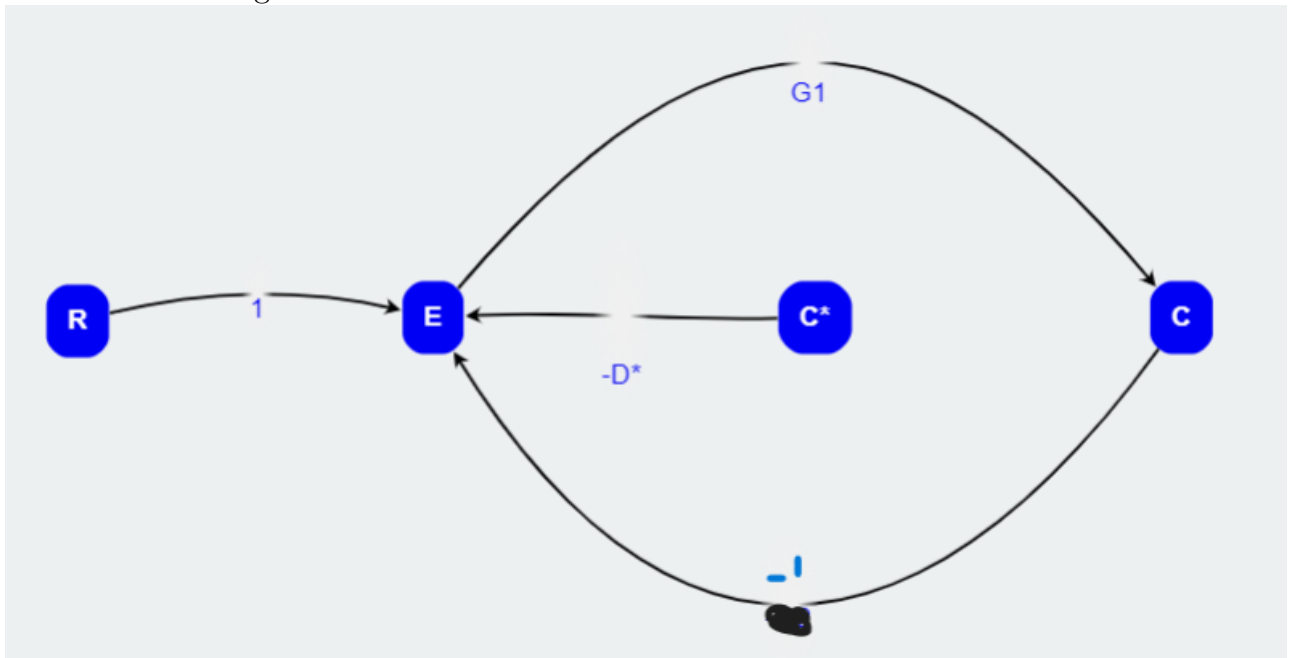# HW3 Solutions

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۱   بخش اجباری

Disclaimer:
This is the solution manual to the homework assigned to students of Digital Control - Dr.Talebi.
We do not guarantee that this solution is precise and thorough so please contact your TA to propose
your innovative solutions and/or any probable mistakes.

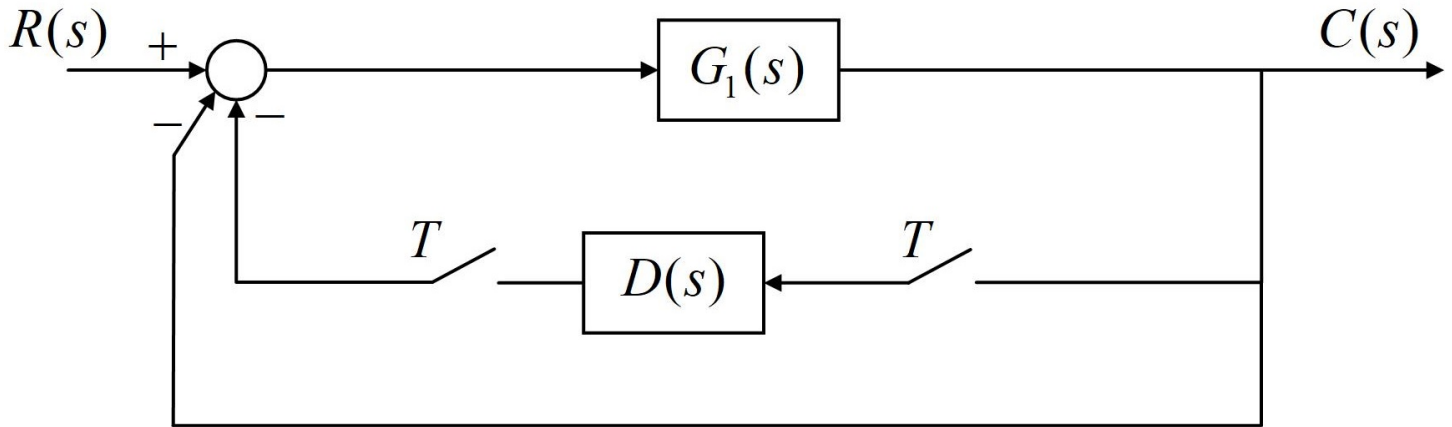First draw the original SFG :

