Dynamic Models and Representations of Reciprocal and Other Gaussian Conditionally Markov Sequences

Reza Rezaie and X. Rong Li

***Abstract*—Conditionally Markov (CM) sequences are powerful mathematical tools for modeling problems. There are several classes of CM sequences, one of which is the reciprocal sequence. To use them in application, we need not only their dynamic models, but also approaches for designing model parameters. Models of two important classes of nonsingular Gaussian (NG) CM sequences, called CM_L and CM_F models, and a model of the NG reciprocal sequence, called reciprocal CM_L model, were presented in our previous work and their applications were discussed. In this paper, these models are studied in more detail and some approaches/guidelines are presented for their parameter design. It is shown that every reciprocal CM_L model can be induced by a Markov model. Then, parameters of every reciprocal CM_L model is obtained from those of the Markov model. Also, it is shown that a NG CM_L (CM_F) sequence can be represented by a sum of a NG Markov sequence and an uncorrelated NG vector. This (necessary and sufficient) representation provides some insight and guideline for designing parameters of a CM_L (CM_F) model. In addition, this representation makes a key concept of CM sequences explicit. From the CM viewpoint, a representation is also obtained for NG reciprocal sequences. This representation is simple and reveals an important property of reciprocal sequences. As a result, the significance of studying reciprocal sequences from the CM viewpoint is demonstrated. A full spectrum of dynamic models from a CM_L model to a reciprocal CM_L model is also presented. This spectrum helps to understand the gradual change from a CM_L model to a reciprocal CM_L model. Moreover, models for intersections of CM classes are presented and their application is pointed out. As a by-product, new representations of some matrices (being characterizations of NG CM_L, CM_F, and reciprocal sequences) are obtained. Some examples are presented for illustration.**

Keywords: Conditionally Markov, reciprocal, Markov, Gaussian, dynamic model, characterization.

I. Introduction

Consider stochastic sequences defined over $[0, N] = \{0, 1, \dots, N\}$. For convenience, let the index be time. A sequence is Markov if and only if (iff) conditioned on the state at any time k, the segment before k is independent of the segment after k. A sequence is reciprocal iff conditioned on the states at any two times k_1 and k_2, the segment inside the interval $[k_1, k_2]$ is independent of the two segments outside $[k_1, k_2]$. In other words, inside and outside are independent given the boundaries. A sequence is CM_F (CM_L) over $[k_1, k_2]$ iff conditioned on the state at time k_1 (k_2), the sequence is Markov over $[k_1 + 1, k_2]$ ($[k_1, k_2 - 1]$) [1]–[2]. The Markov sequence is a special reciprocal sequence and the reciprocal sequence is a special CM sequence.

The authors are with the Department of Electrical Engineering, University of New Orleans, New Orleans, LA 70148. Email addresses are `rrezaie@uno.edu` and `xli@uno.edu`.

Markov processes have been widely studied and used for modeling problems. However, they are not general enough in some cases [3]–[13], and more general processes are needed. Reciprocal processes are a generalization of Markov processes, however, they do not offer an approach to generalizing Markov processes for different problems. CM processes offer such a systematic approach for generalization of Markov processes appropriately for a given problem [1].

Being a motivation for defining reciprocal processes [14], the problem posed by E. Schrodinger [15] about some behavior of particles can be studied in the reciprocal process setting. In [16] reciprocal processes were discussed in the context of stochastic mechanics. In a quantized state space, finite-state reciprocal sequences were used in [3]–[6] for detection of anomalous trajectory patterns, intent inference, and tracking. The approach presented in [7]–[8] for intent inference in an intelligent interactive vehicle's display is implicitly based on the idea of reciprocal processes. In [9], the relation between acausal systems and reciprocal processes was discussed. Applications of reciprocal processes in image processing can be found in [10]–[11]. Some CM sequences were used in [12]–[13] for trajectory modeling and prediction.

Gaussian CM processes were introduced in [17] based on mean and covariance functions, where the processes were assumed nonsingular on the interior of the index (time) interval. [17] considered conditioning at the first time of the CM interval. [18] extended the definition of Gaussian CM processes (presented in [17]) to the general (Gaussian/non-Gaussian) case. In [1] we presented definitions of different (Gaussian/non-Gaussian) CM processes based on conditioning at the first or the last time of the CM interval, studied (stationary/non-stationary) NG CM sequences, and presented their dynamic models and characterizations. Two of these models for two important classes of NG CM sequences (i.e., sequences being CM_L or CM_F over $[0, N]$) are called CM_L and CM_F models. Applications of CM sequences for trajectory modeling in different scenarios were also discussed. In addition, [1]–[2] provided a foundation and preliminaries for studying the reciprocal sequence from the viewpoint of the CM sequence [19]–[20].

Reciprocal processes were introduced in [14] and studied in [21]–[34] and others. [21]–[25] studied reciprocal processes in a general setting. [18] discussed the relationship between the Gaussian CM process and the Gaussian reciprocal process and made an inspiring comment in this regard. [19]–[20] elaborated on the comment of [18] and obtained the relationship between (Gaussian/non-Gaussian) CM and reciprocal processes. It was shown in [18] that a NG continuous-time

CM (including reciprocal) process can be represented in terms of a Wiener process and an uncorrelated NG vector. Following [18], [26]–[27] obtained some results about continuous-time Gaussian reciprocal processes. [28]–[30] presented state evolution models of Gaussian reciprocal processes. In [30], a dynamic model and a characterization of the NG reciprocal sequence were presented. It was shown that the evolution of a reciprocal sequence can be described by a second-order nearest-neighbor model driven by locally correlated dynamic noise [30]. That model is a natural generalization of the Markov model. Due to the dynamic noise correlation and the nearest-neighbor structure, the corresponding state estimation is not straightforward. Recursive estimation of the sequence based on the model presented in [30] was discussed in [31]–[34]. A covariance extension problem for reciprocal sequences was addressed in [35]. Modeling and estimation of finite-state reciprocal sequences were discussed in [36]–[39]. Based on the results of [1]–[2], in [19]–[20] reciprocal sequences were studied from the CM viewpoint. For example, a dynamic model driven by white noise, called a reciprocal CM_L model, of the NG reciprocal sequence was obtained.

Due to its simple structure and whiteness of the dynamic noise, our reciprocal CM_L model is easy to apply. For example, recursive estimation of a reciprocal sequence based on a reciprocal CM_L model is straightforward [19]–[20]. However, it is not clear how parameters of a reciprocal CM_L model can be designed in a problem. An application of CM_L sequences in trajectory modeling with destination information was discussed in [1]. However, a guideline for parameter design of a CM_L model is lacking. An application of a $CM_L \cap [0,k_2]$-CM_L sequence (i.e., a sequence which is CM_L over both $[0,N]$ and $[0,k_2]$) in modeling motion trajectories with a waypoint and a destination was discussed in [1]. However, a dynamic model of $CM_L \cap [0,k_2]$-CM_L sequences, which is necessary for application, has not been presented in the literature. Following [21], [4] obtained a transition probability function of a finite-state reciprocal sequence from a transition probability function of a finite-state Markov sequence in a quantized state space for a problem of intent inference. However, [4] did not discuss if all reciprocal transition probability functions can be obtained from a Markov transition probability function, which is critical for application. Also, it is not always feasible or easy to quantize the state space in some applications. [7]–[8] obtained a transition density of a Gaussian bridging distribution from a Markov transition density. However, [7]–[8] did not show what type of stochastic processes they obtained for modeling their problem of intent inference. In other words, [7]–[8] did not discuss what the type of the obtained transition density was.

The main goal of this paper is three-fold: 1) to present some approaches/guidelines for parameter design of CM_L, CM_F, and reciprocal CM_L models for their application, 2) to obtain a representation of NG CM_L, CM_F, and reciprocal sequences, revealing a key concept behind these sequences, and to demonstrate the significance of studying reciprocal sequences from the CM viewpoint, and 3) to present a full spectrum of dynamic models from a CM_L model to a reciprocal CM_L model and show how models of various intersections of CM classes can be obtained.

The main contributions of this paper are as follows. From the CM viewpoint, we not only show how a Markov model induces a reciprocal CM_L model, but also prove that *every* reciprocal CM_L model can be induced by a Markov model. Then, we give formulas to obtain parameters of the reciprocal CM_L model from those of the Markov model. This approach is more intuitive than a direct parameter design of a reciprocal CM_L model, because one usually has an intuitive understanding of Markov models. In addition, our results make it clear that the transition density obtained in [7]–[8] is actually a reciprocal transition density. A full spectrum of dynamic models from a CM_L model to a reciprocal CM_L model is presented. This spectrum helps to understand the gradual change from a CM_L model to a reciprocal CM_L model. Also, it is demonstrated how dynamic models for intersections of NG CM sequences can be obtained. In addition to their usefulness for application (e.g., application of $CM_L \cap [0,k_2]$-CM_L sequences in trajectory modeling), these models are particularly useful to describe the behavior of a sequence (e.g., a reciprocal sequence) belonging to more than one CM class. Based on a valuable observation, [18] discussed representations of NG continuous-time CM processes (including NG continuous-time reciprocal processes) in terms of a Wiener process and an uncorrelated NG vector. First, we show that the representation presented in [18] is not sufficient for a Gaussian process to be reciprocal (although [18] stated that the representation was sufficient and other papers failed to show that it was not). Then, we obtain a simple (necessary and sufficient) representation for NG reciprocal sequences from the CM viewpoint. As a result, the significance of studying reciprocal sequences from the CM viewpoint is demonstrated. Second, inspired by [18], we show that a NG CM_L (CM_F) sequence can be represented by a sum of a NG Markov sequence and an uncorrelated NG vector. This (necessary and sufficient) representation makes a key concept of CM sequences clear and provides some insight for parameter design of CM_L and CM_F models based on those of a Markov model and an uncorrelated NG vector. Third, we study the obtained representations of NG CM_L, CM_F, and reciprocal sequences in detail and, as a by-product, obtain new representations of some matrices, which are characterizations of NG CM_L, CM_F, and reciprocal sequences.

A preliminary conference version of this paper is [40], where the results were presented without proof. In this paper, we present all proofs and detailed discussion. Other significant results beyond [40] include the following. The notion of a CM_L model *induced* by a Markov model is defined and such a model is studied in Subsection III-A (Definition 3.3, Corollary 3.4, and Lemma 3.1). Dynamic models are obtained for intersections of CM classes (Proposition 3.5 and Proposition 3.6). Uniqueness of the representation of a CM_L (CM_F) sequence (as a sum of a NG Markov sequence and an uncorrelated NG vector) is proved (Corollary 4.3). Such a representation is also presented for reciprocal sequences (Proposition 4.4 and Proposition 4.6). Due to its usefulness for application, the notion of a CM_L model *constructed* from a Markov model is introduced and is compared with that of a CM_L model *induced* by a Markov model (Section IV). As a by-product, representations of some matrices (being characterizations of CM sequences) are obtained in Corollary 4.2 and 4.5.

The paper is organized as follows. Section II reviews some definitions and results required for later sections. In Section III, a reciprocal CM_L model and its parameter design are discussed. Also, it is shown how dynamic models for intersections of CM classes can be obtained. In Section IV, a representation of NG CM_L (CM_F) sequences are presented and parameter design of CM_L and CM_F models is discussed. Some illustrative examples are presented in Section V. Section VI contains a summary and conclusions.

