Lecture 23: Tests for Independence in Two-Way Tables

Chapter 6.4

Today's Example

Google is always tinkering with its search ranking algorithm. Say we want to compare the following 3 algorithms:

1. the current version
2. test algorithm 1
3. test algorithm 2

Today's Example

They measure user satisfaction with the results for a particular search with the `new search` variable:

- no new search: User clicked on a result. Suggests user is satisfied with result.
- new search: User did not click on a result and tried a new related search. Suggests user is dissatisfied with result.

Today's Example

So we have two categorical variables:

- `algorithm`: current, test 1, or test 2
- `new search`: yes or no

Are they independent? i.e. independent of which algorithm is used, do we have the same levels of new search?

Today's Example

Say we observe the following contingency table:

	algorithm			
new search	Current	Test 1	Test 2	Total
No new search	4000	2000	2000	8000
New search	1000	500	500	2000
Total	5000	2500	2500	10000

For all 3 algorithms, there is a new search $\frac{1}{5}$ of the time.

They are independent: regardless of which algorithm used, the proportion of new searches stays the same.

Today's Example

Now say instead we observed the following results:

	algorithm			
new search	Current	Test 1	Test 2	Total
No new search	4000	2500	1500	8000
New search	1000	0	1000	2000
Total	5000	2500	2500	10000

In this case, they are dependent: depending on which `algorithm` used, the proportion of new searches is different.

Hypothesis Test

We test at the $\alpha = 0.05$ significance level:

$$\begin{aligned} H_0 : &\quad \text{the algorithms each perform equally well} \\ \text{vs } H_A : &\quad \text{the algorithms do not perform equally well} \end{aligned}$$

i.e. are the categorial variables `algorithm` and `new search` independent?

Different Names

The following all refer to the same test: χ^2 test for

- two-way tables
- i.e. contingency tables
- independence of two categorical variables
- homogeneity: are the algorithms homogeneous in their performance?

Example from Textbook

Let's make the values match the example from the textbook on page 284:

new search	algorithm Current	Test 1	Test 2	Total
No new search	3511	1749	1818	7078
New search	1489	751	682	2922
Total	5000	2500	2500	10000

Example from Textbook

Before we start, let's make each column reflect a proportion and not a count.

new search	algorithm Current	Test 1	Test 2	Total
No new search	0.7022	0.6996	0.7272	0.7078
New search	0.2978	0.3004	0.2728	0.2922
Total	1	1	1	1

If all algorithms performed the same, we'd expect

- 0.7078 for all 3 values in the top row
- 0.2922 for all 3 values in the bottom row

Are we observing what we expect? i.e. What is the degree of this deviation?

What's Expected

We expect:

new search	algorithm Current	Test 1	Test 2	Total
No new search				$7078 = 0.7078 \times 10000$
New search				$2922 = 0.2922 \times 10000$
Total	5000	2500	2500	10000

What's Expected

We expect:

new search	algorithm Current	Test 1	Test 2	Total
No new search			$1769.5 = 0.7078 \times 2500$	7078
New search			$730.5 = 0.2922 \times 2500$	2922
Total	5000	2500	2500	10000

What's Expected

We expect:

new search	algorithm Current	Test 1	Test 2	Total
No new search		$1769.5 = 0.7078 \times 2500$	1769.5	7078
New search		$730.5 = 0.2922 \times 2500$	730.5	2922
Total	5000	2500	2500	10000

What's Expected

We expect:

new search	algorithm Current	Test 1	Test 2	Total
No new search	$3539 = 0.7078 \times 5000$	1769.5	1769.5	7078
New search	$1461 = 0.2922 \times 5000$	730.5	730.5	2922
Total	5000	2500	2500	10000

Observed vs. Expected

Expected Counts:

new search	algorithm Current	Test 1	Test 2	Total
No new search	3539	1769.5	1769.5	7078
New search	1461	730.5	730.5	2922
Total	5000	2500	2500	10000

