Quantum Hall EffectS and Topology

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Abstract

In this brief article we explain the quantum Hall effects (QHE's) from two different points of views: First we show how quantum states can reproduce the effects. Furthermore, we show how the fractional quantum Hall effect (FQHE) arises from a Berry phase and predict the existence of an emergent statistical field. Later in the text, we develop the Chern-Simons theory, a theory of gauge fields in 3 dimensions and use this theory to revisit the problems of QHE's. At the end of the article, we give a brief and heuristical presentation of the relation between Chern-Simons theory and knot theory and relate this with the existence of just two statistics in 4 dimensions, bosonic and fermionic.

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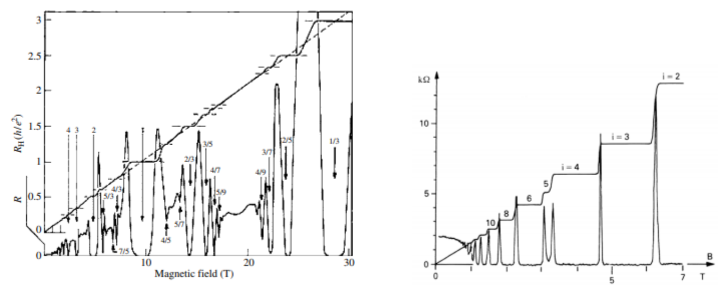
Part I
Introduction

1 Short Historical Context

In 1879 Edward Hall discovered the Hall effect while he was working on his Ph.D. thesis. In these effects, we have the emergence of a voltage, called Hall voltage across a conductor material and in the presence of a perpendicular magnetic field. This voltage implies a conductivity and this conductivity is linear with the magnetic field.

In a 2 dimensional material in the presence of a strong magnetic field and low temperatures, Klaus von Klitzing observed an exactly quantized Hall conductivity fig.(1.1b), and won the Nobel prize in 1985 for this discovery. This quantization appears in plateaux in the experimental data. This effect is called the Integer quantum Hall effect.

Very shortly after, Tsui, Stormer, and Gossard found the plateaux quantized but for fractional numbers fig.(1.1a). Laughlim “explained” this effect later and argued that this effect arises from the interactions between electrons and proposed a miracle wave function to attack this problem. The 1998 Nobel prize was awarded by Tsui, Stormer, and Laughlim for these discoveries.



(a) Fractional quantum Hall effect (b) Integer quantum Hall effect

Figure 1.1: The experimental data of the fractional and integer quantum Hall effects respectively

2 Drude Model (classical theory)

Our goal is to study the transport of electrons (or charge carriers) in a 2d material in the presence of external electromagnetic fields. In order to do that, we will suppose in this text that in some way, charge carriers are constrained in 2d by some potential. We can suppose that the spins of the particles are neglected, i.e the magnetic field is strong enough to not consider the spins-spins interactions. First, let’s orient our 2d system in a way that the magnetic field is in the normal direction. Now we turn on an electric field parallel to the material, like as shown in figure (2.1). Because of our hypothesis, we don’t need more than that.

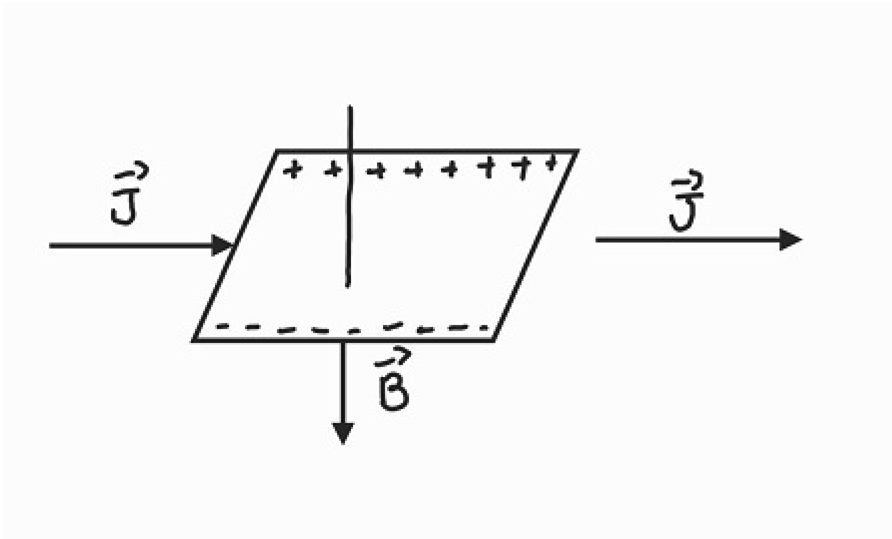


Figure 2.1: System of interest

The magnetic field will deflect the charge carriers and this effect will be responsible for the accumulation of charges in the edges. If we wait, the charges will cancel the action of the magnetic field. We are interested in the transportation after this "thermalization".

After this thermalization, the equations of motion of the charge carriers are:[1]

$$0 = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B} - \mu\frac{\mathbf{v}}{\tau} \tag{2.1}$$

where μ is the effective mass of the particle and e is the charge. If we use the system of interest and take $\mathbf{B}$ in the $\hat{k} = \hat{e}_3$ direction and $\mathbf{J} = -ne\mathbf{v}$ [2] in the $\hat{i} = \hat{e}_1$ direction, we have:

$$eE_i = \epsilon^{ij3}\frac{J_j}{n}(B\hat{e}_3) + \mu\frac{J_i}{\tau ne}. \tag{2.2}$$

Defining the Levi-Civita in 2d by $\epsilon^{ij} = \epsilon^{ij3}\hat{e}_3$, and the conductivity in the absence of the magnetic field by $\sigma_D = \frac{ne^2\tau}{\mu}$, we find

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \frac{1}{\sigma_D}\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}\begin{pmatrix} J_1 \\ J_2 \end{pmatrix}. \tag{2.3}$$

Here, to simplify notation I will define the cyclotron frequency $\omega_B = \frac{eB}{\mu}$. We can recognize this as *Ohm's Law.* So the (tensorial) resistivity is

$$[\rho] = \frac{1}{\sigma_D}\begin{pmatrix} 1 & \frac{eB}{\mu}\tau \\ -\frac{eB}{\mu}\tau & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}. \tag{2.4}$$

In 2d we have an interest fact. The resistance given by $R = \rho L^{d-2}$ is clearly equal to the resistivity, so when we perform a measurement, we measure these two quantities.