II. Definitions and Preliminaries

We consider stochastic sequences defined over the interval $[0, N]$, which is a general discrete index interval. For convenience this discrete index is called time. Also, we consider:

$$[i,j] \triangleq \{i, i+1, \ldots, j-1, j\}, \quad i<j$$
$$[x_k]_i^j \triangleq \{x_k, k \in [i,j]\}, \quad [x_k] \triangleq [x_k]_0^N$$
$$i,j,k_1,k_2,l_1,l_2 \in [0,N]$$

where k in $[x_k]_i^j$ is a dummy variable. $[x_k]$ is a stochastic sequence. The symbol "\" is used for set subtraction. $C_{i,j}$ is a covariance function and $C_i \triangleq C_{i,i}$. C is the covariance matrix of the whole sequence $[x_k]$. For a matrix A, $A_{[r_1:r_2,c_1:c_2]}$ denotes its submatrix consisting of (block) rows r_1 to r_2 and (block) columns c_1 to c_2 of A. Also, 0 may denote a zero scalar, vector, or matrix, as is clear from the context. $F(\cdot|\cdot)$ denotes the conditional cumulative distribution function (CDF). The abbreviations ZMNG and NG are used for "zero-mean nonsingular Gaussian" and "nonsingular Gaussian".

A. Definitions and Notations

Formal (measure-theoretic) definitions of CM (including reciprocal) sequences can be found in [1], [21], [20]. Here, we present definitions in a simple language.

A sequence $[x_k]$ is $[k_1,k_2]$-$CM_c, c \in \{k_1,k_2\}$ (i.e., CM over $[k_1,k_2]$) iff conditioned on the state at time k_1 (or k_2), the sequence is Markov over $[k_1+1,k_2]$ ($[k_1,k_2-1]$). The above definition is equivalent to the following lemma [1].

Lemma 2.1. *$[x_k]$ is $[k_1,k_2]$-$CM_c, c \in \{k_1,k_2\}$, iff $F(\xi_k|[x_i]_{k_1}^j, x_c) = F(\xi_k|x_j,x_c)$ for every $j,k \in [k_1,k_2], j<k$, where $\xi_k \in \mathbb{R}^d$ and d is the dimension of x_k.*

The interval $[k_1,k_2]$ of the $[k_1,k_2]$-CM_c sequence is called the *CM interval* of the sequence.

Remark 2.2. *We consider the following notation ($k_1<k_2$)*

$$[k_1,k_2]\text{-}CM_c = \begin{cases} [k_1,k_2]\text{-}CM_F & \text{if } c=k_1 \\ [k_1,k_2]\text{-}CM_L & \text{if } c=k_2 \end{cases}$$

where the subscript "F" or "L" is used because the conditioning is at the first or the last time of the CM interval.

Remark 2.3. *When the CM interval of a sequence is the whole time interval, it is dropped: the $[0,N]$-CM_c, $c \in \{0,N\}$, sequence is called CM_c.*

A CM_0 sequence is CM_F and a CM_N sequence is CM_L. For different values of k_1, k_2, and c, there are different classes of CM sequences. For example, CM_F and $[1,N]$-CM_L are two classes. By a $CM_F \cap [1,N]$-CM_L sequence we mean a sequence being both CM_F and $[1,N]$-CM_L. We define that every sequence with a length smaller than 3 (i.e., $\{x_0,x_1\}$, $\{x_0\}$, and $\{\}$) is Markov. Similarly, every sequence is $[k_1,k_2]$-CM_c, $|k_2-k_1|<3$. So, CM_L and $CM_L \cap [k_1,N]$-CM_F, $k_1 \in [N-2,N]$ are equivalent.

Lemma 2.4. *$[x_k]$ is reciprocal iff $F(\xi_k|[x_i]_0^j,[x_i]_l^N) = F(\xi_k|x_j,x_l)$ for every $j,k,l \in [0,N]$ ($j<k<l$), where $\xi_k \in \mathbb{R}^d$, and d is the dimension of x_k.*

B. Preliminaries

We review some results required in later sections from [1], [20], [30], [41].

Theorem 2.5. *$[x_k]$ is reciprocal iff it is $[k_1,N]$-CM_F, $\forall k_1 \in [0,N]$, and CM_L.*

Definition 2.6. *A symmetric positive definite matrix is called CM_L if it has form* (1) *and CM_F if it has form* (2)*:*

$$\begin{bmatrix} A_0 & B_0 & 0 & \cdots & 0 & 0 & D_0 \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & D_1 \\ 0 & B_1' & A_2 & B_2 & \cdots & 0 & D_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & D_{N-2} \\ 0 & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_0' & D_1' & D_2' & \cdots & D_{N-2}' & B_{N-1}' & A_N \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} A_0 & B_0 & D_2 & \cdots & D_{N-2} & D_{N-1} & D_N \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & 0 \\ D_2' & B_1' & A_2 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{N-2}' & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & 0 \\ D_{N-1}' & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_N' & 0 & 0 & \cdots & 0 & B_{N-1}' & A_N \end{bmatrix} \tag{2}$$

Here A_k, B_k, and D_k are matrices in general. We call both CM_L and CM_F matrices CM_c. A CM_c matrix is CM_L for $c=N$ and CM_F for $c=0$.

Theorem 2.7. *A NG sequence with covariance matrix C is: (i) CM_c iff C^{-1} is CM_c, (ii) reciprocal iff C^{-1} is cyclic (block) tri-diagonal (i.e. both CM_L and CM_F), (iii) Markov iff C^{-1} is (block) tri-diagonal.*

Corollary 2.8. *A NG sequence with the inverse of covariance matrix $C^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is*

(i) $[0,k_2]$-CM_c ($k_2 \in [1,N-1]$) iff $\Delta_{A_{22}}$ has the CM_c form, where

$$\Delta_{A_{22}} = A_{11} - A_{12}A_{22}^{-1}A_{12}' \tag{3}$$

$A_{11} = A_{[1:k_2+1,1:k_2+1]}$, $A_{22} = A_{[k_2+2:N+1,k_2+2:N+1]}$, *and* $A_{12} = A_{[1:k_2+1,k_2+2:N+1]}$.

(ii) $[k_1, N]$-CM_c $(k_1 \in [1, N-1])$ *iff* $\Delta_{A_{11}}$ *has the* CM_c *form, where*

$$\Delta_{A_{11}} = A_{22} - A'_{12} A_{11}^{-1} A_{12} \tag{4}$$

$A_{11} = A_{[1:k_1,1:k_1]}$, $A_{22} = A_{[k_1+1:N+1,k_1+1:N+1]}$, *and* $A_{12} = A_{[1:k_1,k_1+1:N+1]}$.

A positive definite matrix A is called a $[0, k_2]$-CM_c ($[k_1, N]$ -CM_c) matrix if $\Delta_{A_{22}}$ ($\Delta_{A_{11}}$) in (3) ((4)) has the CM_c form.

Theorem 2.9. *A ZMNG* $[x_k]$ *is* CM_c*,* $c \in \{0, N\}$*, iff it obeys*

$$x_k = G_{k,k-1} x_{k-1} + G_{k,c} x_c + e_k, \quad k \in [1, N] \setminus \{c\} \tag{5}$$

where $[e_k]$ *(*$Cov(e_k) = G_k$*) is a zero-mean white NG sequence, and boundary condition*[1]

$$x_c = e_c, \quad x_0 = G_{0,c} x_c + e_0 \text{ (for } c = N) \tag{6}$$

or equivalently[2]

$$x_0 = e_0, \quad x_c = G_{c,0} x_0 + e_c \text{ (for } c = N) \tag{7}$$

Theorem 2.10. *A ZMNG* $[x_k]$ *is reciprocal iff it satisfies* (5) *along with* (6) *or* (7)*, and*

$$G_k^{-1} G_{k,c} = G'_{k+1,k} G_{k+1}^{-1} G_{k+1,c} \tag{8}$$

$\forall k \in [1, N-2]$ *for* $c = N$*, and* $\forall k \in [2, N-1]$ *for* $c = 0$*. Moreover, for* $c = N$*,* $[x_k]$ *is Markov iff in addition to* (8)*, we have* $G_0^{-1} G_{0,N} = G'_{1,0} G_1^{-1} G_{1,N}$ *for* (6)*, or equivalently* $G_N^{-1} G_{N,0} = G'_{1,N} G_1^{-1} G_{1,0}$ *for* (7)*. Also, for* $c = 0$*,* $[x_k]$ *is Markov iff in addition to* (8)*, we have* $G_{N,0} = 0$*.*

Remark 2.11. *A* CM_c *model of a reciprocal/Markov sequence is called a reciprocal/Markov* CM_c *model. In this way, we distinguish between the reciprocal model of [30] and our reciprocal* CM_c *model presented in [20]–[19].*

Lemma 2.12. *A ZMNG* $[y_k]$ *is Markov iff it obeys*

$$y_k = M_{k,k-1} y_{k-1} + e_k, \quad k \in [1, N] \tag{9}$$

where $y_0 = e_0$ *and* $[e_k]$ *(*$Cov(e_k) = M_k$*) is a zero-mean white NG sequence.*

III. Dynamic Models of Reciprocal and Intersections of CM Classes

A. Reciprocal Sequences

By Theorem 2.10, one can determine whether a CM_c model is for a reciprocal sequence or not. In other words, it gives the required conditions on the parameters of a CM_c model to design a reciprocal CM_c model. However, Theorem 2.10 does not provide an approach for designing the parameters. Theorem 3.2 below provides such an approach. First, we need a lemma.

[1]Note that (7) means that for $c = N$ we have $x_0 = e_0$, $x_N = G_{N,0} x_0 + e_N$, and $x_k = G_{k,k-1} x_{k-1} + G_{k,N} x_N + e_k, k \in [1, N-1]$. Also, for $c = 0$ we have $x_0 = e_0$ and $x_k = G_{k,k-1} x_{k-1} + G_{k,0} x_0 + e_k, k \in [1, N]$. Likewise for (6).

[2]e_0 and e_N in (7) are not necessarily the same as e_0 and e_N in (6). Just for simplicity we use the same notation.

Lemma 3.1. *The set of reciprocal sequences modeled by a reciprocal* CM_L *model* (5) *with parameters* $(G_{k,k-1}, G_{k,N}, G_k)$*,* $k \in [1, N-1]$ *includes Markov sequences.*

Proof. By Theorem 2.10, (5) (for $c = N$) satisfying (8) with (6) models a reciprocal sequence. By Theorem 2.7, C^{-1} of such a sequence is cyclic (block) tri-diagonal given by (1) with $D_1 = \cdots = D_{N-2} = 0$ and

$$D_0 = G'_{1,0} G_1^{-1} G_{1,N} - G_0^{-1} G_{0,N} \tag{10}$$

(see (60) in [1] for calculation of D_0 given by (10)).

Now, consider a reciprocal sequence governed by (5) satisfying (8) with the parameters $(G_{k,k-1}, G_{k,N}, G_k), k \in [1, N-1]$, and boundary condition (6) with the parameters $G_{0,N}$, G_0, and G_N, where

$$G_{0,N} = G_0 G'_{1,0} G_1^{-1} G_{1,N} \tag{11}$$

meaning that $D_0 = 0$. This reciprocal sequence is Markov (Theorem 2.7). Note that since for every possible value of the parameters of the boundary condition the sequence is nonsingular reciprocal modeled by the same reciprocal CM_L model, choice (11) is valid. Thus, there exist Markov sequences belonging to the set of reciprocal sequences modeled by a reciprocal CM_L model (5) with the parameters $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$. □

Theorem 3.2. *(Markov induced* CM_L *model) A ZMNG* $[x_k]$ *is reciprocal iff it can be modeled by a* CM_L *model* (5)–(6) *(for* $c = N$*) induced by a Markov model* (9)*, that is, iff the parameters* $(G_{k,k-1}, G_{k,N}, G_k)$*,* $k \in [1, N-1]$*, of the* CM_L *model* (5)–(6) *of* $[x_k]$ *can be determined by the parameters* $(M_{k,k-1}, M_k)$*,* $k \in [1, N]$*, of a Markov model* (9)*, as*

$$G_{k,k-1} = M_{k,k-1} - G_{k,N} M_{N|k-1} \tag{12}$$

$$G_{k,N} = G_k M'_{N|k} C_{N|k}^{-1} \tag{13}$$

$$G_k = (M_k^{-1} + M'_{N|k} C_{N|k}^{-1} M_{N|k})^{-1} \tag{14}$$

where $M_{N|k} = M_{N,N-1} \cdots M_{k+1,k}$*,* $M_{N|N} = I$*,* $C_{N|k} = \sum_{n=k}^{N-1} M_{N|n+1} M_{n+1} M'_{N|n+1}$*,* $k \in [1, N-1]$*, where* $M_{k,k-1}$*,* $k \in [1, N]$*, are square matrices, and* M_k*,* $k \in [1, N]$*, are positive definite with the dimension of* x_k*.*

Proof. First, we show how (12)–(14) are obtained and prepare the setting for our proof.

