Pinwheels: Orienting the Line Graph of the Complete Graph

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Abstract

We define a pinwheel to be a specific kind of orientation of the line graph of a complete graph. In studying pinwheels, we find that every pinwheel is strongly connected, contains $c((n-3)!)^{\binom{n}{2}}$ many Euler tours, and that Seymour's Conjecture is true for every vertex in any pinwheel. Also, we give tight bounds on the number of 3-cycles possible, the number of spanning trees, and specific eigenvalues in any particular pinwheel, as well as characterize pinwheels of diameter 2.

1. Introduction

Eulerian graphs, line graphs, complete graphs, and tournaments are various families of graphs that are studied extensively in graph theory. In this paper, we introduce a new family of graphs, called pinwheels, and present their properties often in connection with other types of graphs. We will outline elementary properties, such as the number of vertices and edges, as well as more complex properties including diameter, Eulerian tours, and eigenvalues of the adjacency matrix. All terms and notation unique to pinwheels are defined in the paper, but for general graph theory terms, see [3].

To begin our study, consider the line graph.

Definition 1.1: The *line graph, L(G)*, is a graph relating to a graph G, its vertex set being the edge set of G, two vertices of L(G) being adjacent if the corresponding edges share a vertex in G.

A pinwheel, defined in the next section, is based on the line graph of the complete graph on n elements, L(K_n). Given the set, $[n]=\{1,2,...,n\}$, the line graph of the complete graph is a graph with vertices labeled by all possible two-element subsets of [n] with edges connecting every two vertices containing a common element.

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Some fundamental properties of L(K_n) follow from the following definition.

Definition 1.2: A *triplet* Δxyz is the subgraph of L(K_n) induced by the three vertices *xy*, *xz*, and *yz* where *x,y,z* are in [*n*].

The number of triplets in L(K_n) is $\binom{n}{3}$ as that is the number of ways to choose exactly three distinct elements of [*n*], where each combination will form a triplet of L(K_n). Because a vertex is labeled by a combination of two distinct elements of [*n*], the vertex set, denoted $[n]^2 = \{xy \mid x, y \in [n]$ and $x<y\}$, has cardinality $\binom{n}{2}$. For settings in which we do not know whether *x*<*y*, we write {*x*,*y*} as a name of which of *xy* or *yx* is a vertex. The number of edges, $|E(PWC_n)|$, is $3\binom{n}{3}$, since each PWC_n has exactly $\binom{n}{3}$ triplets each containing three edges that only appear in one triplet of that PWC_n (as an edge connects two vertices, and two triplets can have at most one vertex in common). With these properties in mind, we can now define the pinwheel family of graphs

2. Defining Pinwheels

A pinwheel graph is simply a specific type of orientation of the line graph of a complete graph.

Definition 3.2: We define a *pinwheel* to be an oriented L(K_n) in which every triplet forms a directed 3-cycle.

In the triplet $\Delta 123$, for eaxmple, the edge joining 12 and 13 may be directed such that $12 \to 13$ or $13 \to 12$ as long as the triplet remains cyclic. That is, either, $12 \to 13 \to 23 \to 12$ or $13 \to 12 \to 23 \to 13$. The set of pinwheel graphs for a specific *n* is denoted by PWC_n.

For each *n*, there are numerous pinwheels in PWC_n due to the different possibilities for direction. A PWC_4 is portrayed in figure 1.

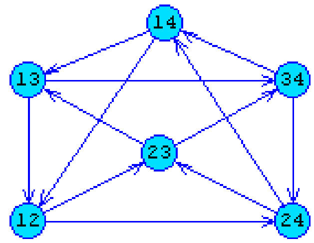


Figure 1

In Figure 1, the triplets are $\Delta 123$, $\Delta 134$, $\Delta 124$, and $\Delta 234$. Note that between any three vertices that do not make up a triplet, there may or may not exist edges between them, let alone a 3-cycle.

3. Basic Properties

We have discovered several properties of PWC$_n$'s. For any PWC$_n$, the outdegree (the number of out-edges) of any vertex equals the indegree (the number of in-edges) of that vertex, since all of the edges in any PWC$_n$ are chosen to form cycles in the triplets (so that for each edge leading into a vertex, there exists an edge leading out of it). The outdegree for any vertex $xy \in [n]^2$ is $d^+(xy) = n - 2$. This is because any vertex xy has an edge joining it in L(K$_n$) with every vertex in the set $\{\{x,a\} \mid x,a \in [n], a \neq x, y\} \cup \{\{b, y\} \mid b, y \in [n], b \neq x, y\}$ and this set has cardinality 2(n-2); this value is the degree of the vertex xy in L(K$_n$) so dividing by 2 (since $d\ (xy) = d^+(xy) + d^-(xy)$ and $d^+(xy) = d^-(xy)$) gives the outdegree of the vertex. Following from this explanation, it is clear that the outdegree is the same for every vertex in PWC$_n$.

The number of PWC$_n$ digraphs for a given n is $2^{\binom{n}{3}}$, as there are two possible directions for each of the $\binom{n}{3}$ triplets. We define a *PWC$_k$ in a PWC$_n$* where k<n to be some PWC$_k$ induced by k elements in [n]. The number of PWC$_k$'s contained in PWC$_n$ is the number of ways to choose k elements, namely $\binom{n}{k}$.

The number of triplets that one vertex, *ab*, is contained in is *n*-2 since for a triplet Δabc we must choose one more element, *c*, out of the remaining *n*-2 elements of [*n*] to complete the triplet.

The next several results, while still basic in nature, require more computation to support.

Proposition 3.1: *The number of triangles, or sets of three pairwise adjacent vertices in any* PWC_n *is* $(n-2)\binom{n}{3}$.

Proof: We know there are $\binom{n}{3}$ triplets. We also know that every non-triplet triangle will have vertices that all share one element of [*n*] (to be proven later), such as the triangle induced by *ax*, *ay*, and *az*. So, the number of triangles that all share the element *a*, for example, would be $\binom{n-1}{3}$ because there are *n*-1 elements of [*n*] excluding the element *a*. Since there are n choices for *a*, there will be $n\binom{n-1}{3}$ non-triplet triangles. Therefore, the number of triangles in any PWC_n is $\binom{n}{3}+n\binom{n-1}{3}$, which simplifies to $(n-2)\binom{n}{3}$.

