A Comprehensive Proof of the Theory of Characteristic Modes

Derived from the foundational paper by R. F. Harrington and J. R. Mautz

IEEE Transactions on Antennas and Propagation, Vol. AP-19, No. 5, September 1971

Abstract

This document provides a detailed, step-by-step derivation of the theory of characteristic modes for conducting bodies. Each concept, equation, and theorem from the foundational paper is proven from first principles to ensure a complete and self-contained explanation. The proofs follow the original operator-based formulation.

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1 The Operator Formulation of Electromagnetic Problems

1.1 The Integral Equation for Currents

We begin with the time-harmonic Maxwell's equations, assuming an $e^{j\omega t}$ time dependence:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{1}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J} \tag{2}$$

The electric field $\mathbf{E}$ can be expressed in terms of the magnetic vector potential $\mathbf{A}$ and the scalar electric potential Φ:

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi \tag{3}$$

For a surface current $\mathbf{J}$ on a surface S, the potentials at a field point $\mathbf{r}$ are given by integrals over the source points $\mathbf{r}'$:

$$\mathbf{A}(\mathbf{r}) = \mu \oint_S \mathbf{J}(\mathbf{r}')\psi(\mathbf{r},\mathbf{r}')\,ds' \tag{4}$$

$$\Phi(\mathbf{r}) = \frac{-1}{j\omega\epsilon} \oint_S \nabla' \cdot \mathbf{J}(\mathbf{r}')\psi(\mathbf{r},\mathbf{r}')\,ds' \tag{5}$$

where $\psi(\mathbf{r},\mathbf{r}')$ is the free-space Green's function:

$$\psi(\mathbf{r},\mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{6}$$

We define a linear operator, L, that maps a surface current $\mathbf{J}$ to the electric field it produces:

$$L(\mathbf{J}) = j\omega\mathbf{A}(\mathbf{J}) + \nabla\Phi(\mathbf{J}) \quad [26] \tag{7}$$

On a perfect electric conductor (PEC) surface S, the tangential component of the total electric field (impressed field $\mathbf{E}^i$ plus scattered field $\mathbf{E}^{scat} = L(\mathbf{J})$) must be zero:

$$[\mathbf{E}^i + L(\mathbf{J})]_{\tan} = 0 \tag{8}$$

This gives the fundamental operator equation for the current $\mathbf{J}$ on S:

$$[L(\mathbf{J})]_{\tan} = -[\mathbf{E}^i]_{\tan} \quad [21] \tag{9}$$

1.2 The Impedance Operator Z and its Properties

We define an **impedance operator** Z as the tangential component of the L operator [37]:

$$Z(\mathbf{J}) = [L(\mathbf{J})]_{\tan} \tag{10}$$

We also define a symmetric product for two vector functions on S [34]:

$$\langle \mathbf{B}, \mathbf{C} \rangle = \oint_S \mathbf{B} \cdot \mathbf{C} \, ds \tag{11}$$

Theorem 1.1 (Symmetry of Z)**.** *The operator Z is symmetric, i.e., $\langle \mathbf{B}, Z\mathbf{C} \rangle = \langle Z\mathbf{B}, \mathbf{C} \rangle$ [39].*

Proof. The symmetry of Z is a direct consequence of the Lorentz reciprocity theorem. For two currents $\mathbf{J}_m$ and $\mathbf{J}_n$ on S, the theorem implies $\oint_S \mathbf{E}_m \cdot \mathbf{J}_n \, ds = \oint_S \mathbf{E}_n \cdot \mathbf{J}_m \, ds$. Since $\mathbf{E}_{m,\tan} = Z(\mathbf{J}_m)$ and $\mathbf{J}_n$ is purely tangential, this becomes:

$$\oint_S Z(\mathbf{J}_m) \cdot \mathbf{J}_n \, ds = \oint_S Z(\mathbf{J}_n) \cdot \mathbf{J}_m \, ds$$

In our symmetric product notation, this is $\langle \mathbf{J}_n, Z\mathbf{J}_m \rangle = \langle \mathbf{J}_m, Z\mathbf{J}_n \rangle$, which proves symmetry. □