Equation (2.4) gives us two predictions: The resistivity $\rho_{xx} = \frac{1}{\sigma_D}$ is a constant when we vary the field B; in contrast, the transverse resistivity $\rho_{xy} = \frac{1}{ne}B$ have a linear dependence with respect to B, as can be seen in figure (2.2). We call $R_H \equiv \frac{1}{ne}$ the **Hall coefficient**, it is the inclination of the curve $\rho_{xy} \times B$.

When we go to quantum mechanics unfortunately this predictions are not quite corrects. Before we talk about that, I will introduce some topics that will be require in later parts.

Notes

[1]The last term is due to the collisions of the electrons with the lattice. τ is called scattering time, and is interpreted as the time between two collisions of the particles with the lattice

[2]n is the density of particles; this notation can be a problem in the following sections, so we must be careful.

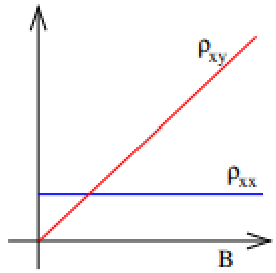


Figure 2.2: Classical predictions.

3 Berry Phases and Berry Connections

In a totally different context (at least now) of the first part, we will now talk about Berry phases in quantum mechanics.

In quantum mechanics we define a physical state up to multiplicative constants. We can neglect the phase in must of the cases, but Berry has shown that when we have adiabatic changes, phases have observable consequences.

Let's suppose the Hamiltonian of the system of interest is given by $H(x^a, \mathbf{R})$, where we write the collective parameters[3] as $\mathbf{R}$. x^a are dynamical variables[4]. Suppose that the parameters vary adiabatically with time, i.e $\mathbf{R} = \mathbf{R}(t)$.

We have the Schrodinger eq.

$$H(\mathbf{R})|\psi(t)\rangle = i\frac{d}{dt}|\psi(t)\rangle. \tag{3.1}$$

We assume that the system at $t = 0$ is in the n-th excited state, i.e $|\psi(0)\rangle = |n, \mathbf{R}(0)\rangle$ (we suppose no level crossing allowed), where

$$H(\mathbf{R}(0))|n, \mathbf{R}(0)\rangle = E_n(\mathbf{R}(0))|n, \mathbf{R}(0)\rangle. \tag{3.2}$$

How does this system evolve? Despite the answer, it looks it should be something like

$$|\psi(t)\rangle = \exp\left[-i\int_0^t E_n(s)ds\right]|n, \mathbf{R}(t)\rangle, \tag{3.3}$$

but this isn't right. This fact is easily checked when we put (3.3) in (3.1)[5]. To fix this problem, we can try the anzatz

$$|\psi(t)\rangle = e^{i\eta(t)} \times \exp\left[-i\int_0^t E_n(s)ds\right]|n, \mathbf{R}(t)\rangle. \tag{3.4}$$

Inserting (3.4) in (3.1) the LHS is

$$H(\mathbf{R}(t))|\psi(t)\rangle = E_n(\mathbf{R}(t))|\psi(t)\rangle$$

and the RHS is

$$i\frac{d}{dt}|\psi(t)\rangle = \left[-\frac{d\eta_n(t)}{dt} + E_n(\mathbf{R}(t))\right]|\psi(t)\rangle$$

$$+e^{i\eta(t)} \times \exp\left[-i\int_0^t E_n(s)ds\right]\frac{d}{dt}|n, \mathbf{R}(t)\rangle.$$

Then equating these two sides, we find

$$\frac{d\eta_n(t)}{dt} = i\langle n, \mathbf{R}(t)|\frac{d}{dt}|n, \mathbf{R}(t)\rangle \tag{3.5}$$

$$\eta_n(t) = i\int_0^t \langle n, \mathbf{R}(t)|\frac{d}{dt}|n, \mathbf{R}(t)\rangle dt = i\int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \langle n, \mathbf{R}(t)|\nabla_{\mathbf{R}}|n, \mathbf{R}(t)\rangle d\mathbf{R}. \tag{3.6}$$

If we suppose a periodic path C in the space of parameters we get

$$\eta_n(t) = i\oint_C \langle n, \mathbf{R}(t)|\nabla_{\mathbf{R}}|n, \mathbf{R}(t)\rangle d\mathbf{R}. \tag{3.7}$$

(3.7) doesn't necessarily vanish. If we define the exterior derivative in the manifold defined via the parameters by: $d = (\partial/\partial R^\mu)\, dR^\mu$, we easily define the Berry connection

$$\mathcal{A} = \langle \mathbf{R}|\,(d|\mathbf{R}\rangle)\,. \tag{3.8}$$

(3.7) is written as

$$\eta_n(t) = i \oint_C \mathcal{A}. \tag{3.9}$$

If S is a surface which is bounded by the loop C, we can use the generalized Stoke's theorem and

$$\eta_n(t) = i \int_S d\mathcal{A}.$$

$d\mathcal{A}$ defines the Berry curvature, or field strength of $\mathcal{A}$

$$\mathcal{F} = d\mathcal{A} = (d\langle \mathbf{R}|) \wedge (d|\mathbf{R}\rangle)\,,$$

where I've used the fact that $d^2 = 0$.

So when we have a periodic adiabatic change, the wave function acquires a phase given by

$$\exp\left[i \int_S \mathcal{F}\right].$$

We can shift the energy levels to rid off the phase acquired by the time evolution. We get

$$|\mathbf{R}(t), t\rangle = e^{i\eta(t)}|\mathbf{R}(t)\rangle.$$

A very mathematical approach to this topic can be found in [14].

