Macroeconomics - Notes(models)

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RBC model

Solution Outline

- **Households**
  1. Solve the utility maximization problem by maximizing with respect to consumption, investments, labor and future capital (bonds might also appear in some settings)
- **Firms**
  1. Solve the profit maximization problem by maximizing with respect to current capital and labor. (The firm is not a monopoly so it does not get to set prices !)
- **Steady state**
  1. Write the equations for the non-stochastic steady state (no time index)
  2. Proceed to log linearize equilibrium equations using non-stochastic steady state equations [a]

[a] See Isabel and Gustavo's notes for the details

1.1 Set-up

1.1.1 Households

- Goal : Maximize the expected utility of the household over its lifetime by picking a path defining optimal consumption, leisure, investment and capital stock at each period depending on the state of the economy (technology shocks)
- Objective function :

$$\max_{C_t,N_t,I_t} E_0[\beta^t(\frac{C_t^{1-\gamma}}{1-\gamma} - \theta\frac{N_t^{1-\psi}}{1+\psi}] \tag{1}$$

- Constraints :

$$\begin{cases} \text{Budget : } C_t + I_t = W_tN_t + r_tK_t \\ \text{Capital Accumulation : } K_{t+1} = (1-\delta)K_t + I_t \\ \text{Time Endowment : } N_t = 1 - L_t \end{cases} \tag{2}$$

1.1.2 Firm

- Goal : Maximize the firm's profit through choosing the optimal combination/level of factor endowments at each period t
- Objective function : (no constraints here !)

$$\max_{K_t,N_t} \pi_t = Y_t - \tilde{R}_tK_t - W_tN_t \tag{3}$$

1.1.3 Steady state

Equilibrium Equations System

- **Euler Equation :** $C_t^{-\gamma} = E_t[\beta C_{t+1}^{-\gamma}(r_{t+1} + 1 - \delta)]$
- **Work/Leisure tradeoff :** $\theta N_t^{-\psi} = W_tC_t^{-\gamma}$
- **Time endowment constraint :** $L_t = 1 - N_t$
- **Flow budget constraint :** $C_t + I_t = Y_t$ [obtained from the firm's profit equation given that profit is null, combined with BC in equation (2)]
- **Capital accumulation constraint :** $K_{t+1} = I_t + (1-\delta)K_t$
- **Firm's MPK :** $\alpha\frac{Y_t}{K_t} = r_t$
- **Firm's MPN :** $(1-\alpha)\frac{Y_t}{N_t} = W_t$
- **Production function :** $Y_t = Z_tK_t^{\alpha}N_t^{1-\alpha}$
- **Law of motion of productivity/technology :** $ln(Z_t) = \rho ln(Z_{t-1}) + \epsilon_t$

Non-stochastic steady state

- Goal : We wish to find the point around wish the economy fluctuates to see how shocks create deviations from this equilibrium (how fast does the economy converge back to its steady state etc..)
- Method : see "Tricks & Results" part

Log-linearization

- Goal : Transform our system of non-linear equations in a system of linear ones. This helps greatly for our analysis (for instance it helps with moment calculation as we have a linear expectation term)
- Limitations : Log linearization is only valid if
  1. We only have small deviations from steady state
  2. The model does not have significant nonlinearities
- Method : See "Tricks & Results" part [1]

1.2 Tricks and results

1.2.1 Households

1. Simplify the constraint : Take the capital accumulation constraint and express I_t as a function of the other variables. Then substitute I_t in the budget constraint to get rid of investments.

$$\textbf{New constraint :}\ W_tN_t + r_tK_t + (1-\delta)K_t = C_t + K_{t+1} \tag{4}$$

