Theory of Characteristic Modes for Conducting Bodies

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Abstract—A theory of characteristic modes for conducting bodies is developed starting from the operator formulation for the current. The mode currents form a weighted orthogonal set over the conductor surface, and the mode fields form an orthogonal set over the sphere at infinity. It is shown that the modes are the same ones introduced by Garbacz to diagonalize the scattering matrix of the body. Formulas for the use of these modes in antenna and scatterer problems are given. For electrically small and intermediate size bodies, only a few modes are needed to characterize the electromagnetic behavior of the body.

I. Introduction

Characteristic modes have long been used in the analysis of radiation and scattering by conducting bodies whose surfaces coincide with coordinate surfaces of coordinate systems in which the Helmholtz equation is separable. Recently Garbacz [1] has shown that similar modes can be defined for conducting bodies of arbitrary shape. He approached the problem by diagonalizing the scattering matrix. This led him to the conclusion that the mode currents are real and the tangential electric mode field is of constant phase over the surface of the body. Garbacz, Turpin, and Wickliff [1], [3], [4] used this property to find the characteristic currents in a few cases, but they did not obtain convenient formulas for computing the mode currents in general.

In this paper we approach the problem from the alternative viewpoint of diagonalizing the operator relating the current to the tangential electric field on the body. By choosing a particular weighted eigenvalue equation, we obtain the same modes as defined by Garbacz. Our approach leads to a simpler derivation of the theory and to explicit formulas for determining the mode currents and fields. For clarity, we summarize the complete theory of such modes, although much of the theory is given explicitly or implicitly in Garbacz [1].

II. Characteristic Currents

Consider the problem of one or more conducting bodies, defined by the surface S, in an impressed electric field $\boldsymbol{E}^i$. An operator equation for the current J on S is [6]

$$[L(J) - \boldsymbol{E}^i]_{\tan} = 0 \tag{1}$$

Manuscript received October 13, 1970; revised March 1, 1971. This work was supported in part by the NSF under Grant GK4227, and by the Air Force Cambridge Research Laboratories under Contract F19628-68-C-0180.

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where the subscript "tan" denotes the tangential components on S. The operator L is defined by

$$L(J) = j\omega A(J) + \nabla\Phi(J) \tag{2}$$

$$A(J) = \mu \oiint_S J(r')\psi(r,r')\, ds' \tag{3}$$

$$\Phi(J) = \frac{-1}{j\omega\epsilon} \oiint_S \nabla'\cdot J(r')\psi(r,r')\, ds' \tag{4}$$

$$\psi(r,r') = \frac{\exp(-jk\,|\,r - r'\,|)}{4\pi\,|\,r - r'\,|}. \tag{5}$$

Here r denotes a field point, r' a source point, and ϵ, μ, and k the permittivity, permeability, and wavenumber, respectively, of free space. Physically, $-L(J)$ gives the electric intensity $\boldsymbol{E}$ at any point in space due to the current J on S. In an antenna problem, the impressed field $\boldsymbol{E}^i$ is the negative of the tangential component of $\boldsymbol{E}$ over S, assumed known. In a scattering problem, the impressed field $\boldsymbol{E}^i$ is due to known sources external to S.

We define the symmetric product of two vector functions $\boldsymbol{B}$ and $\boldsymbol{C}$ on S as

$$\langle B,C\rangle = \oiint_S \boldsymbol{B}\cdot\boldsymbol{C}\, ds. \tag{6}$$

The product $\langle B^*,C\rangle$, where the asterisk denotes complex conjugate, defines an inner product for the complex Hilbert space of all square-integrable vector functions on S. The operator appearing in (1) has the dimensions of impedance, and we introduce the notation

$$Z(J) = [L(J)]_{\tan}. \tag{7}$$

That Z is a symmetric operator, i.e., $\langle B,ZC\rangle = \langle ZB,C\rangle$, follows from the reciprocity theorem [5]. However, Z is not a Hermitian operator, i.e., $\langle B^*,ZC\rangle \neq \langle Z^*B^*,C\rangle$. Because Z is symmetric, its Hermitian parts are real and given by

$$R = \frac{1}{2}(Z + Z^*) \tag{8}$$

$$X = \frac{1}{2j}(Z - Z^*). \tag{9}$$

Now $Z = R + jX$, where R and X are real symmetric operators. Furthermore, R is positive semidefinite, since the power radiated by a current J on S is $\langle J^*,RJ\rangle \geq 0$. If no resonator fields exist internal to S, then R is positive definite, i.e., all currents radiate some power, however small.