Notes

[3]These parameters are external entries, like some magnetic field, flashing lights and etc.
[4]These are what we usually want, like spins, the position, etc.
[5]Where we implicit assume $H(\mathbf{R}(t))|n, \mathbf{R}(t)\rangle = E_n(\mathbf{R}(t))|n, \mathbf{R}(t)\rangle$.

4 Quantum Hall Effects

In section 2 we made a prediction: we found the equations (2.4) that tell us how the resistivity behave in function of the magnetic field B. But experimentally rather distinct things happen.

Integer QHE

The first effect tell us that in the real word the resistivity (consequently the conductivity) has plateaux. These plateaux appear in

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{N}, \quad N \in \mathbb{Z}. \tag{4.1}$$

Furthermore, ρ_{xx} is null and picked in the position of the plateaux. The experimental data is shown in fig. 1.1b.
We call N filling level or filling factor and all positive integers are allowed.

But this is not the whole story.

Fractional QHE

This is the second conflicting result between experimentation and theoretical predictions. This effect looks like the former but the quantization and plateaux are rather different. In the fractional quantum Hall effect (FQHE) the resistivity is given by the same formula, i.e

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}, \quad \nu \in \mathbb{Q}. \tag{4.2}$$

The behave of ρ_{xx} appears in this case too. The experimental data is shown in fig. 1.1a.
In contrast with the IQHE, the filling factors are not all the fractional numbers. The ν's take values of the form $\nu = \frac{1}{2}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{7}, \frac{4}{3}, \frac{5}{3}, \cdots$. Excluding the first, all denominators are odd, and this has to do with the fact that electrons are fermions, something that will become clear later.

We will treat these two effects in a quantum mechanical perspective, but the goal of this paper is quantum field theory so, in the following subsections we will just present a taste of the QM treatment and focus on what will be useful to us.

4.1 IQHE

To treat the IQHE we will neglect the interactions between electrons. The electron in the electronic gas couple minimally with the electromagnetic field, so the Hamiltonian of a single electron is

$$H = \frac{1}{2\mu}\left(\mathbf{p} - e\mathbf{A}\right)^2. \tag{4.3}$$

Because of the gauge freedom, we can solve the equation for eigenvalues and eigenstates of this Hamiltonian in a lot of ways. In order to make the life simple, lets choose $A_y = Bx$ and $A_x = 0$. Then in differential operator form, H is given by

$$H = \frac{1}{2\mu}\left[\partial_x^2 + (\partial_y - eA_y)^2\right]. \tag{4.4}$$

In order to find the solution of eigenvalues and eigenstates of the (4.4), we try the anzatz $\psi_{k,n}(x,y) = e^{iky} f_n(x)$ (it is a smart choice, since the form of the gauge potential give us a transnational invariance in the y-direction). Now defining the magnetic length as $l = \sqrt{\frac{1}{eB}}$[6]

$$\frac{\omega_B}{2}\left[-l^2\partial_x^2 + \left(\frac{x}{l} - lk\right)^2\right] f_n(x) = E_n f_n(x). \tag{4.5}$$

(4.5) is just a displaced oscillator. Then defining $x_k = l^2 k$, we find

$$\psi_{n,k}(x,y) = e^{iky} H_n\left(\frac{x}{l} - lk\right) e^{-\frac{(x-x_k)^2}{2l^2}} \tag{4.6}$$

$$E_n = \omega_B\left(n + \frac{1}{2}\right). \tag{4.7}$$

n is called a Landau level. Each Landau level (L.l) has many electrons states, since there are many possible k's and x_k's. The momentum k is quantized in the y direction that has for example length W

$$k_m = m\frac{2\pi}{W}. \tag{4.8}$$

The quantum number m has a maximum. Indeed, because (4.5) is a displaced harmonic oscillator and the length of the material is L in x direction, the maximum displacement is $x_{k_{max}} = L$, then for a fixing L.l we have a maximum m

$$m_{max} = \frac{LW}{2\pi l^2}. \tag{4.9}$$

Restoring the $\hbar$ in the definition of the magnetic length, we have

$$m_{max} = \frac{LWBe}{h}, \tag{4.10}$$

where we can identify LWB as the flux Φ of the magnetic field in the material. Defining the flux quantum $\Phi_0 = 2\pi\frac{\hbar}{e}$, we have

$$m_{max} = \frac{\Phi}{\Phi_0}. \tag{4.11}$$

Then the density of L.l in the material is given by $n_B = \frac{m_{max}}{LW} = \frac{eB}{2\pi\hbar}$ and the number of filled L.ls is easily found

$$N = \frac{n}{n_B} \in \mathbb{Z}. \tag{4.12}$$

Then the transverse resistivity is given by:

$$\rho_{xy} = \frac{1}{ne}B = \frac{1}{Nn_B e}B = \frac{2\pi\hbar}{Ne^2}. \tag{4.13}$$

This is exactly the IQHE. In part II we will rederive this result by a QFT approach. An explanation for the plateaux can be found at [11, 16].

4.2 FQHE

The FQHE arises from interactions. In addition to the external field $\mathbf{A}$,we have internal interactions and the Hamiltonian of the system is given by (4.3) plus interaction terms between the particles. Laughlin proposed a wave function for the ground state of this Hamiltonian in the symmetric gauge $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ given by:

$$\langle z_1, z_2, \cdots | m\rangle = \prod_{j<k} (z_j - z_k)^m \exp\left[-\frac{1}{4l^2}\sum_i |z_i|^2\right] \tag{4.14}$$

where $z_j = x_j + iy_j$. The interpretation of m will become clear in a second, but we can conclude now that m must be odd in order to make the state described a fermionic state. If we compare (4.13) with (4.2), we can naively interpret the FQHE as an IQHE in particles with fractional filling factor. One way to find this fractional factor is to endow the “particles” that fill the Landau levels with fractional charges. We will call these particles “quasi-particles” and “quasi-holes”.