Given the square matrices $M_{k,k-1}, k \in [1, N]$, and the positive definite matrices $M_k, k \in [1, N]$, there exists a ZMNG Markov sequence $[y_k]$ (Lemma 2.12):

$$y_k = M_{k,k-1} y_{k-1} + e_k^M, \quad k \in [1, N], \quad y_0 = e_0^M \tag{15}$$

where $[e_k^M]$ is a zero-mean white NG sequence with covariances $M_k, k \in [0, N]$.

Since every Markov sequence is CM_L, we can obtain a CM_L model of $[y_k]$ as

$$y_k = G_{k,k-1} y_{k-1} + G_{k,N} y_N + e_k^y, \quad k \in [1, N-1] \tag{16}$$

where $[e_k^y]$ is a zero-mean white NG sequence with covariances $G_k, k \in [1, N-1], G_0^y, G_N^y$, and boundary condition

$$y_N = e_N^y, \quad y_0 = G_{0,N}^y y_N + e_0^y \tag{17}$$

Parameters of (16) can be obtained as follows. By (15), we have $p(y_k|y_{k-1}) = \mathcal{N}(y_k; M_{k,k-1}y_{k-1}, M_k)$. Since $[y_k]$ is Markov, we have, for $\forall k \in [1, N-1]$,

$$\begin{aligned} p(y_k|y_{k-1}, y_N) &= \frac{p(y_k|y_{k-1})p(y_N|y_k, y_{k-1})}{p(y_N|y_{k-1})} \\ &= \frac{p(y_k|y_{k-1})p(y_N|y_k)}{p(y_N|y_{k-1})} \\ &= \mathcal{N}(y_k; G_{k,k-1}y_{k-1} + G_{k,N}y_N, G_k) \end{aligned} \tag{18}$$

and it turns out that $G_{k,k-1}$, $G_{k,N}$, and G_k are given by (12)–(14) [42], where for $i \in [0, N-1]$ we have $p(y_N|y_i) = \mathcal{N}(y_N; M_{N|i}y_i, C_{N|i})$.

Now, we construct a sequence $[x_k]$ modeled by the same model (16) as

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, \quad k \in [1, N-1] \tag{19}$$

where $[e_k]$ is a zero-mean white Gaussian sequence with nonsingular covariances G_k, and boundary condition

$$x_N = e_N, \quad x_0 = G_{0,N}x_N + e_0 \tag{20}$$

but with different parameters of the boundary condition (i.e., $(G_N, G_{0,N}, G_0) \neq (G^y_N, G^y_{0,N}, G^y_0)$). By Theorem 2.9, $[x_k]$ is a ZMNG CM_L sequence. (Note that parameters of (16) and (19) are the same $(G_{k,k-1}, G_{k,N},\ G_k, k \in [1, N-1])$, but parameters of (17) $(G^y_{0,N}, G^y_0, G^y_N)$ and (20) $(G_{0,N}, G_0, G_N)$ are different.)

Sufficiency: we prove sufficiency; that is, a CM_L model with the parameters (12)–(14) is a reciprocal CM_L model. It suffices to show that the parameters (12)–(14) satisfy (8), and consequently $[x_k]$ is reciprocal. Substituting (12)–(14) in (8), for the right hand side of (8), we have

$$\begin{aligned} G'_{k+1,k}G^{-1}_{k+1}G_{k+1,N} &= M'_{N|k}C^{-1}_{N|k+1} - M'_{N|k}C^{-1}_{N|k+1}M_{N|k+1} \\ &\cdot (M^{-1}_{k+1} + M'_{N|k+1}C^{-1}_{N|k+1}M_{N|k+1})^{-1}M'_{N|k+1}C^{-1}_{N|k+1} \end{aligned}$$

and for the left hand side of (8), we have $G^{-1}_k G_{k,N} = M'_{N|k}C^{-1}_{N|k} = M'_{N|k}(C_{N|k+1} + M_{N|k+1}M_{k+1}M'_{N|k+1})^{-1}$, where from the matrix inversion lemma it follows that (8) holds. Therefore, $[x_k]$ is reciprocal. So, equations (5)–(6) with (12)–(14) model a ZMNG reciprocal sequence.

Necessity: Let $[x_k]$ be ZMNG reciprocal. By Theorem 2.10 $[x_k]$ obeys (5)–(6) with (8). By Lemma 3.1, the set of reciprocal sequences modeled by a reciprocal CM_L model contains Markov and non-Markov sequences (depending on the parameters of the boundary condition). So, a sequence modeled by a reciprocal CM_L model and a boundary condition determined as in the proof of Lemma 3.1 (i.e., satisfying (11)) is actually a Markov sequence whose C^{-1} is (block) tri-diagonal (i.e., (1) with $D_0 = \cdots = D_{N-2} = 0$). Given this C^{-1}, we can obtain parameters of a Markov model (15) $(M_{k,k-1}, k \in [1, N],\ M_k, k \in [0, N])$ of a Markov sequence with the given C^{-1} as follows. C^{-1} of a Markov sequence can be calculated in terms of parameters of a Markov CM_L model or in terms of parameters of a Markov model. Equating these two formulations of C^{-1}, parameters of the Markov model are obtained in terms of parameters of the Markov CM_L. Thus, for $k = N-2, N-3, \ldots, 0$,

$$M^{-1}_N = A_N \tag{21}$$
$$M_{N,N-1} = -M_N B'_{N-1} \tag{22}$$
$$M^{-1}_{k+1} = A_{k+1} - M'_{k+2,k+1}M^{-1}_{k+2}M_{k+2,k+1} \tag{23}$$
$$M_{k+1,k} = -M_{k+1}B'_k \tag{24}$$
$$M^{-1}_0 = A_0 - M'_{1,0}M^{-1}_1 M_{1,0} \tag{25}$$

where

$$A_0 = G^{-1}_0 + G'_{1,0}G^{-1}_1 G_{1,0} \tag{26}$$
$$A_k = G^{-1}_k + G'_{k+1,k}G^{-1}_{k+1}G_{k+1,k}, k \in [1, N-2] \tag{27}$$
$$A_{N-1} = G^{-1}_{N-1} \tag{28}$$
$$A_N = G^{-1}_N + \sum_{k=0}^{N-1} G'_{k,N}G^{-1}_k G_{k,N} \tag{29}$$
$$B_k = -G'_{k+1,k}G^{-1}_{k+1}, k \in [0, N-2] \tag{30}$$
$$B_{N-1} = -G^{-1}_{N-1}G_{N-1,N} \tag{31}$$

Following (18) to get a reciprocal CM_L model from this Markov model, we have (12)–(14).

What remains to be proven is that the parameters of the model obtained by (12)–(14) are the same as those of the CM_L model calculated directly based on the covariance function of $[x_k]$. By Theorem 2.9, the model constructed from (12)–(14) is a valid CM_L model. In addition, given a CM_L matrix (a positive definite cyclic (block) tri-diagonal matrix is a special CM_L matrix) as the C^{-1} of a sequence, the set of parameters of the CM_L model and boundary condition of the sequence is unique (it can be seen by the almost sure uniqueness of a conditional expectation [1]). Thus, the parameters (12)–(14) must be the same as those obtained directly from the covariance function of $[x_k]$. Thus, a ZMNG reciprocal sequence $[x_k]$ obeys (5)–(6) with (12)–(14). □

Note that by matrix inversion lemma, (14) is equivalent to $G_k = M_k - M_k M'_{N|k}(C_{N|k} + M_{N|k}M_k M'_{N|k})^{-1}$.

Note that Theorem 3.2 holds true for every combination of the parameters, i.e., square matrices $M_{k,k-1}$ and positive definite matrices $M_k, k \in [1, N]$. By (12)–(14), parameters of every reciprocal CM_L model are obtained from $M_{k,k-1}, M_k, k \in [1, N]$, which are parameters of a Markov model (9). This is particularly useful for parameter design of a reciprocal CM_L model. We explain it for the problem of motion trajectory modeling with destination information as follows. Such trajectories can be modeled by combining two key assumptions: (i) the moving object follows a Markov model (9) (e.g., a nearly constant velocity model) without considering the destination information, and (ii) the destination distribution is known (which can be different from the destination distribution of the Markov model in (i)). Note that we assume the destination distribution is known in theory, but in reality an approximate distribution can be used. Now, (by (i)) let $[y_k]$ be Markov modeled by (15) (e.g., a nearly constant velocity model without considering the destination information) with parameters $M_{k,k-1}, k \in [1, N], M_k, k \in [1, N]$. $[y_k]$ can be also modeled by a CM_L model (16)–(17). By the Markov property, parameters of (16) are obtained as (12)–(14) based on (18). Next, we construct $[x_k]$ modeled by (19)–(20). By Theorem 2.9, $[x_k]$ is a CM_L sequence. Since parameters of (20) are arbitrary, $[x_k]$ can have any joint distribution of x_0 and x_N. So, $[y_k]$ and $[x_k]$ have the same CM_L model ((16) and (19)) (i.e., the same transition (18)), but $[x_k]$ can have any joint distribution of the states at the endpoints. In other words, $[x_k]$ can model any origin and destination. Therefore, combining the two assumptions (i) and (ii) above naturally leads to a CM_L sequence $[x_k]$ whose CM_L model is the same as that of $[y_k]$ while the former can model any origin and destination. Thus, model (19) with (12)–(14) is the desired model for destination-directed trajectory modeling based on (i) and (ii) above. Note that for estimation (tracking/prediction) the non-causality of the CM_L model requires *information* about x_N (e.g., $p(x_N)$), which is available. So, the model is totally applicable.

Markov sequences modeled by the same reciprocal model of [30] were studied in [29]. This is an important topic in the theory of reciprocal processes [21]. In the following, Markov sequences modeled by the same CM_L model (5) are studied and determined. Following the notion of a reciprocal transition density derived from a Markov transition density [21], a CM_L model *induced* by a Markov model is defined as follows. A Markov sequence can be modeled by either a Markov model (9) or a CM_L model (5). Such a CM_L model is called the CM_L model *induced* by the Markov model since parameters of the former can be obtained from those of the latter (see (18) or (21)–(31)). Definition 3.3 for the Gaussian case.

Definition 3.3. *Consider a Markov model* (9) *with parameters* $M_{k,k-1}, k \in [1, N], M_k, k \in [1, N]$. *The* CM_L *model* (5) *with parameters* $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$, *given by* (12)–(14) *is called the* CM_L *model induced by the Markov model.*

Corollary 3.4. *A* CM_L *model* (5) *is for a reciprocal sequence iff it can be so induced by a Markov model* (9).

Proof. See our proof of Theorem 3.2. □

By the proof of Theorem 3.2, given a reciprocal CM_L model (5) (satisfying (8)), we can choose a boundary condition satisfying (11) and then obtain a Markov model (9) for a Markov sequence that obeys the given reciprocal CM_L model (see (21)–(31)). Since parameters of the boundary condition (i.e., $G_{0,N}$, G_0, and G_N) satisfying (11) can take many values, there are many such Markov models and their parameters are given by (21)–(25).

