پرهام رضائی ۴۰۰۱۰۸۴۷

سوال ۱)

$$f_{XY}(x,y) \overset{\text{مستقل اند}}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

$$\begin{matrix} A = X \\ B = \frac{X}{Y} \end{matrix} \Rightarrow \begin{matrix} X = A \\ Y = \frac{A}{B} \end{matrix} \Rightarrow \text{Jacobian} = \begin{bmatrix} 1 & 0 \\ \frac{1}{B} & \frac{-A}{B^2} \end{bmatrix} = \frac{-A}{B^2} - 0 \Rightarrow |\text{Jacobian}| = \frac{|A|}{B^2}$$

$$f_{AB}(a,b) = f_{XY}\left(a, \frac{a}{b}\right) |\text{Jacobian}| = \frac{1}{2\pi} e^{-\frac{\left(a^2 + \frac{a^2}{b^2}\right)}{2}} \frac{|a|}{b^2} = \frac{|a|}{2\pi b^2} e^{\frac{-a^2(1+b^2)}{2b^2}}$$

$-\infty \leq a, b \leq \infty$

ب) روش اول:

$$f_{\frac{X}{Y}}(v) = \int_{-\infty}^{\infty} f_{X,\frac{X}{Y}}(u,v)\, du = \int_{-\infty}^{\infty} \frac{|u|}{2\pi v^2} e^{\frac{-u^2(1+v^2)}{2v^2}} du \overset{\text{تقارن}}{=} 2\int_{0}^{\infty} \frac{u}{2\pi v^2} e^{-\frac{u^2(1+v^2)}{2v^2}}$$

تعریف $c = \frac{-(1+v^2)}{2v^2}$ :

$$= \frac{2}{2\pi v^2} \int_0^{\infty} u e^{cu^2} du = \frac{1}{\pi v^2} \left( \frac{e^{cu^2}}{2c} \Big|_0^{\infty} \right) = \frac{1}{2c\pi v^2}(0-1) = \frac{-1}{2c\pi v^2}$$

$c$ منفی است $\Rightarrow e^{-\infty} = 0$

$$= \frac{-1}{2\pi v^2 \left(\frac{-(1+v^2)}{2v^2}\right)} = \frac{-2v^2}{-2\pi v^2(1+v^2)} = \frac{1}{\pi(1+v^2)} \Rightarrow f_{\frac{X}{Y}}(t) = \frac{1}{\pi(1+t^2)} \quad -\infty < t < \infty \Rightarrow$$

توزیع کوشی

ب) روش دوم

اثبات توزیع زیر

$$F_{\frac{X}{Y}}(t) = P\left(\frac{X}{Y} \leq t\right) = P(X \leq t|Y|)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t|y|} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \left( \int_{-\infty}^{t|y|} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right) dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \Phi(t|y|)\, dy \overset{\text{تقارن}}{=} \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{y^2}{2}} \Phi(t|y|)\, dy$$

$$f_{\frac{X}{Y}}(t) = F'_{\frac{X}{Y}}(t) = \frac{\partial}{\partial t}\left( \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{y^2}{2}} \Phi(t|y|)\, dy \right) =$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial}{\partial t}\left(e^{-\frac{y^2}{2}} \Phi(t|y|)\right) dy = \sqrt{\frac{2}{\pi}} \int_0^{\infty} y e^{-\frac{y^2}{2}} \varphi(ty)\, dy$$

$$= \frac{1}{\pi} \int_0^{\infty} y e^{-\frac{y^2}{2} - \frac{t^2 y^2}{2}} dy = \frac{1}{\pi} \int_0^{\infty} y e^{-\frac{(1+t^2)}{2} y^2}$$

$$u = \frac{(1+t^2)y^2}{2} \Rightarrow du = (1+t^2)\, y\, dy \Rightarrow \; = \frac{1}{\pi(1+t^2)}$$

$$f_{\frac{X}{Y}}(t) = \frac{1}{\pi(1+t^2)} \sim \text{Cauchy distribution}$$

حل ۳-

$P(Y_1 < 0) = 0$, $P(Y_2 < 0) = 0$ $\Rightarrow$ فرض می‌کنیم $Y_1, Y_2 \geq 0$ از (*).