We need to find a consistent wave function for quasi-holes. I will postulate

$$\langle z_1, z_2, \cdots | \eta\rangle \equiv \psi_m^{+\eta}(z_1, z_2, \cdots) = \mathcal{N}_+ \prod_{i=1} (z_i - \eta) \times \langle z_1, z_2, \cdots | m\rangle, \tag{4.15}$$

where η is the position of the quasi-hole that we will treat like parameters of the wave function and $\mathcal{N}_+ = \sqrt{1/\langle \eta|\eta\rangle}$ is a normalization factor. To see the consistence of (4.15) and the meaning of m, we need to realize that the wave function (4.14) describes the fractional felling L.l $\nu = \frac{1}{m}$. The equation (4.15) represent a quasi-hole because in the position η the density of electrons is 0, so we hope that the charge carried by the particle described by (4.15) is a positive charge q.

For M quasi-holes in positions $\eta_1, \eta_2, \cdots, \eta_M$, the wave function is

$$\psi_m^{+\eta_1,\eta_2,\cdots}(z_1, z_2, \cdots) \sim \prod_{j=1}^{M}\prod_{i=1} (z_i - \eta_j) \times \langle z_1, z_2, \cdots | m\rangle,$$

if we have m quasi holes in the position η, then

$$\psi_m^{+\eta_1,\eta_2,\cdots}(z_1, z_2, \cdots) \sim \prod_{i=1} (z_i - \eta)^m \times \langle z_1, z_2, \cdots | m\rangle \tag{4.16}$$

and comparing (4.16) and (4.14), we find that this is just an electron in position η. The wave function of quasi-particles is more subtle and is given by

$$\psi_m^{-\eta}(z_1, z_2, \cdots) = \mathcal{N}_- \prod_{i=1} (2\frac{\partial}{\partial z_i} - \bar{\eta}) \times \langle z_1, z_2, \cdots | m\rangle, \tag{4.17}$$

and the charge of a quasi-particle has the same minus sign of the electron.

We hope that $q = -\frac{e}{m}$. To show this, lets move a quasi-hole around a circle of radius R and find the Berry phase which here will be called $\gamma(t)$. This Berry phase must match with the phase given by the Aharanov-Bohm effect. When we use the Aharanov-Bohm effect, we have

$$\frac{q}{\hbar}\oint \mathbf{A}.d\mathbf{l} = 2\pi \frac{q}{e}\frac{\Phi}{\Phi_0}. \tag{4.18}$$

From the Berry phase side of calculation, we must find

$$\frac{d\gamma}{dt} = \langle \psi_m^{+\eta} | \frac{d}{dt} \psi_m^{+\eta} \rangle. \tag{4.19}$$

In order to find (4.19), we can apply ln in (4.15) and perform the time derivative, we find

$$\frac{d}{dt}\psi_m^{+\eta} = \sum_{i=1} \frac{d}{dt} \ln\left(z_i - \eta(t)\right) \psi_m^{+\eta} \implies$$

$$\frac{d\gamma}{dt} = \langle \psi_m^{+\eta} | \sum_{i=1} \frac{d}{dt} \ln\left(z_i - \eta(t)\right) | \psi_m^{+\eta} \rangle. \tag{4.20}$$

If we use the density of electrons in the presence of a quasi-hole $\rho^{\eta}(z) = \langle \psi_m^{+\eta} | \sum_{i=1} \delta\left(z_i - z\right) | \psi_m^{+\eta} \rangle$,we have

$$\frac{d\gamma}{dt} = i \int dxdy \rho^{\eta}(z) \frac{d}{dt} \ln\left(z - \eta(t)\right)$$

where $z = x + iy$. Now we use chain rule and integrating in η (in the complex plane) using Cauchy's integral formula, we find

$$\gamma = i2\pi i \times \overbrace{\int dxdy \rho^{\eta}(z)}^{\langle n \rangle_R}.$$

where $\langle n \rangle_R$ is the mean number of particles in the circle of radius R.

Because of the meaning of the integral, we can do the following approximation: we can approximate the integral by the total number of electrons that contribute in the conduction band, i.e total number of electrons in the L.l[7], but we already derived this number, (4.11) divided by the felling factor (we are dealing with fractional QHE), i.e

$$\gamma = -2\pi \frac{1}{m} \frac{\Phi}{\Phi_0}. \tag{4.21}$$

Now, equating (4.18) and (4.21), we find $q = -\frac{e}{m}$. This result is interesting and confirms what we already said: The charge of the quasi-hole has different sign of the electron charge and is a fraction of the electron charge.

To derive the statistics of the quasi-particles we use the wave function of two quasi-particles:

$$\psi_m^{+\eta_a,\eta_b}(z_1, z_2, \cdots) = \mathcal{N}_{a,b} \prod_{i=1} (z_i - \eta_a)(z_i - \eta_b) \times \langle z_1, z_2, \cdots | m \rangle. \tag{4.22}$$

We can follow the same steps of the above derivation, but in this case we have two different behaviors: If we choose the quasi-hole a to move around a circle of radius R,the particle b can be inside the circle or outside the circle. In the first case nothing changes, but in the second case, the mean number $\langle n \rangle_R$ is $\langle n \rangle_R \sim \frac{1}{m}\frac{\Phi}{\Phi_0} - \frac{1}{m}$, i.e we need to count the quasi-holes inside the circle. Then the phase is shifted

$$\Delta\gamma = \frac{2\pi}{m}. \tag{4.23}$$

(4.23) is a statistical phase (happens when we interchange two quasi-particles). We can find this statistical phase using the Aharanov-Bohm effect in an emergent statistical gauge field **a** exactly in the same way we did to find the fractional charge. We will treat this problem in a QFT approach in the next section using this statistical gauge field. A similar derivation of this results are find in [6, 16] and the result agrees with the one presented in [7].

Notes

[6] We are making $\hbar = 1$ unless stated otherwise.

