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① الف)

Adam's law

$$\begin{cases} E(X|Z)=g(Z) \longrightarrow E(g(Z))=E(E(X|Z))=E(X) \\ E(Y|Z)=h(Z) \longrightarrow E(h(Z))=E(E(Y|Z))=E(Y) \end{cases}$$

$* \; E(h(X)Y|X)=h(X)E(Y|X)$

چراکه هنگام ثابت بودن X

$h(X)$ حکم ثابت دارد و از E

بیرون میاید،

$$\mathrm{cov}(X,Y|Z)=E\big((X-g(Z))(Y-h(Z))\,|\,Z\big)=$$

$$E(XY-Yg(Z)-Xh(Z)+g(Z)h(Z)|Z)=E(XY|Z)-E(Yg(Z)|Z)-E(Xh(Z)|Z)$$
$$+E(g(Z)h(Z)|Z)$$

$$=E(XY|Z)-\overset{*}{g}(Z)E(Y|Z)-\overset{*}{h}(Z)E(X|Z)+g(Z)\overset{*}{h}(Z)E(1|Z)=$$

$$E(XY|Z)-\cancel{g(Z)h(Z)}-h(Z)g(Z)+\cancel{g(Z)h(Z)}=$$

$$E(XY|Z)-E(X|Z)E(Y|Z) \quad \square$$

ب)

$$① \; E(\mathrm{cov}(X,Y|Z)) \overset{\text{الف}}{=} E\big(E(XY|Z)-\underbrace{E(X|Z)}_{g(Z)}\underbrace{E(Y|Z)}_{h(Z)}\big)=$$

Adam's law

$$E(E(XY|Z))-E(E(X|Z)E(Y|Z))=E(XY)-E(g(Z)h(Z))$$

$$② \; \mathrm{cov}(E(X|Z),E(Y|Z))=\mathrm{cov}(g(Z),h(Z))=E\big((g(Z)-\overbrace{E(g(Z))}^{①})(h(Z)-\overbrace{E(h(Z))}^{②})\big)$$

$$=E\big((g(Z)-E(X))(h(Z)-E(Y))\big)=E(g(Z)h(Z)-g(Z)E(Y)-h(Z)E(X)+E(X)E(Y))$$

$$\Rightarrow ①+②=E(XY)-\cancel{E(g(Z)h(Z))}+\cancel{E(g(Z)h(Z))}-E(Y)E(\overbrace{g(Z)}^{①})-E(X)E(\overbrace{h(Z)}^{②})$$
$$+E(X)E(Y)$$

$$=E(XY)-E(Y)E(X)-E(X)E(Y)+E(X)E(Y)=E(XY)-E(X)E(Y)$$

$$=E\big((X-E(X))(Y-E(Y))\big)=\mathrm{Cov}(X,Y)$$

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ج)

$E(Y|X)=g(X)$

① $E(\mathrm{var}(Y|X)) = E\left(E(Y^2|X) - E(Y|X)^2\right) = \overbrace{E(E(Y^2|X))}^{\text{Adam's law}} - E(g(X)^2)$

$= E(Y^2) - E(g(X)^2)$

Adam's law $\Rightarrow E(E(Y|X)) = E(Y)$

② $\mathrm{var}(E(Y|X)) = E(g(X)^2) - \overbrace{E(g(X))}^{\text{Adam's law}}{}^2 = E(g(X)^2) - E(Y)^2$

$\Rightarrow$ ① + ② $= E(Y^2) - \cancel{E(g(X)^2)} + \cancel{E(g(X)^2)} - E(Y)^2 = E(Y^2) - E(Y)^2 = \mathrm{Var}(Y)$

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X_i ها مستقل $\quad E X_i = 2$

$Var(X_i) = 1$

$n = 50$

$Y = X_1 + \cdots + X_n \qquad \bar{X} = \frac{X_1 + \cdots + X_n}{n} \Rightarrow \mu_{\bar{X}} = \frac{n \mu_{X_i}}{n} = \mu_{X_i} = 2$

$Y = n \bar{X}$

مستقل

$Var(\bar{X}) = Var\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} Var(\sum X_i) = \frac{1}{n^2} \sum Var(X_i) = \frac{n \times 1}{n^2} = \frac{1}{n}$

