HW2 Solutions

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۱ بخش مقدماتی (۳۵ نمره)

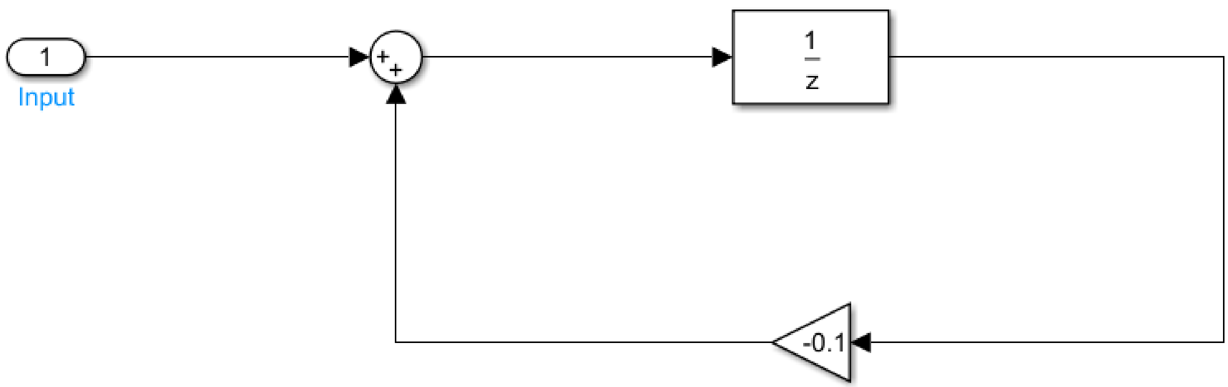
Disclaimer:
This is the solution manual to the homework assigned to students of Digital Control - Dr.Talebi.
We do not guarantee that this solution is precise and thorough so please contact your TA to propose your innovative solutions and/or any probable mistakes.

سوال اول

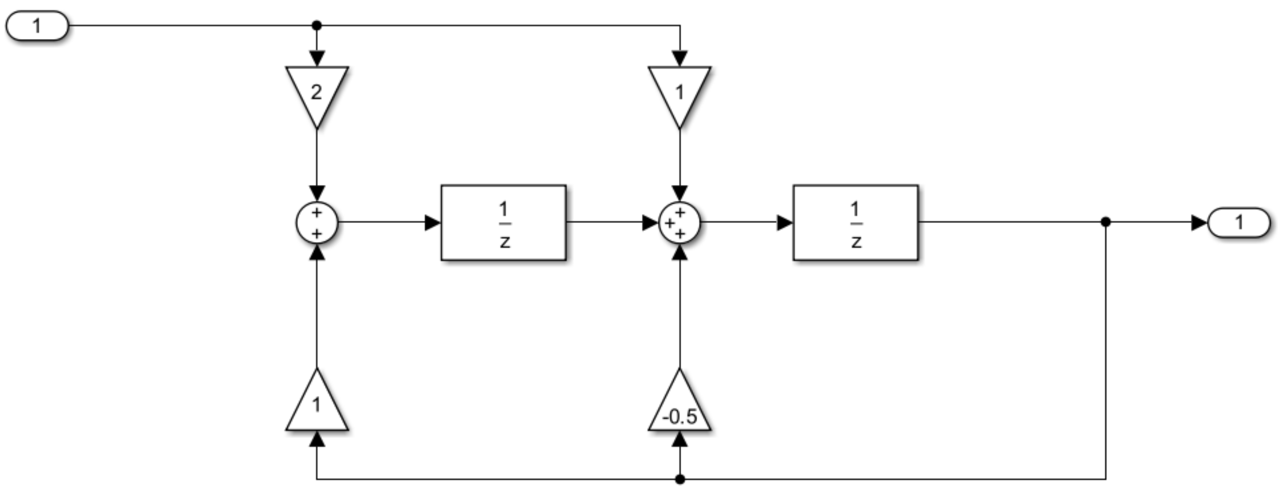
Solution:
$\frac{X(z)}{z} = \frac{2z^2+1}{(z-2)^2(z-1)} = \frac{9}{(z-2)^2} - \frac{1}{z-2} + \frac{3}{z-1}$
then
$X(z) = \frac{9z^{-1}}{(1-2z^{-1})^2} - \frac{1}{1-2z^{-1}} + \frac{3}{1-z^{-1}}$
: At last we have
$x(k) = 9k(2^{k-1}) - 2^k + 3$

سوال دوم

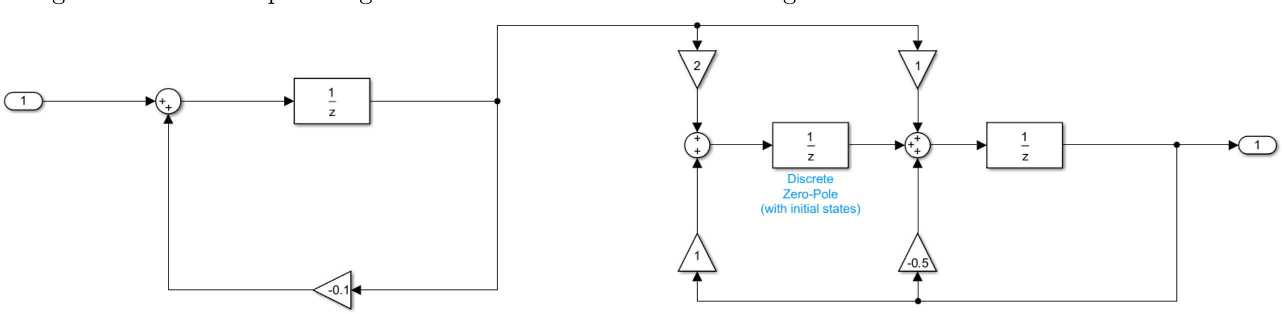
Firstly : Decompose the Transfer function to the following :
$G(z) = \frac{1}{z-0.1} \frac{z-2}{z^2-0.5z+1}$
Realize the first part:
$G_{FP} = \frac{z^{-1}}{1-0.1z^{-1}}$



Now Realize the second part:
$G_{SP} = \frac{z^{-1}-2z^{-2}}{1-0.5z-1+z-2}$
Use Observable Canonical :

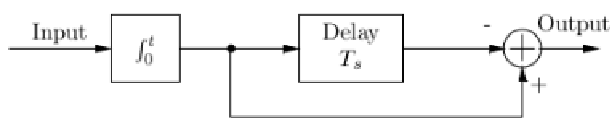


Bring the two realized parts together to have the realized block diagram.



