Destination-Directed Trajectory Modeling and Prediction Using Conditionally Markov Sequences

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Abstract—In some problems there is information about the destination of a moving object. An example is an airliner flying from an origin to a destination. Such problems have three main components: an origin, a destination, and motion in between. To emphasize that the motion trajectories end up at the destination, we call them *destination-directed trajectories*. The Markov sequence is not flexible enough to model such trajectories. Given an initial density and an evolution law, the future of a Markov sequence is determined probabilistically. One class of conditionally Markov (CM) sequences, called the CM_L sequence (including the Markov sequence as a special case), has the following main components: a joint endpoint density (i.e., an initial density and a final density conditioned on the initial) and a Markov-like evolution law. This paper proposes using the CM_L sequence for modeling destination-directed trajectories. It is demonstrated how the CM_L sequence enjoys several desirable properties for destination-directed trajectory modeling. Some simulations of trajectory modeling and prediction are presented for illustration.

Keywords: Trajectory modeling and prediction, conditionally Markov (CM) sequence, dynamic model, destination-directed trajectory.

I. INTRODUCTION

Markov processes have been widely used in many applications. A Markov process has two main components: an initial density and an evolution law. Consider the problem of trajectory modeling and prediction with destination information, e.g., an airliner's trajectory from an origin to a destination. Such a problem has three main components: an origin, motion, and a destination, for which a Markov process does not fit since it can not take the destination information into account. In other words, the destination density of a Markov process is completely determined by its initial density and evolution law. So, a more general process is needed for modeling destination-directed trajectories.

Trajectory modeling and prediction with an intent or a destination has been studied in the literature. Some intent-based trajectory prediction approaches for air traffic control (ATC) can be found in [1]–[6]. [1]–[3] presented some trajectory prediction approaches based on hybrid estimation aided by intent information. Also, in [4] the interacting multiple model (IMM) approach was used for trajectory prediction, where a higher weight was assigned to the model with the closest heading towards the waypoint. [5] presented an approach for trajectory prediction using an inferred intent based on a database. [6] discussed the use of waypoint information for trajectory prediction in ATC. Ship trajectories were modeled by a Gauss-Markov model in [7], where predictive information was incorporated. In [8], a boundary value linear system was used for modeling trajectories with a specific destination. After quantizing the state space, [9]–[11] used finite-state reciprocal sequences for intent inference and trajectory modeling. A problem with quantized state space is the complexity of the corresponding estimation algorithms. So, the complexity of the algorithms used in [9]–[11] was also addressed. [12]–[13] used bridging distributions for the purpose of intent inference, for example, in selecting an icon on an in-vehicle interactive display. A general class of stochastic sequences that naturally models destination-directed trajectories is missing in the literature. The goal of this paper is to start from the main components of destination-directed trajectories and develop a framework and a class of stochastic sequences that naturally models such trajectories.

Inspired by [14], Gaussian CM sequences were studied in [15]–[17]. There are several classes of CM sequences, one of which is called CM_L [15]. A sequence defined over the interval $[0, N]$ is CM_L if and only if (iff) conditioned on the state at time N, the sequence is Markov over $[0, N-1]$. The subscript "L" is used because the conditioning is at the *last* time of the interval. A dynamic model (called CM_L model) governing the nonsingular Gaussian (NG) CM_L sequence was presented in [15]. Reciprocal processes were introduced in [18] connected to a problem posed by E. Schrodinger [19]–[20]. Later, reciprocal processes were studied more in [21]–[39]. Every reciprocal sequence is a CM_L sequence [16]. A dynamic model of the NG reciprocal sequence was presented in [36], which shows that the evolution of the NG reciprocal sequence can be described by a second-order nearest-neighbor model driven by a locally correlated dynamic noise. This model can be seen as a generalization of the Markov model. However, due to the nonwhiteness of the dynamic noise and the nearest-neighbor structure, it is not necessarily easy to apply. That is why different

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approaches have been presented for recursive estimation of reciprocal sequences based on that model [36]–[39]. In [16] the reciprocal sequence was studied from the CM viewpoint and (based on the CM_L model) a model, called a reciprocal CM_L model, was presented for the evolution of the NG reciprocal sequence. This model is driven by white dynamic noise and is simple and easily applicable.