$X = \frac{Y_1}{Y_2}$

نامساوی ثابت می‌ماند

$t \geq 0$: $F_X(t) = P\left(\frac{Y_1}{Y_2} \leq t\right) \overset{(*)}{=} P(Y_1 \leq t Y_2)$

$t < 0 \Rightarrow Y_1 < 0 \text{ or } Y_2 < 0 \Rightarrow P(t) = 0$

$$F_X(t) = \int_0^\infty \int_0^{ty} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \, dx \, dy = \lambda_1 \lambda_2 \int_0^\infty e^{-\lambda_2 y} \int_0^{ty} e^{-\lambda_1 x} dx \, dy =$$

$$\lambda_1 \lambda_2 \int_0^\infty e^{-\lambda_2 y} \left. \frac{e^{-\lambda_1 x}}{-\lambda_1} \right|_0^{ty} dy = \lambda_1 \lambda_2 \int_0^\infty e^{-\lambda_2 y} \left(-\frac{1}{\lambda_1}\right)\left(e^{-ty\lambda_1} - 1\right) dy =$$

$$-\lambda_2 \left( \int_0^\infty e^{-\lambda_2 y - ty\lambda_1} dy - \int_0^\infty e^{-\lambda_2 y} dy \right) = -\lambda_2 \left( \left. \frac{e^{\overset{\text{منفی}}{(-\lambda_2 - t\lambda_1)} y}}{-\lambda_2 - t\lambda_1} \right|_0^\infty - \left. \frac{e^{-\lambda_2 y}}{-\lambda_2} \right|_0^\infty \right)$$

$$= -\lambda_2 \left( \frac{0 - 1}{-\lambda_2 - t\lambda_1} - \frac{0 - 1}{-\lambda_2} \right) = -\lambda_2 \left( \frac{1}{\lambda_2 + t\lambda_1} - \frac{1}{\lambda_2} \right) = 1 - \frac{\lambda_2}{\lambda_2 + t\lambda_1} = \frac{t\lambda_1}{\lambda_2 + t\lambda_1}$$

$$\Rightarrow f_X(t) = \frac{\partial F_X(t)}{\partial t} = \left(\frac{t\lambda_1}{\lambda_2 + t\lambda_1}\right)' = \left(1 - \lambda_2(\lambda_2 + t\lambda_1)^{-1}\right)' = -\lambda_2 \left((\lambda_2 + t\lambda_1)^{-1}\right)' = -\lambda_2 \left[ -(\lambda_2 + t\lambda_1)^{-2} (\lambda_2 + t\lambda_1)' \right]$$

$$= +\lambda_2 (\lambda_2 + t\lambda_1)^{-2} \lambda_1 = \boxed{\frac{\lambda_1 \lambda_2}{(\lambda_2 + t\lambda_1)^2}} \Rightarrow \begin{cases} 0 & x < 0 \\ \dfrac{\lambda_1 \lambda_2}{(\lambda_2 + x\lambda_1)^2} & x \geq 0 \end{cases}$$

ب)

$$P(Y_1 \leq Y_2) = \int_0^\infty \int_0^y \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \, dx \, dy =$$

$$\lambda_1 \lambda_2 \int_0^\infty e^{-\lambda_2 y} \int_0^y e^{-\lambda_1 x} dx \, dy = \lambda_1 \lambda_2 \int_0^\infty e^{-\lambda_2 y} \left. \frac{e^{-\lambda_1 x}}{-\lambda_1} \right|_0^y dy$$

$$= \lambda_1 \lambda_2 \int_0^\infty e^{-\lambda_2 y} \left(-\frac{1}{\lambda_1}\right)\left(e^{-\lambda_1 y} - e^0\right) dy = -\lambda_2 \int_0^\infty e^{-\lambda_2 y} \left(e^{-\lambda_1 y} - 1\right) dy$$

