Lecture 16: Sample Size and Power

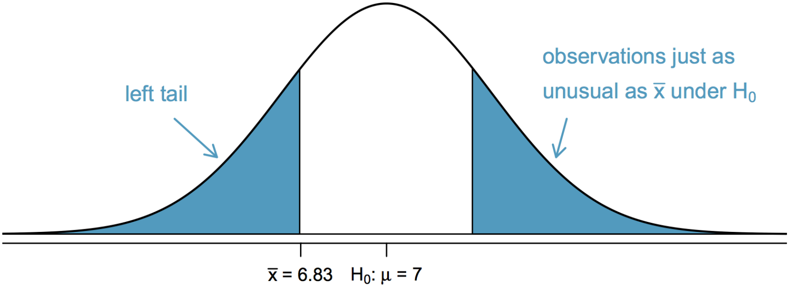
Chapter 4.6

Two-Sided Alternative Hypothesis

Say instead we had a two-sided alternative hypothesis:

- $H_0 : \mu = 7$
- $H_A : \mu \neq 7$

The the p-value would be double: $2 \times 0.007 = 0.014$. Picture:



Setting α

Say Dr. Quack is conducting a hypothesis tests. They start with $\alpha = 0.05$.

Say they conduct the test and get a p-value $= 0.09$. They then publish a paper that says: "having used an $\alpha = 0.10$, we reject the null hypothesis and declare our results to be significant."

What's not honest about this approach?

Ronald Fisher, the creator of p-values, never intended for them to be used this way. Rather he said a small p-value should lead to further investigation.

Goals for Today

- More in depth discussion of
  - 10% sampling rule
  - Skew condition to check to use the normal model
- How big a sample size do I need?
- Statistical power
- Statistical vs practical significance

10% Sampling Rule

Question: Why do we have the rule that says our sample size n should be less than 10% of the population size N?

Intuition: Shouldn't we always sample as many people as we can?

Answer: Yes, if we only care about the mean. If we also care about the SE, then we need to be careful.

Explanation: Recall from HW5 Q1, sampling from a rooms that are half male and half female but with $N = 10$ and $N = 10000$.

10% Sampling Rule

The finite population correction (FPC) to the SE of point estimates accounts for the sampling without replacement:

$$SE = \frac{\sigma}{\sqrt{n}} \times \sqrt{\frac{N-n}{N-1}} = \frac{\sigma}{\sqrt{n}} \times FPC$$

Say we have $N = 10000$.

- Let $n = 100$ i.e. 1%, then

$$FPC = \sqrt{\frac{10000-100}{10000-1}} = 0.995$$

- Let $n = 5000$ i.e. 50%, then

$$FPC = \sqrt{\frac{10000-5000}{10000-1}} = 0.707$$

10% Sampling Rule

$$SE = \frac{\sigma}{\sqrt{n}} \times \sqrt{\frac{N-n}{N-1}} = \frac{\sigma}{\sqrt{n}} \times FPC$$

We've been ignoring the FPC. So when

- If n is relatively small, the FPC is close to 1, so not a problem.
- If n is relatively large, the FPC goes to 1. i.e. $\frac{\sigma}{\sqrt{n}}$ is not the true SE.

Conclusion: By capping $n \leq 10\%$ of N, we have a rule of thumb for keeping the FPC "close" to 1.

10% Sampling Rule

We can tie the conceptual and mathematical notions of sampling:

Conceptual: If we sample everybody in our study population, then we don't need statistics because we know the true μ exactly.

and

Mathematical: If $n = N \Rightarrow FPC = \sqrt{\frac{N-n}{N-1}} = 0$, hence the corrected SE is $\frac{\sigma}{\sqrt{n}} \times \sqrt{\frac{N-n}{N-1}} = 0$.

i.e. if we repeat the procedure many times (sample everybody), we get the same value each time.

i.e. the sampling distribution is just one point: the true μ.

10% Sampling Rule

Question: Why do we care that our SE is correct?

Answer: If not

- the SE in confidence intervals is off
- the z-scores of $\overline{x}$ have the wrong denominator

Skew Condition to Check to Use Normal Model

Throughout the book, they talk about the condition for $\overline{x}$ being nearly normal and using s in place of σ in $SE = \frac{\sigma}{\sqrt{n}}$:

- On page 164: the population distribution is not strongly skewed
- On page 167: the data are not strongly skewed
- On page 168: the distribution of sample observations is not strongly skewed
- On page 185: the population data are not strongly skewed

Skew Condition to Check to Use Normal Model

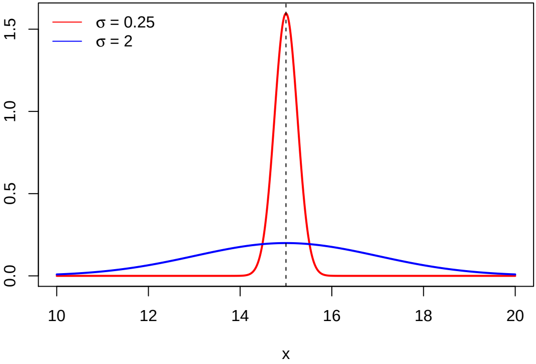
However, they all mean the same thing:

1. The true population distribution from which you are drawing your sample observations/data $x_1, \ldots, x_n$ is not too skewed.
2. The histogram (visual estimate) of the sample observations/data $x_1, \ldots, x_n$ is not too skewed.