The main elements of the CM_L sequence are a joint endpoint density (in other words, an initial density and a final density conditioned on the initial) and an evolution law. Having a Markov-like property, the evolution law of the CM_L sequence is simple (and thus easily applicable) and conceptually appealing for modeling destination-directed trajectories. The CM_L sequence can model well the main elements of destination-directed trajectories.

The main contribution of this paper is that it proposes the use of the CM_L sequence for destination-directed trajectory modeling. It demonstrates that the Markov sequence is not flexible for modeling destination-directed trajectories. Then, it shows how the CM_L sequence can naturally model such trajectories. Also, it discusses that the CM_L sequence enjoys desirable properties for modeling destination-directed trajectories. Simulations for destination-directed trajectory modeling and prediction are presented. This paper builds upon our previous results about CM and reciprocal sequences [15]–[17] to show their application in trajectory modeling.

In Section II, the CM_L sequence and its model are presented for trajectory modeling. In Section III, the CM_L model parameter design is discussed. Simulations are presented in Section IV. In Section V, desirable properties of the CM_L sequence for destination-directed trajectory modeling and prediction are discussed.

II. DESTINATION-DIRECTED TRAJECTORY MODELING USING CM_L SEQUENCE

We consider the following notation for index (time) interval and stochastic sequences: $[x_k]_i^j \triangleq \{x_k, k \in [i,j]\}, i < j$, and $[x_k] \triangleq [x_k]_0^N$, where $[i,j] \triangleq \{i, i+1, \cdots, j-1, j\}$. Also, ZMNG and NG stands for "zero-mean nonsingular Gaussian" and "nonsingular Gaussian", respectively. The symbol "\" is used for set subtraction. $F(\cdot|\cdot)$ is a conditional cumulative distribution function (CDF). $p(\cdot)$ and $p(\cdot|\cdot)$ denote probability density function (PDF) and conditional PDF.

Definition 2.1: $[x_k]$ is Markov if $F(\xi_k|[x_i]_0^j) = F(\xi_k|x_j)$, $\forall j, k \in [0, N], j < k, \forall \xi_k \in \mathbb{R}^d$, where d is the dimension of x_k.

To model the trajectory of a moving object without considering its destination, there are two main components: the origin and the evolution. Accordingly, a Markov sequence is determined by two components: an initial density and an evolution law. Sample paths of a Markov sequence can be used for modeling such trajectories. For example, a nearly constant velocity/acceleration/turn (with white noise) motion model is a Markov model. A dynamic model for the evolution of the Gaussian Markov sequence is as follows.

Lemma 2.2: A ZMNG $[x_k]$ is Markov iff

$$x_k = M_{k,k-1}x_{k-1} + e_k, k \in [1, N], \quad x_0 = e_0 \quad (1)$$

and $[e_k]$ ($M_k = \text{Cov}(e_k)$) is a zero-mean white NG sequence.

Markov sequences are not flexible enough for modeling an origin, evolution, and a destination. This is because the destination density of a Markov sequence is determined by its initial density and evolution law. Therefore, a more general class of stochastic sequences is needed.

Let destination-directed trajectories be modeled as the sample paths of a sequence $[x_k]$. In probability theory, one can interpret the main elements of destination-directed trajectories as follows. The origin (destination) is modeled by a density function of x_0 (x_N). The relationship between the origin and the destination is modeled by their joint density, i.e., joint density of x_0 and x_N. Since the destination of the trajectories (i.e., density of x_N) is known, the evolution law can be modeled as a conditional density (over the space of sample paths) given the state at destination x_N. Different choices of this conditional density correspond to different evolution laws. The simplest choice is conditioned on x_N the density being equal to the product of its marginals: $p([x_k]_0^{N-1}|x_N) = \prod_{k=0}^{N-1} p(x_k|x_N)$. However, this choice of the conditional density is often too simple to be suitable. Then, the next step is to choose the conditional density corresponding to the Markov sequence: $p([x_k]_0^{N-1}|x_N) = p(x_0|x_N) \prod_{k=1}^{N-1} p(x_k|x_{k-1}, x_N)$. This evolution law corresponds to the CM_L sequence. The main elements of a CM_L sequence $[x_k]$ are: a joint density of x_0 and x_N (in other words, an initial density and a final density conditioned on the initial, or equivalently, the other way round) and an evolution law, where the evolution law is conditionally Markov (conditioned on x_N). The above argument naturally leads to CM_L sequences for modeling destination-directed trajectories. Following the same argument, we can consider more general and complicated evolution laws, if necessary. For example, the conditional law (density) (conditioned on x_N) can be higher order Markov instead of the first order Markov. Therefore, by choosing the conditional law, all destination-directed trajectory models can be obtained.

