. If A = $3k+3$, then we are done when $k > 1$ since A = **0**. If not, we make three arbitrary moves and show that the remaining graph and coloring are unwinnable. Let A′ be the new number of vertices of color 0 after making three moves. The minimum value that A′ can have is obtained when two of our moves were made inside the A-many vertices. Note that we cannot move three times in the A-many vertices because the first move cannot be among them. (Note later, when A = $\boldsymbol{1}_c$ or $\boldsymbol{2}_c$ we still will have color 0 at some point in making the three moves.) So,

$$A' \geq A - 2 \geq \frac{2(3k+3)+3}{3} - 2 = \frac{2(3k)+3}{3}.$$

Since $A' \geq \dfrac{2(3k)+3}{3}$, we can apply the induction hypothesis, and thus, $(K_{3k+3}, c) \notin W$. The proof follows similarly if $\boldsymbol{1}_c$ or $\boldsymbol{2}_c$ is greater than or equal to $\dfrac{2(3k+3)+3}{3}$.

To prove the converse of the statement, we prove the contrapositive. Let Z be the maximum number of vertices of the same color in (K_{3k+3}, c) and let Z' be the maximum number of vertices of the same color in (K_{3k}, c'), the resulting complete graph after making three moves in a manner specified later. Consider (K_{3k+3}, c) such that $Z < \dfrac{2(3k+3)+3}{3}$.

Subcase 1: Suppose that there is the same number Z of vertices of each color in (K_{3k+3}, c). We click on a vertex of color 1 in each of our three moves. We have then removed one vertex from each of the original color sets. So Z' is equal to $Z - 1$. Since the sizes of each of the original sets were equal, they were all of size $Z = \dfrac{3k+3}{3}$. Therefore,

$$Z' = Z - 1 = \frac{3k+3}{3} - 1 = \frac{3k}{3} < \frac{2(3k)+3}{3}.$$

Since $Z' < \dfrac{2(3k)+3}{3}$, we can apply the induction hypothesis and obtain $(K_{3k+3}, c) \in W$.

Subcase 2: Suppose there is the same number Z of vertices of two colors in (K_{3k+3}, c) with less than Z of the third color. Then $Z \leq \dfrac{3k+3}{2}$. We make three moves, moving twice in one set of Z-many and once in the other set of Z-many so as to drop Z by one. Therefore,

$$Z' = Z - 1 \leq \frac{3k+3}{2} - 1 < \frac{2(3k)+3}{3}.$$

Since $Z' < \dfrac{2(3k)+3}{3}$, we can apply the induction hypothesis. Therefore $(K_{3k+3}, c) \in W$.

Subcase 3: Suppose there are Z-many of only one color in (K_{3k+3}, c). The sizes for the other two color sets are one set of at most $Z - 1$ and the remaining set being of at most $Z - 2$ (since $n \equiv 0 (\text{mod } 3)$). Therefore $Z \leq k + 2$. We click twice in the Z-many and once in a color set with the next highest size. So,

$$Z' \leq Z - 2 \leq k + 2 - 2 = k < \frac{2(3k)+3}{3}.$$

Since $Z' < \dfrac{2(3k)+3}{3}$, we apply the induction hypothesis, and therefore $(K_{3k+3}, c) \in W$.

Case 2: $n \equiv 1 \pmod 3$.

Assume the statement is true for $n = 1,4,\ldots,3k+1$ for some integer k. Consider (K_{3k+4}, c). We first suppose that $\boldsymbol{0}_c \geq \dfrac{2(3k+4)+1}{3}$ and show that $(K_{3k+4}, c) \notin W$. If $\boldsymbol{0}_c = 3k+4$, we are done since $c = \mathbf{0}$. If $\boldsymbol{0}_c < 3k+4$, we make three arbitrary moves and show that the remaining graph and coloring are unwinnable. Because the maximum number of moves we can make inside the $\boldsymbol{0}_c$ many vertices is two, $\boldsymbol{0}_{c'} \geq \boldsymbol{0}_c - 2$, where $\boldsymbol{0}_{c'}$ is the number of vertices of color 0 in K_{3k+1}. So,

$$\boldsymbol{0}_{c'} \geq \boldsymbol{0}_c - 2 \geq \frac{2(3k+4)+1}{3} - 2 = \frac{2(3k+1)+1}{3}.$$

Therefore, since $\boldsymbol{0}_{c'} \geq \dfrac{2(3k+1)+1}{3}$, we can apply the induction hypothesis and obtain that $(K_{3k+4}, c) \notin W$.

Now suppose that $\boldsymbol{1}_c$ or $\boldsymbol{2}_c > \dfrac{2(3k+4)+1}{3}$. Let $A = \boldsymbol{1}_c$ and assume that $A > \dfrac{2(3k+4)+1}{3}$. If $A = 3k+4$, we are done since after making two moves, all the remaining vertices have color 0. If $A < 3k+4$, we make three moves. Because the maximum number of moves we can make inside the A-many vertices is two, $A' \geq A - 2$. Therefore,

$$A' \geq A - 2 > \frac{2(3k+4)+1}{3} - 2 = \frac{2(3k+1)+1}{3}.$$

Thus, since $A' > \dfrac{2(3k+1)+1}{3}$, we can apply the induction hypothesis and get that $(K_{3k+4}, c) \notin W$. The proof follows similarly for $\boldsymbol{2}_c > \dfrac{2(3k+4)+1}{3}$.

