Pset 4 - Theory

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1 Problem 1

Let's show that $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \xrightarrow{P} \alpha$: We know that $\hat{\beta} \xrightarrow{P} \beta$ (consistent estimator) Using the Law of Large Number, we obtain :

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i \xrightarrow{P} E(y_i) = E(\alpha + \beta x_i + u_i) \tag{1}$$

Similarly, based on the Law of Large Number, we also get :

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} x_i \xrightarrow{P} E(x_i) \tag{2}$$

Considering result (2) and the consistency of $\hat{\beta}$, we can use the Slutsky theorem to assert that :

$$\hat{\beta}\bar{x} \xrightarrow{P} \beta E(x_i) \tag{3}$$

Lastly, using Slutsky with results (1) and (3) this time, we get :

$$\hat{\alpha} \xrightarrow{P} E(\alpha + \beta x_i + u_i) - \beta E(x_i) = \alpha + E(u_i) \tag{4}$$

As $E(u_i|x_i) = 0$ is a condition for $\hat{\beta}$ to be consistent and since $E(u_i|x_i) = 0 \Longrightarrow E(u_i) = 0$, we have :

$$\hat{\alpha} \xrightarrow{P} \alpha \tag{5}$$

So $\hat{\alpha}$ is a consistent estimator of α !

2 Problem 2

We start by proving that :

$$\hat{u}_i = (u_i - \bar{u}) - \sum_{j=1}^{k}(\hat{\beta}_j - \beta_j)(x_{ij} - \hat{x}_j) \tag{6}$$

We know that :

$$\begin{aligned}\hat{u}_i &= y_i - \hat{y}_i \\ &= (\beta_0 + \sum_{j=1}^{k}\beta_j x_{ij} + u_i) - (\hat{\beta}_0 + \sum_{j=1}^{k}\hat{\beta}_j x_{ij}) \\ &= (\beta_0 - \hat{\beta}_0) + \sum_{j=1}^{k}(\beta_j - \hat{\beta}_j)x_{ij} + u_i\end{aligned} \tag{7}$$

Now that we have (7), we can take the sum over all i and divide by n:

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{k}\hat{u}_i &= (\beta_0 - \hat{\beta}_0) + \sum_{j=1}^{k}(\beta_j - \hat{\beta}_j)[\frac{1}{n}\sum_{i=1}^{n}x_{ij}] + \bar{u} \\ \underbrace{\bar{\hat{u}}}_{=0} &= (\beta_0 - \hat{\beta}_0) + \sum_{j=1}^{k}(\beta_j - \hat{\beta}_j)\bar{x_j} + \bar{u} \\ \hat{u}_i - 0 &= \hat{u}_i - (\beta_0 - \hat{\beta}_0) - \sum_{j=1}^{k}(\beta_j - \hat{\beta}_j)\bar{x_j} - \bar{u} \\ \hat{u}_i &= (\beta_0 - \hat{\beta}_0) + \sum_{j=1}^{k}(\beta_j - \hat{\beta}_j)x_{ij} + u_i - (\beta_0 - \hat{\beta}_0) - \sum_{j=1}^{k}(\beta_j - \hat{\beta}_j)\bar{x_j} - \bar{u} \\ \hat{u}_i &= (u_i - \bar{u}) - \sum_{j=1}^{k}(\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j)\end{aligned} \tag{8}$$

Now, using result (8) :

$$\hat{\sigma}^2 = \frac{1}{n-k-1}\sum_{i=1}^{n}[(u_i - \bar{u}) - \sum_{j=1}^{k}(\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j)]^2 \tag{9}$$

Let's expand this squared sum to better identify which terms tends to what :

$$\hat{\sigma}^2 = \frac{1}{n-k-1}\sum_{i=1}^{n}\left[(u_i - \bar{u})^2 - 2(u_i - \bar{u})\sum_{j=1}^{n}(\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j) + \left(\sum_{j=1}^{k}(\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j)\right)^2\right] \tag{10}$$

We know that $E[x_i u_i] = 0$ and that $E[x_i x_i']$ is non singular. Thus, the conditions are satisfied for the OLS estimator to be consistent. In other words, we can assert that :

$$\forall j, \hat{\beta}_j \xrightarrow{P} \beta_j \tag{11}$$

Using Slutsky with result (11), we can now state :

$$\sum_{j=1}^{k}(\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j) \xrightarrow{P} \sum_{j=1}^{k}(\beta_j - \beta_j)(x_{ij} - \bar{x}_j) = 0 \tag{12}$$

Similarly, using the continuous mapping theorem with $g : x \mapsto x^2$, we get :

$$\left(\sum_{j=1}^{k} (\hat{\beta}_j - \beta_j)(x_{ij} - \bar{x}_j) \right)^2 \xrightarrow{P} 0^2 = 0 \tag{13}$$

Finally, apllying the Law of Large number to the leftmost term we get :

$$\frac{1}{n-k-1} \sum_{i=1}^{n} (u_i - \bar{u})^2 \approx \frac{1}{n} \sum_{i=1}^{n} (u_i - \bar{u})^2 \xrightarrow{P} E[(u_i - \bar{u})^2] = Var(u_i) \tag{14}$$

Thus, applying the Slutsky theorem with results (10), (12), (13) and (14), we get :

$$\hat{\sigma}^2 \xrightarrow{P} Var(u_i) + 0 + 0 = \sigma^2 \tag{15}$$

Thus we can safely state that $\hat{\sigma}^2$ is a consistent estimator for σ^2.