A Complete, Self-Contained Proof of the Theory of Degrees of Freedom for Radiating Systems

Derived from the work of M. Gustafsson [3]

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Abstract

This document provides a comprehensive, step-by-step mathematical proof of the core concepts presented in the paper "Degrees of Freedom for Radiating Systems" [3]. The objective is to derive every major theorem and formula from fundamental principles, making the paper's theoretical framework fully self-contained. We begin with a rigorous treatment of Weyl's Law, define the necessary mathematical tools such as Vector Spherical Harmonics, connect these concepts to communication channel capacity via the derivation of radiation modes and water-filling, culminate in the paper's main result on the asymptotic NDoF, and conclude with implications for inverse source problems.

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1 Notation and Preliminaries

λ, k Wavelength and wavenumber ($k = 2\pi/\lambda$).

$\dagger$ Conjugate (Hermitian) transpose, denoted by a dagger (†).

Ω A spatial region occupied by the radiating system.

w_d Volume of the unit d-ball, $w_d = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$.

$\mathbf{I}$ Column vector of expansion coefficients for the electric current density $\mathbf{J}$.

$\mathbf{f}$ Column vector of expansion coefficients for the radiated far-field.

$\mathbf{U}$ The linear operator (matrix) mapping source currents to far-field coefficients: $\mathbf{f} = -\mathbf{U}\mathbf{I}$.

$\mathbf{R}_0, \mathbf{R}_\rho, \mathbf{R}$ Radiation, material loss, and total resistance matrices.

ρ_n, ν_n Eigenvalue and efficiency of the n-th radiation mode.

$A_s(\hat{\mathbf{k}})$ The geometric shadow area of Ω when viewed from direction $\hat{\mathbf{k}}$.

$\langle\cdot\rangle$ Average over all spatial directions and polarizations.

N_1 The Number of Degrees of Freedom (NDoF).

(τ, l, m) Multi-index for Vector Spherical Harmonics (VSH).

2 Vector Spherical Harmonics (VSH)

Definition 2.1 (Vector Spherical Harmonics)**.** The VSH, denoted $\mathbf{Y}_{\tau lm}(\hat{\mathbf{r}})$, are defined from scalar spherical harmonics $Y_{lm}(\hat{\mathbf{r}})$ as follows:

$$\mathbf{Y}_{1lm}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}}\nabla_S Y_{lm}(\hat{\mathbf{r}})$$
$$\mathbf{Y}_{2lm}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} \times \mathbf{Y}_{1lm}(\hat{\mathbf{r}})$$

where ∇_S is the surface gradient on the unit sphere. They form a complete, orthonormal basis for tangential vector fields on the sphere.

3 Weyl's Law and Propagating Modes

Theorem 3.1 (Weyl's Law)**.** For a region $\Omega \subset \mathbb{R}^d$, the number of eigenvalues $N(\nu)$ of the negative Laplacian operator $(-\nabla^2)$ with Dirichlet boundary conditions is asymptotically given by:

$$N_{Wd}(\nu) \approx \frac{w_d|\Omega|\nu^{d/2}}{(2\pi)^d}$$

Proof. The proof first considers a simple domain and then extends to an arbitrary one.

1. **Rectangular Domain:** For a d-dimensional box, the eigenfunctions are sinusoids, and the allowed wave vectors form a grid in k-space. The number of modes with eigenvalue less than ν (i.e., $|\mathbf{k}|^2 < \nu$) is found by counting the grid points inside a hypersphere octant of radius $\sqrt{\nu}$. This gives the desired formula by relating the number of points to the ratio of the k-space volume to the volume-per-mode.

2. **Extension to Arbitrary Domains:** The extension relies on the Dirichlet-Neumann bracketing principle. The eigenvalues of an operator are related to the calculus of variations. For the Dirichlet problem, the n-th eigenvalue can be found by minimizing a Rayleigh quotient over an n-dimensional space of functions that are zero on the boundary. If we have two domains $\Omega_{in} \subset \Omega$, any test function on Ω_{in} can be extended by zero to be a valid test function on Ω. This means the space of test functions for Ω_{in} is a subspace of that for Ω, which implies by the min-max principle that $\nu_n(\Omega) \leq \nu_n(\Omega_{in})$. By tiling space with small cubes and defining Ω_{in} as the union of cubes inside Ω, and Ω_{out} as the union of cubes intersecting Ω, we get the bound $N(\nu, \Omega_{out}) \leq N(\nu, \Omega) \leq N(\nu, \Omega_{in})$. As the cube size goes to zero, the volumes of Ω_{in} and Ω_{out} both approach $|\Omega|$, and the leading term of the count is recovered by the squeeze theorem.