سوال سوم

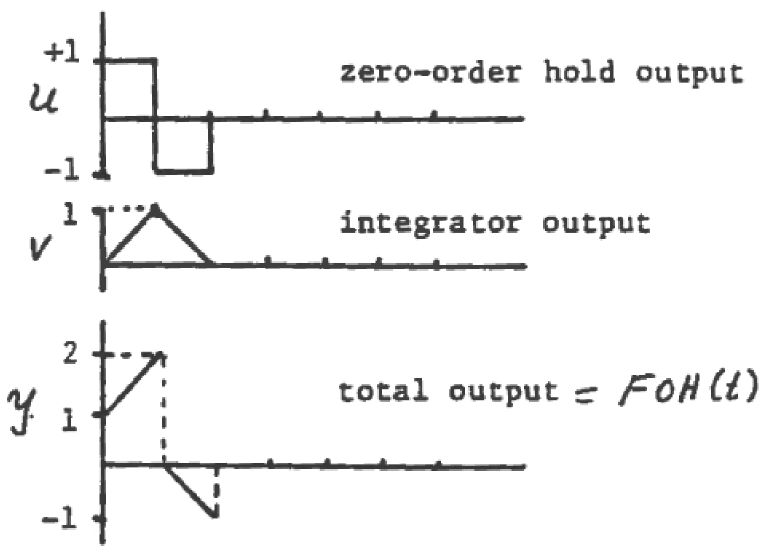
$-\frac{1}{s}U(s)e^{-sT} + \frac{1}{s}U(s) = Y(s)$
The transfer function would be:
$G(s) = \frac{1-e^{-sT}}{s}$
This is the ZOH transfer function.

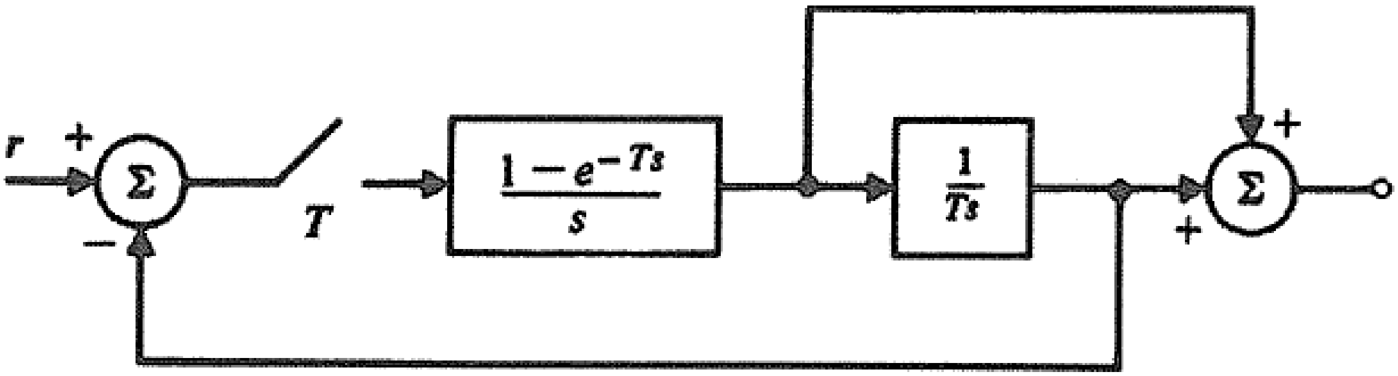


شکل ۱: شکل سوال سوم

سوال چهارم

The key here is to compute the signal out of the zero-order hold and, from that, the output of the integrator. The signals are





شکل ۲: شکل سوال چهارم

سوال پنجم

Based on what we have seen in Section 3.4 of Digital control systems by C.L.Phillips (2015) we have:

$$E^*(s) = \sum_{\substack{at\,poles\\ of\,E(\lambda)}} [residues\,of\,E(\lambda)\frac{1}{1-\varepsilon^{-T(s-\lambda)}}] \tag{۱}$$

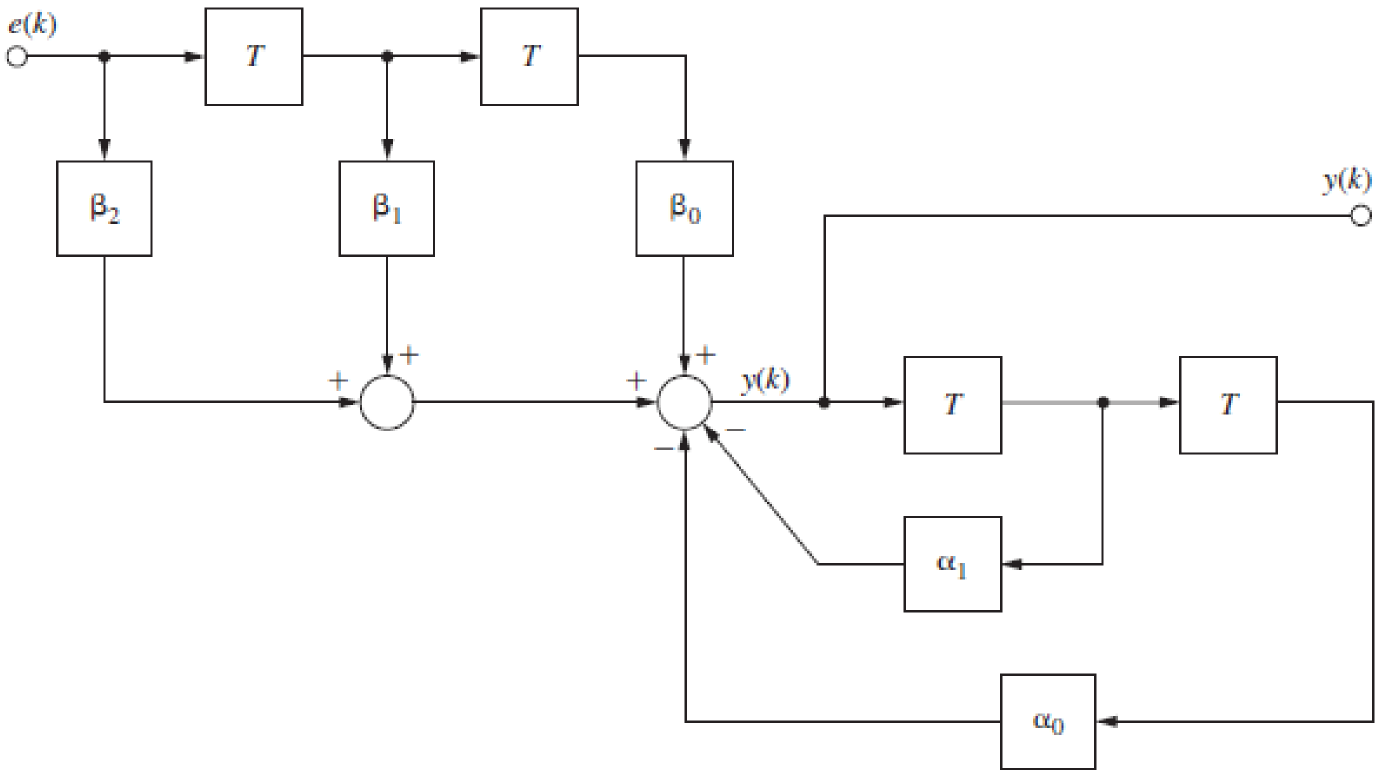
The Starred transform would be as noted bellow:
$\frac{2}{1-e^{-Ts}} + \frac{-1}{1-e^{T(s+1)}}$

