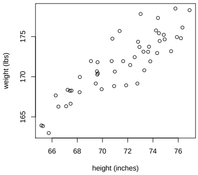
Lecture 24: Linear Regression Part I

Chapter 7.1-7.2

Questions for Today

Say we have the height/weight of 50 individuals and we display the scatterplot/bivariate plot of the seemingly linear relationship:



Questions:

- What is the "best" fitting line through these points?
- What do we mean by "best"?

Regression

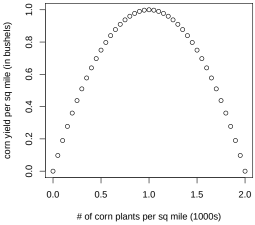
There are many types of regression, all in order to estimate the relationship between variables. We start by considering simple

linear regression (SLR):

- a single explanatory variable / independent variable / predictor variable x
- an outcome variable / dependent variable y
- a presumed linear relationship between them

Example of Non-Linear Relationship

At first as you plant more corn plants, you have higher yield, but past a certain point plants fight for limited resources and they die.



Modeling x and y Linearly

The SLR model assumes that the relationship between x and y can be modeled by a line:

$$y = \beta_0 + \beta_1 x$$

where

- β_0 is the unknown intercept parameter
- β_1 is the unknown slope parameter

Procedure

Based on n pairs of observations (x_i, y_i)

1. Compute point estimates
   - b_0 of parameter β_0
   - b_1 of parameter β_1
2. Associate standard errors SE_{b_0} and SE_{b_1}
3. For both the intercept and slope
   - Build confidence intervals
   - Do hypothesis test

$$\begin{aligned} & H_0 : \beta = 0 \\ \text{vs} \quad & H_A : \beta \neq 0 \end{aligned}$$

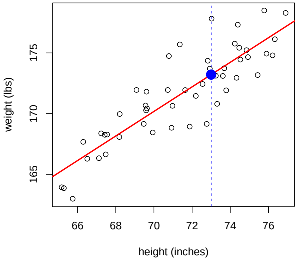
The equation

$$\widehat{y} = b_0 + b_1 x$$

is called the least squares line where $\widehat{y}$ is the fitted/predicted value.

Fitted Value

Here $\widehat{y} = 100 + 0.99x$. Thus for $x = 73$, $\widehat{y} = 173.22$:



Residuals

Residuals are what's leftover: leftover variation in the data unexplained by the model:

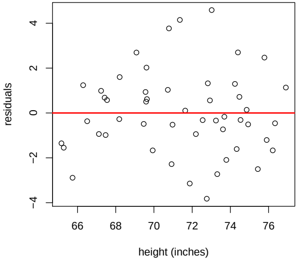
$$
\begin{aligned}
\text{Residual} &= \text{Data} - \text{Fit} \\
e_i &= y_i - \widehat{y}_i
\end{aligned}
$$

where e_i is the residual of the i^{th} observation (x_i, y_i).

We can think of the e_i's as deviations from the model. The smaller the deviations, the better the fit.

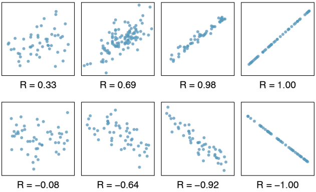
Residual Plot

Residual plots: take previous plot and flatten the red line by subtracting $\widehat{y}$ from y.



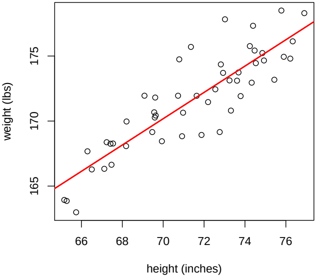
Correlation Coefficient

The correlation coefficient R is a value between $[-1, 1]$ that measures the strength of the linear relationship between x and y.



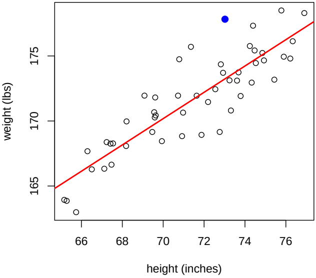
Best Fitting Line

What does "best fitting line" mean?



