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$$f_{XY}(x,y) = \begin{cases} 6e^{-2x-3y} & x,y \geq 0 \\ 0 & o.w \end{cases}$$

الف)

لم کلی:

$$f_{XY}(x,y) = h(x)g(y)$$

که h و g توابعی نامنفی اند $\Rightarrow$ X و Y مستقل اند.

$$\int_{-\infty}^{\infty} h(x)\,dx = c \Rightarrow \int_{-\infty}^{\infty} \frac{h(x)}{c}dx = 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} h(x)g(y)\,dx = g(y)\int_{-\infty}^{\infty} h(x)\,dx = cg(y)$$

انتگرال pmf $1 = \Rightarrow$

$$\int_{-\infty}^{\infty} cg(y)dy = 1 \Rightarrow \int_{-\infty}^{\infty} g(y)\,dy = \frac{1}{c}$$

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{-\infty}^{\infty} h(x)g(y)dy = h(x)\int_{-\infty}^{\infty} g(y)\,dy = \frac{h(x)}{c}$$

$$\Rightarrow f_X(x)f_Y(y) = \frac{h(x)}{c}cg(y) = h(x)g(y) = f_{XY}(x,y)$$ $\Rightarrow$ مستقل اند.

$$\begin{cases} 6e^{-2x-3y} \\ 0 \end{cases} \rightarrow h(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & o.w \end{cases}$$

$$g(y) = \begin{cases} 3e^{-3y} & y \geq 0 \\ 0 & o.w \end{cases}$$

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حل دوم:

$x \geq 0$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{-\infty}^{0} 0\,dy + \int_{0}^{\infty} 6e^{-2x-3y}\,dy$$

$$= 6e^{-2x}\int_0^{\infty} e^{-3y}\,dy = 6e^{-2x} \left. -\frac{1}{3}e^{-3y}\right|_0^{\infty} = 6e^{-2x}\left(0+\frac{1}{3}\right) = 2e^{-2x}$$

$x<0$: $\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{-\infty}^{\infty} 0\,dy = 0$

$y \geq 0$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} 6e^{-2x-3y}\,dx =$$

$$6e^{-3y}\int_0^{\infty} e^{-2x} = 6e^{-3y}\left.\left(-\frac{1}{2}e^{-2x}\right)\right|_0^{\infty} = 6e^{-3y}\left(0+\frac{1}{2}\right) = 3e^{-3y}$$

$y<0$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{-\infty}^{\infty} 0\,dx = 0$$

$$\Rightarrow f_X(x)f_Y(y) = \begin{cases} 3e^{-3y} \times 0 = 0 & y\geq 0 \quad x<0 \\ 6e^{-3y-2x} & y\geq 0 \quad x \geq 0 \\ 0 \times 2e^{-2y} = 0 & y<0 \quad x\geq 0 \end{cases} = \begin{cases} 6e^{-3y-2x} & y,x\geq 0 \\ 0 & \text{o.w} \end{cases} = f_{XY}(x,y)$$

مستقل اند.

ب) Z, Y مستقل

$E(Y|Z) = E(Y)$

$E(Y|X>2)$: استقلال $\rightarrow P(y|X>2) = P(y)$

$$= \int_{-\infty}^{\infty} y\,P(y|x>2) = \int_{-\infty}^{\infty} y\,P(y) = E(Y) = \int_{-\infty}^{\infty} y\,f_Y(y) =$$

$$\int_{-\infty}^{0} y\times 0\,dy + \int_0^{\infty} y(3e^{-3y})\,dy = -\int_0^{\infty} -3e^{-3y}y\,dy = -\left(ye^{-3y} + \frac{1}{3}e^{-3y}\right)\Bigg|_0^{\infty}$$

رشد بیشتر از y ← $0 = \infty \times 0$

$$= +\frac{1}{3} \rightsquigarrow \text{Expo}(3)$$

امید نمایی

ج)

$$S = P(X > Y) = \frac{3}{5}$$

$$S = \int_{-\infty}^{\infty} \int_{y}^{\infty} f_{XY}(x,y)\, dx\, dy$$

استقلال $\quad S = \int_{-\infty}^{\infty} \int_{y}^{\infty} f_X(x) f_Y(y)\, dx\, dy =$

$$S = \int_{-\infty}^{\infty} f_Y(y) \left(\int_{y}^{\infty} f_X(x)\, dx \right) dy$$

$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \rightarrow 0 \leq f_X(x), \Rightarrow$ انتگرال، هر بازه‌ای متناهی است. پس $0 = 0 \times C$

$$\text{if } y < 0 \Rightarrow f_Y(y) = 0 \rightarrow S = 0 + \int_{0}^{\infty} f_Y(y) \left(\int_{y}^{\infty} f_X(x)\, dx \right) dy$$

$y \geq 0 \Rightarrow$

$$\int_{0}^{\infty} 3e^{-3y} \left(\int_{y}^{\infty} 2e^{-2x}\, dx \right) dy = 6 \int_{0}^{\infty} e^{-3y} \left(\frac{e^{-2x}}{-2} \right) \Big|_{y}^{\infty} dy = 6 \int_{0}^{\infty} e^{-3y} \left(0 + \frac{e^{-2y}}{2} \right) dy$$

$$= \frac{6}{2} \int_{0}^{\infty} e^{-5y}\, dy = 3 \left(\frac{-e^{-5y}}{5} \right) \Big|_{0}^{\infty} = 3 \left(0 + \frac{1}{5} \right) = \frac{3}{5}$$