Conversely, we assume that $\boldsymbol{0}_c < \dfrac{2(3k+4)+1}{3}$ and $\boldsymbol{1}_c$ and $\boldsymbol{2}_c \leq \dfrac{2(3k+4)+1}{3}$ and show that $(K_{3k+4}, c) \in W$. There are three cases.

Subcase 1: Suppose $\boldsymbol{0}_c$, $\boldsymbol{1}_c < \frac{2(3k+4)+1}{3}$ and $\boldsymbol{2}_c = \frac{2(3k+4)+1}{3}$. Since $\boldsymbol{2}_c = \frac{2(3k+4)+1}{3}$, either $\boldsymbol{0}_c$ or $\boldsymbol{1}_c \leq \frac{(3k+4)-1}{3}$. Note that both $\boldsymbol{0}_c$ and $\boldsymbol{1}_c$ cannot equal $\frac{(3k+4)-1}{3}$. So assume that $\boldsymbol{0}_c \leq \frac{(3k+4)-1}{3}$ and $\boldsymbol{1}_c < \frac{(3k+4)-1}{3}$. We click three times: twice in the $\boldsymbol{2}_c$-many and once in the $\boldsymbol{0}_c$-many. So, in the remaining colored graph we will have

$$\boldsymbol{0}_{c'}, \leq \frac{(3k+4)-1}{3} - 1 < \frac{2(3k+1)+1}{3}.$$

Also,

$$\boldsymbol{1}_{c'}, < \frac{(3k+4)-1}{3} < \frac{2(3k+1)+1}{3}.$$

Finally,

$$\boldsymbol{2}_{c'} = \frac{2(3k+4)+1}{3} - 2 = \frac{6k+3}{3} = \frac{2(3k+1)+1}{3}.$$

Since $\boldsymbol{0}_{c'}$, $\boldsymbol{1}_{c'} < \frac{2(3k+1)+1}{3}$ and $\boldsymbol{2}_{c'} \leq \frac{2(3k+1)+1}{3}$, we apply the induction hypothesis and obtain that $(K_{3k+4}, c) \in W$. The proof follows similarly if $\boldsymbol{1}_c \leq \frac{(3k+4)-1}{3}$ and $\boldsymbol{0}_c < \frac{(3k+4)-1}{3}$.

Subcase 2: Let $\boldsymbol{0}_c$, $\boldsymbol{2}_c < \frac{2(3k+4)+1}{3}$ and $\boldsymbol{1}_c = \frac{2(3k+4)+1}{3}$. This case follows similar to subcase 1.

Subcase 3: Let $\boldsymbol{0}_c$, $\boldsymbol{1}_c$, $\boldsymbol{2}_c < \frac{2(3k+4)+1}{3}$. Since n ≡ 1 (mod 3) we have two cases.

(*i*). Suppose there is the same number Z of vertices of two colors in (K_{3k+4}, c). (Note that there is less than Z of the third color since $n \equiv 1 \pmod 3$). Then $Z \leq \frac{3k+4}{2}$. We move twice in one set of Z-many and once in the other set of Z-many. Then

$$Z' \leq Z - 1 \leq \frac{3k+4}{2} - 1 < \frac{2(3k+1)+1}{3}.$$

Since $Z' < \frac{2(3k+1)+1}{3}$, we apply the induction hypothesis and obtain that $(K_{3k+4}, c) \in W$.

(ii). Suppose there are Z-many of only one color in (K_{3k+4}, c). The sizes for the other two color sets are at most $Z - 1$. So $Z \le \frac{3k+6}{3}$. We click on a vertex of color 1 each time so as to reduce Z by one. Therefore,

$$Z' \le Z - 1 \le \frac{3k+6}{3} - 1 < \frac{2(3k+1)+1}{3}.$$

Since $Z' < \frac{2(3k+1)+1}{3}$, we can apply the induction hypothesis. Thus $(K_{3k+4}, c) \in W$.

Case 3: $n \equiv 2 \pmod 3$.

Assume the statement is true for $n = 2, 5, \ldots, 3k+2$ for some integer k. Consider (K_{3k+5}, c). We first suppose that $\boldsymbol{0}_c$ or $\boldsymbol{2}_c \ge \frac{2(3k+5)+2}{3}$ and show that $(K_{3k+5}, c) \notin W$. Let A = $\boldsymbol{0}_c$ and assume A $\ge \frac{2(3k+5)+2}{3}$. If A $= 3k+5$, we are done. If not, we make three arbitrary moves and show that the remaining graph and coloring are unwinnable. Let A′ be the number of vertices of color 0 after making the three moves. The minimum value that A′ can be is A – 2, which is obtained when two of our three moves were made inside the A-many vertices. So,

$$\mathrm{A}' \ge \mathrm{A} - 2 \ge \frac{2(3k+5)+2}{3} - 2 = \frac{2(3k+2)+2}{3}.$$

Since A′ $\ge \frac{2(3k+2)+2}{3}$, we can apply the induction hypothesis. Therefore $(K_{3k+5}, c) \notin W$. The proof follows similarly if $\boldsymbol{2}_c \ge \frac{2(3k+5)+2}{3}$.