۲ بخش متوسط (۳۵ نمره)

حل دو سوال از این بخش الزامی است.

سوال ششم

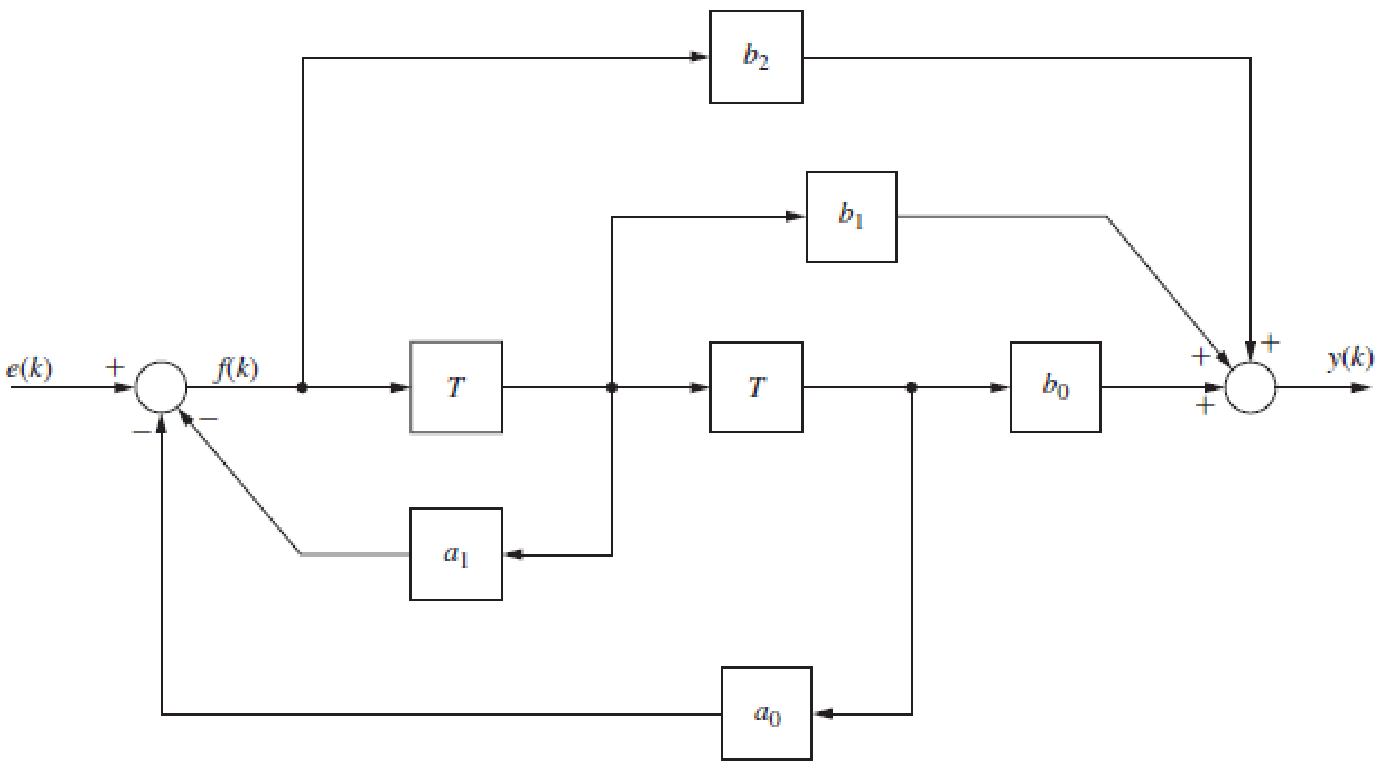
For the first part you can substitute k in the given range to calculate what is asked.
Second Part:
recall that we had :
$y(k+2) = Z^2Y(z) - Zy(1) - y(0)$
now take z transform from the difference equation:
$Z^2Y(z) - \frac{3}{4}ZY(z) + \frac{1}{8}Y(z) = E(z)$
$(Z^2 - \frac{3}{4}Z + \frac{1}{8})Y(z) = E(z)$
$\frac{Y(z)}{E(z)} = \frac{1}{Z^2-\frac{3}{4}Z+\frac{1}{8}}$
The transfer function is unstable.