□

4 Capacity, Losses, and Radiation Modes

Theorem 4.1 (Channel Capacity with Power Constraint)**.** The capacity of the MIMO channel $\mathbf{f} = -\mathbf{UI} + \mathbf{n}$ is found by solving:

$$C = \max_{\mathrm{Tr}(\mathbf{RP})=1, \mathbf{P} \geq 0} \log_2(\det(\mathbf{I} + \gamma \mathbf{UPU}^\dagger))$$

Proof. The problem diagonalizes in the basis of radiation modes, reducing to the maximization of $\sum_n \log_2(1 + \gamma \nu_n p_n)$ subject to $\sum_n p_n = 1$ and $p_n \geq 0$. This is solved using a Lagrangian:

$$\mathcal{L}(\{p_n\}, \mu) = \sum_n \log_2(1 + \gamma \nu_n p_n) - \mu \left(\sum_n p_n - 1 \right)$$

Setting the partial derivative with respect to p_n to zero yields the stationarity condition:

$$\frac{\partial \mathcal{L}}{\partial p_n} = \frac{1}{\ln 2} \frac{\gamma \nu_n}{1 + \gamma \nu_n p_n} - \mu = 0 \quad \Longrightarrow \quad p_n = \frac{1}{\mu \ln 2} - \frac{1}{\gamma \nu_n}$$

Incorporating the positivity constraint $p_n \geq 0$ gives the water-filling solution:

$$p_n = \max \left(0, \frac{1}{\mu_0} - \frac{1}{\gamma \nu_n} \right)$$

where the constant "water level" $\mu_0 = \mu \ln 2$ is chosen to satisfy the total power constraint $\sum_n p_n = 1$. □

5 The Asymptotic NDoF and Shadow Area

Theorem 5.1 (Average Maximum Effective Area)**.** The maximum partial effective area, averaged over all directions and polarizations, is:

$$\langle \max A_{eff} \rangle = \frac{\lambda^2}{8\pi} \sum_{n=1}^{\infty} \nu_n$$

Proof. The proof requires evaluating $\langle |a_n|^2 \rangle$. Following [3, Appendix B], for an incident plane wave with coefficient normalization $a_n = 4\pi j^{\tau-1-l} \hat{\mathbf{e}} \cdot \mathbf{Y}_n(\hat{\mathbf{k}})$, this average becomes:

$$\langle |a_n|^2 \rangle = \frac{(4\pi)^2}{8\pi^2} \int_{S^2} \int_{\text{pol}} |\hat{\mathbf{e}} \cdot \mathbf{Y}_n(\hat{\mathbf{k}})|^2 \, d\Omega_{\hat{\mathbf{e}}} \, d\Omega_{\hat{\mathbf{k}}}$$

The inner polarization integral evaluates to $\pi |\mathbf{Y}_n(\hat{\mathbf{k}})|^2$. The outer integral over the sphere, by VSH orthonormality, is 1. Combining constants yields $\langle |a_n|^2 \rangle = 2\pi$. Substituting this into the expression for $\langle \max A_{eff} \rangle$ gives the desired result. □

5.1 The Main Result: NDoF from Shadow Area

1. **Asymptotic Behavior of Radiation Modes:** As illustrated in [3, Figs. 5, 10], for electrically large, low-loss objects, the efficiencies $\{\nu_n\}$ bifurcate, allowing the approximation $\sum \nu_n \approx N_1$. This gives $\langle \max A_{eff} \rangle \approx \frac{\lambda^2}{8\pi} N_1$.

2. **High-Frequency Limit and the Optical Theorem:**

   Theorem 5.2 (Optical Theorem)**.** The total power extinguished from an incident beam, $P_{ext} = P_{sca} + P_{abs}$, is related to the imaginary part of the vector scattering amplitude $\mathbf{f}(\hat{\mathbf{k}})$ in the forward direction by $\sigma_{ext} = P_{ext}/I_{inc} = (4\pi/k) \operatorname{Im}\{\hat{\mathbf{e}}_{inc} \cdot \mathbf{f}(\hat{\mathbf{k}}_{inc})\}$.