Proposition 3.2: *Any vertex in* PWC_n *is contained in exactly* n^2-4n+4 *triangles.*

Proof: Let $ab\in[n]^2$. We know that it is contained in *n*-2 triplets from a previous comment. We also know that *ab* is adjacent to every vertex in the set $\{\{a,x\},\{b,y\}\in[n]^2\mid x,y\in[n]\setminus\{a\}-\{b\}\}$. The vertex *ab* will be adjacent to *n*-2 vertices of the form {*a*,*x*} and *n*-2 vertices of the form {*b*,*y*}. Since there exists a non-triplet triangle between any three "{*a*,*x*}" vertices and any three "{*b*,*y*}" vertices, *ab* will be contained in $\binom{n-2}{2}$ triangles of vertices containing the element a and $\binom{n-2}{2}$ triangles of vertices containing the element b. Therefore, any vertex will be contained in $(n-2)+2\binom{n-2}{2}=n^2-4n+4$ triangles.

Proposition 3.3: *The independence number of* PWC_n *is* $\lfloor \frac{n}{2} \rfloor$.

Proof: We know that there exists an edge between every two vertices that share an element of $[n]$. Therefore, the independence number is the largest number of vertices in $[n]^2$ having no elements in common in $[n]$, which equals $\lfloor \frac{n}{2} \rfloor$.

Note that Propositions 3.2 and 3.3 hold as well for $L(K_n)$.

Now, with these elementary properties understood, we present more involved results in the following sections.

4. Euler Tours

An *Euler tour* is a walk in a directed graph that traverses each edge exactly once. We include the following proposition that is used in our proof that any PWC_n has an Euler tour.

Proposition 4.1: *Every* PWC_n *is strongly connected.*

Proof: We consider the two possible cases for pairs of vertices.

Case 1: Let $ab, cd \in [n]^2$, where only three of the elements *a, b, c, d* are distinct. Consider the cycle in the triplet that contains the *ab* and *cd*. There will be either a one or two-step path from *ab* to *cd*.

Case 2: Let $ef, gh \in [n]^2$, where *e, f, g,* and *h* are distinct. In the triplet Δefg, there exists a one or two step path from *ef* to *fg*, and in the triplet Δfgh there exists a one or two step path from *fg* to *gh*. Therefore, composing these two paths together, there exists a walk from *ef* to *gh*.

Now we can show the existence of an Euler tour.

Theorem 4.2: *There exists an Euler tour in any PWC$_n$.*

Proof: According to [3, Exercise 1.4.19], a digraph is Eulerian (has an Euler tour) if and only if $d^+(v) = d^-(v)$ for each vertex v in the digraph and the underlying graph has at most one nontrivial component. We will verify these two conditions. In any PWC$_n$ the outdegree and the indegree of every vertex v is $(n-2)$, so $d^+(v) = d^-(v)$ for each vertex v in the digraph. From Proposition 4.1, we know that all PWC$_n$'s are strongly connected so there is only one nontrivial component in any PWC$_n$. Therefore, in any PWC$_n$, there exists an Euler tour.

With the knowledge that an Euler tour exists in any PWC$_n$, we go on to count the number of Euler tours within a given PWC$_n$.

Proposition 4.3: *In a PWC$_n$ digraph, the number of Eulerian tours is* $c((n-3)!)^{\binom{n}{2}}$*, where c counts the number of in-trees to, or the out-trees from, any vertex.*

Proof: It is stated in [3, Theorem 2.2.28] that in an Eulerian digraph with $d_i = d^+(v) = d^-(v)$, the number of Eulerian circuits is $c\prod_i (d_i - 1)!$. In any PWC$_n$ the outdegree of every vertex is $n-2$ and the number of vertices is $\binom{n}{2}$. Because the number being multiplied is now constant for any particular n, we can write the expression for PWC$_n$ in particular as $c((n-3)!)^{\binom{n}{2}}$.

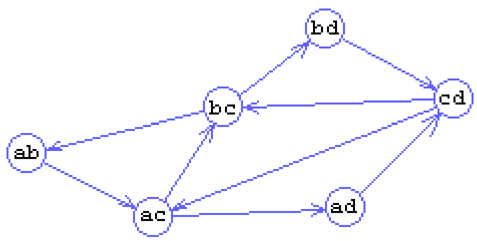
Note that the value for c varies for each distinct orientation of PWC$_n$, but it can be calculated as given in [3, Theorem 2.2.21]. Given a PWC$_n$, G, let $Q^- = D^- - A'$ where D^- is a diagonal matrix of the outdegrees in G, and the i,jth-entry of A' is the number of edges from v_j to v_i. The number of spanning out-trees of G rooted at one v_i, the value of c, is the value of each cofactor in the ith row of Q^-.

5. Measuring Diameter

Another result, which follows from the fact that PWC_n is strongly connected, concerns diameter. In order to measure the diameter, we will first consider distance.

Theorem 5.1: *For any PWC_n, the distance between any two vertices is at most three. For vertices that share an element, the distance is at most two.*

Proof: Let $ab, cd \in [n]^2$ of some PWC_n. If *a, b, c,* and *d* are only three distinct elements, then there will exist a one or two-step path through the triplet that contains both vertices. Suppose *a, b, c,* and *d* are distinct. Then, either $ab \rightarrow ac$ or $ac \rightarrow bc$, so without loss of generality assume $ab \rightarrow ac$. We can form a triplet between the vertex *ac* and each of *ab* and *cd*. Through these two triplets, there will be at most a three-step path from *ab* to *cd* via *ac* as in the figure.



Because the distance between any two vertices will always be either one, two, or three, we can deduce that the diameter will be two or three for each PWC_n (we know the diameter cannot be one as there is not always an edge between any two vertices). We can efficiently compile for any PWC_n whether the diameter is two or is three. In the following results, we will show first that it is possible to have diameter two, and second characterize when the diameter is two versus three.