شکل ۳: شکل سوال هفتم

سوال هفتم

Based on the block diagram we have:
$y(k) = \beta_2 e(k) + \beta_1 e(k-1) + \beta_0 e(k-2) - \alpha_1 y(k-1) - \alpha_0 y(k-2)$
Take z transform:
$(1 + \alpha_1 Z^{-1} + \alpha_0 Z^{-2})Y(z) = (\beta_2 + \beta_1 Z^{-1}\beta_0 Z^{-2})E(z)$
Find the transfer function:
$\frac{Y(z)}{E(z)} = \frac{(\beta_2+\beta_1 Z^{-1}+\beta_0 Z^{-2})}{(1+\alpha_1 Z^{-1}+\alpha_0 Z^{-2})}$
Which would be equal to :
$\frac{Y(z)}{E(z)} = \frac{(\beta_2 Z^2+\beta_1 Z+\beta_0)}{(Z^2+\alpha_1 Z+\alpha_0)}$
So we would have:
$\beta_2 = 2, \beta_1 = -2.4, \beta_0 = 0.72$
$\alpha_1 = -1.4, \alpha_0 = 0.98$



شکل ۴: شکل سوال هشتم

سوال هشتم

The block diagram represents the following difference equation:
$f(k) = -\alpha_1 f(k-1) - \alpha_0 f(k-2) + e(k)$
$y(k) = b_2 f(k) + b_1 f(k-1) + b_0 f(k-2)$
Now take the z-transform:
$F(z) = (-\alpha_1 Z^{-1} + -\alpha_0 Z^{-2})F(z) + E(z)$
$Y(z) = (b_2 + b_1 Z^{-1} + b_0 Z^{-2})F(z)$
Now derive the transfer function:
$\frac{F(z)}{E(z)} = \frac{1}{1+\alpha_1 Z^{-1}+\alpha_0 Z^{-2}}$
$\frac{Y(z)}{F(z)} = (b_2 + b_1 Z^{-1} + b_0 Z^{-2})$
$\frac{Y(z)}{E(z)} = \frac{b_2+b_1 Z^{-1}+b_0 Z^{-2}}{1+\alpha_1 Z^{-1}+\alpha_0 Z^{-2}}$
Which is equivalent to :
$\frac{Y(z)}{E(z)} = \frac{b_2 Z^2+b_1 Z+b_0}{Z^2+\alpha_1 Z+\alpha_0}$
$b_2 = 2, b_1 = -2.4, b_0 = 0.72$
$\alpha_1 = -1.4, \alpha_0 = 0.98$

سوال نهم

Recall the following equation:
$G(z) = (1 - z^{-1})\, Z\{\frac{G(s)}{s}\}$
using the equation we have the following ZOH-equavalent transfer function:
$\frac{1}{2} + \frac{-e(1-Z^{-1})}{1-e^{-T}Z^{-1}} + \frac{\frac{1}{2}e^{2}(1-Z^{-1})}{1-e^{-2T}Z^{-1}}$
Where T must be substituted to 0.5

سوال دهم

recall that :

$$\cos\omega n \overset{*}{\longleftrightarrow} [\frac{e^{sT}(e^{sT} - \cos\omega)}{e^{2sT} - 2e^{sT}\cos\omega + 1}] \tag{۲}$$

now calculate it for the two given functions
1)$[\frac{e^{sT}(e^{sT} - \cos 4\pi T)}{e^{2sT} - 2e^{sT}\cos 4\pi T + 1}]$
2)$[\frac{e^{sT}(e^{sT} - \cos 16\pi T)}{e^{2sT} - 2e^{sT}\cos 16\pi T + 1}]$
It is now evident why the starred transform are the same.

۳ بخش تکمیلی (۳۰ نمره)

حل دو سوال از این بخش الزامی است.