Now suppose $\boldsymbol{1}_c > \frac{2(3k+5)+2}{3}$. We show that $(K_{3k+5}, c) \notin W$. If $\boldsymbol{1}_c = 3k+5$, we are done. If not, we make three arbitrary moves and show the remaining graph and coloring are unwinnable. Since we can move at most two times inside the $\boldsymbol{1}_c$-many vertices,

$$\boldsymbol{1}_{c'} \ge \boldsymbol{1}_c - 2 > \frac{2(3k+5)+2}{3} - 2 = \frac{2(3k+2)+2}{3}.$$

We apply the induction hypothesis, since $\boldsymbol{1}_{c'} > \frac{2(3k+2)+2}{3}$. Thus, $(K_{3k+5}, c) \notin W$.

Conversely, we assume that $\boldsymbol{0}_c$ and $\boldsymbol{2}_c < \frac{2(3k+5)+2}{3}$ and $\boldsymbol{1}_c \leq \frac{2(3k+5)+2}{3}$ and show that $(K_{3k+5}, c) \in W$. There are two cases.

Subcase 1: Let $\boldsymbol{0}_c$, $\boldsymbol{2}_c < \frac{2(3k+5)+2}{3}$ and $\boldsymbol{1}_c = \frac{2(3k+5)+2}{3}$. Since $\boldsymbol{1}_c = \frac{2(3k+5)+2}{3}$, either $\boldsymbol{0}_c$ or $\boldsymbol{2}_c \leq \frac{(3k+5)-2}{3}$. Note that both $\boldsymbol{0}_c$ and $\boldsymbol{2}_c$ cannot equal $\frac{(3k+5)-2}{3}$. So assume that $\boldsymbol{0}_c \leq \frac{(3k+5)-2}{3}$ and $\boldsymbol{2}_c < \frac{(3k+5)-2}{3}$. We click three times: twice in the $\boldsymbol{1}_c$-many and once in the $\boldsymbol{0}_c$-many. So, in the remaining colored graph we will have

$$\boldsymbol{0}_{c'} \leq \frac{(3k+5)-2}{3} - 1 < \frac{2(3k+2)+2}{3}.$$

Also,

$$\boldsymbol{2}_{c'} < \frac{(3k+5)-2}{3} < \frac{2(3k+2)+2}{3}.$$

$$\boldsymbol{1}_{c'} = \boldsymbol{1}_c - 2 = \frac{2(3k+5)+2}{3} - 2 = \frac{2(3k+2)+2}{3}.$$

Since $\boldsymbol{0}_{c'}$, $\boldsymbol{2}_{c'} < \frac{2(3k+2)+2}{3}$ and $\boldsymbol{1}_{c'} \leq \frac{2(3k+2)+2}{3}$, we can apply the induction hypothesis and obtain that $(K_{3k+5}, c) \in W$. The proof follows similarly if $\boldsymbol{2}_c \leq \frac{(3k+5)-2}{3}$ and $\boldsymbol{0}_c < \frac{(3k+5)-2}{3}$.

Subcase 2: Let $\boldsymbol{0}_c$, $\boldsymbol{1}_c$, $\boldsymbol{2}_c < \frac{2(3k+5)+2}{3}$. Since $n \equiv 2 \pmod 3$, we have two cases.

(*i*). Suppose there is the same number Z of vertices of two colors in (K_{3k+5}, c). (Note that there is less than Z of the third color since $n \equiv 2 \pmod 3$). Then $Z \leq \frac{3k+5}{2}$. We move twice in one set of Z-many and once in the other set of Z-many. So,

$$Z' \leq Z - 1 \leq \frac{3k+5}{2} - 1 < \frac{2(3k+2)+2}{3}.$$

Since $Z' < \frac{2(3k+2)+2}{3}$, we apply the induction hypothesis. Therefore, $(K_{3k+5}, c) \in W$.

(ii). Suppose there are Z-many of only one color in (K_{3k+5}, c). The sizes for the other two color sets are either at most $Z-2$, or one of size $Z-1$ and the other of size at most $Z-3$ (since $n \equiv 2 \pmod 3$). So $Z \le \frac{3k+9}{3}$. We click twice in the Z-many vertices and once in a set of the next highest size. Therefore,

$$Z' \le Z - 2 \le \frac{3k+9}{3} - 2 < \frac{2(3k+2)+2}{3}.$$

Since $Z' < \frac{2(3k+2)+2}{3}$, we apply the induction hypothesis and get that $(K_{3k+5}, c) \in W$. □

V. *X*-Coloring Modulo 3

Perhaps one of our most exciting finds, the discovery of the $\boldsymbol{X}$-coloring modulo 3, helps us to give an example of an unwinnable coloring for any given graph. A graph G has the $\boldsymbol{X}$-*coloring* if for each vertex v,

$$\boldsymbol{X}(v) = \begin{cases} 0, & \text{if } \deg(v) \equiv 0 \pmod 3 \\ 1, & \text{if } \deg(v) \equiv 2 \pmod 3 \\ 2, & \text{if } \deg(v) \equiv 1 \pmod 3 \end{cases}.$$

Theorem 5.1: *If an undirected graph G has the* $\boldsymbol{X}$*-coloring then* $(G, \boldsymbol{X}) \notin W$.

