Would Quantum Mechanics be a geometrical theory?

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Motivation

- Try To understand some features of QM.

1) Equivalent physical states differ by a phase

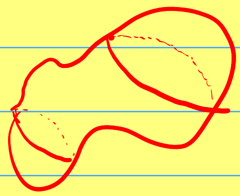
(Why ?)

$|\psi\rangle \sim \lambda|\psi\rangle \,, \; \lambda \in U(1)$

2) A smooth passage from classical To quantum

Classic system $\xrightarrow{???}$ Quantum system

3) Boundaries (confinement) and quantization

 $\xrightarrow{?}$ $|\psi\rangle$

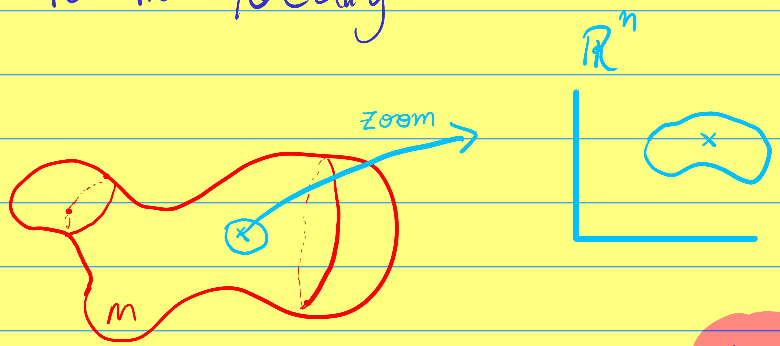
- Building of classical system of a quantum with no classical analog

(What ???)

A naive introduction to Manifolds

We know calculus on $\mathbb{R}^n$, but the universe is more than that

Def: (Manifold) Mathematical struture very similar to $\mathbb{R}^n$ locally



$\dim M = n$

The "zoom" is performed by a local map called coordinates (local)

$$\phi : M \to V \subset \mathbb{R}^n$$

$$p \longmapsto \phi(p) = (x^1(p), \ldots, x^n(p))$$

(ϕ, V) is a chart

A set of charts is an Atlas.

All the definitions on manifolds are independent of coordinates

Physics → All physical laws are independent of the choice of the description

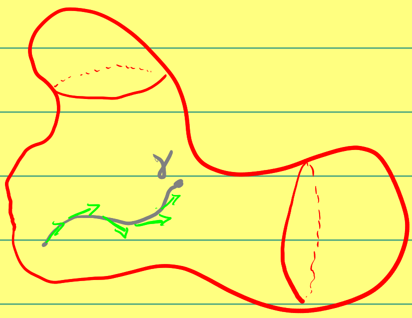
Vectors:

Let $\gamma : [a,b] \longrightarrow M$ be a curve on M

Let $f : M \longrightarrow \mathbb{R}$ be a function over M

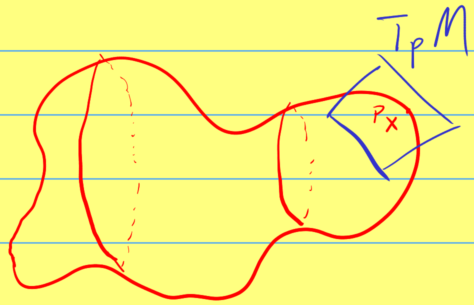
$$\frac{d}{dt} f(\gamma(t)) = \frac{d x^\mu(\gamma(t))}{dt} \frac{\partial f}{\partial x^\mu}$$

$$= \left[X^\mu \partial_\mu \right]\Big|_p f \qquad X^\mu\Big|_p = \frac{dx^\mu}{dt}\Big|_p$$



Def: Tangent vectors are differential operators

tangent vectors on $p \in M$ form a vector space T_pM, base $\{\partial_\mu\} = \{\hat{e}_\mu\}$



$$TM = \bigcup_{\forall p \in M} T_pM$$

↳ tangent bunde

Vector fields leave in TM

$X = X^\mu \partial_\mu$

↳ is defined in all M !!!