Definition 5.2: We define A_n to be an oriented line graph of K_n such that for any three elements $a,b,c \in [n]$, whenever $a<b<c$, $ab \rightarrow ac$, $ac \rightarrow bc$, and $bc \rightarrow ab$

Lemma 5.3: *For each n, there exists some PWC_n for which the diameter is 2.*

Proof: We know that A_n is a PWC_n because every triplet is ordered in a three cycle by the definition. Also, be the definition, each PWC_4 in A_n will be isomorphic to A_4. We also know that since any A_4 will have diameter 2 (as all of its non-triplet triangles are not three cycles) every A_4 in A_n will have diameter 2. Since any two vertices *ab, cd* are contained in a subgraph of A_n isomorphic to A_4, we can get from any vertex of A_n to another copy in two steps or less within that A_4 copy, making A_n have diameter 2. A_n is a PWC_n, therefore, A_n is the desired sort of PWC_n.

Theorem 5.4: *The diameter of any PWC_n is*

a. *$diam(PWC_n)=3$ if there exists a PWC_4 within that PWC_n such that $diam(PWC_4)=3$, &*
b. *$diam(PWC_n)=2$ if for all PWC_4 in PWC_n $diam(PWC_4)=2$.*

Proof: a. If the diameter of a PWC_4 subgraph in PWC_n is three, then there are two vertices $ab, cd \in \mathrm{PWC}_4$, where *a*, *b*, *c*, *d* are distinct, for which within that subgraph dist(*ab*,*cd*)=3. Assume for contradiction that the diameter of PWC_n is two. Then there would be some vertex in PWC_n not in PWC_4 that connects our two vertices. But this says that our connecting vertex would have to have something in common with both *ab* and *cd*, which means the connecting vertex would appear in that PWC_4. This contradicts the fact that $diam(PWC_4)=3$. Therefore, the diameter of PWC_n is three.

b. Let D be some PWC_n, and assume that the diameter of every PWC_4 in D is two. Then, for any two vertices, say $st, uv \in$ D (the unit case being when *s, t, u, v* are disjoint) we can look at the PWC_4 induced by the set $\{s, t, u, v\}$. Since the diameter of every PWC_4 is 2, there must be a one or two-step path between the vertices *st* and *uv*. Therefore, the diameter of PWC_n is two.

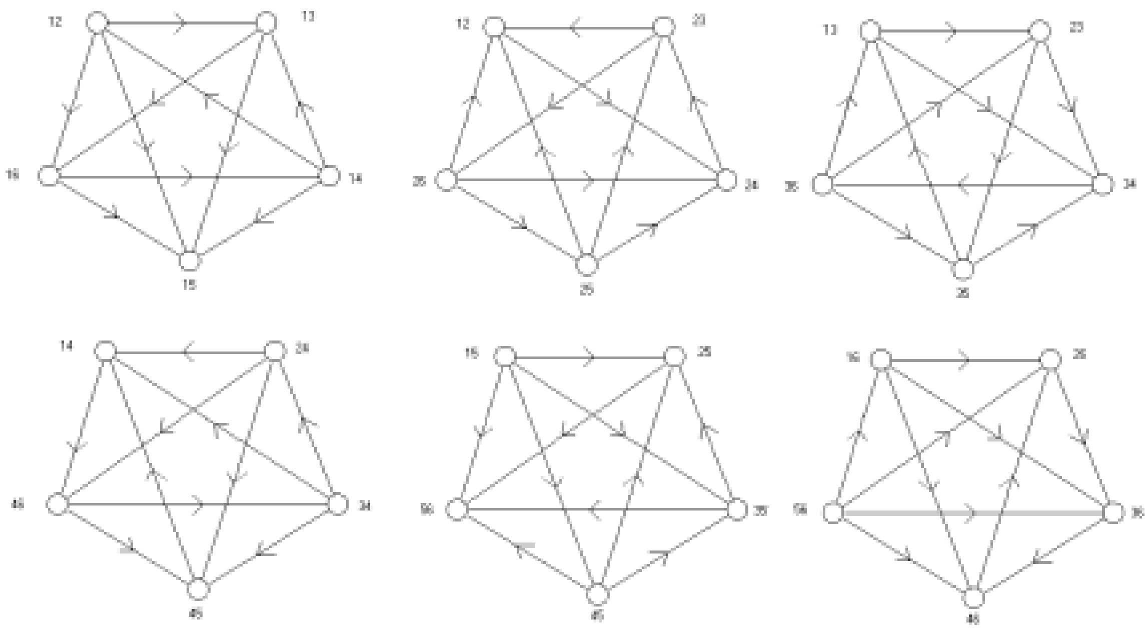
It is easy to see that because every PWC_4 in PWC_n must have diameter two for PWC_n to have diameter two, as *n* increases, the probability of having a PWC_n with diameter two decreases rapidly. Although we could not find the specific probability of this occurring, we were able to determine an upper bound.

Proposition 5.5: *The probability of a PWC$_n$ having diameter 2 is at most* $\left(\frac{7}{8}\right)^{\binom{\lfloor n/2 \rfloor}{2}}$.

Proof: We find a large number of edgewise-independent PWC$_4$ subgraphs within a given PWC$_n$. To be independent from all other PWC$_4$ subgraphs in a set, any given PWC$_4$ can share at most two elements with any other PWC$_4$. So, we will arrange the elements so that they come in pairs, where 1 and 2 are always in the same PWC$_4$, and the same for 3 and 4, 5 and 6, (n-1) and n for $1,2,3,4,5,6, n-1, n \in [n]$. There will be $\lfloor n/2 \rfloor$ of these pairs. The number of ways to combine these pairs then becomes $\binom{\lfloor n/2 \rfloor}{2}$. From the previous theorem, we know that for any PWC$_n$ to have diameter 2, all of its PWC$_4$ subgraphs must have diameter 2. Since there is a $\left(\frac{7}{8}\right)$ probability of any given PWC$_4$ to have diameter 2 and since these events are independent,, we know that the likelihood of all these PWC$_4$ subgraphs to have diameter 2 is $\left(\frac{7}{8}\right)^{\binom{\lfloor n/2 \rfloor}{2}}$, so the probability that all PWC$_4$ subgraphs have diameter 2 is no larger.

