Lecture 22: Chi-Square Tests for Goodness-of-Fit

Chapter 6.3

Question for Today

Say we had $n = 100$ people picked as jurors, we expect the breakdown to be:

Race	White	Black	Hispanic	Other	Total
Registered Voters	72%	7%	12%	9%	100%
Representation	72	7	12	9	$n = 100$

Question for Today

Say we observe the following. Is there a bias? i.e. a non-random mechanism?

Race	White	Black	Hispanic	Other	Total
Registered Voters	72%	7%	12%	9%	100%
Representation	75	6	11	8	$n = 100$

Chi-Square Tests

Chi-square χ^2 tests allow us to compare

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i.e. What is the "goodness" of the fit of the observed counts to the expected counts?

The Data

Let's use $n = 275$ people. Assuming the same proportions as above, we compute the expected counts. Ex: $198 = 275 \times 0.72$.

Race	White	Black	Hispanic	Other	Total
Expected Counts	198	19.25	33	24.75	275

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Observed Counts	205	26	25	19	275

Hypothesis Test in General

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\begin{array}{ll}
 & H_0 : \text{The data are consistent with the specified distribution.} \\
\text{vs} & H_A : \text{The data are not consistent with the specified distribution.}
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H_0 can also be stated: the data are a random sample from the distribution and any differences of observed vs expected reflect natural sampling variation.

Hypothesis Test in Our Case

$$
\begin{aligned}
& H_0 : \text{the jurors are randomly sampled i.e. there is no racial bias} \\
\text{vs} \quad & H_A : \text{the jurors are not randomly sampled i.e. there is racial bias}
\end{aligned}
$$

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3. ANOVA:
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   - null distribution: F-distribution with $df_1 = k - 1$ and $df_2 = n - k$
4. Goodness-of-fit:
   - test statistic: χ^2-statistic
   - null distribution: χ^2 distribution with $df = k - 1$

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Note when:

- observed $=$ expected $\Rightarrow Z = 0$
- observed $>$ expected $\Rightarrow Z > 0$
- observed $<$ expected $\Rightarrow Z < 0$

The Z's measure deviations.

Deviations

Now treat $+$'ve and $-$'ve differences as the same by squaring Z:

$$
\begin{aligned}
Z^2 &= \left(\frac{\text{observed} - \text{expected}}{\sqrt{\text{expected}}}\right)^2 \\
&= \frac{(\text{observed} - \text{expected})^2}{\text{expected}}
\end{aligned}
$$

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$$Z^2 = \left(\frac{\text{observed} - \text{expected}}{\sqrt{\text{expected}}}\right)^2 = \frac{(\text{observed} - \text{expected})^2}{\text{expected}}$$

Why square it and not absolute value it? It's easier to do calculus on x^2 than $|x|$.

Chi-Square Test Statistic

Finally sum all values of Z^2. This is the chi-square test statistic for one-way tables.

$$\chi^2 = \sum_{i=1}^{k} Z_i^2 = \sum_{i=1}^{k} \frac{(\text{observed}_i - \text{expected}_i)^2}{\text{expected}_i}$$

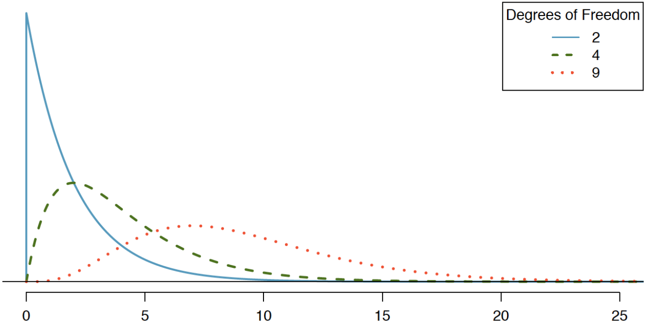
Chi-Square Test Statistic

In the case of the jury data, we have 4 groups: white, black, hispanic, and other:

$$\begin{aligned}
\chi^2 &= Z_w^2 + Z_b^2 + Z_h^2 + Z_o^2 \\
&= \frac{(205 - 198)^2}{198} + \ldots + \ldots + \frac{(19 - 24.75)^2}{24.75} \\
&= 5.89
\end{aligned}$$

p-values

We compare the test statistic to a χ^2 distribution with $df = k - 1$ degrees of freedom.
Note: not $df = n - 1$ like with t-test.



p-values

The p-value is the area to the right of the test statistic. Use p.412:

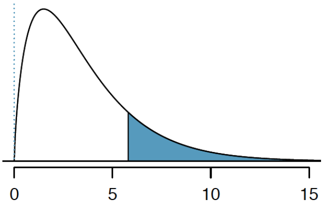


Figure B.2: Areas in the chi-square table always refer to the right tail.

Upper tail		0.3	0.2	0.1	0.05	0.02	0.01	0.005	0.001
df	2	2.41	3.22	4.61	5.99	7.82	9.21	10.60	13.82
	3	3.66	4.64	6.25	7.81	9.84	11.34	12.84	16.27
	4	4.88	5.99	7.78	9.49	11.67	13.28	14.86	18.47
	5	6.06	7.29	9.24	11.07	13.39	15.09	16.75	20.52

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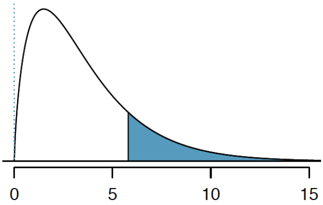

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	4	4.88	5.99	7.78	9.49	11.67	13.28	14.86	18.47
	5	6.06	7.29	9.24	11.07	13.39	15.09	16.75	20.52

In our case, $df = k - 1 = 3$, and $\chi^2 = 5.89$, which is in between $(4.64, 6.25)$, so p-value is in between $(0.1, 0.2)$. Not overwhelming evidence against H_0.

Hypothetical Scenarios

Say we have two hypothetical scenarios of observed counts:

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- For all 4 groups, say observed = expected, then

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  hence p-value = 1.

- Say we observed 275 others and 0 for the rest, then

$$\chi^2 = 2786.11 + 84.46 + 189.57 + 12588.11 = 15648.25$$

  hence p-value = 0.

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2. Sample size: Similarly like with proportions, we need at least 5 cases in each scenario (each cell in the table)
3. Degrees of freedom: We need at least $df = 2$, i.e. $k \geq 3$

Next Time

We look at chi-square tests for two-way tables to test for independence. i.e. are two variables independent from each other?