The idea of obtaining a reciprocal evolution law from a Markov evolution law was used in [15], [21], and later for finite-state reciprocal sequences in [3], [38]. Our contributions are different. First, our reciprocal CM_L model above is from the CM viewpoint. Second, Theorem 3.2 not only induces a reciprocal CM_L model by a Markov model, but also shows that *every* reciprocal CM_L model can be induced by a Markov model (by necessity and sufficiency of Theorem 3.2). This is important for application of reciprocal sequences (i.e., parameter design of a reciprocal CM_L model) because one usually has an intuitive understanding of Markov models (see the above explanation for trajectory modeling with reciprocal sequences). Third, our proof of Theorem 3.2 is constructive and shows how a given reciprocal CM_L model can be induced by a Markov model. Fourth, our constructive proof of Theorem 3.2 gives all possible Markov models by which a given reciprocal CM_L model can be induced. Note that only one CM_L model can be induced by a given Markov model (it can be verified by (21)–(31)). However, a given reciprocal CM_L model can be induced by many different Markov models. This is because (11) holds for many different choices of parameters of the boundary condition (i.e., $G_{0,N}$, G_0, and G_N) each of which leads to a Markov model with parameters given by (21)–(25) (see the proof of necessity of Theorem 3.2). By Theorem 3.2, one can see that the transition density of the bridging distribution used in [7]–[8] is a reciprocal transition density.

B. Intersections of CM Classes

In some applications sequences with more than one CM property (i.e., belonging to more than one CM class) are desired. An example is trajectories with a waypoint and a destination information. Assume not only the destination density (at time N), but also the density of the state at a time $k_2 (< N)$ is known (i.e., waypoint information). First, consider only the waypoint information at k_2 (without destination information). In other words, we know the state density at k_2 but not after. With a CM evolution law between 0 and k_2, such trajectories can be modeled as a $[0, k_2]$-CM_L sequence. Now, consider only the destination information (density) without waypoint information. Such trajectories can be modeled as a CM_L sequence. Then, trajectories with a waypoint and a destination information can be modeled as a sequence being both $[0, k_2]$-CM_L and CM_L, denoted as $CM_L \cap [0, k_2]$-CM_L. In other words, the sequence has both the CM_L property and the $[0, k_2]$-CM_L property. Studying the evolution of other sequences belonging to more than one CM class, for example $CM_L \cap [k_1, N]$-CM_F, is also useful for studying reciprocal sequences. The NG reciprocal sequence is equivalent to $CM_L \cap CM_F$ [19]. Proposition 3.5 below presents a dynamic model of $CM_L \cap [k_1, N]$-CM_F sequences, based on which one can see a full spectrum of models from a CM_L sequence to a reciprocal sequence.

Proposition 3.5. *A ZMNG* $[x_k]$ *is* $CM_L \cap [k_1, N]$-CM_F *iff it obeys* (5)–(6) *with* ($\forall k \in [k_1 + 1, N - 2]$)

$$G_k^{-1} G_{k,N} = G'_{k+1,k} G_{k+1}^{-1} G_{k+1,N} \tag{32}$$

Proof. A ZMNG CM_L sequence has a CM_L model (5)–(6) (Theorem 2.9). Also, a NG sequence is $[k_1, N]$-CM_F iff its C^{-1} has the $[k_1, N]$-CM_F form (Corollary 2.8). Then, a sequence is $CM_L \cap [k_1, N]$-CM_F iff it obeys (5)–(6), where C^{-1} of the sequence has the $[k_1, N]$-CM_F form, which is equivalent to (32) (see [1] for calculation of C^{-1} in terms of parameters of a CM_L model). □

Proposition 3.5 shows how models change from a CM_L model to a reciprocal CM_L model for $k_1 = 0$ (compare (32) and (8) (for $c = N$)). Note that CM_L and $CM_L \cap [k_1, N]$-CM_F, $k_1 \in [N - 2, N]$ are equivalent (Subsection II-A).

Following the idea of the proof of Proposition 3.5, we can obtain models for intersections of different CM classes, for example $CM_c \cap [k_1, k_2]$-$CM_c \cap [m_1, m_2]$-CM_c sequences. However, the above approach does not lead to simple results in some cases, e.g., $CM_L \cap [0, k_2]$-CM_L sequences. A different way of obtaining a model for $CM_L \cap [0, k_2]$-CM_L sequences is presented in Proposition 3.6.

Proposition 3.6. *A ZMNG $[x_k]$ is $CM_L \cap [0, k_2]$-CM_L iff*

$$x_k = G_{k,k-1}x_{k-1} + G_{k,k_2}x_{k_2} + e_k, k \in [1, k_2 - 1] \quad (33)$$

$$x_{k_2} = e_{k_2}, \quad x_0 = G_{0,k_2}x_{k_2} + e_0 \quad (34)$$

$$x_N = \sum_{i=0}^{k_2} G_{N,i}x_i + e_N \quad (35)$$

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, k \in [k_2 + 1, N - 1] \quad (36)$$

where $[e_k]$ ($Cov(e_k) = G_k$) is a zero-mean white NG sequence,

$$G'_{N,j}G_N^{-1}G_{N,i} = 0 \quad (37)$$

$$G_l^{-1}G_{l,k_2} = G'_{l+1,l}G_{l+1}^{-1}G_{l+1,k_2} + G'_{N,l}G_N^{-1}G_{N,k_2} \quad (38)$$

$j = 0, \ldots, k_2-3$, $i = j+2, \ldots, k_2-1$, *and* $l = 0, \ldots, k_2-2$.

Proof. Necessity: Let $[x_k]$ be a ZMNG $CM_L \cap [0, k_2]$-CM_L sequence. Let $p(\cdot)$ and $p(\cdot|\cdot)$ be its density and conditional density, respectively. Then,

$$x_{k_2} \sim p(x_{k_2}) \quad (39)$$

$$x_0 \sim p(x_0|x_{k_2}) \quad (40)$$

Since $[x_k]$ is $CM_L \cap [0, k_2]$-CM_L, it is $[0, k_2]$-CM_L. Thus, for $k \in [1, k_2 - 1]$,

$$x_k \sim p(x_k|x_0, \ldots, x_{k-1}, x_{k_2}) = p(x_k|x_{k-1}, x_{k_2}) \quad (41)$$

Also, since $[x_k]$ is CM_L, for $k \in [k_2 + 1, N]$,

$$x_N \sim p(x_N|x_0, \ldots, x_{k_2}) \quad (42)$$

$$x_k \sim p(x_k|x_0, \ldots, x_{k-1}, x_N) = p(x_k|x_{k-1}, x_N) \quad (43)$$

According to (39)–(40), we have $x_{k_2} = e_{k_2}$ and $x_0 = G_{0,k_2}x_{k_2} + e_0$, where e_0 and e_{k_2} are uncorrelated ZMNG with nonsingular covariances G_0 and G_{k_2}, $G_{0,k_2} = C_{0,k_2}C_{k_2}^{-1}$, $G_{k_2} = C_{k_2}$, $G_0 = C_0 - C_{0,k_2}C_{k_2}^{-1}C'_{0,k_2}$, and C_{l_1,l_2} is the covariance function of $[x_k]$. For $k \in [1, k_2 - 1]$, by (41), we have $x_k = G_{k,k-1}x_{k-1} + G_{k,k_2}x_{k_2} + e_k$, $G_k = \text{Cov}(e_k)$ (see [1]), $[G_{k,k-1}, G_{k,k_2}] = [C_{k,k-1}, C_{k,k_2}] \begin{bmatrix} C_{k-1} & C_{k-1,k_2} \\ C_{k_2,k-1} & C_{k_2} \end{bmatrix}^{-1}$, and $G_k = C_k - [C_{k,k-1}, C_{k,k_2}] \begin{bmatrix} C_{k-1} & C_{k-1,k_2} \\ C_{k_2,k-1} & C_{k_2} \end{bmatrix}^{-1} [C_{k,k-1}, C_{k,k_2}]'$

For $k \in [k_2+1, N]$, by (42), we have $x_N = \sum_{i=0}^{k_2} G_{N,i}x_i + e_N$, $G_N = \text{Cov}(e_N)$, $[G_{N,0}, G_{N,1}, \ldots, G_{N,k_2}] = C_{[N+1:N+1,1:k_2+1]}(C_{[1:k_2+1,1:k_2+1]})^{-1}$, and $G_N = C_N - C_{[N+1:N+1,1:k_2+1]}(C_{[1:k_2+1,1:k_2+1]})^{-1}C'_{[N+1:N+1,1:k_2+1]}$. Here, $C_{[r_1:r_2,c_1:c_2]}$ denotes the submatrix of the covariance matrix C of $[x_k]$ including the block rows r_1 to r_2 and the block columns c_1 to c_2 [3].

[3]Note that C is an $(N + 1) \times (N + 1)$ matrix for a scalar sequence.

By (43), we have $x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, k \in [k_2 + 1, N - 1]$, $G_k = \text{Cov}(e_k)$, $[G_{k,k-1}, G_{k,N}] = [C_{k,k-1}, C_{k,N}] \begin{bmatrix} C_{k-1} & C_{k-1,N} \\ C_{N,k-1} & C_N \end{bmatrix}^{-1}$, and $G_k = C_k - [C_{k,k-1}, C_{k,N}] \begin{bmatrix} C_{k-1} & C_{k-1,N} \\ C_{N,k-1} & C_N \end{bmatrix}^{-1} [C_{k,k-1}, C_{k,N}]'$. In the above, $[e_k]$ is a zero-mean white NG sequence with covariances G_k.

Now we show that (37)–(38) hold. We construct C^{-1} of the whole sequence $[x_k]$ and obtain (37)–(38) from the fact that C^{-1} is both CM_L and $[0, k_2]$-CM_L. $[x_k]_0^{k_2}$ obeys (33)–(34). So, by Theorem 2.9, $[x_k]_0^{k_2}$ is CM_L. Then, by Theorem 2.7, $(C_{[1:k_2+1,1:k_2+1]})^{-1}$ is CM_L for every value of parameters of (33)–(34) (i.e., C^{-1} is $[0, k_2]$-CM_L). C^{-1} of $[x_k]$ is calculated by stacking (33)–(36) as follows. We have

$$\mathcal{G}x = e \quad (44)$$

where $x \triangleq [x'_0, x'_1, \ldots, x'_N]'$, $e \triangleq [e'_0, e'_1, \ldots, e'_N]'$, $\mathcal{G} = \begin{bmatrix} \mathcal{G}_{11} & 0 \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix}$,

$$\mathcal{G}_{21} = \begin{bmatrix} 0 & \cdots & 0 & -G_{k_2+1,k_2} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -G_{N,0} & \cdots & -G_{N,k_2-1} & -G_{N,k_2} \end{bmatrix}$$

$\mathcal{G}_{11}$ is

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G_{0,k_2} \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,k_2} \\ 0 & -G_{2,0} & I & 0 & \cdots & -G_{2,k_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{k_2-1,k_2-2} & I & -G_{k_2-1,k_2} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

and $\mathcal{G}_{22}$ is

$$\begin{bmatrix} I & 0 & \cdots & 0 & -G_{k_2+1,N} \\ -G_{k_2+2,k_2+1} & I & 0 & \cdots & -G_{k_2+2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -G_{N-1,N-2} & I & -G_{N-1,N} \\ 0 & \cdots & 0 & 0 & I \end{bmatrix}$$

Then,

$$C^{-1} = \mathcal{G}'G^{-1}\mathcal{G} \quad (45)$$

where $G = \text{diag}(G_0, G_1, \ldots, G_N)$. Since $[x_k]$ is CM_L, C^{-1} has the CM_L form, which is equivalent to (37)–(38).

