$X_i \sim Bern(\frac{1}{6})$ ①

$$\Rightarrow \quad X_1 + \dots + X_n \sim Binom(n, \frac{1}{6})$$

$$\xrightarrow{\text{iid}} \quad Z = \frac{\bar{X} - p}{\sqrt{\frac{pq}{n}}} \sim N(0,1)$$

$$P\left(-z_{\alpha/2} \le \frac{\bar{X} - p}{\sqrt{\frac{pq}{n}}} \le z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\Rightarrow P\left(p - \sqrt{\frac{pq}{n}}\, z_{\alpha/2} \le \bar{X} \le \sqrt{\frac{pq}{n}}\, z_{\alpha/2} + p\right) = 1 - \alpha$$

$$P\left(np - \sqrt{npq}\, z_{\alpha/2} \le \sum X_i \le \sqrt{npq}\, z_{\alpha/2} + np\right) = 1 - \alpha$$

$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \qquad z_{0.025} = 1.96$

$$\Rightarrow P\left(np - \sqrt{npq}\,(1.96) \le \sum X_i \le \sqrt{npq}\,(1.96) + np\right) = 95\%$$

$n = 500$
$p = \frac{1}{6} \Rightarrow q = \frac{5}{6}$

$$P\left(\frac{250}{3} - \sqrt{\frac{2500}{36}}\,(1.96) \le \sum X_i \le \sqrt{\frac{2500}{36}}\,(1.96) + \frac{500}{6}\right) = 95\%$$

$$\Rightarrow P\left(\frac{250}{3} - \frac{50}{6}(1.96) \le \sum X_i \le \frac{50}{6}\, 1.96 + \frac{250}{3}\right.$$

$$\Rightarrow P(67 \le \sum X_i \le 99.\bar{6}) = 95\%$$

$$\Rightarrow P(t \in [67, 99]) = 95\%$$

۲

الف)

$X_i \sim U(0,\theta)$

$$P(x_1,\dots,x_n;\theta) = \prod f_{X_i}(x_i) = \begin{cases} \left(\frac{1}{\theta}\right)^n & 0 \le x_1,\dots,x_n \le \theta \\ 0 & \text{o.w} \end{cases}$$

$$f_{X_i}(x_i) = \frac{1}{\theta - 0} \quad \text{if } 0 \le x_i \le \theta$$

$A = \left(\frac{1}{\theta}\right)^n \Rightarrow A = \theta^{-n} \Rightarrow \frac{dA}{d\theta} = -n\theta^{-(n+1)} \xrightarrow{\theta > 0} \frac{dA}{d\theta} < 0 \rightarrow$ با افزایش θ کاهش می‌یابد

و پس ماکسیمم، کمترین θ است اما $\theta \ge x_i$ وگرنه عبارت ۰ کمینه است می‌شود.

$\Rightarrow \hat{\theta}_{ML} = \max(X_1,\dots,X_n) \leftarrow$ کلی

$\hookrightarrow \max(x_1,\dots,x_n)$

ب)

$$B(\hat{\theta}) = E(\hat{\theta}) - \theta = E(\max(X_1,\dots,X_n)) - \theta$$

order statistics: $f_{X_{(n)}}(x) = n f_X(x) F_X(x)^{n-1}$

$$f_{X_{(n)}}(x) = n \frac{1}{\theta} \left(\frac{x}{\theta}\right)^{n-1} = \frac{n x^{n-1}}{\theta^n}$$

$$\Rightarrow E(X_{(n)}) = \int_{-\infty}^{\infty} x f_{X_{(n)}}(x) = 0 + \int_0^{\theta} \frac{n x^n}{\theta^n} dx = \frac{n}{\theta^n} \int_0^{\theta} x^n dx = \frac{n}{\theta^n}\left(\frac{\theta^{n+1}}{n+1} - \frac{0^{n+1}}{n+1}\right)$$

$$= \frac{n\theta}{n+1}$$

$$\Rightarrow B(\hat{\theta}) = E(\hat{\theta}) - \theta = \frac{n\theta}{n+1} - \theta = \frac{-\theta}{n+1} \neq 0 \Rightarrow \text{biased}$$

$X_1, \dots, X_n$

توزیع فرض μ , σ^2

می دانیم $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ $\qquad E(S^2) = \sigma^2$

$\hat{\theta} = \bar{X}^2$

$$E(\bar{X}^2) = var(\bar{X}) + E(\bar{X})^2 = \frac{\sigma^2}{n} + \mu^2$$

$$\Rightarrow E(\underbrace{\bar{X}^2 - \frac{S^2}{n}}) = E(\bar{X}^2) - \frac{E(S^2)}{n} = \frac{\sigma^2}{n} + \mu^2 - \frac{\sigma^2}{n} = \mu^2$$

$\Rightarrow$ تخمین گر غیر بایاس

الف)

3 بلدرچین علامت‌دار

از بین ۶ تاش ۲ تاش علامت‌دار

$$w = 3$$

$$X \sim HGeom(n, w)$$

$$\to \underset{\text{in } m}{p(X=k)} = \binom{w}{k}\binom{n-w}{m-k} \Big/ \binom{n}{m}$$

$$\rightsquigarrow L(x_1; \theta) = \frac{\binom{3}{2}\binom{\theta-3}{4}}{\binom{\theta}{6}} = \frac{3\,\frac{\cancel{(\theta-3)}\cancel{(\theta-4)}\cancel{(\theta-5)}(\theta-6)}{\cancel{4!}}}{\frac{\theta(\theta-1)(\theta-2)\cancel{(\theta-3)}\cancel{(\theta-4)}\cancel{(\theta-5)}}{\cancel{6!}\ 30}}$$

$$= 90 \overbrace{\frac{\theta-6}{\theta(\theta-1)(\theta-2)}}^{A} \to \frac{dL}{d\theta} = 90\frac{dA}{d\theta} \rightsquigarrow \frac{dA}{d\theta} \overset{?}{=} 0 \qquad * \theta \geq 3$$

$$\frac{dA}{d\theta} = \frac{-2\theta^3 + 21\theta^2 - 36\theta + 12}{(\theta-2)^2(\theta-1)^2\theta^2} \Rightarrow -2\theta^3 + 21\theta^2 - 36\theta + 12 = 0$$