[7] We can see this by a semi-classical interpretation of L.l, that fix each L.l to a quantized radius of the path followed by a particle in the presence of a magnetic field

Part II
QHE and Topology

In this section we introduce the concept of a topological quantum field theory (TQFT) and use these concepts to once again solve the QHE's problem. To be more specific we will use a particular TQFT called Chern-Simons theory to treat the QHE's.

5 Topology and Chern-Simons Theory

A TQFT is QFT that doesn't depend on the metric of the base manifold (if we talk in the language of fiber bundles). We can do a precise definition of TQFT telling about a set of observables that do not depend on the metric, but this is not necessary to what we want. For us a TQFT is a QFT whose action does not depend on the metric; we will call this action a topological action.

An action for a field theory in (2+1)D of the form

$$S = \int d^{2+1}x \epsilon^{\mu\nu\lambda} \left(a_\mu b_\nu c_\lambda\right), \tag{5.1}$$

is a topological action. To prove this fact we just need to remember how the covariant vector transforms under a diffeomorphism, i.e

$$a_\mu(x) = \frac{\partial x'^\lambda}{\partial x^\mu} a'_\lambda(x'). \tag{5.2}$$

Now is straightforward to show that

$$\epsilon^{\mu\nu\lambda} a_\mu(x) b_\nu(x) c_\lambda(x) = \det\left[\frac{\partial x'}{\partial x}\right] \epsilon^{\sigma\tau\rho} a'_\sigma(x') b'_\tau(x') c'_\rho(x'). \tag{5.3}$$

On the other hand, the element of volume transforms as $d^{2+1}x' = \det\left[\frac{\partial x'}{\partial x}\right] d^{2+1}x$, then the net effect is

$$d^3x \epsilon^{\mu\nu\lambda} a_\mu(x) b_\nu(x) c_\lambda(x) = d^3x' \epsilon^{\sigma\tau\rho} a'_\sigma(x') b'_\tau(x') c'_\rho(x'). \tag{5.4}$$

(5.4) tells that we don't need to insert $\sqrt{-g}$ in the action to make it invariant, so

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0. \tag{5.5}$$

The Chern-Simons (CS) action is of the form (5.1) with $a_\mu = c_\mu$ and $b_\nu = \partial_\nu$, i.e

$$S_{CS} \equiv CS = \frac{k}{4\pi} \int d^{2+1}x \epsilon^{\mu\nu\lambda} \left(a_\mu \partial_\nu a_\lambda\right). \tag{5.6}$$

CS is clearly a topological action, is not a good classical theory, because it is not gauge invariant. If we take the gauge transformation $a_\mu \to a_\mu + \partial_\mu \chi$, we have

$$\delta(CS) = \frac{k}{4\pi} \int d^{2+1}x \epsilon^{\mu\nu\lambda} \partial_\mu \left(\chi \partial_\nu a_\lambda\right), \tag{5.7}$$

where we cancel other terms because $\partial_\nu \partial_\mu$ is symmetric and $\epsilon^{\mu\nu\lambda}$ anti-symmetric. If M is our manifold space-time and ∂M the boundaries, we can use the generalized Stoke's theorem and write

$$\delta(CS) = \frac{k}{4\pi} \int_M d^{2+1}x \epsilon^{\mu\nu\lambda} \partial_\mu \left(\chi \partial_\nu a_\lambda\right) = \frac{k}{4\pi} \int_{\partial M} d\sigma_\mu \left(\epsilon^{\mu\nu\lambda} \chi \partial_\nu a_\lambda\right) \tag{5.8}$$

where $d\sigma_\mu$ is the oriented surface element. (5.8) is a boundary term, but is a very troublesome boundary term. It does not disappear as we are used to see. To better understand what this means, lets take a gauge transformation and focus on the matter content of the theory. By a gauge transformation, the matter field transforms as $\psi \to e^{ie\chi/\hbar}\psi$, this is a single valued transformation but χ doesn't need to be single valued. We should consider χ defined only mod 2π, i.e

$$\chi \sim \chi + 2\pi \frac{\hbar}{e}. \tag{5.9}$$

So because of this fact χ is not single valued, the boundary term (5.8) won't vanish as usual. *Quantization makes things more interesting.*

Notes

[6]We are making $\hbar = 1$ unless stated otherwise.
[7]We can see this by a semi-classical interpretation of L.l, that fix each L.l to a quantized radius of the path followed by a particle in the presence of a magnetic field

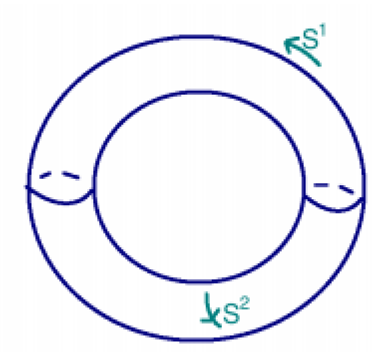


Figure 6.1: The space-time manifold $S^2 \times S^1$ for the system.

6 Chern-Simons and QHE's

In order to use CS theory in QHE's we will quantize this theory and do things right. The first step we will take is to consider as spatial domain the S^2 sphere and a periodic time S^1; this is, for example, the case when we want to deal with a statistical theory and calculate the partition function of the theory $Tr\left[e^{-\beta H}\right]$ using Euclidean time. A sketch of the manifold $S^2 \times S^1$ is fig. (6.1).

If we choose the gauge $\chi = 2\pi \frac{\hbar}{e}\tau$, where τ is the Euclidean time, the requirement that $e^{i\chi e/\hbar}$ is single valued is satisfied.

