Lecture 29: Bayesian Statistics

Chapter 2.2.7

Recall from Lecture 3.3: Conditional Probability

The conditional probability of an event A given the event B, is defined by

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

This is read as "the probability of A given B" or "the probability of A conditional on B."

Back to Midterm: New Notation

Two possible outcomes for hypothesis test:

- "reject H_0 in favor of H_A" $= \oplus$'ve result
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with performance measures:

- $\alpha = 0.05 = \text{Pr}(\text{Reject } H_0 \text{ when } H_0 \text{ true}) = \text{Pr}(\oplus | H_0)$
- Power
  $= 1 - \beta = 0.8 = \text{Pr}(\text{Reject } H_0 \text{ when } H_A \text{ true}) = \text{Pr}(\oplus | H_A)$

Back to Midterm

Say H_A is true 10% of the time.

So

- $\Pr(H_A) = 0.1$
- $\Pr(H_0) = 1 - \Pr(H_A) = 1 - 0.1 = 0.9$

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We conduct 1000 hypotheses of H_0 vs H_A, so

- H_A is true 100 times
- H_0 is true 900 times

Back to Midterm

So recall from the midterm we have the following 2×2 table of possible outcomes:

		Test conclusion	
		$\ominus$	$\oplus$
Truth	H_0 true	$(1 - 0.05) \times 900 = 855$	$0.05 \times 900 = 45$
	H_A true	$(1 - 0.8) \times 100 = 20$	$0.8 \times 100 = 80$

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  i.e. What is $\Pr(H_0|\ominus)$? $\frac{855}{855+20} = 97.7\%$

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  $\Pr(H_A|\oplus) = \frac{320}{320+30} = 91.4\%$?

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- Of the $\ominus$'s, what prop'n was right?
  $\Pr(H_0|\ominus) = \frac{570}{570+80} = 87.7\%$

How Reliable Are Your Test Results?

For the exact same hypothesis testing machine we get

	$\Pr(H_A \mid \oplus)$	$\Pr(H_0 \mid \ominus)$
$P(H_A) = 10\%$	64%	97.7%
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As quoted in the Economist article: *By and large, scientists want surprising results, and so they test hypotheses that are normally pretty unlikely and often very unlikely.*

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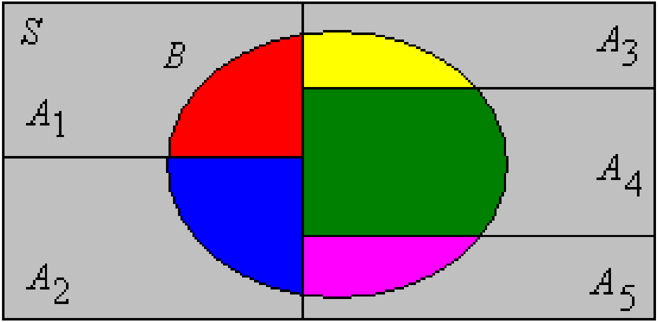
Notice the flip between A_i and B.

Illustration

- The sample sample S is the overall grey box
- $A_1, \ldots, A_5$ are the five blocks that partition S.
- B is the oval



Tailored to our Situation

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Then by Bayes Theorem, the probability that a $\oplus$ result is right is

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$$\begin{aligned} \Pr(H_A|\oplus) &= \frac{\Pr(\oplus|H_A)\Pr(H_A)}{\Pr(\oplus|H_A)\Pr(H_A) + \Pr(\oplus|H_0)\Pr(H_0)} \\ &= \frac{(1-\beta) \times \Pr(H_A)}{(1-\beta) \times \Pr(H_A) + \alpha \times \Pr(H_0)} \end{aligned}$$

Notions of both type I error rate and power (AKA type II error rate) are included!

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Back to initial example where $\alpha = 0.05$, $1 - \beta = 0.8$, $\Pr(H_A) = 0.10$

$$\Pr(H_A|\oplus) = \frac{0.8 \times 0.1}{0.8 \times 0.1 + 0.05 \times 0.9} = 0.64$$

Similarly

$$\begin{aligned}
\Pr(H_0|\ominus) &= \frac{\Pr(\ominus|H_0)\Pr(H_0)}{\Pr(\ominus|H_A)\Pr(H_A) + \Pr(\ominus|H_0)\Pr(H_0)} \\
&= \frac{(1-\alpha) \times \Pr(H_0)}{\beta \times \Pr(H_A) + (1-\alpha) \times \Pr(H_0)} \\
&= \frac{0.95 \times 0.9}{0.2 \times 0.1 + 0.95 \times 0.9} = 0.977
\end{aligned}$$

The Debate

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Why isn't everybody taking into account $P(H_A)$ when testing H_0 vs H_A?

