Gaussian Conditionally Markov Sequences: Algebraically Equivalent Dynamic Models

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***Abstract*—The conditionally Markov (CM) sequence contains different classes, including Markov, reciprocal, and so-called CM_L and CM_F (two CM classes defined in our previous work). Markov sequences are special reciprocal sequences, and reciprocal sequences are special CM_L and CM_F sequences. Each class has its own forward and backward dynamic models. The evolution of a CM sequence can be described by different models. For a given problem, a model in a specific form is desired or needed, or one model can be easier to apply and better than another. Therefore, it is important to study the relationship between different models and to obtain one model from another. This paper studies this topic for models of nonsingular Gaussian (NG) CM_L, CM_F, reciprocal, and Markov sequences. Two models are *probabilistically equivalent (PE)* if their stochastic sequences have the same distribution, and are *algebraically equivalent (AE)* if their stochastic sequences are path-wise identical. A unified approach is presented to obtain an AE forward/backward CM_L/CM_F/reciprocal/Markov model from another such model. As a special case, a backward Markov model AE to a forward Markov model is obtained. While existing results are restricted to models with nonsingular state transition matrices, our approach is not. In addition, a simple approach is presented for studying and determining Markov models whose sequences share the same reciprocal/CM_L model.**

Keywords: Conditionally Markov, reciprocal, dynamic model, probabilistically equivalent, algebraically equivalent.

I. Introduction

Consider stochastic sequences (i.e., discrete-time processes) defined over $[0,N] = (0,1,\ldots,N)$. A sequence is Markov if and only if (iff) conditioned on the state at any time j, the segment before j is independent of the segment after j. A sequence is reciprocal iff conditioned on the states at any two times j and l, the segment inside the interval (j,l) is independent of the two segments outside $[j,l]$. In other words, inside and outside are independent given the boundaries. Note that this holds for every $j,l \in [0,N]$, $j < l$. Now, if this is only true for $l = N$ (and every $j \in [0,N-1]$), then the sequence is CM_L. A sequence is CM_F (CM_L) iff conditioned on the state at time 0 (N), the sequence is Markov over $[1,N]$ ($[0,N-1]$) [1]. So, obviously reciprocal sequences are special CM_L/CM_F sequences and not vice versa. Also, every Markov sequence is a reciprocal sequence.

Markov processes (and their dynamic models[1]) have been widely studied and used for modeling random problems [2]–[8]. For many problems, however, they are not general enough [9]–[21], and more general processes are needed. The reciprocal process is a generalization of the Markov process. The CM process is a more natural generalization of the Markov process. It includes the reciprocal process as a special case. So, the set of CM processes is very large and it includes many classes.

Reciprocal and CM processes have found significant applications. In a quantized state space, [9]–[13] used finite-state reciprocal sequences for detecting anomalous trajectory patterns, intent inference, and tracking. [14] extended the results of [13] to the Gaussian case. The work of [15]–[16] for intent inference, e.g., in an intelligent interactive vehicle's display, can be interpreted in the reciprocal process setting. [17] studied the relationship between acausal systems and reciprocal processes. Application of reciprocal processes in image processing can be found in [18]–[19]. In [20]–[21], CM sequences were used for motion trajectory modeling with waypoint and destination information.

In theory, Gaussian CM processes were introduced in [22] based on mean and covariance functions. [23] extended the definition of Gaussian CM processes (presented in [22]) to the general (Gaussian/non-Gaussian) case. In [1], we defined other (Gaussian/non-Gaussian) CM processes, studied (stationary/non-stationary) NG CM sequences, obtained dynamic models and characterizations of CM_L and CM_F sequences, and discussed their applications. Reciprocal processes were introduced in [24]. [25]–[27] studied reciprocal processes in a general setting. Based on a valuable observation, [23] commented on the relationship between Gaussian CM and Gaussian reciprocal processes. [28] elaborated on the comment of [23] and presented a relationship between the CM process and the reciprocal process for the general (Gaussian/non-Gaussian) case. [29]–[30] presented and studied a dynamic model of NG reciprocal sequences. [28] and [31]–[32] studied reciprocal sequences from the CM viewpoint and developed dynamic models, called reciprocal CM_L and reciprocal CM_F models, with white dynamic noise for the NG reciprocal sequence. A characterization of the NG Markov sequence was presented in [33]. [34] considered modeling and estimation of finite-state reciprocal sequences.

The evolution of a Markov sequence can be modeled by a Markov, reciprocal, or CM_L model[2]. Similarly, a reciprocal sequence can have a model in the form of the one in [29] or in the form of a CM_L (CM_F) model of [28]. Therefore,

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[1]A dynamic model is an equation that describes the state evolution of the process.

[2]By a "model", we may mean a model with or without its boundary condition, as is clear from the context.

a CM sequence can have more than one model. One model can be easier to apply than another for an application. For example, the reciprocal CM_L model of [28] is easier to apply than the reciprocal model of [29] for trajectory modeling with destination information [20]. The dynamic noise is white for the former but colored for the latter. But the reciprocal model of [29] can be useful for some other purposes since it generalizes a Markov model in a nearest-neighbor structure. In addition, a Markov model is simpler than a reciprocal or CM_L model. So, for a Markov sequence, a Markov model is more desired than a reciprocal or CM_L model. Moreover, sometimes only a forward model (FM) is available when a backward model (BM) is required. So, it is important to determine one model from another.

In some cases, *probabilistic equivalence* is not sufficient because it is only about distributions, not a sample path. The two-filter smoothing approach is an example, which needs the relationship in dynamic noise and boundary values[3] between an FM and a BM for them to share an identical sample path of the sequence [35]–[37]. In other words, *algebraically equivalent (AE)* Markov FM and BM are required. To our knowledge, there is no general and unified approach to determining AE Markov, reciprocal, or CM models in the literature.

Motivated by the two-filter smoothing approach, determination of a Markov BM from a Markov FM has been the topic of several papers [38]–[42]. The Markov FM and BM derived in [38]–[40] are PE, but not AE. The Markov BM presented in [41] is AE only to FMs with nonsingular state transition matrices. For models with a singular state transition matrix, [41] only provided a PE BM. Later papers followed the approach of [41] and, to our knowledge, there is no Markov BM that is AE to an FM with a singular state transition matrix in the literature. As a result, we can not check the required conditions of a two-filter smoother for such a Markov model.

Given a Markov model, [29] determined an AE reciprocal model. However, [29] did not present a unified approach for determining other AE CM models.

An important question in the theory of reciprocal processes regards Markov processes sharing the same reciprocal evolution law [30], [25]. Given a reciprocal model of [29], [30] discussed determination of Markov sequences sharing the same reciprocal model. Also, given a reciprocal transition density, [25] determined the required conditions on the joint endpoint distribution for the process to be Markov. It is desired to have a simple approach for studying and determining Markov models whose sequences share the same reciprocal/CM_L/CM_F model. This is not only useful for understanding the relationships between the models/sequences, but also helpful for application of the models. [31]–[32] discussed CM_L models *induced* by Markov models for trajectory modeling with destination information, and showed that using a Markov model to induce a CM_L model is useful for parameter design of the latter. Also, it was shown that a reciprocal CM_L model can be induced by any Markov model whose sequence obeys the given reciprocal CM_L model (and some boundary condition). So, it is desired to determine all such Markov models and to study their relationship. But a simple approach for this purpose is lacking in the literature.

This paper makes the following main contributions. Relationships between CM_L, CM_F, reciprocal, and Markov models for NG sequences are studied. The notion of AE models is defined versus PE ones. Then, a general and unified approach for linear models is presented, based on which from one such model, any AE model can be obtained. The presented approach is simple and not restricted to the above models. As a special case, a Markov BM that is AE to a Markov FM is obtained. Unlike [41], this approach works for both singular and nonsingular state transition matrices. So, the required conditions in the derivation of two-filter smoothing can be verified for all Markov models. The reciprocal model of [29] AE to a Markov model is obtained as a special case of our result. A simple approach is presented for studying and determining Markov models whose sequences share the same reciprocal/CM_L model.

A preliminary and short conference version of this paper is [43], where only Propositions 3.1 and 3.2 (with proof), Proposition 3.3 (without proof), and examples of Section IV were presented. Significant results beyond those of [43] include the following. A proof of Proposition 3.3 is presented. In Section V, two approaches are elaborated for obtaining models AE to a reciprocal model; parameters of PE models are discussed. In Section VI, a simple approach is presented for studying Markov models whose sequences share the same reciprocal/CM_L model. Appendices give details for determining AE models.

The paper is organized as follows. Section II reviews definitions and models of CM_L, CM_F, reciprocal, and Markov sequences. Also, definitions of PE and AE models are presented. Section III presents a unified approach for determining AE models. Section IV discusses two AE models obtained based on the approach of Section III. Section V discusses some points regarding AE models. Section VI presents a simple approach for studying Markov models whose sequences share the same reciprocal/CM_L model. Section VII presents an illustrative example. Section VIII contains a summary and conclusions. Some details of the approach of Section III are presented in appendices.

II. Conventions, Definitions, and Preliminaries

A. Conventions

The following conventions are used:

$$[i,j] \triangleq (i, i+1, \dots, j-1, j), \quad i<j$$
$$[x_k]_i^j \triangleq (x_i, x_{i+1}, \dots, x_j), \quad [x_k] \triangleq [x_k]_0^N$$
$$x \triangleq [x_0', x_1', \dots, x_N']', \quad C = \text{Cov}(x)$$

where k in $[x_k]_i^j$ is a dummy variable. The symbols "$\setminus$" and " $'$ " are used for set subtraction and matrix transposition, respectively. $F(\cdot|\cdot)$ denotes a conditional cumulative distribution function (CDF). "nonsingular Gaussian" is abbreviated as NG. The term "boundary value" is used for vectors in equations as a "boundary condition". Throughout the paper, we consider the

[3]For a forward (backward) Markov model, a boundary value means an initial (final) value.

models reviewed in Subsection II-C. Also, this paper considers only zero-mean NG sequences. A Gaussian sequence $[x_k]$ is nonsingular if its covariance matrix C is nonsingular.

B. Definitions

Definition 2.1. *$[x_k]$ is $CM_c, c \in \{0, N\}$, if* [4]

$$F(\xi_k|[x_i]_0^j, x_c) = F(\xi_k|x_j, x_c) \quad (1)$$

$\forall j, k \in [0, N], j < k$, $\forall \xi_k \in \mathbb{R}^d$ [5].

