Lecture 8: Normal Distribution

Chapter 3.1

Goals for Today

- Define the normal distribution in terms of its parameters
- Review: $\frac{2}{3}$ / 95% / 99.7% rule
- Standardizing normal observations to z-scores

Normal Distribution

Normal distributions:

1. are symmetric
2. are unimodal and bell-shaped
3. have area under the curve 1

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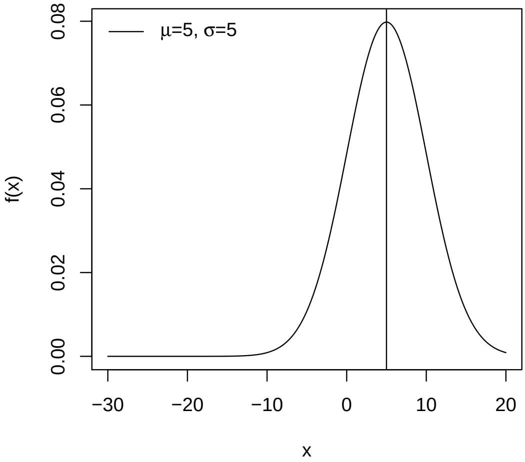
Recall these were the population mean and the population SD.

Normal Distribution

μ (mean) specifies the center, σ (standard deviation) the spread.



Normal Example

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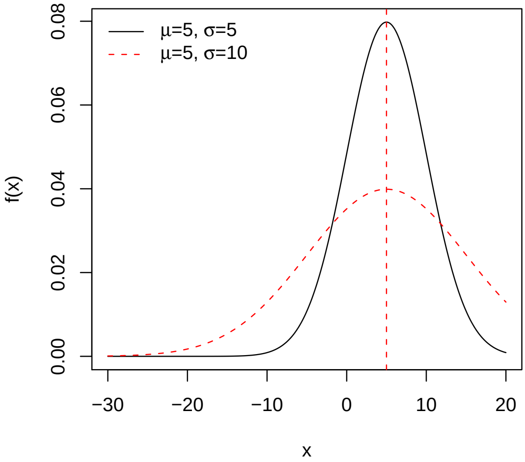

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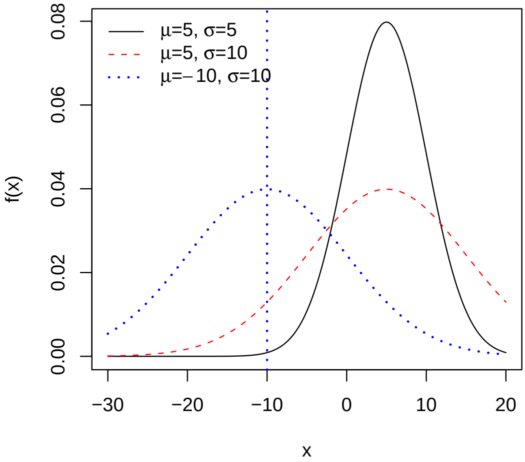
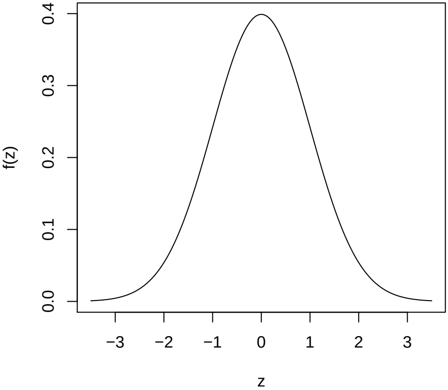


Standardized Normal Distribution

If $\mu = 0$ and $\sigma = 1$, this is the standard normal distribution:



Rules of Thumb

Recall if a distribution is normal, then:

1. Approx. $\frac{2}{3}$'s of the data are within ± 1 SD of the mean
2. Approx. 95% of the data are within ± 2 SD of the mean