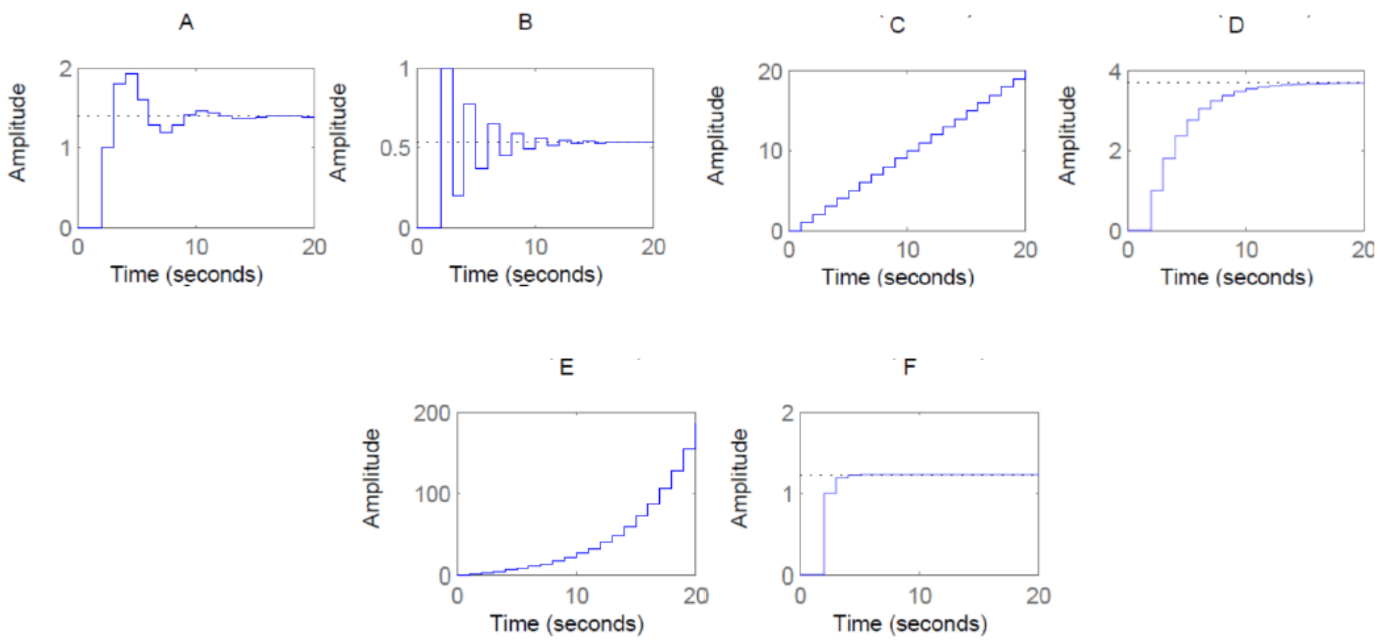
سوال یازدهم

The main idea of solving this kind of questions is paying attention to the stability and Final Value Theorem. Note that the given figures are the step response of the transfer functions. Meaning that you take into account a $\frac{1}{1-Z^{-1}}$ term multiplied by the transfer function.
Based on what is said we now begin to identify the unstable outputs.
• G_4 is unstable with having repeated poles on z = 1. This is C.
• G_6 is unstable for having a poles on z = 1.2. This is E.
The rest are stable.
•Poles of G_1 are -0.1 and -0.7. We know that negative poles cause the step response to be oscillatory so both A and B could be the answer. In order to determine which the answer is, we use final value theorem which equals to 0.534759 this closer to B.
•Poles of G_2 are 0.1 and 0.7. Both F and D could be the answer. In order to determine exactly which the answer is, we use final value theorem which equals to 3.70 this closer to D.
•Poles of G_3 are 0.1 and 0.1. Both F and D could be the answer. In order to determine exactly which the answer is, we use final value theorem which equals to 1.23 this closer to F.
•Poles of G_5 are 0.4 - 0.6i and 0.4 + 0.6i. It is in the stable region. In order to determine exactly which the answer is, we use final value theorem which equals to 1.38 this is A.

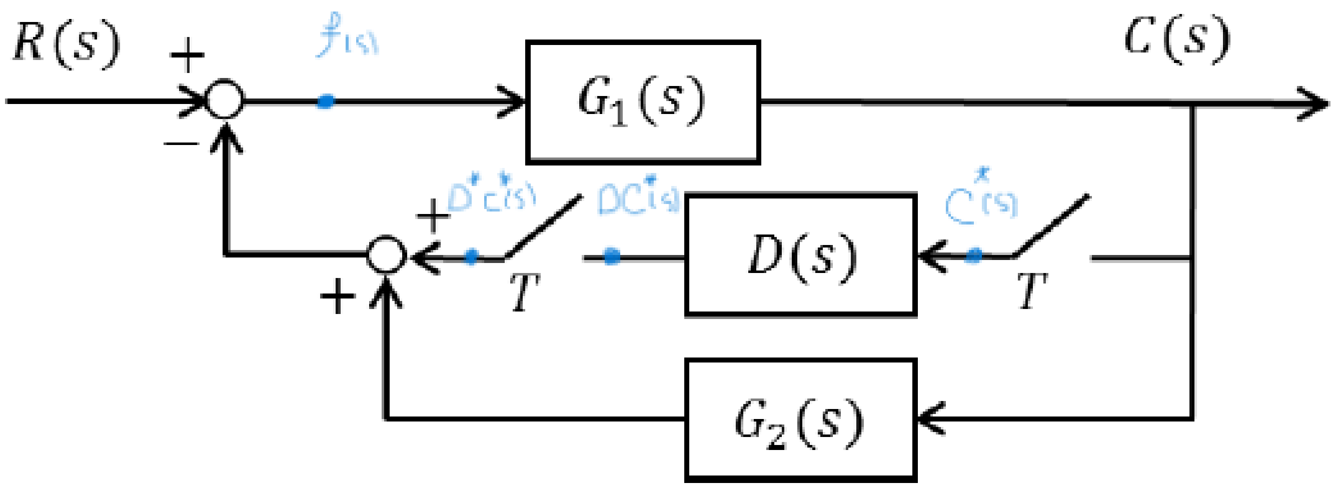
$$G_{۱}(z) = \frac{۱}{z^{۲} + ۰.۸\square + ۰.۰۷} \qquad G_{۴}(z) = \frac{۱}{z - ۱}$$

$$G_{۲}(z) = \frac{۱}{z^{۲} - ۰.۸\square + ۰.۰۷} \qquad G_{۵} = \frac{۱}{z^{۲} - ۰.۸\square + ۰.۵۲}$$

$$G_{۳}(z) = \frac{۱}{z^{۲} - ۰.۲\square + ۰.۰۱} \qquad G_{۶}(z) = \frac{۱}{z - ۱.۲}$$



شکل ۵: شکل سوال یازدهم



شکل ۶: شکل سوال دوازدهم

سوال دوازدهم

As you can see in the picture below we could write:

$C(s) = G_1(s)f(s)$ *

$f(s) = R(s) - D^*(s)C^*(s) - G_2(s)C(s)$

$C(s) = \frac{R(s)G_1(s)}{1+G_1(s)G_2(s)} - \frac{G_1D^*(s)C^*(s)}{1+G_1(s)G_2(s)}$ **

$C^*(s) = (\frac{R(s)G_1(s)}{1+G_1(s)G_2(s)})^* - (\frac{G_1}{1+G_2})^* D^*(s)C^*(s)$

$C^*(s)(1 + (\frac{G_1}{1+G_2})^* D^*(s)) = (\frac{R(s)G_1(s)}{1+G_1(s)G_2(s)})^*$

$C^*(s) = \frac{(\frac{R(s)G_1(s)}{1+G_1(s)G_2(s)})^*}{(1+(\frac{G_1}{1+G_2})^* D^*(s))}$

Now use the equation derived above inside the ** equation and then use it in the * equation to get the following:

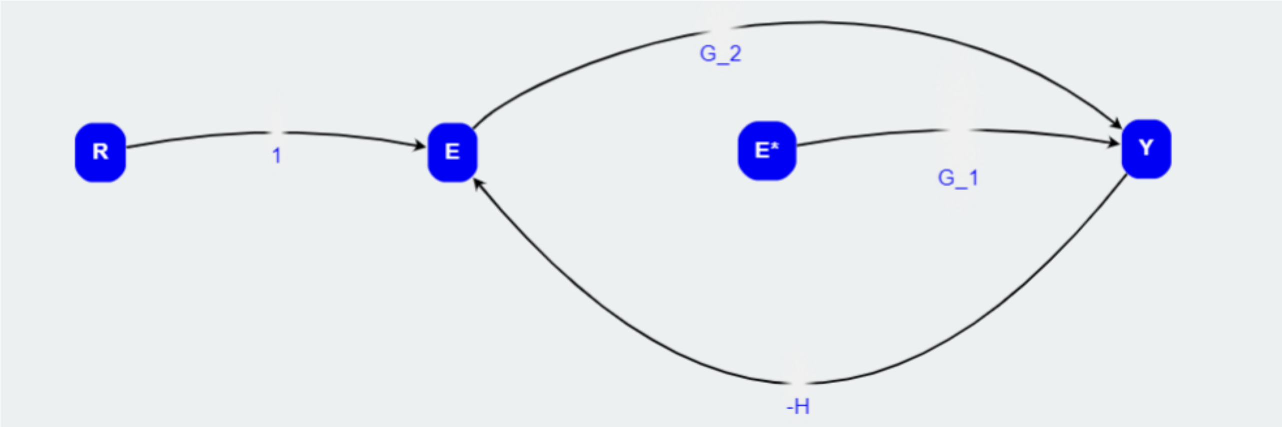
$C(s) = \frac{R(s)G_1(s)}{1+G_1(s)G_2(s)} - \frac{G_1D^*(s)(\frac{(\frac{R(s)G_1(s)}{1+G_1(s)G_2(s)})^*}{(1+(\frac{G_1}{1+G_2})^* D^*(s))})}{1+G_1(s)G_2(s)}$

Similarly:

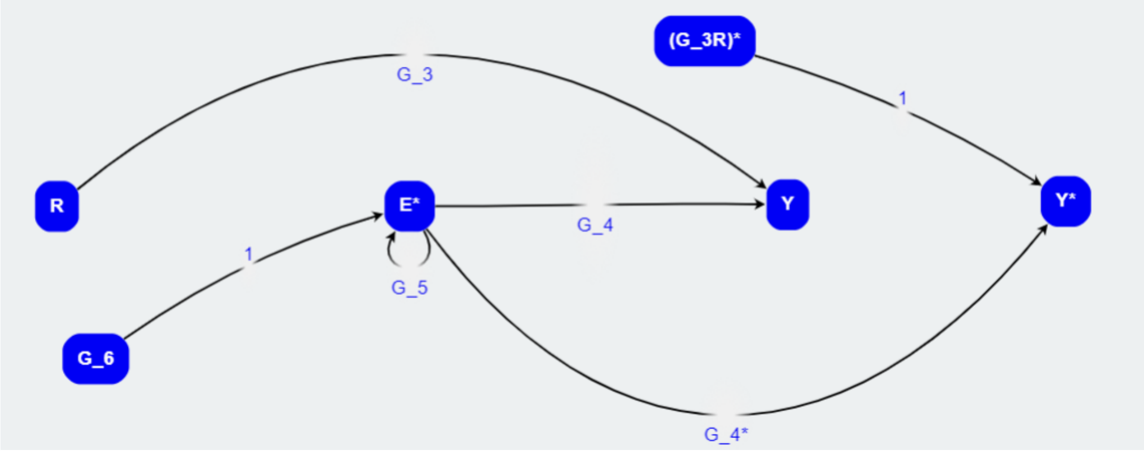
$C(z) = \frac{R(z)G_1(z)}{1+G_1(z)G_2(z)} - \frac{G_1D^*(z)(\frac{(\frac{R(z)G_1(z)}{1+G_1(z)G_2(z)})^*}{(1+(\frac{G_1}{1+G_2})^* D^*(z))})}{1+G_1(z)G_2(z)}$

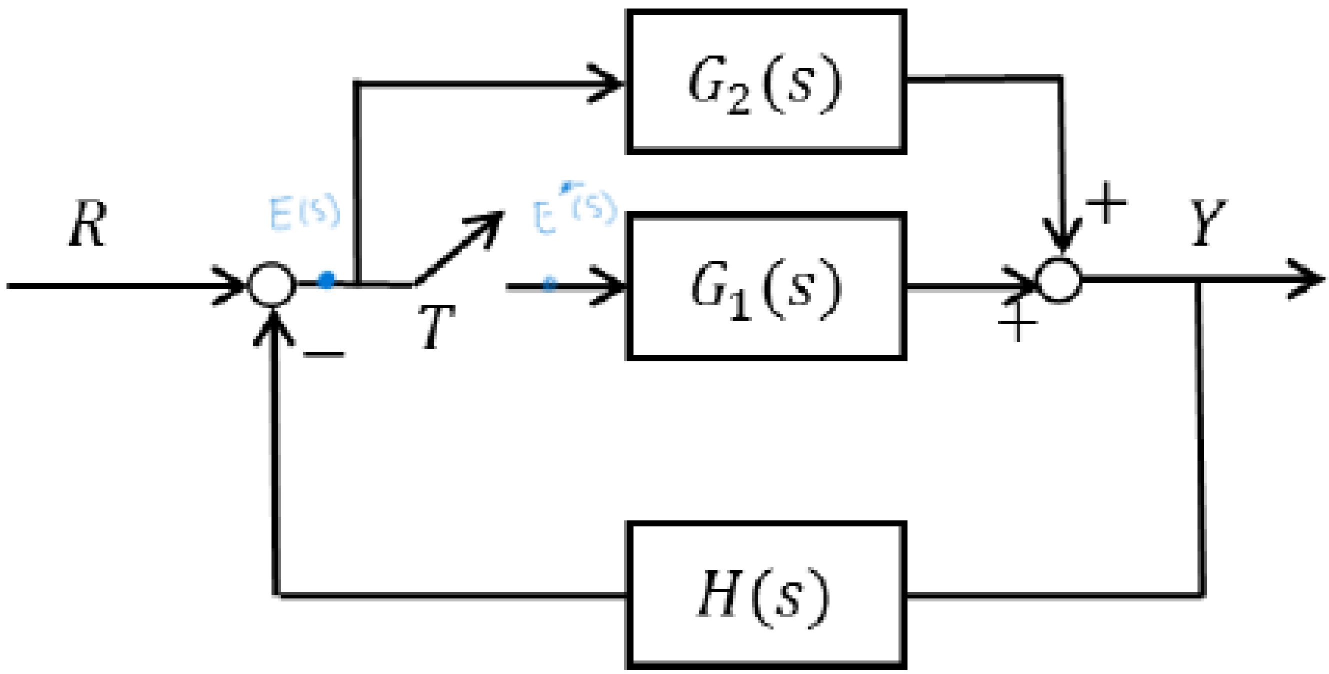
سوال سیزدهم

As you can see in the picture below we could write:
$E(s) = R(s) - H(s)Y(s)$
$Y(s) = G_2(s)E(s) + G_1(s)E^*(s)$
Draw the original SFG:



