Lecture 23: Tests for Independence in Two-Way Tables

Chapter 6.4

Previously... Chi-Square Tests

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Chi-square χ^2 tests allow us compare observed counts to expected counts.

We compute a χ^2 statistic, and then compare it to the χ^2 distribution to compute *p*-values.

$$\chi^2 = \frac{(\mathsf{obs}_1 - \mathsf{exp}_1)^2}{\mathsf{exp}_1} + \ldots + \frac{(\mathsf{obs}_k - \mathsf{exp}_k)^2}{\mathsf{exp}_k}$$

Previously... Assumptions for Chi-Square Test

1. Independence: Each case is independent of the each other
2. Sample size/distribution: We need at least 5 cases in each scenario i.e. each cell in the table
3. Degrees of freedom: We need at least $df = 2$, i.e. $k \geq 3$

Today's Example

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1. The current version
2. test 1: test algorithm that boosts the search rank of sites with pictures of cats
3. test 2: test algorithm that boosts the search rank of sites with pictures of dogs

Today's Example

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Among the many metrics for user satisfaction with the results for a particular query (search term) is the `new search` variable:

- No new search: User clicked on a result. User is likely satisfied with result.
- New search: User did not click on a result and tried a new related search. User is likely unsatisfied with result.

Today's Example

So we have two categorical variables:

- Categorial variable `algorithm` (current, test 1, and test 2)
- Binary Categorial variable `new search` (yes/no)

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So we have two categorical variables:

- Categorial variable `algorithm` (current, test 1, and test 2)
- Binary Categorial variable `new search` (yes/no)

Are they independent? i.e. for different algorithms, do we have different levels of new search?

Today's Example

Let's select queries to evaluate each algorithm and organize them in a contingency table:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search				
New search				
Total	5000	2500	2500	10000

Today's Example

Say we observed the following results:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	4000	2000	2000	8000
New search	1000	500	500	2000
Total	5000	2500	2500	10000

Today's Example

Say we observed the following results:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	4000	2000	2000	8000
New search	1000	500	500	2000
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We observe that for all 3 algorithms, there is no new search $\frac{4}{5}$ of the time.

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We observe that for all 3 algorithms, there is no new search $\frac{4}{5}$ of the time.

In this case, `algorithm` and `new search` are independent: regardless of which `algorithm` used, the proportion of new searches stays the same.

Today's Example

Now say instead we observed the following results:

	algorithm			
new search	Current	Test 1	Test 2	Total
No new search	4000	2500	1500	8000
New search	1000	0	1000	2000
Total	5000	2500	2500	10000

Today's Example

Now say instead we observed the following results:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	4000	2500	1500	8000
New search	1000	0	1000	2000
Total	5000	2500	2500	10000

In this case, `algorithm` and `new search` are not independent: depending on which `algorithm` used, the proportion of new searches is different.

Hypothesis Test

How now test at the $\alpha = 0.05$ significance level:

$$\begin{aligned} H_0: &\quad \text{The algorithms each perform equally well} \\ \text{vs } H_A: &\quad \text{The algorithms do not perform equally well} \end{aligned}$$

i.e. are the categorial variables `algorithm` and `new search` independent?

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i.e. are the categorial variables `algorithm` and `new search` independent?

We can do this via χ^2 tests: comparing observed vs expected counts.

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The following all refer to the same test:

- χ^2 test for two-way tables
- χ^2 test for independence of two categorical variables
- χ^2 test for homogeneity: are the algorithms homogeneous in their performance?
- χ^2 test for contingency tables

Example from Textbook

Let's make the values match the example from the textbook:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	3511	1749	1818	7078
New search	1489	751	682	2922
Total	5000	2500	2500	10000

Example from Textbook

Before we start, let's make each column reflect a proportion and not a count.

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	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	0.7022	0.6996	0.7272	0.7078
New search	0.2978	0.3004	0.2728	0.2922
Total	1	1	1	1

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If all algorithms performed the same, we'd expect

- 0.7078 for all 3 values in the top row
- 0.2922 for all 3 values in the bottom row

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Total	1	1	1	1

If all algorithms performed the same, we'd expect

- 0.7078 for all 3 values in the top row
- 0.2922 for all 3 values in the bottom row

The question is: what is the degree of this deviation?

χ^2 Statistic

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As just shown, the overall proportions were 0.7078 and 0.2978.

i.e. under the null hypothesis, we expect these proportions to hold for all algorithms.

What's Expected

We expect:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search				$7078 = 0.7078 \times 10000$
New search				$2922 = 0.2922 \times 10000$
Total	5000	2500	2500	10000

What's Expected

We expect:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search			$1769.5 = 0.7078 \times 2500$	7078
New search			$730.5 = 0.2922 \times 2500$	2922
Total	5000	2500	2500	10000

What's Expected

We expect:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search		$1769.5 = 0.7078 \times 2500$	1769.5	7078
New search		$730.5 = 0.2922 \times 2500$	730.5	2922
Total	5000	2500	2500	10000

What's Expected

We expect:

new search	algorithm Current	Test 1	Test 2	Total
No new search	$3539 = 0.7078 \times 5000$	1769.5	1769.5	7078
New search	$1461 = 0.2922 \times 5000$	730.5	730.5	2922
Total	5000	2500	2500	10000