6. Tournament Theory

The structure and orientation of PWC$_n$'s can be displayed and studied using the theory of tournaments, leading to the discovery of new information about this family of graphs. To present PWC$_n$'s using the tournament structure, we consider the fact that each PWC$_n$ decomposes into n tournaments each with n-1 vertices. For each $a \in [n]$, we denote T_{n-1}^{a} the subgraph of the PWC$_n$ induced by the set of vertices {*ab* or *ba*: b $\in [n] \backslash a$}. This subgraph is clearly a tournament. Every vertex, *ab*, is in the two T_{n-1}'s, namely T_{n-1}^{a} (with the vertices of the form *ax* or *xa* with $x \in [n] \setminus a$) and T_{n-1}^{b} (with vertices of the form *by* or *yb* with $y \in [n] \setminus b$). Because there is no edge between *ab* and *cd* if *a, b, c, d*$\in [n]$ are all distinct, every edge between *ab* and some other vertex must be in one of the two T_{n-1}'s that contain vertex *ab*. So each edge of PWC$_n$ is in one and only one $T_5^{\,a}$. A PWC$_6$ decomposes into 6 T_{n-1}^{x}'s as is seen below.



Here is a more complex property of this decomposition of PWC_n.

Proposition 6.1: *Every non-triplet triangle of any PWC_n is contained entirely in exactly one T_{n-1}^{x} where $x \in [n]$.*

Proof: Assume for contradiction that there exists a non-triplet triangle that is not completely contained in a T_{n-1}^{x} where $x \in [n]$. Let *ab* be a vertex in the triangle. Without loss of generality, we choose the second vertex of the triangle, a vertex with one element in common with *ab,* to be *ac*. In order for the triangle to be part of more than one tournament, the third vertex must not include the element *a*. It must however have an element in common with each of the other two vertices, leaving *bc* as the only possible choice for the third vertex. But the triangle between those three vertices is a triplet so we reach a contradiction and can state that every non-triplet triangle of any PWC_n is contained entirely in a T_{n-1}^{x} where $x \in [n]$.

Since every triplet has no element common to all three vertices, note that no triplet is a subgraph of any one T_{n-1}.

Using this proposition we can determine bounds for the number of 3-cycles in any PWC_n.

Theorem: *The number of 3-cycles in any orientation of PWC_n is bounded such that*

$$\binom{n}{3} \leq \textit{ the number of 3-cycles } \leq \binom{n}{3} + nm'$$

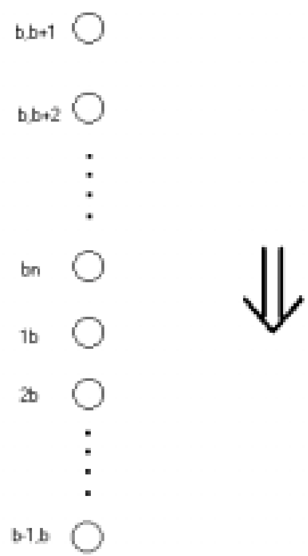
where $m' = \begin{cases} 1/24\,((n-1)^3 - (n-1)), & n = \text{even} \\ 1/24((n-1)^3 - 4(n-1)), & n = \text{odd} \end{cases}$. *Moreover, there exists an orientation of PWC_n with exactly $\binom{n}{3}$ 3-cycles, and an orientation with exactly $\binom{n}{3} + nm'$ where $m' = \begin{cases} 1/24\,((n-1)^3 - (n-1)), & n = \text{even} \\ 1/24((n-1)^3 - 4(n-1)), & n = \text{odd} \end{cases}$ 3-cycles, i.e. the bounds given above are tight.*

Proof: We begin by proving that (a) the number of 3-cycles is at least $\binom{n}{3}$ and (b) the number of 3-cycles is at most $\binom{n}{3} + nx$, where $x = \begin{cases} 1/24\,((n-1)^3 - (n-1)), & n = \text{even} \\ 1/24((n-1)^3 - 4(n-1)), & n = \text{odd} \end{cases}$. Then we prove that (c) $\binom{n}{3}$ is a tight lower bound and (d) $\binom{n}{3} + n * x$ where $x = \begin{cases} 1/24\,((n-1)^3 - (n-1)), & n = \text{even} \\ 1/24((n-1)^3 - 4(n-1)), & n = \text{odd} \end{cases}$ is an tight upper bound.

(a). We know that the number of triplets in any PWC_n is $\binom{n}{3}$ from the basic properties. By the definition of a PWC_n, each of its triplets is a 3-cycle. Therefore, there must be at least $\binom{n}{3}$ 3-cycles in every PWC_n.

(b). From proposition 6.1 we know that every non-triplet triangle is in a T_{n-1}^{x} where $x \in [n]$. So the maximum number of 3-cycles will be upper bounded by the maximum number of 3-cycles in a single tournament, multiplied by the number of tournaments and then added to the number of 3-cycles in the triplets, or $\binom{n}{3} + nm$ where m is the maximum number of 3-cycles in each of the tournaments. According to [2, pp.9], the number of 3-cycles in a tournament T_n is at most $\begin{cases} 1/24\,(n^3 - n), & n = \text{odd} \\ 1/24(n^3 - 4n), & n = \text{even} \end{cases}$, and equality holds for regular tournaments, where regular means that $d^+(v) = d^-(v)$ for all vertices v in T_n when n is odd and $d^+(v) = d^-(v) \pm 1$ for all vertices v in T_n when n is even. We are considering tournaments on n-1 vertices, so the expression is modified and the maximum number of 3-cycles in a T_{n-1} in PWC_n is $\begin{cases} 1/24((n-1)^3 - (n-1)), & n = \text{even} \\ 1/24((n-1)^3 - 4(n-1)), & n = \text{odd} \end{cases}$. So the maximum number of 3-cycles in an orientation of PWC_n is at most $\binom{n}{3} + nm$ where $m = \begin{cases} 1/24((n-1)^3 - (n-1)), & n = \text{even} \\ 1/24((n-1)^3 - 4(n-1)), & n = \text{odd} \end{cases}$.

(c). Consider the digraph A_n, an orientation of PWC_n, defined in Definition 6.2. Take an arbitrary T_{n-1}^{b} with vertices of the form bx or xb, for any $x \in [n]$. Consider any $i \in \{1,2,\ldots,b\text{-}1\}$ and $j \in \{b+1,\ldots,n\}$. From this $i < b < j$, and so, from the definition of A_n, $bj \rightarrow ib$. Let i_1 and i_2 both be elements of $\{1,2,\ldots,b\text{-}1\}$ such that $i_1 < i_2$. Then, from the definition of A_n, $i_1 b \rightarrow i_2 b$. Let j_1 and j_2 both be elements of $\{b+1,\ldots,n\}$ such that $j_1 < j_2$. Then, from the definition of A_n, $bj_1 \rightarrow bj_2$. Using these rules to organize T_{n-1}^{b}, we see that it is a transitive tournament, with vertex $\{b,b+1\}$ having outdegree n-2 and the vertex $\{b\text{-}1,b\}$ having outdegree of zero as in the figure below



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So, in A_n, T_{n-1}^{x} is transitive for every $x \in [n]$. In a transitive tournament there are no cycles, so the only 3-cycles in A_n are the triplets. Therefore, the number of 3-cycles in A_n is $\binom{n}{3}$, and $\binom{n}{3}$ is a tight lower bound.