Sufficiency: We need to show that a sequence modeled by (33)-(38) is $CM_L \cap [0, k_2]$-CM_L, that is, its C^{-1} has both CM_L and $[0, k_2]$-CM_L forms. Since $[x_k]_0^{k_2}$ obeys (33)–(34), $(C_{[1:k_2+1,1:k_2+1]})^{-1}$ has the CM_L form for every choice of parameters of (33)–(34) (Theorem 2.9 and Theorem 2.7). So, $[x_k]$ governed by (33)-(38) is $[0, k_2]$-CM_L. Also, C^{-1} can be calculated by (45). It can be seen that (37)–(38) is equivalent to C^{-1} having the CM_L form. Thus, a sequence modeled by (33)-(38) is $CM_L \cap [0, k_2]$-CM_L. The Gaussianity of $[x_k]$ follows clearly from linearity of (33)-(36). Also, $[x_k]$ is nonsingular due to (45), the nonsingularity of $\mathcal{G}$, and the positive definiteness of G. $\square$

IV. Representations of CM and Reciprocal Sequences

A representation of NG continuous-time CM processes in terms of a Wiener process and an uncorrelated NG vector was presented in [18]. Inspired by [18], we show that a NG CM_c sequence can be represented by a sum of a NG Markov sequence and an uncorrelated NG vector. We also show how to use a NG Markov sequence and an uncorrelated NG vector to construct a NG CM_c sequence.

Proposition 4.1. *A ZMNG $[x_k]$ is CM_c iff it can be represented as*

$$x_k = y_k + \Gamma_k x_c, \quad k \in [0,N] \setminus \{c\} \tag{46}$$

where $[y_k] \setminus \{y_c\}$ [4] is a ZMNG Markov sequence, x_c is a ZMNG vector uncorrelated with $[y_k] \setminus \{y_c\}$, and Γ_k are some matrices.

Proof. Let $c = N$. Necessity: It is shown that a ZMNG CM_L $[x_k]$ can be represented as (46). $[x_k]$ obeys

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, \quad k \in [1, N-1] \tag{47}$$

$$x_0 = G_{0,N}x_N + e_0 \tag{48}$$

$$x_N = e_N \tag{49}$$

where $[e_k]$ ($G_k = \text{Cov}(e_k)$) is zero-mean white NG.

According to (48), we consider $y_0 = e_0$ and $\Gamma_0 = G_{0,N}$. So, $x_0 = y_0 + \Gamma_0 x_N$. For $k \in [1, N-1]$, we have

$$\begin{aligned} x_k &= G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k \\ &= G_{k,k-1}(y_{k-1} + \Gamma_{k-1}x_N) + G_{k,N}x_N + e_k \\ &= G_{k,k-1}y_{k-1} + e_k + (G_{k,k-1}\Gamma_{k-1} + G_{k,N})x_N \end{aligned}$$

By induction, $[x_k]$ can be represented as $x_k = y_k + \Gamma_k x_N$, $k \in [0, N-1]$, where for $k \in [1, N-1]$, $y_k = M_{k,k-1}y_{k-1} + e_k$, $M_{k,k-1} = G_{k,k-1}$, $\Gamma_k = G_{k,k-1}\Gamma_{k-1} + G_{k,N}$, $y_0 = e_0$, $\Gamma_0 = G_{0,N}$, and x_N is uncorrelated with the Markov sequence $[y_k]_0^{N-1}$, because x_N is uncorrelated with $[e_k]_0^{N-1}$.

What remains is to show the nonsingularity of $[y_k]_0^{N-1}$ and the random vector x_N. Since the sequence $[x_k]$ is nonsingular, x_N is nonsingular. Also, we have $y_0 = e_0$. In addition, the covariances $G_k, k \in [0, N]$, are nonsingular. Thus, $M_k = \text{Cov}(e_k)$, $k \in [0, N-1]$, are all nonsingular. Similar to (45), we have $C^{\mathsf{y}} = \text{Cov}(\mathsf{y}) = \mathfrak{M}^{-1}M\mathfrak{M}'^{-1}$, where $\mathsf{y} = [y_0', y_1', \ldots, y_{N-1}']'$, $M = \text{diag}(M_0, M_1, \ldots, M_{N-1})$ and $\mathfrak{M}$ is a nonsingular matrix. Therefore, $[y_k]_0^{N-1}$ is nonsingular because $\mathcal{M}$ and $\mathfrak{M}$ are nonsingular.

Sufficiency: We show that given a ZMNG Markov sequence $[y_k]_0^{N-1}$ uncorrelated with a ZMNG vector x_N, $[x_k]$ constructed as $x_k = y_k + \Gamma_k x_N, k \in [0, N-1]$ is a ZMNG CM_L sequence, where Γ_k are some matrices. Therefore, it suffices to show that $[x_k]$ obeys (5)–(6). Since $[y_k]_0^{N-1}$ is a ZMNG Markov sequence, it obeys (Lemma 2.12) $y_k = M_{k,k-1}y_{k-1} + e_k$, $k \in [1, N-1]$, $y_0 = e_0$, where $[e_k]_0^{N-1}$ is a zero-mean white NG sequence with covariances M_k.

[4] For $c = N$, $[y_k] \setminus \{y_c\} = [y_k]_0^{N-1}$, and for $c = 0$, $[y_k] \setminus \{y_c\} = [y_k]_1^N$.

We have $x_0 = y_0 + \Gamma_0 x_N$. So, consider $G_{0,N} = \Gamma_0$. Then, for $k \in [1, N-1]$, we have

$$\begin{aligned} x_k &= y_k + \Gamma_k x_N = M_{k,k-1}y_{k-1} + e_k + \Gamma_k x_N \\ &= M_{k,k-1}x_{k-1} + (\Gamma_k - M_{k,k-1}\Gamma_{k-1})x_N + e_k \end{aligned} \tag{50}$$

We consider $G_{k,k-1} = M_{k,k-1}$ and $G_{k,N} = \Gamma_k - M_{k,k-1} \cdot\Gamma_{k-1}$. Covariances M_k, $k \in [0, N-1]$ and $\text{Cov}(x_N)$ are nonsingular. So, covariances $G_k = \text{Cov}(e_k), k \in [0, N]$ (let $e_N = x_N$), are all nonsingular. So, $[x_k]$ is nonsingular (it can be shown similar to (45)). Thus, by (50), it can be seen that $[x_k]$ obeys (5) (note that $[e_k]$ is white). So, $[x_k]$ is a ZMNG CM_L sequence.

For $c = 0$ we have a parallel proof. So, we skip the details and only present some results required later. Necessity: Let $c = 0$. The proof is based on the CM_F model. Let $[x_k]$ be a ZMNG CM_F sequence governed by (5) and (7) (for $c = 0$). It is possible to represent $[x_k]$ as (46) with the Markov sequence $[y_k]_1^N$ governed by $y_k = M_{k,k-1}y_{k-1} + e_k$, $k \in [2, N]$, where for $k \in [2, N]$, $M_{k,k-1} = G_{k,k-1}$, $\Gamma_1 = 2G_{1,0}$, $\Gamma_k = G_{k,k-1}\Gamma_{k-1} + G_{k,0}$.

Sufficiency: Let $[y_k]_1^N$ be a ZMNG Markov sequence governed by $y_k = M_{k,k-1}y_{k-1} + e_k$, $k \in [2, N]$, where $[e_k]_1^N$ (let $y_1 = e_1$) is a zero-mean white NG sequence with covariances M_k. Also, let x_0 be a ZMNG vector uncorrelated with the sequence $[y_k]_1^N$. It can be shown that the sequence $[x_k]$ constructed by (46) obeys (5) and (7) (for $c = 0$), where for $k \in [2, N]$, $G_{k,k-1} = M_{k,k-1}$, $G_{1,0} = \frac{1}{2}\Gamma_1$, and $G_{k,0} = \Gamma_k - M_{k,k-1}\Gamma_{k-1}$. □

Proposition 4.1 makes a key concept behind the NG CM_c sequence clear, that is, every NG CM_c sequence can be represented as a sum of two components: a NG Markov sequence and an uncorrelated NG vector. As a result, it provides some insight and guideline for design of CM_c models in application. Below we explain the idea for designing a CM_L model for motion trajectory modeling with destination information. A CM_L model is more general than a reciprocal CM_L model. Consequently, the following guideline for CM_L model design includes the approach of Theorem 3.2 as a special case. The guideline is as follows. First, a Markov model (e.g., a nearly constant velocity model) with the given origin distribution (without considering other information) is considered. The sequence modeled by this model is $[y_k]_0^{N-1}$ in (46). Assuming the destination (distribution of x_N) is known. Then, based on Γ_k, the Markov sequence $[y_k]_0^{N-1}$ is modified to satisfy the available information in the problem (e.g., about the general form of trajectories) leading to the desired trajectories $[x_k]$ which end up at the destination. A direct attempt for designing parameters of a CM_L model for this problem is hard. However, the above guideline makes parameter design easier and intuitive. In addition, one can learn Γ_k (which shows the impact of the destination) from a set of trajectories. In the following, the representation of Proposition 4.1 is studied further to provide insight and tools for its application [12].

The following representation of CM_c matrices is a by-product of Proposition 4.1.

Corollary 4.2. *Let C be an $(N+1)d \times (N+1)d$ positive definite block matrix (with $(N+1)$ blocks in each row/column and each block with dimension $d \times d$). C^{-1} is CM_c iff*

$$C = B + \Gamma D \Gamma' \tag{51}$$

where D is a $d \times d$ positive definite matrix and (i) for $c = N$, $B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma = \begin{bmatrix} S \\ I \end{bmatrix}$, (ii) for $c = 0$, $B = \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} I \\ S \end{bmatrix}$, where $(B_1)^{-1}$ is $Nd \times Nd$ block tri-diagonal, S is $Nd \times d$, and I is the $d \times d$ identity matrix.

Proof. Let $c = N$. Necessity: By Theorem 2.7, for every CM_L matrix C^{-1}, there exists a ZMNG CM_L sequence $[x_k]$ with the covariance C. By Proposition 4.1, we have

$$x = y + \Gamma x_N \tag{52}$$

where $x \triangleq [x'_0, x'_1, \ldots, x'_{N-1}, x'_N]$, $\mathrm{y} \triangleq [y'_0, y'_1, \ldots, y'_{N-1}]'$, $y \triangleq [\mathrm{y}', 0]'$, $S \triangleq [\Gamma'_0, \Gamma'_1, \ldots, \Gamma'_{N-1}]'$, $\Gamma \triangleq [S', I]'$, and $[y_k]_0^{N-1}$ is a ZMNG Markov sequence uncorrelated with the ZMNG vector x_N. Then, by (52), we have

$$\mathrm{Cov}(x) = \mathrm{Cov}(y) + \Gamma \mathrm{Cov}(x_N) \Gamma' \tag{53}$$

because y and x_N are uncorrelated. Then, (53) leads to (51), where $C \triangleq \mathrm{Cov}(x)$, $B \triangleq \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} = \mathrm{Cov}(y)$, $B_1 \triangleq \mathrm{Cov}(\mathrm{y})$, $D \triangleq \mathrm{Cov}(x_N)$, and by Theorem 2.7, $(B_1)^{-1}$ is block tri-diagonal. Therefore, for every CM_L matrix C^{-1} we have (51).