Definition 2.3: $[x_k]$ is CM_L if $F(x_k|[x_i]_0^j, x_N) = F(x_k|x_j, x_N)$, $\forall j, k \in [0, N-1], j < k, \forall \xi_k \in \mathbb{R}^d$, where d is the dimension of x_k.

In other words, $[x_k]$ is CM_L iff $[x_k]_{j+1}^{N-1}$ and $[x_k]_0^{j-1}$ are independent conditioned on x_j and x_N ($\forall j \in [1, N-2]$). A CM_L model is as follows [15].

Theorem 2.4: A ZMNG $[x_k]$ is CM_L iff

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, k \in [1, N-1] \quad (2)$$

$$x_N = e_N, \quad x_0 = G_{0,N}x_N + e_0 \quad (3)$$

and $[e_k]$ ($G_k = \text{Cov}(e_k)$) is a zero-mean white NG sequence.

For trajectory modeling we need non-zero-mean sequences. A non-zero-mean NG sequence $[x_k]$ is CM_L (Markov) iff its zero-mean part is governed by (2)–(3) ((1)). For simplicity, we present the results for the zero-mean case; however, in the simulation non-zero-mean sequences are considered. The CM_L model governing the non-zero-mean NG CM_L sequences considered in the simulation is as follows. Let μ_0 (μ_N) and C_0 (C_N) be the mean and covariance of the origin (destination) density. Also, let

$C_{0,N}$ be the cross-covariance of the states at the origin and the destination. We have $x_N \sim \mathcal{N}(\mu_N, C_N)$. Then, $x_0 = \mu_0 + G_{0,N}(x_N - \mu_N) + e_0$, where $G_{0,N} = C_{0,N}C_N^{-1}$, $G_0 = C_0 - C_{0,N}C_N^{-1}(C_{0,N})'$, and $G_N = C_N$. Then, the state evolution for $k \in [1, N-1]$ is governed by (2).

In (2), x_N is generated first and then followed by the other states. Therefore, the model is not causal. However, we should notice that for estimation (tracking/prediction), the non-causality of model (2) requires *information* about x_N (i.e., $p(x_N)$), which is available. Therefore, this model is totally applicable.

III. CM_L MODEL PARAMETER DESIGN FOR DESTINATION-DIRECTED TRAJECTORY MODELING

To use the CM_L model for the evolution of destination-directed trajectories, we need an approach for design of its parameters. In the following, such an approach is presented [17]. Before doing so, we review the reciprocal sequence as a special case of the CM_L sequence.

Lemma 3.1: $[x_k]$ is reciprocal iff $F(\xi_k | [x_i]_0^j, [x_i]_l^N) = F(\xi_k | x_j, x_l)$, $\forall j, k, l \in [0, N]$ $(j < k < l)$, $\forall \xi_k \in \mathbb{R}^d$, where d is the dimension of x_k.

Theorem 3.2 below presents the conditions under which the CM_L model governs a reciprocal sequence [16]. Such a CM_L model is called a *reciprocal CM_L* model.

Theorem 3.2: A ZMNG $[x_k]$ is reciprocal iff it is governed by (2)–(3) and $G_k^{-1}G_{k,N} = G_{k+1,k}'G_{k+1}^{-1}G_{k+1,N}$, $k \in [1, N-2]$. Moreover, $[x_k]$ is Markov iff additionally we have $G_0^{-1}G_{0,N} = G_{1,0}'G_1^{-1}G_{1,N}$.