Proof: Suppose G has the $\boldsymbol{X}$-coloring and suppose v is the final vertex in any supposedly winning sequence of moves on G. If $\boldsymbol{X}(v) = 0$ initially, then $deg(v) \equiv 0 \pmod 3$. This means v would have changed color $3n$ times for some integer n, while following the supposed winning sequence. Therefore, it is still of color 0 after following all but the last move of the purportedly winning sequence, so we cannot win. If $\boldsymbol{X}(v) = 1$ initially, then $deg(v) \equiv 2 \pmod 3$. Then v changed color some $3n + 2$ times when following the winning sequence, which means it is now of color 0. Thus we cannot win. Finally, if $\boldsymbol{X}(v) = 2$ initially, then $deg(v) \equiv 1 \pmod 3$. So v changed color some $3n + 1$ times while following the winning sequence. Thus it is now of color 0 and we cannot win. Therefore, no matter what vertex v of G is the final vertex in the sequence, $(G, \boldsymbol{X})$ is unwinnable. □

This theorem is important because it applies to every type of graph. A similar proof can also be made for an ***X***-coloring on digraphs replacing the word degree with in-degree. These concepts will be introduced in section (VII). It also is very useful in the statement and proof of the characterization of winnable colorings for our next type of graph, the complete bipartite graph.

VI. Complete Bipartite Graphs

A natural follow-up to the complete graph is the complete bipartite graph. A graph G is *bipartite* if "$V(G)$ is the union of two disjoint (possibly empty) independent sets." A *complete bipartite graph* is a "simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets." The complete bipartite graph with partite sets of sizes m and n is denoted $K_{m,n}$. This type also has the nice inductive quality in which deleting a vertex results in another complete bipartite graph.

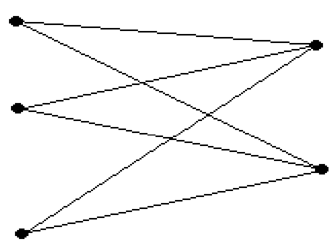


Figure 6.1: an example of a complete bipartite graph on 5 vertices, $K_{3,2}$

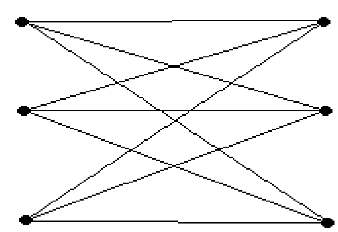


Figure 6.2: an example of a complete bipartite graph on 6 vertices, $K_{3,3}$

We must first define a couple of terms before we are able to continue with a relevant lemma relating to the ***X***-coloring that will help introduce a theorem about complete bipartite graphs. A vertex v is *bad* if $deg(v) + c(v) \equiv 0$ (mod3). A vertex v is *good* if $deg(v) + c(v)$ is not congruent to 0 (mod3). This is equivalent to saying that a vertex is bad if it has a color like that of the ***X***-coloring, and a vertex is good if it does not have a color like that of the ***X***-coloring.

Lemma 6.1: *In a graph or digraph G, a good vertex v stays good through any sequence of moves on any other vertices of G, and a bad vertex v stays bad through any sequence of moves on any other vertices on G.*

Proof: Every time a vertex changes color, it loses one in-degree. Thus, for x moves that affect the color or in-degree of v out of all the moves made in changing from (G, c) to (G', c'),
$deg_{G'}(v) = deg_G(v) - x$ while $c'(v) = c(v) + x$. So, when v is bad,
$deg(v) + c'(v) = deg(v) - x + c(v) + x = deg(v) + c(v) \equiv 0 \pmod 3$.

Likewise, when v is *good*, $deg(v') + c(v') = deg(v) - x + c(v) + x = deg(v) + c(v)$ which is not congruent to 0 (mod3). ☐

Theorem 6.1: *Suppose $m, n \geq 2$.*
Suppose $m, n \equiv 0 \pmod 3$. Then $(K_{m,n}, c) \notin W$ iff c is $\boldsymbol{0}$ or has $m + n - 1$ vertices of color 0 and one vertex of color 1 or 2.
Suppose either m or n is not congruent to 0 (mod3). Then $(K_{m,n}, c) \notin W$ iff $c = \boldsymbol{0}$ or $c = \boldsymbol{X}$.