Sufficiency: Let $(B_1)^{-1}$ be an $Nd \times Nd$ block tri-diagonal matrix, D be a $d \times d$ positive definite matrix, and S be an $Nd \times d$ matrix. By Theorem 2.7, for every $Nd \times Nd$ block tri-diagonal matrix $(B_1)^{-1}$, there exists a Gaussian Markov sequence $[y_k]_0^{N-1}$ with $(C^{\mathrm{y}})^{-1} = (B_1)^{-1}$, where $C^{\mathrm{y}} = \mathrm{Cov}(\mathrm{y})$ and $\mathrm{y} = [y'_0, y'_1, \ldots, y'_{N-1}]'$. Also, given a $d \times d$ positive definite matrix D, there exists a Gaussian vector x_N with $\mathrm{Cov}(x_N) = D$. Let x_N and $[y_k]_0^{N-1}$ be uncorrelated. By Proposition 4.1, $[x_k]$ constructed by (52) is a CM_L sequence. Also, by Theorem 2.7, C^{-1} of $[x_k]$ is a CM_L matrix. With $C \triangleq \mathrm{Cov}(x)$, (51) follows from (53). Thus, for every block tri-diagonal matrix $(B_1)^{-1}$, every positive definite matrix D, and every matrix S, C^{-1} is a CM_L matrix. The proof for $c = 0$ is similar. □

Corollary 4.3. *For every CM_c sequence, the representation* (46) *is unique.*

Proof. Let $c = N$, and $[x_k]$ be a CM_L sequence governed by (5) with parameters $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$, and (6) with the parameters $(G_{0,N}, G_0, G_N)$. By Proposition 4.1, $[x_k]$ can be represented as (46). Parameters (denoted by $M_{k,k-1}, k \in [1, N-1]$, $M_k, k \in [0, N-1]$) of the Markov model (9) of $[y_k]_0^{N-1}$, covariance of x_N denoted by D, and the matrices Γ_k, $k \in [0, N-1]$, can be calculated in terms of the parameters of the CM_L model as follows (see the proof of Proposition 4.1):

$$D = G_N, \quad \Gamma_0 = G_{0,N} \tag{54}$$

$$M_k = G_k, \quad k \in [0, N-1] \tag{55}$$

$$M_{k,k-1} = G_{k,k-1}, \quad k \in [1, N-1] \tag{56}$$

$$\Gamma_k = G_{k,k-1}\Gamma_{k-1} + G_{k,N}, \quad k \in [1, N-1] \tag{57}$$

Now, assume that there exists a different representation of the form (46) for $[x_k]$. Denote parameters of the corresponding Markov model by $\tilde{M}_{k,k-1}, k \in [1, N-1]$, $\tilde{M}_k, k \in [0, N-1]$, and the weight matrices by $\tilde{\Gamma}_k$, $k \in [0, N-1]$ (covariance of x_N is D). By the proof of Proposition 4.1, parameters of the corresponding CM_L model are

$$G_{0,N} = \tilde{\Gamma}_0, \quad G_N = D \tag{58}$$

$$G_{k,k-1} = \tilde{M}_{k,k-1}, \quad k \in [1, N-1] \tag{59}$$

$$G_{k,N} = \tilde{\Gamma}_k - \tilde{M}_{k,k-1}\tilde{\Gamma}_{k-1}, \quad k \in [1, N-1] \tag{60}$$

$$G_k = \tilde{M}_k, \quad k \in [0, N-1] \tag{61}$$

Parameters of a CM_L model of a CM_L sequence are unique [1]. Comparing (54)-(57) and (58)-(61), it can be seen that the parameters $\tilde{M}_{k,k-1}, k \in [1, N-1]$, $\tilde{M}_k, k \in [0, N-1]$, and $\tilde{\Gamma}_k$, $k \in [0, N-1]$, are the same as $M_{k,k-1}, k \in [1, N-1]$, $M_k, k \in [0, N-1]$, and Γ_k, $k \in [0, N-1]$. In other words, parameters of the representation (46) are unique. Uniqueness of (46) for $c = 0$ can be proven similarly. □

Based on a valuable observation, [18] discussed the relationship between Gaussian CM and Gaussian reciprocal processes. Then, based on the obtained relationship, [18] presented a representation of NG reciprocal processes. It was shown in [20] that the relationship between Gaussian CM and Gaussian reciprocal processes presented in [18] was incomplete, that is, the presented condition was not sufficient for a Gaussian process to be reciprocal (although [18] stated that it was sufficient and other papers failed to show that it was not). Then, the relationship between CM and reciprocal processes for the general (Gaussian/non-Gaussian) case was presented in [20]–[19] (Theorem 2.5 above). Also, it was shown that CM_L in Theorem 2.5 was the missing part in the results of [18]. Consequently, it can be seen that the representation presented in [18] is not sufficient for a NG process to be reciprocal and its missing part is the representation of CM_L processes.

In the following, we present a simple necessary and sufficient representation of NG reciprocal sequences from the CM viewpoint. It demonstrates the significance of studying reciprocal sequences from the CM viewpoint.

Proposition 4.4. *A ZMNG $[x_k]$ is reciprocal iff it can be represented as both*

$$x_k = y_k^L + \Gamma_k^L x_N, \quad k \in [0, N-1] \tag{62}$$

$$x_k = y_k^F + \Gamma_k^F x_0, \quad k \in [1, N] \tag{63}$$

where $[y_k^L]_0^{N-1}$ and $[y_k^F]_1^N$ are ZMNG Markov sequences, x_N and x_0 are ZMNG vectors uncorrelated with $[y_k^L]_0^{N-1}$ and $[y_k^F]_1^N$, respectively, and Γ_k^L and Γ_k^F are some matrices.

Proof. A NG $[x_k]$ is reciprocal iff it is both CM_L and CM_F (Theorem 2.7). On the other hand, $[x_k]$ is CM_L (CM_F) iff it can be represented as (62) ((63)) (Proposition 4.1). So, $[x_k]$ is reciprocal iff it can be represented as both (62) and (63). □

By (62)–(63) the relation between sample paths of the two Markov sequences is $y_k^L+\Gamma_k^L x_N = y_k^F+\Gamma_k^F x_0$, $k \in [1, N-1]$, $y_0^L + \Gamma_0^L x_N = x_0$, $x_N = y_N^F + \Gamma_N^F x_0$.

The following representation of cyclic block tri-diagonal matrices is a by-product of Proposition 4.4.

Corollary 4.5. *Let C be an $(N + 1)d \times (N + 1)d$ positive definite block matrix (with $(N+1)$ blocks in each row/column and each block with dimension $d \times d$). Then, C^{-1} is cyclic block tri-diagonal iff*

$$C = B^L + \Gamma^L D^L (\Gamma^L)' = B^F + \Gamma^F D^F (\Gamma^F)' \tag{64}$$

where D^L and D^F are $d \times d$ positive definite matrices, $B^L = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma^L = \begin{bmatrix} S_1 \\ I \end{bmatrix}$, $B^F = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}$, $\Gamma^F = \begin{bmatrix} I \\ S_2 \end{bmatrix}$, $(B_1)^{-1}$ and $(B_2)^{-1}$ are $Nd \times Nd$ block tri-diagonal, S_1 and S_2 are $Nd \times d$, and I is the $d \times d$ identity matrix.

Proof. Necessity: Let C^{-1} be a positive definite cyclic block tri-diagonal matrix. So, C^{-1} is CM_L and CM_F. Then, by Corollary 4.2 we have (64). Sufficiency: Let a positive definite matrix C be written as (64). By Corollary 4.2, C^{-1} is CM_L and CM_F and consequently cyclic block tri-diagonal. □

The reciprocal sequence is an important special CM_L (CM_F) sequence. So, it is important to know under what conditions the representation (46) is for a reciprocal sequence.

Proposition 4.6. *Let $[y_k] \setminus \{y_c\}$, $c \in \{0, N\}$, be a ZMNG Markov sequence, $y_k = M_{k,k-1}y_{k-1} + e_k, k \in [1, N] \setminus \{a\}$,*

$$a = \begin{cases} 1 & \text{if } c = 0 \\ N & \text{if } c = N \end{cases}, \quad r = \begin{cases} 1 & \text{if } c = 0 \\ N-1 & \text{if } c = N \end{cases}$$

where $[e_k] \setminus \{e_c\}$ is a zero-mean white NG sequence with covariances M_k (for $c = 0$ we have $e_1 = y_1$; for $c = N$ we have $e_0 = y_0$). Also, let x_c be a ZMNG vector with a covariance C_c uncorrelated with the Markov sequence $[y_k] \setminus \{y_c\}$. Let $[x_k]$ be constructed as

$$x_k = y_k + \Gamma_k x_c, \quad k \in [0, N] \setminus \{c\} \tag{65}$$

where Γ_k are some matrices. Then, $[x_k]$ is reciprocal iff $\forall k \in [1, N-1] \setminus \{r\}$,

$$M_k^{-1}(\Gamma_k - M_{k,k-1}\Gamma_{k-1}) = M'_{k+1,k}M_{k+1}^{-1}(\Gamma_{k+1} - M_{k+1,k}\Gamma_k) \tag{66}$$

Moreover, $[x_k]$ is Markov iff in addition to (66), we have

$$(M_0)^{-1}\Gamma_0 = M'_{1,0}M_1^{-1}(\Gamma_1 - M_{1,0}\Gamma_0), \ (\textit{for } c = N) \tag{67}$$

$$\Gamma_N - M_{N,N-1}\Gamma_{N-1} = 0, \ (\textit{for } c = 0) \tag{68}$$

Proof. By Proposition 4.1, $[x_k]$ constructed by (65) is a CM_c sequence. Parameters of the CM_L model (i.e., $c = N$) are calculated by (58)–(61) (($\tilde{M}_{k,k-1}, k \in [1, N-1]$, $\tilde{M}_k, k \in [0, N-1]$ and $\tilde{\Gamma}_k, k \in [0, N-1]$ are replaced by $M_{k,k-1}, k \in [1, N-1]$, $M_k, k \in [0, N-1]$, and $\Gamma_k, k \in [0, N-1]$). Parameters of the CM_F model (i.e., $c = 0$) are calculated as $G_{k,k-1} = M_{k,k-1}$, $k \in [2, N]$, $G_k = M_k$, $k \in [1, N]$, $G_{1,0} = \frac{1}{2}\Gamma_1$, $G_0 = D$, $G_{k,0} = \Gamma_k - M_{k,k-1}\Gamma_{k-1}$, $k \in [2, N]$. Then, by Proposition 2.10, the CM_c sequence $[x_k]$ is reciprocal iff (66) holds. Also, $[x_k]$ is Markov iff in addition to (66), (67) holds for $c = N$ and (68) for $c = 0$. □

Due to their importance in design of CM_c dynamic models, the main elements of representation (46) are formally defined.

Definition 4.7. *In (46), $[y_k] \setminus \{y_c\}$ is called an underlying Markov sequence and the Markov model (without considering the initial condition) is called an underlying Markov model. Also, $[x_k]$ is called a CM_c sequence constructed from an underlying Markov sequence and the CM_c model (without considering the boundary condition) is called a CM_c model constructed from an underlying Markov model.*

Corollary 4.8. *For CM_c models, having the same underlying Markov model is equivalent to having the same $G_{k,k-1}, G_k$, $\forall k \in [1, N] \setminus \{a\}$ ($a = N$ for $c = N$, and $a = 1$ for $c = 0$).*

Proof. Given a Markov model with parameters $M_{k,k-1}, M_k, k \in [1, N]\setminus\{a\}$, by our proof of Proposition 4.1, parameters of a CM_c model constructed from the Markov model are $G_{k,k-1} = M_{k,k-1}$, $G_{k,c} = \Gamma_k - M_{k,k-1}\Gamma_{k-1}, G_k = M_k, k \in [1, N] \setminus \{a\}$. Clearly all CM_c models so constructed have the same $G_{k,k-1}, G_k, k \in [1, N] \setminus \{a\}$.

For a CM_c model with the parameters $G_{k,k-1}, G_{k,c}, G_k$, $\forall k \in [1, N] \setminus \{a\}$, parameters of its underlying Markov model are uniquely determined as (see the proof of Proposition 4.1)

$$M_{k,k-1} = G_{k,k-1}, \quad M_k = G_k, \quad k \in [1, N] \setminus \{a\} \tag{69}$$

So, CM_c models with the same $G_{k,k-1}, G_k, \forall k \in [1, N]\setminus\{a\}$, are constructed from the same underlying Markov model. □

In the following, we try to distinguish between two concepts which are both useful in the application of CM_c and reciprocal sequences: 1) a CM_L model *induced* by a Markov model (Definition 3.3) and 2) a CM_L model *constructed* from its underlying Markov model (Definition 4.7).

By Theorem 3.2, a CM_L model induced by a Markov model is actually a reciprocal CM_L model. In other words, non-reciprocal CM_L models can not be so induced (with (12)–(14)) by any Markov model. By Corollary 3.4, every reciprocal CM_L model can be induced by a Markov model. However, the corresponding Markov model is not unique. In addition, every Markov sequence modeled by a Markov model is also modeled by the CM_L model induced by the Markov model.