$C = G_1 E$

$E = R - C - D^* C^*$

$C = \frac{G_1 R}{G_1 + 1} - \frac{D^* C^*}{1 + G_1}$

Derive C^* as follows:

$C^* = \left(\frac{G_1 R}{1+G_1}\right)^* - \left(\frac{G_1}{1+G_1}\right)^* D^* C^*$

$C^* = \frac{\left(\frac{G_1 R}{1+G_1}\right)^*}{\left(1 + \left(\frac{G_1}{1+G_1}\right)^* D^*\right)}$

$C = \frac{G_1 R}{G_1 + 1} - \frac{D^* \left(\frac{\left(\frac{G_1 R}{1+G_1}\right)^*}{\left(1 + \left(\frac{G_1}{1+G_1}\right)^* D^*\right)}\right)}{1 + G_1}$

شکل ۱: شکل سوال اول

سوال دوم

Recall the equivalent for forward discretization :

$s \rightarrow \frac{z-1}{T}$

we must substitute it in the transfer function to get the following discrete transfer function:

$H(z)_{forward} = 10\frac{z-0.75}{z+1.5}$

from the original transfer function we have:

$\frac{s+1}{0.1s+1} \xrightarrow{s \rightarrow j\omega} \frac{j\omega+1}{0.1j\omega+1} \frac{j\omega+1}{0.1j\omega+1} = \frac{\sqrt{\omega^2+1}}{\sqrt{(0.1\omega)^2+1}} \angle \tan^{-1}\omega - \tan^{-1}0.1\omega$

$\omega = 3 \rightarrow \varphi = \tan^{-1}3 - \tan^{-1}0.3 \simeq 55°$

$H(z)_{forward} = 10\frac{z-0.75}{z+1.5} \xrightarrow{z=e^{j\omega T}} 10\frac{e^{j\omega T}-0.75}{e^{j\omega T}+1.5}$

$10\frac{\cos\omega T - 0.75 + j\sin\omega T}{\cos\omega T + 1.5 + j\sin\omega T}$

$\varphi = \tan^{-1}\frac{\sin\omega T}{\cos\omega T - 0.75} - \tan^{-1}\frac{\sin\omega T}{\cos\omega T + 1.5}$

$\varphi = \tan^{-1}\frac{\sin 0.75}{\cos 0.75 - 0.75} - \tan^{-1}\frac{\sin 0.75}{\cos 0.75 + 1.5}$

$\varphi = \tan^{-1}\frac{.68}{.0183} - \tan^{-1}\frac{0.68}{0.23168} = 88.46 - 16.94 = 71.52$

Discretize using the Backward rule to get :

$s \rightarrow \frac{z-1}{Tz}$

$H(z)_{Backward} = 3.57\frac{z-0.8}{z-0.286}$

$H(z) = 3.57\frac{z-0.8}{z-0.286} \xrightarrow{z \rightarrow e^{j\omega T}} 3.57\frac{e^{j\omega T}-0.8}{e^{j\omega T}-0.286}$

$3.57\frac{\cos\omega T - 0.8 + j\sin\omega T}{\cos\omega T - 0.286 + j\sin\omega T}$

$\varphi = \tan^{-1}\frac{\sin\omega T}{\cos\omega T - 0.8} - \tan^{-1}\frac{\sin\omega T}{\cos\omega T - 0.286}$

$\varphi = \tan^{-1}\frac{\sin 0.75}{\cos 0.75 - 0.8} - \tan^{-1}\frac{\sin 0.75}{\cos 0.75 - 0.286}$

