The Continuum Hypothesis is True... for all Practical Purposes

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Abstract

Let G be an uncountable subset of the real numbers. It is shown that if G is a closed set, open set, or F_σ set, then the cardinality of G is the same as the cardinality of the real numbers. In particular, the Continuum Hypothesis is true for the basic sets in the construction of the Borel sets.

1 Introduction

Set theory is the language of mathematics. Everything mathematicians know and use today can be traced back to set theory and the eight axioms of Ernst Zermelo and Abraham Fraenkel, denoted ZF. These basic suppositions are used to develop the natural, rational, and real numbers and, thus, mathematics. Without a foundation, it would be impossible to define any concepts or formulate any theorems or lemmas. More importantly, it is necessary to establish undefined concepts in order to prevent a never-ending cycle of philosophic speculation and doubt about our processes. Due to the intuitive and powerful nature of ZF, set theorists agree upon these axioms without formal justification. Fortunately, there are models, or structures, of set theory in which ZF is true. Seeing ZF work in a model helps to give credibility to the system of axioms.

Furthermore, there are models that are consistent with some of the axioms and not with others. These models illustrate the relative independence of the axioms. Similarly, there are statements that are independent of ZF. That is, there exist models of ZF where the statement is true and there exist models of

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ZF where the statement is false. Statements of this kind are not necessarily intuitive; therefore, they are called undecidable questions.

A highly controversial undecidable question that was first proposed by Zermelo in 1904 is the Axiom of Choice (AC): If $\mathcal{X}$ is a pairwise disjoint family of nonempty sets, then there is a set C which consists of one and only one element from each set in $\mathcal{X}$. In 1935, Kurt Gödel showed that ZF plus AC is consistent provided that ZF is. Also, in 1963, Paul Cohen showed that ZF and not AC are consistent if ZF is. Together, these results show that AC is an undecidable question. It should be noted that AC is not needed when $\mathcal{X}$ is a finite collection of sets. However, AC makes it possible to choose elements in infinite collections without having a defined rule for doing so. Some mathematicians do not accept this axiom, but others do since it is needed to prove many theorems, including several that are obviously true. We will accept AC in this paper and use it along with ZF, forming our set theoretic axiomatic system Zermelo-Fraenkel-Choice, or ZFC.

This paper will focus on another debated undecidable question in mathematics, the Continuum Hypothesis (CH): If A is an infinite subset of the real numbers then either $|A| = |\mathbb{N}|$ or $|A| = |\mathbb{R}|$. Georg Cantor showed in 1874 that there is indeed more than one level of infinity. That is, there is a low-level "countable infinity" and a high-level "uncountable infinity." Cantor suggested that the actual number of real numbers may be the infinity immediately following countable infinity. Since he could not prove this outright, CH was born, named from the fact that the reals represent a linear continuum.

Paralleling the way AC was shown to be independent, Gödel showed that CH is consistent with ZF, and Cohen showed that not CH is also consistent with ZF. Thus, CH is independent of ZF and is also an undecidable question. However, CH is so controversial that it has never been accepted as an axiom of set theory; it has remained an independent idea. It should also be noted that AC and CH are consistent with each other; Gödel reached this result in 1938. He also showed that every model of ZF has a sub-model in which AC and CH both hold.

Much research has also been done to determine the sizes of different sets in mathematics. For example, the set of irrational numbers and the set of all functions from the natural numbers to the natural numbers each have the same cardinality as the reals. We will prove that CH holds for open sets, closed sets, and F_σ sets. These results are the basis for proving that CH is true for the collection of sets known as the Borel sets. This is important, because the Borel sets are commonly used in practical applications.

2 Preliminaries

Before beginning our proof, we will familiarize the reader with some necessary definitions and theorems, as well as a description of the Borel sets.

Definition 2.1 *A set $G \subseteq \mathbb{R}$ is* **open** *if for all x in G, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq G$.*

Definition 2.2 *A set $F \subseteq \mathbb{R}$ is* **closed** *if for all sequences $\{x_n\}_{n=1}^{\infty} \subseteq F$, such that $\{x_n\}$ converges to x, then $x \epsilon F$. (Alternatively, F is closed if its complement is open.)*

Definition 2.3 *A set z is an* **ordinal** *if and only if*
1) $x \epsilon y \epsilon z \Rightarrow x \epsilon z$ (z is transitive)
2) $x \epsilon y \epsilon w \epsilon z \Rightarrow x \epsilon w$ (Every element of z is transitive).
If α is an ordinal and there is no such ordinal β with $\alpha = \beta + 1$, then α is a **limit ordinal**.

