Homology of erodivisors

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Abstract

Let be a commutative ring with unity. We define a semisimplicial abelian group based on the structure of the semigroup of ideals of and investigate various properties of the homology groups of the associated chain complex.

Let be a commutative ring with unity. The set ( ) of zero-divisors in a ring does not possess any obvious algebraic structure consequently, the study of this set has often involved techniques and ideas from outside algebra. Several recent attempts, among them [2], [3] have focused on studying the so-called *divisrgraph* $_R$, whose vertices are the zero-divisors of , with xy being an edge if and only if xy 0. This object $_R$ is somewhat unwieldy in that it has many symmetries for example, if u 2 is any unit, then x 7! ux induces a (graph) automorphism of $_R$. One way of treating this issue – following an idea of Lauve [5] – is to work with the *idal divisrgraph* I_R. In eect, one replaces zero-divisors of by proper ideals with nonzero annihilator this is the approach adopted by the authors in [1]. Such a perspective also has its shortcomings for instance, it does not adequately detect the phenomenon of there being three distinct proper ideals I K in with IK 0, but I 60, IK 60, K 60.

In this paper we adopt a dierent philosophy, using a new type of homology to study ( ) and capture the situation described above. Roughly speaking, if we denote by $_n$( ) the free abelian group generated by the set of (1)-tuples (I_0 : : : I_n) of distinct ideals of such that I_0 : : : I_n 60, there are obvious maps $_n$( ) ! $_{n-1}$( ) obtained by forgetting one of the factors. This gives ( ) the structure of a semi-simplicial abelian group hence we may speak of its associated chain complex ( ). Our homology groups H ( ) are then defined as the homology groups of a certain quotient of $_:$( ). The idea behind this construction was sketched by Lauve in [5], although the precise definition is due to the authors.

After giving a precise definition of these homology groups $H_\ast(\ )$, we study the group $H_0(\ )$ in depth and compute $H_1(\mathbb{Z}/p^r\mathbb{Z})$ when p is a prime and r 1 is an integer. e then give some conditions on sucient to ensure that $H_n(\ )$ 0 for n 0. In the last section we consider the *Euler characteristic* $(\ )$ $\sum_{n=0}^{1}(\ 1)^n \mathrm{rk}\, H_n(\ )$. Using some ideas from partition theory, we prove the surprising result that $(\mathbb{Z}/p^r\mathbb{Z})$ is always either 0, 1, or 2, depending on the value of r relative to the "pentagonal" numbers $m(3m\ 1)/2$ and the related numbers $m(3m\ 1)/2$. e also derive formulas for the Euler characteristic for some other special types of finite rings.

The authors would like to thank Aaron Lauve, David Anderson, Dennis Keeler, Neil O. Smith, Siamak Yassemi and Tao Jiang for useful conversations and references. They are also deeply grateful to Dan Pritikin for his invaluable insight in elucidating the combinatorics of the calculation of the Euler characteristic in Theorem 6.1. Finally, they thank Miami University for financial support of the second author's research during the summer of 2003.

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Let be a commutative ring and P the set of proper ideals of . For each n 0, let $S_n(\ )$ be the set of ordered $(n\ 1)$-tuples $(I_0 \ldots I_n)$, where $I_0 \ldots I_n$ are distinct proper ideals of and $I_0 I_1 \ldots I_n \neq 0$ let $S_{\ 1}(\ )$ be a set consisting of one element. If there is no danger of ambiguity, we simply write S_n instead of $S_n(\ )$. Observe that for each i, 0 i n, there is a "face map" $^n_i : S_n \to S_{n\ 1}$ defined by $^n_i(I_0 \ldots I_n)\ (I_0 \ldots I_i \ldots I_n)$. Moreover, $S_0(\ )$ $\emptyset$ if and only if is a field, so when is not a field, there is a unique "augmentation" map $S_0(\ ) \to S_{\ 1}(\ )$. Now for each n 1, let $_n$ be the free abelian group generated by S_n. e denote by $[I_0 \ldots I_n]$ the basis element corresponding to $(I_0 \ldots I_n) \in S_n$. Likewise, the various face maps ^n_i extend $\mathbb{Z}$-linearly to maps $^n_i : {}_n \to {}_{n\ 1}$ moreover, if $S_0 \neq \emptyset$, there is a unique $\mathbb{Z}$-linear map $: {}_0 \to {}_{\ 1}$ $\mathbb{Z}$ defined by $(\sum n_i (I_i))$ $\sum n_i$. Thus, there is a semisimplicial abelian group:

$$.(\ ) : \quad \ldots \to \quad {}_1 \to \quad {}_0$$

with augmentation $: {}_0 \to \mathbb{Z}$ if is not a field.

This in turn gives rise to an (augmented) chain complex in the standard manner by taking an alternating sum of face maps. For each n 0, define $_n$ $\sum_{i=0}^{n}(\ 1)^i\ ^n_i$ then we have a complex:

L. I and distinct ppridals f . Thn $[I]$ and $[\ ]$ hav th sa class in $H_0(\ )$ if and nly if I and li in th sa cntd pnnt f th graph Γ_R.