$\Rightarrow \quad \frac{\bar{X} - \mu}{\sigma} \dot\sim Z \sim N(0,1)$

$\Rightarrow \quad \frac{\bar{X} - 2}{\sqrt{\frac{1}{n}}} = (\bar{X} - 2)\sqrt{50} \dot\sim Z \Rightarrow \quad \bar{X} = \frac{Z}{\sqrt{50}} + 2 \Rightarrow$

$Y = 50 \bar{X} = \sqrt{50} Z + 100$

$\Rightarrow$

$P(90 \leq Y \leq 110) = P(90 \leq \sqrt{50} Z + 100 \leq 110) =$

$P(-10 \leq \sqrt{50} Z \leq 10) = P\left(\frac{-10}{5\sqrt{2}} \leq Z \leq \frac{10}{5\sqrt{2}}\right) =$

$P(-\sqrt{2} \leq Z \leq \sqrt{2}) = \Phi(\sqrt{2}) - \Phi(-\sqrt{2}) = \Phi(\sqrt{2}) - (1 - \Phi(\sqrt{2}))$

$= 2\Phi(\sqrt{2}) - 1$

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۳) الف)

$$E(X)=\sum_{x=1}^{\infty} xP(X=x) \underset{\downarrow}{\geq} \sum_{x=1}^{k} xP(X=x) \overset{\text{نزولی}}{\geq} \sum_{x=1}^{k} xP(X=k)$$

$\underbrace{x}_{\geq 0}\underbrace{P(X=x)}_{\geq 0} \geq 0$

$$=P(X=k)\sum_{x=1}^{k} x = P(X=k)\frac{k(k+1)}{2}$$

$$\Rightarrow \frac{2E(X)}{k(k+1)} \geq P(X=k) \qquad k^2<k^2+k \Rightarrow \frac{1}{k^2} > \frac{1}{k(k+1)}$$

$$\Rightarrow \frac{2E(X)}{k^2} \geq \frac{2E(X)}{k(k+1)} \geq P(X=k) \quad \boxed{\checkmark}$$

ب) X متغیر تصادفی پیوسته نامنفی با pdf نزولی

$$E(X)=\int_0^{\infty} x f_X(x)\,dx = \int_0^{k} x f_X(x)\,dx + \int_k^{\infty} x f_X(x)\,dx \geq \int_0^{k} x f_X(x)\,dx = A$$

$x f_X(x) \geq 0 \Rightarrow$ است ≥ 0

اثبات نزولی بودن $0<x\leq k$

$$f_X(x)\geq f_X(k) \Rightarrow x f_X(x) \geq x f_X(k)$$

$$\Rightarrow E(X)=A \geq \int_0^{k} x f_X(k)\,dx = f_X(k)\int_0^{k} x\,dx = f_X(k)\,\frac{x^2}{2}\Big|_0^{k} = f_X(k)\frac{k^2}{2}$$

$$\Rightarrow E(X) \geq \frac{k^2}{2} f_X(k) \underset{kx>0}{\overset{k\to x}{\Longrightarrow}} f(x) \leq \frac{2E(X)}{x^2} \quad \checkmark$$

اعداد را $a_1,\dots,a_n$ ، خطاها $U_1,\dots,U_n$

$$U_i \sim Unif(-0.5, 0.5) \to \mu = \frac{b+a}{2} = \frac{0.5+(-0.5)}{2} = 0$$

$$\hookrightarrow var = \frac{(b-a)^2}{12} = \frac{(0.5-(-0.5))^2}{12} = \frac{1}{12}$$

تفاوت مجموعها $= \left| \sum_{i=1}^{n}(a_i+U_i) - \sum_{i=1}^{n} a_i \right| = \left| \sum_{i=1}^{n} U_i \right|$ ⊛ جنبل

$$\bar{U} = \sum_{i=1}^{n} U_i \Big/ n \longrightarrow \mu = 0$$

$$\hookrightarrow var = \frac{\sigma^2}{n} = \frac{1}{12n}$$

$$\Rightarrow \frac{\bar{U}-\mu}{\sigma} \dot{\sim} N(0,1) = Z \Rightarrow \frac{\bar{U}-0}{\frac{1}{\sqrt{12n}}} = Z$$

$$\Rightarrow \frac{Z}{\sqrt{12n}} = \bar{U} \qquad \sum U_i = n\bar{U} = \frac{nZ}{\sqrt{12n}} = \frac{\sqrt{n}\,Z}{\sqrt{12}}$$

⊛ $P\left(\left|\sum_{i=1}^{n} U_i\right| > 3\right) = 1 - P\left(\left|\sum_{i=1}^{n} U_i\right| \le 3\right)$