We use the notation ω_1 to denote the first uncountable ordinal - ordinals are well ordered by definition.

Theorem 2.4 $|\mathbb{R}| = 2^{\omega}$. *(This is called the continuum, and is often denoted by c.)*

Definition 2.5 $|A| \leq |B|$ *if there exists an injection from A to B.*

Theorem 2.6 *(Cantor-Bernstein or Schröder-Bernstein) $|A| \leq |B|$ and $|B| \leq |A|$ implies $|A| = |B|$.*

The following is a description of how the Borel subsets of the real line are actually constructed:

First, let $\mathcal{B}_0$ be the collection of open sets. Then, let $\mathcal{B}_1$ be all the complements of elements of $\mathcal{B}_0$ together with $\mathcal{B}_0$. Now suppose that $\alpha < \omega_1$. If α is a limit ordinal or has the form $\lambda + 2m$, where λ is a limit ordinal and $m \epsilon \omega$, then let $\mathcal{B}_\alpha = \{\cup_{n\epsilon\omega} A_n |$ for each n, $A_n \epsilon \mathcal{B}_\beta$ for some $\beta < \alpha\}$. If $\alpha = \lambda + 2m + 1$, where λ is a limit ordinal, then let $\mathcal{B}_\alpha = \mathcal{B}_{\lambda+2m} \cup \{X \backslash A |\ A \epsilon \mathcal{B}_{\lambda+2m}\}$. Ultimately, $\mathcal{B} = \cup_{\alpha<\omega_1} \mathcal{B}_\alpha$ and the elements of $\mathcal{B}$ are the Borel sets.

Notice that $\mathcal{B}_1$ is the set of all open sets and closed sets. In addition to these, $\mathcal{B}_2$ will also contain the new sets formed by taking countable unions of closed sets. These are called F_σ sets. F_σ sets will also include all countable unions of open sets and closed sets since every open set can be written as a countable union of closed sets. Also, $\mathcal{B}_2$ will include the countable unions of open sets since such unions will just be open.

3 Open Sets

Since we know that any open set is uncountable, the first result of this paper is that CH holds for open sets.

Lemma 3.1 *If G is an element of $\mathcal{B}_0$, then $|G| = |\mathbb{R}|$.*

Proof. Let G be an element of $\mathcal{B}_0$. Define the function $f : G \to \mathbb{R}$ by $f(x) = x$. Clearly f is an injection, so $|G| \leq |\mathbb{R}|$. Now, we want to show that $|\mathbb{R}| \leq |G|$. Since G must contain an open interval, it will suffice to show that any open interval has the same cardinality as the real numbers. Define the function $g : \mathbb{R} \to (a, b)$ by $g(x) = \left(\frac{b-a}{\pi}\right) \arctan(x) + \frac{a+b}{2}$. To show that g is injective, let x and y be arbitrary elements of $\mathbb{R}$, and suppose $g(x) = g(y)$. Hence, $\left(\frac{b-a}{\pi}\right) \arctan(x) + \frac{a+b}{2} = \left(\frac{b-a}{\pi}\right) \arctan(y) + \frac{a+b}{2}$. Thus, $\arctan(x) = \arctan(y)$. Since arctan is an inverse function, it is an injection. Hence $x = y$, and it follows that g is an injection. Therefore, $|\mathbb{R}| \leq |G|$, and by the Cantor-Bernstein Theorem, $|G| = |\mathbb{R}|$. ■

4 Closed Sets

Any infinite closed set is either countable or uncountable. If it is countable, then by definition, the set has the same cardinality as the natural numbers. If it is uncountable, we show that the set has the same cardinality as the real numbers. First, it is necessary to prove that any uncountable set can be divided at a point to form two uncountable subsets.

Lemma 4.1 *For any uncountable set A, there exists an $x \epsilon \mathbb{R}$ such that the $(-\infty, x) \cap A$ and $(x, \infty) \cap A$ are both uncountable.*

Proof. Let A be an uncountable set. Indirectly, suppose for all $x \epsilon \mathbb{R}$, either $(-\infty, x) \cap A$ or $(x, \infty) \cap A$ is countable. Notice that both cannot be countable, since A is uncountable. Let (r_n) be an enumeration of $\mathbb{Q}$, the rational numbers. For every $n \epsilon \mathbb{N}$, let A_n be $(-\infty, r_n) \cap A$ if $(-\infty, r_n) \cap A$ is countable, otherwise let $A_n = (r_n, \infty) \cap A$. We will show $A \setminus \cup_{n=1}^{\infty} A_n$ is countable, thus proving A is countable. If $A \setminus \cup_{n=1}^{\infty} A_n = \emptyset$, then A is countable. If not, there exists $y \epsilon A \setminus \cup_{n=1}^{\infty} A_n$. Now, observe that for any $y \epsilon A$,

$$A = [\cup_{r_n < y} (-\infty, r_n) \cap A] \cup \{y\} \cup [\cup_{r_n > y} (r_n, \infty) \cap A].$$