We decompose Z into its real and imaginary Hermitian parts, which are real operators because Z is symmetric [45]:

$$Z = R + jX \quad \text{where} \quad R = \frac{1}{2}(Z + Z^*) \quad \text{and} \quad X = \frac{1}{2j}(Z - Z^*) \quad [42, 43] \tag{12}$$

Theorem 1.2 (Positive Semi-Definiteness of R)**.** *The operator R is positive semi-definite [46].*

Proof. The time-averaged power radiated by a current $\mathbf{J}$ is given by $P_{rad} = \frac{1}{2}\mathrm{Re}\{\langle \mathbf{J}^*, Z\mathbf{J} \rangle\}$.

$$\langle \mathbf{J}^*, Z\mathbf{J} \rangle = \langle \mathbf{J}^*, (R + jX)\mathbf{J} \rangle = \langle \mathbf{J}^*, R\mathbf{J} \rangle + j\langle \mathbf{J}^*, X\mathbf{J} \rangle$$

Since R and X are real symmetric operators, $\langle \mathbf{J}^*, R\mathbf{J} \rangle$ and $\langle \mathbf{J}^*, X\mathbf{J} \rangle$ are real. Thus, $P_{rad} = \frac{1}{2}\langle \mathbf{J}^*, R\mathbf{J} \rangle$. Since radiated power is always non-negative, we must have $\langle \mathbf{J}^*, R\mathbf{J} \rangle \geq 0$. □

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2 The Characteristic Mode Eigenvalue Problem

2.1 Derivation of the Fundamental Eigenvalue Equation

To find basis functions ($\mathbf{J}_n$) that diagonalize Z, we solve a generalized eigenvalue problem. The key insight is to choose the weighting operator to be R, which ensures orthogonality of the radiation patterns [54].

$$Z(\mathbf{J}_n) = \nu_n R(\mathbf{J}_n) \quad [51] \tag{13}$$

Substituting $Z = R + jX$ and defining the eigenvalue as $\nu_n = 1 + j\lambda_n$ [58]:

$$\begin{aligned} (R + jX)(\mathbf{J}_n) &= (1 + j\lambda_n)R(\mathbf{J}_n) \quad [57] \\ R(\mathbf{J}_n) + jX(\mathbf{J}_n) &= R(\mathbf{J}_n) + j\lambda_n R(\mathbf{J}_n) \end{aligned} \tag{14}$$

Canceling terms, we arrive at the **fundamental real-valued eigenvalue equation for characteristic modes**:

$$X(\mathbf{J}_n) = \lambda_n R(\mathbf{J}_n) \quad [61] \tag{15}$$

Since R and X are real operators, the eigenvalues λ_n and the corresponding eigencurrents $\mathbf{J}_n$ must be real [63].

2.2 Orthogonality and Normalization of Eigencurrents

Theorem 2.1 (Orthogonality of Eigencurrents)**.** *The eigencurrents* $\mathbf{J}_n$ *are orthogonal with respect to the* R *and* X *operators for distinct eigenvalues [65, 66].*

Proof. Consider two distinct modes, m and n ($\lambda_m \neq \lambda_n$):

$$X(\mathbf{J}_m) = \lambda_m R(\mathbf{J}_m) \tag{16}$$
$$X(\mathbf{J}_n) = \lambda_n R(\mathbf{J}_n) \tag{17}$$

Take the symmetric product of (16) with $\mathbf{J}_n$ and (17) with $\mathbf{J}_m$:

$$\langle \mathbf{J}_n, X\mathbf{J}_m \rangle = \lambda_m \langle \mathbf{J}_n, R\mathbf{J}_m \rangle \tag{18}$$
$$\langle \mathbf{J}_m, X\mathbf{J}_n \rangle = \lambda_n \langle \mathbf{J}_m, R\mathbf{J}_n \rangle \tag{19}$$

By the symmetry of R and X, the left-hand sides are equal, so the right-hand sides must be equal. Using symmetry again on the right-hand side:

$$(\lambda_m - \lambda_n)\langle \mathbf{J}_m, R\mathbf{J}_n \rangle = 0$$

Since $\lambda_m \neq \lambda_n$, we must have $\langle \mathbf{J}_m, R\mathbf{J}_n \rangle = 0$. It immediately follows that $\langle \mathbf{J}_m, X\mathbf{J}_n \rangle = 0$ for $m \neq n$. □

