Lecture 10: Bernoulli and Geometric Random Variables

Chapter 3.3-3.5

Goals for Today

Define

- Bernoulli random variables
- Geometric random variables

Mathematical Definition of a Bernoulli Random Variable

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Random variables are described in terms of their distribution.

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- Coin flips: heads vs tails
- Medical test (for a disease): positive vs negative
- Rolling a die and getting a 6 vs not getting a 6

Bernoulli Distribution

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- Coin flips: heads vs tails
- Medical test (for a disease): positive vs negative
- Rolling a die and getting a 6 vs not getting a 6

In each case we can define the outcomes to be success vs failure. No moral judgement; just labels.

Bernoulli Distribution

Say we have trials where we have two outcomes: either a "success" or a "failure". Classic example: coin flips have $p = 0.5$ of heads, if we define heads as the success.

- probability p of a "success." Denote successes with a "1."
- probability $1 - p$ of a "failure." Denote failures with a "0."

Definition of a Bernoulli Random Variable

If X is a random variable that takes value

- 1 with probability of success p
- 0 with probability of failure $1 - p$

then X is a Bernoulli random variable with mean and standard deviation:

$$\begin{aligned} \mu &= p \\ \sigma &= \sqrt{p(1-p)} \end{aligned}$$

Intuition Behind σ

Sample Proportion

Say you repeat n instances of a Bernoulli random variable. You end up with a sample $x_1, \ldots, x_n$

The sample proportion $\widehat{p}$ (p-hat) is the sample mean of these observations. i.e.

$$\widehat{p} = \frac{\#\text{ of successes}}{\#\text{ of trials}} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

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- A success as rolling a 6.
  So $P(X = 1) = P(\text{success}) = p = \frac{1}{6}$.
- A failure as rolling anything else.
  So $P(X = 0) = P(\text{failure}) = 1 - p = \frac{5}{6}$.

Back to Lecture 3.1: Population vs Sample Values

	True Population Value	Sample Value
Mean	μ	$\overline{x}$
Variance	σ^2	s^2
Standard Deviation	σ	s
Proportion	p	$\widehat{p}$

Back to Lecture 3.1: Population vs Sample Values

	True Population Value	Sample Value
Mean	μ	$\overline{x}$
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Proportion	p	$\widehat{p}$

The sample proportion $\widehat{p}$ is a specific kind of sample mean for Bernoulli random variables, which estimates p, a specific kind of population mean.

Scenario

Question: Say

- the San Francisco Giants have equal probability $p = 0.6$ of winning any game
- games are independent

It's the beginning of the season. What is the probability that they don't win their first game until the 5th game of the season?

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For this to happen, there must be 4 loses in the first 4 games AND a win in the 5th game:

$$\begin{aligned} P(\text{1st W in 5th game}) &= P(\text{4 loses}) \times P(\text{win}) \\ &= (P(\text{loss}))^4 \times P(\text{win}) \\ &= (1-p)^4 \times p \\ &= 0.4^4 \times 0.6 = 0.01536. \end{aligned}$$

Geometric Random Variables

Geometric Distribution: If the probability of a success in any trial is p, the trials are independent, then the probability of finding the first success on the n^{th} trial is given by

$$(1-p)^{n-1}p$$

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Also

$$\begin{aligned} \mu &= \frac{1}{p} \\ \sigma^2 &= \frac{1-p}{p^2} \\ \sigma &= \frac{\sqrt{1-p}}{p} \end{aligned}$$

Intuition Behind μ

Think about μ: $\frac{1}{p}$ is the average number of trials we need until the first success.

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- Say $p = 0.5$. Then $\mu = \frac{1}{0.5} = 2$
- Say $p = 0.001$. Then $\mu = \frac{1}{0.001} = 1000$

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- Say $p = 0.001$. Then $\mu = \frac{1}{0.001} = 1000$

In the first case, the probability of a success is lower, so we expect on average it will take more trials until the first success.