For the forward (backward) direction, a CM_0 sequence is called CM_F (CM_L). The subscript "F" ("L") is used because the conditioning is on the state at the *first* (*last*) time in the forward (backward) direction. Similarly, a CM_N sequence is called CM_L (backward CM_F). The evolution of a sequence can be modeled by an FM or a BM. The forward direction is the default. The backward direction will be made explicit.

Definition 2.2. *$[x_k]$ is reciprocal if $F(\xi_k|[x_i]_0^j, [x_i]_l^N) = F(\xi_k|x_j, x_l)$, $\forall j, k, l \in [0, N]$, $j < k < l$, $\forall \xi_k \in \mathbb{R}^d$.*

Definition 2.3. *$[x_k]$ is Markov if $F(\xi_k|[x_i]_0^j) = F(\xi_k|x_j)$, $\forall j, k \in [0, N]$, $j < k$, $\forall \xi_k \in \mathbb{R}^d$.*

C. Preliminaries: Dynamic Models and Characterizations

Let $[x_k]$ be a zero-mean NG sequence.

1) Markov Model: $[x_k]$ is Markov iff

$$x_k = M_{k,k-1}x_{k-1} + e_k^M, k \in [1, N] \quad (2)$$

where $x_0 = e_0^M$ and $[e_k^M]$ ($M_k = \text{Cov}(e_k^M)$) is a zero-mean white NG sequence. We have

$$\mathcal{M}x = e^M, \quad e^M = [(e_0^M)', (e_1^M)', \ldots, (e_N^M)']' \quad (3)$$

where $\mathcal{M}$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -M_{1,0} & I & 0 & \cdots & 0 & 0 \\ 0 & -M_{2,1} & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -M_{N-1,N-2} & I & 0 \\ 0 & 0 & 0 & \cdots & -M_{N,N-1} & I \end{bmatrix}$$

From (3), the inverse of the covariance matrix of $[x_k]$ is

$$C^{-1} = \mathcal{M}'M^{-1}\mathcal{M} \quad (4)$$

where $M = \text{Cov}(e^M) = \text{diag}(M_0, M_1, \ldots, M_N)$. C^{-1} is (block) tri-diagonal as

$$\begin{bmatrix} A_0 & B_0 & 0 & \cdots & 0 & 0 & 0 \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_1' & A_2 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & 0 \\ 0 & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & B_{N-1}' & A_N \end{bmatrix} \quad (5)$$

2) Backward Markov Model: $[x_k]$ is Markov iff

$$x_k = M_{k,k+1}^B x_{k+1} + e_k^{BM}, k \in [0, N-1] \quad (6)$$

where $x_N = e_N^{BM}$ and $[e_k^{BM}]$ ($M_k^B = \text{Cov}(e_k^{BM})$) is a zero-mean white NG sequence.

We have

$$\mathcal{M}^B x = e^{BM}, \quad e^{BM} = [(e_0^{BM})', \ldots, (e_N^{BM})']' \quad (7)$$

$$C^{-1} = (\mathcal{M}^B)'(M^B)^{-1}\mathcal{M}^B \quad (8)$$

where $M^B = \text{Cov}(e^{BM}) = \text{diag}(M_0^B, \ldots, M_N^B)$, C^{-1} is (block) tri-diagonal, and $\mathcal{M}^B$ is the nonsingular matrix

$$\begin{bmatrix} I & -M_{0,1}^B & 0 & \cdots & 0 & 0 \\ 0 & I & -M_{1,2}^B & 0 & \cdots & 0 \\ 0 & 0 & I & -M_{2,3}^B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I & -M_{N-1,N}^B \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \quad (9)$$

3) Reciprocal Model [29]: $[x_k]$ is reciprocal iff

$$R_k^0 x_k - R_k^- x_{k-1} - R_k^+ x_{k+1} = e_k^R, \quad k \in [1, N-1] \quad (10)$$

where $[e_k^R]_1^{N-1}$ is a zero-mean colored Gaussian sequence with $E[e_k^R(e_k^R)'] = R_k^0$, $k \in [1, N-1]$, $E[e_k^R(e_{k+1}^R)'] = -R_k^+$, $k \in [1, N-2]$, $E[e_k^R(e_j^R)'] = 0, |k-j| > 1$, $R_k^+ = (R_{k+1}^-)'$, $k \in [1, N-2]$ and boundary condition (i) or (ii) below, with parameters of (10) and either boundary condition leading to a nonsingular sequence.

(i) The first type:

$$R_0^0 x_0 - R_0^- x_N - R_0^+ x_1 = e_0^R \quad (11)$$

$$R_N^0 x_N - R_N^- x_{N-1} - R_N^+ x_0 = e_N^R \quad (12)$$

where $E[e_0^R(e_1^R)'] = -R_0^+$, $E[e_N^R(e_0^R)'] = -R_N^+$, $E[e_0^R(e_0^R)'] = R_0^0$, $E[e_0^R(e_k^R)'] = 0$, $k \in [2, N-1]$, $E[e_N^R(e_k^R)'] = 0$, $k \in [1, N-2]$, $E[e_N^R(e_N^R)'] = R_N^0$, $E[e_{N-1}^R(e_N^R)'] = -R_{N-1}^+$, $(R_0^-)' = R_N^+$, $(R_N^-)' = R_{N-1}^+$, $(R_1^-)' = R_0^+$.

(ii) The second type: $[x_0', x_N']' \sim \mathcal{N}(0, \text{Cov}([x_0', x_N']'))$, which can be written as

$$x_0 = e_0^R, \quad x_N = R_{N,0}x_0 + e_N^R \quad (13)$$

or equivalently

$$x_N = e_N^R, \quad x_0 = R_{0,N}x_N + e_0^R \quad (14)$$

where e_0^R and e_N^R are uncorrelated zero-mean NG vectors[6] with covariances R_0^0 and R_N^0, and uncorrelated with $[e_k^R]_1^{N-1}$.

Consider (10) and boundary condition[7] (11)–(12) with appropriate parameters leading to a nonsingular sequence. Then,

$$\mathfrak{R}x = e^R, \quad e^R = [(e_0^R)', \ldots, (e_N^R)']' \quad (15)$$

$$C^{-1} = \mathfrak{R}'R^{-1}\mathfrak{R} = R \quad (16)$$

[4] $F(\xi_k|x_j) = P\{x_k^1 \leq \xi_k^1, x_k^2 \leq \xi_k^2, \ldots, x_k^d \leq \xi_k^d|x_j\}$, where for example x_k^1 and ξ_k^1 are the first entries of the vectors x_k and ξ_k, respectively. Similarly for other CDFs.

[5] d is the dimension of x_k

[6] e_0^R and e_N^R (and their covariances) in (13) are not necessarily the same as those in (14) or in the first boundary condition. Likewise for e_0 and e_N in (20) and (21). We use the same notation for simplicity.

[7] Boundary condition (ii) is discussed only in Section V. In all other sections, we consider boundary condition (i).

where $R = \text{Cov}(e^R) = \mathfrak{R}$ and $\mathfrak{R}$ is

$$\begin{bmatrix} R_0^0 & -R_0^+ & 0 & \cdots & 0 & -R_0^- \\ -R_1^- & R_1^0 & -R_1^+ & 0 & \cdots & 0 \\ 0 & -R_2^- & R_2^0 & -R_2^+ & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -R_{N-1}^- & R_{N-1}^0 & -R_{N-1}^+ \\ -R_N^+ & 0 & 0 & \cdots & -R_N^- & R_N^0 \end{bmatrix} \tag{17}$$

Since the sequence is nonsingular, so is (17) [29]. Then, $C^{-1} = R$ is (block) cyclic tri-diagonal, denoted by

$$\begin{bmatrix} A_0 & B_0 & 0 & \cdots & 0 & 0 & D_0 \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_1' & A_2 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & 0 \\ 0 & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_0' & 0 & 0 & \cdots & 0 & B_{N-1}' & A_N \end{bmatrix} \tag{18}$$

A reciprocal model is symmetric. So, its forward and backward versions are the same.

Remark 2.4. *In this paper, model* (10) *(with either boundary condition) is called a reciprocal model, to be distinguished from our reciprocal CM_L and CM_F models, presented below.*

4) CM_c Models [1], [28]: $[x_k]$ is CM_c, $c \in \{0, N\}$, iff

$$x_k = G_{k,k-1}x_{k-1} + G_{k,c}x_c + e_k, \quad k \in [1, N] \setminus \{c\} \tag{19}$$

where $[e_k]$ ($G_k = \text{Cov}(e_k)$) is a zero-mean white NG sequence, and boundary condition[8]

$$x_0 = e_0, \quad x_c = G_{c,0}x_0 + e_c \text{ (for } c = N) \tag{20}$$

or equivalently

$$x_c = e_c, \quad x_0 = G_{0,c}x_c + e_0 \text{ (for } c = N) \tag{21}$$

For $c = 0$, we have a CM_F model. Then,

$$\mathcal{G}^F x = e^F, \quad e^F \triangleq [e_0', \dots, e_N']' \tag{22}$$

$$C^{-1} = (\mathcal{G}^F)'(G^F)^{-1}\mathcal{G}^F \tag{23}$$

where $G^F = \text{Cov}(e^F) = \text{diag}(G_0, \dots, G_N)$ and $\mathcal{G}^F$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -2G_{1,0} & I & 0 & \cdots & 0 & 0 \\ -G_{2,0} & -G_{2,1} & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -G_{N-1,0} & 0 & \cdots & -G_{N-1,N-2} & I & 0 \\ -G_{N,0} & 0 & 0 & \cdots & -G_{N,N-1} & I \end{bmatrix}$$

C^{-1} is a CM_F matrix defined as follows.

[8]Note that (20) means that for $c = N$ we have $x_0 = e_0$ and $x_N = G_{N,0}x_0 + e_N$, and for $c = 0$ we have $x_0 = e_0$. Likewise for (21).

Definition 2.5. *A symmetric positive definite matrix is CM_F if it has the following form*

$$\begin{bmatrix} A_0 & B_0 & D_2 & \cdots & D_{N-2} & D_{N-1} & D_N \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & 0 \\ D_2' & B_1' & A_2 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{N-2}' & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & 0 \\ D_{N-1}' & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_N' & 0 & 0 & \cdots & 0 & B_{N-1}' & A_N \end{bmatrix} \tag{24}$$

Here A_k, B_k, and D_k are matrices in general.

