Fuzzy Voronoi Diagram

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Abstract. In this paper, with first introduce a new extension of Voronoi diagram. We assume Voronoi sites to be fuzzy sets and then define Voronoi diagram for this kind of sites, and provide an algorithm for computing this diagram for fuzzy sites. In the next part of the paper we change sites from set of points to set of fuzzy circles. Then we define the *fuzzy Voronoi diagram* for such sets and introduce an algorithm for computing it.

Key words: Fuzzy Voronoi Diagram, Voronoi Diagram, Fuzzy Voronoi Cell, Fuzzy Geometry, Fuzzy Set

1 Introduction

Fuzzy objects becomes to be focused after 1965 when Zadeh introduced Fuzzy set for the first time and after that it becomes a part of other fields. In this paper we work on Fuzzy Voronoi diagrams. It's an important task for two reasons. First that this diagram will be helpful in Fuzzy spaces, and somehow in Probabilistic spaces. Second is that Voronoi diagram is used in other fields and even other sciences. So defining this kind of diagram will solve the same problems when other fields switch context to fuzzy ones. Voronoi diagram is studied in some extensions. These extensions are based on changing the meter of the space, dimension of the space or sites of the diagram [1], [2], [3]. Also some other Voronoi diagrams are introduced, like weighted Voronoi diagram in [4] and approximate version of Voronoi diagram in [5]. In this paper we first change the set of sites to fuzzy set and then to set of fuzzy circles, and introduce Voronoi diagrams for these types of sets and provider algorithms for computing them.

Definition 1. *Let P be a discrete subset of a metric space like X. For every point p in P, the set of all points x in X which their distance from p is lower (or equal) to other points of P is said to its Voronoi cell (or Diricle domain) and be shown by $V(p)$. In mathematical words:*

$$V(p) = \{x \in X \,|\forall q \in P \,[d(x,p) \leq d(x,q)]\,\} \tag{1}$$

Definition 2. *Let P be a discrete subset of a metric space like X. Voronoi diagram of P will be the set of all Voronoi cells of its points, which is shown by $V(P)$. Members of P also called Voronoi cite. In mathematical words:*

$$V(P) = \{V(p) \,|p \in P\} \tag{2}$$

Because of topological properties of this set we can equivalently define the Voronoi diagram as set of boundary of it. In this paper we work on the second definition. Also this boundary could be the points which have the property that, they have maximum distance from the nearest site(s). If we assume $\mathbb{R}^2$ as the space and use Euclidian meter, this boundary would be some line segments. An example of such diagrams is shown in Figure 1.

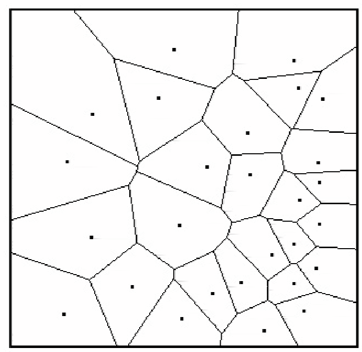


Fig. 1. A Sample Voronoi Diagram for a Set of Points.

2 Fuzzy Voronoi Diagram

Definition 3. *Let $\tilde{P}$ be any fuzzy set. The non-fuzzy image of it will be shown by P (with the same Alphabet, without tilde), and will be defined as follows:*

$$P = \left\{ x \,\middle|\, \exists \lambda \ (x, \lambda) \in \tilde{P} \right\} \tag{3}$$

Definition 4. *Let to be a fuzzy subset of R2. Fuzzy Voronoi cell of a point will be defined as follows:*

$$\tilde{V}(\tilde{P}) = \left\{ (x, \chi_P(x)) \,\middle|\, \forall \tilde{q} \in \tilde{P} \, [d(x, p) \leq d(x, q)] \right\} \tag{4}$$

Definition 5. *Let $\tilde{P}$ be a fuzzy subset of $\mathbb{R}^2$. Fuzzy Voronoi diagram of it will be defined as follows. Let (x, α) be a member of boundary of fuzzy Voronoi cell of* *a point p in P, (x, φ) will be a member of fuzzy Voronoi diagram of $\tilde{P}$ in which:*

$$\varphi = \top_{(x,\alpha)\in\tilde{V}(\tilde{p})\wedge\tilde{p}\in\tilde{P}}(\alpha) \tag{5}$$

Note: In this paper we use $\top$ and $\perp$ for T-Norm and S-Norm.

