Bayesian Inference for an Inverse Heat Conduction Problem

1 Problem Description

We consider an inverse heat conduction problem on the unit square $\Omega = (0, 1)^2$, governed by the Laplace equation

$$-\Delta u = 0 \quad \text{in } \Omega,$$

with homogeneous Dirichlet boundary conditions on three sides and an unknown temperature profile $f(y)$ prescribed on the left boundary $x = 0$. Noisy interior measurements of the temperature field are provided at scattered locations.

The goal is to infer the unknown boundary function $f(y)$ together with uncertainty quantification.

2 Boundary Parameterization

The unknown boundary condition is represented using a finite expansion of Gaussian radial basis functions (RBFs),

$$f(y) = \sum_{j=1}^{k} \alpha_j \exp\left(-\frac{(y - y_j^{bc})^2}{2\ell^2}\right),$$

with $k = 15$ equispaced centers $y_j^{bc} \in [0, 1]$ and width $\ell = 0.15$. This choice ensures sufficient flexibility while enforcing smoothness consistent with the elliptic nature of the PDE.

3 Forward Model

For a given coefficient vector $\alpha \in \mathbb{R}^k$, the Laplace equation is solved using a finite difference scheme. The resulting solution is evaluated at the measurement locations using a Gaussian mollifier to account for grid discretization effects. The resulting forward map is linear in α and can be written as

$$\mathbf{u} = G\alpha + \varepsilon,$$

where $G \in \mathbb{R}^{m\times k}$ is the forward matrix and ε denotes measurement noise.

4 Bayesian Formulation

We adopt a Bayesian framework with additive Gaussian noise,

$$\varepsilon \sim \mathcal{N}(0, \sigma^2 I),$$

where σ is provided with the data. A zero-mean Gaussian prior is placed on the coefficients,

$$\alpha \sim \mathcal{N}(0, \sigma_\alpha^2 I), \quad \sigma_\alpha = 0.5,$$

which regularizes the inverse problem and penalizes unphysically large boundary fluxes.

Under these assumptions, the posterior distribution is Gaussian with closed-form expressions for the mean and covariance,

$$\Sigma_{\text{post}} = \left(G^\top \Sigma_{\text{noise}}^{-1} G + \Sigma_{\text{prior}}^{-1}\right)^{-1}, \quad \mu_{\text{post}} = \Sigma_{\text{post}} G^\top \Sigma_{\text{noise}}^{-1} \mathbf{u}.$$

5 Results

The posterior mean of $f(y)$ provides a smooth reconstruction of the boundary temperature, while pointwise credible intervals quantify uncertainty. The reconstructed boundary accurately explains the interior measurements, as demonstrated by the strong agreement between measured and predicted temperatures.

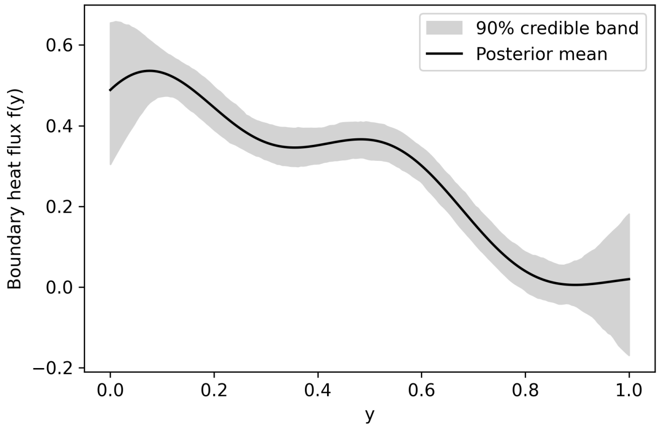

Figure 1: Posterior mean and 90% credible band for the boundary temperature $f(y)$.

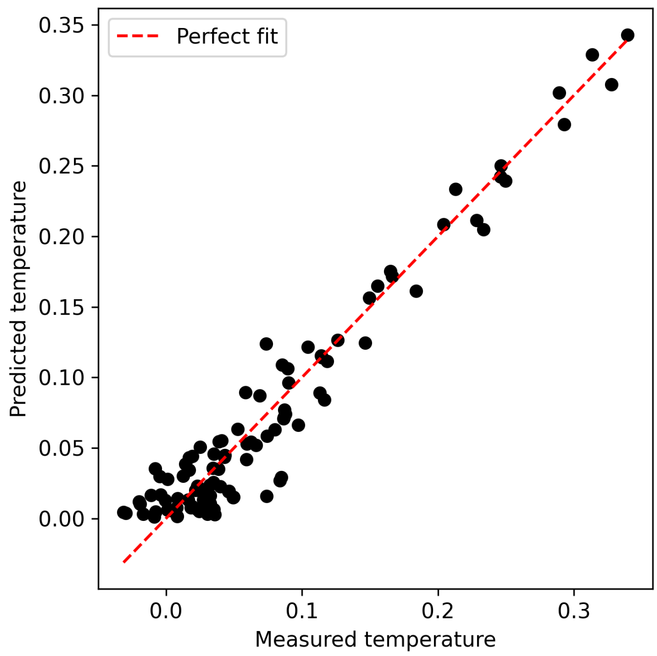

Figure 2: Measured vs. predicted interior temperatures using the posterior mean.

6 Conclusion

The inverse heat problem admits an exact linear-Gaussian Bayesian solution under the chosen parameterization. The approach provides both a stable reconstruction and principled uncertainty quantification, highlighting regions where the data is informative and where it is not.