2. Derive the FOC with respect to C_t to obtain the shadow value λ_t of increasing consumption.

$$\textbf{Expected result :}\ C_t^{-\gamma} = \lambda_t \tag{5}$$

3. Derive the FOC with respect to K_t to get the Euler Equation. Replace λ_t and λ_{t+1} using (1)

$$\textbf{Expected result :}\ \underbrace{C_t^{-\gamma}}_{=\lambda_t} = E_t[\beta \underbrace{C_{t+1}^{-\gamma}}_{=\lambda_{t+1}}(r_{t+1} + 1 - \delta)] \tag{6}$$

4. Derive the FOC with respect to N_t to get the leisure/work tradeoff equation.

$$\textbf{Expected result :}\ \theta N_t^{-\psi} = W_tC_t^{-\gamma} \tag{7}$$

1.2.2 Firm

1. Derive the FOC with respect to K_t to find the marginal product of capital (MPK)

$$\textbf{Expected result :}\ \alpha\frac{Y_t}{K_t} = \tilde{R}_t\ [2] \tag{8}$$

2. Derive the FOC with respect to N_t to find the marginal product of labor (MPN)

$$\textbf{Expected result :}\ (1-\alpha)\frac{Y_t}{N_t} = W_t \tag{9}$$

[1] For additional details, please refer to Gustavo and Isabel's notes
[2] α corresponds to the share of production due to capital (generally fixed to 2/3)

1.2.3 Steady state

Non-stochastic steady state

1. To obtain the system of non-stochastic steady state equation you need to remove all time/period indexes from the variables and rearrange each equation when needed
2. For the law of motion of productivity : The ϵ term disappears as there are no shocks in steady state
    - BEWARE : (assuming $\rho \neq 1$) you can't divide by $log(Z)$ as we can't tell whether it is equal to 0. Therefore move log on the same side and divide by the constant to end up with $Z = 1$

Non-Stochastic Steady State System

- **Euler Equation :** $1 = \beta(r + 1 - \delta) = \beta R$
- **Work/Leisure tradeoff :** $\theta N^{-\psi} = W C^{-\gamma}$
- **Time endowment constraint :** $L = 1 - N$
- **Flow budget constraint :** $C + I = Y$
- **Capital accumulation constraint :** $I = \delta K$
- **Firm's MPK :** $\alpha \frac{Y}{K} = r$
- **Firm's MPN :** $(1 - \alpha)\frac{Y}{N} = W$
- **Production function :** $ZK^{\alpha}N^{1-\alpha}$
- **Law of motion of productivity/technology :** $Z = 1$

Log-Linearization

1. We replace the variables by their exponential forms ($X_t = Xe^{\hat{X}_t}$)
2. We cancel out constant terms using steady state equations
3. If we have multiplications of variables by one another we use Taylor

Taylor Expansion Theorem

Taylor's Theorem: For $f \in C^{n+1}(I)$ where I is an open interval containing a:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \text{ where } \xi \text{ lies between } a \text{ and } x.$$

Conditions:

- $f^{(n+1)}$ exists on I
- f is continuous on $[a, x]$

4. Otherwise, using the log is sufficient to find back what we need
5. Note : for the log linearization of the EE, it is useful to use the following expression which is also obtained through log-linearization $R\hat{R}_{t+1} = r\hat{r}_{t+1}$ [3]

[3] See Gustavo's note, p.10

Log-Linearized System

- **Euler Equation :** $\hat{C}_t = E_t[\hat{C}_{t+1} - \frac{1}{\gamma}\hat{R}_{t+1}]$
- **Work/Leisure tradeoff :** $\psi\hat{N}_t = \hat{W}_t - \gamma\hat{C}_t$
- **Time endowment constraint :** $\hat{N}_t = -\frac{L}{N}\hat{L}_t$
- **Flow budget constraint :** $\frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t = \hat{Y}_t$
- **Capital accumulation constraint :** $\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t$
- **Interest rate (firm's MPK):** $\hat{r}_t = \hat{Y}_t - \hat{K}_t$
- **wage (Firm's MPN) :** $\hat{W}_t = \hat{Y}_t - \hat{N}_t$
- **Production function :** $\hat{Y}_t = \hat{Z}_t + \alpha\hat{K}_t + (1-\alpha)\hat{N}_t$
- **Law of motion of productivity/technology :** $\hat{Z}_t = \rho\hat{Z}_{t-1} + \epsilon_t$ [4]
- **Gross interest rate** $\hat{R}_t = \frac{r}{R}\hat{r}_t$