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سوال ۲

ب

$$f_{XY}(x,y) = \begin{cases} 1 & 0 \le y \le 1 \ , \ |x| \le y \\ 0 & o.w \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \underbrace{\int_{-\infty}^{0} f_{XY}(x,y)\,dy}_{0} + \int_0^1 f_{XY}(x,y)\,dy + \underbrace{\int_1^{\infty} f_{XY}(x,y)\,dy}_{0}$$

$$= \int_0^1 f_{XY}(x,y)\,dy = \underbrace{\int_0^{\min(|x|,1)} f_{XY}(x,y)\,dy}_{0} + \int_{\min(|x|,1)}^{1} f_{XY}(x,y)\,dy =$$

$y<|x|$ چون ←

$$= \int_{\min(|x|,1)}^{1} 1\,dy = \{1 - \min(|x|,1)\}$$

$$f_X(x) = \begin{cases} 0 & |x|>1 \\ 1-|x| & |x| \le 1 \end{cases}$$

$$\text{if } y \notin [0,1] : \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{-\infty}^{\infty} 0\,dx = 0$$

$y \in [0,1]$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \underbrace{\int_{-\infty}^{-y} f_{XY}(x,y)\,dx}_{0} + \int_{-y}^{y} f_{XY}(x,y)\,dx + \underbrace{\int_{y}^{\infty} f_{XY}(x,y)\,dx}_{0} -$$

$|x| \ge y$ $\qquad$ $|x| \ge y$

$$\int_{-y}^{y} f_{XY}(x,y)\,dx = \int_{-y}^{y} 1\,dx = y-(-y) = 2y$$

$$f_Y(y) = \begin{cases} 2y & y \in [0,1] \\ 0 & o.w \end{cases}$$

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$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \underbrace{\int_{-\infty}^{-1} x \times 0\,dx}_{0} + \int_{-1}^{1} x(1-|x|)\,dx + \underbrace{\int_{1}^{\infty} x \cdot 0\,dx}_{0} =$$

$$\int_{-1}^{1}(x - x|x|)\,dx = \int_{-1}^{1} x\,dx - \left(\int_{-1}^{0} -x^2\,dx + \int_{0}^{1} x^2\,dx\right)$$

$$\underbrace{\frac{x^2}{2}\Big|_{-1}^{1}}_{0} - \left(\frac{x^3}{3}\Big|_{0}^{1} - \frac{x^3}{3}\Big|_{-1}^{0}\right) = 0 - \left(\frac{1}{3} - \frac{1}{3}\right) = 0 \Rightarrow E(X) = 0$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{-\infty}^{0} y \times 0\,dy + \int_{0}^{1} y \cdot 2y\,dy + \int_{1}^{\infty} y \times 0\,dy = 2\int_{0}^{1} y^2\,dy = 2\frac{y^3}{3}\Big|_{0}^{1} = \frac{2}{3}$$

$$\Rightarrow E(Y) = \frac{2}{3}$$

$$\mathrm{Cov}(X,Y) = E\big((X - E(X))(Y - E(Y))\big) = E\big(XY - X\underbrace{E(Y)}_{\frac{2}{3}} - \underbrace{E(X)}_{0}Y + \underbrace{E(X)}_{0}E(Y)\big) =$$

$$E\left(XY - \frac{2}{3}X\right) = E(XY) - \frac{2}{3}\underbrace{E(X)}_{0} = E(XY)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} y \left(\int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\right) dy$$

$$\int_{-\infty}^{0} y\left(\int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\right)dy + \int_{0}^{1} y \int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy + \int_{1}^{\infty} y \int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy$$

$$= \underbrace{\int_{-\infty}^{0} y\left(\int_{-\infty}^{\infty} x \times 0\,dx\right)dy}_{0} + \int_{0}^{1} y \int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy + \int_{1}^{\infty} y \underbrace{\int_{-\infty}^{\infty} x \cdot 0\,dx}_{0}\,dy =$$

$$= \int_{0}^{1} y \underbrace{\int_{-\infty}^{-y} x \times 0\,dx}_{0}\,dy + \int_{0}^{1} y \int_{-y}^{y} x \cdot 1\,dx\,dy + \int_{0}^{1} y \underbrace{\int_{y}^{\infty} x \cdot 0\,dx}_{0}\,dy =$$

$$= \int_{0}^{1} y \int_{-y}^{y} x\,dx\,dy = \int_{0}^{1} y\, \frac{x^2}{2}\Big|_{-y}^{y}\,dy = \int_{0}^{1} y\, \underbrace{\frac{y^2 - (-y)^2}{2}}_{0} = \int_{0}^{1} 0 = 0$$

$$\Rightarrow \boxed{\mathrm{Cov}(X,Y) = 0}$$

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ب)

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & o.w \end{cases} \qquad f_X(x) = \begin{cases} 0 & |x| > 1 \\ 1-|x| & |x| \le 1 \end{cases} \qquad f_{XY}(x,y) = \begin{cases} 1 & 0 \le y \le 1 \ , \ |x| \le y \\ 0 & o.w \end{cases}$$

اگر مستقل باشد $\forall x,y :\ f_X(x) f_Y(y) = f_{XY}(x,y)$

$0 \le |x| \le y \le 1 :\ f_Y(y) f_X(x) = 2y(1-|x|) \overset{\text{باید}}{=} f_{XY}(x,y) = 1$

y ثابت بگیریم $\rightarrow$ $2y(1-|x|)$ درباره x متغیر است اما باید ثابت باشد در x $\Rightarrow$ مستقل نیستند