Lecture 11: Binomial and Poisson Random Variables

Chapter 3.3-3.5

Goals for Today

Define

- Binomial random variables
- Poisson random variables

Binomial Distribution

So say now, instead of $P(\text{1st W in 5th game}) = P(\text{LLLLW})$, we want the probability that they win exactly one out of the five games. Five ways:

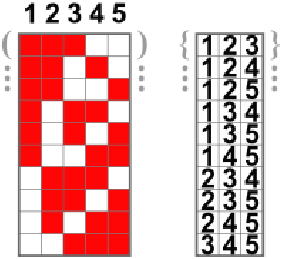
Pattern	Probability	Equals
WLLLL	$p \times (1-p)^4$	$= p \times (1-p)^4$
LWLLL	$(1-p) \times p \times (1-p)^3$	$= p \times (1-p)^4$
LLWLL	$(1-p)^2 \times p \times (1-p)^2$	$= p \times (1-p)^4$
LLLWL	$(1-p)^3 \times p \times (1-p)$	$= p \times (1-p)^4$
LLLLW	$(1-p)^4 \times p$	$= p \times (1-p)^4$

Each pattern (book calls it scenario) has the same probability regardless of order by independence, and there are 5 ways to choose the pattern.

So $P(\text{win exactly one out of five})$ is
$5 \times p \times (1-p)^4 = 5 \times 0.4^4 \times 0.6 = 0.0768$

Step Back... Example of n choose k

Say I give you $n = 5$ balls labeled 1 thru 5. How many different ways can you choose $k = 3$ of them?



As we see, 10 ways.

Step Back... n choose k in General

Say I give you n balls labeled 1 thru n. How many different ways can you choose k of them?

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This is read n choose k.

In example: $n = 5$ and $k = 3$

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = \frac{120}{12} = 10$$

Note that $0! = 1$

Binomial Distribution

Suppose the probability of a single trial being a success is p. Then the probability of observing exactly k successes in n independent trials is given by:

$$\begin{aligned} P(\text{exactly } k \text{ successes}) &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \end{aligned}$$

The mean, variance, and SD are:

$$\mu = np \qquad \sigma^2 = np(1-p) \qquad \sigma = \sqrt{np(1-p)}$$

Conditions for Binomial Distribution

1. The trials are independent.
2. The number of trials n is fixed
3. Each trial outcome can be classified as a failure or a success
4. The probability of a success p is the same for each trial

Back to Soccer Example

The Portland Timbers have equal probability p $= 0.6$ of winning any particular soccer game. We want the probability that they win exactly one out of the five games. Five ways:

Pattern	Probability	Equals
WLLLL	$p \times (1-p)^4$	$= p \times (1-p)^4$
LWLLL	$(1-p) \times p \times (1-p)^3$	$= p \times (1-p)^4$
LLWLL	$(1-p)^2 \times p \times (1-p)^2$	$= p \times (1-p)^4$
LLLWL	$(1-p)^3 \times p \times (1-p)$	$= p \times (1-p)^4$
LLLLW	$(1-p)^4 \times p$	$= p \times (1-p)^4$

Letting a win be a “success”:

$$
\begin{aligned}
P(k = 1 \text{ win}) &= \binom{n}{k} p^k (1-p)^{n-k} = \frac{5!}{1! \times 4!} 0.6 \times 0.4^4 \\
&= 5 \times 0.6 \times 0.4^4 = 0.0768
\end{aligned}
$$

Back to Soccer Example

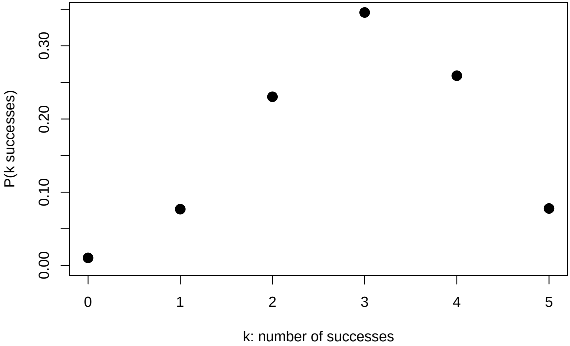
What about the probability that they win all their games! i.e. $k = 5$:

$$
\begin{aligned}
P(k = 5 \text{ wins}) &= \binom{n}{k} p^k (1-p)^{n-k} = \binom{5}{5} 0.6^5 (1-0.6)^0 \\
&= \frac{5!}{5! \times 0!} 0.6^5 \times 1 = 0.08
\end{aligned}
$$

What about the probability that they at win at least one game?

$$
\begin{aligned}
P(\text{at least } k = 1 \text{ wins}) &= P(k = 1 \text{ win}) + \ldots + P(k = 5 \text{ wins}) \\
&= 1 - P(\text{k=0 wins}) \\
&= 1 - \frac{5!}{0! \times 5!} 0.6^0 \times 0.4^5 = 1 - 0.01024 \\
&= 0.98976
\end{aligned}
$$

Back to Soccer Example



Poisson Distribution

Say you want to count the number of rare events in a large population over a unit of time. Examples:

- the number of car accidents at a particular intersection on a given week
- the number of ambulance calls on any given day in Portland
- the number of soldiers in the Prussian army killed accidentally by horse kick from 1875 to 1894

The Poisson distribution helps us describe the number of such events that will occur in a short unit of time for a fixed population if the individuals within the population are independent.

Poisson Distribution

Suppose we are watching for rare events and the number of observed events follows a Poisson distribution with rate λ

$$P(\text{observe k rare events}) = \frac{\lambda^k e^{-\lambda}}{k!}$$

where k may take a value 0, 1, 2, ... where $e \approx 2.718$.

The mean and SD are λ and $\sqrt{\lambda}$.

Conditions for Poisson Distribution

A random variable may be Poisson distributed if

1. The event in question is rare
2. The population is large
3. The events occur independently of each other

Exercise 3.47 on Page 158

A coffee shop serves an average of 75 customers per hour during the morning rush. What is the probability that the coffee shop serves 70 customers in one hour during this time of the day?

In this case, $\lambda = 75$ is the rate

$$P(k = 70) = \frac{75^{70} e^{-75}}{70!} = 0.040$$

Type `dpois(x=70, lambda=75)` in `R`

Next Time

Chapter 4: Foundations for Inference

- Variability in estimates $\overline{x}$, $\widehat{p}$, etc.
- In fact, we can associate a distribution to these estimates