Dual Space: the cotangent space (T_p^*M) is the space of linear functionals on T_pM

$df : T_pM \to \mathbb{R} \Rightarrow df \in T_p^*M$

The base of T_p^*M must obey

$$d\varphi^\nu[\partial_\mu] = \delta^\nu{}_\mu$$

Defining the action of T_p^*M on T_pM by:

$$df[X] = X[f]$$

remember $X \in TM$ is a differential operator.

$\{dx^\nu\} = \{\hat{e}^\nu\}$ is a base of T_p^*M

then all $df \in T_p^*M$ have the form

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu$$

⚠ looks the usual definition but it's not

Cotangent bundle : $T^*M = \bigcup_{\forall p \in M} T_p^*M$

Important: If M has dimention n, TM and T^*M are manifolds of dimention $2n$

$u \in TM \Rightarrow u = (p, X_p)$, $p \in M$ e $X_p \in T_pM$

X_p → n components

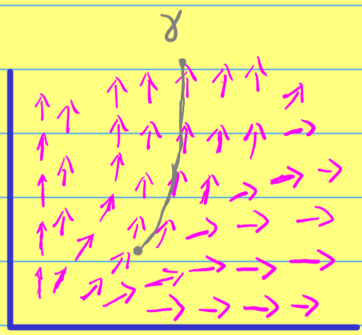
p → $+$ n components

$u = (x^1(p), \dots, x^n(p), X^1_p, \dots, X^n_p) \in \mathbb{R}^n \times T_pM$

The same to $df \in T^*M$

$$df = \left(x^1(p), \dots, x^2(p), \frac{\partial f}{\partial x^1}\Big|_p, \dots, \frac{\partial f}{\partial x^n}\Big|_p\right) \in \mathbb{R}^n \times T_p^* M$$

Flux:



γ is a integral curve of the vector field

$$X = X^\mu \partial_\mu$$

$$\frac{dx^\mu(\gamma(t))}{dt} = X^\mu(x(\gamma(t)))$$

X is tangent to the curve γ.

END OF MATHEMATICAL Review !!!

Classical Mechanics on Manifolds and n-forms

n-forms: A tensor of order (q,r) is an element of $\underbrace{TM \otimes \cdots \otimes TM}_{q\text{-copies}} \otimes \underbrace{T^*M \otimes \cdots T^*M}_{r\text{-copies}}$

Def: A totally antisymmetric $(0,n)$ tensor is called n-form

Ex: 1-form

dx^ν, df

2-form

$$dx^\mu \wedge dx^\nu \equiv dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

$$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

Antisymmetric

$\triangle$ $dx^\mu \otimes dx^\nu \neq dx^\nu \otimes dx^\mu$

$dx^\mu \wedge dx^\nu$ is a natural base for 2-forms $\Omega^2(M)$

Generalization:

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, dx^{\mu_{\pi(1)}} \otimes \cdots \otimes dx^{\mu_{\pi(n)}}$$

define $\Omega^n(M)$

Ex: If $\alpha \in \Omega^n(M)$

$$\alpha = \alpha_{\mu_1, \mu_2, \ldots, \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}$$

antisymmetric tensor

Exterior derivative:

Def: $d: \Omega^n(M) \longrightarrow \Omega^{n+1}(M)$

$$d\alpha = \frac{\partial}{\partial x^\nu} \alpha_{\mu_1 \cdots \mu_n} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

Very Important: $[\partial_\mu, \partial_\nu] = 0 \Rightarrow d^2 = 0$

d generalizes all kinds of derivatives in 3D

i) if $\alpha \in \Omega^0(\mathbb{R}^3)$

$$d\alpha = \frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy + \frac{\partial \alpha}{\partial z} dz$$

ii) if $\alpha \in \Omega^1(\mathbb{R}^3)$

$$d\alpha = \left(\frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y}\right) dx \wedge dy +$$

$$- \left(\frac{\partial \alpha_y}{\partial z} - \frac{\partial \alpha_z}{\partial y}\right) dz \wedge dy +$$

$$+ \left(\frac{\partial \alpha_x}{\partial z} - \frac{\partial \alpha_z}{\partial x}\right) dz \wedge dx$$