An approach for the CM_L model parameter design for modeling destination-directed trajectories is as follows. Such trajectories can be modeled by combining (superimposition of) two key assumptions: (i) the moving object follows a Markov model (1) (e.g., a nearly constant velocity model) without considering the destination information, and (ii) the destination density is known (which can differ from the destination density of the Markov model in (i)). Note that for theoretical purposes we assume the destination density is known. But in a real problem an approximate destination density can be used. The above two assumptions are valid in a real problem of trajectory modeling with destination. Let $[s_k]$ be a Markov sequence governed by (1) (e.g., a nearly constant velocity model). Since every Markov sequence is CM_L, $[s_k]$ can also obey a CM_L model as

$$s_k = G_{k,k-1}s_{k-1} + G_{k,N}s_N + e_k^s, k \in [1, N-1] \quad (4)$$
$$s_N = e_N^s, \quad s_0 = G_{0,N}^s s_N + e_0^s \quad (5)$$

where $[e_k^s]$ is a zero-mean white NG sequence with covariances $G_k, k \in [1, N-1]$, G_0^s, and G_N^s.

Parameters of (4) can be obtained as follows. By (1), we have $p(s_k|s_{k-1}) = \mathcal{N}(s_k; M_{k,k-1}s_{k-1}, M_k)$. Since $[s_k]$ is Markov, we have ($k \in [1, N-1]$)

$$
\begin{aligned}
p(s_k|s_{k-1}, s_N) &= \frac{p(s_k|s_{k-1})p(s_N|s_k, s_{k-1})}{p(s_N|s_{k-1})} \\
&= \frac{p(s_k|s_{k-1})p(s_N|s_k)}{p(s_N|s_{k-1})} \quad (6) \\
&= \mathcal{N}(s_k; G_{k,k-1}s_{k-1} + G_{k,N}s_N; G_k)
\end{aligned}
$$

where $G_{k,k-1}$, $G_{k,N}$, and G_k are obtained as

$$G_{k,k-1} = M_{k,k-1} - G_{k,N}M_{N|k-1} \quad (7)$$
$$G_{k,N} = G_k M_{N|k}' C_{N|k}^{-1} \quad (8)$$
$$G_k = (M_k^{-1} + M_{N|k}' C_{N|k}^{-1} M_{N|k})^{-1} \quad (9)$$
$$M_{N|k} = M_{N,N-1} \cdots M_{k+1,k}, k \in [1, N-1], M_{N|N} = I$$
$$C_{N|k} = \sum_{n=k}^{N-1} M_{N|n+1} M_{n+1} M_{N|n+1}', k \in [1, N-1]$$

$$p(s_N|s_i) = \mathcal{N}(s_N; M_{N|i}s_i, C_{N|i}), i \in [0, N-1]$$

Now, we construct a sequence $[x_k]$ governed by

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, k \in [1, N-1] \quad (10)$$
$$x_N = e_N, \quad x_0 = G_{0,N}x_N + e_0 \quad (11)$$

where $[e_k]$ is a zero-mean white NG sequence with covariances $G_k, k \in [1, N-1]$, G_0, and G_N. Note that (10) and (4) have the same parameters $(G_{k,k-1}, G_{k,N}, G_k, k \in [1, N-1])$, but parameters of (11) $(G_{0,N}, G_0, G_N)$ and parameters of (5) $(G_{0,N}^s, G_0^s, G_N^s)$ are different. Parameters of (11) $(G_{0,N}, G_0, G_N)$ can be chosen arbitrarily (i.e. $G_{0,N}$ can be any matrix with suitable dimension, and G_0 and G_N any positive definite matrix with suitable dimension). Thus, $[x_k]$ can have any joint density of x_0 and x_N. So, $[s_k]$ and $[x_k]$ have the same CM_L model ((4) and (10)) (in other words, the same transition density (6)), but $[x_k]$ can have any joint endpoint density. In other words, any origin and destination of $[x_k]$ can be so modeled. By Theorem 2.4, $[x_k]$ is a CM_L sequence. Therefore, combining assumptions (i) and (ii) above naturally leads to the CM_L sequence $[x_k]$ whose CM_L model is the same as that of $[s_k]$ while the former can model any origin and destination. Thus, model (10) with (7)–(9) is a desirable model for destination-directed trajectory modeling based on (i) and (ii) above. Such a CM_L model is used in Section IV. Since it is obtained from a Markov model, (10) is called a CM_L model *induced* by a Markov model.

The following theorem shows that an induced CM_L model (i.e., (10) with (7)–(9)) is actually a reciprocal CM_L model. So, all its governed sequences $[x_k]$ are reciprocal (for any choice of parameters of boundary condition (11)). Also, it shows that every reciprocal CM_L model can be induced by a Markov model following the above approach [17].