Proof: In either case, the fact that the ***0*** and ***X***-colorings are unwinnable is clear. Consider the case where $m, n \equiv 0 \pmod 3$, and all of the vertices except one are of color 0 and that one is either of color 1 or 2. Then the only first legal move to make is to click on the nonzero vertex. To show that the coloring is unwinnable, it suffices to show that upon making this forced move, the remaining graph is colored with its ***X***-coloring, which we know is unwinnable. As a result of making the forced move, all of the vertices not in the partite set of the clicked vertex will change to color 1 and will also have their degree changed to 2 (mod 3), consistent with the ***X***-coloring. The vertices formerly within the partite set of the clicked vertex will have neither their coloring nor their degree changed, which means that they are still of color 0 and have degree 0 (mod3). So, the vertices are colored as in the ***X***-coloring.

Now we prove that if the starting coloring is not one of those listed above, then the graph is winnable. We do this by induction on the total number of vertices.
BASE CASE: $K_{2,2}$ is easily verified. (Note: $K_{2,2} \approx C_4$. See Theorem 3.1)
INDUCTION STEP: Assume that the claim is true for $m + n = 4,5,\ldots,k\text{-}1$ for some $k \geq 5$, and consider any $K_{m,n}$ for which $m + n = k$. Its coloring c is not one of the unwinnable colorings

listed above. Let side A represent the side with m vertices and side B represent the side with n vertices.

Case 1: The graph is of the form $K_{2,n}$ where $n \geq 3$.

Subcase (*i*): If there is any vertex of color 0 on side B, and if there is a vertex of color 1 or 2 on the side A, then we click on a nonzero vertex on side A.

Subcase (*ii*): If there is a vertex of color 0 on side B, but both vertices on side A have color 0, click once on the vertices on side B, which will give you two vertices of color one on side A. Since you still have a vertex of color 0 on side B, click on side A.

The graph resulting from either subcase (*i*) or subcase (*ii*) is a tree that has at least one leaf of color one. So, by Theorem 2.3, the graph is winnable.

Subcase (*iii*): If side B has no vertices of color zero and if it has at least one vertex v of color 1, click on v. If there are no vertices of color 1 on side B, then they must all be of color 2, so click on a vertex of color 2.

The resulting graph clearly does not have the 0 coloring because there are vertices remaining of either color 1 or color 2 on side B.

The only way that you could have the ***X***-coloring is if you have only vertices of color 1 on side B and bad vertices on side A remaining. This means that the vertex you clicked on must have also been color 1 because otherwise you would have clicked somewhere else. Had this been the case, we would have started with the ***X***-coloring. Since we did not start with the ***X***-coloring, we do not have it now; so, the graph is winnable.

Case 2: We have $K_{m,n}$ with m, $n \geq 3$. Then we begin by clicking on a vertex of $K_{m,n}$ according to the following procedure:

1. If either partite set consists of 2 (mod3) many vertices of color 2 and one vertex of either color 0 or 1, we click on a vertex of color 2 from that partite set first.
2. Otherwise, if there is a partite set in which every vertex is of color 2, we click on a vertex in that partite set first.
3. Otherwise, if there is a vertex whose degree plus its color is congruent to 0 (mod 3), and its color is 1 or 2, then we click on it first. (In other words, if there is a bad vertex of color 1 or 2, click on it.)
4. Otherwise, we click on any vertex that is of color 1 or 2.

Without loss of generality, this procedure specifies that we click on an A side vertex first.

It suffices to show that the coloring/graph resulting from clicking on a vertex is not one of the unwinnable configurations. So, we will first show that it can never be all red, second show that it can never be the $\boldsymbol{X}$-coloring, and third show that when the remaining graph/coloring has m, $n \equiv 0 \pmod 3$, it does not have one vertex that is of color 1 or 2 and the rest of color 0. Note that when m-1, $n \equiv 0$ (mod3), $\boldsymbol{X} = \boldsymbol{0}$.

Step one: In order for the new graph to have the all zero coloring, one partite set of the original graph would have been all of color 2 and changed to color 0. However, if we had had one partite set all of color 2, we would have clicked on it, leaving at least one vertex of color 2. So, there is no way that the remaining graph is the all zero coloring.

Step two: Assume that the resulting graph is the $\boldsymbol{X}$-coloring.

Then all of its remaining vertices are bad, which means that they started out bad. So, the only vertex that could have started out good was the vertex v that was clicked on. Now, if we had begun with a configuration like that of type 1 in our procedure, we certainly do not have the $\boldsymbol{X}$-coloring now. If it had been like that of type 2 from our procedure, v would have to be the same as the other vertices in its partite set, which means that v was a bad vertex. Now we are at type 3, which means that v must have been bad because otherwise we would have clicked on another vertex first since all of the other vertices were bad; the only exception to this is if all of the vertices except v had been red, which means that the original graph would have been the case where m, $n \equiv 0$ (mod3) and all of the vertices are of color 0 except one of color 1 or 2. This is a contradiction because we couldn't have started with that configuration. So, all of the vertices must have been bad, which means that we started with the $\boldsymbol{X}$-coloring, which is a contradiction. Therefore, we do not have the $\boldsymbol{X}$-coloring now.