iii) if $\alpha \in \Omega^2(\mathbb{R}^3)$

$$d\alpha = \left(\frac{\partial \alpha_{yz}}{\partial x} + \frac{\partial \alpha_{zx}}{\partial y} + \frac{\partial \alpha_{xy}}{\partial z}\right) dx \wedge dy \wedge dz$$

iv) if $\alpha \in \Omega^3(\mathbb{R}^3)$

$\boxed{d\alpha = 0}$ exact form

Stokes Theorm

$$\oint_{\partial\Sigma} \alpha = \int_{\Sigma} d\alpha$$

Contraction: $i_X : \Omega^n(M) \longrightarrow \Omega^{n-1}(M)$

$$i_X \alpha = X^{\nu} \alpha_{\nu\mu_2 \cdots \mu_n} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}$$

notation: $i_X \alpha = X \lrcorner\, \alpha$

Classical Mechanics

$Q \to$ conf. space — Manifold

$T^*Q \to$ phase space — Manifold, cot. bundle

Newton eq.

$$\frac{d^2 q_i(t)}{dt} = -\frac{1}{m}\frac{\partial V(q)}{\partial q_i}$$

$$p_i = m\frac{dq_i(t)}{dt} \Rightarrow \begin{cases} \dfrac{dq_i}{dt} = \dfrac{p_i}{m} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial V(q)}{\partial q_i} \end{cases}$$ Hamilton eq.

Solutions of Hamilton eq. are integral curves in T^*Q (the phase space).

$\hookrightarrow$ Local coordinates

$(q_1, q_2, \dots, q_n, p_1, \dots, p_n)$

generated by the Hamiltonian vector field

$$\Xi_H = \frac{1}{m} p_i \overset{\hat{e}_{q_i}}{\frac{\partial}{\partial q_i}} - \frac{\partial V}{\partial q_i} \overset{\hat{e}_{p_i}}{\frac{\partial}{\partial p_i}}$$

$$H = \frac{1}{2m} p^2 + V$$

By definition

$$i_{\Xi_H} \omega = -dH \qquad (*)$$

where ω is a 2-form on T^*Q

$$-\omega = dq \wedge dp \qquad \text{symplect 2-form}$$

So the solution of Newton eq. is equivalente to integrate a flux defined by $(*)$

The symplectic 2-form is exact

$$d\omega = 0 \Rightarrow \qquad \omega = d\Theta$$

symplectic potential

$$\Theta = p\, dq$$

We can generalize the hamiltonian flux

$H \to f$

$$\Xi_f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}$$

Defines the Poisson brahket

$$\Xi_f(g) = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} = -\{f, g\}$$

If $\gamma(t)$ is an integral curve of Ξ_f, by definition

$$\frac{d}{dt} g(\gamma(t)) = -\{f, g\}(\gamma(t))$$

$\Rightarrow$ $\{f, g\} = 0$ if $g = cte$ along $\gamma(t)$

$\Rightarrow$ H is a conserved function along Ξ_H

$$\{H, H\} = 0$$

Pre - Quantization

We need to define geometrical strutures, such that

$$Q_{pre} : f \longrightarrow \hat{f}$$

function in M $f(q,p)$

Quantum operator on $L^2(M)$

C.1) is Linear $\quad Q_{pre}(f+g) = Q_{pre}(f) + Q_{pre}(g)$

C.2) Constant operator $\quad Q_{pre}(cte) \propto \mathbb{1}$

C.3) Dirac Quantization condition

$$-i\hbar\, Q_{pre}(\{f,g\}) = [\,Q_{pre}(f),\, Q_{pre}(g)\,]$$

(incertainty principles)