Theorem 3.3: A ZMNG $[x_k]$ is reciprocal iff it obeys (2)–(3), where $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$, are given by (7)–(9), $M_{k,k-1}$, $k \in [1, N]$, are square matrices, and M_k, $k \in [1, N]$, are positive definite matrices with the dimension of x_k.

The idea of obtaining a reciprocal evolution law from a Markov evolution law was used in [19], [24], and later for finite-state reciprocal sequences in [9], [35]. However, first, our reciprocal CM_L model induced as above (Theorem 3.3) is from the viewpoint of the CM sequence. Second, Theorem 3.3 shows that *every* reciprocal CM_L model can be induced by a Markov model (i.e, necessity and sufficiency).

The above approach is only one method for the CM_L model parameter design for trajectory modeling. A more general approach for the CM_L model parameter design was sketched in [17], based on a representation of a CM_L sequence in terms of a Markov sequence.

IV. SIMULATION

Destination-directed trajectories are modeled by a NG CM_L sequence. Model (2) can be written as ($k \in [1, N-1]$)

$$\begin{bmatrix} x_k \\ x_N \end{bmatrix} = \begin{bmatrix} G_{k,k-1} & G_{k,N} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ x_N \end{bmatrix} + \begin{bmatrix} e_k \\ 0 \end{bmatrix} \quad (12)$$

Also, the measurement equation is $z_k = Hx_k + v_k, k \in [1, N], H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, where $[v_k]_1^N$ ($\text{Cov}(v_k) = \text{diag}(100, 100)$) is a zero-mean white NG sequence uncorrelated with $[x_k]$. Results from linear system theory can be used for trajectory filtering and prediction. We skip them.

Consider a two-dimensional scenario, where the state of a moving object at time k is $x_k = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}]_k'$ with position $[\mathsf{x}, \mathsf{y}]'$ and velocity $[\dot{\mathsf{x}}, \dot{\mathsf{y}}]'$. The origin and destination densities are assumed to be Gaussian. Means and covariances of the origin (μ_0 and C_0) and destination (μ_N and C_N) densities are $\mu_0 = [2000, 70, 5000, 0]'$, $C_0 = \text{diag}(1000, 10, 1000, 10)$, $\mu_N = [130000, 70, 10000, 0]'$, and $C_N = \text{diag}(1000, 10, 1000, 10)$. Only for simplicity the origin and the destination are assumed to be statistically independent (i.e., $G_{0,N} = 0$ in (3)). In fact, the CM_L modeling is much more powerful if $G_{0,N} \neq 0$ in (3) (to model the relationship between the origin and the destination). We consider a CM_L model induced by a Markov model (Theorem 3.3). Parameters of the corresponding Markov model (1) are $M_{k,k-1} = \text{diag}(F, F), M_k = \text{diag}(Q, Q), k \in [1, N]$, where $F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$, $Q = q \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix}$, $T = 15$ second, $q = 0.01$, and $N = 100$. Fig. 1 compares the Markov model and the CM_L model for trajectory modeling (the 50 solid lines are trajectories of the CM_L sequence and the 50 dash lines are trajectories of the Markov sequence). Both sequences model the origin well. Also, near the origin the two models differ a little. However, later their difference grows. This is due to the poor performance of the Markov model in incorporating the destination information. Fig. 2 shows the log of the average Euclidean errors (AEE) [40] of the predicted estimates of the position obtained based on the CM_L and the Markov model. The AEE of the position predicted estimate ($\text{AEE}_{k+n|k}^p$) is $\frac{1}{M} \sum_{i=1}^{M} \sqrt{(\mathsf{x}_{k+n} - \hat{\mathsf{x}}_{k+n|k})^2 + (\mathsf{y}_{k+n} - \hat{\mathsf{y}}_{k+n|k})^2}$, where $[\mathsf{x}_{k+n}, \mathsf{y}_{k+n}]'$ is the true position at $k + n$ ($k + n = 10, \ldots, 100$) and $[\hat{\mathsf{x}}_{k+n|k}, \hat{\mathsf{y}}_{k+n|k}]'$ is its prediction using
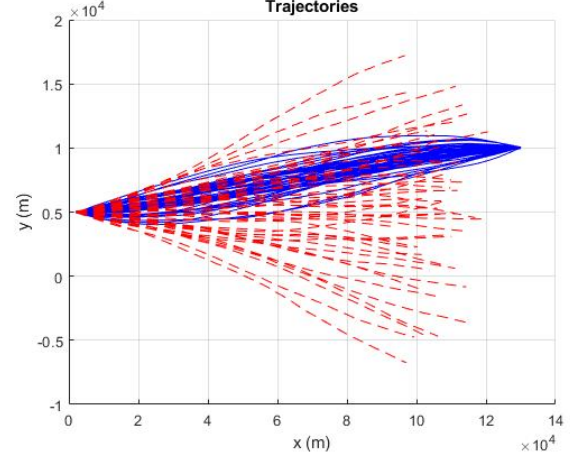