Next consider the eigenvalue equation

$$Z(J_n) = \nu_n M(J_n) \tag{10}$$

where ν_n are eigenvalues, J_n are eigenfunctions, and M is a weight operator to be chosen. The eigenfunctions for any choice of symmetric M will diagonalize Z, but only the choice $M = R$ also gives orthogonality of the radiation patterns. Hence, we choose $M = R$ and set $Z = R + jX$ in (10), obtaining

$$(R + jX)(J_n) = \nu_n R(J_n). \tag{11}$$

We next let

$$\nu_n = 1 + j\lambda_n \tag{12}$$

and cancel the common term $R(J_n)$ in (11), obtaining

$$X(J_n) = \lambda_n R(J_n). \tag{13}$$

Both X and R are real symmetric operators. Hence, all eigenvalues λ_n and eigenfunctions J_n must be real. The J_n must also satisfy the usual orthogonality relationships

$$\langle J_m, RJ_n \rangle = 0$$
$$\langle J_m, XJ_n \rangle = 0$$
$$\langle J_m, ZJ_n \rangle = 0 \tag{14}$$

where $m \neq n$. Furthermore, since the J_n are real, the orthogonality relationships are also valid for inner products, i.e.,

$$\langle J_m^*, RJ_n \rangle = 0$$
$$\langle J_m^*, XJ_n \rangle = 0$$
$$\langle J_m^*, ZJ_n \rangle = 0 \tag{15}$$

where $m \neq n$. The choice of $\{J_n\}$ as basis functions therefore simultaneously leads to diagonal matrix representations of R, X, and Z. We shall call these J_n the *characteristic currents* or *eigencurrents* of the conducting body defined by S.

So far the eigencurrents are of indeterminate amplitude. Each eigencurrent which radiates can be normalized according to

$$\langle J_n^*, RJ_n \rangle = 1 \tag{16}$$

i.e., it radiates unit power. Each eigencurrent associated with an internal resonance cannot be so normalized, but they are not needed for radiation problems. When normalized according to (16), the orthogonality relationships (14) and (15) can be combined with (16) to give

$$\langle J_m, RJ_n \rangle = \langle J_m^*, RJ_n \rangle = \delta_{mn}$$
$$\langle J_m, XJ_n \rangle = \langle J_m^*, XJ_n \rangle = \lambda_n \delta_{mn}$$
$$\langle J_m, ZJ_n \rangle = \langle J_m^*, ZJ_n \rangle = (1 + j\lambda_n)\delta_{mn} \tag{17}$$

where δ_{mn} is the Kronecker delta (0 if $m \neq n$, and 1 if $m = n$). For further theory we assume the eigencurrents to be normalized. If unnormalized currents are used, the factor $\langle J_n, RJ_n \rangle$ must be properly introduced into the theory.

III. Characteristic Fields and Patterns

The electric field $\boldsymbol{E}_n$ and the magnetic field $\boldsymbol{H}_n$ produced by an eigencurrent $\boldsymbol{J}_n$ on S will be called the *characteristic fields* or *eigenfields* corresponding to $\boldsymbol{J}_n$. The set of all $\boldsymbol{E}_n$ or $\boldsymbol{H}_n$ form a Hilbert space of all fields throughout space produced by currents on S. We obtain orthogonality relationships for the characteristic fields from those for characteristic currents by means of the complex Poynting theorem [5]. Explicitly, the complex power balance for currents $\boldsymbol{J}$ on S is given by

$$P = \langle J^*, ZJ \rangle = \langle J^*, RJ \rangle + j\langle J^*, XJ \rangle$$
$$= \oint\!\!\oint_{S'} \boldsymbol{E} \times \boldsymbol{H}^* \cdot d\boldsymbol{s} + j\omega \iiint_{\tau'} (\mu \boldsymbol{H} \cdot \boldsymbol{H}^* - \epsilon \boldsymbol{E} \cdot \boldsymbol{E}^*) \, d\tau \tag{18}$$

where S' is any surface enclosing S and τ' is the region enclosed by S'. Equation (18) is a Hermitian quadratic form, for which the associated Hermitian bilinear form is

$$P(J_m, J_n) = \langle J_m^*, ZJ_n \rangle. \tag{19}$$

If $\boldsymbol{J}_m$ and $\boldsymbol{J}_n$ are eigencurrents, then the orthonormality relationships (17) apply, and we have from (19) and Maxwell's equations,

$$\oint\!\!\oint_{S'} \boldsymbol{E}_m \times \boldsymbol{H}_n^* \cdot d\boldsymbol{s} + j\omega \iiint_{\tau'} (\mu \boldsymbol{H}_m \cdot \boldsymbol{H}_n^* - \epsilon \boldsymbol{E}_m \cdot \boldsymbol{E}_n^*) \, d\tau = (1 + j\lambda_n)\delta_{mn}. \tag{20}$$

This equation can be separated into real and imaginary parts to give orthogonality relationships similar to the first two of (17), if desired.

If the body S is of finite extent, and if S' is chosen to be the sphere at infinity (S_∞ of Fig. 1), then (20) gives orthogonality relationships for radiation patterns and fields. On S_∞ the characteristic fields are of the form of outward traveling waves, i.e.,

$$\boldsymbol{E}_n = \eta \boldsymbol{H}_n \times \boldsymbol{n} = \frac{-j\omega\mu}{4\pi r} \exp(-jkr) \boldsymbol{F}_n(\theta, \phi). \tag{21}$$