The natural try

$$Q_{pre}(f) = -i\hbar X_f$$

Is linear by definition C.1 ✓

The Dirac condition follows C.3 ✓

$$[i\hbar X_f, i\hbar X_g] = i\hbar(i\hbar X_{\{f,g\}})$$

$Q_{pre}(Cte) = 0$ C.2 ✗

Don't works

Second Try

$$Q_{pre}(f) = -i\hbar X_f + f$$

is linear C.1 ✓

$Q_{pre}(cte) = cte \cdot \mathbb{1}$ C.2 ✓

Dirac condition C.3 ✗
fails

$$[Q_{pre}(f), Q_{pre}(g)] \neq -i\hbar Q[\{f, g\}]$$

Last try

symplectic potential

$$Q_{pre}(f) = -i\hbar\left(\vec{\omega}_f - \frac{i}{\hbar} i_{\vec{\omega}_f}(\Theta)\right) + f$$

$$= -i\hbar\left(\vec{\omega}_f - \frac{i}{\hbar}\Theta(\vec{\omega}_f)\right) + f$$

is Linear C.1 ✓

$Q_{pre}(cte) = cte \cdot \mathbb{1}$ C.2 ✓

Proof of C.3

$$\left(\hat{\hat{f}} - \frac{i}{\hbar} \Theta(\hat{\hat{f}}) \right) = \nabla_{\hat{\hat{f}}}$$

Covariant Derivative

$$[\nabla_X , \nabla_Y] = \nabla_{[X,Y]} - \frac{i}{\hbar} d\Theta(X,Y)$$

symplectic curvature

$$= \nabla_{[X,Y]} - \frac{i}{\hbar} \omega(X,Y)$$

remember $\omega = dp \wedge dq$, so in particular

$$[\nabla_{\hat{\hat{f}}} , \nabla_{\hat{\hat{g}}}] = \nabla_{\{f,g\}} - \frac{i}{\hbar} \{f,g\}$$

is straightfoward conclud C.3 ✓

We find our pre-quantization.

A poroblem appear. Θ is not well defined

$\Theta + d\lambda$ is a good symplectic potential too

$$d(\Theta + d\lambda) = d\Theta + \underset{\overset{\shortparallel}{0}}{d^2\lambda} = \omega$$

This is GOOD

if we have $\Theta^1 - \Theta^2 = d\lambda$ and a unitary transformation from $L^2(M)$ to it self

$$U_\lambda \Psi = e^{i\frac{\lambda}{\hbar}} \Psi,$$

then

$$U_\lambda^{-1} \nabla_X^1 U_\lambda = \nabla_X^2 \qquad \text{Unitarily equivalent}$$

Where $\nabla_X^i = X - \frac{i}{\hbar} \Theta^i(X)$

Proof: $X(e^{i\frac{\lambda}{\hbar}} \Psi) = e^{i\frac{\lambda}{\hbar}} X(\Psi) + e^{i\frac{\lambda}{\hbar}} \frac{i}{\hbar} X(\lambda) \Psi$

- $X(\lambda) = (d\lambda)(X) = \Theta^1(X) - \Theta^2(X)$

$\Rightarrow \quad \nabla_X^1(e^{i\frac{\lambda}{\hbar}} \Psi) = e^{i\frac{\lambda}{\hbar}} \nabla_X^2(\Psi)$

in particular

$$U_\lambda Q_{pre}^1(f) U_\lambda^{-1} = Q_{pre}^2(f)$$

$\Rightarrow$ ψ and $e^{i\frac{\lambda}{\hbar}}\psi$ are equivalents

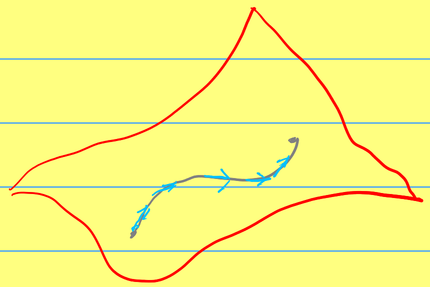
$\Rightarrow$ $h^2(M)$ is projective space

(space of rays)

When can we quantize our theory?

suppose s parallel to a curve

$\gamma : [0,s] \longrightarrow T^*Q$



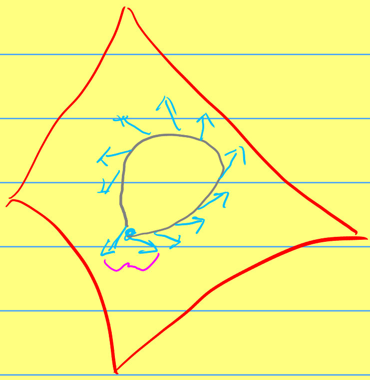
then

$$\nabla_{\dot\gamma} s = 0$$

$$= \left(\frac{ds}{dt} - \frac{i}{\hbar}\Theta(\dot\gamma)\, s \right)$$

$$\Rightarrow \quad s(t) = \exp\left[\frac{it}{\hbar}\int_\gamma \Theta\right] s(0)$$

