Foundational Proofs of the Theory of Characteristic Modes

Based on Harrington & Mautz

1 Proof of the Symmetry of the Impedance Operator Z

The symmetry of the operator $\mathbf{Z}$ is a direct consequence of the Lorentz Reciprocity Theorem.

Thesis For any two currents $\mathbf{B}$ and $\mathbf{C}$ on a surface S, the operator $\mathbf{Z}$ is symmetric, satisfying:

$$\langle \mathbf{B}, \mathbf{ZC} \rangle = \langle \mathbf{ZB}, \mathbf{C} \rangle \tag{1}$$

where the symmetric product is defined as $\langle \mathbf{A}, \mathbf{B} \rangle = \int_S \mathbf{A} \cdot \mathbf{B}\, ds$. [cite: 1108]

Proof Let a current $\mathbf{B}$ on S produce an electric field $\mathbf{E}_B$, and a current $\mathbf{C}$ on S produce $\mathbf{E}_C$. The reciprocity theorem states:

$$\int_S \mathbf{C} \cdot \mathbf{E}_B\, ds = \int_S \mathbf{B} \cdot \mathbf{E}_C\, ds \tag{2}$$

The electric field generated by a current $\mathbf{J}$ is given by $\mathbf{E} = -L(\mathbf{J})$. [cite: 1105] The operator $\mathbf{Z}$ is the tangential component of L. [cite: 1111] Substituting this into the theorem:

$$\int_S \mathbf{C} \cdot (-\mathbf{ZB})\, ds = \int_S \mathbf{B} \cdot (-\mathbf{ZC})\, ds \tag{3}$$

Using the symmetric product notation, this becomes $\langle \mathbf{C}, -\mathbf{ZB} \rangle = \langle \mathbf{B}, -\mathbf{ZC} \rangle$. By linearity, we prove the symmetry:

$$\langle \mathbf{C}, \mathbf{ZB} \rangle = \langle \mathbf{B}, \mathbf{ZC} \rangle \quad \blacksquare \tag{4}$$

2 Proof of Real Eigenvalues (λ_n) and Eigencurrents ($\mathbf{J}_n$)

This proof derives from the generalized eigenvalue equation using the real symmetric operators $\mathbf{R}$ and $\mathbf{X}$.

Thesis The eigenvalues λ_n and eigencurrents $\mathbf{J}_n$ that satisfy the equation $\mathbf{X}(\mathbf{J}_n) = \lambda_n \mathbf{R}(\mathbf{J}_n)$ are purely real.

Proof of Real Eigenvalues Take the complex inner product of the eigenvalue equation with $\mathbf{J}_n$:

$$\langle \mathbf{J}_n^*, \mathbf{XJ}_n \rangle = \langle \mathbf{J}_n^*, \lambda_n \mathbf{RJ}_n \rangle = \lambda_n \langle \mathbf{J}_n^*, \mathbf{RJ}_n \rangle \tag{5}$$

The operators $\mathbf{R}$ and $\mathbf{X}$ are Hermitian, and a property of Hermitian operators is that their quadratic forms, $\langle \psi^*, \mathbf{A}\psi \rangle$, are always real numbers. Since $\langle \mathbf{J}_n^*, \mathbf{XJ}_n \rangle$ and $\langle \mathbf{J}_n^*, \mathbf{RJ}_n \rangle$ are both real, their ratio must be real. Thus, λ_n must be real. $\blacksquare$

Proof of Real Eigencurrents The eigenvalue equation can be written as $(\mathbf{X} - \lambda_n \mathbf{R})\mathbf{J}_n = 0$. Since $\mathbf{X}$, $\mathbf{R}$, and λ_n are all real, the operator $(\mathbf{X} - \lambda_n \mathbf{R})$ is a real symmetric operator. A linear homogeneous equation with a real operator can always possess a set of purely real eigenfunctions $\mathbf{J}_n$. $\blacksquare$