Every CM_L model can be constructed from its underlying Markov model, which is unique (Corollary 4.3). So, an underlying Markov model plays a fundamental role in constructing a CM_L model. However, an underlying Markov sequence is not modeled by the constructed CM_L model. Actually, an underlying Markov sequence is defined over $[0, N-1]$ while the constructed CM_L sequence is defined over $[0, N]$.

The underlying Markov model of a reciprocal CM_L model induced by a Markov model is determined as follows. Let $M_{k,k-1}, M_k$, $\forall k \in [1, N]$, be the parameters of a Markov model (9). Parameters of the reciprocal CM_L model induced by this Markov model are calculated by (12)–(14). Then, by (69), parameters of the underlying Markov model denoted by $(F_{k,k-1}, Q_k)$, $\forall k \in [1, N-1]$, are $F_{k,k-1} =$

$M_{k,k-1} - (Q_k M'_{N|k} C^{-1}_{N|k}) M_{N|k-1}$ and $Q_k = (M_k^{-1} + M'_{N|k} C^{-1}_{N|k} M_{N|k})^{-1}$, where for $k \in [1, N-1]$, $M_{N|k} = M_{N,N-1} \cdots M_{k+1,k}$, $C_{N|k} = \sum_{n=k}^{N-1} M_{N|n+1} M_{n+1} M'_{N|n+1}$, and $M_{N|N} = I$.

V. Illustrative Examples

A CM_L model induced by a Markov model was presented in Section III and its application for modeling trajectories with destination information was discussed. In the following, some illustrative examples are presented. Consider a two-dimensional scenario, where the state of a moving object at time k is $x_k = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}]'_k$ with position $[\mathsf{x}, \mathsf{y}]'$ and velocity $[\dot{\mathsf{x}}, \dot{\mathsf{y}}]'$. To model trajectories between an origin and a destination, we use a CM_L model induced by a (nearly constant velocity) Markov model (9) (Theorem 3.2) with $M_{k,k-1} = \text{diag}(F, F)$ and $M_k = \text{diag}(Q, Q), k \in [1, N]$, where $F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$, $Q = q \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix}$, $T = 15$ second, $q = 0.01$, and $N = 100$. For illustration, we consider different origin and destination densities with the same covariances $C_0 = C_N = \text{diag}(A, A)$, $A = \begin{bmatrix} 1000 & 40 \\ 40 & 10 \end{bmatrix}$, and the same cross-covariance $C_{0,N} = \text{diag}(B, B)$, $B = \begin{bmatrix} 800 & 20 \\ 20 & 7 \end{bmatrix}$, in all cases, but different means: **case I:** $\mu_0 = [2000, 70, 5000, 0]'$ and $\mu_N = [80000, 70, 7000, 0]'$, **case II:** $\mu_0 = [2000, 70, 5000, 0]'$ and $\mu_N = [140000, 70, 7000, 0]'$, and **case III:** $\mu_0 = [2000, 70, 5000, 10]'$ and $\mu_N = [130000, 70, 1000, -10]'$. Dynamic models of ZMNG Markov and CM_L sequences are given in Lemma 2.12 and Theorem 2.9, respectively. A nonzero-mean sequence is Markov (CM_L) iff its zero-mean part obeys the model of Lemma 2.12 (Theorem 2.9). The initial condition of the Markov model $y_k = M_{k,k-1} y_{k-1} + e_k^M$, ($e_k^M \sim \mathcal{N}(0, M_k)$), $k \in [1, N]$, with the above parameters is $y_0 = e_0^M$, where $e_0^M \sim \mathcal{N}(\mu_0, C_0)$. The boundary condition of the CM_L model $x_k = G_{k,k-1} x_{k-1} + G_{k,N} x_N + e_k$, ($e_k \sim \mathcal{N}(0, G_k)$), $k \in [1, N-1]$, is $x_N = e_N$ and $x_0 = \mu_0 + C_{0,N} C_N^{-1}(x_N - \mu_N) + e_0$, where $e_N \sim \mathcal{N}(\mu_N, C_N)$ and $e_0 \sim \mathcal{N}(0, C_0 - C_{0,N} C_N^{-1} C'_{0,N})$, and the parameters of the model are given by (12)–(14).

Fig. 1 shows trajectories generated by the CM_L model for case I. Fig. 2 compares these trajectories with those of the Markov model for case I over the same time interval. Figs. 3 and 4 compare trajectories generated by Markov and CM_L models for case II and III, respectively (the 50 solid lines are trajectories of the CM_L sequence and the 50 dash lines are trajectories of the Markov sequence). Both sequences model the origin well and near the origin the two models differ little. However, later their difference grows. This is due to the effect of the destination information in the CM_L model.

To study the behavior of the sequences, x components of velocity for the generated sequences are shown in Figs. 5 and 6 for case I and II, respectively. Note that the time duration for all cases and sequences is $[0, N]$. Since the destination is closer to the origin in case I than in case II, the x component of velocity of the CM_L sequence decreases (increases) on

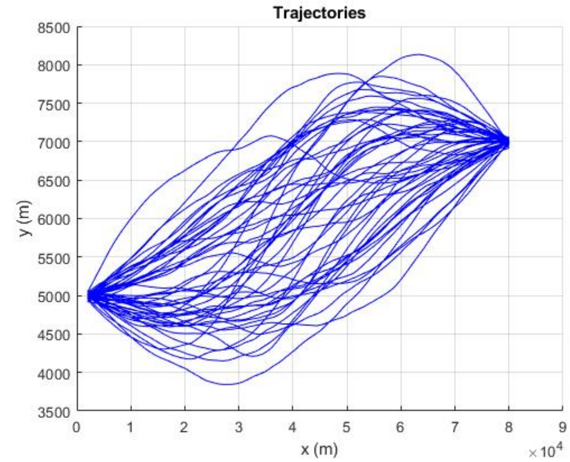

Figure 1. CM_L model for trajectory modeling with destination (case I).

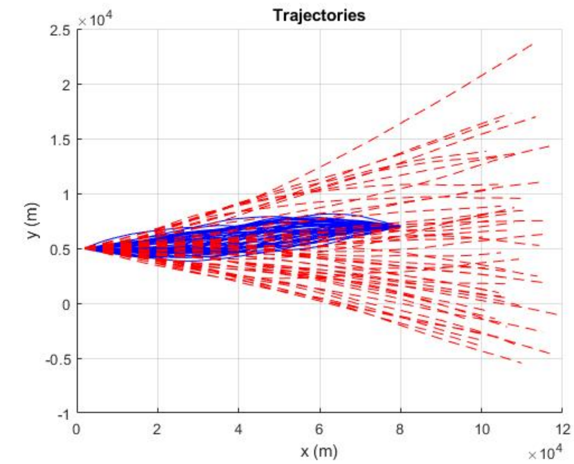

Figure 2. CM_L (solid lines) vs. Markov (dash lines) model for trajectory modeling with destination (case I).

the way in case I (case II) and again increases (decreases) to satisfy the position and velocity at destination (Figs. 5 and 6). Also, Figs. 7 and 8 show the x and y components of velocity, respectively, for case III. All changes of velocity components are intuitive by comparing Markov and CM_L trajectories in Figs. 2, 3, and 4. Take case III as an example. The x-position mean of the destination is 130000 while the x-position at the end of Markov trajectories is around 110000. The x-velocity means in case III at the origin and the destination are the same. So, the x-velocity for CM_L sequences should be greater than that of Markov on the way (Fig. 7) to satisfy the x-position at the destination (note that the x-velocity for the Markov sequence does not change much overall). The y-velocity is decreasing because the y-velocity mean at the destination is -10 and the y-position mean at the destination is 1000, which is smaller than that of the origin. One can also interpret trajectories of Figs. 2, 3, and 4 based on (46).

VI. Summary and Conclusions

Dynamic models for different classes of nonsingular Gaussian (NG) conditionally Markov (CM) sequences have been

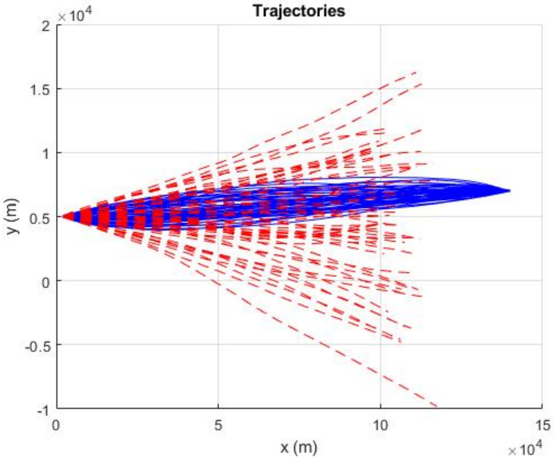

Figure 3. CM_L (solid lines) vs. Markov (dash lines) model for trajectory modeling with destination (case II).

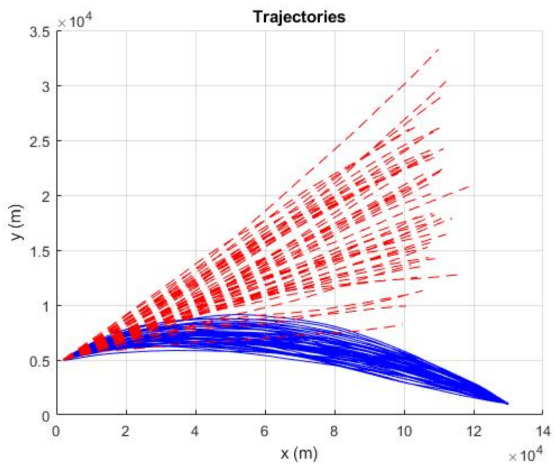

Figure 4. CM_L (solid lines) vs. Markov (dash lines) model for trajectory modeling with destination (case III).

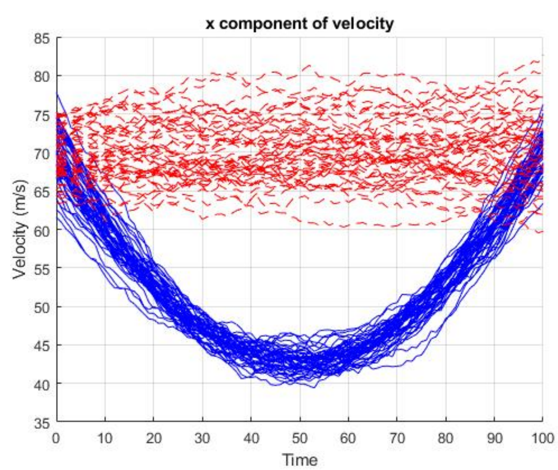

Figure 5. x component of velocity for CM_L (solid lines) vs. Markov (dash lines) model (case I).

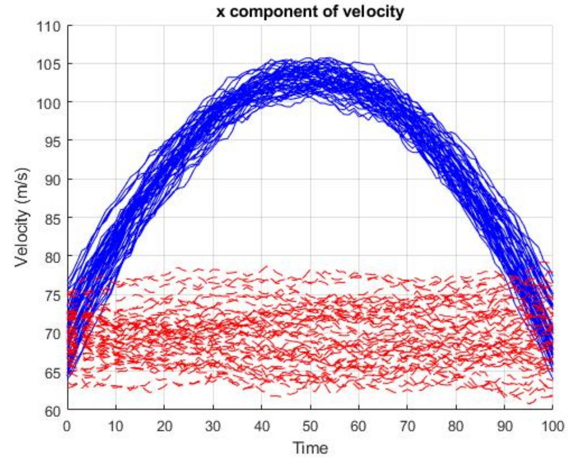

Figure 6. x component of velocity for CM_L (solid lines) vs. Markov (dash lines) model (case II).