For all $r_n < y$, $(-\infty, r_n) \cap A$ is countable because $y \notin \cup_{n=1}^{\infty} A_n$. Similarly, for all $r_n > y$, $(r_n, \infty) \cap A$ is countable. Thus $A \setminus \cup_{n=1}^{\infty} A_n = \{y\}$. Since $A \setminus \cup_{n=1}^{\infty} A_n$ is countable, A is countable, which is a contradiction. ■

Corollary 4.2 *For any uncountable set A, there exist $a, b \in \mathbb{R}$, where $a < b$, such that $(-\infty, a) \cap A$ and $(b, \infty) \cap A$ are both uncountable.*

Proof. Apply Lemma 4.1 twice. ■

With this result, we have the necessary tools to show that an uncountable closed set has the same cardinality as the real numbers.

Lemma 4.3 *If F is an uncountable element of $\mathcal{B}_1$, then $|F| = |\mathbb{R}|$.*

Proof. Since we have already proven this lemma for open sets, we can assume that F is closed. Without loss of generality, let $F \subseteq [0, 1]$ since $(0, 1)$ is isomorphic to $\mathbb{R}$. Define the function $f : F \to \mathbb{R}$ by $f(x) = x$. Clearly, f is an injection, so $|F| \leq |\mathbb{R}|$. Now, we want to show that $|\mathbb{R}| \leq |F|$. To obtain this result, we use the following construction which is based on the construction of the Cantor set in reference[**?**]. Let T be the set of all finite sequences of 0's and 1's. We construct a system of closed intervals $H = \{H_t | t \in T\}$ as follows:

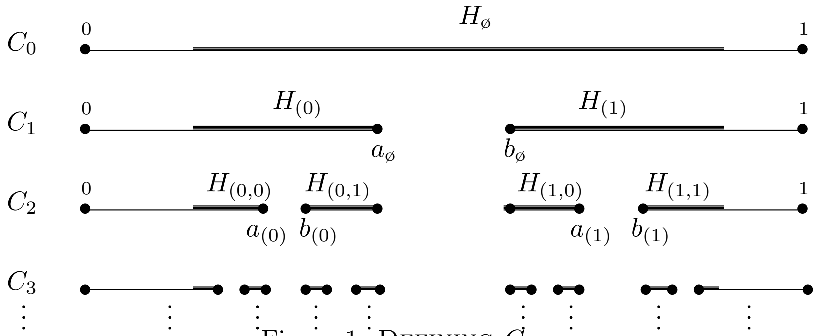

Figure 1: Defining C_F

Let

$$H_\emptyset = [0, 1] \cap F,$$

and choose $a_\emptyset, b_\emptyset \in H_\emptyset$ so that both $(0, a_\emptyset) \cap F$ and $(b_\emptyset, 1) \cap F$ are uncountable. This is possible because of Corollary 4.2.

Now let

$$H_{(0)} = [0, a_\emptyset] \cap F$$

and let

$$H_{(1)} = [b_\emptyset, 1] \cap F.$$

Next, choose $a_{(0)}, b_{(0)} \in H_{(0)}$ so that $(0, a_{(0)}) \cap H_{(0)}$ is uncountable and $(b_{(0)}, 1) \cap H_{(0)}$ is uncountable. Then, choose $a_{(1)}, b_{(1)} \in H_{(1)}$ in the same way.

Let

$$\begin{aligned} H_{(0,0)} &= [0, a_{(0)}] \cap H_{(0)}, \\ H_{(0,1)} &= [b_{(0)}, 1] \cap H_{(0)}, \\ H_{(1,0)} &= [0, a_{(1)}] \cap H_{(1)}, \\ H_{(1,1)} &= [b_{(1)}, 1] \cap H_{(1)}. \end{aligned}$$

In general, choose $a_{(x_1,x_2,\ldots x_n)}, b_{(x_1,x_2,\ldots x_n)} \in H_{(x_1,x_2,\ldots x_n)}$ so that $(0, a_{(x_1,x_2,\ldots x_n)}) \cap H_{(x_1,x_2,\ldots x_n)}$ is uncountable and $(b_{(x_1,x_2,\ldots x_n)}, 1) \cap H_{(x_1,x_2,\ldots x_n)}$

is uncountable. Then let

$$H_{(x_1,x_2,...x_n,0)} = [0, a_{(x_1,x_2,...x_n)}] \cap H_{(x_1,x_2,...x_n)}$$

and let

$$H_{(x_1,x_2,...x_n,1)} = [b_{(x_1,x_2,...x_n)}, 1] \cap H_{(x_1,x_2,...x_n)}.$$