3 Proof of Weighted Orthogonality

Thesis For two distinct modes m and n ($\lambda_m \neq \lambda_n$), the eigencurrents are orthogonal with respect to both $\mathbf{R}$ and $\mathbf{X}$.

$$\langle \mathbf{J}_m, \mathbf{RJ}_n \rangle = 0 \tag{6}$$

$$\langle \mathbf{J}_m, \mathbf{XJ}_n \rangle = 0 \tag{7}$$

Proof Consider the eigenvalue equations for modes m and n:

$$\mathbf{X}(\mathbf{J}_m) = \lambda_m \mathbf{R}(\mathbf{J}_m) \tag{8}$$
$$\mathbf{X}(\mathbf{J}_n) = \lambda_n \mathbf{R}(\mathbf{J}_n) \tag{9}$$

Take the symmetric product of (8) with $\mathbf{J}_n$ and (9) with $\mathbf{J}_m$:

$$\langle \mathbf{J}_n, \mathbf{X}\mathbf{J}_m \rangle = \lambda_m \langle \mathbf{J}_n, \mathbf{R}\mathbf{J}_m \rangle \tag{10}$$
$$\langle \mathbf{J}_m, \mathbf{X}\mathbf{J}_n \rangle = \lambda_n \langle \mathbf{J}_m, \mathbf{R}\mathbf{J}_n \rangle \tag{11}$$

Due to the symmetry of $\mathbf{R}$ and $\mathbf{X}$, the left-hand sides of (10) and (11) are equal. Therefore, the right-hand sides are equal:

$$\lambda_m \langle \mathbf{J}_m, \mathbf{R}\mathbf{J}_n \rangle = \lambda_n \langle \mathbf{J}_m, \mathbf{R}\mathbf{J}_n \rangle \tag{12}$$

Rearranging gives $(\lambda_m - \lambda_n)\langle \mathbf{J}_m, \mathbf{R}\mathbf{J}_n \rangle = 0$. Since $\lambda_m \neq \lambda_n$, it must be that $\langle \mathbf{J}_m, \mathbf{R}\mathbf{J}_n \rangle = 0$. Substituting this back into (11) shows that $\langle \mathbf{J}_m, \mathbf{X}\mathbf{J}_n \rangle = 0$. ■

4 Proof of the Physical Interpretation of λ_n

Thesis λ_n is proportional to the difference between the time-average stored magnetic and electric energy for that mode.

Proof The complex Poynting theorem is given by:

$$\langle \mathbf{J}^*, \mathbf{Z}\mathbf{J} \rangle = \oint_{S'} \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{s} + j\omega \iiint_{\tau'} (\mu \mathbf{H} \cdot \mathbf{H}^* - \epsilon \mathbf{E} \cdot \mathbf{E}^*)\, d\tau \tag{13}$$

For a normalized eigencurrent $\mathbf{J}_n$, we have $\langle \mathbf{J}_n^*, \mathbf{R}\mathbf{J}_n \rangle = 1$ and $\langle \mathbf{J}_n^*, \mathbf{X}\mathbf{J}_n \rangle = \lambda_n$. Therefore, the left-hand side is $\langle \mathbf{J}_n^*, \mathbf{Z}\mathbf{J}_n \rangle = 1 + j\lambda_n$.

The real part of the integral term on the right is the radiated power, which is 1 due to normalization. The expression becomes: [cite: 1168, 1169]

$$1 + j\lambda_n = (1) + j\omega \iiint (\mu |\mathbf{H}_n|^2 - \epsilon |\mathbf{E}_n|^2)\, d\tau \tag{14}$$

Equating the imaginary parts of this equation gives the physical meaning of λ_n:

$$\lambda_n = \omega \iiint (\mu |\mathbf{H}_n|^2 - \epsilon |\mathbf{E}_n|^2)\, d\tau \quad ■ \tag{15}$$

This shows λ_n is proportional to the difference between stored magnetic energy ($W_m \propto \int \mu |H|^2 d\tau$) and stored electric energy ($W_e \propto \int \epsilon |E|^2 d\tau$).