(d). Consider an orientation B_n of PWC_n which is oriented according to the following rules. In the triplet Δabc where $a < b < c$, if $a+b+c$ is even then the edges are oriented so that $ab \to ac \to bc \to ab$. If $a+b+c$ is odd, the edges are oriented so that $bc \to ac \to ab \to bc$. Take any T_{n-1}^{x} within B_n. There are two possibilities for *x;* it could be odd or it could be even.

Case 1: Let *x* be odd. Consider a particular vertex $\{d,x\}$ of T_{n-1}^{x}. Since the remaining vertices $\{a,x\}$ of T_{n-1}^{x} ($a \neq d$) consistently contain x, here we may briefly denote vertex $\{a,x\}$ simply as *a*. To complete the proof that T_{n-1}^{x} is regular, it suffices to show that the outdegree of $\{d,x\}$ within T_{n-1}^{x} equals $\left\lfloor \frac{n-2}{2} \right\rfloor$ or $\left\lceil \frac{n-2}{2} \right\rceil$ for each *d*

Case 1.1: *d* is odd and *x<d.* Vertex *d* has an edge out to every odd numbered vertex *a* such that *a<d,* and to every even numbered vertex *e* such that *d<e.* It has an edge in from every even numbered vertex *b,* such that *b<d,* and from every odd numbered vertex *f* such that *d<f.* Thus the outdegree of *d* equals number of odd numbers less than *d* plus the number of even numbers greater than *d* and less than or equal to *n.* So $d^{+}(d) = \frac{x-1}{2} + \frac{(d-1)-(x+1)}{2} + \left\lfloor \frac{n-(d-1)}{2} \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor$.

Case 1.2: *d* is odd and *d<x.* Much as in Case 1.1, $d^{+}(d) = \frac{d-1}{2} + \frac{(x-2)-(d-1)}{2} + \left\lceil \frac{n-(x-1)}{2} \right\rceil = \left\lceil \frac{n-2}{2} \right\rceil$.

Case 1.3: *d* is even and *x<d.* Vertex *d* has an edge out to every even numbered vertex *a* such that *a<d,* and to every odd numbered vertex *e* such that *d<e.* It has an edge in from every odd numbered vertex *b* such that *b<d,* and from every even numbered vertex f such that *d<f.* Thus the outdegree of *d* equals the number of even numbered vertices less than *d* plus the number of odd numbered vertices greater than *d.* Thus $d^{+}_{T_{n-1}^{x}}(d) = \frac{x-1}{2} + \frac{(d-1)-(x+1)}{2} + \left\lceil \frac{n-(d-1)}{2} \right\rceil = \left\lceil \frac{n-2}{2} \right\rceil$.

Case 1.4: *d* is even and *d<x.* Much as Case 1.3, $d^{+}_{T_{n-1}^{x}}(d) = \frac{d-1}{2} + \frac{(x-2)-(d-1)}{2} + \left\lfloor \frac{n-(x-1)}{2} \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor$.

Case 2: Let x be even.

Case 2.1: d is odd and $x<d$. Vertex d has an edge out to every even numbered vertex a such that $a<d$, and to every odd numbered vertex e such that $d<e$. It has an edge in from every odd numbered vertex b such that $b<d$, and from every even numbered vertex f such that $d<f$. This is equivalent to Case 1.3 above.

Case 2.2: d is odd and $d<x$. Following the same rules as Case 2.1, this is equivalent to Case 1.4 above.

Case 2.3: d is even and $x<d$. Vertex d has an edge out to every odd numbered vertex a such that $a<d$, and to every even numbered vertex e such that $d<e$. It has an edge in from every even numbered vertex b, such that $b<d$, and from every odd numbered vertex f such that $d<f$. This is equivalent to Case 1.1 above.

Case 2.4: d is even and $d<x$. Following the same rules as Case 2.3, this is equivalent to Case 1.2 above.

In each case, outdegree of $\{d,x\}$ within T_{n-1}^{x} equals $\left\lfloor \frac{n-2}{2} \right\rfloor$ or $\left\lceil \frac{n-2}{2} \right\rceil$, so every T_{n-1}^{x} in B_n is regular. From part (b) of this proof we know that if a tournament is regular the equality holds and "the # of 3 cycles in one regular T_{n-1}" $= \begin{cases} 1/24\,((n-1)^3-(n-1)), & n=\text{even} \\ 1/24((n-1)^3-4(n-1)), & n=\text{odd} \end{cases}$. All n tournaments T_{n-1}, of B_n are regular, so the number of 3-cycles in B_n is $\binom{n}{3}+nm$ where $m=\begin{cases} 1/24\,((n-1)^3-(n-1)), & n=\text{even} \\ 1/24((n-1)^3-4(n-1)), & n=\text{odd} \end{cases}$. Therefore, $\binom{n}{3}+nm$ where $m=\begin{cases} 1/24\,((n-1)^3-(n-1)), & n=\text{even} \\ 1/24((n-1)^3-4(n-1)), & n=\text{odd} \end{cases}$, is an tight upper bound on the number of 3-cycles in a PWC_n.

Following easily from part c. of the above proof, we can count the number of PWC_n's that have $\binom{n}{3}$ many 3-cycles.