Here $\eta = (\mu/\epsilon)^{1/2}$ is the intrinsic impedance of space, $\boldsymbol{n}$ is the unit radial vector on S_∞, and (θ, ϕ) are the angular coordinates of position on S_∞. The complex vector $\boldsymbol{F}_n$ of (21) is called the *characteristic pattern* or *eigenpattern* corresponding to the eigencurrent $\boldsymbol{J}_n$. Adding (20) to its conjugate with m and n interchanged, we find that

$$\frac{1}{\eta} \oint\!\!\oint_{S_\infty} \boldsymbol{E}_m \cdot \boldsymbol{E}_n^* \, ds = \delta_{mn}. \tag{22}$$

Hence, the characteristic far fields form an orthonormal set in the Hilbert space of all square-integrable vector functions on S_∞. We can also express (22) in terms of the

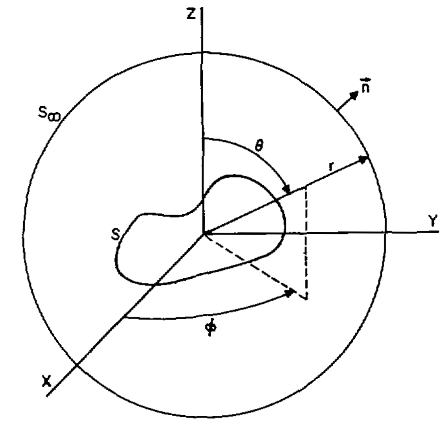

Fig. 1. Surfaces and coordinates.

characteristic magnetic field as

$$\eta \oint\!\!\oint_{S_\infty} \boldsymbol{H}_m \cdot \boldsymbol{H}_n^* \, ds = \delta_{mn}. \tag{23}$$

Finally, subtracting (20) from its conjugate with m and n interchanged, we obtain the orthogonality relationship

$$\omega \iiint (\mu \boldsymbol{H}_m \cdot \boldsymbol{H}_n^* - \epsilon \boldsymbol{E}_m \cdot \boldsymbol{E}_n^*) \, d\tau = \lambda_n \delta_{mn} \tag{24}$$

where the integration extends over all space. For $m = n$, (24) states that λ_n is 2ω times the total stored magnetic energy minus the total stored electric energy. (This assumes normalization according to $\langle \boldsymbol{J}_n, R\boldsymbol{J}_n \rangle = 1$.)

IV. Modal Solutions

A modal solution for the current $\boldsymbol{J}$ on a conducting body can be obtained by using the eigencurrents as both expansion and testing functions in the method of moments [6]. Following this procedure, we assume $\boldsymbol{J}$ to be a linear superposition of the mode currents

$$\boldsymbol{J} = \sum_n \alpha_n \boldsymbol{J}_n \tag{25}$$

where the α_n are coefficients to be determined. Substituting (25) into the operator equation (1), and using the linearity of L, we obtain

$$[\sum_n \alpha_n L \boldsymbol{J}_n - \boldsymbol{E}^i]_{\tan} = 0. \tag{26}$$

Next, the inner product of (26) with each $\boldsymbol{J}_m$ in turn is taken, giving the set of equations

$$\sum_n \alpha_n \langle \boldsymbol{J}_m, Z\boldsymbol{J}_n \rangle - \langle \boldsymbol{J}_m, \boldsymbol{E}^i \rangle = 0 \tag{27}$$

where $m = 1,2,\cdots$. Here we have put $L_{\tan} = Z$, and dropped the subscript "tan" on $\boldsymbol{E}^i$. Because of the orthogonality relationship (17), (27) reduces to

$$\alpha_n (1 + j\lambda_n) = \langle \boldsymbol{J}_n, \boldsymbol{E}^i \rangle. \tag{28}$$

The right-hand side of (28) is called the *modal excitation coefficient*:

$$V_n^i = \langle \boldsymbol{J}_n, \boldsymbol{E}^i \rangle = \oint\!\!\oint_S \boldsymbol{J}_n \cdot \boldsymbol{E}^i \, ds. \tag{29}$$

Substituting for α_n from (28) into (25), we have the modal solution for the current $\boldsymbol{J}$ on S:

$$\boldsymbol{J} = \sum_n \frac{V_n^i \boldsymbol{J}_n}{1 + j\lambda_n}. \tag{30}$$

If the eigencurrents $\boldsymbol{J}_n$ are not normalized according to (16), the term $1 + j\lambda_n$ in (30) should be replaced by $(1 + j\lambda_n) \langle \boldsymbol{J}_n, R\boldsymbol{J}_n \rangle$.

The fields are linearly related to the currents, and hence can also be expressed in modal form. Explicitly, these forms are

$$\boldsymbol{E} = \sum_n \frac{V_n^i \boldsymbol{E}_n}{1 + j\lambda_n} \tag{31}$$

$$\boldsymbol{H} = \sum_n \frac{V_n^i \boldsymbol{H}_n}{1 + j\lambda_n} \tag{32}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the fields from $\boldsymbol{J}$ everywhere in space. Again, if the eigencurrents are not normalized, the term $1 + j\lambda_n$ must be replaced by $(1 + j\lambda_n) \langle \boldsymbol{J}_n, R\boldsymbol{J}_n \rangle$.