For $c = N$, we have a CM_L model. Then,

$$\mathcal{G}^L x = e^L, \quad e^L \triangleq [e_0', \dots, e_N']' \tag{25}$$

$$C^{-1} = (\mathcal{G}^L)'(G^L)^{-1}\mathcal{G}^L \tag{26}$$

where $G^L = \text{Cov}(e^L) = \text{diag}(G_0, \dots, G_N)$, $\mathcal{G}^L$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,N} \\ 0 & -G_{2,1} & I & 0 & \cdots & -G_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{N-1,N-2} & I & -G_{N-1,N} \\ -G_{N,0} & 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

for (20), and $\mathcal{G}^L$ for (21) is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G_{0,N} \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,N} \\ 0 & -G_{2,0} & I & 0 & \cdots & -G_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{N-1,N-2} & I & -G_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

C^{-1} is a CM_L matrix defined as follows.

Definition 2.6. *A symmetric positive definite matrix is CM_L if it has the following form*

$$\begin{bmatrix} A_0 & B_0 & 0 & \cdots & 0 & 0 & D_0 \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & D_1 \\ 0 & B_1' & A_2 & B_2 & \cdots & 0 & D_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & D_{N-2} \\ 0 & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_0' & D_1' & D_2' & \cdots & D_{N-2}' & B_{N-1}' & A_N \end{bmatrix} \tag{27}$$

Remark 2.7. *$[x_k]$ is reciprocal iff it obeys* (19) *along with* (20) *or* (21) *and*

$$G_k^{-1}G_{k,c} = G_{k+1,k}'G_{k+1}^{-1}G_{k+1,c} \tag{28}$$

$\forall k \in [1, N-2]$ for $c = N$, or $\forall k \in [2, N-1]$ for $c = 0$. Moreover, for $c = N$, $[x_k]$ is Markov iff in addition to (28)*, we have $G_0^{-1}G_{0,N} = G_{1,0}'G_1^{-1}G_{1,N}$ for* (21)*, or equivalently $G_N^{-1}G_{N,0} = G_{1,N}'G_1^{-1}G_{1,0}$ for* (20)*. Also, for $c = 0$, $[x_k]$ is Markov iff $G_{N,0} = 0$ in addition to* (28)*.*

By Remark 2.7, a reciprocal sequence may obey a CM_c model. Such a CM_c model is called a reciprocal CM_c model.

5) Backward CM_c Models: $[x_k]$ is CM_c, $c \in \{0, N\}$, iff

$$x_k = G^B_{k,k+1} x_{k+1} + G^B_{k,c} x_c + e^B_k, k \in [0, N-1] \setminus \{c\} \tag{29}$$

where $[e^B_k]$ ($G^B_k = \text{Cov}(e^B_k)$) is a zero-mean white NG sequence, and boundary condition

$$x_N = e^B_N, \quad x_c = G^B_{c,N} x_N + e^B_c \text{ (for } c = 0) \tag{30}$$

or equivalently

$$x_c = e^B_c, \quad x_N = G^B_{N,c} x_c + e^B_N \text{ (for } c = 0) \tag{31}$$

For $c = 0$, we have a backward CM_L model. Then,

$$\mathcal{G}^{BL} x = e^{BL}, \quad e^{BL} = [(e^B_0)', \dots, (e^B_N)']' \tag{32}$$

$$C^{-1} = (\mathcal{G}^{BL})'(G^{BL})^{-1}\mathcal{G}^{BL} \tag{33}$$

where C^{-1} is a CM_F matrix, $G^{BL} = \text{Cov}(e^{BL}) = \text{diag}(G^B_0, \dots, G^B_N)$, $\mathcal{G}^{BL}$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G^B_{0,N} \\ -G^B_{1,0} & I & -G^B_{1,2} & \cdots & 0 & 0 \\ -G^B_{2,0} & 0 & I & -G^B_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -G^B_{N-1,0} & 0 & \cdots & 0 & I & -G^B_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

for (30), and $\mathcal{G}^{BL}$ for (31) is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -G^B_{1,0} & I & -G^B_{1,2} & \cdots & 0 & 0 \\ -G^B_{2,0} & 0 & I & -G^B_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -G^B_{N-1,0} & 0 & \cdots & 0 & I & -G^B_{N-1,N} \\ -G^B_{N,0} & 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

For $c = N$, we have a backward CM_F model. Then,

$$\mathcal{G}^{BF} x = e^{BF}, \quad e^{BF} = [(e^B_0)', \dots, (e^B_N)']' \tag{34}$$

$$C^{-1} = (\mathcal{G}^{BF})'(G^{BF})^{-1}\mathcal{G}^{BF} \tag{35}$$

where C^{-1} is a CM_L matrix, $G^{BF} = \text{Cov}(e^{BF}) = \text{diag}(G_0, \dots, G_N)$, and $\mathcal{G}^{BF}$ is the nonsingular matrix

$$\begin{bmatrix} I & -G^B_{0,1} & 0 & \cdots & 0 & -G^B_{0,N} \\ 0 & I & -G^B_{1,2} & \cdots & 0 & -G^B_{1,N} \\ 0 & 0 & I & -G^B_{2,3} & \cdots & -G^B_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I & -2G^B_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{36}$$

Remark 2.8. *$[x_k]$ is reciprocal iff it obeys* (29) *along with* (30) *or* (31) *and* $(G^B_{k+1})^{-1}G^B_{k+1,c} = (G^B_{k,k+1})'(G^B_k)^{-1}G^B_{k,c}$, $\forall k \in [1, N-2]$ *for* $c = 0$, *or* $\forall k \in [0, N-3]$ *for* $c = N$. *Moreover, for* $c = 0$, $[x_k]$ *is Markov iff we also have* $(G^B_0)^{-1}G^B_{0,N} = (G^B_{N-1,0})'(G^B_{N-1})^{-1}G^B_{N-1,N}$ *for* (30), *or* $(G^B_N)^{-1}G^B_{N,0} = (G^B_{N-1,N})'(G^B_{N-1})^{-1}G^B_{N-1,0}$ *for* (31); *for* $c = N$, $[x_k]$ *is Markov iff we also have* $G^B_{0,N} = 0$.

Forward and backward CM_L (CM_F) models have similar structures. They differ only in the time direction.

For a Markov model, $[e^M_k]^N_1$ is the dynamic noise and e^M_0 is the initial value. Likewise for other models.

Let $[x_k]$ be a CM sequence obeying any of the above models. Then,

$$Tx = v, \quad v = [v'_0, \dots, v'_N]' \tag{37}$$

where the vector v consists of the dynamic noise and the boundary values. The matrix T is determined by parameters of the corresponding model. T is nonsingular for all models (i.e., forward/backward Markov, reciprocal, CM_L, and CM_F) considered. (Note that since $[x_k]$ is assumed nonsingular, T is nonsingular for the reciprocal model.)

Definition 2.9. *Two models $T_1 x = v$ and $T_2 y = w$ are PE if x and y have the same distribution.*

Definition 2.10. *Two models $T_1 x = v$ and $T_2 y = w$ are AE if $x = y$.*

Definition 2.11. *Two models are equivalent if they are either PE or AE.*

III. Determination of Algebraically Equivalent Models: A Unified Approach

By Definitions 2.9, 2.10, and 2.11 equivalence is mutual: if model 2 is equivalent to model 1, so is model 1 to model 2.

To determine a PE model, we need to fix its parameters. Thus, we have the following proposition.

Proposition 3.1. *Any two models considered*

$$T_1 x = v \tag{38}$$

$$T_2 y = w \tag{39}$$

are PE iff

$$T'_2 P_2^{-1} T_2 = T'_1 P_1^{-1} T_1 \tag{40}$$

where $v = [v'_0, \dots, v'_N]'$ and $w = [w'_0, \dots, w'_N]'$ are the vectors of the dynamic noise and boundary values with covariances Cov$(v) = P_1$ and Cov$(w) = P_2$.

Proof. For the sequence obeying model (38) we have $C^{-1} = T'_1(P_1)^{-1}T_1$ because $E[(T_1 x)(T_1 x)'] = E[vv']$. Similarly, for the sequence obeying (39), we have $C^{-1} = T'_2(P_2)^{-1}T_2$. Two models are PE iff their sequences have the same covariance matrix; thus we have (40). □

Due to the special structures of T_1, P_1, T_2, and P_2, parameters of model 2 can be easily obtained from parameters of model 1 using (40) (see Appendix A for more details). Then, P_2 and T_2 are known. Note that parameters of model 2 so calculated are unique. This can be easily verified based on (40) for all models (see Appendix A). This uniqueness also follows from the definition of conditional expectation.

Clearly, AE models are PE. The next proposition relates dynamic noise and boundary values for two PE models to be AE.

Proposition 3.2. *Two PE models* (38) *and* (39) *are AE if*

$$T'_2(P_2)^{-1} w = T'_1(P_1)^{-1} v \tag{41}$$

Proof. Let P_2, T_2, P_1, and T_1 be given (Proposition 3.1). Given model (38), we show how (41) leads to an AE model (39). First, we show that w has the desired covariance P_2. By (41), we have $T_2'(P_2)^{-1}\text{Cov}(w)(P_2)^{-1}T_2 = T_1'(P_1)^{-1}\text{Cov}(v)(P_1)^{-1}T_1$. From $\text{Cov}(v) = P_1$ and (40) it follows that $\text{Cov}(w) = P_2(T_2')^{-1}T_2'(P_2)^{-1}T_2(T_2)^{-1}P_2 = P_2$. Thus, w is the required vector.

Now we show that (41) implies that models (38) and (39) generate the same sample path of the sequence. We have

$$\begin{aligned}T_1'(P_1)^{-1}T_1 y &\overset{(40)}{=} T_2'(P_2)^{-1}T_2 y \overset{(39)}{=} T_2'(P_2)^{-1}w \\ &\overset{(41)}{=} T_1(P_1)^{-1}v \overset{(38)}{=} T_1(P_1)^{-1}T_1 x \quad \Longrightarrow \ y = x\end{aligned}$$

So, (39) and (38) are algebraically equivalent. □

By Propositions 3.1 and 3.2, given a model, one can construct an AE model. For two AE models, how are the sample paths of their dynamic noise and boundary values related? The next proposition answers this question.

Proposition 3.3. *For two AE models considered*

$$T_1 x = v \tag{42}$$

$$T_2 y = w \tag{43}$$

the sample paths of v and w are related by (41)*, where $v = [v_0', \dots, v_N']'$ and $w = [w_0', \dots, w_N']'$ are vectors of the dynamic noise and boundary values with covariances $Cov(v) = P_1$ and $Cov(w) = P_2$, and the nonsingular matrices T_1 and T_2 are determined by the model parameters.*

Proof. Algebraic equivalence (i.e., $x = y$) of (42) and (43) yields

$$T_2^{-1}w = T_1^{-1}v \tag{44}$$

It follows from the equivalence of (42) and (43) that

$$C^{-1} = T_1'P_1^{-1}T_1 = T_2'P_2^{-1}T_2 \tag{45}$$

Then, using (44) and (45), we have $(T_2'P_2^{-1}T_2)T_2^{-1}w = (T_1'P_1^{-1}T_1)T_1^{-1}v$, which leads to (41). □

Remark 3.4. (41) *is equivalent to* (44).