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New-Keynesian model

Solution Outline

- **Households**
  1. Intratemporal : Solve the expenditure minimization problem for Household by minimizing with respect to consumption of a given good at a given period ($C_s(i)$)
  2. Intertemporal : Solve the utility maximization problem by maximizing expected utility with respect to aggregate consumption, labor and bonds
- **Firms**
  1. Flexible prices : Solve the profit maximization problem, assuming that prices can be adjusted at each period, by maximizing with respect to the price of a given good at a given period ($P_t(i)$)
  2. Sticky prices : Solve the profit maximization problem, assuming prices have a non null probability of carrying over to the next period, by maximizing with respect to the price of a given good at the period the price is set - right before the carryover($P_t(i)$)
- **Steady State**
  1. Log Linearization : same process as with RBC with a little twist for adjusted prices[a]
  2. Phillips curve : find the equation from the phillips curve based based on the log-linearized system. It shows how current inflation depends on expected future inflation and current economic conditions (marginal cost)

[a] See "Results & Methods" part for the details

[4] be careful, we keep ϵ as is here, we do not log-linearize it

2.1 Set-up

2.1.1 Households

A) The Expenditure Minimization Problem

- Goal : Minimize total expenditures across all goods while achieving a certain level of aggregate consumption
- Objective function :

$$\min_{C_s(j)} \int_0^1 P_s(i)C_s(i)di \tag{10}$$

- Constraint :

$$C_s = (\int_0^1 C_s(i)^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}} \text{ with } C_s \text{ the desired aggregate consumption level}^{5} \tag{11}$$

B) The Utility Maximization Problem

- Goal : maximize expected utility over the lifetime of the consumer
- objective function :

$$\max_{N_s,C_s,B_s} E_t[\sum_{s=t}^{\infty} \beta^{s-t}(\frac{C_s^{1-\gamma}-1}{1-\gamma} - \phi\frac{N_s^{1+\psi}}{1+\psi})]^{6} \tag{12}$$

- Constraints :

$$\begin{cases} \text{Budget : } R_{s-1}B_{s-1} + W_sN_s + D_s = T_s + B_s + P_sC_s \text{ with } T_s = \text{taxes} \\ \text{Initial condition : } B_{t-1} = 0 \\ \text{TVC : } \lim_{s\to\infty} B_s(\Pi_{i=0}^s R_{s-i}) = 0 \end{cases} \tag{13}$$

2.1.2 Firms

A) The Profit Maximization Problem : Flexible Prices

- Goal : set optimal prices at each period so as to maximize the companies profit (monopoly behavior : does not take prices as given)
- Objective function :

$$\max_{P_s(i),N_s(i)} \pi_s(i) = (1+\tau)P_s(i)Y_s(i) - W_sN_s(i) \tag{14}$$

- Constraints :

$$\begin{cases} \text{Production function : } Y_s(i) = A_sN_s(i)^{7} \\ \text{Technology flow : } ln(A_s) = \rho_A ln(A_{s-1}) + \epsilon_s^{a\ 8} \\ \text{Monopoly : } Y_s(i) = C_s(i) = (\frac{P_s}{P_s(i)})^{\theta}C_s^{\ 9} \end{cases} \tag{15}$$

B) The Profit Maximization Problem : Sticky Prices

[6] θ is the elasticity of substitution across goods (the lower it is, the more market power firms will have

[6] γ is the inverse of IES (intertemporal elasticity of substitution), ψ the inverse of the Frisch elasticity of labor supply(the lower it is, the less responsive the household is to wage changes) and ϕ governs the degree of wage rigidity