$$= -\lambda_2 \left( \int_0^\infty e^{-(\lambda_2 + \lambda_1) y} dy - \int_0^\infty e^{-\lambda_2 y} dy \right) = -\lambda_2 \left( \left. \frac{e^{\overset{\text{منفی}}{(-\lambda_2 - \lambda_1)} y}}{-\lambda_2 - \lambda_1} \right|_0^\infty - \left. \frac{e^{\overset{\text{منفی}}{-\lambda_2} y}}{-\lambda_2} \right|_0^\infty \right)$$

$$= -\lambda_2 \left( \frac{0 - 1}{-\lambda_2 - \lambda_1} - \frac{0 - 1}{-\lambda_2} \right) = -\lambda_2 \left( \frac{1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_2} \right) = 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$\Rightarrow P(Y_1 \leq Y_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Sum = 1

implies $P(Y_2 \leq Y_1) = \frac{\lambda_2}{\lambda_1 + \lambda_2}$

حل ۴ -

$$* \quad E\left(e^{t(a+bX)}\right) = e^{at} E\left(e^{btX}\right) = e^{at} M_X(bt)$$

جمع n متغیر تصادفی برنولی داریم ← $X \sim Binomial(n,p)$ که میدانیم
(با توزیع یکسان)

$$Z = \frac{X-\mu}{\sigma} \quad \Leftarrow \quad E(X) = np \qquad Var(X) = npq \qquad q = 1-p$$

طبق ستاره * داریم که:

$$E\left(e^{\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right)t}\right) = e^{-\frac{\mu}{\sigma}t} E\left(e^{\frac{X}{\sigma}t}\right)$$

$$M_X\left(\frac{t}{\sigma}\right) = E\left(e^{\frac{X}{\sigma}t}\right) = \sum_{x=0}^{n} e^{\frac{tx}{\sigma}} \binom{n}{x} p^x (1-p)^{n-x} = \left(e^{\frac{t}{\sigma}}p + (1-p)\right)^n = \left(1 + p\left(e^{\frac{t}{\sigma}} - 1\right)\right)^n$$

$$\Rightarrow M_X(t) = e^{-\frac{\mu}{\sigma}t}\left(1 + p\left(e^{\frac{t}{\sigma}} - 1\right)\right)^n$$

اثبات *:

$$E\left(e^{t(a+bX)}\right) = \sum_{x:\, a+bX=x} e^{t\overset{x}{\cancel{(a+bx)}}} P(a+bX = x) =$$

$$\sum e^{t(a+bx)} P(X=x) = e^{at} \sum e^{btx} P(X=x)$$

$E(X|Y)=A \qquad E(Y|X)=B \qquad \operatorname{cov}(X+A,Y+B)-\operatorname{cov}(A,B) \overset{?}{=} 3\operatorname{cov}(X,Y)$

حل:

$$\operatorname{cov}(X+A,Y+B)=\operatorname{cov}(X,Y+B)+\operatorname{cov}(A,Y+B)=\operatorname{cov}(X,Y)+\operatorname{cov}(X,B)+\operatorname{cov}(A,Y)+\operatorname{cov}(A,B)$$

$$\Rightarrow \operatorname{cov}(X+A,Y+B)-\operatorname{cov}(A,B)=\operatorname{cov}(X,Y)+\operatorname{cov}(X,B)+\operatorname{cov}(A,Y) \overset{?}{=} 3\operatorname{cov}(X,Y)$$

$$\Leftrightarrow \operatorname{cov}(X,B)+\operatorname{cov}(A,Y) \overset{?}{=} 2\operatorname{cov}(X,Y)$$

$$\operatorname{cov}(X,B)=E\big((X-E(X))(B-E(B))\big)=E\big((X-E(X))(E(Y|X)-E(Y))\big)=E\big(XE(Y|X)-XE(Y)-E(X)E(Y|X)+E(X)E(Y)\big)$$

Adam's law: $E(E(X|Y))=E(X)$

$E(E(Y|X))=E(Y)$

$$\operatorname{cov}(A,Y)=E\big((A-E(A))(Y-E(Y))\big)=E\big((E(X|Y)-E(X))(Y-E(Y))\big)=E\big(YE(X|Y)-YE(X)-E(Y)E(X|Y)+E(X)E(Y)\big)$$