Step 3: For the final case, we will assume that the coloring of the new graph is 1 vertex of color 1 or 2 with the rest of color 0 and that m, $n \equiv 0$ (mod3). There are two ways to get this configuration:

1) We began with a vertex of color 1 or 2 along with 0 (mod 3) vertices of color 0 on side B and a vertex of color 0 or 1 along with 2 (mod 3) vertices of color 2 on side A, and we clicked on the vertex of color 1 or 2 on side B. This is a contradiction, however, because the procedure at the beginning said to click on side A in this situation.

2) We began with side B consisting of 2 vertices each of color 1 or 2 along with 2 (mod 3) vertices of color 0 and side A consisting of 0 (mod 3) vertices of color 2, and we clicked on

one of the vertices on side B that are of color 1 or 2. This, however, is also a contradiction because we were supposed to click on side A in this situation.

Now since we have $K_{m,n}$, and we know that it is not one of the unwinnable configurations, we can appeal to the induction hypothesis and the graph is winnable. □

VII. Acyclic Digraphs

We moved next to examine some types of directed graphs. We begin with the simplest type: acyclic digraphs. It is necessary to first present some definitions that relate to this type of graph. A *directed graph* or *digraph* G is "a triple consisting of a vertex set, edge set, and a function assigning each edge an ordered pair of vertices." A graph is *acyclic* if it has no directed cycles. The *out-degree* of a vertex v in a digraph, denoted $deg^{+}(v)$, is "the number of edges of which it is the tail." The *in-degree* of a vertex v in a digraph, denoted $deg^{-}(v)$, is "the number of edges of which it is the head." A *source* in a digraph is a vertex v such that $deg^{-}(v) = 0$. We now present a theorem about the nature of acyclic digraphs, which is necessary to understand the proofs that follow about the winnability of digraphs modulo 3.

Theorem 7.1: *A digraph G on n vertices is acyclic if and only if the vertices of G can be labeled as $v_1, v_2, \ldots, v_n$ such that each edge of G from v_i to v_j satisfies $i < j$.* (See exercise 1.4.14 in [2].)

Theorem 7.2: *Any acyclic digraph is winnable if and only if every source has color 1 or 2.*

Proof: We proceed by induction on the number of vertices n.

BASE CASE: The cases $n = 1$ and $n = 2$ are easily verified.

INDUCTIVE STEP: Assume the statement is true for 1,2,...,k for some integer k. Consider an acyclic digraph G with $k+1$ vertices. Suppose that G has a source of color 0. Then this source will never change color since its in-degree is 0. Thus the acyclic digraph is unwinnable.

Now assume that every source in G has color 1 or 2. We want to show that (G, c) is winnable. By Theorem 7.1, we can order the vertices 1,2,...,$k+1$ in such a way that each edge from a vertex v to a vertex w is such that $v < w$.

We click on the largest numbered vertex v having non-zero color. For each vertex w for which (v, w) is not an edge, w undergoes no change in color nor change in whether it is a source, so among such vertices w there is still no source of color 0. For each vertex w for which (v, w) is an edge, we know that $v < w$. Therefore, $c(w) = 0$ by the maximality of w. Therefore, after clicking at v such a w is not a source of color 0 since its color is 1.

So, after clicking at v we have an acyclic digraph (since G was acyclic) with k vertices in which all the sources have non-zero color. Applying the induction hypothesis, this digraph on k vertices is winnable, and therefore the original acyclic digraph with $k+1$ vertices is winnable. □

VIII. Directed Cycles

Directed cycles, denoted by DC_n are the directed counterpart of cycles. They follow nicely from acyclic digraphs, much like undirected cycles follow nicely from trees because after clicking on a vertex in a directed cycle, we get a directed path, which is a type of acyclic digraph.

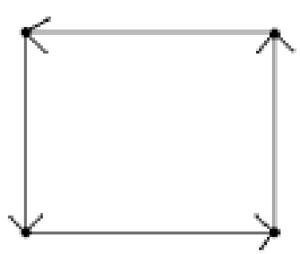


Figure 8.1: an example of a directed cycle with 4 vertices, DC_4

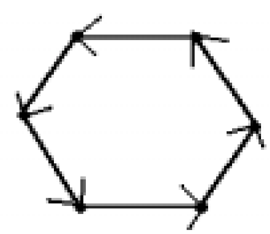


Figure 8.2: an example of a directed cycle with 6 vertices, DC_6

We characterize which colorings are winnable on directed cycles.

Theorem 8.1: *(DC_n, c) is unwinnable if and only if $c = \mathbf{0}$ or $c = \mathbf{2}$.*

Proof: (DC_n, c) is clearly unwinnable when $c = \mathbf{0}$. Suppose $c = \mathbf{2}$ and click on any vertex. The resulting graph is an acyclic digraph with a source of color 0, which is unwinnable from Theorem 7.2. Thus, the graph is unwinnable.

