Lecture 4.1: Normal Distribution

Chapter 3.1

Goals for Today

- Define the normal distribution in terms of its parameters
- Review: $\frac{2}{3}$ / 95% / 99.7% rule
- Standardizing normal observations to z-scores

Normal Distribution

From text page 118:

Many variables are nearly normal, but none are exactly normal. Thus the normal distribution, while not perfect for any single problem, is very useful for a variety of problems. We will use it in data exploration and to solve important problems in statistics.

Normal Distribution

Normal distributions:

1. are symmetric
2. are unimodal
3. are bell-shaped
4. have area under the curve 1

Normal Distribution

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Recall from Lecture 3.1, these were the population mean and the population SD. No sampling for now...

Normal Distribution

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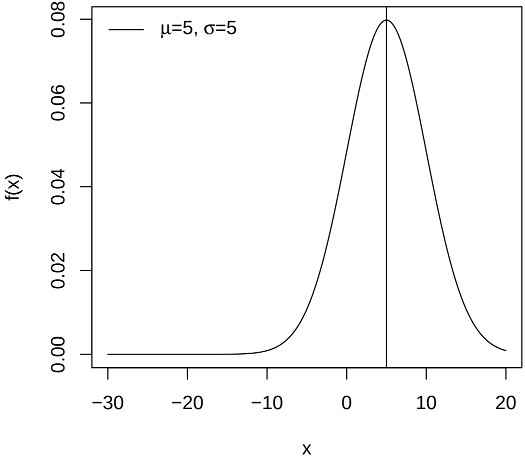

Normal Example

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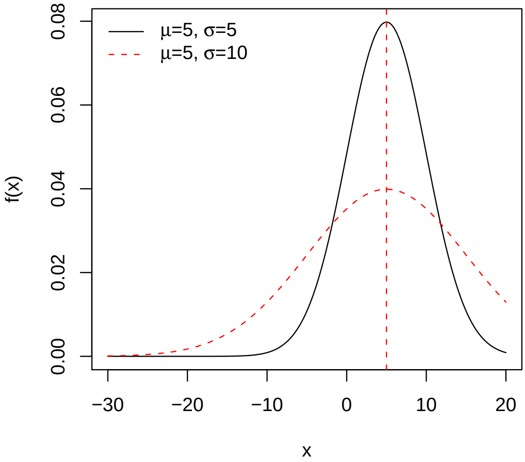

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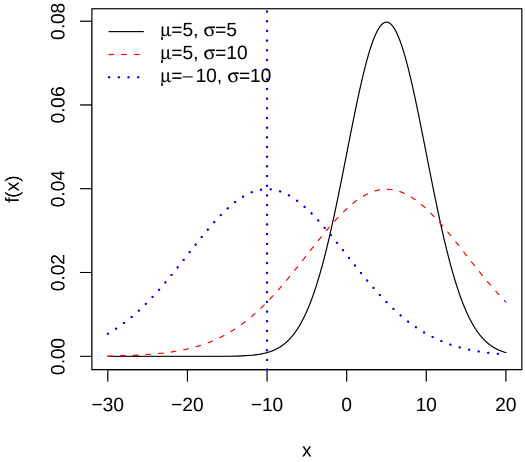
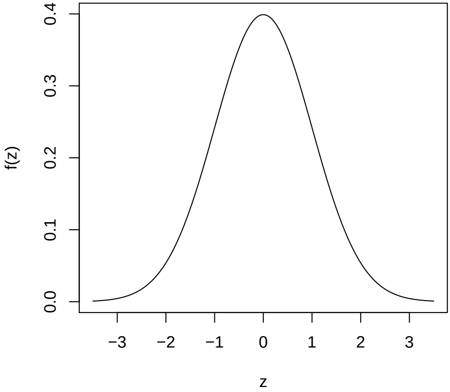


Standardized Normal Distribution

When $\mu = 0$ and $\sigma = 1$, we call this the standard normal distribution:

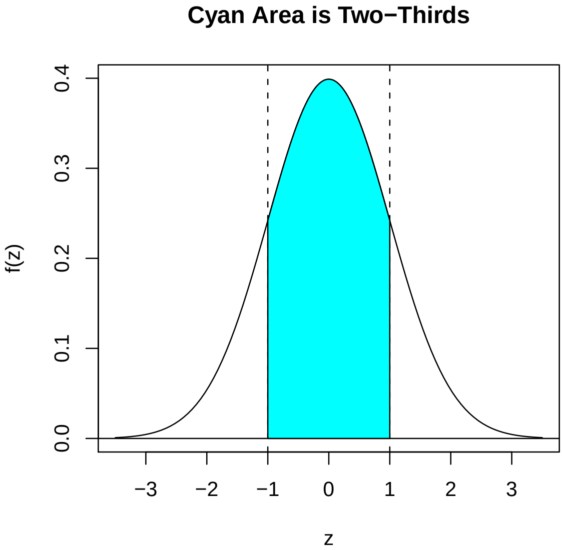


Back to Lecture 3.1

If a distribution is normal, then:

1. Approximately $\frac{2}{3}$'s of the data are within 1 standard deviation of the mean (book says 68%)
2. Approximately 95% of the data are within 2 standard deviations of the mean
3. Approximately 99.7% of the data are within 3 standard deviations of the mean

Ex: Standard Normal $\mu = 0$, $\sigma = 1$



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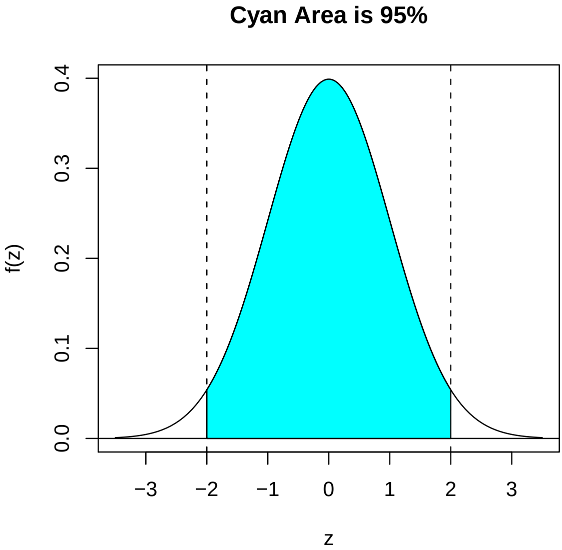

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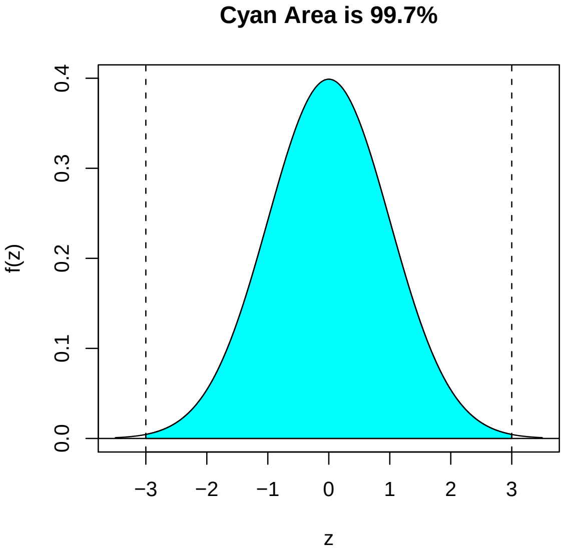

Motivating Example

From text: Say Ann scores 1800 on the SAT and Tom scores 24 on the ACT.

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	SAT	ACT
Mean μ	1500	21
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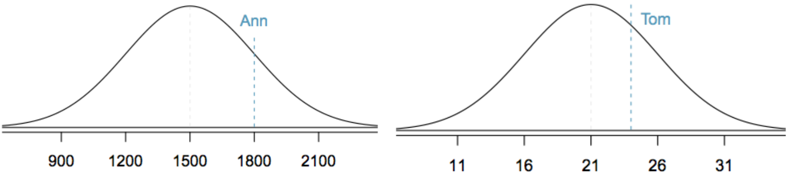

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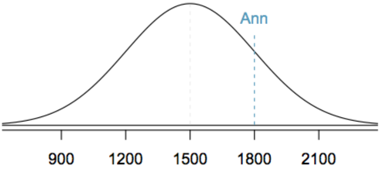

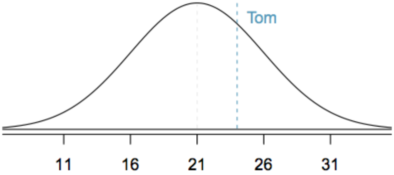

Question: How do we numerically compare their relative performance?

z-scores

The z-score of an observation x is the number of standard deviations it falls above or below the mean. We compute the z-score for an observation x that follows a distribution with mean μ and SD σ:

$$z = \frac{x - \mu}{\sigma}$$

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The z-score is also called the standardized observation.