[9] A_s is productivity/technology at period s

[9] ϵ represents technology shocks, ρ represents determines how much of the previous period's technology level carries over to the current period (if 0 shocks are short-lived

[9] The firm behaves monopolistically so it equates quantities produced with quantities demanded

- Goal : find the optimal price that maximizes profits over time given the fact that price can't be adjusted for a certain number of periods
- Problem :

$$X_t(i) = \underset{P_t(i)}{\arg\max}\, E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}\frac{C_s^{-\gamma}/P_s}{C_t^{-\gamma}/P_t}((1+\tau)P_t(i) - \frac{W_s}{A_s})(\frac{P_t(i)}{P_s})^{-\theta}C_s]\ {}^{10\ 11} \tag{16}$$

2.1.3 Planner

- objective : allocate consumption to the representative household and labor to the firms in a way that maximizes the well-being of the household. The allocation it chooses is the efficient allocation and can be compared to the allocation obtained in the decentralized case to identify inefficiencies.
- objective function :

$$\max_{C_t,N_t} U(C_t, N_t) \tag{17}$$

- constraints :

$$\begin{cases} \textbf{Aggregate Consumption Constraint : } C_t = \left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}} \\ \textbf{Production Function for each good i : } C_t(i) = A_t N_t(i), \forall i \in [0,1] \\ \textbf{Aggregate Labor Constraint : } N_t = \int_0^1 N_t(i) di \end{cases} \tag{18}$$

- results : at optimality households consume the same quantity of all goods, allocate the same amount of labor to all firms (why ? : 1) goods enter utility function in the same way, 2)concave utility) and the marginal disutility of labor equals its marginal benefit.
- potential inefficiencies in decentralized equilibrium :
  1. Price stickiness
  2. Market power (monopolies)
  3. Price dispersion (linked to price stickiness but may be problematic even in the period t-1)

[11] alternative notation : $\Lambda_s = \beta^{s-t}(\frac{C_s^{-\gamma}/P_s}{C_t^{-\gamma}/P_t})$ and $\forall s, \tilde{c_s}(i) = (\frac{P_t(i)}{P_s})^{-\theta}C_s$

[11] Λ_s is named the relevant stochastic discount factor for nominal payoff, $\tilde{c_s}(i)$ is the demand for good i conditional on the price of this good not varying over time

2.1.4 Steady state

Equilibrium Equations System

- Demand for a differentiated good : $C_s(i) = (\frac{P_s}{P_s(i)})^{\theta} C_s$
- Euler Equation (inter-temporal) : $E_t\left[\beta \frac{R_s}{\Pi_{s+1}} C_{s+1}^{-\gamma}\right] = C_s^{-\gamma}$
- Labor supply : $\phi N_s^{\psi} = \frac{C_s^{-\gamma}}{P_s} W_s$
- Labor demand (per firm) : $N_t(i) = \frac{1}{A_t}(\frac{P_t(i)}{P_t})^{-\theta} C_t$
- Labor market clearing : $N_t = \int_0^1 N_t(i)di$
- Price index : $P_t = (\int_0^1 P_t(i)^{1-\theta}di)^{\frac{1}{1-\theta}}$
- Adjusted price (sticky) : $X_t(i) = \frac{\theta}{(\theta-1)(1+\tau)} \frac{E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}\frac{W_s}{A_s}P_s^{\theta}C_s^{1-\gamma}]}{E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}P_s^{\theta-1}C_s^{1-\gamma}]}$
- Flexible price : $P_t(i) = \frac{1}{1+\tau}\frac{\theta}{\theta-1}\frac{W_t}{A_t}$
- Taylor rule (Monetary policy) : $R_t = R\Pi_t^{\phi_\pi} e^{\epsilon_t^r}$

Non-stochastic steady state and Log-linearization

- Goal : same as with the RBC model
- Method : see "Tricks & Results" part

Phillips curve

- Goal : show current inflation depends on expected future inflation and current economic conditions (marginal cost)
- Method (how to get it) : see "Tricks & Results" part