Suppose $c \neq \mathbf{0}$ and $c \neq \mathbf{2}$. It is easily shown that there exists a vertex v of color 1 or 2 whose out-neighbor w is either color 1 or color 0. Click on v. The graph resulting from clicking on v is an acyclic digraph with w as its sole source, that source being of color 1 or 2. This is winnable by Theorem 7.2. □

IX. Tournaments

To conclude our work on directed graphs, we investigate tournaments. The simplest form of a tournament, a transitive tournament, happens to be an acyclic digraph and therefore conforms to the pattern of Theorem 7.2. We proceed with a couple of definitions about tournaments. An *orientation* of a graph G is "a digraph D obtained from G by choosing an orientation ($x \rightarrow y$ or $y \rightarrow x$) for each edge $xy \in E(G)$." A *tournament* is "an orientation of a complete graph." A *transitive tournament* is a tournament in which if there is an edge from v to w and another edge from w to y then there must be and edge from v to y. A *one-major-upset tournament* on n vertices, $OMUT_n$, where the vertices are ordered according to Theorem 7.1, is the result of reversing the direction of the edge between vertex n and vertex 1.

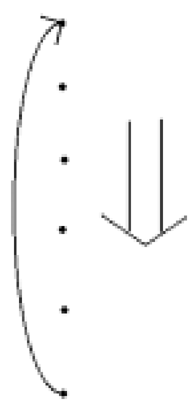


Figure 9.1: an example of a one major upset tournament on 6 vertices, $OMUT_6$

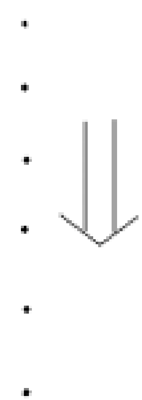


Figure 9.2: an example of a transitive tournament on 6 vertices, TRANS T_6

Note: If there is no edge drawn joining two vertices, then it is to be understood that there is an edge between the vertices, that edge oriented in the direction of the double arrow.

We now prove a theorem about the one-major-upset tournament.

Theorem 9.1: *($OMUT_n$, c) is unwinnable if and only if c =* $\mathbf{0}$ *or* $\mathbf{X}$.

Proof: We proceed by induction on the number of vertices n.

BASE CASE: The case when $n = 3$ is (DC_3, c), which was already handled in Theorem 8.1.

INDUCTIVE STEP: Suppose the claim is true for $n = 3,4,\ldots,k$ for some integer $k \geq 3$. Consider a coloring c for $OMUT_{k+1}$.

First, if $c = \boldsymbol{0}$ or $\boldsymbol{X}$ we know that $(OMUT_{k+1}, c)$ is not winnable.

Suppose $c \neq \boldsymbol{0}$ or $\boldsymbol{X}$. Let the vertices of $OMUT_{k+1}$ be $v_1,v_2,\ldots,v_n$ with there being an edge directed from i to j if and only if either

a) $i < j$ and $(i, j) \neq (1, k+1)$, or

b) $i = k+1, j = 1$.

Case 1: $c(v_1) = 0$, $c(v_{k+1}) = 1$ or 2. Click first at v_{k+1}. The remaining digraph is acyclic and its only source, v_1 , now has color 2. By Theorem 7.2 the remaining graph is winnable, so $(OMUT_{k+1}, c)$ is winnable.

Case 2: $c(v_1) = 0$, $(v_{k+1}) = 0$. Since c is not equal to $\boldsymbol{0}$**,** there exists v_i with color 1 or 2 where $1 < i < k+1$. Click on v_i, turning the color of v_{k+1} to 1 and leaving the color of v_1 as 0. Now click on v_{k+1} (changing the color of v_1 from 0 to 1). The remaining graph is an acyclic digraph and its only source, namely v_1, is of color 1. By Theorem 7.2, this graph is winnable, and therefore so is $(OMUT_{k+1}, c)$ using this winning strategy.

Case 3: $c(v_1) = 1$ or 2, $c(v_2) = 0$ or 1. Click on v_1. The remaining graph is an acyclic digraph with its only source, namely v_2, having color 1 or 2. By Theorem 7.2 it is winnable. We can conclude then that $(OMUT_{k+1}, c)$ is also winnable, using this winning strategy.

Case 4: $c(v_1) = 1$, $c(v_2) = 2$. Click on v_2. Now we have $OMUT_k$ with some coloring c'. Since $c'(v_1) = 1$ we know that c' is not $\boldsymbol{X}$ or $\boldsymbol{0}$**.** Then by the induction hypothesis, $(OMUT_k, c')$ is winnable and using this winning strategy we conclude that $(OMUT_{k+1}, c)$ is also winnable.