Finally, if the reciprocity theorem [5] is used, alternative expressions for the modal excitation coefficients are obtained. For example, if $\boldsymbol{E}^i$ is produced by an electric current $\boldsymbol{J}^i$, then reciprocal to (29) we have

$$V_n^i = \iiint \boldsymbol{E}_n \cdot \boldsymbol{J}^i \, d\tau \tag{33}$$

where the integration extends over the impressed currents. Similarly, if $\boldsymbol{E}^i$ is produced by a magnetic current $\boldsymbol{M}^i$, then reciprocal to (29) we have

$$V_n^i = -\iiint \boldsymbol{H}_n \cdot \boldsymbol{M}^i \, d\tau. \tag{34}$$

More generally, if $\boldsymbol{E}^i$ is produced by both electric currents $\boldsymbol{J}^i$ and magnetic currents $\boldsymbol{M}^i$, then V_n^i is given by the sum of (33) and (34).

V. Linear Measurements

Any scalar ρ linearly related to the current, i.e., a linear functional of the current, will be called a *linear measurement* of the current. Two examples of linear measurements are 1) a component of the current at some point on S, or 2) a component of the field ($\boldsymbol{E}$ or $\boldsymbol{H}$) at some point in space. Every linear functional of $\boldsymbol{J}$ can be expressed as

$$\rho = \langle \boldsymbol{E}^m, \boldsymbol{J} \rangle \tag{35}$$

where $\boldsymbol{E}^m$ is a given vector function, usually an electric field on S. For example, if ρ is the jth component of the field E_j^J from J, then (35) becomes [5], [6]

$$E_j^J = \langle \boldsymbol{E}^j, \boldsymbol{J} \rangle \tag{36}$$

where $\boldsymbol{E}^j$ is the electric field on S produced by a j-directed electric dipole $Il = 1$ placed at the field point. If the jth component of $\boldsymbol{H}$ were desired, then a unit magnetic dipole would be placed at the field point, and so on.

If the modal solution (30) is substituted into the general measurement formula (35), there results

$$\rho = \sum_n \frac{V_n{}^i V_n{}^m}{1 + j\lambda_n} \tag{37}$$

where $V_n{}^m$ is the *modal measurement coefficient*

$$V_n{}^m = \langle J_n, E^m \rangle = \oint\!\!\oint_S J_n \cdot E^m \, ds. \tag{38}$$

Note that $V_n{}^m$ is of the same functional form as the excitation coefficient $V_n{}^i$ given by (29). Hence, (37) is a symmetric bilinear functional of $\boldsymbol{E}^i$ (the impressed field, or excitation) and of $\boldsymbol{E}^m$ (the measurement field, or adjoint excitation). Of course, the symmetry of (37) is a consequence of the symmetry of the original operator Z.

Reciprocal forms for the measurement coefficients, analogous to (33) and (34) for excitation coefficients, can also be written. For example, if the source of $\boldsymbol{E}^m$ is electric current $\boldsymbol{J}^m$, then

$$V_n{}^m = \iiint \boldsymbol{E}_n \cdot \boldsymbol{J}^m \, d\tau \tag{39}$$

analogous to (33). If the source of $\boldsymbol{E}^m$ is magnetic current $\boldsymbol{M}^m$, then

$$V_n{}^m = -\iiint \boldsymbol{H}_n \cdot \boldsymbol{M}^m \, d\tau \tag{40}$$

analogous to (34). Finally, if $\boldsymbol{E}^m$ is produced by both a $\boldsymbol{J}^m$ and an $\boldsymbol{M}^m$, the measurement coefficient $V_n{}^m$ is given by the sum of (39) and (40).

VI. Application to Radiation and Scattering Problems

Two important specializations of the general theory are 1) radiation from apertures in conducting bodies and 2) plane-wave scattering by conducting bodies. Explicit formulas for these two cases are given in this section. Other problems, such as antennas in the vicinity of conductors and near-field measurements, are also special cases of the general formulas, but they are not considered explicitly.

Consider a conducting body of surface S in which one or more apertures exist, as suggested by Fig. 2. There are sources internal to S which produce a tangential electric field $\boldsymbol{E}_{\tan}$ (assumed known) over the apertures. Then $\boldsymbol{E}^i = -\boldsymbol{E}_{\tan}$ is the impressed field, and the mode excitation coefficients (29) become

$$V_n{}^i = -\oint\!\!\oint_S \boldsymbol{J}_n \cdot \boldsymbol{E}_{\tan} \, ds. \tag{41}$$

The radiation pattern for the aperture is then given by the modal solution (31). For computation, we must deal with one number at a time, say some component of $\boldsymbol{E}$ at a particular position (θ,ϕ) on S_∞. For this, we place a unit electric dipole $\boldsymbol{Il} = \boldsymbol{u}_m$ at (θ,ϕ) on S_∞ and evaluate the modal measurement coefficient by (38) and (39). This gives

$$V_n{}^m = \oint\!\!\oint_S \boldsymbol{J}_n \cdot \boldsymbol{E}^m \, ds = \boldsymbol{E}_n \cdot \boldsymbol{u}_m \tag{42}$$

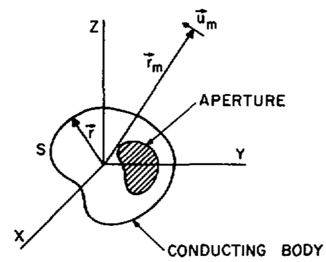

Fig. 2. Aperture antenna.