Although (44) looks simpler, for constructing AE models, (41) is preferred because P_1 and P_2 in (41) for the models considered are block diagonal, and their inverses can be easily calculated, while calculation of the inverses of T_1 and T_2 in (44) is not straightforward in general.

Theorem 3.5 follows from Propositions 3.2 and 3.3.

Theorem 3.5. *Two PE models* (38) *and* (39) *are AE iff* (41) *holds.*

The uniqueness of parameters of equivalent models was discussed after the proof of Proposition 3.1. The relationship between the dynamic noise of two AE models is unique since (41) is the same as (44) which is the same as Definition 2.10. So, we have the following remark.

Remark 3.6. *By* (40) *and* (41) *(for a given form, i.e., Markov, reciprocal, CM_L, or CM_F) the probabilistically/algebraically equivalent model is unique.*

IV. Algebraically Equivalent Models: Examples

Following Propositions 3.1 and 3.2, AE models can be obtained. Two such examples are presented in this section, and more in appendices. Appendix A shows how parameters of PE models can be uniquely determined from each other (Proposition 3.1). Appendix B shows how the dynamic noise and boundary values of AE models are related (Proposition 3.2).

A. Forward and Backward Markov Models

By (40), parameters of a Markov BM (6) are obtained from those of a Markov FM (2). For $k = 2, 3, \dots, N$,

$$\begin{aligned}(M_0^B)^{-1} &= M_0^{-1} + M_{1,0}'M_1^{-1}M_{1,0} &(46)\\ M_{0,1}^B &= M_0^B M_{1,0}'M_1^{-1} &(47)\\ (M_{k-1}^B)^{-1} &= M_{k-1}^{-1} + M_{k,k-1}'M_k^{-1}M_{k,k-1} - \\ &\quad (M_{k-2,k-1}^B)'(M_{k-2}^B)^{-1}M_{k-2,k-1}^B &(48)\\ M_{k-1,k}^B &= M_{k-1}^B M_{k,k-1}'M_k^{-1} &(49)\\ (M_N^B)^{-1} &= M_N^{-1} - (M_{N-1,N}^B)'(M_{N-1}^B)^{-1}M_{N-1,N}^B &(50)\end{aligned}$$

By (41), the dynamic noise and boundary values of the two models are related by

$$\begin{aligned}(M_0^B)^{-1}e_0^{BM} &= M_0^{-1}e_0^M - M_{1,0}'M_1^{-1}e_1^M &(51)\\ (M_k^B)^{-1}e_k^{BM} &= (M_{k-1,k}^B)'(M_{k-1}^B)^{-1}e_{k-1}^{BM} + M_k^{-1}e_k^M \\ &\quad -M_{k+1,k}'M_{k+1}^{-1}e_{k+1}^M, k \in [1, N-1] &(52)\\ (M_N^B)^{-1}e_N^{BM} &= (M_{N-1,N}^B)'(M_{N-1}^B)^{-1}e_{N-1}^{BM} + M_N^{-1}e_N^M &(53)\end{aligned}$$

By these equations, given a BM, one can obtain its AE FM.

As discussed in [35]–[37], the two-filter smoother is based on fusing two estimates obtained from a forward filter and a backward filter. To obtain the smoothing estimate at time k, the forward/backward filter gives an estimate using all measurements before/after k. The forward/backward filter is based on an FM/BM. The estimate of the forward/backward filter is a function of the dynamic noise (and boundary values) of the FM/BM. To optimally fuse the two estimates, it is necessary to verify whether there is any correlation between the two estimates. Therefore, it is necessary to have a relationship in the dynamic noise between the FM and the BM. In other words, AE forward and backward models are required. Note that the noise relationship between the FM and the BM is not clear for PE models. So, PE models are not useful in deriving the two-filter smoother. The only existing approach to determining AE Markov FM and BM was presented in [41]. As clarified in [41], in the case of singular state transition matrices, its approach does not work: its FM and BM are not AE, but only PE. Our (46)–(50) and (51)–(53) give AE FM and BM no matter if the state transition matrix is singular or nonsingular: (40) and (41) work no matter if $M_{k,k-1}$ and $M_{k,k+1}^B$ are singular or nonsingular. Note that there is no inverse of transition matrices in (46)–(50) or (51)–(53). Also, the approach of [41] is only for Markov models. Our approach is simple yet determines AE models for all models considered.

B. Reciprocal CM_L and Reciprocal Models

By (40), parameters of a reciprocal model are obtained from those of a reciprocal CM_L model. For (19)–(20), parameters of the reciprocal model are

$$R_0^0 = G_0^{-1} + G_{1,0}' G_1^{-1} G_{1,0} + G_{N,0}' G_N^{-1} G_{N,0} \tag{54}$$
$$R_k^0 = G_k^{-1} + G_{k+1,k}' G_{k+1}^{-1} G_{k+1,k}, k \in [1, N-2] \tag{55}$$
$$R_{N-1}^0 = G_{N-1}^{-1} \tag{56}$$
$$R_N^0 = G_N^{-1} + \sum_{k=1}^{N-1} G_{k,N}' G_k^{-1} G_{k,N} \tag{57}$$
$$R_k^+ = G_{k+1,k}' G_{k+1}^{-1}, k \in [0, N-2] \tag{58}$$
$$R_{N-1}^+ = G_{N-1}^{-1} G_{N-1,N} \tag{59}$$
$$R_0^- = -G_{1,0}' G_1^{-1} G_{1,N} + G_{N,0}' G_N^{-1} \tag{60}$$

and for (19) and (21) we have (55)–(56), (58)–(59), and

$$R_0^0 = G_0^{-1} + G_{1,0}' G_1^{-1} G_{1,0} \tag{61}$$

$$R_N^0 = G_N^{-1} + \sum_{k=1}^{N-1} G_{k,N}' G_k^{-1} G_{k,N} + G_{0,N}' G_0^{-1} G_{0,N} \tag{62}$$
$$R_0^- = G_0^{-1} G_{0,N} - G_{1,0}' G_1^{-1} G_{1,N} \tag{63}$$

By (41), the dynamic noise and boundary values of the two models are related by: for (19)–(20),

$$e_0^R = G_0^{-1} e_0 - G_{1,0}' G_1^{-1} e_1 - G_{N,0}' G_N^{-1} e_N \tag{64}$$
$$e_k^R = G_k^{-1} e_k - G_{k+1,k}' G_{k+1}^{-1} e_{k+1}, k \in [1, N-2] \tag{65}$$
$$e_{N-1}^R = G_{N-1}^{-1} e_{N-1} \tag{66}$$
$$e_N^R = -\sum_{k=1}^{N-1} G_{k,N}' G_k^{-1} e_k + G_N^{-1} e_N \tag{67}$$

and for (19) and (21), replace (64) and (67) by

$$e_0^R = G_0^{-1} e_0 - G_{1,0}' G_1^{-1} e_1 \tag{68}$$
$$e_N^R = -\sum_{k=1}^{N-1} G_{k,N}' G_k^{-1} e_k + G_N^{-1} e_N - G_{0,N}' G_0^{-1} e_0 \tag{69}$$

By these equations, one can obtain an AE reciprocal CM_L model from a reciprocal model. This is important because a reciprocal CM_L model is easier to apply than a reciprocal model [20]. For example, estimation based on a CM_L model is straightforward, but several papers were devoted to estimation based on a reciprocal model [44]–[48].

V. More About Algebraically Equivalent Models

A. Models AE to a Reciprocal Model

This section presents two approaches for determining models AE to a reciprocal model (10) (along with (14)), or the other way round. The same approach works for boundary condition (13).

We first show how to determine parameters of a reciprocal model (10) PE to a reciprocal CM_L model (19) and (21). Model (10) is obtained based on conditional expectations [29], so its parameters are as given in Subsection IV-B for an NG reciprocal sequence (i.e., with a given covariance matrix). (21) and (14) are the same since they are both obtained from the joint density of x_0 and x_N, which is the same for both reciprocal and reciprocal CM_L models.

Similarly, from parameters of a reciprocal model (10) and (14), we can uniquely determine parameters of its PE reciprocal CM_L model (19) and (21). Also, by (40), parameters of other PE models can be determined.

AE models are discussed next.

1) The First Approach: We show that the unified approach of Section III (i.e., (41)) works for models AE to a reciprocal model (10) and (14).

First, we determine the structure of T, P, and ξ in (37) for model (10). We have

$$\mathfrak{R}_r x = e^r \tag{70}$$

where $e^r \triangleq [(e_0^R)', \ldots, (e_N^R)']'$ and

$$\mathfrak{R}_r = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & -R_{0,N} \\ -R_1^- & R_1^0 & -R_1^+ & \cdots & 0 & 0 \\ 0 & -R_2^- & R_2^0 & -R_2^+ & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -R_{N-1}^- & R_{N-1}^0 & -R_{N-1}^+ \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{71}$$

It is nonsingular because its submatrix of the block rows and columns 2 to N is nonsingular since (17) is nonsingular. Its nonsingularity can be verified by the determinant of a partitioned matrix [49]. Also, the covariance of e^r is

$$R_r = \begin{bmatrix} R_0^0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & R_1^0 & -R_1^+ & \cdots & 0 & 0 \\ 0 & -R_2^- & R_2^0 & -R_2^+ & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -R_{N-1}^- & R_{N-1}^0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & R_N^0 \end{bmatrix} \tag{72}$$

which is likewise nonsingular because model (10) is independent of boundary condition [29].

With (71) and (72), models AE to (10) and (14) can be obtained by (41).

2) The Second Approach: In the first approach, $(R_r)^{-1}$ is required in (41), which is not desirable since R_r is not block diagonal. In the following, we present a simple relationship in dynamic noise and boundary values between a reciprocal model and an AE reciprocal CM_L model.

It suffices to construct a reciprocal CM_L model AE to a reciprocal model. Then, by Proposition 3.2 other AE models can be obtained. We show that (73) below makes a PE reciprocal model AE to a reciprocal CM_L model (19) and (21):

$$e^r = T_{R|CM_L} e \tag{73}$$

where $T_{R|CM_L}$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & G_1^{-1} & -G'_{2,1}G_2^{-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & G_2^{-1} & -G'_{3,2}G_3^{-1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & G_3^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & G_{N-1}^{-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{74}$$

where $e \triangleq [e'_0, \ldots, e'_N]'$ is the vector of dynamic noise and boundary values of the reciprocal CM_L model and $e^r \triangleq [(e_0^R)', \ldots, (e_N^R)']'$ is that of the reciprocal model of [29]. Following [29], part of (73) was used in [14].