Change the equations so that Y is written in terms of R:
$Y(s) = G_2(s)(R(s) - H(s)Y(s)) + G_1(s)E^*(s)$
$Y(s) + G_2(s)H(s)Y(s) = G_2(s)R(s) + G_1(s)E^*(s)$
$(1 + G_2(s)H(s))Y(s) = G_2(s)R(s) + G_1(s)E^*(s)$
$Y(s) = \frac{G_2(s)R(s)}{(1+G_2(s)H(s))} + \frac{G_1(s)E^*(s)}{(1+G_2(s)H(s))}$
$E(s) = \frac{R(s)}{1+G_2(s)H(s)} - \frac{G_1(s)H(s)}{1+G_2(s)H(s)}E^*(s)$
$E^*(s) = (\frac{R(s)}{1+G_2(s)H(s)})^* - (\frac{G_1(s)H(s)}{1+G_2(s)H(s)})^*E^*(s)$
Compute the starred transform:
$Y^*(s) = (\frac{G_2(s)R(s)}{(1+G_2(s)H(s))})^* + (\frac{G_1(s)}{(1+G_2(s)H(s))})^*E^*(s)$
Three equations that help us draw the modified SFG:
$Y(s) = \frac{G_2(s)R(s)}{(1+G_2(s)H(s))} + \frac{G_1(s)E^*(s)}{(1+G_2(s)H(s))}$
$Y^*(s) = (\frac{G_2(s)R(s)}{(1+G_2(s)H(s))})^* + (\frac{G_1(s)}{(1+G_2(s)H(s))})^*E^*(s)$
$E^*(s) = (\frac{R(s)}{1+G_2(s)H(s)})^* - (\frac{G_1(s)H(s)}{1+G_2(s)H(s)})^*E^*(s)$
Let's write it in a more summarized manner:
$Y(s) = G_3R(s) + G_4E^*(s)$
$Y^*(s) = (G_3R(s))^* + G_4^*E^*(s)$
$E^* = G_6 + G_5E^*(s)$
It is obvious that we have:
$G_3 = (\frac{G_2(s)}{(1+G_2(s)H(s))})$
$G_4 = (\frac{G_1(s)}{(1+G_2(s)H(s))})$
$G_5 = (\frac{G_1(s)H(s)}{1+G_2(s)H(s)})^*$
$G_6 = (\frac{R(s)}{1+G_2(s)H(s)})^*$





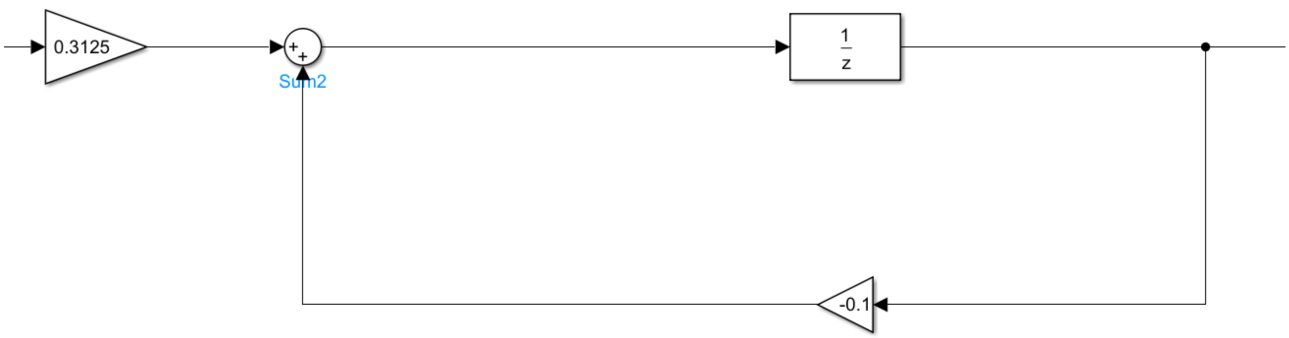
شکل ۷: شکل سوال سیزدهم

ادامه سوال سیزدهم

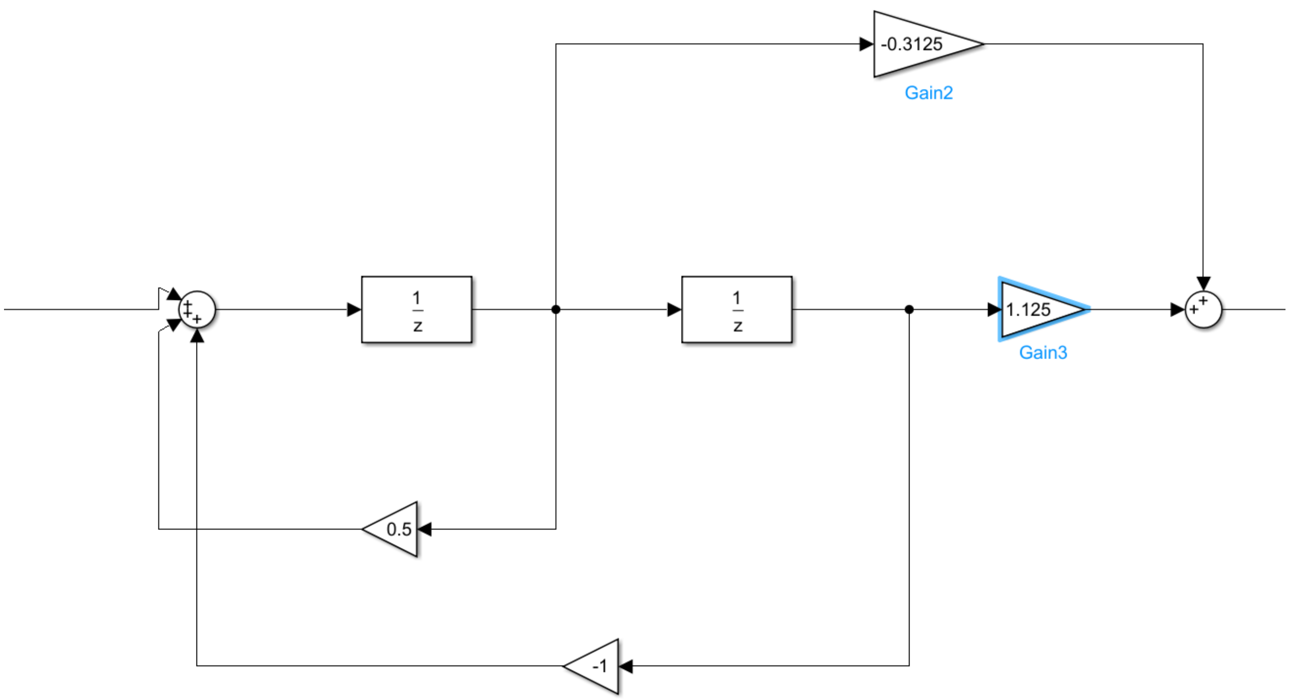
In the previous page, you can see the modified SFG.
Obviously, you could derive $\frac{Y(s(z))}{R(s(z))}$ but not the $\frac{Y(s)}{R^*(s)}$