Theorem 1. *Non-fuzzy image of fuzzy Voronoi diagram of a fuzzy set of sites is Voronoi diagram of non-fuzzy image of the set of sites.*

Proof. Let (x, α) be a member of fuzzy Voronoi diagram of $\tilde{P}$. So this point is member of at least two fuzzy Voronoi cell like Voronoi cells of $\tilde{p_{i_1}}$ and $\tilde{p_{i_2}}$. Based on Definition 3, $d(x, p_{i_1}) = d(x, p_{i_2})$ and also

$$\forall \tilde{q} \in \tilde{P}\left[d(x, p_{i_1}) \leq d(x, q)\right] \tag{6}$$

So we have simliar result for P:

$$\forall q \in P\left[d(x, p_{i_1}) \leq d(x, q)\right] \tag{7}$$

And because of Definition 2, x would be a member of $V(P)$.

Reverse assume that x be a member of $V(P)$. So there must exists sites $p_{i_1}, \ldots, p_{i_n}$ $(n \geq 2)$ such that x be a member of their Voronoi cells, and without loss of generality assume $p_{i_1}, \ldots, p_{i_n}$ be all of such sites. So for any j in the range, $\left(\chi_P\left(p_{i_j}\right)\right)$ would be a member of $\tilde{V}(\tilde{p_{i_j}})$. So based one Definition 5:

$$\left(x, \top_{1\leq j\leq n}\left(\chi_P\left(p_{i_j}\right)\right)\right) \in \tilde{V}(\tilde{P}) \tag{8}$$

And also $\chi_P\left(p_{i_j}\right) > 0$ holds for all j which garanties:

$$\top_{1\leq j\leq n}\left(\chi_P\left(p_{i_j}\right)\right) > 0 \tag{9}$$

3 Algorithm for Fuzzy Voronoi Diagram

According to Theorem 1 we can compute Fuzzy Voronoi diagram of a fuzzy set easily. We can compute the classic Voronoi diagram for the non-Fuzzy image of set of sites according to Fortune algorithm [6] and while computing the diagram we can compute the degree of membership for each line segment in $O(1)$ because any line segment will be boundary of two Voronoi cell so computing of the $\top$ will cost $O(1)$ time. So the total time of algorithm would be $O(n.\log(n)$ which the time of Fortune algorithm is.

4 Fuzzy Voronoi Diagram for Fuzzy Circles

In the previous section we introduce an algorithm for computing Fuzzy Voronoi diagram for Fuzzy set of points. In this section we extend the definition of Voronoi diagram to cover a larger set of objects not only points. Assume that the members of the set of sites become itself sets. It means that each site is a set of points which can be any geometric object like circle, rectangle and etc. This extension of Voronoi diagram for no-Fuzzy objects is studied in many researches like [4], [5], [7]. But the Voronoi diagram of a set of Fuzzy objects is not much similar to non-Fuzzy version like previous sections. To clarify the difference, consider the following example. Assume that we have only two sites which both are circles with equal radius which both have continues characteristic function. And assume that the degree of membership converges to zero in boundary of circles. Now let we want to have non-fuzzy Voronoi diagram for non-fuzzy image of these sites, it's clear that the diagram would be bisector of their centers. But if we want to convert it to fuzzy one, like previous section; we see that the degree of membership of this line in the diagram would be zero because it made from points which membership degree is lower than any (according to the continuity of the membership functions which was assumed) so the line wouldn't be part of the diagram!

The problem occurs because the Voronoi diagram of fuzzy objects should be created using all of its points and according to their degree of membership. So we would have a diagram like Figure 2 for such sites.

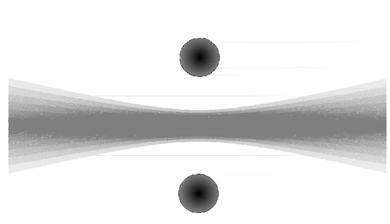


Fig. 2. Fuzzy Voronoi Diagram of two Fuzzy Circle.