Let $[e_k]$ be white (since it is for a reciprocal CM_L model). We show that $[e_k^R]$ has the properties of reciprocal dynamic noise and boundary values. By (74), the covariance of $[e_k^R]_1^{N-1}$ is cyclic tridiagonal. So, $[e_k^R]_1^{N-1}$ can serve as dynamic noise of a reciprocal model (10). It is a function of $[e_k]_1^{N-1}$ with $e_0^R = e_0$ and $e_N^R = e_N$. Then, since $[e_k]$ is white, $[e_k^R]_1^{N-1}$ is uncorrelated with e_0^R and e_N^R and consequently with x_0 and x_N. Therefore, $[e_k^R]_1^{N-1}$ can serve as reciprocal dynamic noise, and e_0^R and e_N^R as boundary values.

Now, we show that (73) leads to the same sample path of the sequence obeying the reciprocal CM_L model and the reciprocal model. From (73), we have

$$e_k^R = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1}, k \in [1, N-2] \tag{75}$$

Substituting e_k and e_{k+1} of the CM_L model (19) into (75), after some manipulation, we get

$$\begin{aligned} e_k^R = (G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k})x_k - \\ G_k^{-1}G_{k,k-1}x_{k-1} - G'_{k+1,k}G_{k+1}^{-1}x_{k+1} + \\ (-G_k^{-1}G_{k,N} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,N})x_N \end{aligned} \tag{76}$$

Using (28), (76) becomes

$$\begin{aligned} e_k^R = (G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k})x_k - \\ G_k^{-1}G_{k,k-1}x_{k-1} - G'_{k+1,k}G_{k+1}^{-1}x_{k+1} \end{aligned} \tag{77}$$

(77) has the properties (the structure and parameters) of (10) and thus can serve as a reciprocal model for $k \in [1, N-2]$. In addition, for $k = N-1$, based on (73) we have

$$e_{N-1}^R = G_{N-1}^{-1}e_{N-1} \tag{78}$$

Substituting e_{N-1} of (19), we have

$$\begin{aligned} e_{N-1}^R = G_{N-1}^{-1}x_{N-1} - G_{N-1}^{-1}G_{N-1,N-2}x_{N-2} - \\ G_{N-1}^{-1}G_{N-1,N}x_N \end{aligned} \tag{79}$$

(79) can serve as a reciprocal model for $k = N-1$. So, by (77), (79) and since (21) and (14) are identical, (73) leads to the same sample path of the sequence obeying the two models. In other words, the two models are AE.

Next, from a reciprocal model (10) and (14), we construct its AE reciprocal CM_L model. Calculation of the parameters of (19) and (21) from those of (10) and (14) was discussed above. So, $T_{R|CM_L}$ is known. First, we show that e in (73) has a (block) diagonal covariance matrix, i.e, $[e_k]$ is white (which is the case for a reciprocal CM_L model). According to (10) and (14), e_0^R and e_N^R are uncorrelated, and uncorrelated with $[e_k^R]_1^{N-1}$. By (73), we have $e_0 = e_0^R$ and $e_N = e_N^R$. Also, $[e_k]_1^{N-1}$ are linear combinations of $[e_k^R]_1^{N-1}$. So, e_0 and e_N are mutually uncorrelated and uncorrelated with $[e_k]_1^{N-1}$. Therefore, we only need to show that $[e_k]_1^{N-1}$ is white. The covariance of $[(e_1^R)', \ldots, (e_{N-1}^R)']'$ is $(R_r)_{[2:N,2:N]}$, (i.e., the $[2:N, 2:N]$ submatrix of R_r (72)). By (73), we have

$$\begin{aligned} &(R_r)_{[2:N,2:N]} = \\ &(T_{R|CM_L})_{[2:N,2:N]}(\text{Cov}(e))_{[2:N,2:N]}(T_{R|CM_L})'_{[2:N,2:N]} \end{aligned} \tag{80}$$

Let C be the covariance matrix of the reciprocal sequence. Now calculate C^{-1} based on the reciprocal CM_L model (19) and (21) (Appendix A). Then, we have the decomposition

$$\begin{aligned} &(C^{-1})_{[2:N,2:N]} = \\ &\quad (T_{R|CM_L})_{[2:N,2:N]}G_{[2:N,2:N]}(T_{R|CM_L})'_{[2:N,2:N]} \end{aligned} \tag{81}$$

where $G_{[2:N,2:N]} = \text{diag}(G_1, \ldots, G_{N-1})$. By (16)–(17), we have $(R_r)_{[2:N,2:N]} = (C^{-1})_{[2:N,2:N]}$. Comparing (81) and (80), we have $(\text{Cov}(e))_{[2:N,2:N]} = G_{[2:N,2:N]}$, meaning that $[e_k]_1^{N-1}$ is white. So, $[e_k]$ is white.

Next, we show that (73) leads to algebraic equivalence of the reciprocal model and the reciprocal CM_L model. (73) for $k = N-1$ is

$$e_{N-1}^R = G_{N-1}^{-1}e_{N-1} \tag{82}$$

Using e_{N-1}^R from the reciprocal model (10), we obtain

$$R_{N-1}^0 x_{N-1} - R_{N-1}^- x_{N-2} - R_{N-1}^+ x_N = G_{N-1}^{-1}e_{N-1}$$

Expressing R_{N-1}^0, R_{N-1}^-, and R_{N-1}^+ of the reciprocal model in terms of parameters of the reciprocal CM_L model (specifically (56), (58), (59)) yields

$$\begin{aligned} G_{N-1}^{-1}x_{N-1} - (G_{N-1}^{-1}G_{N-1,N-2})x_{N-2} - (G_{N-1}^{-1}G_{N-1,N})x_N \\ = G_{N-1}^{-1}e_{N-1} \end{aligned}$$

which leads to

$$x_{N-1} - G_{N-1,N-2}x_{N-2} - G_{N-1,N}x_N = e_{N-1} \tag{83}$$

Clearly (83) is a CM_L model (19) for $k = N-1$ with e_{N-1} related to e_{N-1}^R by (82). Then, By (73), for $k \in [1, N-2]$, we have

$$e_k^R = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1} \tag{84}$$

Substituting e_k^R of the reciprocal model (10) into (84) yields

$$\begin{aligned} R_k^0 x_k - R_k^- x_{k-1} - R_k^+ x_{k+1} = \\ G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1} \end{aligned} \tag{85}$$

Substituting e_{k+1} from the reciprocal CM_L model (19) into (85), we obtain

$$\begin{aligned} (G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k})x_k - G_k^{-1}G_{k,k-1}x_{k-1} - \\ G'_{k+1,k}G_{k+1}^{-1}x_{k+1} = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}(x_{k+1} - \\ G_{k+1,k}x_k - G_{k+1,N}x_N) \end{aligned} \tag{86}$$

After manipulation, (86) becomes

$$G_k^{-1}x_k - G_k^{-1}G_{k,k-1}x_{k-1} - G'_{k+1,k}G_{k+1}^{-1}G_{k+1,N}x_N = G_k^{-1}e_k \quad (87)$$

Using (28) for the coefficient of x_N in (87), (87) leads to

$$x_k - G_{k,k-1}x_{k-1} - G_{k,N}x_N = e_k \quad (88)$$

This is a CM_L model (19) for $k \in [1, N-2]$ with $[e_k]_1^{N-2}$ related to $[e_k^R]_1^{N-2}$ by (73). Also, the two models have identical boundary conditions. So, (73) forces the two models to have the same sample paths. That is, using (73), the reciprocal model and the reciprocal CM_L model are AE.

B. Parameters of PE Markov and Reciprocal Models

By (40), parameters of PE models can be uniquely determined (Appendix A). In some cases given parameters of a model, one can calculate parameters of a PE model in different ways. Due to the uniqueness, the apparently different results must be the same. For example, in the following we consider an approach (different from (40)) for calculating parameters of a reciprocal model PE to a Markov model. Then, we show that the results are actually the same as those of Appendix A.

Given a Markov model (2) of $[x_k]$, by (40), parameters of a PE reciprocal model (10) are (Appendices A-D and A-C), for $k \in [1, N-1]$,

$$R_k^0 = M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k} \quad (89)$$

$$R_k^+ = M'_{k+1,k}M_{k+1}^{-1} \quad (90)$$

$$R_k^- = M_k^{-1}M_{k,k-1} \quad (91)$$

Parameters of the reciprocal model (10) can be also obtained as follows. The transition density of $[x_k]$ is

$$p(x_k|x_{k-1}) = \mathcal{N}(x_k; M_{k,k-1}x_{k-1}, M_k) \quad (92)$$

By the Markov property, we have

$$\begin{aligned} p(x_k|x_{k-1}, x_{k+1}) &= \frac{p(x_k|x_{k-1})p(x_{k+1}|x_k)}{p(x_{k+1}|x_{k-1})} \\ &= \mathcal{N}(x_k; R_{k,k-1}x_{k-1} + R_{k,k+1}x_{k+1}, R_k) \end{aligned}$$

Then, we define r_k as

$$r_k = x_k - R_{k,k-1}x_{k-1} - R_{k,k+1}x_{k+1} \quad (93)$$

where the covariance of r_k is R_k and

$$\begin{aligned} R_{k,k-1} &= M_{k,k-1} - (M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k})^{-1} \\ &\quad \cdot M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k}M_{k,k-1} \\ R_{k,k+1} &= (M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k})^{-1}M'_{k+1,k}M_{k+1}^{-1} \\ R_k &= (M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k})^{-1} \end{aligned}$$

Pre-multiplying both sides of (93) by R_k^0 (which is nonsingular), we obtain

$$R_k^0x_k = R_k^0R_{k,k-1}x_{k-1} + R_k^0R_{k,k+1}x_{k+1} + R_k^0r_k \quad (94)$$

By the uniqueness of parameters, we must have $R_k^0R_{k,k-1} = R_k^-$, $R_k^0R_{k,k+1} = R_k^+$, and $\text{Cov}(R_k^0r_k) = R_k^0$. Comparing the parameters of (94) with (89), (90), and (91), it is not clear that $R_k^0R_{k,k-1} = R_k^-$, which, however, can be verified using

$$(M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k})^{-1}(M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k})M_{k,k-1} = M_{k,k-1}$$

VI. Markov Models and Reciprocal/CM_L Models

An important question in the theory of reciprocal processes is about Markov processes sharing by the same reciprocal evolution law [30], [25]. It is desired to determine Markov transition models (i.e., without the initial condition) of Markov sequences, which obey a reciprocal CM_L model (and an arbitrary boundary condition). Also, given two Markov transition models, whether their sequences share the same CM_L transition model (i.e., without boundary conditions). Studying such issues will gain a better understanding of the models and sequences, and is useful for their application. For example, [31] discussed CM_L models *induced* by Markov models for trajectory modeling with destination information, and showed that inducing a CM_L model by a Markov model is useful for parameter design of a reciprocal CM_L model, and that a reciprocal CM_L model can be induced by any Markov model whose sequences obey the given reciprocal CM_L model (and some boundary condition). So, it is desired to determine all such Markov models and their relationships. In the following, a simple approach is presented for studying and determining different Markov models whose sequences share the same reciprocal/CM_L model.