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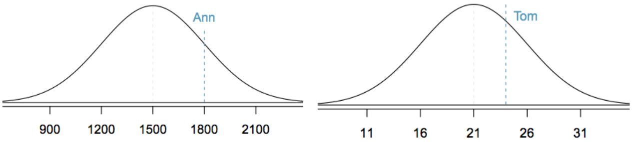

- Ann scored 1800. $z = \frac{1800-1500}{300} = +1$ standard deviation from the mean
- Tom scored 24. $z = \frac{24-21}{5} = +0.6$ standard deviation from the mean

So Ann did relatively better.

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So we can compare observations from any normally distributed data with (μ, σ)

i.e. we've standardized the observations to make them comparable.

Percentiles

Recall from Lecture 3.1: A percentile (%'ile) indicates the value below which a given percentage of observations in a group of observations fall.

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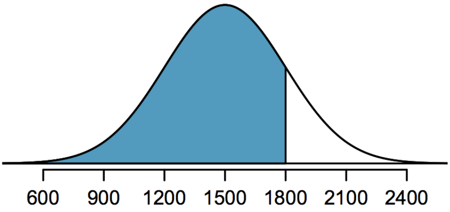
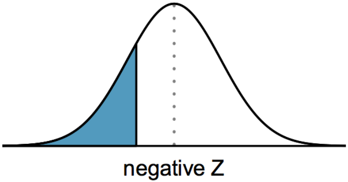
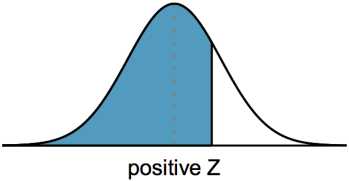
Question: What %'ile is Ann's SAT score of 1800?
i.e. what is the blue shaded area?



Percentiles

Because the total area under the curve is 1, the area to the left of z represents the %'ile of the observation:





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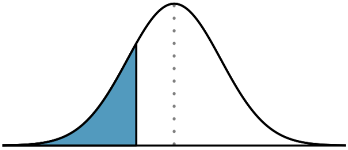

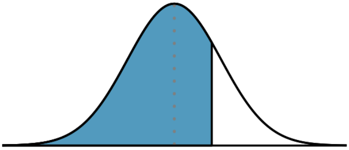

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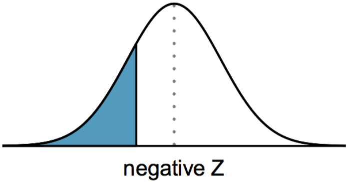

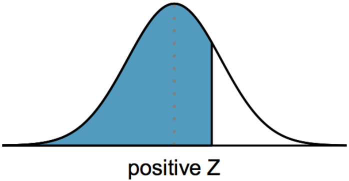

- The blue shaded area on the left plot will be less than 0.5 i.e. %'iles less than 50%
- The blue shaded area on the right plot will be greater than 0.5 i.e. %'iles greater than 50%

Normal Probability Table

A normal probability table allows you to:

- identify the %'ile corresponding to a z-score
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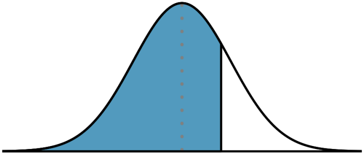
The normal probability tables on page 409 represent z-scores and %'iles corresponding:



Normal Probability Table

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
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From textbook:

- Red case: We are given a z-score of 0.43. A lookup tells us the area to the left of z=0.43 is 0.6664, i.e. the 66th %'ile

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From textbook:

- Red case: We are given a z-score of 0.43. A lookup tells us the area to the left of z=0.43 is 0.6664, i.e. the 66th %'ile
- Blue case: We want the z-score that is the 80th %'ile. We do a reverse lookup: the closest value on the table is 0.7995, which corresponds to a z-score of 0.84.

Back to Ann and Tom

- Since Ann had a z-score of 1.0, her %'ile is 0.8413. (1.0 row, 0.00 column)
  i.e. She did better than 84.13% of SAT test takers.

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- Since Tom had a z-score of 0.6, his %'ile is 0.7257. (0.6 row, 0.00 column)
  i.e. He did better than 72.57% of ACT test takers

Next Time

Next time we will:

- Re-iterate the motivation for doing all this.
- Go over an example using z-scores.
- Evaluating the normal approximation.