خطی بودن امید ریاضی

$$\Rightarrow \operatorname{cov}(X,B)+\operatorname{cov}(A,Y)=E\big(XE(Y|X)+YE(X|Y)-XE(Y)-YE(X)-E(X)E(Y|X)-E(Y)E(X|Y)+2E(X)E(Y)\big)$$

$$\overset{?}{=} 2\operatorname{cov}(X,Y)=2E\big((X-E(X))(Y-E(Y))\big)=2E\big(XY-XE(Y)-YE(X)+E(X)E(Y)\big)=E\big(2XY-2XE(Y)-2YE(X)+2E(X)E(Y)\big)$$

انتقال

$$\xrightarrow{} \Leftrightarrow E\big(XE(Y|X)+YE(X|Y)+\underbrace{XE(Y)}_{\text{constant}}+\underbrace{YE(X)}_{\text{constant}}-\underbrace{E(X)E(Y|X)}_{\text{constant}}-\underbrace{E(Y)E(X|Y)}_{\text{constant}}-2XY\big)=0$$

$$\Leftrightarrow E\big(XE(Y|X)+YE(X|Y)-2XY\big)+\underline{E(Y)E(X)}+\underline{E(X)E(Y)}-\underline{E(X)E(Y)}-\underline{E(Y)E(X)}=0$$

$$\Leftrightarrow E\big(XE(Y|X)+YE(X|Y)-2XY\big)=0$$

$$\Leftrightarrow E(E(XY|X))+E(E(YX|Y))-2E(XY)=0$$

$E(h(X)Y|X)=h(X)E(Y|X)$ طبق

چون برای $X=x$ $h(x)=h(x)$

آدام

$$\Leftrightarrow E(XY)+E(XY)-2E(XY)=0$$

$$\Leftrightarrow 0=0 \quad \boxed{\checkmark}$$

اثبات : Adam's law

$$E(E(X|Y)) = E\left(\sum_x x \, P(X=x|Y)\right) = \sum_y \left[\sum_x x \, P(X=x|Y=y)\right] \cdot P(Y=y)$$

$$= \sum_y \sum_x x \, P(X=x, Y=y) \overset{\text{finite}}{=} \sum_x \sum_y x \, P(X=x, Y=y) = \sum_x x \sum_y P(X=x, Y=y)$$

$$= \sum_x x \, P(X=x) = E(X)$$

---

$* \, E(Y|X) = f(X)$

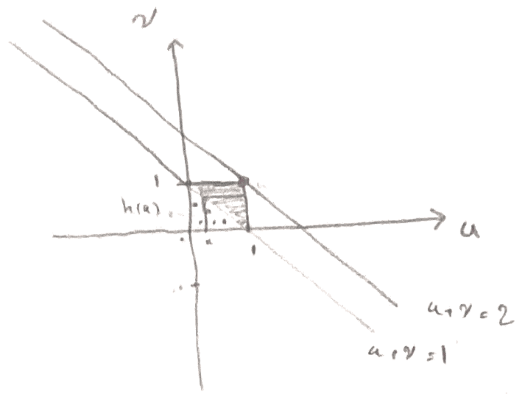
$$* \quad E(\underbrace{h(X)}Y|X) = h(X)E(Y|X)$$

که $X$ را فرض کردیم

پس برخورد با $h(X)$ مثل ثابت است.

-۶

حل تصویری:




احتمال هر نقطه ناحیه ۱ برابر $\frac{1}{2}$
و ناحیه ۲ برابر $\frac{3}{2}$
و مابقی ۰ است.