This skew is a problem that might affect the normality of $\overline{x}$ unless n is large.

Sample Size: Thought Experiment

Say you have two distributions with $\mu = 15$ but different σ.



Which of the two distributions do you think will require a bigger *n* to estimate μ "well"?

Margin of Error

Recall our formula for a 95% confidence interval:

$$\left[\overline{x} - 1.96\frac{s}{\sqrt{n}},\ \overline{x} + 1.96\frac{s}{\sqrt{n}}\right]$$

The margin of error is half the width of the CI. Say we knew the true standard deviation σ, then

$$\text{Margin of Error: } 1.96\frac{\sigma}{\sqrt{n}}$$

Identify n for a Desired Margin of Error

To estimate the necessary sample size n for a maximum desired margin of error m, we set

$$m \geq z^* \frac{\sigma}{\sqrt{n}}$$

where z^* is the critical value chosen to correspond to the desired confidence level. Ex. for a 95% CI, $z^* = 1.96$.

Solve for n.

Identify n for a Desired Margin of Error

Since

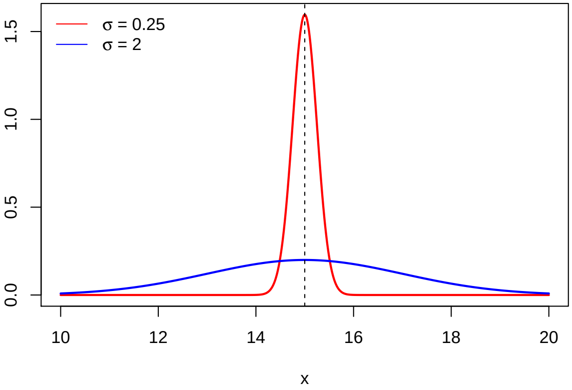
$$
\begin{aligned}
m &\geq z^* \frac{\sigma}{\sqrt{n}} \\
\sqrt{n} &\geq z^* \frac{\sigma}{m} \\
n &\geq \left(z^* \frac{\sigma}{m} \right)^2
\end{aligned}
$$

So

- As σ goes up, you need more n
- As z^* goes up, i.e. higher confidence level, you need more n
- As the desired margin of error goes up, you don't need as much n

Back to Thought Experiment

For the same desired maximal margin of error m and same confidence level, we need a larger n to estimate the mean of the blue curve:



Type II Error Rate and Power

The significance level α associated with a hypothesis test is the type I error rate: the rate at which we reject H_0 when it is true.

The type II error rate β is the rate at which we fail to reject H_0 when H_A is true. $1 - \beta$ is called the statistical power.

Statistical Power $=$ P(Rejecting H_0 when H_A is true)

Type II Error Rate and Power

Say we are conducting $N = A + B + C + D$ hypothesis tests.

		Test conclusion	
		do not reject H_0	reject H_0 in favor of H_A
Truth	H_0 true	A	B
	H_A true	C	D

- The Type I Error rate is the rate $\alpha = \frac{B}{A+B}$ at which B occurs given H_0 is true.
- The Type II Error is the rate $\beta = \frac{C}{C+D}$ at which C occurs given H_A is true.
- The power is the rate $1 - \beta = 1 - \frac{C}{C+D} = \frac{D}{C+D}$ at which D occurs given H_A is true.

Practical vs Statistical Significance

When rejecting the null hypothesis, we call this a statistically significant result. But statistically significant results aren't always practically significant.

Example: say we are comparing the average exam score of men μ_M and women μ_W. We can do a two-sample test (Chapter 5):

- $H_0 : \mu_M - \mu_F = 0$ (same average exam score)
- $H_A : \mu_M - \mu_F \neq 0$ (different average exam score)

Practical vs Statistical Significance

Say for very large n_M & n_F we observe $\overline{x}_M = 19.0002$ and $\overline{x}_F = 19.0001$.

The point estimate of $\mu_M - \mu_F$ is $\overline{x}_M - \overline{x}_F = 0.0001$. This difference is near negligible, it is still possible to "reject H_0 at an α-significance level."

However, the 95% confidence interval on the difference might look like

$$[0.00005, 0.00015]$$

Practical vs Statistical Significance

Moral of the story

- Hypothesis tests with "rejections of H_0" focus almost entirely on statistical significance.
- Confidence intervals allow you to also focus on practical significance.