If γ is a loop



$$s[1] = \exp\left[\frac{i}{\hbar}\oint_{\gamma} \alpha\right] s[0]$$

by stoke theorm and unity of ODE

$$s[1] = \exp\left[\frac{i}{\hbar}\int_{\Sigma} \omega\right] s[0] = \exp\left[-\frac{i}{\hbar}\oint_{\Sigma'} \omega\right] s[0]$$

because of orientation

$$\Rightarrow \quad \exp\left[\frac{i}{\hbar}\int_{\Sigma\cup\Sigma'} \omega\right] = 1$$

$$\Rightarrow \quad \frac{1}{2\pi\hbar}\int_{\Sigma\cup\Sigma'} \omega \in \mathbb{Z}$$

boundaries

I) The quantization procedure to projective spaces (equivalent descriptions)

$$|\psi\rangle \sim \lambda|\psi\rangle$$ equivalents physical states

II) Boundaries give us the answer

Quantizable or not Quantizable

Note: the condition $\frac{1}{2\pi\hbar}\int_{\Sigma\cup\Sigma'} w \in \mathbb{Z}$ is the Bohr-Somerfield quantization of the Hydrogen Atom

$$\oint p_i \, dq_i = 2\pi n \hbar$$

Now lets Q_{pre} apply in q_i and p_i

Akward

$$Q_{pre}(q_i) = q_i + i\hbar \frac{\partial}{\partial p_i}$$ ✗

$$Q_{pre}(p_i) = -i\hbar \frac{\partial}{\partial q_i}$$ ✓

Quantization and polarizations

The last equation give us the wrong result.
We need to look for some restriction.

Wave functions are functions of (q, p)

$\Psi = \Psi(q, p)$ Wrong

We must have

$\langle q | \Psi \rangle = \Psi(q)$ or $\langle p | \Psi \rangle = \Psi(p)$

Polarization: Fix a gauge θ. The position space is a subspace of $L^2(M)$, such that

$$\nabla_{\frac{\partial}{\partial p_i}} \Psi = 0 \qquad \forall i$$

The momentum space is a subspace of $L^2(M)$, such that

$$\nabla_{\frac{\partial}{\partial q_i}} \Psi = 0 \qquad \forall i$$

If we fix the gauge $\Theta = p\, dq_i$ we have

$$\nabla_{\frac{\partial}{\partial p_i}} = \frac{\partial}{\partial p_i}$$

$$\nabla_{\frac{\partial}{\partial q_i}} = \frac{\partial}{\partial q_i} - \frac{i}{\hbar} p_i$$

Then the position space is the Hilbert space of functions independents of p

$$\Psi(p,q) = \phi(q)$$

Functions of momentum space have the form

$$\Psi(p,q) = e^{\frac{i}{\hbar} p \cdot q} \phi(p)$$ (Prove by direct substitution)

If we apply the quantum operators

$$Q_{pre}(q_i) = q_i + i\hbar \frac{\partial}{\partial p_i}$$

$$Q_{pre}(p_i) = -i\hbar \frac{\partial}{\partial q_i}$$

1º) in position space

$$Q_{\cancel{pre}}(q_i)\,\Psi(q) = q_i\,\Psi(q_i)$$ ✓

$$Q_{\cancel{pre}}(p_i)\,\Psi(q) = -i\hbar \frac{\partial}{\partial q_i}\Psi(q)$$ ✓

2º) momentum space

$$Q_{\cancel{pre}}(q_i)\left[e^{i\frac{qp}{\hbar}}\phi(p)\right] = e^{i\frac{qp}{\hbar}}\left(i\hbar \frac{\partial \phi(p)}{\partial p_i}\right)$$ ✓

$$Q_{\cancel{pre}}(p_i)\left[e^{i\frac{qp}{\hbar}}\phi(p)\right] = e^{i\frac{qp}{\hbar}}\left(p_i\,\phi(p)\right)$$ ✓