The eigencurrents are normalized such that each mode radiates unit power [74]:

$$\langle \mathbf{J}_n^*, R\mathbf{J}_n \rangle = 1 \tag{20}$$

Combining orthogonality and normalization gives the complete orthonormal set of relations [78, 79, 80]:

$$\langle \mathbf{J}_m^*, R\mathbf{J}_n \rangle = \delta_{mn} \tag{21}$$
$$\langle \mathbf{J}_m^*, X\mathbf{J}_n \rangle = \lambda_n \delta_{mn} \tag{22}$$
$$\langle \mathbf{J}_m^*, Z\mathbf{J}_n \rangle = (1 + j\lambda_n)\delta_{mn} \tag{23}$$

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3 Modal Solutions and Field Properties

3.1 Modal Expansion for Current and Fields

Any current $\mathbf{J}$ on S can be expanded in the basis of eigencurrents [125]:

$$\mathbf{J} = \sum_n \alpha_n \mathbf{J}_n \tag{24}$$

Substituting this into $Z(\mathbf{J}) = \mathbf{E}^i$ and taking the symmetric product with $\mathbf{J}_m$:

$$\sum_n \alpha_n \langle \mathbf{J}_m, Z\mathbf{J}_n \rangle = \langle \mathbf{J}_m, \mathbf{E}^i \rangle \quad [131]$$

Using the orthogonality relation $\langle \mathbf{J}_m, Z\mathbf{J}_n \rangle = (1 + j\lambda_n)\delta_{mn}$, the sum collapses:

$$\alpha_m(1 + j\lambda_m) = \langle \mathbf{J}_m, \mathbf{E}^i \rangle \equiv V_m^i \quad [133, 135]$$

The term V_n^i is the **modal excitation coefficient**. The solution for the current is:

$$\mathbf{J} = \sum_n \frac{V_n^i \mathbf{J}_n}{1 + j\lambda_n} \quad [137] \tag{25}$$

The fields are given by corresponding expansions [142, 143]:

$$\mathbf{E} = \sum_n \frac{V_n^i \mathbf{E}_n}{1 + j\lambda_n} \quad , \quad \mathbf{H} = \sum_n \frac{V_n^i \mathbf{H}_n}{1 + j\lambda_n} \tag{26}$$

3.2 Physical Interpretation of λ_n and Field Orthogonality

Theorem 3.1 (Physical Meaning of λ_n)**.** *The eigenvalue λ_n is 2ω times the net time-averaged stored energy (magnetic minus electric) for mode n [119].*

Proof. The complex power for a single normalized mode $\mathbf{J}_n$ is $\langle \mathbf{J}_n^*, Z\mathbf{J}_n \rangle = 1 + j\lambda_n$. From the Poynting theorem, the imaginary part of complex power is related to the difference in stored energies:

$$\mathrm{Im}\{\langle \mathbf{J}_n^*, Z\mathbf{J}_n \rangle\} = \lambda_n = 2\omega \iiint_{\text{space}} \left(\frac{\mu}{2}|\mathbf{H}_n|^2 - \frac{\epsilon}{2}|\mathbf{E}_n|^2 \right) d\tau \quad [24, 116]$$

A positive λ_n implies a mode with dominant magnetic energy (inductive), while a negative λ_n implies dominant electric energy (capacitive) [417, 418]. A mode with $\lambda_n = 0$ is resonant [419]. □

Theorem 3.2 (Far-Field Orthogonality)**.** *The characteristic far-fields $\mathbf{E}_n$ form an orthonormal set on the sphere at infinity, S_∞ [104].*

Proof. From the Poynting theorem applied to two modes m and n:

$$\oint_{S_\infty} (\mathbf{E}_m \times \mathbf{H}_n^*) \cdot d\mathbf{s} = \langle \mathbf{J}_m^*, Z\mathbf{J}_n \rangle = (1 + j\lambda_n)\delta_{mn} \quad [94, 95]$$