Relationships between different models (and their boundary conditions) can be studied based on the entries of C^{-1} calculated from the models and their boundary conditions. Some entries of C^{-1} depend on model parameters only and others depend also on boundary conditions (Appendix A). Proofs of the following results are based on Appendix A.

The next proposition gives conditions for Markov sequences of Markov models to share the same reciprocal model.

Proposition 6.1. *Two Markov sequences having by Markov models* (2) *with parameters* $M_{k,k-1}^{(i)}$, $M_k^{(i)}$, $k \in [1, N]$, $i = 1, 2$, *share the same reciprocal model* (10) *iff*

$$\begin{aligned} &(M_k^{(1)})^{-1} + (M_{k+1,k}^{(1)})'(M_{k+1}^{(1)})^{-1}M_{k+1,k}^{(1)} = \\ &(M_k^{(2)})^{-1} + (M_{k+1,k}^{(2)})'(M_{k+1}^{(2)})^{-1}M_{k+1,k}^{(2)}, k \in [1, N-1] \end{aligned} \quad (95)$$

$$(M_{k+1,k}^{(1)})'(M_{k+1}^{(1)})^{-1} = (M_{k+1,k}^{(2)})'(M_{k+1}^{(2)})^{-1}, k \in [0, N-1] \quad (96)$$

Proof. Two sequences share the same reciprocal model iff their C^{-1} (18) have the same entries $A_1, A_2, \ldots, A_{N-1}, B_0, B_1, \ldots, B_{N-1}$. So, two Markov sequences having Markov models with parameters $M_{k,k-1}^{(i)}$, $M_k^{(i)}$, $k \in [1, N]$, $i = 1, 2$, share the same reciprocal model iff (95)–(96) hold. □

Sequences having any Markov model (2) satisfying

$$R_k^0 = M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k}, k \in [1, N-1] \quad (97)$$

$$R_k^+ = M'_{k+1,k}M_{k+1}^{-1}, k \in [0, N-1] \quad (98)$$

share a given reciprocal model (with some boundary condition) (see Proposition 6.1). Therefore, all Markov models whose sequences share a reciprocal model are determined.

Proposition 6.2. *Two sequences share the same reciprocal model* (10) *iff they share the same reciprocal CM_L model* (19) *($c = N$).*

Proof. Two sequences share the same reciprocal model (10) (reciprocal CM_L model (19) ($c = N$)) iff their C^{-1} (18) have the same entries $A_1, \dots, A_{N-1}$ $, B_0, \dots, B_{N-1}$, that is, iff they share the same reciprocal CM_L model (19) ($c = N$). □

By Proposition 6.2 and (97)–(98) we can determine all Markov models whose sequences share a reciprocal CM_L model (19). All we need to do is to replace the model parameters in (97)–(98) (i.e., R_k^0 and R_k^+) with the corresponding (block) entries of the C^{-1} calculated from the parameters of (19) (see Subsection IV-B or Appendix A).

The following proposition determines conditions for two Markov sequences sharing the same reciprocal transition model to share the same Markov transition model.

Proposition 6.3. *Two Markov sequences sharing the same reciprocal model* (10) *share the same Markov model* (2) *iff for the parameters of* (12) *we have*

$$(R_N^0)^{(1)} = (R_N^0)^{(2)} \tag{99}$$

or equivalently $M_N^{(1)} = M_N^{(2)}$, where the superscripts (1) *and* (2) *correspond to the first and the second sequence.*

Proof. Two reciprocal sequences share the same reciprocal model iff their C^{-1} (18) have the same entries $A_1, \dots, A_{N-1}, B_0, \dots,$ B_{N-1}. Two Markov sequences share the same Markov model iff their C^{-1} (5) have the same entries $A_1, \dots, A_N$, $B_0, \dots, B_{N-1}$. So, two Markov sequences sharing the same reciprocal model share the same Markov model iff they have the same A_N, i.e., (99) holds (see (100)). □

More general relationships between different models considered can be studied based on the entries of C^{-1} calculated from the models and the boundary conditions. In general, we can obtain conditions for two sequences sharing the same transition model to share the same transition model of different type.

VII. Illustrative Examples

In Section III, we presented an approach for determining equivalent models. In the introduction we pointed out trajectory modeling as an application of CM (including reciprocal) sequences. Below, we clarify that for trajectory modeling with destination information a (reciprocal) CM_L model (19) (for $c = N$) [28] is much better than a reciprocal model (10) [29], and so given a reciprocal model, it is better to obtain its equivalent CM_L model.

As stated in the introduction, after quantizing the state space, [9]–[12] used finite state reciprocal sequences for trajectory modeling with destination information. However, such quantization is not always feasible–it can be computationally prohibitive. So, a continuous-state space may be desired, as for a Gaussian sequence. The structure of the reciprocal model of [29] (see (10)) is less than desirable for trajectory modeling with destination information. By (10), the state at current time k depends on the state at the previous time $k-1$ and the next time $k+1$, which in reality is unknown. Also, the noise in (10) is colored, which makes estimation (filtering/prediction/smoothing) difficult. Several papers tried hard to find a recursive estimator based on (10) (e.g., [29]–[48]). In [1] and [28] we presented the CM_L sequence for trajectory modeling with destination information. Also, we presented for it a dynamic model called a CM_L model and a special CM_L model (called a reciprocal CM_L model) for reciprocal sequences. Our (reciprocal) CM_L model is suitable for trajectory modeling with destination information. By (19) (for $c = N$), the state at the current time k depends on the state at the previous time $k-1$ and the state at the last time (i.e., destination). This is fine because information about the destination is available. Also, unlike (19), the noise in our (reciprocal) CM_L model is white, and thus estimation is straightforward. So, our (reciprocal) CM_L model is more suitable for application (than (10)). Therefore, given a reciprocal model (10), it is important to determine the equivalent CM_L model. Then, we can use the equivalent CM_L model in application.

In the following, a CM_L sequence is used to model trajectories of an object moving from an origin to a destination. Such trajectories can be modeled by combining two key assumptions [32]: (i) the moving object follows a Markov model (e.g., a nearly constant velocity model) without considering the destination information, and (ii) the joint origin and destination distribution is known (which can be different from that of the Markov model in (i)). If the joint distribution is not known, an approximating can be used. Now, let $[y_k]$ be Markov modeled by (2). $[y_k]$ can be also modeled by a CM_L model (19). By the Markov property, parameters of (19) are obtained based on $p(y_k|y_{k-1}, y_N) = \frac{p(y_k|y_{k-1})p(y_N|y_k)}{p(y_N|y_{k-1})} = \mathcal{N}(y_k; G_{k,k-1}y_{k-1} + G_{k,N}y_N, G_k)$, where $G_{k,k-1} = M_{k,k-1} - G_{k,N}M_{N|k}M_{k,k-1}$, $G_{k,N} = G_k M'_{N|k} C^{-1}_{N|k}$, $G_k = (M_k^{-1} + M'_{N|k} C^{-1}_{N|k} M_{N|k})^{-1}$, $M_{N|k} = M_{N,N-1} \cdots M_{k+1,k}$, $M_{N|N} = I$, and $C_{N|k} = \sum_{n=k}^{N-1} M_{N|n+1} M_{n+1} M'_{N|n+1}$, $k \in [1, N-1]$. This CM_L model (19) is called the Markov-induced CM_L model [32]. Now, by the choice of parameters ($G_{N,0}$, G_0, and G_N) of the boundary condition (20), the Markov-induced CM_L model describes a CM_L sequence $[x_k]$ that can have any origin and destination distribution. Assume the state of a moving object at time k is $x_k = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}]'_k$ with position $[\mathsf{x}, \mathsf{y}]'$ and velocity $[\dot{\mathsf{x}}, \dot{\mathsf{y}}]'$. We consider a CM_L model induced by a (nearly constant velocity) Markov model with parameters $M_{k,k-1} = \text{diag}(F, F)$ and $M_k = \text{diag}(Q, Q)$, $k \in [1, N]$, where $F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$, $Q = q \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix}$, $T = 15$ second, $q = 0.01$, and $N = 100$. The origin has the mean $\mu_0 = [2000, 60, 5000, 5]'$ and covariance $C_0 = \text{diag}(A, A)$, $A = \begin{bmatrix} 1000 & 40 \\ 40 & 10 \end{bmatrix}$, the destination has the mean $\mu_N = [100000, 60, 2000, 5]'$ and covariance $C_N = C_0$, and their

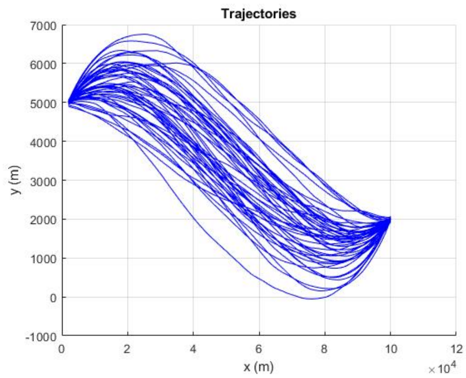

Figure 1. CM_L trajectories from an origin to a destination.