These procedure give us the rights quantum spaces

Aplication to second Quantization

Kähler Manifold: Symplectic Manifold (M,ω) with local coordinates $\{q^a, p_a\}$ $a=1,\cdots,n$

A complex struture

$$J\frac{\partial}{\partial q_i} = \frac{\partial}{\partial p_i} \quad ; \quad J\frac{\partial}{\partial p_i} = -\frac{\partial}{\partial q_i}$$

$$J^2 = -\mathbb{1}$$

Complex manifolds have holomorfic coordinates

$$z^a = p_a + i q^a \quad ; \quad \bar{z}^a = p_a - i q^a .$$

J splits the tangent space into 2

$$T_pM = T_p^+M \oplus T_p^-M$$

$$T_p^{\pm}M = \{ Z \in T_pM \mid JZ = \pm i Z \}$$

T_p^+M base is $\partial_a \equiv \frac{\partial}{\partial z^a} = \frac{1}{2}\left(\frac{\partial}{\partial p_a} - i\frac{\partial}{\partial q^a}\right)$

$T_{\bar{p}}M$ base is $\partial_{\bar{a}} \equiv \frac{\partial}{\partial \bar{z}^a} = \frac{1}{2}\left(\frac{\partial}{\partial p_a} + i\frac{\partial}{\partial q^a}\right)$

We can define partial exterior derivatives, such that

$$d = \partial + \bar{\partial} \qquad [\partial, \bar{\partial}] = 0$$

if $d\Omega = 0$, then

$$\Omega = i\partial\bar{\partial}K$$

K is called Kähler scalar

if we have $\omega = dp_a \wedge dq^a$ in holomofic coordinates

$$\omega = \frac{i}{2} dZ^a \wedge d\bar{Z}^a = -i\partial\bar{\partial}K \quad ; \quad K = \frac{\bar{Z}^a Z^a}{2} = \frac{\bar{Z}Z}{2}$$

we can choose the gauge

$$\Theta = -i\partial K = -\frac{i}{2}\bar{Z}^a dZ^a$$

This is a convenient choise the polarization

$$\nabla_{\partial_{\bar a}} \Psi = 0$$

because $\Theta(\partial_{\bar a}) = 0 \Rightarrow \dfrac{\partial \Psi}{\partial \bar z^a} = 0$

$\therefore\ \Psi(\bar z, z) = \phi(z)$ holomorphic functions

By a gauge transformation $\Theta' = \Theta + i\hbar\, dK$

we know $\Psi(\bar z, z) = e^{-\bar z z} \phi(z)$

The scalar product is

$$\langle \Psi_1, \Psi_2 \rangle = \int \phi^\dagger \phi\, e^{-\bar z z} \left(\frac{d\bar z^a dz^a}{2\pi\hbar} \right)^n < \infty$$

coherent states

Because ϕ is holomorphic, we can expand

$$\phi = \phi_0 + \phi_a z^a + \phi_{ab} z^a z^b + \cdots$$

Symmetric subspace $SH = \mathbb{C} \oplus SH_1 \oplus SH_2 \oplus \cdots$
of H Boson Fock space

When we apply the quantization procedure to z^a, $\bar{z}^a$, $z^a\bar{z}^a$ we have

$$Q(z^a) = z^a \equiv a^\dagger$$

$$Q(\bar{z}^a) = \hbar\partial_a \equiv a$$

$$Q(z^a\bar{z}^a) = \hbar z^a\partial_a \equiv a^\dagger a$$

${z^a}^n$ is the number eigenvector and n is the number eigenvalue

$e^{-\bar{z}z}$ is the coherent state

Conclusions / comments / Question

- QM as consequence of symplectic curvature
- Equivalent physical states = U(1) gauge theory
- Boundaries = Quantization
- The theory is not complete
- Is a open area
- Can we thing in a deeper QM?

G-gauge theory

- The total wave function has something to do with Wigner distributions?

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