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If I and are in the same connected component of Γ_R, then there is some path I $_0$ $_1$ $:::$ $_n$ connecting I and , where the $_i$ are ideals such that for each i 0 $:::$ n 1, $_i$ $_{i+1}$ $6\!\!0$. This directly implies that $\sum_{i=0}^{n-1}[\ _i\ \ _{i+1}]$ is an element of T_1, and by direct calculation we see that

$$_0(\sum_{i=0}^{n-1}[\ _i\ \ _{i+1}])\ [\ \ _0]\ \ [\ _n]\ [\ I]\ \ [\ ]$$

Hence $[I]$ $[\ ]$ in $H_0(\ )$.

Conversely, suppose $[I]$ and $[\ ]$ define the same class in $H_0(\ )$. Then $[I]$ $[\ ]$ $_0(\sum_{i=0}^{n}[\ _i\ B_i])$ $\sum_{i=0}^{n}[\ _i]$ $[B_i]$ where $_i$, B_i are distinct proper ideals of and $_i B_i$ 6 $;$. Let n be the smallest integer for which this is possible. e prove by induction on n that, after suitable reordering of the $_i$ and B_i, there is a path in Γ_R from I to .

e may assume without loss of generality that $_0$ I and B_n . If B_0 , then I $6\!\!0$ and we are done. Otherwise, assume B_0 6 that is, n 0. Since

$$[I]\ \ [\ ]\ [\ I]\ \ [B_0]\ [\ \ _1]\ \ [B_1]\ \ :::\ [\ \ _n]\ \ [B_n]$$

is a relation in a free abelian group, we may assume without loss of generality that $_1$ B_0. Then, adding $[B_0]$ $[I]$ to both sides of this equation, we get

$$[B_0]\ \ [\ ]\ [\ \ _1]\ \ [B_1]\ \ :::\ [\ \ _n]\ \ [B_n]$$

so by induction there is a path in Γ_R from B_0 to . Since $_0B_0$ $6\!\!0$, this means that f $_0$ B $_0$g is an edge in Γ_R, and hence that there is a path from $_0$ I to .

$\in$ *L. $I_1 ::: I_n$ distinct ppridals f lying in tally distinct cntd pnnts f Γ_R. Thn th classs f $[I_1] :::$ $[I_n]$ ar linarly indpndnt in $H_0(\ )$.*

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If is a field, the assertion is trivial. Otherwise, let $C_1 ::: C_r$ be the components of Γ_R. Suppose the class of $\sum_{i=1}^{n} c_i[I_i]$ in $H_0(\ )$ is 0. e may assume that each I_i lies in component C_i of Γ_R. Now *t. n 0 an intgr An Int 2 T_1 is alld an n ict (siply a* circuit*) if thr ist ppridals $I_1 ::: I_n$ f sh that*

$$[I_1 I_2] \quad ::: [I_{n} \; _1 I_n] \; [I_n I_1]$$

A 3 ict is alld a triangle

Clearly the definition has been chosen to reect the fact that in the above context, $I_1 \quad I_2 \quad ::: I_n \quad I_1$ is a circuit in the graph $H_{\mathbb{Z}=p^r\mathbb{Z}}$. The analysis of Ker $_0$ proceeds by a sequence of lemmas.

$\mathcal{L} \in$ ynt fi 2 K $_0$ ay ittn

$$fi \quad \sum_{k=1}^{m} \; k$$

hwra h $_k$ is a ict.

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The proof is by induction on the number of symbols in fi. If fi 0, the claim is clear. Otherwise, let $fi \quad \sum_{j=1}^{r}[\; _j \; B_j]$ with r chosen to be as small as possible. e may assume that there is no pair of integers (j_1, j_2), 1 j_1 j_2 r such that $_{j_1}$ B_{j_2} and $_{j_2}$ B_{j_1}, for then we may use the relation I [ ] [I] in T_1 to simplify the expression for fi and obtain a relation with smaller r.

Since fi 2 Ker $_0$, we have:

$$0 \quad _0(fi) \quad _0(\sum_{j=1}^{r}[\; _j \; B_j]) \quad \sum_{j=1}^{r}[\; _j] \quad [B_j]$$

Since this is a relation in the (free abelian) group T_0, it follows that there is some j such that B_1 $_j$. ithout loss of generality we may assume that j 2. By the previous discussion, it follows that $_1$ 6 B_2. Now it must be the case that there is some j such that B_2 $_j$ without loss of generality, we assume that j 3. Continue this procedure until one reaches s r such that B_s $_1$. Then

$$fi_1 \quad [\; _1 B_1] \; [B_1 B_2] \quad ::: [B_{s} \; _2 B_{s} \; _1] \; [B_{s} \; _1 \quad _1]$$

is a circuit in T_1. By induction, fi fi_1 is a sum of circuits in T_1 hence fi itself is a sum of circuits.

$\mathcal{L}$ yn n ict in T_1 $T_1(\mathbb{Z}p^r\mathbb{Z})$ ay ittn as a s f triangls.