Lecture 11: Binomial and Poisson Random Variables

Chapter 3.3-3.5

Goals for Today

Define

- Binomial random variables
- Poisson random variables

Binomial Distribution

Say instead of $P(\text{1st W in 5th game})$, we want the probability that they win exactly one out of the five games. Five ways:

Pattern	Probability	Equals
WLLLL	$p \times (1-p)^4$	$= p \times (1-p)^4$
LWLLL	$(1-p) \times p \times (1-p)^3$	$= p \times (1-p)^4$
LLWLL	$(1-p)^2 \times p \times (1-p)^2$	$= p \times (1-p)^4$
LLLWL	$(1-p)^3 \times p \times (1-p)$	$= p \times (1-p)^4$
LLLLW	$(1-p)^4 \times p$	$= p \times (1-p)^4$

Binomial Distribution

Each pattern has the same probability regardless of order by independence and there are 5 ways to choose the pattern. So

$$
\begin{aligned}
P(\text{exactly one win out of five}) &= 5 \times p \times (1-p)^4 \\
&= 5 \times 0.4^4 \times 0.6 = 0.0768
\end{aligned}
$$

Step Back... Example of n choose x

Say I give you $n = 3$ balls labeled 1 thru 3. How many different ways can you choose $x = 2$ of them? 3 ways:

$$(1, 2), \; (1, 3), \text{ and } \; (2, 3)$$

Step Back... n choose x in General

Say I give you n balls labeled 1 thru n. How many different ways can you choose x of them? This is read n choose x:

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

In example: $n = 3$ and $x = 2$

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3 \times 2 \times 1}{(2 \times 1)(1)} = \frac{6}{2} = 3$$

Note that $0! = 1$

Binomial Distribution

Suppose the probability of a single trial being a success is p. Let X be the random number of successes observed in n independent trials. The probability of observing x successes is:

$$\begin{aligned} P(X = x) &= \binom{n}{x} p^x (1-p)^{n-x} \\ &= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \end{aligned}$$

The mean, variance, and SD are:

$$\mu = np \qquad \sigma^2 = np(1-p) \qquad \sigma = \sqrt{np(1-p)}$$

Conditions for Binomial Distribution

1. The trials are independent.
2. The number of trials n is fixed
3. Each trial outcome can be classified as a failure or a success
4. The probability of a success p is the same for each trial

Back to Soccer Example

Probability of exactly one win?

Pattern	Probability	Equals
WLLLL	$p \times (1-p)^4$	$= p \times (1-p)^4$
LWLLL	$(1-p) \times p \times (1-p)^3$	$= p \times (1-p)^4$
LLWLL	$(1-p)^2 \times p \times (1-p)^2$	$= p \times (1-p)^4$
LLLWL	$(1-p)^3 \times p \times (1-p)$	$= p \times (1-p)^4$
LLLLW	$(1-p)^4 \times p$	$= p \times (1-p)^4$

Letting a win be a "success":

$$
\begin{aligned}
P(X=1) &= \binom{n}{x} p^x (1-p)^{n-x} = \frac{5!}{1! \times 4!} 0.6 \times 0.4^4 \\
&= 5 \times 0.6 \times 0.4^4 = 0.0768
\end{aligned}
$$

Back to Soccer Example

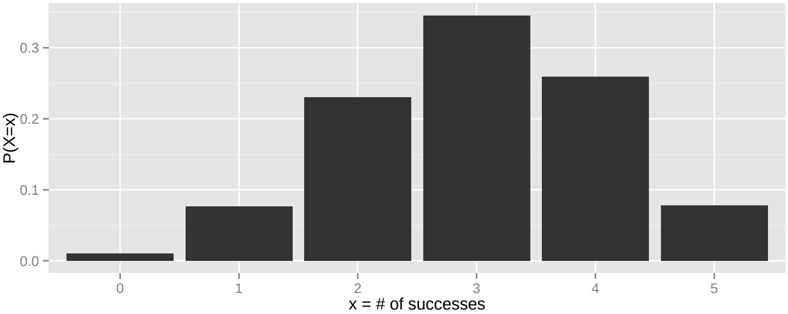
What is the probability that they win all their games! i.e. $X = 5$:

$$
\begin{aligned}
P(X = 5) &= \binom{n}{x} p^x (1-p)^{n-x} = \binom{5}{5} 0.6^5 (1-0.6)^0 \\
&= \frac{5!}{5! \times 0!} 0.6^5 \times 1 = 0.08
\end{aligned}
$$

What is the probability that they at win at least one game?

$$
\begin{aligned}
P(X \geq 1) &= P(X = 1) + \ldots + P(X = 5) \\
&= 1 - P(X = 0) \\
&= 1 - \frac{5!}{0! \times 5!} 0.6^0 \times 0.4^5 = 1 - 0.01024 \\
&= 0.98976
\end{aligned}
$$

Back to Soccer Example



Poisson Distribution

Say you want to count the number of rare events in a large population over a unit of time. Ex:

- # of car accidents at an intersection on a given week
- # of ambulance calls on any given day in Portland
- # of soldiers in the Prussian army killed accidentally by horse kick from 1875 to 1894

The Poisson distribution helps us model such counts.

Poisson Distribution

A random number X of the count of the number of events follows a Poisson distribution with rate λ

$$P(\text{we observe x rare events}) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where x may take a value 0, 1, 2, ... where $e \approx 2.718$.

The mean and SD are λ and $\sqrt{\lambda}$.

Conditions for Poisson Distribution

A random variable may be Poisson distributed if

1. The event in question is rare
2. The population is large
3. The events occur independently of each other

Exercise 3.47 on Page 158

A coffee shop serves an average of 75 customers per hour during the morning rush. Let X be the (random) number of customers that the coffee shop serves in one hour at this time of the day.

What is the probability $X = 70$?

Exercise 3.47 on Page 158

In this case, $\lambda = 75$ is the rate

$$P(X = 70) = \frac{75^{70} e^{-75}}{70!} = 0.040$$

We can do this in R via `dpois(x=70, lambda=75)` in `R`

Next Time

Chapter 4: Foundations for Inference

- Variability in estimates $\overline{x}$, $\widehat{p}$, etc.
- In fact, we can associate a distribution to these estimates