Corollary: *There are at least n! orientations of PWC_n that have exactly* $\binom{n}{3}$ *many 3-cycles.*

Proof: Let $f:[n]\to[n]$ be a bijection, i.e. a permutation of {1,2,…n}, there being n! many distinct such permutations *f*. To each *f* we associate digraph A_n' with vertex set $[n]^2$ by the following rule. Whenever $ab\to cd$ in A_n, we include in A_n' the edge $f(a)f(b)\to f(c)f(d)$, and we include no other edges on A_n'. Clearly f is an isomorphism between A_n and A_n'. Also, since the vectors of transitive T_{n-1}' have distinct outdegrees within T_{n-1}', each of these permutations f results in a different A_n' associated with f. Thus by this process we generate at least n! orientations of PWC_n each having the desired number of 3-cycles. The only way to have the minimum number of 3-cycles is to have 3-cycles occur only in the triplets. As the following shows, the only way to have no 3-cycles in any of the T_{n-1}'s is to have every T_{n-1} be transitive. Suppose for contradiction that T_{n-1}^x is not transitive for some $x\in[n]$. This tournament will have at least one strong component containing more than one vertex. Since there is only one edge between any two vertices, it must have more than two vertices. Any strong component with 3-or more vertices has a 3-cycle. So T_{n-1}^x is transitive for every $x\in[n]$.

For reasons of brevity, we omit the rest of the proof along these lines, we make the stronger claim that there are exactly *n*! many PWC_n graphs with only $\binom{n}{3}$ many 3-cycles, or equivalently, that each PWC_n with only $\binom{n}{3}$ many 3-cycles is isomorphic to A_n.

7. Seymour's Conjecture

A famous unsolved problem in graph theory, that of the validity of Seymour's Conjecture for all directed graphs, remains an unsolved problem. However, we can show that Seymour's Conjecture holds true for all PWC_n's.

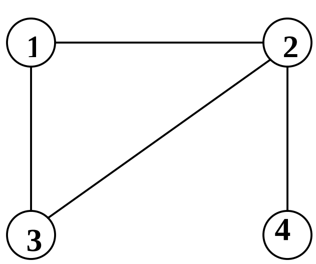
Theorem 7.1: *As in Seymour's Conjecture, in any PWCn there exists a vertex x for which* $d^{++}(x)\ge 2d^{+}(x)$.

Proof: To prove Seymour's Conjecture holds, we need only to look at the structure of any given PWC_n. Take any vertex $ab \in [n]^2$. We know this vertex is contained in *n*-2 triplets, each forming a 3-cycle. The vertex has one out-neighbor per triplet, the outdegree thus being *n*-2. Of course, all of the outneighbors will defeat the third vertex on each of their respective triplets making that third vertex reachable from *ab* in two steps but not one. Since no vertex other than *ab* is in more than one of the triplets containing *ab*, it follows that $d^{++}(ab) \geq 2d^{+}(ab)$. So in fact each vertex of PWC_n satisfies the condition of the conjecture. Therefore, Seymour's Conjecture holds.

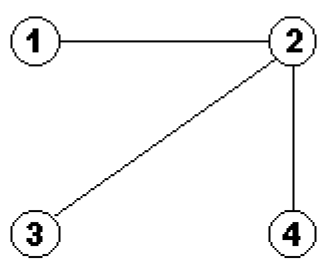
8. Linear Algebra

This section covers linear algebraic facts applied to the line graph $L(K_n)$ and PWC_n and other related material. This includes counting spanning trees, predicting an eigenvalue based on the digraph, and eigenvalues of acyclic digraphs.

Definition 8.1: A spanning tree, **τ**, of an undirected graph, G is a connected acyclic subgraph containing all the vertices of the graph. An example of this is:



G



τ(G)

Definition 8.2: An adjacency matrix is a square matrix (A) containing entries that are 0 or 1 where, $A_{i,j} = \begin{cases} 1, p_i \to p_j \\ 0, p_i \not\to p_j \end{cases}$

Definition 8.3: For G on *n* vertices, Spec L (G) is a list of the *n* eigenvalues, $\lambda, \lambda_{1,} \lambda_{2\ldots}\lambda_n$ of the adjacency matrix A of G.

Theorem 8.4: The number of spanning trees of the line graph of the complete graph K_n is

$$n^{n-2} 2^{\frac{1}{2}n(n-3)+1} (n-1)^{\frac{1}{2}n(n-3)-1} .$$

Proof: Biggs [1, pp.19-20] works out Spec L (K_n) as follows, with the first row giving the value, of the distinct eigenvalues, and the second row giving the number of times each eigenvalue appears with multiplicity. The spectrum of a in question is Spec L (K_n) =

$$\begin{pmatrix} 2n-4 & n-4 & -2 \\ 1 & n-1 & \frac{1}{2}n(n-3) \end{pmatrix}.$$

West [3, Theorem 8.6.28] states that if G is a k-regular connected simple graph with eigenvalues k = $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$, then

$$\tau(G) = \frac{(k-\lambda_2)(k-\lambda_3)(k-\lambda_4)\ldots(k-\lambda_n)}{n}$$

, and the denominator equaling the number of vertices in G. Since L(K_n) is a k-regular connected simple graph with $\binom{n}{2}$ many vertices, we can use this result to find the number of spanning trees of L(K_n). Thus,

$$\tau(L(k_n)) = \frac{(2n-4-(n-4))^{n-1}((2n-4)-(-2))^{\frac{1}{2}n(n-3)}}{\binom{n}{2}} = \frac{n^{n-1}(2n-2)^{\frac{1}{2}n(n-3)}}{\frac{n(n-1)}{2}} =$$

$$\frac{n^{n-1}(2)^{\frac{1}{2}n(n-3)}(n-1)^{\frac{1}{2}n(n-3)}}{\frac{n(n-1)}{2}} = \left(\frac{n^{n-1}}{n}\right)\left(\frac{(n-1)^{\frac{1}{2}n(n-3)}}{n-1}\right)(2)^{\frac{1}{2}n(n-3)}(2) = n^{n-2}(n-1)^{\frac{1}{2}n(n-3)-1}(2)^{\frac{1}{2}n(n-3)=1} .$$

Concerning directed graphs, the theory of their eigenvalues is more complicated. In particular, the eigenvalues may be complex. But at least we can use the following well-known fact.

Theorem 8.5: *If digraph D is k- out regular, then k is an eigenvalue for D (i.e. for A (D)).*

Proof: Let *A(D)* be an adjacency matrix for a k-out regular digraph D. Since D is k-out regular, each row of *A(D)* has k entries equal to one and all the other entries zeros. Consider a column vector, $\vec{v}$, consisting of all ones. When *A(D)* is multiplied by $\vec{v}$ in the resulting product the column vector resulting has each entry equal to k. So, $A(D) * \vec{v} = k\,\vec{v}$, proving k is an eigenvalue for A(D).