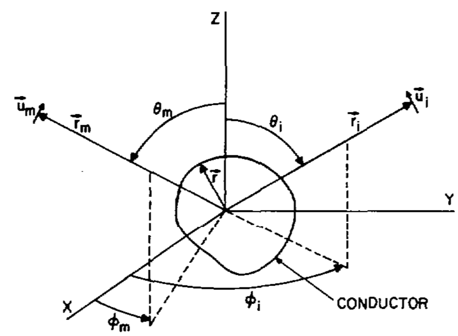

Fig. 3. Conducting scatterer.

where $\boldsymbol{E}^m$ is the field produced by the distant dipole. Explicitly, in the vicinity of S the dipole field is [5]

$$\boldsymbol{E}^m = \frac{-j\omega\mu}{4\pi r_m} \exp(-jkr_m)[\boldsymbol{u}_m \exp(-j\boldsymbol{k}_m \cdot \boldsymbol{r})]. \tag{43}$$

Here $\boldsymbol{k}_m$ is the vector propagation constant of the wave from $\boldsymbol{Il} = \boldsymbol{u}_m$ and $\boldsymbol{r}_m$ is the position vector to $\boldsymbol{Il}$ (see Fig. 2). Now (42) becomes

$$V_n{}^m = \frac{-j\omega\mu}{4\pi r_m} \exp(-jkr_m) \oint\!\!\oint_S \boldsymbol{J}_n \cdot \boldsymbol{u}_m \exp(-j\boldsymbol{k}_m \cdot \boldsymbol{r}) \, ds. \tag{44}$$

Substituting this into (31) dotted into $\boldsymbol{u}_m$, we have

$$\boldsymbol{E} \cdot \boldsymbol{u}_m = \frac{-j\omega\mu}{4\pi r_m} \exp(-jkr_m) \sum_n \frac{V_n{}^i R_n{}^m}{1 + j\lambda_n} \tag{45}$$

where the $V_n{}^i$ are given by (41) and

$$R_n{}^m = \oint\!\!\oint_S \boldsymbol{J}_n \cdot \boldsymbol{u}_m \exp(-j\boldsymbol{k}_m \cdot \boldsymbol{r}) \, ds \tag{46}$$

are the plane-wave measurement coefficients. Equation (45) provides a convenient formula for computations.

Next, consider a conducting body of surface S in a plane-wave scattering problem, as suggested by Fig. 3. The impressed field is the unit plane wave

$$\boldsymbol{E}^i = \boldsymbol{u}_i \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}) \tag{47}$$

where $\boldsymbol{u}_i$ is the polarization vector and $\boldsymbol{k}_i$ is the propagation vector. The excitation coefficients (29) are now

$$V_n{}^i = R_n{}^i = \oint\!\!\oint_S \boldsymbol{J}_n \cdot \boldsymbol{u}_i \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}) \, ds. \tag{48}$$

Note that this is of the same functional form as the plane-wave measurement coefficients (46), hence the notation

$R_n{}^i$ for (48). The determination of the scattered field at some measurement position (θ_m,ϕ_m) is the same problem as the determination of the radiation field in the antenna problem. Hence, the scattered field in the direction (θ_m,ϕ_m) is given by (45) with $V_n{}^i$ replaced by $R_n{}^i$, or

$$\boldsymbol{E}\cdot\boldsymbol{u}_m = \frac{-j\omega\mu}{4\pi r_m}\exp(-jkr_m)\sum_n \frac{R_n{}^iR_n{}^m}{1+j\lambda_n}. \tag{49}$$

A commonly used parameter in plane-wave scattering problems is the echo area, defined as [5]

$$\sigma = 4\pi r_m{}^2 \,|\, \boldsymbol{E}\cdot\boldsymbol{u}_m \,|^2. \tag{50}$$

Substituting from (49) into (50), we obtain

$$\sigma = \frac{\omega^2\mu^2}{4\pi}\left|\sum_n \frac{R_n{}^iR_n{}^m}{1+j\lambda_n}\right|^2. \tag{51}$$

Note that σ is a function of the polarization of the incident wave $\boldsymbol{u}_i$ and of the measurement wave $\boldsymbol{u}_m$, as well as of the coordinates (θ_i,ϕ_i) of the incident wave direction and (θ_m,ϕ_m) of the measurement direction.