الف)

$\Rightarrow$ if $u \notin [0,1]$ : $f_U(u) = 0$

else:

$\frac{h(u)}{1} = \frac{1-u}{1} \Rightarrow h(u) = 1-u \Rightarrow f_U(u) = \frac{1}{2}(1-u) + \frac{3}{2}u = \frac{1}{2} + u$

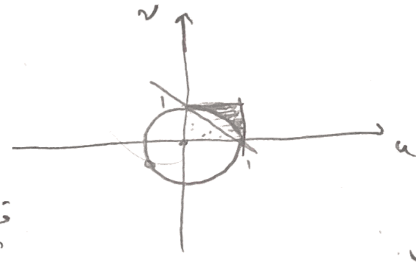
روش دیگر: $f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv \overset{\text{if } u\notin[0,1]}{=} \int_{-\infty}^{\infty} 0\,dv = 0$ $\quad f(u,v) = 0 \;\; v \notin [0,1] \rightarrow \int_1^{2-u} \frac{3}{2}\,dv = 0$

$\Rightarrow = \int_0^1 f_{U,V}(u,v)\,dv = \int_0^{\overset{\text{کمتر از ۱}}{1-u}} \frac{1}{2}\,dv + \int_{1-u}^{\overset{\text{بیشتر از ۱}}{2-u}} \frac{3}{2}\,dv = \int_0^{1-u} \frac{1}{2}\,dv + \int_{1-u}^{1} \frac{3}{2}\,dv$

$= \frac{v}{2}\Big|_0^{1-u} + \frac{3v}{2}\Big|_{1-u}^{1} = \frac{1-u}{2} + \left(\frac{3}{2} - \frac{3-3u}{2}\right) = \frac{1-u}{2} + \frac{3u}{2} = u + \frac{1}{2}$

ب)

$P(X^2+Y^2 \geq 1)$



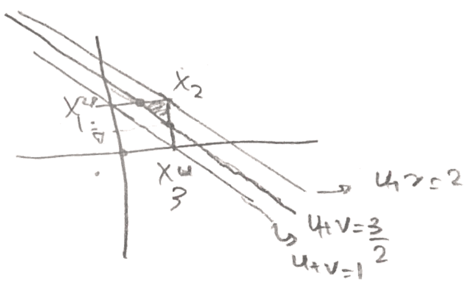

یعنی خارج دیسک و روی دیسک کشیده شده.

یعنی مساحت ناحیه ضرب در احتمال آن که $\frac{3}{2}$ است

$A = \underbrace{1\times 1}_{\text{مربع}} - \underbrace{\frac{1^2\pi}{4}}_{\text{قطاع } \frac{1}{4} \text{ دایره}} = 1 - \frac{\pi}{4}$

$\Rightarrow P(X^2+Y^2 \geq 1) = \left(1-\frac{\pi}{4}\right)\frac{3}{2} = \frac{3}{2} - \frac{3\pi}{8}$

$P(X+Y \leq \frac{3}{2})$




$P(X+Y > \frac{3}{2}) + P(X+Y \leq \frac{3}{2}) = 1$

$\Rightarrow P(X+Y > \frac{3}{2})$ را حساب می‌کنیم،

یعنی مساحت ناحیه ضرب در احتمال هر نقطه که برابر $\frac{3}{2}$ است پس طبق تالس به نسبت $\frac{1}{2}$ داریم

$\frac{[A]}{[X_1\hat{X_2}X_3]} = \frac{1}{4} \Rightarrow \overset{\text{ناحیه}}{[A]} = \frac{1}{4} \times \frac{1\times 1}{2} = \frac{1}{8} \Rightarrow P(X+Y > \frac{3}{2}) = \frac{1}{8} \times \frac{3}{2} = \frac{3}{16}$

$\Rightarrow P(X+Y \leq \frac{3}{2}) = 1 - \frac{3}{16} = \frac{13}{16}$

$P(X+Y \leq \frac{3}{2})$

$0 \leq x \leq 1 \Rightarrow \frac{3}{2} \geq \frac{3}{2} - x \geq 0{,}5$

$$\int_0^1 \int_0^{\frac{3}{2}-x} f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_0^{1-x} f_{X,Y}(x,y)\, dy\, dx + \int_0^1 \int_{1-x}^{\min(1, \frac{3}{2}-x)} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^1 \frac{y}{2}\Big|_0^{1-x} dx + \int_0^{0{,}5} \int_{1-x}^{1} \frac{3}{2}\, dy\, dx + \int_{0{,}5}^{1} \int_{1-x}^{\frac{3}{2}-x} \frac{3}{2}\, dy\, dx$$