Definition 6. *Let $\tilde{P}$ be a finite family of fuzzy subsets of $\mathbb{R}^2$, then $\tilde{x}$ would be a member of its fuzzy Voronoi diagram $\tilde{V}(\tilde{P})$, iff there exists $\tilde{p_i}$ and $\tilde{p_j}$ in $\tilde{P}$ such that the distances of x from at least one pair of points like x_i and x_j in them be equal and also $\tilde{p_i}$ and $\tilde{p_j}$ be two of the most nearest sites to x. The degree of membership of x in $\tilde{V}(\tilde{P})$ would be $\perp$ of all such pairs.*

Definition 7. *A fuzzy subset of $\mathbb{R}^2$ is called a fuzzy circle iff there exists a disk in $\mathbb{R}^2$ such that:*

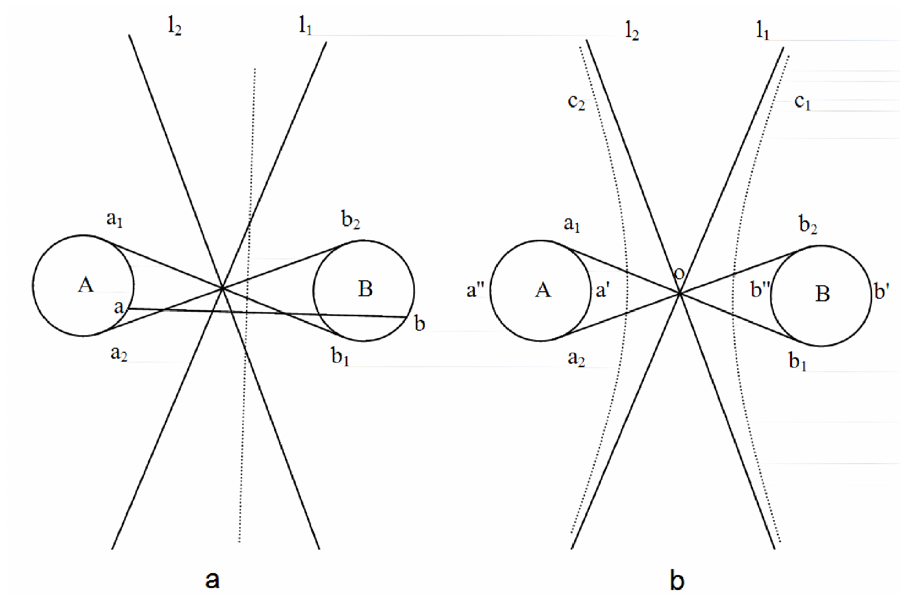
- *All points of the set are member of the disk.*
- *In any open neighborhood of a boundary point of the disk, there exists at least one point of the fuzzy set (with membership degree greater than zero).*

Note: If we remove the second condition from the above definition any bounded subset of $\mathbb{R}^2$ becomes a fuzzy circle.

It is the time to compute the Voronoi diagram for fuzzy circles. In this paper we assume all circles have the same radius and this section we assume that they boundary points have non zero degree of membership which can be simply proved that is not important because of the second condition of the Definition 7 and also connectedness of $\mathbb{R}^2$.

As the Definition 6 says the Voronoi diagram of two set will be fuzzy union of Voronoi diagram of pair of points which each one belongs to one the sets. But this sentence doesn't lead to an algorithm because this operation is not a computable operation. So we should analyze those diagrams to provide an algorithm.

In this part we introduce a method for computing the boundary of fuzzy Voronoi diagram of two fuzzy circles. We assume that this boundary will be created by boundary points of the circle and by Theorem 2 we prove this assumption. Let A and B be two fuzzy circle (Fig. 4.a and let a and b be a pair of point on their boundary. Slope of their connector line is clearly between slope of $\overline{a_1b_1}$ and $\overline{a_2b_2}$ which common tangents of these circles are. So slope of their bisectors would be between l_1's and l_2's. So these bisectors in a bounded interval is out of the area which is between l_1 and l_2.



Also the function which maps pair of points of A and B to their bisector is differentionable so that's enough to consider the boundary points, the farthest lines from o, which are made by (a', b') and (a'', b'') (Figure 4.b). These pairs make two lines which are perpendicular on connector line of center of circles and their distance is equal to diameter of the circles. And according to the result of above paragraph (bisectors are inside the area between l_1 and l_2 except in a bounded interval) and also continuity of the mapping function of pairs of points to the lines, we can assume a shape like dotted one in the Figure **??**.b.

Now we should compute the boundary of the diagram not by introducing its shape! Let (A_x, A_y) and (B_x, B_y) be centers of circles and both radius be r. First we must compute points a_1, a_2, b_1 and b_2 which is simply done in $O(1)$. Then we must iterate arcs $a_1a'a_2$ and $b_1b'b_2$ simultaneously and compute their bisector lines. We use parametrization for simpler computation. For simplifying of the result let both circles be on a horizontal line. As it shown in Figure 4 while the angle changes on A from m to $2\pi - m$, the angle on B is changing from $\pi + m$ to $\pi - m$ and so:

$$a_x = A_x + r.\cos(t_1) \tag{10}$$
$$a_y = A_y + r.\sin(t_1) \tag{11}$$
$$b_x = B_x + r.\cos(t_2) \tag{12}$$
$$b_y = B_y + r.\sin(t_2) \tag{13}$$

in which $t_2 = \frac{m}{m-\pi}(t_1 - m) + \pi + m$ and $m \leq t_1 \leq 2\pi - m$.