Now we can perform the integral (5.8) using the standard notation $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$,

$$\begin{aligned}\delta(CS) &= \frac{k}{4\pi}\left[\int d^{2+1}\partial_\mu\left(\chi\epsilon^{\mu\nu\lambda}f_{\nu\lambda}\right)\right] \\ &= \frac{k}{4k}\left\{\int dx^0dx^1dx^2\left[\partial_0\left(\chi f_{12}\right)+\partial_1\left(\chi f_{20}\right)+\partial_2\left(\chi f_{01}\right)\right]\right\}\end{aligned} \tag{6.1}$$

Choosing the gauge $A_0 = s$ where s is constant, we have (performing the derivatives and doing partial integration)

$$\begin{aligned}\delta(CS) &= \frac{k}{2\pi}\int dx^0dx^1dx^2\partial_0\left(\chi f_{12}\right) \\ &= \frac{k}{2\pi}\left[\int_{S^2} dx^1dx^2 f_{12}\right]\left(2\pi\frac{\hbar}{e}\right)\end{aligned} \tag{6.2}$$

where in the second equivalence we integrate over the periodic time and restore $\hbar$. The integral that appears in (6.2) is just the flux of the magnetic field in the sphere[8], and is given by the Dirac quantization $n\frac{\hbar}{e}$ where n is a positive integer. In the end of calculation

$$\delta(CS) = k\frac{\hbar^2}{e^2}2\pi.$$

In quantum mechanics, the action appears just in the path integral in $\exp\frac{i}{\hbar}S$, then if $\delta(CS)/\hbar = N2\pi$ where N is an integer, the quantum theory of the Chern-Simons action is acceptable. So we conclude that in a quantum theory, k is quantized and is

$$k = N\frac{e^2}{\hbar}. \tag{6.3}$$

Now we will try to apply this in the QHE's and see what we're going to achieve.

6.1 Integer QHE

In context of QHE, the external electromagnetic field is just a background and not a dynamical field of the theory. In 3D spatial dimensions, we have a sort of contributions of the external field. If we denote the 3D spatial domain by W_3, examples of these contributions are ferromagnetism and ferroeletricity, and electric and magnetic susceptibilities given respectively by

$$S' = \int_{W_3\times\mathbb{R}} \vec{a}.\vec{E} + \vec{b}.\vec{B},$$

$$S'' = \int_{W_3\times\mathbb{R}} \alpha_{ij}E_iE_j + \beta_{ij}B_iB_j.$$

The above actions are allowed in (3+1)D space time because they are explicitly gauge invariant, in fact they have just a dependence over the electric and magnetic fields, making it straightforward to prove gauge invariance from this point of view. When we look to "our" 2D spatial domain in a finite temperature (so we have a QFT approach with periodic euclidean time), we can allow the CS theory.

It's important to note that these actions are not fundamental actions, but just effective actions that appear after we integrate other degrees of freedom in the path integral approach. Then introducing a coupling of the CS effective action with the current J^μ, we have

$$S_{eff}\left[A\right] = CS + \int d^{2+1}x \left(J^\mu A_\mu\right) + \cdots . \tag{6.4}$$

By looking the form of the coupling, we can find the mean value of the current:

$$\mathcal{Z}_{eff}[A] = e^{i\frac{S_{eff}[A]}{\hbar}} \implies \langle J_\mu\rangle = -i\hbar\frac{1}{\mathcal{Z}_{eff}}\frac{\delta\mathcal{Z}_{eff}}{\delta A_\mu} = \frac{\delta S}{\delta A_\mu}. \tag{6.5}$$

When we apply the equation (6.5) to our electromagnetic background using the quantization (6.3), we easily find for component $F_{0i} = E_i$ of electromagnetic strength field

$$\langle J_i\rangle = \frac{N}{2\pi}\frac{e^2}{\hbar}\epsilon_{ij}E_j \tag{6.6}$$

but this equation exactly reproduces the integer QHE like we want.

Now we need to look to the FQHE in the QFT point of view, but we have some clues of what we need to do.

6.2 Fractional QHE

In the end of section 4.2 we claimed that one can reproduce the FQHE by introducing a statistical gauge field a_μ. In the end of the last sections we concluded that we can use CS theory to reproduces the QHE. So we can guess that the FQHE must arise from a statistical gauge field with CS action. Here we must be careful, as the statistical field emerges from quantum dynamics of the particles, so a_μ must be a dynamical field.

This statistical field must couple with the particles and with the electromagnetic potential field A_μ. In order to keep the topological character of the theory, we must have the action

$$S = \int d^{2+1}x \left(\frac{e^2}{2\pi\hbar}\epsilon^{\nu\mu\rho}a_\nu\partial_\mu A_\rho - \frac{me^2}{4\pi\hbar}\epsilon^{\nu\mu\rho}a_\nu\partial_\mu a_\rho + j^\mu a_\mu + \cdots\right) \tag{6.7}$$

where the first term is the coupling with the external electromagnetic field and the 3^{th} term is the coupling with the particles. m is clearly an integer by the work we already did. The functional integral is then

$$\mathcal{Z}[a,A] = \int \mathcal{D}a \exp\left[\frac{i}{\hbar}S\left[a,A\right]\right].$$

We need to integrating out the field a, but this is easy. Defining the shift in the current j_μ by $\tilde{j}_\mu = j_\mu - \left(\frac{e^2}{2\pi\hbar}\right)\epsilon^{\mu\nu\rho}\partial_\nu A_\rho$ we can use the functional Gaussian integration and end up with the effective action

$$\mathcal{Z}_{eff}[A] = \mathcal{N}\exp\left[S_{eff}\left[A\right]\right] \implies S_{eff} = \frac{\pi\hbar}{me^2}\int d^{2+1}x\left[\tilde{j}_\mu\left(\frac{\epsilon^{\mu\nu\rho}\partial_\nu}{\partial^2}\right)\tilde{j}_\rho\right] \tag{6.8}$$

where $\mathcal{N}$ is just a normalization that arises from the integration of a. This effective action contain 3 terms:

1. An CS type term, like one we already dealt with:

$$S_{CS.eff} = \frac{e^2}{4\pi\hbar m}\int d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho); \tag{6.9}$$

2. A source term:

$$S_s = \frac{1}{m}\int d^3x A_\mu j^\mu; \tag{6.10}$$

3. An interaction term:

$$S_{int} = \frac{\pi\hbar}{me^2}\int d^3y\int d^3x\left[j_\mu(x)\left(\frac{\epsilon^{\mu\nu\rho}\partial_\nu}{\partial^2}\right)j_\rho(y)\right]. \tag{6.11}$$

Using the eq. (6.5), we find from the first term

$$\langle J_i \rangle = \frac{1}{2\pi m} \frac{e^2}{\hbar} \epsilon_{ij} E_j, \tag{6.12}$$

until now ν is just an integer. This reproduces the fractional QHE, but we need to prove that m is odd. We will do this later.