بازه 1,6 تا 8,455 صعودی

و سپس نزولی است. پس بین 8 یا 9 است.

$$\theta = 8 : A = \frac{2}{8 \times 7 \times 6} = \frac{1}{168} \qquad \theta = 9 : A = \frac{3}{9 \times 8 \times 7} = \frac{1}{168}$$

که برابرند پس تعداد کل بلدرچین‌ها تخمین 8,455 است که 8 یا 9 است.

(هردو L برابر)

(a)

$$X_i = \eta + \epsilon_i$$

$$E(\epsilon_i) = 0 \qquad var(\epsilon_i) = \sigma_i^2$$

$$\hat{\eta} = \alpha_1 x_1 + \dots + \alpha_n x_n$$

$$B(\hat{n}) = 0 \Rightarrow E(\hat{\eta}) = \eta \qquad E(\hat{\eta}) = E\left(\sum \alpha_i x_i\right) =$$

$$\sum \alpha_i E(x_i) = \sum \alpha_i E(\eta + \epsilon_i) = \sum \alpha_i (\overset{\eta}{\cancel{E(\eta)}} + \overset{0}{\cancel{E(\epsilon_i)}}) = n \sum \alpha_i$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} \alpha_i = 1}$$

$$var(\hat{\eta}) = var\left(\sum_{i=1}^{n} \alpha_i x_i\right) \overset{\text{مستقل}}{=} \sum_{i=1}^{n} var(\alpha_i x_i) = \sum_{i=1}^{n} \alpha_i^2 \, var(x_i)$$

$$= \sum_{i=1}^{n} \alpha_i^2 \, var(\underbrace{\eta}_{\text{ثابت}} + \epsilon_i) = \sum_{i=1}^{n} \alpha_i^2 \, var(\epsilon_i) = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2$$

به دنبال مینیمم کردن $\sum_{i=1}^{n} \sigma_i^2 \alpha_i^2$ با $\sum_{i=1}^{n} \alpha_i = 1$ است.

$$g(\alpha_1, \dots, \alpha_n) = \sum \alpha_i - 1 \qquad f(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^{n} \sigma_i^2 \alpha_i^2$$

$$\nabla g = (1, 1, \dots, 1) \neq 0$$

$$\xrightarrow[lm]{} \nabla f = \lambda \nabla g$$

$$\nabla f \rightarrow \frac{\partial f}{\partial \alpha_j} = 2 \sigma_j^2 \alpha_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{\partial}{\partial \alpha_j} (\sigma_i^2 \alpha_i^2)$$

$$\frac{\partial}{\partial \alpha_j}\left(\sigma_i^2 \alpha_i^2\right) = \sigma_i^2 \frac{\partial}{\partial \alpha_j}\left(1-\alpha_1-\dots-\hat{\alpha_i}-\dots-\alpha_n\right)^2 =$$

$$2\sigma_i^2 \alpha_i (-1) = -2\sigma_i^2 \alpha_i \Rightarrow \frac{\partial f}{\partial \alpha_j} = 2\sigma_j^2 \alpha_j - \sum_{\substack{i=1 \\ i \neq j}}^{n} 2\sigma_i^2 \alpha_i$$

$(1,1,\dots,1)$

$$\Rightarrow \lambda \nabla g = \nabla f \Rightarrow \forall i \quad 2\sigma_j^2 \alpha_j - 2\sum_{\substack{i=1 \\ i \neq j}}^{n} \sigma_i^2 \alpha_i = \lambda$$

$$\Rightarrow 2\sigma_j^2 \alpha_j = 2\sum_{\substack{i=1 \\ i \neq j}}^{n} \sigma_i^2 \alpha_i + \lambda \overset{\frac{\lambda}{2}=c}{\Rightarrow} 2\sigma_j^2 \alpha_j = \sum_{i=1}^{n} \sigma_i^2 \alpha_i + C$$

$\forall i$ RHS

$$\Rightarrow \sigma_1^2 \alpha_1 = \sigma_i^2 \alpha_i \Rightarrow \alpha_i = \frac{\sigma_1^2}{\sigma_i^2} \alpha_1$$

$$\Rightarrow \sum \alpha_i = 1 \Rightarrow \sum \frac{\sigma_1^2}{\sigma_i^2} \alpha_1 = 1 \Rightarrow \alpha_1 \sigma_1^2 \sum \frac{1}{\sigma_i^2} = 1$$

$$\Rightarrow \alpha_1 = \frac{1}{\sum_{i=1}^{n} \frac{\sigma_1^2}{\sigma_i^2}} \quad \overset{\times \frac{\sigma_1^2}{\sigma_j^2}}{\Rightarrow} \quad \alpha_j = \frac{1}{\sum_{i=1}^{n} \frac{\sigma_j^2}{\sigma_i^2}}$$

$$* \; 2\sigma_1^2 \alpha_1 - \frac{2}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}} \qquad \sum_{i=1}^{n} \sigma_i^2 \alpha_i = \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$$

$$\Rightarrow c = \frac{2-n}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$$