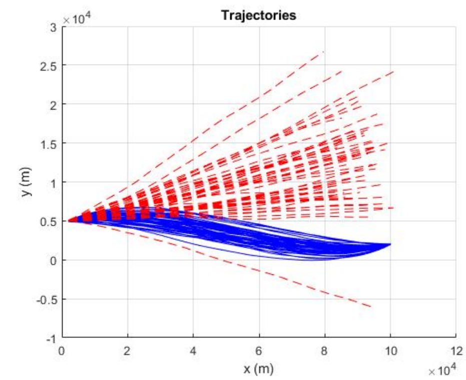

Figure 2. CM_L (solid lines) and Markov (dash lines) trajectories.

cross-covariance is $C_{N,0} = \text{diag}(B, B)$, $B = \begin{bmatrix} 800 & 20 \\ 20 & 7 \end{bmatrix}$. We have $G_0 = C_0$, $G_N = C_N - C_{N,0}C_0^{-1}C'_{N,0}$, and $G_{N,0} = C_{N,0}C_0^{-1}$ [1]. Fig. 1 shows CM_L trajectories from the origin to the destination. Fig. 2 shows Markov trajectories (dash red lines) and CM_L trajectories (solid blue lines). As illustrated above, CM_L trajectories can model any destination. Also, by (19), filtering/prediction/smoothing is straightforward by defining $s_k = [x'_k, x'_N]'$ since s_k, $k \in [0, N-1]$, is Markov.

VIII. Summary and Conclusions

Conditionally Markov (CM) sequences are powerful tools for problem modeling. Markov, reciprocal, CM_L, and CM_F are some CM classes. Their dynamic models are important for application. Relationships between models of different classes of nonsingular Gaussian CM sequences have been studied. The notion of algebraically equivalent models has been defined and a unified approach has been presented for determining algebraically or probabilistically equivalent models. This approach is simple and nonrestrictive (e.g., no need to assume nonsingularity for the matrix coefficients of the models, as the existing results do).

An important question in the theory of reciprocal processes is regarding Markov processes sharing the same reciprocal evolution law. A simple approach has been presented for studying and determining Markov models whose sequences share the same reciprocal/CM_L model.

Appendix A
Probabilistically Equivalent Models

Parameters of PE models can be calculated by (40). Since there are several different models, to save space, it suffices to show i) how to write the entries of the inverse of the covariance matrix of a sequence, C^{-1}, in terms of the parameters of the model and boundary condition of the sequence, ii) given C^{-1}, how to calculate parameters of a model and its boundary condition from the entries of C^{-1}. Then, based on (i) and (ii), from parameters of a model and its boundary condition, parameters of any PE model and its boundary condition can be uniquely determined.

A. CM_L Sequences

1) Forward CM_L Model ($c = N$): For (19):

$$\begin{aligned}
A_k &= G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k}, k \in [1, N-2] \\
A_{N-1} &= G_{N-1}^{-1} \\
B_k &= -G'_{k+1,k}G_{k+1}^{-1}, k \in [0, N-2] \\
B_{N-1} &= -G_{N-1}^{-1}G_{N-1,N} \\
D_k &= -G_k^{-1}G_{k,N} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,N}, k \in [1, N-2]
\end{aligned}$$

for boundary condition (20):

$$\begin{aligned}
A_0 &= G_0^{-1} + G'_{1,0}G_1^{-1}G_{1,0} + G'_{N,0}G_N^{-1}G_{N,0} \\
A_N &= G_N^{-1} + \sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}G_{k,N} \\
D_0 &= G'_{1,0}G_1^{-1}G_{1,N} - G'_{N,0}G_N^{-1}
\end{aligned}$$

and for (21):

$$\begin{aligned}
A_0 &= G_0^{-1} + G'_{1,0}G_1^{-1}G_{1,0} \\
A_N &= G_N^{-1} + \sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}G_{k,N} + G'_{0,N}G_0^{-1}G_{0,N} \\
D_0 &= -G_0^{-1}G_{0,N} + G'_{1,0}G_1^{-1}G_{1,N}
\end{aligned}$$

2) Backward CM_F Model ($c = N$): For (29)–(30):

$$\begin{aligned}
A_0 =& (G_0^B)^{-1} \\
A_{k+1} =& (G_{k,k+1}^B)'(G_k^B)^{-1}G_{k,k+1}^B + (G_{k+1}^B)^{-1} \\
& k \in [0, N-2] \\
A_N =& \sum_{k=0}^{N-2} (G_{k,N}^B)'(G_k^B)^{-1}G_{k,N}^B \\
& +4(G_{N-1,N}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B + (G_N^B)^{-1} \\
B_k =& -(G_k^B)^{-1}G_{k,k+1}^B, k \in [0, N-2]
\end{aligned}$$

$$\begin{aligned}
B_{N-1} =& (G_{N-2,N-1}^B)'(G_{N-2}^B)^{-1}G_{N-2,N}^B \\
& -2(G_{N-1}^B)^{-1}G_{N-1,N}^B \\
D_0 =& -(G_0^B)^{-1}G_{0,N}^B \\
D_k =& (G_{k-1,k}^B)'(G_{k-1}^B)^{-1}G_{k-1,N}^B - (G_k^B)^{-1}G_{k,N}^B \\
& k \in [1, N-2]
\end{aligned}$$

Lemma A.1. *Parameters of CM_L model* (19) *along with* (20) *or* (21) *and backward CM_F model* (29)–(30) *of a zero-mean NG CM_L sequence with the inverse of its covariance matrix equal to any given CM_L matrix* (27) *can be uniquely determined as follows:*

(i) CM_L model (19) *($c = N$):*

$$G_{N-1}^{-1} = A_{N-1}, \quad G_{1,0} = -G_1 B_0'$$
$$G_{N-1,N} = -G_{N-1} B_{N-1}$$
$$\begin{cases} k = N-1, \dots, 2: \\ G_{k,k-1} = -G_k B_{k-1}' \\ G_{k-1}^{-1} = A_{k-1} - G_{k,k-1}'(G_k)^{-1} G_{k,k-1} \\ G_{k-1,N} = G_{k-1} G_{k,k-1}' G_k^{-1} G_{k,N} - G_{k-1} D_{k-1} \end{cases}$$

Parameters of the boundary condition are: for (20)

$$G_N^{-1} = A_N - \sum_{k=1}^{N-1} G_{k,N}' G_k^{-1} G_{k,N}$$
$$G_{N,0} = G_N G_{1,N}' G_1^{-1} G_{1,0} - G_N D_0'$$
$$G_0^{-1} = A_0 - G_{1,0}' G_1^{-1} G_{1,0} - G_{N,0}' G_N^{-1} G_{N,0}$$

and for (21)*:*

$$G_0^{-1} = A_0 - G_{1,0}' G_1^{-1} G_{1,0}$$
$$G_{0,N} = G_0 G_{1,0}' G_1^{-1} G_{1,N} - G_0 D_0$$
$$G_N^{-1} = A_N - \sum_{k=1}^{N-1} G_{k,N}' G_k^{-1} G_{k,N} - G_{0,N}' G_0^{-1} G_{0,N}$$

(ii) Backward CM_F model (29)–(30) *($c = N$):*

$$(G_0^B)^{-1} = A_0$$
$$\begin{cases} k = 0, 1, \dots, N-2: \\ G_{k,k+1}^B = -G_k^B B_k \\ (G_{k+1}^B)^{-1} = A_{k+1} - (G_{k,k+1}^B)'(G_k^B)^{-1} G_{k,k+1}^B \end{cases}$$
$$G_{0,N}^B = -G_0^B D_0$$
$$\begin{cases} k = 1, 2, \dots, N-2: \\ G_{k,N}^B = G_k^B (G_{k-1,k}^B)'(G_{k-1}^B)^{-1} G_{k-1,N}^B - G_k^B D_k \end{cases}$$
$$2G_{N-1,N}^B = G_{N-1}^B (G_{N-2,N-1}^B)'(G_{N-2}^B)^{-1} G_{N-2,N}^B - G_{N-1}^B B_{N-1}$$
$$(G_N^B)^{-1} = A_N - \sum_{i=0}^{N-2} (G_{i,N}^B)'(G_i^B)^{-1} G_{i,N}^B - 4(G_{N-1,N}^B)'(G_{N-1}^B)^{-1} G_{N-1,N}^B$$

B. CM_F Sequences

1) CM_F Model ($c = 0$): For (19)–(20):

$$A_0 = G_0^{-1} + \sum_{k=2}^{N} G_{k,0}'(G_k)^{-1} G_{k,0} + 4G_{1,0}' G_1^{-1} G_{1,0}$$

$$A_k = G_{k+1,k}'(G_{k+1})^{-1} G_{k+1,k} + G_k^{-1}, k \in [1, N-1]$$
$$A_N = G_N^{-1}$$
$$B_0 = G_{2,0}' G_2^{-1} G_{2,1} - 2G_{1,0}' G_1^{-1}$$
$$B_k = -G_{k+1,k}'(G_{k+1})^{-1}, k \in [1, N-1]$$
$$E_k = G_{k+1,0}' G_{k+1}^{-1} G_{k+1,k} - G_{k,0}' G_k^{-1}, k \in [2, N-1]$$
$$E_N = -G_{N,0}' G_N^{-1}$$

2) Backward CM_L Model ($c = 0$): For model (29):

$$A_1 = (G_1^B)^{-1}$$
$$A_k = (G_{k-1,k}^B)'(G_{k-1}^B)^{-1} G_{k-1,k}^B + (G_k^B)^{-1}, k \in [2, N-1]$$
$$B_0 = -(G_{1,0}^B)'(G_1^B)^{-1}$$
$$B_k = -(G_k^B)^{-1} G_{k,k+1}^B, k \in [1, N-1]$$
$$E_k = (G_{k-1,0}^B)'(G_{k-1}^B)^{-1} G_{k-1,k}^B - (G_{k,0}^B)'(G_k^B)^{-1}$$
$$k \in [2, N-1]$$

for boundary condition (30):

$$A_0 = (G_0^B)^{-1} + \sum_{k=1}^{N-1} (G_{k,0}^B)'(G_k^B)^{-1} G_{k,0}^B$$
$$A_N = (G_{N-1,N}^B)'(G_{N-1}^B)^{-1} G_{N-1,N}^B + (G_N^B)^{-1} + (G_{0,N}^B)'(G_0^B)^{-1} G_{0,N}^B$$
$$E_N = (G_{N-1,0}^B)'(G_{N-1}^B)^{-1} G_{N-1,N}^B - (G_0^B)^{-1} G_{0,N}^B$$

and for (31):

$$A_0 = (G_0^B)^{-1} + \sum_{k=1}^{N-1} (G_{k,0}^B)'(G_k^B)^{-1} G_{k,0}^B + (G_{N,0}^B)'(G_N^B)^{-1} G_{N,0}^B$$
$$A_N = (G_{N-1,N}^B)'(G_{N-1}^B)^{-1} G_{N-1,N}^B + (G_N^B)^{-1}$$
$$E_N = (G_{N-1,0}^B)'(G_{N-1}^B)^{-1} G_{N-1,N}^B - (G_{N,0}^B)'(G_N^B)^{-1}$$