In this example, we assumed we knew the true $P(H_A)$. In real life however, we don't.

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- Bayesian Statistics: the true θ is a distribution of values that reflects our belief in the plausibility of different values.

Specific Example

Concrete example: the Probability of Flipping Heads

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1. A prior distribution $\Pr(\theta)$. It reflects our prior belief about θ.
2. The likelihood function $\Pr(X|\theta)$. This is the mechanism that generates the data.
3. A posterior distribution $\Pr(\theta|X)$. We update our belief about θ after observing data X.

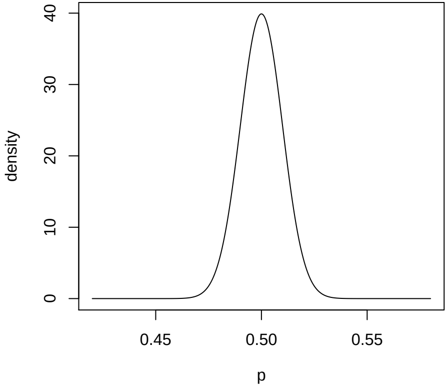
$$\Pr(\theta|X) = \frac{\Pr(X|\theta) \cdot \Pr(\theta)}{\Pr(X)}$$

The Issue: The Bayesian Procedure

Where do you come up with $\Pr(\theta)$? It's completely subjective! You decide!

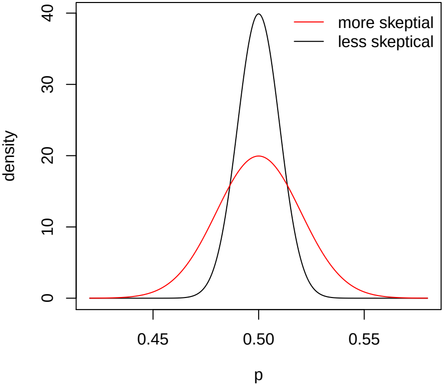
Prior Distribution

This distribution can reflect someone's prior belief of p.



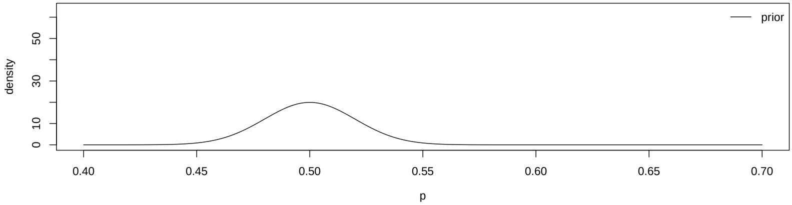
Prior Distribution

Say someone is more skeptical that $p = 0.5$, we can lower it.