اختلاف مجموعها

$$P\left(\left|\sum_{i=1}^{n} U_i\right| \le 3\right) = P\left(-3 \le \sum_{i=1}^{n} U_i \le 3\right)$$

$$= P\left(-3 \le \frac{\sqrt{n}\,Z}{\sqrt{12}} \le 3\right) \underset{n=50}{=} P\left(-3 \le \frac{5\sqrt{2}\,Z}{2\sqrt{3}} \le 3\right)$$

$$= P\left(\frac{-3\sqrt{6}}{5} \le Z \le \frac{3\sqrt{6}}{5}\right) = \Phi\left(\frac{3\sqrt{6}}{5}\right) - \Phi\left(\frac{-3\sqrt{6}}{5}\right) =$$

$$\Phi\left(\frac{3\sqrt{6}}{5}\right) - \left(1 - \Phi\left(\frac{3\sqrt{6}}{5}\right)\right) = 2\Phi\left(\frac{3\sqrt{6}}{5}\right) - 1$$

$$\Rightarrow P\left(\left|\sum_{i=1}^{n} U_i\right| > 3\right) = 1 - \left(2\Phi\left(\frac{3\sqrt{6}}{5}\right) - 1\right) = 2\left(1 - \Phi\left(\frac{3\sqrt{6}}{5}\right)\right)$$

$$= 2\,\Phi\left(\frac{-3\sqrt{6}}{5}\right)$$

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(5) زمان اجرای iام را X_i

$$\Rightarrow \quad \bar{X} = \frac{X_1 + \dots + X_n}{n} \begin{cases} \to \mu \\ \to var = \frac{\sigma^2}{n} = \frac{4}{n} \end{cases}$$

$$\Rightarrow \quad \frac{\bar{X} - \mu}{\sigma} \;\dot\sim\; Z \sim \mathcal{N}(0,1)$$

$$\frac{\bar{X} - \mu}{\frac{2}{\sqrt{n}}} = \frac{\sqrt{n}}{2}(\bar{X} - \mu) \;\dot\sim\; Z$$

$$P(|Z| \leq 2 \times 1) \geq 95\%$$

[illegible]
68٪، 95٪، 99.7٪

$$\Rightarrow P\left(\frac{\sqrt{n}}{2}|\bar{X} - \mu| \leq 2\right) \geq 95\%$$

$$\Rightarrow P\left(|\bar{X} - \mu| \leq \frac{4}{\sqrt{n}}\right) \geq 95\%$$

95٪ صحت
دنبال ۲ اطمینان 0.5 + هستیم ⇒

دنبال $P(|\bar{X} - \mu| \leq 0.5) \geq P\left(|\bar{X} - \mu| \leq \frac{4}{\sqrt{n}}\right) \geq 95\%$

پس
$$\Rightarrow \quad 0.5 \geq \frac{4}{\sqrt{n}} \Rightarrow \sqrt{n} \geq \frac{4}{0.5} = 8 \Rightarrow n \geq 64$$

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$$b = \frac{\sigma^2}{a} > 0$$ چون اجتماع با حالت منفی آمد خلق بزرگتر

⑥

$$P(X \geq a) = P(\underbrace{X+b}_{\geq 0} \geq a+b) \leq P((X+b)^2 \geq (a+b)^2) \underset{\text{Markov inequality}}{\leq} \frac{E(X+b)^2}{(a+b)^2}$$

$$= \frac{E(X^2 + b^2 + 2bX)}{(a+b)^2} = \frac{E(X^2) + b^2 + 2b\overbrace{E(X)}^{0}}{(a+b)^2} = \frac{E(X^2) + b^2}{(a+b)^2} = A$$

$$\sigma^2 = Var(X) = E(X^2) - \underbrace{E(X)}_{0}{}^2 = E(X^2)$$

$$\Rightarrow P(X \geq a) \leq A = \frac{\sigma^2 + b^2}{(a+b)^2} = \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{\left(a + \frac{\sigma^2}{a}\right)^2} = \frac{\sigma^2 \left(\frac{\cancel{a^2 + \sigma^2}}{\cancel{a^2}} \right)}{\frac{(a^2+\sigma^2)^{\cancel{2}}}{\cancel{a^2}}} = \frac{\sigma^2}{\sigma^2 + a^2}$$

$$\Rightarrow P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$