The second term tells us about the fractional charge of the "effective particles".

The last term tells us about the fractional statistics of the "effective particles". These effective particles are the quasi-particles that we see in section 4.

To understand how this last term give rise to the fractional statistics, we can note that[9]: If we have two particles in positions x_1^i and x_2^i in the far past and these particles move around each other and end up respectively in x_1^f and x_2^f in the far future, the amplitude for this configuration is,

$$W = \int \mathcal{D}A \langle x_1^f, x_2^f | P | x_1^i, x_2^i \rangle e^{\frac{i}{\hbar} S_{eff}[A]}.$$

where, P is the propagator. For our proposes it's enough to analyse two cases: when the particles come back to x_1^i and x_2^i, i.e, $x_1^i = x_1^f$ and $x_2^i = x_2^f$and when the particles are exchanged, i.e, $x_1^i = x_2^f$ and $x_2^i = x_1^f$.

The propagator is usually proportional to $\frac{1}{-(\partial_\mu + iA_\mu)^2} = \frac{1}{-D^2} = \int_0^\infty d\alpha \exp\left[+i\alpha D^2\right]$, and when sandwiched by bras and kets, we will have a factor $\sim \exp\left[i\int A_\mu dx^\mu\right]$.

If we denote the amplitude for exchange by W_e and the amplitude for the direct path by W_d, we can show that these amplitudes will be related by

$$W_d = W_e \langle \exp\left[i \oint_\Gamma A_\mu dx^\mu\right] \rangle, \tag{6.13}$$

where Γ is the difference of paths between the direct process and the exchange process when we identify the end of the paths. We can write the exponential as

$$\exp\left[i \oint_\Gamma A_\mu dx^\mu\right] = \exp\left[i \int d^3x A_\mu j^\mu\right]. \tag{6.14}$$

Now we need to deal with the average. Because we are in (2+1)D the effective action need to be the CS action, and we can implement the same technique as before, shift the field, integrating out, and find

$$\begin{aligned} \langle \exp\left[i \oint_\Gamma A_\mu dx^\mu\right] \rangle &= \langle \exp\left[i \oint_\Gamma A_\mu dx^\mu\right] \rangle_{CS} \\ &= \exp\left[\frac{\pi}{m} \int d^3y \int d^3x \left[j_\mu(x) \left(\frac{\epsilon^{\mu\nu\rho}\partial_\nu}{\partial^2}\right) j_\rho(y)\right].\right] \end{aligned} \tag{6.15}$$

Using the current for discrete particles $j^0 = \sum \delta(x_i - x_i(\tau))$ and $\vec{j}(x_i, \tau) = \sum \dot{x}_i \delta(x_i - x_i(\tau))$, we will have

$$\langle \exp\left[i \oint_\Gamma A_\mu dx^\mu\right] \rangle = \exp\left[i \frac{\pi}{m}\right]. \tag{6.16}$$

This reproduces the result of (4.23) without an annoying factor of 2. This can be understood by the following argument: In this last experiment, we can interpret like in the first fig of (6.2), the black paths represent an direct process and the red represent an exchange process. In contrast (4.23) it was obtained when we move particle 1 around particle 2, exactly the second diagram in (6.2). Now, when we identify the end of the paths (glue the top of the 3^{th} plane and the bottom of the first plane), the paths in the second diagram cross with each other the double of times of the first, and we can see that we have the flux of the particle 1 in the particle 2 and the flux of particle 2 in particle 1.

Now, because of the statistic factor (6.16) we can understand why m, must be odd. When we have k quasi-particles and these particles move around them self and the other particles, we will have an statistical factor given by $\frac{\pi}{m}k^2$. We start the analyses with fermions and in the development of the theory we broke these fermions in m parts, then when we take these m peaces and put all together in a bag, we must have a fermion again. So the statistical factor of this bag is $m\pi$, and m must be odd in order to recover the fermionic statistics $\psi(1,2) = -\psi(2,1)$.

Now we can ask the following question: can we deform these diagrams into a non knoted/linked diagram, that can be shrunk to a point? The answer is NO! To clarify, lets give some more thought on this question.

There are a lot of references about Chern-Simons theory and QHE's; the ones we used the most were [21, 20, 15, 16]. The last derivation that shows us the emergent statistics is done in details in [7].

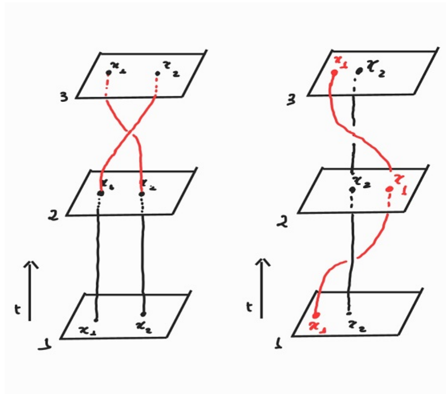


Figure 6.2: First diagram represents the last mental experiment, when the black lines represent the direct process and the red lines represent the exchange process and we have an overall path (the collage of red with black). The second diagram represents the mental experiement of section 4.2 and give us phase (4.23).

Notes

[8]It's very hard to do this experimentally, but we just need the possibility of doing it.
[9]Here in order to simplify notation, let's back to $\hbar = e = 1$.