In the far-field, $\mathbf{H}_n = \frac{1}{\eta}(\hat{\mathbf{r}} \times \mathbf{E}_n)$. The integral becomes $\frac{1}{\eta} \oint_{S_\infty} (\mathbf{E}_m \cdot \mathbf{E}_n^*) ds$. Taking the real part of the equation:

$$\frac{1}{\eta} \oint_{S_\infty} \mathbf{E}_m \cdot \mathbf{E}_n^* \, ds = \delta_{mn} \quad [102] \tag{27}$$

□

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4 Applications and Advanced Formulations

4.1 Plane-Wave Scattering

For an incident plane wave $\mathbf{E}^i = \mathbf{u}_i e^{-j\mathbf{k}_i \cdot \mathbf{r}}$ [226], the excitation coefficient becomes:

$$V_n^i = \oint_S \mathbf{J}_n \cdot (\mathbf{u}_i e^{-j\mathbf{k}_i \cdot \mathbf{r}}) \, ds \equiv R_n^i \quad [234] \tag{28}$$

The scattered far-field in a direction (θ_m, ϕ_m) with polarization $\mathbf{u}_m$ is then given by [241]:

$$\mathbf{E}^s \cdot \mathbf{u}_m = \frac{-j\omega\mu}{4\pi r_m} e^{-jkr_m} \sum_n \frac{R_n^i R_n^m}{1 + j\lambda_n} \tag{29}$$

where R_n^m is a similar coefficient for a wave incident from the measurement direction.

4.2 Diagonalization of the Scattering Matrix

The scattering operator S relates an incoming wave E_{in} to the resulting outgoing wave E_{out} such that $E_{out} = SE_{in}$ [340]. We choose the characteristic fields $\mathbf{E}_n$ as the basis for outgoing waves and their conjugates $\mathbf{E}_n^*$ for incoming waves [345, 348].

Theorem 4.1 (Diagonalization of S)**.** *In the basis of characteristic fields, the scattering matrix $[S]$ is diagonal [406].*

Proof. Consider an impressed field composed of a single mode and its standing wave counterpart, $\mathbf{E}^i = \mathbf{E}_m + \mathbf{E}_m^*$ [386]. Using the convention $\mathbf{E}_{\tan} = -Z(\mathbf{J})$, the excitation coefficients are:

$$V_n^i = \langle \mathbf{J}_n, -(Z\mathbf{J}_m + Z^*\mathbf{J}_m) \rangle = -\langle \mathbf{J}_n^*, (Z + Z^*)\mathbf{J}_m \rangle$$

Using orthogonality, this becomes:

$$V_n^i = -((1 + j\lambda_m)\delta_{nm} + (1 - j\lambda_m)\delta_{nm}) = -2\delta_{nm} \quad [390]$$

Only the m-th coefficient is non-zero, $V_m^i = -2$. The scattered field is therefore:

$$\mathbf{E}^s = \sum_n \frac{V_n^i \mathbf{E}_n}{1 + j\lambda_n} = \frac{-2\mathbf{E}_m}{1 + j\lambda_m} \quad [392]$$

The incident wave that produces the impressed field is $E_{in} = \mathbf{E}_m^*$. The total outgoing wave is $E_{out} = E_{in,\text{transmitted}} + E_{scat} = \mathbf{E}_m + \mathbf{E}^s$.

$$E_{out} = \mathbf{E}_m + \frac{-2\mathbf{E}_m}{1 + j\lambda_m} = \mathbf{E}_m \left(1 - \frac{2}{1 + j\lambda_m}\right) = \mathbf{E}_m \left(\frac{1 + j\lambda_m - 2}{1 + j\lambda_m}\right)$$

$$E_{out} = -\frac{1 - j\lambda_m}{1 + j\lambda_m}\mathbf{E}_m$$

An incoming wave $\mathbf{E}_m^*$ produces an outgoing wave proportional only to $\mathbf{E}_m$. This shows that the scattering matrix $[S]$ is diagonal in this basis, with elements S_n:

$$S_n = -\frac{1 - j\lambda_n}{1 + j\lambda_n} \quad [406] \tag{30}$$

This completes the proof, demonstrating that the complex scattering process is decoupled into a series of independent scalar modal responses governed by the real eigenvalues λ_n. □