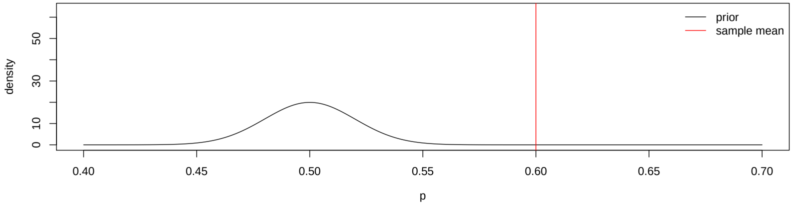
The Bayesian Procedure

Say we have the following prior belief centered at $p = 0.5$



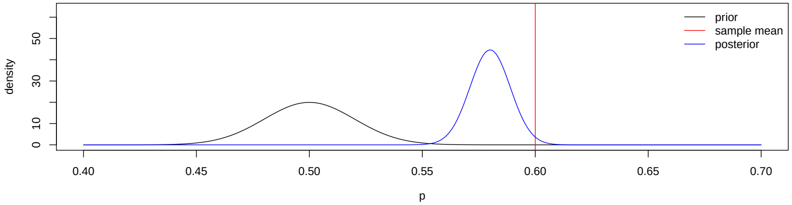
The Bayesian Procedure

Say we collect data, represented by the red line, suggesting $p = 0.6$



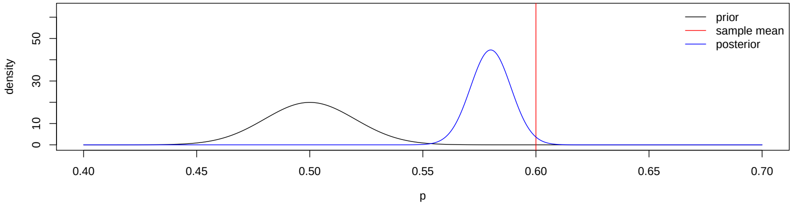
The Bayesian Procedure

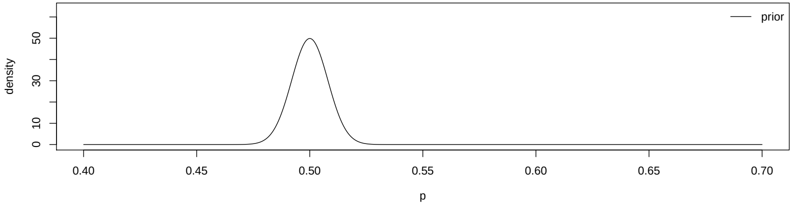
We then update our belief, as reflected in the posterior distribution!



The Bayesian Procedure

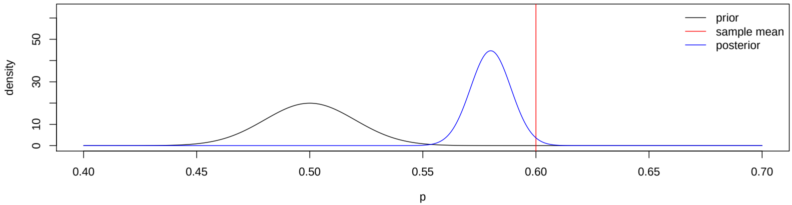
Now say we have a stronger prior belief that $p = 0.5$

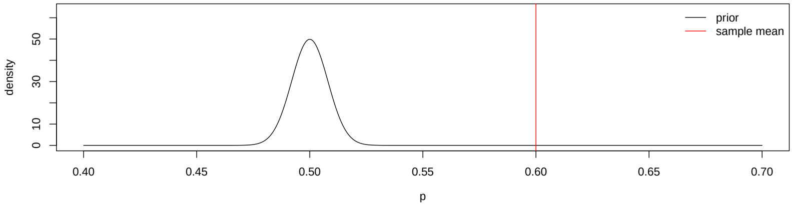




The Bayesian Procedure

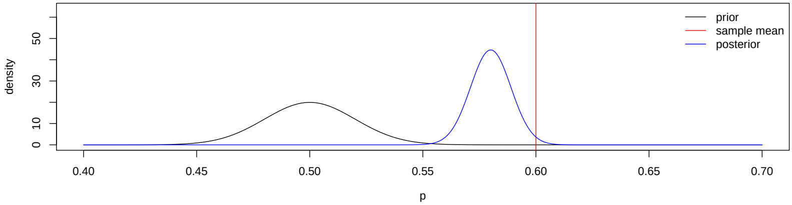
Say we observed the same data (as represented in red).

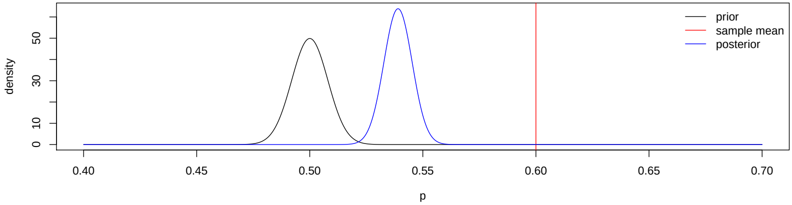




The Bayesian Procedure

The posterior in this case is pulled left due to the sharper prior.





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Back to Hypothesis Testing Machine example. In order to use such a procedure in real life, we would need to specify some prior belief $\Pr(H_A)$ that H_A is true.