VII. Dyadic Representations

Any bilinear functional can be represented in terms of a dyadic operator, the Dirac bra-ket notation being well suited for this purpose. In the modal solution for the current, let $E^i\rangle$ denote the tangential component of the impressed $\boldsymbol{E}$ on S, and $J\rangle$ a current on S. The characteristic currents are denoted by $J_n\rangle$ or $\langle J_n$. Then (30) becomes

$$J\rangle = \sum_n \frac{J_n\rangle\langle J_n,E^i\rangle}{1+j\lambda_n} \tag{52}$$

where we have used (29) for the excitation coefficients. Similarly, if $\langle E^m$ denotes the tangential component of the measurement field $\boldsymbol{E}^m$ on S, the general linear functional (37) becomes

$$\rho = \sum_n \frac{\langle E^m,J_n\rangle\langle J_n,E^i\rangle}{1+j\lambda_n} \tag{53}$$

where we have used (38) for the measurement coefficients. It is evident from (53) that

$$Y = Z^{-1} = \sum_n \frac{J_n\rangle\langle J_n}{1+j\lambda_n} \tag{54}$$

is a dyadic representation for the inverse operator to Z, called the spectral form of $Y = Z^{-1}$. In terms of (54), we can write (52) as

$$J\rangle = YE^i\rangle \tag{55}$$

which is the inverse equation to our starting equation $ZJ\rangle = E^i\rangle$. Similarly, we can write (53) as

$$\rho = \langle E^m,YE^i\rangle. \tag{56}$$

The inverse to this equation is

$$\rho = \langle J^m,ZJ^i\rangle \tag{57}$$

where $\langle J^m$ is the current on S excited by the measurement field $\langle E^m$ and $J^i\rangle$ is the current on S excited by the impressed field $E^i\rangle$.

If the impressed and measurement fields are produced by electric currents, we can use the reciprocal formulas (33) and (39) for the excitation and measurement coefficients. For this, we introduce the bilinear product

$$\{A,B\} = \iiint \boldsymbol{A}\cdot\boldsymbol{B}\,d\tau \tag{58}$$

where the integration is over all space. Now let $E\}$ denote the electric field $\boldsymbol{E}$ everywhere in space, and $J^i\}$ the impressed sources everywhere in space. In terms of the mode fields $E_n\}$ or $\{E_n$, we can now write the electric field (31) as

$$E\} = \sum_n \frac{E_n\}\{E_n,J^i\}}{1+j\lambda_n} \tag{59}$$

where we have used (33) for the excitation coefficients. Similarly, if $\{J^m$ denotes the source of the measurement field everywhere in space, the general linear functional (37) becomes

$$\rho = \sum_n \frac{\{J^m,E_n\}\{E_n,J^i\}}{1+j\lambda_n} \tag{60}$$

where we have used (39) for the measurement coefficients. It is now evident that

$$\Gamma = \sum_n \frac{E_n\}\{E_n}{1+j\lambda_n} \tag{61}$$

is a dyadic operator in spectral form. In terms of (61), we can write (59) as

$$E\} = \Gamma J^i\} \tag{62}$$

from which it is evident that Γ is a type of the Green's function. Explicitly, it gives the field $\boldsymbol{E}$ due to $\boldsymbol{J}$ on S (sometimes called the scattered field) when the conducting body S is excited by impressed sources $\boldsymbol{J}^i$ elsewhere in space. Similarly, (60) can be written as

$$\rho = \{J^m,\Gamma J^i\} \tag{63}$$

which is an alternative form for the general bilinear functional ρ.

A development similar to the preceding applies for the $\boldsymbol{H}$ field if the impressed and measurement fields are produced by magnetic currents. To summarize, let $H_n\}$ or $\{H_n$ represent the magnetic modal fields, and define the magnetic dyadic operator

$$\hat{\Gamma} = \sum_n \frac{H_n\}\{H_n}{1+j\lambda_n}. \tag{64}$$

Now, letting $M^i\}$ denote an impressed magnetic current, analogous to (59) and (62) we have

$$H\} = -\hat{\Gamma}M^i\} = -\sum_n \frac{H_n\}\{H_n,M^i\}}{1+j\lambda_n}. \tag{65}$$

The minus sign is due to that appearing in (40). It is evident that $-\hat{\Gamma}$ is a magnetic Green's function, giving the field $\boldsymbol{H}$ due to $\boldsymbol{J}$ on S when the conducting body is excited by impressed sources $M^i\}$ elsewhere in space. Letting $\{M^m$ denote a measurement magnetic current, analogous to (60) and (63) we have

$$\rho = \{M^m, \hat{\Gamma} M^i\} = \sum_n \frac{\{M^m, H_n\}\{H_n, M^i\}}{1 + j\lambda_n}. \qquad (66)$$

This is the most general form for a bilinear functional when both the impressed and measurement currents are magnetic.

If both electric and magnetic currents exist, it is convenient to use a six-component formulation for the problem [6]. In this case, field vectors $\psi = (\boldsymbol{E}, \boldsymbol{H})$ and source vectors $K = (\boldsymbol{J}, \boldsymbol{M})$ are defined, and equations (62) and (65) combined into a single six-component equation. We have no use at present for this generalization, and will not pursue it further.