Figure 1. CM_L (solid lines) vs. Markov (dash lines) model for trajectory modeling with destination.
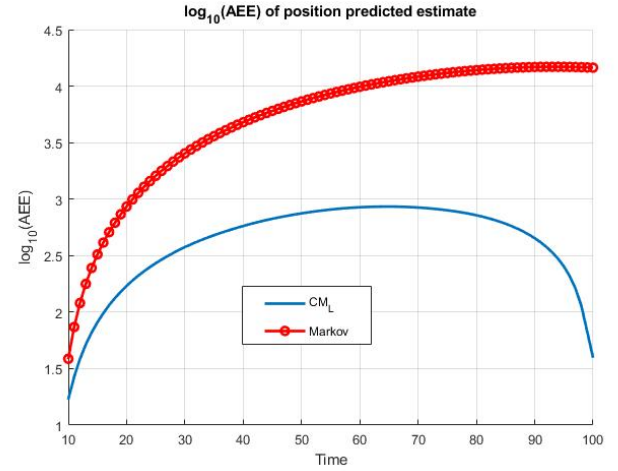

Figure 2. Logarithm of AEE of position prediction ($\log_{10}(\text{AEE}_{9+n|9}^p)$).

measurements up to time $k = 9$, and $M = 1000$ is the number of Monte Carlo runs. The ratio of $\text{AEE}_{100|9}^p$ of the Markov model over $\text{AEE}_{100|9}^p$ of the CM_L model $\frac{\text{AEE}_{100|9}^p(\text{Markov})}{\text{AEE}_{100|9}^p(CM_L)}$ is 368.13, which is huge.

V. CONCLUSIONS

We call motion trajectories that end up at a specific destination, *destination-directed trajectories*. The main components of these trajectories are: an origin, a destination, and evolution. The main components of the CM_L sequence are a joint endpoint density (i.e., an initial density and conditioned on it a destination density) and a Markov-like evolution law. The CM_L sequence is conceptually appealing, flexible, and simple to apply for modeling destination-directed trajectories. First, the CM_L sequence is a good fit for destination-directed trajectory modeling and prediction. Second, it outperforms Markov-based trajectory models very much. Third, the CM_L sequence is a powerful tool, capable of modeling different scenarios. For example, assume that according to the air

traffic control near an airport, airliners are supposed to enter the terminal area from some prespecified direction. This information can be used to specify a destination density of a CM_L sequence. Fourth, the CM_L sequence can model the correlation between the states of an airliner at the origin and the destination airports. Fifth, there is no restriction on the parameters of the CM_L dynamic model, which makes its analysis easily possible. Also, since there is no restriction on the parameters, they can be easily determined, e.g., based on an optimality criterion for trajectory design. It is possible based on a representation of a CM_L sequence in terms of a Markov sequence presented in [17]. Sixth, the CM_L dynamic model provides a natural way for the update (uncertainty reduction) of the available information about the destination of an airliner as more measurements are received. In addition, CM sequences provide a simple framework for trajectory modeling in different scenarios, e.g., a CM sequence was proposed in [41] for trajectory modeling with waypoint information.

REFERENCES

[1] I. Hwang and C. E. Seah. Intent-Based Probabilistic Conflict Detection for the Next Generation Air Transportation System. *Proceedings of the IEEE*. Vol. 96, No. 12, pp. 2040-2059, 2008.