Let $C_n = \cup\{H_t | t$ is a finite sequence of length n in $T\}$. Now let $C_F = \cap_{n\in\omega} C_n$. Notice, each C_n is closed since it is the finite union of sets H_t, and each H_t is closed since it is the intersection of a closed interval with a closed set. Thus, C_F is closed and $C_F \subseteq F$. Since $|\mathbb{R}| = 2^\omega$, and 2^ω is the set of all infinite sequences of 0's and 1's, we can map the real numbers into C_F by mapping 2^ω into C_F. Now, let f be an infinite sequence of 0's and 1's, where $f|_n$ is the finite sequence consisting of the first n terms of f. Let $H_f = \cap_{n=1}^\infty H_{f|_n}$. Notice that $\{H_{f|_n}|$ for all $n \in \omega\}$ is a set of nested intervals. That is, $H_{f|_1} \supseteq H_{f|_2} \supseteq H_{f|_3}. \ . \ .$ Furthermore, each H_f is non-empty because any limit l of a sequence of points chosen from each $H_{f|_n}$ must be in H_f. Also, because F is closed, l must be in F. This shows that we can map any infinite sequence of 0's and 1's to a point in C_F by mapping f to a point in H_f. We now show that this mapping is one-to-one. Choose two infinite sequences $f = (f_1, f_2, ...)$ and $g = (g_1, g_2, ...)$, and let n be the first place where $f_n \neq g_n$. This means that $H_{f|_n} \cap H_{g|_n} = \emptyset$. Therefore, $H_f \cap H_g = \emptyset$ and f is mapped to a different limit point than g. Thus, $|\mathbb{R}| \leq |C_F| \leq |F|$, which means $|\mathbb{R}| = |F|$. ■

5 F_σ Sets

So far, we have shown that CH is true for open sets and closed sets. All that remains to be shown in this paper is that CH is true for countable unions of open and closed sets, or F_σ sets.

Theorem 5.1 *If J is an uncountable element of $\mathcal{B}_2$, then $|J| = |\mathbb{R}|$.*

Proof. Since 2 is of the form $\lambda + 2m$, where λ is the limit ordinal 0 and $m = 1$, then $\mathcal{B}_2 = \{\cup_{n\epsilon\omega} A_n | \forall n\epsilon\omega,\ A_n \epsilon B_\lambda$ for some $\lambda < 2\}$ as defined in our construction of the Borel sets. In particular, $\mathcal{B}_2 = \{\cup_{n\epsilon\omega} A_n | \forall n\epsilon\omega,\ A_n \epsilon B_1\}$. Let J be an uncountable element of $\mathcal{B}_2$. Then, $J = \cup_{n\epsilon\omega} A_n$ where $A_n \epsilon B_1$. Suppose A_n is countable for all $n\epsilon\omega$, then J is countable. This is a contradiction. So there exists $n\epsilon\omega$ such that A_n is uncountable. Since $A_n \in \mathcal{B}_1$, we have already shown that $|A_n| = |\mathbb{R}|$. Hence $|\mathbb{R}| \leq |J|$, because $A \subseteq J$. Recall that $J \subseteq \mathbb{R}$, so we have $|J| \leq |\mathbb{R}|$. Therefore $|J| = |\mathbb{R}|$. ■

6 Conclusion

The theorem that CH is true for Borel sets was first proven by P. Alexandrov and was published in a French journal in 1916. This result is particularly interesting, because the Borel sets are the only sets we use in engineering and most areas of mathematics. For this reason, the controversy of the Continuum Hypothesis is rarely addressed. The Borel sets are also the smallest collection of sets needed for probability theory, since complements and countable unions and intersections are crucial concepts in this field.

We have illustrated the first critical stages of a full proof of CH for the Borel sets, and the remainder can be obtained using the Principle of Mathematical Induction. This is a difficult task, but we plan to continue to work to develop a complete proof. Our future research also includes exploring alternate constructions of the Borel sets and obtaining further results based on the closed set construction that we have introduced. We would also like to explore instances in which CH fails. Another interesting result is that c, the continuum, can only be equal to certain numbers since it does not have countable cofinality. Future research could possibly include exploring this notion along with searching for consistent results for the size of c.

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