Lemma A.2. *Parameters of CM_F model* (19)–(20) *and backward CM_L model* (29) *along with* (30) *or* (31) *of a zero-mean NG CM_F sequence with the inverse of its covariance matrix equal to any given CM_F matrix* (24) *can be uniquely determined as follows:*

(i) CM_F model (19)–(20)*:*

$$G_N^{-1} = A_N$$
$$\begin{cases} k = N, N-1, \dots, 2: \\ G_{k,k-1} = -G_k B_{k-1}' \\ G_{k-1}^{-1} = A_{k-1} - G_{k,k-1}'(G_k)^{-1} G_{k,k-1} \end{cases}$$
$$G_{N,0} = -G_N E_N'$$
$$\begin{cases} k = N-1, N-2, \dots, 2: \\ G_{k,0} = G_k G_{k+1,k}' G_{k+1}^{-1} G_{k+1,0} - G_k E_k' \end{cases}$$
$$2G_{1,0} = G_1 G_{2,1}' G_2^{-1} G_{2,0} - G_1 B_0'$$
$$G_0^{-1} = A_0 - \sum_{k=2}^{N} G_{k,0}' G_k^{-1} G_{k,0} - 4G_{1,0}' G_1^{-1} G_{1,0}$$

(ii) Backward CM_L model (29) *($c = 0$):*

$$(G_1^B)^{-1} = A_1$$
$$\begin{cases} k = 1, 2, \dots, N-2: \\ G_{k,k+1}^B = -G_k^B B_k \\ (G_{k+1}^B)^{-1} = A_{k+1} - (G_{k,k+1}^B)'(G_k^B)^{-1} G_{k,k+1}^B \end{cases}$$
$$G_{N-1,N}^B = -G_{N-1}^B B_{N-1}$$
$$G_{1,0}^B = -G_1^B B_0'$$
$$\begin{cases} k = 2, \dots, N-1: \\ G_{k,0}^B = G_k^B (G_{k-1,k}^B)'(G_{k-1}^B)^{-1} G_{k-1,0}^B - G_k^B E_k' \end{cases}$$

Parameters of the boundary condition are: for (30)

$$\begin{aligned}(G_0^B)^{-1} &= A_0 - \sum_{k=1}^{N-1}(G_{k,0}^B)'(G_k^B)^{-1}G_{k,0}^B\\ G_{0,N}^B &= G_0^B(F_{N-1,0}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B - G_0^B E_N\\ (G_N^B)^{-1} &= A_N - (G_{N-1,N}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B\\ &\quad - (G_{0,N}^B)'(G_0^B)^{-1}G_{0,N}^B\end{aligned}$$

and for (31)*:*

$$\begin{aligned}(G_N^B)^{-1} =&A_N - (G_{N-1,N}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B\\ (G_0^B)^{-1} =&A_0 - \sum_{k=1}^{N-1}(G_{k,0}^B)'(G_k^B)^{-1}G_{k,0}^B\\ &-(G_{N,0}^B)'(G_N^B)^{-1}G_{N,0}^B\\ G_{N,0}^B =&G_N^B(G_{N-1,N}^B)'(G_{N-1}^B)^{-1}G_{N-1,0}^B - G_N^B E_N'\end{aligned}$$

C. Reciprocal Sequences

For reciprocal model (10) along with (11)–(12):

$$R_k^0 = A_k, k \in [0,N] \tag{100}$$
$$R_k^+ = (R_{k+1}^-)' = -B_k, k \in [0,N-1] \tag{101}$$
$$R_0^- = (R_N^+)' = -D_0 \tag{102}$$

Model (10) with (13) or (14) was discussed in Section V.

D. Markov Sequences

1) Markov Model (2)*:*

$$\begin{aligned}A_0 =&M_0^{-1} + M_{1,0}'M_1^{-1}M_{1,0}, \quad A_N = M_N^{-1}\\ A_k =&M_k^{-1} + M_{k+1,k}'M_{k+1}^{-1}M_{k+1,k}, k \in [1,N-1]\\ B_k =&-M_{k+1,k}'M_{k+1}^{-1}, k \in [0,N-1]\end{aligned}$$

2) Backward Markov Model (6)*:*

$$\begin{aligned}A_0 =&(M_0^B)^{-1}\\ A_k =&(M_k^B)^{-1} + (M_{k-1,k}^B)'(M_{k-1}^B)^{-1}M_{k-1,k}^B, k \in [1,N-1]\\ A_N =&(M_N^B)^{-1} + (M_{N-1,N}^B)'(M_{N-1}^B)^{-1}M_{N-1,N}^B\\ B_k =&-(M_k^B)^{-1}M_{k,k+1}^B, k \in [0,N-1]\end{aligned}$$

Lemma A.3. *Parameters of Markov model* (2) *and backward Markov model* (6) *of a zero-mean NG Markov sequence with the inverse of its covariance matrix equal to any given symmetric positive definite (block) tri-diagonal matrix can be uniquely determined as follows:*

(i) Markov model (2)*:*

$$M_N^{-1} = A_N, \quad M_{N,N-1} = -M_N B_{N-1}'$$
$$\begin{cases} k = N-2, N-3, \dots, 0:\\ M_{k+1}^{-1} = A_{k+1} - M_{k+2,k+1}'M_{k+2}^{-1}M_{k+2,k+1}\\ M_{k+1,k} = -M_{k+1}B_k'\end{cases}$$
$$M_0^{-1} = A_0 - M_{1,0}'M_1^{-1}M_{1,0}$$

(ii) Backward Markov model (6)*:*

$$(M_0^B)^{-1} = A_0, \quad M_{0,1}^B = -M_0^B B_0$$
$$\begin{cases} k = 2,3,\dots,N:\\ (M_{k-1}^B)^{-1} = A_{k-1} - (M_{k-2,k-1}^B)'(M_{k-2}^B)^{-1}M_{k-2,k-1}^B\\ M_{k-1,k}^B = -M_{k-1}^B B_{k-1}\end{cases}$$
$$(M_N^B)^{-1} = A_N - (M_{N-1,N}^B)'(M_{N-1}^B)^{-1}M_{N-1,N}^B$$

Appendix B
Algebraically Equivalent Models

Following (41), relationships between some AE models are presented.

A. Reciprocal Model (10) and Markov Model (2)

$$\begin{aligned}e_0^R =&M_0^{-1}e_0^M - M_{1,0}'M_1^{-1}e_1^M, \quad e_N^R = M_N^{-1}e_N^M\\ e_k^R =&M_k^{-1}e_k^M - M_{k+1,k}'M_{k+1}^{-1}e_{k+1}^M, k \in [1,N-1]\end{aligned}$$

B. CM_L Model and Markov Model (2)

(i) CM_L model (19)–(20) ($c = N$):

$$G_0^{-1}e_0 - G_{1,0}'G_1^{-1}e_1 - G_{N,0}'G_N^{-1}e_N = M_0^{-1}e_0^M - M_{1,0}'M_1^{-1}e_1^M \tag{103}$$
$$G_k^{-1}e_k - G_{k+1,k}'G_{k+1}^{-1}e_{k+1} = M_k^{-1}e_k^M - M_{k+1,k}'M_{k+1}^{-1}e_{k+1}^M, k \in [1,N-2] \tag{104}$$
$$G_{N-1}^{-1}e_{N-1} = M_{N-1}^{-1}e_{N-1}^M - M_{N,N-1}'M_N^{-1}e_N^M \tag{105}$$
$$-\sum_{k=1}^{N-1}G_{k,N}'G_k^{-1}e_k + G_N^{-1}e_N = M_N^{-1}e_N^M \tag{106}$$

(ii) CM_L model (19) and (21): we have (104), (105), and

$$G_0^{-1}e_0 - G_{1,0}'G_1^{-1}e_1 = M_0^{-1}e_0^M - M_{1,0}'M_1^{-1}e_1^M \tag{107}$$
$$-\sum_{k=1}^{N-1}G_{k,N}'G_k^{-1}e_k + G_N^{-1}e_N - G_{0,N}'G_0^{-1}e_0 = M_N^{-1}e_N^M \tag{108}$$

C. CM_F Model (19) and Reciprocal Model (10)

$$\begin{aligned}e_0^R =&G_0^{-1}e_0 - 2G_{1,0}'G_1^{-1}e_1 - \sum_{k=2}^{N}G_{k,0}'G_k^{-1}e_k\\ e_1^R =&G_1^{-1}e_1 - G_{2,1}'G_2^{-1}e_2\\ e_k^R =&G_k^{-1}e_k - G_{k+1,k}'G_{k+1}^{-1}e_{k+1}, k \in [2,N-1]\\ e_N^R =&G_N^{-1}e_N\end{aligned}$$

D. CM_L Model and Backward CM_F Model (29)

(i) CM_L model (19)–(20): we have

$$(G_0^B)^{-1}e_0^B = G_0^{-1}e_0 - G_{1,0}'G_1^{-1}e_1 - G_{N,0}'G_N^{-1}e_N \tag{109}$$
$$-(G_{k-1,k}^B)'(G_{k-1}^B)^{-1}e_{k-1}^B + (G_k^B)^{-1}e_k^B = G_k^{-1}e_k - G_{k+1,k}'G_{k+1}^{-1}e_{k+1}, k \in [1,N-2] \tag{110}$$
$$-(G_{N-2,N-1}^B)'(G_{N-2}^B)^{-1}e_{N-2}^B + (G_{N-1}^B)^{-1}e_{N-1}^B = G_{N-1}^{-1}e_{N-1} \tag{111}$$

$$\sum_{k=0}^{N-2}(G^B_{k,N})'(G^B_k)^{-1}e^B_k+2(G^B_{N-1,N})'(G^B_{N-1})^{-1}e^B_{N-1}-$$
$$(G^B_N)^{-1}e^B_N=\sum_{k=1}^{N-1}G'_{k,N}G^{-1}_k e_k+G^{-1}_N e_N \tag{112}$$

(ii) CM_L model (19) and (21): we have (110), (111), and
$$(G^B_0)^{-1}e^B_0=G^{-1}_0 e_0-G'_{1,0}G^{-1}_1 e_1 \tag{113}$$
$$(G^B_N)^{-1}e^B_N-2(G^B_{N-1,N})'(G^B_{N-1})^{-1}e^B_{N-1}-$$
$$\sum_{k=0}^{N-2}(G^B_{k,N})'(G^B_k)^{-1}e^B_k=-\sum_{k=1}^{N-1}G'_{k,N}G^{-1}_k e_k+G^{-1}_N e_N$$
$$-G'_{0,N}G^{-1}_0 e_0 \tag{114}$$

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