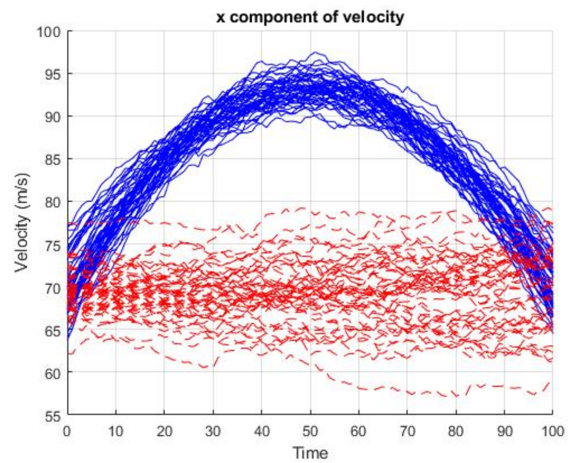

Figure 7. x component of velocity for CM_L (solid lines) vs. Markov (dash lines) model (case III).

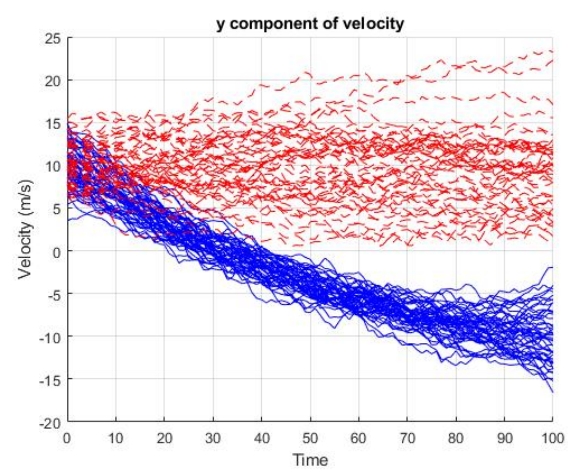

Figure 8. y component of velocity for CM_L (solid lines) vs. Markov (dash lines) model (case III).

studied, and approaches/guidelines for designing their parameters have been presented. Every reciprocal CM_L model can be induced by a Markov model. More specifically, parameters of the former are obtained from those of the latter. In some applications sequences belonging to more than one CM class are desired. It has been shown how dynamic models of such NG CM sequences can be obtained. A spectrum of CM dynamic models has been presented, which makes a gradual change of models from a CM_L model to a reciprocal CM_L model clear. A NG CM_c sequence can be represented by a sum of a NG Markov sequence and an uncorrelated NG vector. This representation reveals a key concept of CM_c sequences. In addition, it is useful for designing CM_L and CM_F models in application. Moreover, a representation of NG reciprocal sequences has been presented from the CM viewpoint, which demonstrates the significance of studying reciprocal sequences from the CM viewpoint. The results of this paper provide some required theoretical tools for application of CM sequences, e.g., trajectory modeling and prediction with destination information.

REFERENCES

[1] R. Rezaie and X. R. Li. Gaussian Conditionally Markov Sequences—Part I: Modeling and Characterization. *IEEE Trans. on Signal Processing*, Under review.

[2] R. Rezaie and X. R. Li. Nonsingular Gaussian Conditionally Markov Sequences. *IEEE Western New York Image and Signal Processing Workshop*. Rochester, USA, Oct. 2018.

[3] M. Fanaswala and V. Krishnamurthy. Detection of Anomalous Trajectory Patterns in Target Tracking via Stochastic Context-Free Grammar and Reciprocal Process Models. *IEEE Jour. of Selected Topics in Signal Processing*, Vol. 7, No. 1, pp. 76-90, 2013.

[4] M. Fanaswala, V. Krishnamurthy, and L. B. White. Destination-aware Target Tracking via Syntactic Signal Processing. *IEEE Inter. Conference on Acoustics, Speech and Signal Processing*, Prague, Czech, May 2011.

[5] L. B. White and F. Carravetta. Normalized Optimal Smoothers for a Class of Hidden Generalized Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 62, No. 12, pp. 6489-6496, 2017.

[6] G. Stamatescu, L. B. White , and R. B. Doust. Track Extraction With Hidden Reciprocal Chains. *IEEE Trans. on Automatic Control*. Vol. 63, No. 4, pp. 1097-1104, 2018.

[7] B. I. Ahmad, J. K. Murphy, S. J. Godsill, P. M. Langdon, and R. Hardy. Intelligent Interactive Displays in Vehicles with Intent Prediction: A Bayesian Framework. *IEEE Signal Processing Magazine*, Vol. 34, No. 2, pp. 82-94, 2017.

[8] B. I. Ahmad, J. K. Murphy, P. M. Langdon, and S. J. Godsill. Bayesian Intent Prediction in Object Tracking Using Bridging Distributions. *IEEE Trans. on Cybernetics*. Vol. 48, No. 1, pp. 215-227, 2018.

[9] A. J. Krener. Reciprocal Processes and the Stochastic Realization Problem for Acausal Systems. *Modeling, Identification, and Robust Control*, C. I. Byrnes and A. Lindquist (editors), Elsevier, 1986.

[10] A. Chiuso, A. Ferrante, and G. Picci. Reciprocal Realization and Modeling of Textured Images. *44th IEEE Conference on Decision and Control*, Seville, Spain, Dec. 2005.

[11] G. Picci and F. Carli. Modelling and Simulation of Images by Reciprocal Processes. *Tenth Inter. Conference on Computer Modeling and Simulation*, Cambridge, UK, Apr. 2008.

[12] R. Rezaie and X. R. Li. Destination-Directed Trajectory Modeling and Prediction Using a Conditionally Markov Sequence. *IEEE Western New York Image and Signal Processing Workshop*, Rochester, USA, Oct. 2018.

[13] R. Rezaie and X. R. Li. Trajectory Modeling and Prediction with Waypoint Information Using a Conditionally Markov Sequence. *56th Allerton Conference on Communication, Control, and Computing*, Illinois, USA, Oct. 2018.

[14] S. Bernstein. Sur les liaisons entre les grandeurs aleatoires. *Verh. des intern. Mathematikerkongr I,* Zurich, 1932.

[15] E. Schrodinger. Uber die Umkehrung der Naturgesetze. *Sitz. Ber. der Preuss. Akad. Wissen., Berlin Phys. Math.* 144, 1931.

[16] B. Levy and A. J. Krener. Stochastic Mechanics of Reciprocal Diffusions. *Jour. of Mathematical Physics*. Vol. 37, No. 2, pp. 769-802, 1996.

[17] C. B. Mehr and J. A. McFadden. Certain Properties of Gaussian Processes and their First-Passage Times. *Jour. of Royal Statistical Society*, (B), Vol. 27, pp. 505-522, 1965.

[18] J. Abraham and J. Thomas. Some Comments on Conditionally Markov and Reciprocal Gaussian Processes. *IEEE Trans. on Information Theory*. Vol. 27, No. 4, pp. 523-525, 1981.

[19] R. Rezaie and X. R. Li. Gaussian Reciprocal Sequences from the Viewpoint of Conditionally Markov Sequences. *Inter. Conference on Vision, Image and Signal Processing*, Las Vegas, USA, Aug. 2018.

[20] R. Rezaie and X. R. Li. Gaussian Conditionally Markov Sequences—Part II: Reciprocal Sequences. *IEEE Trans. on Signal Processing*, Under review.

[21] B. Jamison. Reciprocal Processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, vol. 30, pp. 65-86, 1974.

[22] C. Leonard, S. Rœlly, and J-C Zambrini. Reciprocal Processes. A Measure-theoretical Point of View. *Probability Surveys*, Vol. 11, pp. 237–269, 2014.

[23] G. Conforti, P. Dai Pra, and S. Roelly. Reciprocal Class of Jump Processes. *Jour. of Theoretical Probability*, June 2014.

[24] R. Murr, *Reciprocal Classes of Markov Processes. An Approach with Duality Formulae*, Ph.D. thesis, Universitat Potsdam, 2012.

[25] S. Rœlly. *Reciprocal Processes. A Stochastic Analysis Approach*. In V. Korolyuk, N. Limnios, Y. Mishura, L. Sakhno, and G. Shevchenko, editors, Modern Stochastics and Applications, volume 90 of Optimization and Its Applications, pp. 53–67. Springer, 2014.

[26] J-P Carmichael, J-C Masse, and R. Theodorescu. Representations for Multivariate Reciprocal Gaussian Processes. *IEEE Trans. on Information Theory*, Vol. 34, No. 1, pp. 155-157, 1988.

[27] J-P Carmichael, J-C Masse, and R. Theodorescu. Multivariate Reciprocal Stationary Gaussian Processes. *Jour. of Multivariate Analysis*, 23, pp. 47-66, 1987.

[28] A. J. Krener, R. Frezza, and B. C. Levy. Gaussian Reciprocal Processes and Self-adjoint Stochastic Differential Equations of Second Order. *Stochastics and Stochastic Reports*, Vol. 34, No. 1-2, pp. 29–56, 1991.

[29] B. C. Levy and A. Beghi, Discrete-time Gauss-Markov Processes with Fixed Reciprocal Dynamics. *Jour. of Mathematical Systems, Estimation, and Control*, Vol. 4, No. 3, pp. 1-25, 1994.

[30] B. C. Levy, R. Frezza, and A. J. Krener. Modeling and Estimation of Discrete-Time Gaussian Reciprocal Processes. *IEEE Trans. on Automatic Control*. Vol. 35, No. 9, pp. 1013-1023, 1990.

[31] E. Baccarelli and R. Cusani. Recursive Filtering and Smoothing for Gaussian Reciprocal Processes with Dirichlet Boundary Conditions. *IEEE Trans. on Signal Processing*, Vol. 46, No. 3, pp. 790-795, 1998.

[32] E. Baccarelli, R. Cusani, and G. Di Blasio. Recursive Filtering and Smoothing for Reciprocal Gaussian Processes-pinned Boundary Case. *IEEE Trans. on Information Theory*, Vol. 41, No. 1, pp. 334-337, 1995.

[33] D. Vats and J. M. F. Moura, Recursive Filtering and Smoothing for Discrete Index Gaussian Reciprocal Processes. *43rd Annual Conference on Information Sciences and Systems*, Baltimore, USA, Mar. 2009.

[34] D. Vats and J. M. F. Moura. Telescoping Recursive Representations and Estimation of Gauss–Markov Random Fields. *IEEE Trans. on Information Theory*, Vol. 57, No. 3, pp. 1645-1663, 2011.

[35] F. P. Carli, A. Ferrante, M. Pavon, and G. Picci. A Maximum Entropy Solution of the Covariance Extension Problem for Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 56, No. 9, pp. 1999-2012, 2011.

[36] F. Carravetta and L. B. White. Modelling and Estimation for Finite State Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 57, No. 9, pp. 2190-2202, 2012.

[37] F. Carravetta. Nearest-neighbour Modelling of Reciprocal Chains. *An Inter. Jour. of Probability and Stochastic Processes*, Vol. 80, No. 6, pp. 525-584, 2008.

[38] L. B. White and F. Carravetta. Optimal Smoothing for Finite State Hidden Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 56, No. 9, pp. 2156-2161, 2011.

[39] L B. White and H. X. Vu. Maximum Likelihood Sequence Estimation for Hidden Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 58, No. 10, pp. 2670-2674, 2013.

[40] R. Rezaie and X. R. Li. Models and Representations of Gaussian Reciprocal and Conditionally Markov Sequences. *Inter. Conference on Vision, Image and Signal Processing*, Las Vegas, USA, Aug. 2018.

[41] R. Ackner and T. Kailath. Discrete-Time Complementary Models and Smoothing. *Inter. Jour. of Control*, Vol. 49, No. 5, pp. 1665-1682, 1989.

[42] A. L. Barker, D. E. Brown, and W. N. Martin. Bayesian Estimation and the Kalman Filter. *Computers Mathematics Application*, Vol. 30, No 10, pp. 55-77, 1995.