Question 3.1 (5pts) Solve the following recurrence equations by the master Theorem or any method that we discussed in class

1. $T(n) = 32T(n/4) + n^2\sqrt{n}$
2. $T(n) = 3T(n/9) + n^4 \lg n$
3. $T(n) = 8T(n/4) + n\sqrt{n}$
4. $T(n) = T(n-1) + \lg n$
5. $T(n) = 2T(n/2 - 1)$

Master Theorem: Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence:

$$T(n) = \begin{cases} aT(n/b) + f(n) & \text{if } n > 1, \\ \Theta(1) & \text{if } n = 1. \end{cases}$$

Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq kf(n)$ for some constant $k < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$.

Answer

1. $T(n) = 32T(n/4) + n^2\sqrt{n}$

   given a common form of this recurrence

   $$T(n) = aT(n/b) + f(n)$$

   so, we obtain

   $$a = 32 \qquad b = 4 \qquad f(n) = n^2\sqrt{n} = n^{5/2} = O(n^{5/2})$$

   also we have

   $$\log_b a = \log_4 32 = \log_{2^2} 2^5 = 5/2 \lg 2 = 5/2$$

   it's case 2 of master theorem where if $f(n) = n^{5/2} = \Theta(n^{\log_b a}) = O(n^{5/2})$

   then the complexity is easier to obtain i.e by second case master theorem

   $$T(n) = O(n^{5/2} \lg n)$$

2. $$T(n) = 3T(n/9) + n^4 \lg n$$

   given a common form of this recurrence

   $$T(n) = aT(n/b) + f(n)$$

   so, we obtain

   $$a = 3 \qquad b = 9 \qquad f(n) = n^4 \lg n$$

   also we have

   $$\log_b a = \log_9 3 = 1/2$$

   it seems we need to use the third case of master theorem since $n^{1/2}$ is smaller than $n^4 \lg n$

   $$f(n) = n^4 \lg n = \Omega(n^{1/2+\epsilon}) \text{ choose } \epsilon = 3$$

but we need to check the regularity condition, so the third case master theorem can be applied.

$$af(n/b) = 3f(n/9) = 3\left(\frac{n}{9}\right)^4 \lg\left(\frac{n}{9}\right) \leq \frac{3}{9^4}n^4 \lg n = \frac{3}{9^4}\lg n$$

we get $c = 3/9^4$ then the third case master theorem can be applied, it implies

$$T(n) = \Theta(n^4 \lg n)$$

3. $T(n) = 8T(n/4) + n\sqrt{n}$ given a common form of this recurrence

$$T(n) = aT(n/b) + f(n)$$

so, we obtain

$$a = 8 \qquad b = 4 \qquad f(n) = n^{3/2}$$

also we have $\log_4 8 = 3/2$, which $n^{3/2}$ same as $f(n) = n^{3/2}$ then we use second case of master theorem

$$\therefore T(n) = \Theta(n^{3/2} \lg n)$$

4. $T(n) = T(n-1) + \lg n$

$$\begin{aligned}
T(n) &= T(n-1) + \lg n \\
&= [T(n-2) + \lg(n-1)] + \lg n \\
&= [T(n-3) + \lg(n-2)] + \lg(n(n-1)) \\
&= \vdots \\
&= [T(3) + \lg(4)] + \lg(n!/4!) \\
&= [T(2) + \lg(3)] + \lg(n!/3!) \\
&= [T(1) + \lg(2)] + \lg(n!/2!) \\
&= T(1) + \lg n! \\
&= 1 + \lg n! = O(\lg n!) \\
&\leq 1 + \lg n^n = O(n \lg n)
\end{aligned}$$

which there is a fact that $O(\lg n!) = O(n \lg n)$

5. $T(n) = 2T(n/2 - 1)$ let $n = m + c$

$$\begin{aligned}
T(n) &= 2T(n/2 - 1) \\
T(m+c) &= 2T\left(\frac{m+c}{2} - 1\right) \\
T(m+c) &= 2T(m/2 + c/2 - 1)
\end{aligned}$$

find c such that

$$c = c/2 - 1 \rightarrow c/2 = -1 \rightarrow c = -2$$

and let

$$S(m) = T(m-2) = 2T(m/2 - 2)$$

also

$$S(m/2) = T(m/2 - 2)$$

we obtain

$$S(m) = 2S(m/2)$$

then by master theorem

$$a = 2 \qquad b = 2 \qquad f(n) = 0$$

also we have

$$\log_b a = \log_2 2 = 1$$

then it seems necessary to use the first master theorem case.

$$f(m) = 0 = O(m^{1-\epsilon}) \qquad , \epsilon = 0.5$$

then we get the complexity of $S(m) = \Theta(m) \rightarrow T(n) = S(n+2) = \Theta(n+2) = \Theta(n)$