Lecture 28: Logistic Regression

Chapter 8.4

Binary Outcome Variables

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- $Y_i = 1$ with probability p_i
- $Y_i = 0$ with probability $1 - p_i$

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Logistic regression: we are modeling p_i, the probability associated with the i^{th} observation for $i = 1, \ldots, n$

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However, you may end up fitting p_i's that are either

- less than 0
- greater than 1

Outcome Variable

Rather, what is modeled is the logit transformation or log-odds of p_i

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_{1,i} + \ldots + \beta_k x_{k,i}$$

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Why this transformation? It maps the $[0, 1]$ interval to a $(-\infty, \infty)$ interval.

Outcome Variable

First, convert p_i into odds:

"Two to one odds for event X" $\equiv$ "There is a 66% chance of event X occurring."

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- for $p_i = 0.5 \Rightarrow \log\left(\frac{p_i}{1-p_i}\right) = \log\left(\frac{0.5}{0.5}\right) = 0$

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- for $p_i = 0.5 \Rightarrow \log\left(\frac{p_i}{1-p_i}\right) = \log\left(\frac{0.5}{0.5}\right) = 0$
- for $p_1 = 1 \Rightarrow \log\left(\frac{p_i}{1-p_i}\right) = \log\left(\frac{1}{0}\right)$ [1] $= \log(\infty) = \infty$

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Outcome Variable

Figure 8.14 from page 369

Simple Logistic Regression Example

So say we fit a logistic regression with ($n = 3921$):

- Y_i is `spam`: binary variable of whether message was classified as spam (1 if spam)
- x is `to_multiple`: binary variable indicating if more than one recipient listed

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- Y_i is `spam`: binary variable of whether message was classified as spam (1 if spam)
- x is `to_multiple`: binary variable indicating if more than one recipient listed

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	-2.1161	0.0562	-37.67	0.0000
`to_multiple`	-1.8092	0.2969	-6.09	0.0000

The regression equation is

$$\log\left(\frac{p_i}{1-p_i}\right) = -2.12 - 1.81 \times \texttt{to\_multiple}$$

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is the inverse logit transformation.

So to convert the regression equation to probabilities, we compute

$$p_i = \frac{\exp(\beta_0 + \beta_1 x_{1,i} + \ldots + \beta_k x_{k,i})}{1 + \exp(\beta_0 + \beta_1 x_{1,i} + \ldots + \beta_k x_{k,i})}$$

Fitted Probabilities

To compute the fitted probabilities $\widehat{p}_i$:

- `to_multiple`$= 0$ (only one recipient):

$$\widehat{p}_i = \frac{\exp(-2.12 - 1.81 \times 0)}{1 + \exp(-2.12 - 1.81 \times 0)} = 0.11$$

- `to_multiple`$= 1$ (many recipients):

$$\widehat{p}_i = \frac{\exp(-2.12 - 1.81 \times 1)}{1 + \exp(-2.12 - 1.81 \times 1)} = 0.02$$

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Note: 11% and 2% are not dramatically different. In an ideal world of binary predictors, we'd have fitted probabilities of 100% and 0%.

Fitted Model Using Backwards Regression

The following model was selected in the text using backwards selection using $\alpha = 0.05$.

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	-0.8057	0.0880	-9.15	0.0000
to_multiple?	-2.7514	0.3074	-8.95	0.0000
word winner used?	1.7251	0.3245	5.32	0.0000
special formatting?	-1.5857	0.1201	-13.20	0.0000
'RE:' in subject?	-3.0977	0.3651	-8.48	0.0000
attachment?	0.2127	0.0572	3.72	0.0002
word password used?	-0.7478	0.2956	-2.53	0.0114

Fitted Model Using Backwards Regression

The following variables increase the probability that the email is spam, since $\beta > 0$

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Fitted Model Using Backwards Regression