Now we should compute the points which each line makes on the boundary of the Voronoi diagram. For this purpose we find the intersection two lines which are infinitely near to each other. The next Maple program will compute the point for two neighbor circle.

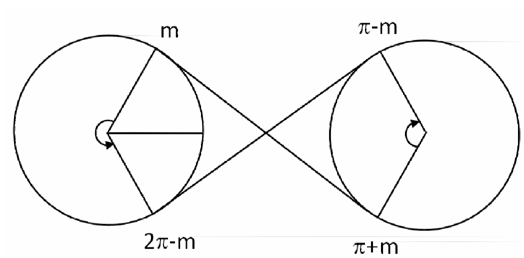


Fig. 3. Arcs on Adjacent Circles

```
ax1:=Ax+r*cos(t);
ay1:=Ay+r*sin(t);
ax2:=Ax+r*cos(t+e);
ay2:=Ay+r*sin(t+e);
bx1:=Bx+r*cos(m*(t-m)/(m-pi)+pi+m);
by1:=By+r*sin(m*(t-m)/(m-pi)+pi+m);
bx2:=Bx-r*cos(m*(t-m)/(m-pi)+pi+m+e);
by2:=By+r*sin(m*(t-m)/(m-pi)+pi+m+e);
```

```
y1:=(ay1+by1)/2+((ax1+bx1)/2-x)*((ax1-bx1)/(ay1-by1));
y2:=(ay2+by2)/2+((ax2+bx2)/2-x)*((ax2-bx2)/(ay2-by2));
x1:=(ax1+bx1)/2+((ay1+by1)/2-y)*((ay1-by1)/(ax1-bx1));
x2:=(ax2+bx2)/2+((ay2+by2)/2-y)*((ay2-by2)/(ax2-bx2));
xte:=solve(y1=y2,x);
yte:=solve(x1=x2,y);
xt:=limit(xte,e=0);
yt:=limit(yte,e=0);
simplify(yt);
simplify(xt);
```

Theorem 2. *The boundary of Voronoi diagram for empty circles and non empty ones are equal.*

Proof. Suppose a and b be a pair of points on circles A and B, and suppose one of them be inside of the circle. Without loss of generality assume that a be the interior point. Assume the connector line of a and b. This line will makes two points on the boundary of A. Assume that these points be a' and a''. Clearly bisector of $\overline{ab}$ will be between bisectors of $\overline{a'b}$ and $\overline{a''b}$. So it completely places inside the Voronoi diagram area and it won't make a point on the boundary of the diagram.

5 Algorithm for Computing Fuzzy Voronoi Diagram for Fuzzy Circles

That's enough to compute the classic Voronoi diagram for the center of circles and then for each line segment in the diagram compute the fuzzy Voronoi diagram as described above and compute the intersection between two neighbors (which costs $O(1)$ for each pair). The correctness of the algorithm follows from Definition 6 in which only the most nearest sites has role in the diagram. For computing the intersection of two neighbors that's enough to compute the above Maple program twice (for example for circle A, B and then for A, C). Let their result be two pair (x1t1,y1t1) and (x2t2, y2t2) and their parameter be t1 and t2 then the following Maple command will compute the value of t1 and t2 in which the intersection happens. So can compute the intersection points from them.

```
solve({x2t2=x1t1,y2t2=y1t1},{t1,t2});
```

6 Conclusion

In this paper we introduce fuzzy Voronoi diagram and studied it for point and circle cells and showed that the diagram can be computed in $O(n \log(n))$ time for both and this will be helpful because of its application in other fields.

Some other works still can be done on this diagram and related topics which some of them are:

- Computing the degree of membership of interior points for fuzzy Voronoi diagram of circles
- Studying the dual of this diagram which may lead to extension of Delony Triangulation
- Studying the fuzzy Voronoi diagram for other shapes
- Studying the fuzzy Voronoi diagram with other meters or in higher dimensions
- [8] creates Voronoi diagram for finding the nearest point for a fuzzy controller maybe using fuzzy Voronoi diagram makes some improvements.

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