$\varphi = \tan^{-1}\frac{0.68}{0.0683} - \tan^{-1}\frac{0.68}{0.4457}$

$\varphi \simeq 40$

سوال سوم

Recall the formula for tustin with prewarping:

$s \to \dfrac{1}{\tan(\frac{\omega_1 T}{2})} \dfrac{z-1}{z+1}$

we must substitute it in the transfer function to get the following discrete transfer function:

$H(z) = 4.89 \dfrac{z-0.768}{z-0.161}$

$H(z) = 4.89 \dfrac{z-0.768}{z+0.161} \xrightarrow{z \to e^{j\omega T}} 4.89 \dfrac{e^{j\omega T}-0.768}{e^{j\omega T}+0.161}$

$4.89 \dfrac{\cos\omega T - 0.768 + j\sin\omega T}{\cos\omega T + 0.161 + j\sin\omega T}$

$\varphi = \tan^{-1} \dfrac{\sin\omega T}{\cos\omega T - 0.768} - \tan^{-1} \dfrac{\sin\omega T}{\cos\omega T + 0.161}$

$\varphi = \tan^{-1} \dfrac{0.68}{.0363} - \tan^{-1} \dfrac{0.68}{0.8926}$

$\varphi = -86.944 - 37.3$

$\varphi = 55.756$

سوال چهارم

Recall the equivalent formula for zero pole matching. Substitute it to get the discrete transfer function as bellow:

$H_{PZ} = 4.150 \dfrac{z-0.779}{z-0.082}$

$H_{PZ} = 4.150 \dfrac{z-0.779}{z-0.082} \xrightarrow{z \to e^{j\omega T}} 4.150 \dfrac{e^{j\omega T}-0.779}{e^{j\omega T}-0.082}$

$4.150 \dfrac{\cos\omega T - 0.779 + j\sin\omega T}{\cos\omega T - 0.082 + j\sin\omega T}$

$\varphi = \tan^{-1} \dfrac{\sin\omega T}{\cos\omega T - 0.779} + \tan^{-1} \dfrac{\sin\omega T}{\cos\omega T - 0.082}$

$\varphi = \tan^{-1} \dfrac{0.68}{-0.047} - \tan^{-1} \dfrac{0.68}{0.6496}$

$\varphi = -86.046 - 46.3097$

$\varphi = 47.64$

Looking back at the calculations of section 4.3.3 of Digital Control of Dynamic Systems by G.Franklin:

Let's choose T=1

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = H x(k) \tag{1}$$

$$\Phi = e^{AT}$$

$$\Gamma = \int_0^T e^{A\eta} d\eta B$$

$$\Phi = L^{-1}\left\{(sI - AT)^{-1}\right\}$$

$$AT = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\Phi = L^{-1}\left\{\begin{pmatrix} \frac{s+1}{s^2+s+1} & \frac{-1}{s^2+s+1} \\ \frac{1}{s^2+s+1} & \frac{s+1.57\times10^{-16}}{s^2+s+1} \end{pmatrix}\right\}$$

The rest is routine and left as an exercise to students.

Follow section 6.1 , equations 6.27 through 6.31 of Digital Control of dynamic systems by G.Franklin for the solution.

۲ بخش امیتازی

سوال هفتم

Solution:

To obtain V_o, we will replace the R-$2R$ network with its Thevenin equivalent circuit, as shown in Fig. 2. The computation of the Thevenin resistance and Thevenin voltage is shown in Figs. 3 and 4, respectively, where repeated use of Thevenin's theorem has been made to simplify the circuit systematically.
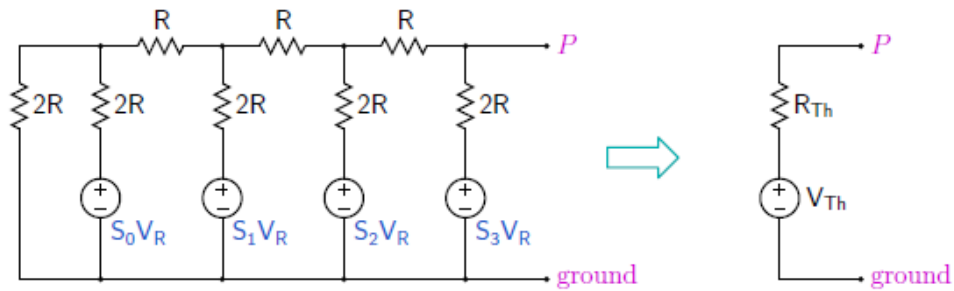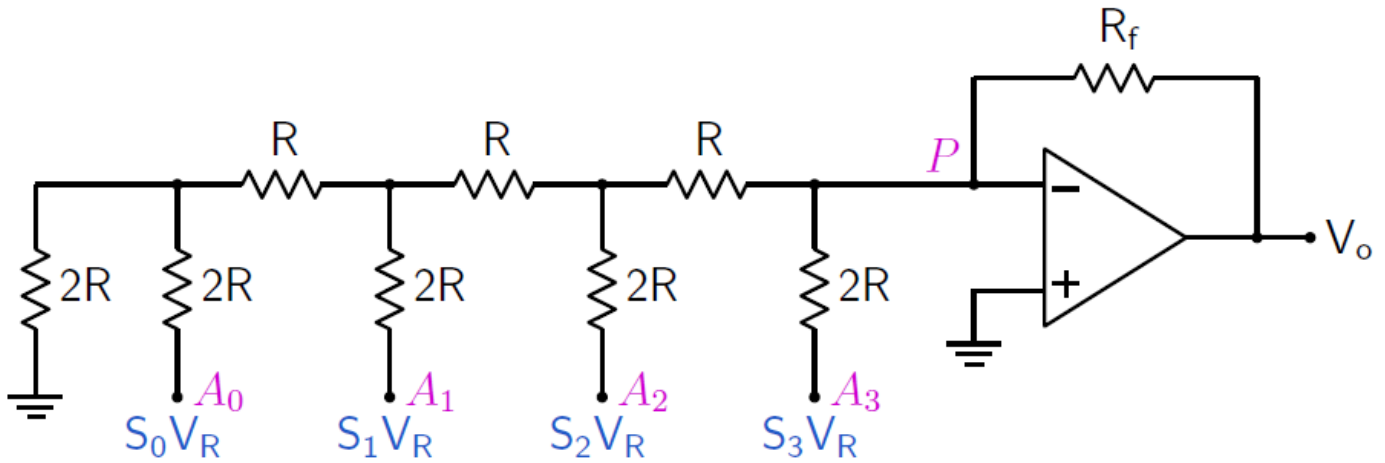

Figure 2: Representation of R-$2R$ ladder with Thevenin equivalent circuit.

Finally, we replace the R-$2R$ network in the original DAC circuit of Fig. 1 with its Thevenin equivalent circuit and obtain the circuit shown in Fig. 5. This circuit is simply an inverting amplifier, and the output voltage is given by

$$V_o = -\frac{R_f}{R} V_{\text{Th}} = -\frac{V_R}{8} = -0.625 \, \text{V}, \tag{1}$$

شکل ۲: شکل سوال هفتم

سوال هشتم

This task must be done in multiple steps:
Firstly, compare 10 which is half of 20 with the input voltage:
$10 < 13.478$
MSB Gets Set we have : 10000000
$10 + 5 > 13.478$
the next bit remains 0 : 10000000
$10 + 2.5 < 13.478$
the next bit gets set : 1010000
$10 + 2.5 + 1.25 > 13.478$
the next bit remains 0 : 10100000
$10 + 2.5 + 0.625 < 13.478$
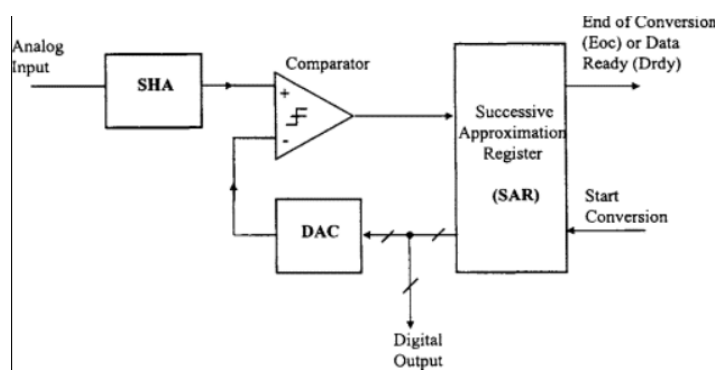the next bit gets set : 10101000
$10 + 2.5 + 0.625 + 0.3125 < 13.478$
the next bit gets set : 10101100
$10 + 2.5 + 0.625 + 0.3125 + 0.15625$
the next bit remains 0 : 10101100

شکل ۳: شکل سوال هشتم