Lecture 29: Bayesian Statistics

Chapter 2.2.7

## Recall Conditional Probability

The conditional probability of an event $A$ given the event $B$, is defined by

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

## New Notation

Two possible outcomes for hypothesis test:
- "reject $H_0$ in favor of $H_A$" $= \oplus$'ve result
- "do not reject $H_0$" $= \ominus$'ve result.

with performance measures:
- $\alpha = 0.05 = \Pr(\text{Reject } H_0 \text{ when } H_0 \text{ true}) = \Pr(\oplus | H_0)$
- Power
  $= 1 - \beta = 0.8 = \Pr(\text{Reject } H_0 \text{ when } H_A \text{ true}) = \Pr(\oplus | H_A)$

## Previously

Say $H_A$ is true 10% of the time.

So
- $\Pr(H_A) = 0.1$
- $\Pr(H_0) = 1 - \Pr(H_A) = 1 - 0.1 = 0.9$

We conduct 1000 hypotheses of $H_0$ vs $H_A$, so
- $H_A$ is true 100 times
- $H_0$ is true 900 times

## Previously

So recall from previously we have the following $2 \times 2$ table of possible outcomes:

| | | Test conclusion | |
|---|---|---|---|
| | | $\ominus$ | $\oplus$ |
| **Truth** | $H_0$ true | $(1 - 0.05) \times 900 = 855$ | $0.05 \times 900 = 45$ |
| | $H_A$ true | $(1 - 0.8) \times 100 = 20$ | $0.8 \times 100 = 80$ |

- Of the $\oplus$'s, what prop'n was right?
  i.e. What is $\Pr(H_A|\oplus)$? $\frac{80}{80+45} = 64\%$?
- Of the $\ominus$'s, what prop'n was right?
  i.e. What is $\Pr(H_0|\ominus)$? $\frac{855}{855+20} = 97.7\%$

## Different Set-Up

Now say for the same machine $H_A$ is true 40% of the time. i.e. $P(H_A) = 0.4$

| | | Test conclusion | |
|---|---|---|---|
| | | $\ominus$ | $\oplus$ |
| **Truth** | $H_0$ true | $(1 - 0.05) \times 600 = 570$ | $0.05 \times 600 = 30$ |
| | $H_A$ true | $(1 - 0.8) \times 400 = 80$ | $0.8 \times 400 = 320$ |

- Of the $\oplus$'s, what prop'n was right?
  $\Pr(H_A|\oplus) = \frac{320}{320+30} = 91.4\%$?
- Of the $\ominus$'s, what prop'n was right?
  $\Pr(H_0|\ominus) = \frac{570}{570+80} = 87.7\%$

# How Reliable Are Your Test Results?

For the exact same hypothesis testing machine we get

|  | $\Pr(H_A|\oplus)$ | $\Pr(H_0|\ominus)$ |
|---|---|---|
| $P(H_A) = 10\%$ | 64% | 97.7% |
| $P(H_A) = 40\%$ | 91.4% | 87.7% |

# How Reliable Are Your Test Results?

The probability that a positive result is right depends on how likely $H_A$ is. Same goes for negative results.

Question 2 from Quiz 9: (Nature article) Say a scientist obtains a p-value of 0.01. An incorrect interpretation of this is that it is the probability of a "false alarm" (type I error)... If one wants to make a statement about this being a false alarm, what additional piece of information is required?

Answer 2: The plausibility of the hypothesis being tested for.

# Bayes Theorem

This brings us to Bayes Theorem:

- Let $A_1, \ldots, A_k$ be $k$ events that are a partition of the sample space $S$.
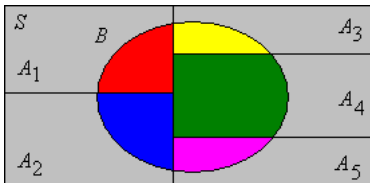- Let $B$ be an event of interest

The Bayes Theorem states:

$$\Pr(A_i|B) = \frac{\Pr(B|A_i) \times \Pr(A_i)}{\sum_{j=1}^{k} \Pr(B|A_j) \times \Pr(A_j)}$$

# Illustration

- The sample sample $S$ is the overall grey box
- $A_1, \ldots, A_5$ are the five blocks that partition $S$.
- $B$ is the oval

## Tailored to our Situation

- The sample sample $S$ is all possible hypotheses
- $H_0$ and $H_A$ partition $S$. i.e. $k = 2$
- Let $B$ be a $\oplus$ result

Then by Bayes Theorem, the probability that a $\oplus$ result is right is

$$
\begin{aligned}
\Pr(H_A|\oplus) &= \frac{\Pr(\oplus|H_A)\Pr(H_A)}{\Pr(\oplus|H_A)\Pr(H_A) + \Pr(\oplus|H_0)\Pr(H_0)} \\
&= \frac{(1-\beta) \times \Pr(H_A)}{(1-\beta) \times \Pr(H_A) + \alpha \times \Pr(H_0)}
\end{aligned}
$$

Notions of both type I error rate and power (AKA type II error rate) are included!