   Proof. The total power flowing out of a large sphere enclosing the scatterer is $P_{out} = \oint \mathbf{S}_{total} \cdot d\mathbf{A}$. The total field is $\mathbf{E} = \mathbf{E}_{inc} + \mathbf{E}_{sca}$. The time-averaged Poynting vector has three terms: $\mathbf{S}_{inc}$, $\mathbf{S}_{sca}$, and an interference term $\mathbf{S}_{int} = \frac{1}{2} \operatorname{Re}\{\mathbf{E}_{inc} \times \mathbf{H}^*_{sca} + \mathbf{E}_{sca} \times \mathbf{H}^*_{inc}\}$. The net power removed from the incident beam is $P_{ext} = -\oint \mathbf{S}_{int} \cdot d\mathbf{A}$. The incident field is a plane wave, e.g., $\mathbf{E}_{inc} = \hat{\mathbf{e}} E_0 e^{ikz}$, and the scattered field is an outgoing spherical wave, $\mathbf{E}_{sca} \sim \mathbf{f}(\hat{\mathbf{k}}) \frac{e^{ikr}}{r}$. Evaluating the integral of $\mathbf{S}_{int}$ over the large sphere via the method of stationary phase shows that the only contribution comes from the forward direction ($\theta = 0$), where the phases of the plane wave and spherical wave match. The result of this integration yields the theorem. In the high-frequency limit, $\sigma_{ext} \to 2A_s$, and for a highly absorbing object, $\sigma_{abs} \approx A_{eff} \to A_s$. □

3. **Conclusion:** Equating the two asymptotic expressions for $\langle \max A_{eff} \rangle$ gives the final result:

$$\frac{\lambda^2}{8\pi} N_1 \approx \langle A_s \rangle \implies \boxed{N_1 \approx \frac{8\pi \langle A_s \rangle}{\lambda^2}}$$

Lemma 5.3 (Cauchy's Mean Cross Section Formula)**.** For any convex body K, the average shadow area is one-quarter of its total surface area A. That is, $\langle A_s \rangle = A/4$.

Proof. The projected area (shadow) of K onto a plane with normal $\hat{\mathbf{u}}$ is $A_s(\hat{\mathbf{u}}) = \int_{\partial K} \max(0, \hat{\mathbf{n}} \cdot \hat{\mathbf{u}})\, dS$, where $\hat{\mathbf{n}}$ is the outward normal at a point on the surface ∂K. To find the average shadow area, we integrate this over the unit sphere S^2 and divide by 4π.

$$\langle A_s \rangle = \frac{1}{4\pi} \int_{S^2} A_s(\hat{\mathbf{u}})\, d\Omega_{\hat{\mathbf{u}}} = \frac{1}{4\pi} \int_{S^2} \left(\int_{\partial K} \max(0, \hat{\mathbf{n}} \cdot \hat{\mathbf{u}})\, dS \right) d\Omega_{\hat{\mathbf{u}}}$$

By Fubini's theorem, we can swap the order of integration:

$$\langle A_s \rangle = \frac{1}{4\pi} \int_{\partial K} \left(\int_{S^2} \max(0, \hat{\mathbf{n}} \cdot \hat{\mathbf{u}})\, d\Omega_{\hat{\mathbf{u}}} \right) dS$$

The inner integral is over all directions $\hat{\mathbf{u}}$. For a fixed $\hat{\mathbf{n}}$, the term $\hat{\mathbf{n}} \cdot \hat{\mathbf{u}}$ is positive over exactly one hemisphere. Let $\hat{\mathbf{n}}$ point along the z-axis. Then $\hat{\mathbf{n}} \cdot \hat{\mathbf{u}} = \cos\theta$. The integral is $\int_0^{2\pi} \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta\, d\phi = 2\pi[\frac{1}{2}\sin^2\theta]_0^{\pi/2} = \pi$. This result is independent of the choice of $\hat{\mathbf{n}}$.

$$\langle A_s \rangle = \frac{1}{4\pi} \int_{\partial K} (\pi)\, dS = \frac{\pi}{4\pi} \int_{\partial K} dS = \frac{1}{4} A$$

□

6 Implications for Inverse Source Problems

The NDoF concept also dictates the stability of inverse problems. Reconstructing the source current $\mathbf{I}$ from noisy measurements $\mathbf{f}$ via Tikhonov regularization leads to a solution for the coefficients of the radiation modes, c_n:

$$c_n = \frac{\rho_n}{\rho_n + \delta} \cdot c_n^{\text{unreg}}$$

The NDoF is the number of modes that can be stably reconstructed (where $\rho_n \gg \delta$).

References

[1] C. F. Bohren and D. R. Huffman, *Absorption and Scattering of Light by Small Particles*. New York: Wiley-Interscience, 1983.

[2] A. L. Cauchy, "Sur la rectification des courbes et la quadrature des surfaces courbes," *Mémoires de l'Académie des sciences de l'Institut de France*, vol. 22, pp. 3-15, 1832.

[3] M. Gustafsson, "Degrees of Freedom for Radiating Systems," *IEEE Transactions on Antennas and Propagation*, vol. 73, no. 2, pp. 1028-1038, Feb. 2025.

[4] G. Kristensson, *Scattering of Electromagnetic Waves by Obstacles*. Edison, NJ: SciTech Publishing, 2016.

[5] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators*. New York: Academic Press, 1978.

[6] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, 2nd ed. Cambridge: Cambridge University Press, 2014.