Observed Counts:

new search	algorithm Current	Test 1	Test 2	Total
No new search	3511	1749	1818	7078
New search	1489	751	682	2922
Total	5000	2500	2500	10000

Chi-Square Statistic

We compute χ^2 test statistic: for all $i = 1, \ldots, 6$ cells

$$\frac{(\text{observed count}_i - \text{expected count}_i)^2}{\text{expected count}_i}$$

$$\begin{aligned}
\text{Row 1, Col 1} &= \frac{(3511 - 3539)^2}{3539} = 0.222 \\
\vdots \quad & \quad \vdots \\
\text{Row 2, Col 3} &= \frac{(682 - 730.5)^2}{730.5} = 3.220
\end{aligned}$$

So

$$\begin{aligned}
\chi^2 &= 0.222 + 0.237 + \ldots + 3.220 \\
&= 6.120
\end{aligned}$$

Chi-Square Distribution

We compare this to a χ^2 distribution to get the p-value. What are the degrees of freedom?

$$\begin{aligned} df &= (\# \text{ of rows - } 1) \times (\# \text{ of columns - } 1) \\ &= (R-1) \times (C-1) \\ &= (2-1) \times (3-1) = 2 \text{ in our case} \end{aligned}$$

Chi-Square Distribution

Looking up 6.120 in the χ^2 table on page 412 on the $df = 2$ row, it would be between 0.05 and 0.01. Since our $\alpha = 0.05$, we reject the null hypothesis and accept the alternative that the algorithms do not perform equally well.

i.e. the `algorithm` and `new search` categorical variables are dependent.

Conditions/Assumptions

Nearly identical to conditions/assumptions for χ^2 tests for goodness-of-fit:

1. Independence: Each case is independent of the other
2. Sample size/distribution: We need at least 5 cases in each scenario i.e. each cell in the table
3. Degrees of freedom: (Different than before) We need $df = (R-1) \times (C-1) \geq 2$.

Why Are They Called Degrees of Freedom?

In the case of χ^2 tests, the degrees of freedom is the number of values needed before you specify all values in the cells of the table.

Why Are They Called Degrees of Freedom? Rows

Each row has $df = 2$ because if we specify 2 values, all values in the row are specified.

Example:

	algorithm			
new search	Current	Test 1	Test 2	Total
No new search	X	Y		7078
New search				2922
Total	5000	2500	2500	10000

then the missing value is $7078 - X - Y$.
i.e. the wiggle room we have is $C - 1$ two cells

Why Are They Called Degrees of Freedom? Columns

Each column has $df = 1$ because if we specify 1 value, all values in the column are specified.

Example:

	algorithm			
new search	Current	Test 1	Test 2	Total
No new search	X			7078
New search				2922
Total	5000	2500	2500	10000

then the missing value is $5000 - X$.
i.e. the wiggle room we have is $R - 1$ one cell

Why Are They Called Degrees of Freedom? Columns

So the overall df is $(C-1) \times (R-1)$, in our case $df = 2$.

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	X	Y		7078
New search				2922
Total	5000	2500	2500	10000

i.e. if we know these two values, we can fill the rest of the table.

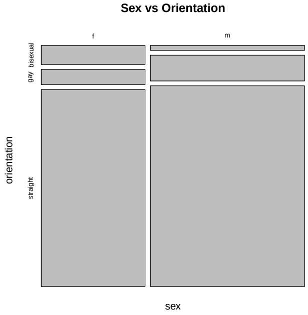
Real-Life Example

For 59,946 OkCupid users in San Francisco CA in June 2012, consider the cross-classification of their sex and sexual orientation via a contingency table:

	Orientation			
Sex	Bisexual	Gay	Straight	Total
Female	1996	1588	20533	24117
Male	771	3985	31073	35829
Total	2767	5573	51606	59946

This is better visualized with a mosaic plot:

Real-Life Example



Real-Life Example

Sex and sexual orientation are not independent: knowing one variable provides information about the other.

$\chi^2 = 1495$ and degrees of freedom $(3-1) \times (2-1) = 2$. The p-value $= 0$.