$$\frac{1}{2}\int_0^1 1-x\, dx + \int_0^{0{,}5} \frac{3y}{2}\Big|_{1-x}^{1} dx + \int_{0{,}5}^{1} \frac{3y}{2}\Big|_{1-x}^{\frac{3}{2}-x} dy\, dx =$$

$$\frac{1}{2}\, 1 - \frac{1}{2}\frac{x^2}{2}\Big|_0^1 = \frac{1}{4}$$

$$\frac{3}{2}\int_0^{0{,}5} x + \frac{3}{2}\int_{0{,}5}^{1} \frac{1}{2}\, dx$$

$$= \frac{1}{4} + \frac{3}{4} x^2\Big|_0^{0{,}5} + \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} + \frac{1}{4} + \frac{3}{16} = \frac{6+4+3}{16} = \frac{13}{16}$$

---

$P(X^2 + Y^2 \geq 1)$

$* \quad x+y \geq 1 \longrightarrow$

اثبات: (میدانیم $X, y \geq 0$)

$\therefore x^2+y^2 < 1 \Leftarrow x^2+y^2 \leq x^2+y^2+2xy = (x+y)^2 < 1$ فرض خلف $x+y<1$

$$\int_0^1 \int_{\sqrt{1-x^2}}^{1} f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_{\sqrt{1-x^2}}^{1} \frac{3}{2}\, dy\, dx = \frac{3}{2}\int_0^1 (1-\sqrt{1-x^2})\, dx =$$

$$\frac{3}{2} x\Big|_0^1 - \frac{3}{2}\int_0^1 \sqrt{1-x^2}\, dx = \frac{3}{2} - \frac{3}{2}\int_0^1 \sqrt{1-x^2}\, dx = \frac{3}{2} - \frac{3\pi}{8}$$

$$\int_0^1 \sqrt{1-x^2}\, dx \qquad \begin{matrix} x = \sin\theta \\ dx = \cos\theta \end{matrix} \qquad \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{\pi}{4} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2\theta\, d\theta = \frac{\pi}{4} + \frac{1}{2}\, \frac{\sin 2\theta}{2}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{4}(\sin\pi - \sin 0) = \frac{\pi}{4}$$

۷-

الف)

$Var(X_1+\dots+X_n) = Var(X_1)+\dots+Var(X_n) + 2\sum_{i<j} cov(X_i,X_j)$

introduction to statistic (stat 110)

اثبات و تفسیر در کتاب

اثبات استقرا:

$Var(X+Y)=Var(X)+Var(Y)+2cov(X,Y)$

$Y=Z+T$ $Var(X+Z+T) = Var(X)+Var(Z)+Var(T)+2cov(Z,T)+2cov(X,Z)+2cov(X,T)$

$\vdots$

همین در کتاب stat chapman

$\Rightarrow var(X_1+\dots+X_n) = n\sigma^2 + 2\binom{n}{2}\eta = n\sigma^2 + n(n-1)\eta$

ب)

$cov(X_m+X_{m+1}+X_{m+2}, X_{m+j}+X_{m+j+1}+X_{m+j+2}) = \sum_{0\le i,k\le 2} cov(X_{m+i}, X_{m+j+k})$

9 terms

if $j=0$: $= \sum_{0\le i\le 2} cov(X_{m+i},X_{m+i}) + \sum_{0\le j\ne i\le 2} cov(X_{m+i},X_{m+j}) = 3\sigma^2+6\eta$

3 term    6 terms

if $j=1$: $cov(X_m+X_{m+1}+X_{m+2}, X_{m+1}+X_{m+2}+X_{m+3}) = 2\sigma^2+7\eta$

2 تا ($X_{m+1},X_{m+1}$) ($X_{m+2},X_{m+2}$)

if $j=2$ $cov(X_m+X_{m+1}+X_{m+2}, X_{m+2}+X_{m+3}+X_{m+4}) = \sigma^2+8\eta$

($X_{m+2},X_{m+2}$) اجت برابر

else:

$= 9\eta$

هیچ جفت برابر نیست.