[2] J. Yepes, I. Hwang, and M. Rotea. An Intent Based Trajectory Prediction Algorithm for Air Traffic Control. *AIAA Guidance, Navigation, and Control Conference*, San Francisco, CA, USA, Aug. 2005.

[3] J. Yepes, I. Hwang, and M. Rotea. Algorithms for Aircraft Intent Inference and Trajectory Prediction. *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 30, No. 2, pp. 370-382, 2007.

[4] Y. Liu and X. R. Li. Intent Based Trajectory Prediction by Multiple Model Prediction and Smoothing. *AIAA Guidance, Navigation, and Control Conference*, Kissimmee, Florida, USA, Jan., 2015.

[5] J. Krozel and D. Andrisani. Intent Inference and Strategic Path Prediction. *AIAA Guidance, Navigation, and Control Conference*, San Francisco, CA, USA, Aug. 2005.

[6] K. Mueller and J. Krozel. Aircraft ADS-B Intent Verification Based on Kalman Tracking Filter. *AIAA Guidance, Navigation, and Control Conference*, Denver, Colorado, USA, Aug. 2000.

[7] D. A. Castanon, B. C. Levy, and A. S. Willsky. Algorithms for Incorporation of Predictive Information in Surveillance Theory. *Int. J. Systems SCI.*, Vol. 16, No. 3, pp. 367-382, 1985.

[8] A. J. Krener. Filtering of Boundary Value Discrete Time Linear Systems. *52nd IEEE Conference on Decision and Control*, Florence, Italy, Dec. 2013.

[9] M. Fanaswala, V. Krishnamurthy, and L. B. White. Destination-Aware Target Tracking Via Syntactic Signal Processing. *IEEE International Conf. on Acoustics, Speech and Signal Processing (ICASSP)*, Prague, Czech Republic, May 2011.

[10] M. Fanaswala, V. Krishnamurthy. Detection of Anomalous Trajectory Patterns in Target Tracking via Stochastic Context-Free Grammer and Reciprocal Process Models. *IEEE Journal of Selected Topics in Signal Processing*, Vol. 7, No. 1, pp. 76-90, 2013.

[11] L. B. White and F. Carravetta. Normalized Optimal Smoothers for a Class of Hidden Generalized Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 62, No. 12, pp. 6489-6496, 2017.

[12] B. I. Ahmad, J. Murphy, P. M. Langdon, R. Hardy, and S. J. Godsill. Destination Inference Using Bridging Distributions. *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, Brisbane, QLD, Australia, Apr. 2015.

[13] B. I. Ahmad, J. K. Murphy, S. J. Godsill, P. M. Langdon, and R. Hardy. Intelligent Interactive Displays in Vehicles with Intent Prediction: A Bayesian framework. *IEEE Signal Processing Magazine*, Vol. 34, No. 2, 2017.

[14] C. B. Mehr and J. A. McFadden, Certain Properties of Gaussian Processes and Their First-Passage Times. *Journal of Royal Statistical Society (B)*, vol. 27, pp. 505-522, 1965.

[15] R. Rezaie and X. R. Li. Nonsingular Gaussian Conditionally Markov Sequences. *IEEE Western New York Image and Signal Processing Workshop*. Rochester, USA, Oct. 2018.

[16] R. Rezaie and X. R. Li. Gaussian Reciprocal Sequences from the Viewpoint of Conditionally Markov Sequences. *International Conference on Vision, Image and Signal Processing (ICVISP)*, Las Vegas, Aug. 2018.

[17] R. Rezaie and X. R. Li. Models and Representations of Gaussian Reciprocal and Conditionally Markov Sequences. *International Conference on Vision, Image and Signal Processing (ICVISP)*, Las Vegas, Aug. 2018.

[18] S. Bernstein. Sur Les Liaisons Entre Les Grandeurs Aleatoires. *Verh. des intern. Mathematikerkongr I*, Zurich, 1932.

[19] E. Schrodinger. Uber Die Umkehrung der Naturgesetze. *Sitz. Ber. der Preuss. Akad. Wissen., Berlin Phys. Math.* 144, 1931.

[20] E. Schrodinger. Theorie Relativiste de L'electron et L'interpretation de la Mechanique Quantique. *Ann. Inst. H. Poincare* 2, 269-310, 1932.

[21] D. Slepian. First Passage Time for a Particular Gaussian Process. *Annals of Mathematical Statistics* 32, pp. 610-612, 1961.