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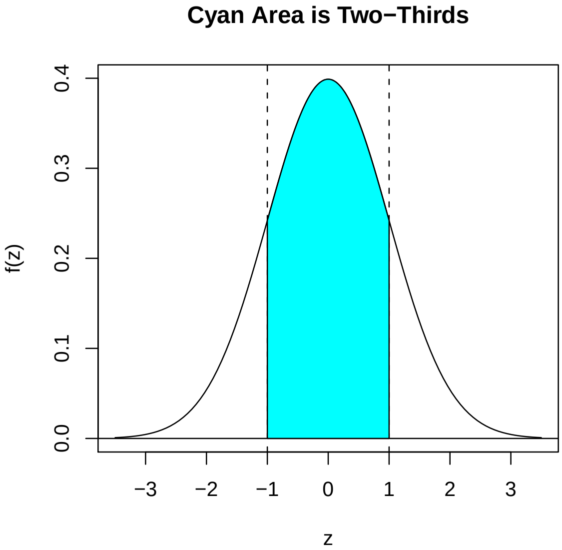
1. Approx. $\frac{2}{3}$'s of the data are within ± 1 SD of the mean
2. Approx. 95% of the data are within ± 2 SD of the mean
3. Also approx. 99.7% of the data are within ± 3 SD of the mean

Ex: Standard Normal $\mu = 0$, $\sigma = 1$



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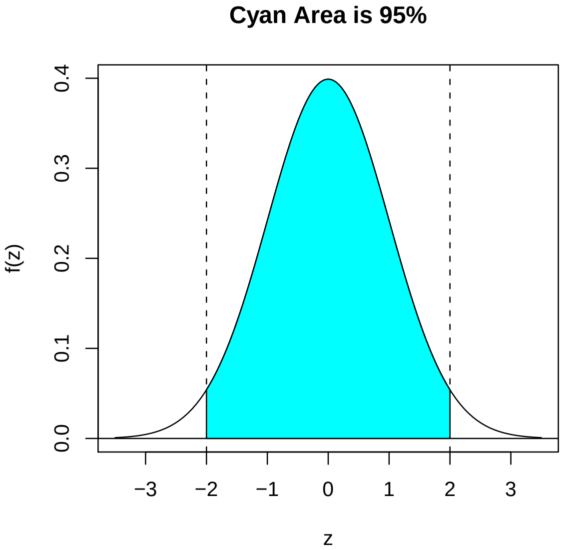

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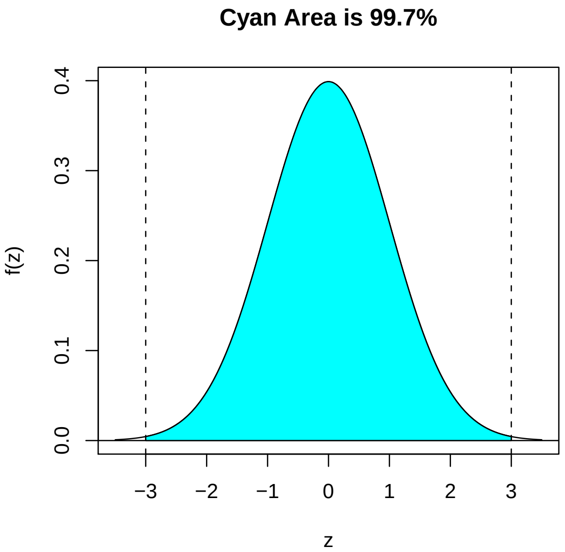

Motivating Example

From text: Say Ann scores 1800 on the SAT and Tom scores 24 on the ACT.

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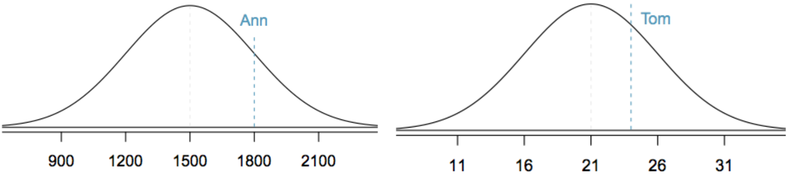
	SAT	ACT
Mean μ	1500	21
SD σ	300	5

Question: Who did relatively better?



z-scores

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The z-score for an observation x that follows a distribution with mean μ and SD σ:

$$z = \frac{x - \mu}{\sigma}$$

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   re-scale the spread of the $x - \mu$ values to be 1 by dividing by σ.

So we can compare observations from any normally distributed data with (μ, σ)

i.e. we've standardized the observations to make them comparable.