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(Intercept)	-0.8057	0.0880	-9.15	0.0000
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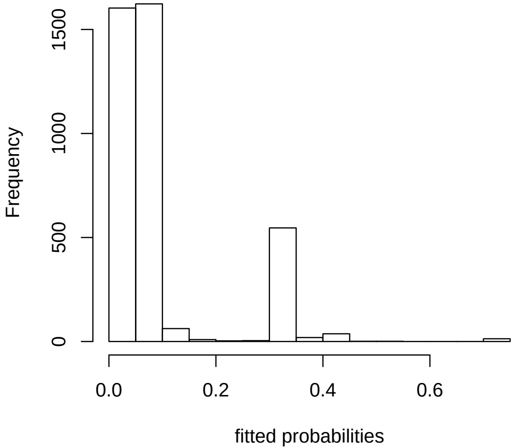
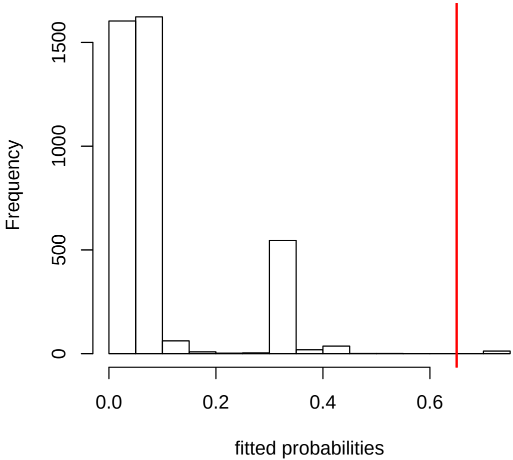
Fitted Probabilities

These are all 3921 fitted probabilities:



Using Cutoffs to Classify Emails as Spam

Say we use a cutoff of 65% to classify an email spam or not:



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		Classification: Not Spam	Classification: Spam
Truth	Not Spam	3351	3
	Spam	357	10

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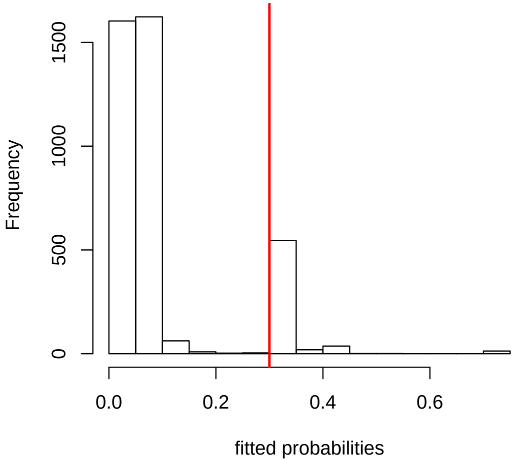
		Classification	
		Not Spam	Spam
Truth	Not Spam	3351	3
	Spam	357	10

- Of the emails classified as spam: $\frac{10}{10+3} = 76\%$ correct
- Of the emails classified not as spam: $\frac{3351}{3351+357} = 90.3\%$ correct

Using Cutoffs to Classify Emails as Spam

Now say we use a cutoff of 30% to classify an email spam or not:



Using Cutoffs to Classify Emails as Spam

Using a cutoff of 30%:

		Classification	
		Not Spam	Spam
Truth	Not Spam	3138	416
	Spam	166	201

Using Cutoffs to Classify Emails as Spam

Using a cutoff of 30%:

		Classification: Not Spam	Classification: Spam
Truth	Not Spam	3138	416
	Spam	166	201

- Of the emails classified as spam: $\frac{201}{201+416} = 32.6\%$ correct
- Of the emails classified not as spam: $\frac{3138}{3138+166} = 95.0\%$ correct

Using Cutoffs to Classify Emails as Spam

Moral of the Story: most classifiers (like hypothesis tests) are never perfect. There will almost always be a trade-off between:

- Type I errors: labeling an email spam when it is not
- Type II errors: failing to label an email as spam when it is

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Please read pages 375 and 376 from the text.

Next Time

Bayes Theorem, Bayesian statistics, False Discovery Rate