سوال چهاردهم

Just like what we had in the second question, but here it is parallel, whereas in the Second question the Series implementation is asked:
$G(z) = \frac{-2.1575}{(z-0.1)} + \frac{2.1875z-1.875}{(z^2-0.5z+1)}$
implement the first transfer function:
$G_1(z) = \frac{-2.1575z^{-1}}{1-0.1z^{-1}}$
$G_2(z) = \frac{2.1875z^{-1}+1.875z^{-2}}{(1-0.5z^{-1}+z^{-2})}$
we implement G_1:

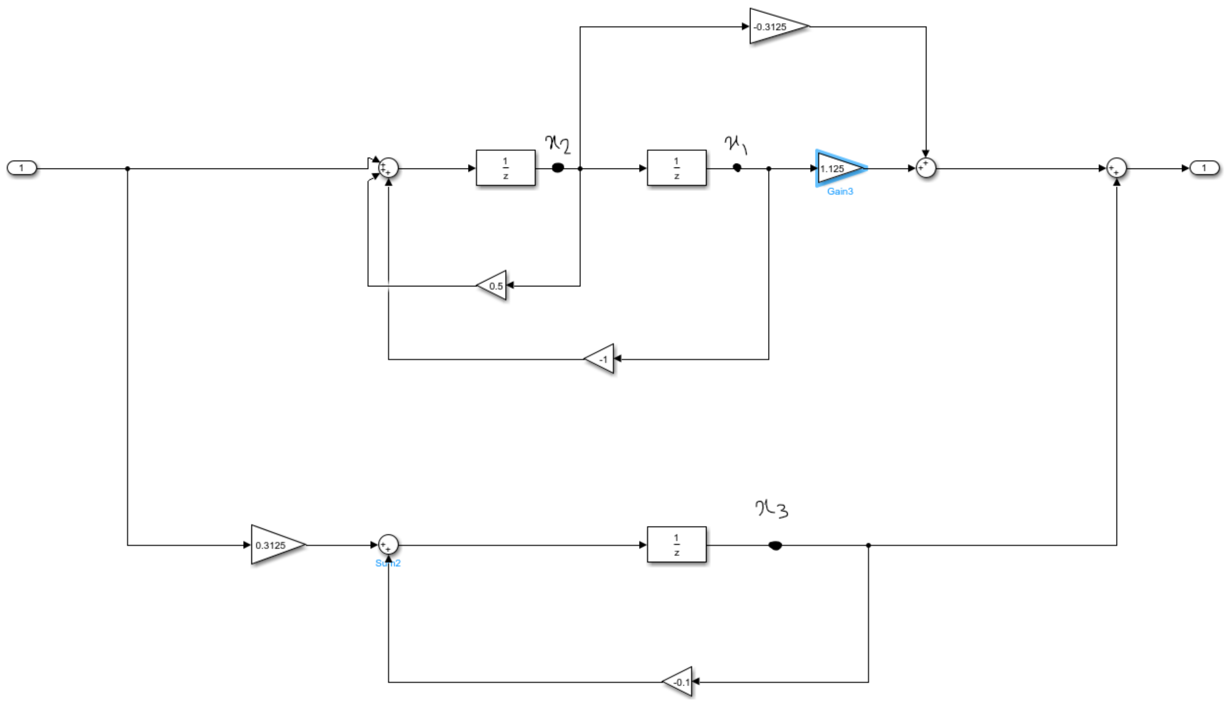


we implement G_2:



ادامه سوال چهاردهم

Now put them together



Now for the second part of the problem

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & -1 & 0 \\ 0 & 0 & -0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0.3125 \end{pmatrix} u \tag{۳}$$

$$y = \begin{pmatrix} 1.125 & -0.3125 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{۴}$$

$$\tag{۵}$$

سوال پانزدهم

Use the following equation again:
$G(z) = (1 - z^{-1})\, Z\{\frac{G(s)}{s}\}$
We get:
$\frac{b}{a} + \frac{a-b}{a} \frac{1-z^{-1}}{1-e^{-aT}z^{-1}}$
which is equal to :
$\frac{1-(\frac{b}{a}e^{-aT}+\frac{a-b}{a})z^{-1}}{1-e^{-aT}z^{-1}}$
Which is always non-minimum phase here is why:
The zero is $(\frac{b}{a}e^{-aT} + \frac{a-b}{a})$ let us rewrite it like bellow:
$\frac{b}{a}(e^{-aT} - 1) + 1$
Since $\frac{b}{a}$ is always negative and similarly $(e^{-aT} - 1)$ is always negative, the first term is always positive. Which means the zero is always greater than 1, hence the system is non-minimum phase.