Back to Example

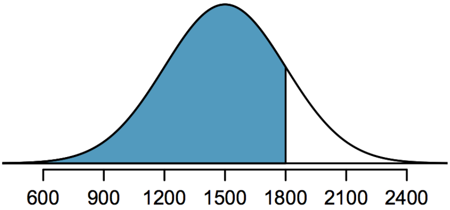
- Ann scored 1800. $z = \frac{1800-1500}{300} = +1$ standard deviation from the mean
- Tom scored 24. $z = \frac{24-21}{5} = +0.6$ standard deviation from the mean

So Ann did relatively better.

Percentiles

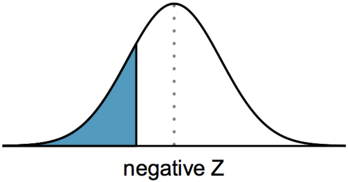
Recall a percentile (%'ile) indicates the value below which a given %'age of observations fall below.

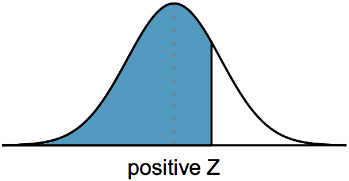
Question: What %'ile is Ann's SAT score of 1800?
i.e. what is the blue shaded area?



Percentiles

Because the total area under the curve is 1, the area to the left of z represents the %'ile of the observation:





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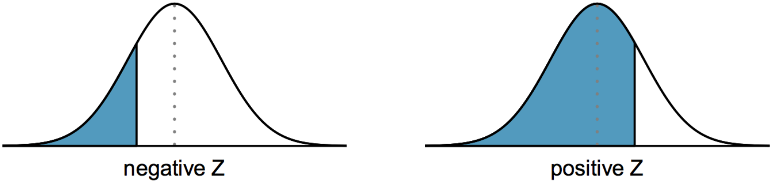

- The blue shaded area on the left plot will be less than 0.5. We have %'iles less than the 50th %'ile.

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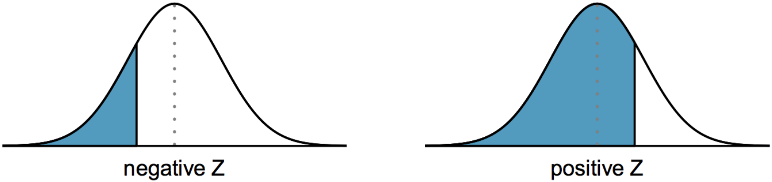

- The blue shaded area on the left plot will be less than 0.5. We have %'iles less than the 50th %'ile.
- The blue shaded area on the right plot will be greater than 0.5. We have %'iles greater than the 50th %'ile.

Normal Probability Table

A normal probability table allows you to:

- identify the %'ile corresponding to a z-score
- or vice versa: the z-score corresponding to a %'ile

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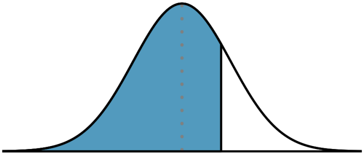
The normal probability tables on page 409 represent z-scores and %'iles corresponding to area to the left:



Normal Probability Table

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

- Red case: Given a z-score of 0.43. A lookup tells us the area to the left of z=0.43 is 0.6664, i.e. the 66th %'ile

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- Red case: Given a z-score of 0.43. A lookup tells us the area to the left of z=0.43 is 0.6664, i.e. the 66th %'ile
- Blue case: We want the z-score that is the 80th %'ile. Reverse lookup: the closest value on the table is 0.7995, i.e. a z-score of 0.84.

Back to Ann and Tom

- Since Ann had a z-score of 1.0, her %'ile is 0.8413. (1.0 row, 0.00 column)
  i.e. She did better than 84.13% of SAT test takers.

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  i.e. She did better than 84.13% of SAT test takers.
- Since Tom had a z-score of 0.6, his %'ile is 0.7257. (0.6 row, 0.00 column)
  i.e. He did better than 72.57% of ACT test takers

Next Time

Next time we will:

- Re-iterate the motivation for the normal curve.
- Go over examples using z-scores.
- Evaluating the normal approximation.