Observed vs. Expected

We expect:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	3539	1769.5	1769.5	7078
New search	1461	730.5	730.5	2922
Total	5000	2500	2500	10000

Observed vs. Expected

We expect:

`new search`	algorithm Current	Test 1	Test 2	Total
No new search	3539	1769.5	1769.5	7078
New search	1461	730.5	730.5	2922
Total	5000	2500	2500	10000

We observed:

`new search`	algorithm Current	Test 1	Test 2	Total
No new search	3511	1749	1818	7078
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Computing Expected Counts in a Two-Way Table

In effect, we compute the expected count for the i^{th} row and j^{th} column via:

$$\text{Expected Count for Row } i \text{, Col } j = \frac{(\text{Row } i \text{ Total}) \times (\text{Column } j \text{ Total})}{\text{Table Total}}$$

Chi-Square Statistic

We compute χ^2 test statistic: for all $i = 1, \ldots, 6$ cells

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$$\begin{aligned}
\text{Row 1, Col 1} &= \frac{(3511 - 3539)^2}{3539} = 0.222 \\
\vdots \quad & \quad \vdots \\
\text{Row 2, Col 3} &= \frac{(682 - 730.5)^2}{730.5} = 3.220
\end{aligned}$$

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\vdots \quad & \quad \vdots \\
\text{Row 2, Col 3} \quad &= \quad \frac{(682 - 730.5)^2}{730.5} = 3.220
\end{aligned}$$

So

$$\begin{aligned}
\chi^2 \quad &= \quad 0.222 + 0.237 + \ldots + 3.220 \\
&= \quad 6.120
\end{aligned}$$

Chi-Square Distribution

We compare this to a χ^2 distribution to get the p-value. What are the degrees of freedom?

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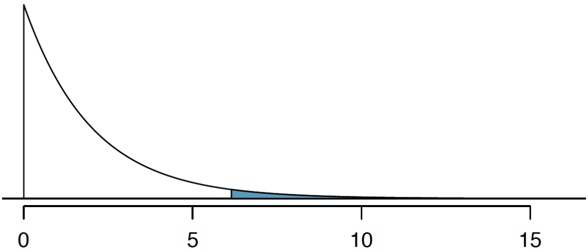
$$
\begin{aligned}
df &= (\#\text{ of rows - 1}) \times (\#\text{ of columns - 1}) \\
&= (R-1) \times (C-1) \\
&= (2-1) \times (3-1) = 2 \text{ in our case}
\end{aligned}
$$

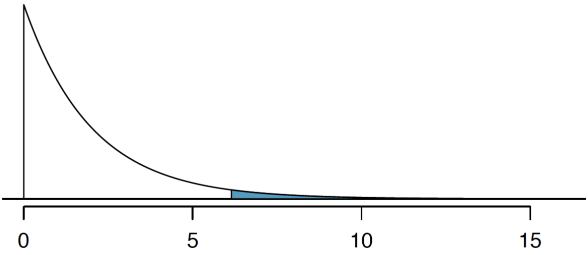
Chi-Square Distribution

Looking up 6.120 in the χ^2 table on page 412 on the *df* $= 2$ row, it would be between 0.05 and 0.02. Since our $\alpha = 0.05$, we reject the null hypothesis and accept the alternative that the algorithms do not perform equally well.



Chi-Square Distribution

Looking up 6.120 in the χ^2 table on page 412 on the $df = 2$ row, it would be between 0.05 and 0.02. Since our $\alpha = 0.05$, we reject the null hypothesis and accept the alternative that the algorithms do not perform equally well.



We can also compute the p-value exactly in R by typing `pchisq(6.120, df=2, lower.tail=FALSE)`. It was 0.047.

Conditions/Assumptions

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Conditions/Assumptions

Nearly identical to conditions/assumptions for χ^2 tests for goodness-of-fit:

1. Independence: Each case is independent of the other
2. Sample size/distribution: We need at least 5 cases in each scenario i.e. each cell in the table
3. Degrees of freedom: (Different than before) We need $df = (R-1) \times (C-1) \geq 2$.

Why Are They Called Degrees of Freedom?

In the case of χ^2 tests, the degrees of freedom is the number of values needed before you specify all values in the cells of the table.

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	algorithm			
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Total	5000	2500	2500	10000

then the missing value is $7078 - X - Y$.
i.e. the wiggle room we have is $C - 1$ two cells

Why Are They Called Degrees of Freedom? Columns

Each column has 1 degree of freedom because once we've specified 1 value, all values in the column are specified.

Say in the 1st column, we specify one (arbitrarily chosen) value:

	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	X			7078
New search				2922
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	algorithm			
`new search`	Current	Test 1	Test 2	Total
No new search	X			7078
New search				2922
Total	5000	2500	2500	10000

then the missing value is $5000 - X$.
i.e. the wiggle room we have is $R - 1$ one cell

Why Are They Called Degrees of Freedom? Columns

So the overall *df* is the row degree of freedom times the column degree of freedom $(C - 1) \times (R - 1)$, in our case $df = 2$.

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	algorithm			
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No new search	X	Y		7078
New search				2922
Total	5000	2500	2500	10000

i.e. if we know these two values, we can fill the rest of the table.

Next Lecture

We kick off Chapter 7: Introduction to Linear Regression.