## Tailored to our Situation

Back to initial example where $\alpha = 0.05$, $1 - \beta = 0.8$, $\Pr(H_A) = 0.10$

$$
\Pr(H_A|\oplus) = \frac{0.8 \times 0.1}{0.8 \times 0.1 + 0.05 \times 0.9} = 0.64
$$

Similarly

$$
\begin{aligned}
\Pr(H_0|\ominus) &= \frac{\Pr(\ominus|H_0)\Pr(H_0)}{\Pr(\ominus|H_A)\Pr(H_A) + \Pr(\ominus|H_0)\Pr(H_0)} \\
&= \frac{(1-\alpha) \times \Pr(H_0)}{\beta \times \Pr(H_A) + (1-\alpha) \times \Pr(H_0)} \\
&= \frac{0.95 \times 0.9}{0.2 \times 0.1 + 0.95 \times 0.9} = 0.977
\end{aligned}
$$

## The Debate

Previously, you applied Bayes Theorem from scratch/intuition.

Why isn't everybody taking into account $P(H_A)$ when testing $H_0$ vs $H_A$?

In this example, we assumed we knew the true $P(H_A)$. In real life however, we don't.

## Statistics In General

Statistics is inferring about some unknown parameter $\theta$.

- Frequentist Statistics: the true $\theta$ is a single value.
- Bayesian Statistics: the true $\theta$ is a distribution of values that reflects our belief in the plausibility of different values.

Ex: Coin Flips

## The Bayesian Procedure

To express our belief about $\theta$ from as a Bayesian, we have:

1. A prior distribution $\Pr(\theta)$. It reflects our prior belief about $\theta$.
2. The likelihood function $\Pr(X|\theta)$. This is the mechanism that generates the data.
3. A posterior distribution $\Pr(\theta|X)$. We update our belief about $\theta$ after observing data $X$.

$$\Pr(\theta|X) = \frac{\Pr(X|\theta) \cdot \Pr(\theta)}{\Pr(X)}$$

## The Issue: The Bayesian Procedure

Where do you come up with $\Pr(\theta)$? It's completely subjective! You decide!

## Prior Distribution

This distribution can reflect someone's prior belief of *p*.
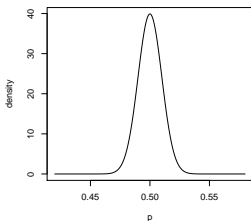
## Prior Distribution

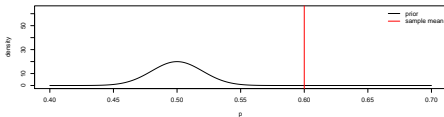Say someone is more skeptical that $p = 0.5$, we can lower it.
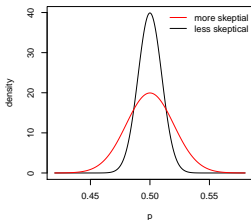
## The Bayesian Procedure

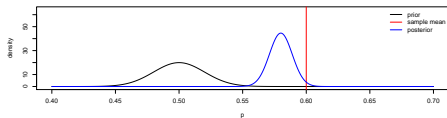Say we have the following prior belief centered at $p = 0.5$
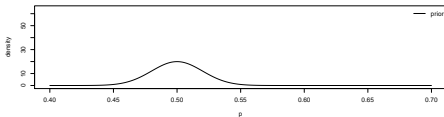
## The Bayesian Procedure

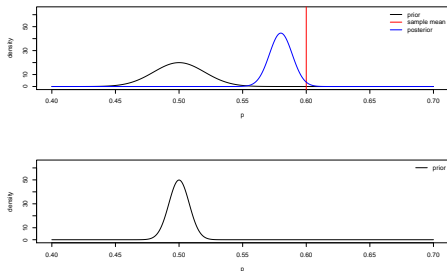Say we collect data, represented by the red line, suggesting $p = 0.6$

## The Bayesian Procedure

We then update our belief, as reflected in the posterior distribution!

## The Bayesian Procedure
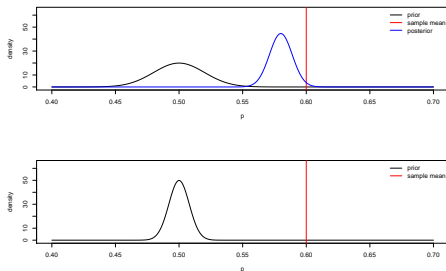
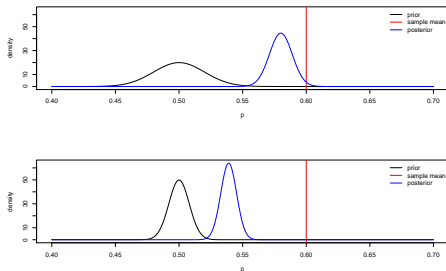Now say we have a stronger prior belief that $p = 0.5$

## The Bayesian Procedure

Say we observed the same data (as represented in red).

## The Bayesian Procedure

The posterior in this case is pulled left due to the sharper prior.

## Back to Debate

Frequentists believe statistics should be completely objective and therefore do not accept the premise of a subjective prior.

Back to Hypothesis Testing Machine example. In order to use such a procedure in real life, we would need to specify some prior belief $\Pr(H_A)$ that $H_A$ is true.