Corollary: *n-2 is an eigenvalue of each* PWC_{n}.

Proof: Knowing that at least one of the eigenvalues of a k-out regular graph is going to be the outdegree, we can say that since *n*-2 is the formula for the out-degree of PWC_n, then it is also an eigenvalue of PWC_n.

There is a theorem, for acyclic graphs, for determining the eigenvalues. But, first we need to know something about the structure of A(D) when D is acyclic.

Theorem 8.6: *The vertices of any acyclic digraph D can be ordered so that the resulting adjacency matrix is upper triangular.*

Proof: Assume D is a finite acyclic digraph.

Claim: D contains a vertex v_1 of outdegree 0. For contradiction, assume the opposite, that each vertex *x* has $d^+(x) \geq 1$. Now choose a longest path in D, ending in a vertex v_1. Then, one of the two cases holds for the final vertex v_1:

Case 1: v_1 has an out edge to a previous vertex in the path. This won't work as it contradicts the acyclic nature of D.

Case 2: v_1.has an out edge to a vertex not in the path. This creates a longer path, contradicting and the maximality of the former path. Therefore, there are no out edges so $d^+(v_1) = 0$. This vertex v_1 now is placed last in an order we are forming, so that the bottom row of the adjacency matrix A(D):

$$A(D)= \begin{array}{c} \\ \\ v_1 \end{array} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, look at the matrix A(D-x_k), where x_k is the vertex that corresponds to A(D) matrix, this A(D-x_k), where D- x_k is a subgraph induced by all vertices of D- x_k, where the bottom vertex of A(D-x_k) must have $d^{+}_{D-x_k}(x) = 0$. Going on we see this is true for A(D-x_{k-1}) and so on. This proves that we can order the vertices of acyclic, so that when we overlap the subgraph adjacency matrices, we get A(D) to be upper triangular .

This proof helps in understanding the following theorem.

Theorem 8.7: *For an acyclic digraph one of the eigenvalues will always be zero.*

If all acyclic digraphs can be arranged such that one vertex will have $d^{-}(x)$ of zero. Then the adjacency matrix will have one row of all zeros. While finding the eigenvalue, compute *det(A-λI$_n$)*, where λ is a variable, *A* is the adjacency matrix, and I_n is the Identity matrix. The *n* roots of the resulting polynomial are the *n* eigenvalues. Multiply I_n by λ resulting in a diagonal of λ's with the other entries zero. After subtracting λI_n from *A*, we get an upper triangular matrix with -λ's down the diagonal.

$$det \begin{bmatrix} -\lambda & x_{1,2} & \cdots & x_{1,(n-1)} & x_{1,n} \\ 0 & -\lambda & \cdots & x_{2,(n-1)} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & x_{(n-1),n} \\ 0 & 0 & \cdots & 0 & -\lambda \end{bmatrix} (-\lambda)^n,$$ whose *n* roots are all zeros.

The determinant will be equal to the product of the diagonal entries. From this, we see λ = 0 is a solution, proving that 0 is one of the eigenvalues of D .

9. Conclusions and Further Research

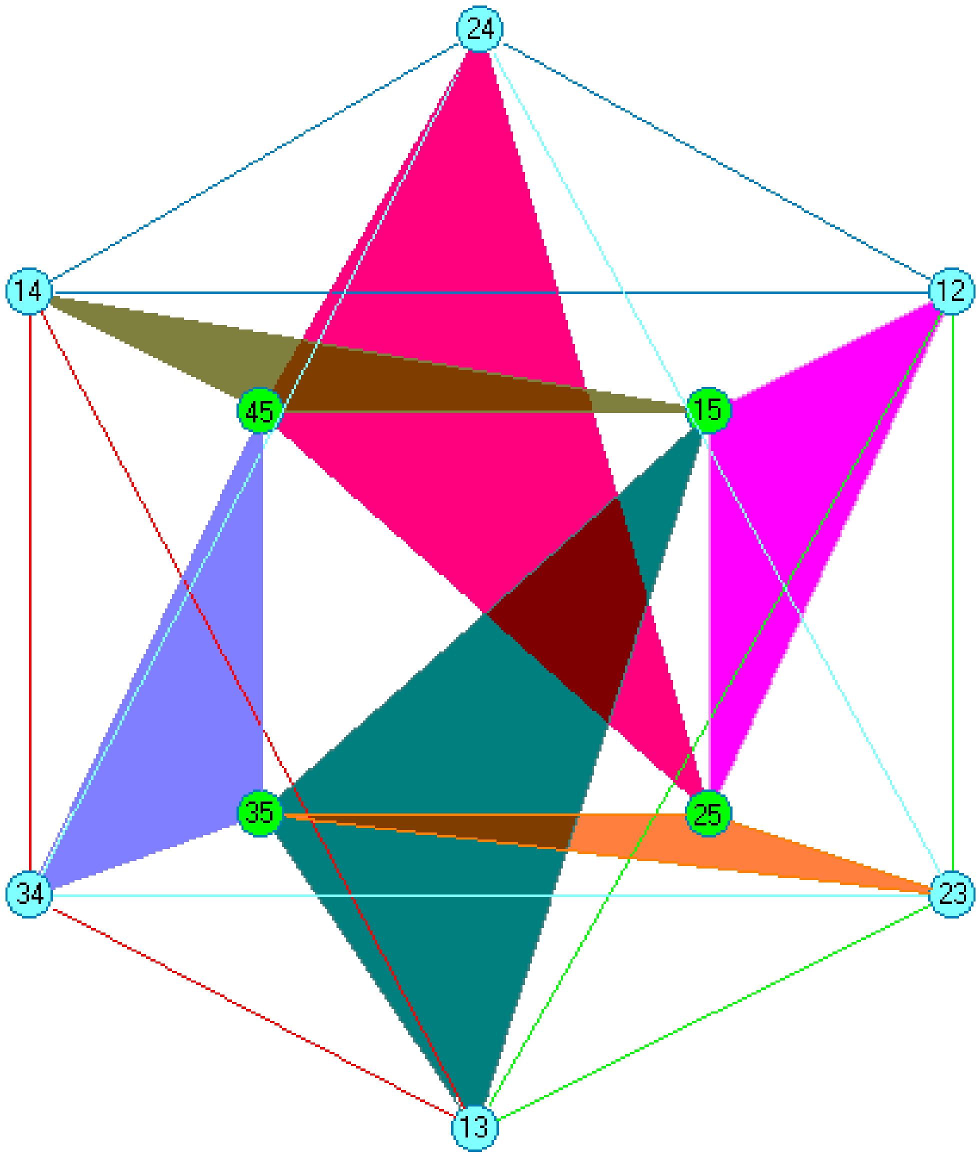
In this paper we have launched study of the pinwheel graph, and begun to exploring some of its properties and characteristics. We have been able to prove such things as the existence of an Euler tour, exact rules for diameter, bounds on the number of 3-cycles, the validity of Seymour's conjecture for pinwheels, and the relationship between adjacency matrices and eigenvalues. Despite these varied results, there is much left to discover about the pinwheel family of graphs.