Case 5: $c(v_1) = 2$ and $c(v_2) = 2$. Click on v_2. The remaining graph is $OMUT_k$, with some coloring c'. Assume for contradiction that $c' = \boldsymbol{X}$ or $\boldsymbol{0}$. It is enough to reach a contradiction since by the induction hypothesis there would be a winning strategy for $(OMUT_k, c')$. We know that = $c'(v_1) = c(v_1) = 2$, so $c' \neq \boldsymbol{0}$ and therefore $c' = \boldsymbol{X}$. Then $c'(v_i) = 0$ if in the remaining graph $deg^{-}(v_i) \equiv 0 \pmod 3$ and $c'(v_i) = 1$ if $deg^{-}(v_i) \equiv 2 \pmod 3$ and $c'(v_i) = 2$ if $deg^{-}(v_i) \equiv 1 \pmod 3$. Moving at v_2 increased the color of each of the vertices $v_3,v_4,\ldots,v_{k+1}$ by one and reduced each of their indegrees by one. Then for v_i, $i = 3,4\ldots k+1$, we have that $c(v_i) = 0$ if $\deg^-(v_i) \equiv 0 \pmod 3$, $c(v_i) = 1$ if $\deg^-(v_i) \equiv 2 \pmod 3$, and $c(v_i) = 2$ if $\deg^-(v_i) \equiv 1 \pmod 3$. Also we have that v_1 and v_2 have in-degree one and $c(v_1) = c(v_2) = 2$ in $OMUT_{k+1}$. Therefore $c = \boldsymbol{X}$ for $OMUT_{k+1}$, a contradiction..

This exhausts all possible cases, so we conclude that our claim is true. □

X. *X*-Coloring Modulo *n*

Our mod 3 research arose from the mod 2 work done in [1]. We naturally pose the question of whether our results generalize to graph colorings in which we allow more than three colors. The following is a case where we found such a generalization, where the rules concerning mod n colorings are as follows. At each move we must click at a vertex of non-zero color, whereupon that vertex is deleted and the colors of its former neighbors are increased by 1 (mod n), the object being to remove all vertices. We define the $\boldsymbol{X}$-coloring on a graph G by $\boldsymbol{X}(v) \equiv$ -indegree(v) (mod n) for each vertex v; that is, $\boldsymbol{X}(v)$ is chosen so that indegree $(v) + \boldsymbol{X}(v) \equiv 0 \pmod n$.

Theorem 10.1: *If a graph has the* $\boldsymbol{X}$*-coloring, then* $G \notin W$.

Proof: Suppose v is the final vertex of any supposedly winning sequence of moves on G. Once all but the last move have been made in this sequence, the color of v has changed $deg^-(v)$ number of times, so the new color of v is $\boldsymbol{X}(v) + deg^-(v) \equiv 0 \pmod n$. Since the color of v is now 0, we cannot click on it; thus, the graph is unwinnable. □

XI. Conclusion and Future Research

In our paper we were able to formalize many of the winning strategies and cases for *Lights Out* (mod 3). Among the discoveries that we made, one of the most important was: the 2,2-procedure (Theorem 2.2), which enabled us to produce half the unwinnable configurations for the trees. The 0,0 procedure, by contrast, was much easier to discover. The fact that there are 2^{n-1} unwinnable initial colorings for a given tree (Theorem 2.4) was also very interesting in itself because just having this fact in the conjecture stage helped us in the process of finding patterns for unwinnable configurations that were not produced by the 0-procedure, eventually leading us to the 2,2 procedure. The discovery that could be named as one of the central theorems of this paper is the ***X***-coloring (Theorem 5.1). This theorem, along with the discoveries made in [1], led us to the generalization in Theorem 10.1. These two theorems are important because they were useful throughout our research in obtaining many of our proofs.

Our research is not finished; there are still many other aspects of the graphs and other types of graphs that could be investigated. We were able to generalize the theorem for the ***X***-coloring to be applicable modulo any n. It would be very interesting to see if more of our theorems could likewise generalize for an arbitrary modulus. There are many other types of graphs that can be used as the "board" in the game, such as more general "path-of-upsets" tournaments and other types of tournaments. We hope that this work will be of interest to those in the math community who enjoy studying the mathematical properties of games, and also to other math students, so that they may get interested in pursuing research in this field of mathematics.

XIII. Acknowledgements

As SUMSRI comes to an end, we would like to thank all the people who, by their help and support, encouraged us to continue our research even at times when we felt we were going nowhere. Special thanks to Dr. Daniel Pritikin, our research advisor, for his time and patience during the entire program. Also to Mr. Charles Hague for his encouragement and help during our most difficult times. We would also like to thank all the SUMSRI staff, especially Ms. Bonita Porter and Ms. Moira Miller, professors, GA's and our peers. This research wouldn't have been possible without the funding from the National Security Agency, National Science Foundation and Miami University. Thank y'all!!!

XIV. References

[1] Craft, Miller and Pritikin, *A Variation of Lights Out*, In progress.

[2] D.B. West, *Introduction to Graph Theory* (Prentice-Hall, New Jersey, 2001).

XV. Web Sites

We include this section for all of those interested in looking in the internet for information regarding the game *Lights Out*. If a website appears in this list, it doesn't mean that the authors of this paper agree with its content.

- http://www.sar.usf.edu/~dwahl/LightsOut.html
    - In this one you will be able to play the game selecting the number of colors, from 2 to 9.
- http://www.whitman.edu/offices_departments/mathematics/lights_out/
- http://gbs.mit.edu/~kbarr/lo/
    - This is the website for The Lights Out Fan Club. JOIN!!!!