Finally, if the electric currents for both the impressed and measurement sources are specialized to unit electric dipoles on the sphere at infinity, we obtain the bilinear scattering dyadic introduced by Garbacz [1], [2]. To be explicit, let the unit incident plane wave be produced by the distant impressed dipole

$$\boldsymbol{I l}_i = \frac{-4\pi r}{j\omega\mu} \exp(jkr) \boldsymbol{u}_i \qquad (67)$$

and let the measurement source be the unit dipole $\boldsymbol{I l}_m = \boldsymbol{u}_m$. Let the ρ of (60) be $\boldsymbol{u}_m \cdot \boldsymbol{E}$, in which case (60) reduces to

$$\boldsymbol{u}_m \cdot \boldsymbol{E} = \frac{-4\pi r}{j\omega\mu} \exp(jkr) \sum_n \frac{(\boldsymbol{u}_m \cdot \boldsymbol{E}_n)(\boldsymbol{E}_n \cdot \boldsymbol{u}_i)}{1 + j\lambda_n}. \qquad (68)$$

The pattern functions $\boldsymbol{F}_n$ are defined by (21), and (68) can be written in terms of them as

$$\boldsymbol{u}_m \cdot \boldsymbol{E} = \frac{-j\omega\mu}{4\pi r} \exp(-jkr) \sum_n \frac{(\boldsymbol{u}_m \cdot \boldsymbol{F}_n)(\boldsymbol{F}_n \cdot \boldsymbol{u}_i)}{1 + j\lambda_n}. \qquad (69)$$

Defining the dyadic pattern operator as

$$\mathfrak{F} = \sum_n \frac{\boldsymbol{F}_n \boldsymbol{F}_n}{1 + j\lambda_n} \qquad (70)$$

we can write (69) as

$$\boldsymbol{u}_m \cdot \boldsymbol{E} = \frac{-j\omega\mu}{4\pi r} \exp(-jkr)(\boldsymbol{u}_m \cdot \mathfrak{F} \cdot \boldsymbol{u}_i). \qquad (71)$$

This is the $\boldsymbol{u}_m$ component of the scattered field due to a $\boldsymbol{u}_i$ polarized incident wave. The echo area, defined by (50), is given by

$$\sigma = \frac{(\omega\mu)^2}{4\pi} |\boldsymbol{u}_m \cdot \mathfrak{F} \cdot \boldsymbol{u}_i|^2. \qquad (72)$$

The dyadic operator $\mathfrak{F}$ is valid only for the far field, not for the near field.

VIII. Scattering and Perturbation Matrices

The scattering matrix was first defined as that matrix which relates the amplitudes of incoming spherical modes to outgoing spherical modes [7]. More generally, the incoming and outgoing waves can be expanded in terms of arbitrary basis functions. We show that if the characteristic fields $\boldsymbol{E}_n$ are chosen as the basis of outgoing waves, and their conjugates $\boldsymbol{E}_n^*$ as the basis of incoming waves, then the scattering matrix is diagonalized.

In a scattering problem the far-zone field can be expressed as the sum of incoming and outgoing waves as

$$\boldsymbol{E} = \boldsymbol{E}_{\text{in}} + \boldsymbol{E}_{\text{out}}. \qquad (73)$$

For a given scatterer, for each incoming wave $\boldsymbol{E}_{\text{in}}$ there is a unique outgoing wave $\boldsymbol{E}_{\text{out}}$. The *scattering operator* is defined to be that which operates on $\boldsymbol{E}_{\text{in}}$ to give $\boldsymbol{E}_{\text{out}}$, i.e.,

$$\boldsymbol{E}_{\text{out}} = S\boldsymbol{E}_{\text{in}}. \qquad (74)$$

Given an outgoing wave $\boldsymbol{E}_{\text{out}}$, the conjugate field $\boldsymbol{E}_{\text{out}}^*$ will be an incoming wave. This is evident from either spherical mode theory, or consideration of L^*, adjoint to L of (2). The characteristic fields $\boldsymbol{E}_n$ are outgoing waves, and we choose them as basis functions for $\boldsymbol{E}_{\text{out}}$, i.e.,

$$\boldsymbol{E}_{\text{out}} = \sum_n b_n \boldsymbol{E}_n. \qquad (75)$$

The conjugates $\boldsymbol{E}_n^*$ are incoming waves, and we choose them as basis functions for $\boldsymbol{E}_{\text{in}}$, i.e.,

$$\boldsymbol{E}_{\text{in}} = \sum_n a_n \boldsymbol{E}_n^*. \qquad (76)$$

The scattering matrix $[S]$ is that which relates the column vector $\bar{b}$ (components b_n) to the column vector $\bar{a}$ (components a_n) according to

$$\bar{b} = [S]\bar{a}. \qquad (77)$$

A field of the form $\boldsymbol{E}_n + \boldsymbol{E}_n^*$ is a source-free field, shown as follows. The wave equation for the field $\boldsymbol{E}_n$ due to a current $\boldsymbol{J}_n$ is

$$\nabla \times \nabla \times \boldsymbol{E}_n - k^2 \boldsymbol{E}_n = -j\omega\mu \boldsymbol{J}_n.$$

The field $\boldsymbol{E}_n^*$ satisfies the conjugate equation. Now if $\boldsymbol{J}_n$ is real, as it is for characteristic currents, then $\boldsymbol{E}_n + \boldsymbol{E}_n^*$ satisfies the source-free wave equation. Hence, in the absence of a body, the field will be a linear superposition of fields of the standing-wave type $\boldsymbol{E}_n + \boldsymbol{E}_n^*$, i.e.,

$$\sum_n a_n (\boldsymbol{E}_n + \boldsymbol{E}_n^*). \qquad (78)$$

It is evident from (75)–(78) that, when no body is present, $\bar{b} = \bar{a}$ and the scattering matrix is the identity matrix.