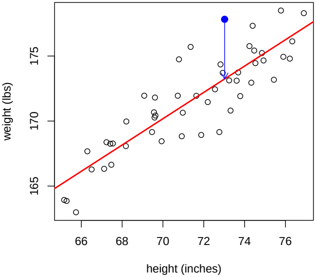
Best Fitting Line

Consider ANY point x_i for $i = 1, \ldots, 50$ (in blue).



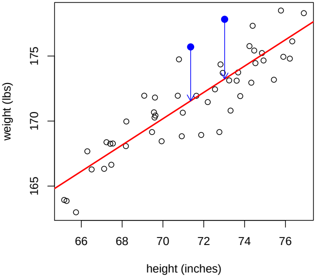
Best Fitting Line

Now consider this point's deviation from the regression line



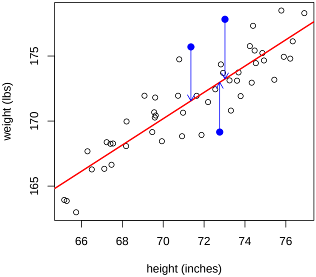
Best Fitting Line

Do this for another point x_i...



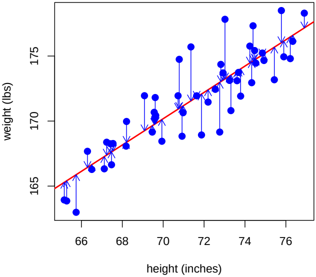
Best Fitting Line

Do this for another point x_i. . .



Best Fitting Line

The regression line minimizes the sum of the squared arrow lengths.



Least Squares

i.e. the regression line minimizes:

$$e_1^2 + e_2^2 + \ldots + e_n^2$$

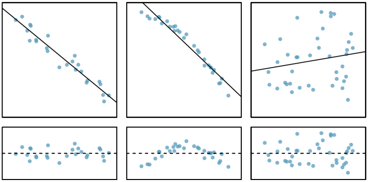
This is called minimizing the least squares criterion.

Conditions for Simple Linear Regression

- Linearity: The data should show a linear trend.
- Independence: The residuals should be independent
- Nearly normal residuals: The residuals e_i must be nearly normal (verify with QQ-plot) with mean 0.
- Constant variability: The variability of points around the least squares line remains roughly constant (i.e. for all values of x).

Behavior of Residuals: 3 Examples

Sample data + regression on top, residual plots on bottom.



- Plots 1 and 3 are roughly linear.
- Plots 1 and 3 have roughly constant variability, but the 3rd plot has higher variability

Finding the Least Squares Line

To find the least squares line we need to find the point estimates:

- The point estimate b_1 of the slope β_1 is

$$b_1 = \frac{s_y}{s_x} R$$

- The regression line always goes through $(\overline{x}, \overline{y})$. We use this fact to find the point estimate of b_0 of the intercept β_0.

Finding the Point Estimate of the Intercept b_0

Given the slope and a point on the line (x_0, y_0), the equation for the line can be written as

$$\begin{aligned} \text{slope} &= \frac{\text{rise}}{\text{run}} = \frac{y - y_0}{x - x_0} \\ y - y_0 &= \text{slope} \times (x - x_0) \end{aligned}$$

So

$$\begin{aligned} & y - \overline{y} = b_1(x - \overline{x}) \\ \text{so} \quad & y = (\overline{y} - b_1\overline{x}) + b_1 x \\ \text{so} \quad & b_0 = \overline{y} - b_1\overline{x} \end{aligned}$$

Measuring the Strength of a Fit

If $R = -1$ or $R = 1$ we have a perfect linear fit between x and y, if $R = 0$ then there is no fit.

However R^2 is a more commonly used measure of the strength of fit. For SLR, it is correlation coefficient squared, but not for other kinds of regression.

R^2 of a linear model describes the proportion of the total variation in y that is explained by the least squares line.

Next Time

- How to interpret regression line parameter estimates
- Categorical Variable for x: male vs female, new vs used, etc.
- Inference for linear regression