[22] B. Jamison. Reciprocal Processes: The Stationary Gaussian Case. *Annals of Mathematical Statistics*, vol.41, No. 5, pp. 1624-1630, 1970.

[23] S. C. Chay, On Quasi-Markov Random Fields, *Journal of Multivariate Analysis*, Vol. 2, pp. 14-76, 1972.

[24] B. Jamison. Reciprocal Processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, Vol. 30, pp. 65-86, 1974.

[25] J-P Carmichael, J-C Masse, and R. Theodorescu, Processus Gaussiens Stationnaires Reciproques sur un Intervalle, *C. R. Acad. Sc. Paris*, t. 295 (27 Sep. 1982).

[26] J. Abraham and J. Thomas. Some Comments on Conditionally Markov and Reciprocal Gaussian Processes. *IEEE Trans. on Information Theory*. Vol. 27, No. 4, pp. 523-525, 1981.

[27] A. J. Krener. Reciprocal Diffusions and Stochastic Differential Equations of Second Order. *Stochastics*, Vol. 24, No. 4, pp. 393-422, 1988.

[28] A. J. Krener, R. Frezza, and B. C. Levy. Gaussian Reciprocal Processes and Self-adjoint Stochastic Differential Equations of Second Order. *Stochastics and Stochastic Reports*, Vol. 34, No. 1-2, pp. 29-56, 1991.

[29] B. C. Levy, A. Beghi, Discrete-time Gauss-Markov Processes with Fixed Reciprocal Dynamics. *Journal of Mathematical Systems, Estimation, and Control*, Vol. 4, No. 3, pp. 1-25, 1994.

[30] A. Beghi, Continuous-time Gauss-Markov Processes with Fixed Reciprocal Dynamics. *Journal of Mathematical Systems, Estimation, and Control*, Vol. 4, No. 4, pp. 1-24, 1994.

[31] J. Chen and H. L. Weinert, A New Characterization of Multivariate Gaussian Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 38, No. 10, pp. 1601-1602 , 1993.

[32] F. Carravetta and L. B. White. Modelling and Estimation for Finite State Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 57, No. 9, pp. 2190-2202, 2012.

[33] L. B. White and F. Carravetta. Optimal Smoothing for Finite State Hidden Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 56, No. 9, pp. 2156-2161, 2011.

[34] F. Carravetta. Nearest-neighbour Modelling of Reciprocal Chains. *An International Journal of Probability and Stochastic Processes*, Vol. 80, No. 6, pp. 525-584, 2008.

[35] L B. White and H. X. Vu. Maximum Likelihood Sequence Estimation for Hidden Reciprocal Processes. *IEEE Trans. on Automatic Control*, Vol. 58, No. 10, pp. 2670-2674, 2013.

[36] B. C. Levy, R. Frezza, and A. Krener. Modeling and Estimation of Discrete-Time Gaussian Reciprocal Processes. *IEEE Trans. on Automatic Control*. Vol. 35, No. 9, pp. 1013-1023, 1990.

[37] E. Baccarelli and R. Cusani. Recursive Filtering and Smoothing for Gaussian Reciprocal Processes with Dirichlet Boundary Conditions. *IEEE Trans. on Signal Processing*, Vol. 46, No. 3, pp. 790-795, 1998.

[38] E. Baccarelli, R. Cusani, and G. Di Blasio. Recursive Filtering and Smoothing for Reciprocal Gaussian Processes-pinned Boundary Case. *IEEE Trans. on Information Theory*, Vol. 41, No. 1, pp. 334-337, 1995.

[39] D. Vats and J. M. F. Moura. Recursive Filtering and Smoothing for Discrete Index Gaussian Reciprocal Processes. *43rd Annual Conference on Information Sciences and Systems*, Baltimore, MD, USA, Mar. 2009.

[40] X. R. Li and Z. Zhao. Evaluation of Estimation Algorithms Part I: Incomprehensive Measures of Performance. *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 42, No. 4, pp. 1340-1358, 2006.

[41] R. Rezaie and X. R. Li. Trajectory Modeling and Prediction with Waypoint Information Using a Conditionally Markov Sequence. *56th Allerton Conference on Communication, Control, and Computing*, Illinois, USA, Oct. 2018.