When a scatterer is present, the outgoing waves are partly due to the impressed field $\boldsymbol{E}^i$ and partly due to the field from the currents $\boldsymbol{J}$ on S, called the scattered field $\boldsymbol{E}^s$. The *perturbation operator* P is defined to be that which operates on $2\boldsymbol{E}_{\text{in}}$ to yield $\boldsymbol{E}^s$, i.e.,

$$\boldsymbol{E}^s = 2P\boldsymbol{E}_{\text{in}}. \qquad (79)$$

The factor 2 was introduced by Garbacz [1] for convenience in other formulas. The field $\boldsymbol{E}^s$ is an outgoing wave, and can be expanded in the $\boldsymbol{E}_n$ as

$$\boldsymbol{E}^s = \sum_n c_n \boldsymbol{E}_n. \tag{80}$$

Expansion (76) is still used for $\boldsymbol{E}_{\text{in}}$. The *perturbation matrix* $[P]$ is that which relates the column vector $\bar{c}$ (components c_n) to the column vector $\bar{a}$ according to

$$\bar{c} = 2[P]\bar{a}. \tag{81}$$

It is evident from the definitions of $[S]$ and $[P]$ that

$$[S] = [I + 2P] \tag{82}$$

where $[I]$ is the identity matrix.

We next show that both $[S]$ and $[P]$ are diagonal matrices, and obtain their elements. The impressed field $\boldsymbol{E}^i$ is a free-space field, and hence must be of the form (78). Because of linearity, it will suffice to show that a single-mode impressed field excites only the corresponding modal current. Hence, we assume an impressed field

$$\boldsymbol{E}^i = \boldsymbol{E}_m + \boldsymbol{E}_m^*. \tag{83}$$

Then the mode excitation coefficients (29) are

$$\begin{aligned} V_n{}^i &= \langle J_n, E_m + E_m^* \rangle = -\langle J_n, ZJ_m + Z^*J_m \rangle \\ &= -(1 + j\lambda_n + 1 - j\lambda_n)\delta_{mn} = -2\delta_{mn}. \end{aligned} \tag{84}$$

Thus, all mode coefficients are zero except $V_m{}^i$, which is -2. From (31) we have

$$\boldsymbol{E}^s = \frac{-2\boldsymbol{E}_m}{1 + j\lambda_m}. \tag{85}$$

Hence, if $\boldsymbol{E}_{\text{in}}$ is $\boldsymbol{E}_m^*$, then $\boldsymbol{E}^s$ contains only $\boldsymbol{E}_m$ as shown by (85). If the incident field contains many modes, as in (78), then the scattered field will contain a sum of terms of the form of (85). Comparing (81) with (85), it is evident that

$$[P] = \begin{bmatrix} \dfrac{-1}{1 + j\lambda_1} & 0 & 0 & \cdots \\ 0 & \dfrac{-1}{1 + j\lambda_2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{86}$$

i.e., $[P]$ is diagonal with elements $-1/(1 + j\lambda_n)$. Finally, from (82) we compute the scattering matrix as

$$[S] = \begin{bmatrix} -\dfrac{1 - j\lambda_1}{1 + j\lambda_1} & 0 & 0 & \cdots \\ 0 & -\dfrac{1 - j\lambda_2}{1 + j\lambda_2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{87}$$

i.e., $[S]$ is diagonal with elements $-(1 - j\lambda_n)/(1 + j\lambda_n)$. These formulas agree with those of Garbacz [1].

IX. Discussion

An extensive theory of the characteristic modes of conducting bodies is developed in this paper, starting from the operator equation for the current on the body. These are the same modes obtained by Garbacz, who started from the scattering matrix. The statement, made several times by Garbacz and Wickliff [1], [4], that the perturbation operator transforms converging modes into diverging modes of the same form, is somewhat misleading. This paper shows that the converging modes are transformed into diverging modes which are the complex conjugate of the converging modes. We have not considered the question of completeness of the sets of mode functions in Hilbert space. Garbacz [1] considers this question, and we find his arguments convincing.

The eigenvalues λ_n range from $-\infty$ to $+\infty$, with those of smallest magnitude being more important for radiation and scattering problems. We therefore order the modes according to $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \cdots$. Also, (24) shows that those modes with positive λ have predominantly stored magnetic energy, while those with negative λ have predominantly stored electric energy. We therefore call those modes with $\lambda > 0$ *inductive modes*, and those with $\lambda < 0$ *capacitive modes*. A mode having $\lambda = 0$ is called an *externally resonant mode*. The modes corresponding to the internal cavity resonances for the conducting surface have $|\lambda| = \infty$, and do not enter into radiation and scattering problems.

We concur with Garbacz's speculation that these modes should prove to be of value, both theoretically and computationally, for radiation and scattering problems. In a companion paper [9] a straightforward method for computing the modes is given. These computations bear out the speculation that, for electrically small and intermediate size bodies, only a few modes are needed to characterize the radiation and scattering properties of the conducting body. This property, coupled with the orthogonality properties of the modes, should make them valuable for synthesis and optimization problems in antenna and scattering theory.

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