Part III

Knot Theory

The theory that answer the last question is called knot theory and was the reason for the first physicist winning the Fields medal, Edward Witten. If we can disentangle, the phase acquired must be

$$\langle \exp\left[i\oint_\Gamma A_\mu dx^\mu\right]\rangle_{CS} = 1. \tag{6.17}$$

Lets call this quantity in the brackets Wilson Loop, and denote by

$$W_i(s) = \exp\left[is\oint_{\Gamma_i} A_\mu dx^\mu\right] \tag{6.18}$$

where s is an integer that labeled the representation of the connection A_μ. Witten showed that $\langle W_i(s)\rangle_{CS}$is an link invariant. This means that if we can deform a link Γ_i (in a 3D manifold) into a link Γ_i', then

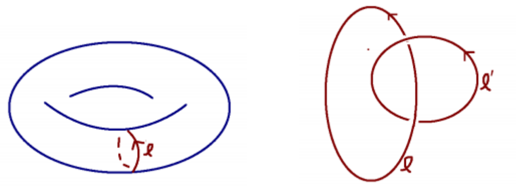
$$\langle W_i(s)\rangle_{CS} = \langle W_i'(s)\rangle_{CS} \tag{6.19}$$

where the $W_i'(s)$ is the Wilson Loop relative to the link Γ_i'. So, if Γ is an link/loop that can be shrunk to a point, we trivially have $W(s)$ =1 and $\langle W(s)\rangle_{CS} = 1$.

Then, the phase (6.16) that arises from an Aharanov-Bohm effect, has two possible causes:

1. The link Γ is defined in a 3 manifold that has non trivial topology, like our periodic time;

2. Another Wilson Loop $W'(s')$ can exist an the paths Γ and Γ' cannot be untangled, and one quasi-particle in the path Γ will be affected by the quasi-particle in the path Γ'. This is the consequence of the statistical fractional phase.



(a) non contratible loop on torus. (b) Link of two loops on $\mathbb{R}^3$.

Figure 6.3: Representation of the two sources of the phase.

But, how we can calculate $V_i(s) = \langle W_i(s) \rangle_{CS}$? To illustrate how mathematicians calculate this objects, lets take the fundamental representation of $SU(2)$. In this representations $V_i(G) = \langle W_i(G) \rangle_{CS}$ are called Jones Polynomials and will be denoted by $J(i)$. All kinds of crosses of links and loops are summarized by fig. (6.4).

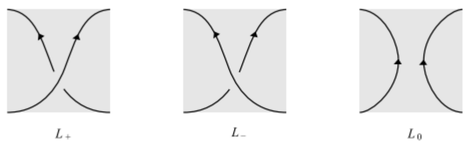
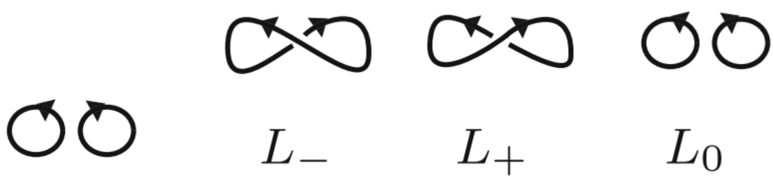


Figure 6.4: Fundamental crosses

Now suppose we want calculate the Jones Polynomial of the link represented in the fig. (6.5a), two unlinked circles. To do that we will cut the "knot" and use L_+, L_- and L_0 to fill this cut. We will obtain the fig. (6.5b).

The Jones Polynomials for these 3 new links must satisfy what we call Skein relation $\gamma V_{L_+}(G) + \beta V_{L_-}(G) + \alpha V_{L_0}(G) = 0$, for some γ, β and α. These coefficients are given by the representation that we look. For our case, $\gamma = t^{-1}$, $\beta = -t$ and $\alpha = \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)$, i.e

$$t^{-1} J(L_+) - t J(L_-) + \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right) J(L_0) = 0. \tag{6.20}$$



(a) Link of interest. (b) Links obtained by the incertion of the fundamental links L_+, L_- and L_0.

Figure 6.5: Example of computation of the Jones Polynomial

We can easily note that the first two links are the unknot link (we are in 3D), then by what we know $J(L_+) = J(L_-) = 1$ (the average of the trivial knot). The 3^{th} knot is exactly what we want, and by using the Skein relation (6.20) , we find

$$J(L_0) = -t^{\frac{1}{2}} - t^{-\frac{1}{2}}.$$

Using these recursion relation we can find the Jones Polynomials of more troubled links.

The relation between Knots and gauge theory (CS theory for our case) explain why in 4D (or 3+1) we have just trivials statistics. When we add an extra dimension, we can allways disentangle the knots. This can be visualized in lower dimentions: When we have the plane without the origin $\mathbb{R}^2/\{0\}$ we cannot shrunk to a point loops that "contain" the origin. When we go to $\mathbb{R}^3/\{0\}$ we can use the extra dimention to move the loops arround the origin and we can shrunk every loop to a point. Now we can take $\mathbb{R}^3$and take off the x direction for example $\mathbb{R}^3/\{x\}$. In this case, loops that contain de x axis cannot be shrunk to a point. Perform the same approach, when we go to $\mathbb{R}^4/\{x\}$ we will be able to move the loops arround de x axis and shrunk avery loop to a point.

Now if we identify the end point of the x axis, we will have fig. (6.3b). This line of thinking show us that in more than 3 dimensions, we have just trivial statistics.

The releation between Chern-Simons and the emergent statistics is briefly discussed in [20] and we can find a very pretty approach in the watershed article [19]. This last article is the responsible for the Fields medal awarded by Edward Witten.

Introductions to the mathematical content of knot theory can be found in [8, 3].

7 Conclusions

We saw in this work the power of quantum field theory in the treatment of the QHE's, when we use the Chern-Simons theory as an effective theory to find the quantization of the conductivity. In the integer quantum Hall effect this was done by an effective theory of electromagnetic field and in factional quantum Hall effect this effective treatment was done by the introduction of an emergent statistical field.

By studying why these fields generate a statistical phase, we connect our theory with the mathematical theory of knots and this enables us to see a very interesting fact about how particle statistics relate to the topology of the space in which they are embedded.

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