Another avenue of research, related to what we have done, would be to change the rules. What if the triplets were directed transitively, rather than our previously required 3-cycles? We do know there are $6^{\binom{n}{2}}$ ways to orient such a "transitive pinwheel", that the average outdegree in these is the number of edges divided by the number of vertices, namely $\dfrac{3\binom{n}{3}}{\binom{n}{2}} = n-2$, and the diameter is sometimes infinite, i.e. these are not always strong. Interestingly enough, this kind of graph does not necessarily have Euler tours, nor is it necessarily out regular. With this in mind, there is much unanswered concerning both the pinwheel family of graphs and its transitive counterpart.

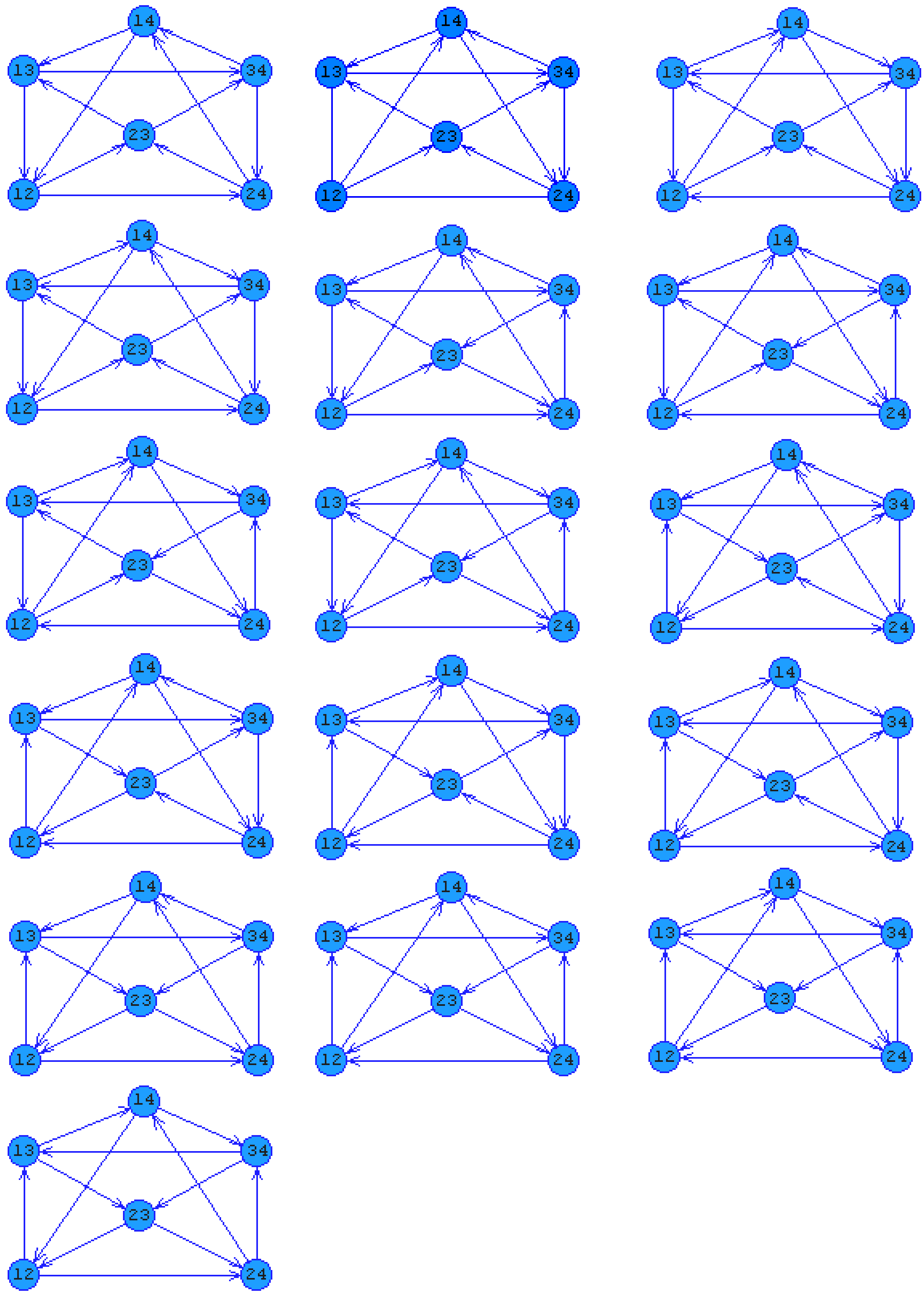
10. Acknowledgements

As we leave our mark as SUMSRI participants, we would like to give special thanks to our research advisor Dr. Daniel Pritikin for his time and encouragement throughout the program. Thanks to Eric Appelt, our graduate assistant, for his patience and for sharing his enthusiasm for the subject. We would also like to thank all of the SUMSRI staff, especially the directors, Ms. Bonita Porter, our short course professors, graduate assistants and fellow participants. We would like to express our gratitude to the National Security Agency, National Science Foundation and Miami University, without whom this program would not exist. To those who encouraged us to come and to those who encouraged us while we were here, thank you.

11. Appendix



1. $L(K_5)$, i.e. an unoriented $PWC_{5,}$ its triplets shaded.



2. The 16 orientations of PWC_4

12. References

[1] N. Biggs, *Algebraic Graph Theory*, (Cambridge University Press, Cambridge), pp.19—20.

[2] J. Moon, *Topics on Tournaments*, (Holt, Rinehart, and Winston, NY,1968).

[3] D. West, *Introduction to Graph Theory*, (Prentice Hall, NJ, 2001).