2.2 Tricks and results

2.2.1 Households

A) The Expenditure Minimization Problem

1. Derive the FOC wrt. $C_s(i)$: be mindful that we are considering a single good in a single period so the integral in the left hand term can be safely ignored (it's as if we were to take a specific term in a sum

$$\textbf{Expected result :}\ C_s(i) = (\frac{\lambda_s}{P_s(i)})^{\theta} C_s \qquad (19)$$

2. Substitute the result obtained for $C_s(i)$ in the constraint to find an expression for λ_s

$$\textbf{Expected result :}\ \lambda_s = P_s \qquad (20)$$ [12]

3. Substitute λ_s in the expression for the optimal $C_s(i)$ that we found earlier

4. Substitute this new expression for $C_s(i)$ in the objective function

Expected result : $TE = P_s C_s$ (21)

B) The Utility Maximization Problem

1. FOC wrt C_s : define λ_s as a function of the rest

Expected result : $\lambda_s = \frac{C_s^{-\gamma}}{P_s}$ (22)

2. FOC wrt N_s (work/leisure tradeoff): equalize marginal cost (hours spent working) with marginal benefit of labor (consumption derived from labor income) → can't improve situation by working more/less

Expected result : $\phi N_s^{\psi} = \frac{C_s^{-\gamma}}{P_s} W_s$ (23)

3. FOC wrt B_s (Euler equation) : use (8) and equalize marginal expected utility of future consumption and marginal benefit of current consumption → can't improve situation by consuming more now or in the future

Expected result : $E_t\left[\beta \frac{R_s}{\Pi_{s+1}} C_{s+1}^{-\gamma}\right] = C_s^{-\gamma}$ [13] (24)

2.2.2 Firms

A) The Profit Maximization Problem : Flexible Prices

1. Rearrange the problem to get rid of $N_s(i)$ and the constraint : to do that replace $C_s(i)$ by its constraint value and use the production function to express $N_s(i)$ as a function of other variables

Expected result : $\max_{P_s(i)} [(P_s(i)(1+\tau) - \frac{W_s}{A_s}] P_s(i)^{-\theta} P_s^{\theta} C_s$ (25)

2. Find the FOC wrt to $P_s(i)$: consumption and price aggregates cancel out, then just rearrange the terms

Expected result : $P_s(i) = \frac{1}{1+\tau} \frac{\theta}{\theta - 1} \frac{W_s}{A_s}$ (26)

B) The Profit Maximization Problem : Sticky Prices

- Deriving the FOC with respect to $P_t(i)$:
  1. Start by canceling out aggregate terms with index t
  2. divide by $P_t(i)^{-\theta-1}$ on each side
  3. Move all terms that do not depend on s outside of the expectation

[12] interpretation : the shadow value of increasing aggregate consumption is equal to the price index

[13] $\Pi_{s+1} = \frac{P_{s+1}}{P_s}$ and corresponds to inflation

4. Put the remaining $P_t(i)$ on one side and rearrange term to express it as a function of all other remaining terms

Expected result : $$X_t(i) = \frac{\theta}{(\theta-1)(1+\tau)} \frac{E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}\frac{W_s}{A_s}P_s^{\theta}C_s^{1-\gamma}]}{E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}P_s^{\theta-1}C_s^{1-\gamma}]} \tag{27}$$

2.2.3 Planner

1. Solving the planners problem : set up the lagrangian with a costate variable for each of the three constraints. Be careful for the production constraint as you need to introduce a constraint for each good.

expected result :
$$\mathcal{L} = U(C_t, N_t) + \lambda_t \left[C_t - \left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} \, di \right)^{\frac{\theta}{\theta-1}} \right]$$
$$+ \int_0^1 \mu_t(i) \left[C_t(i) - A_t \, N_t(i) \right] di + \nu_t \left[N_t - \int_0^1 N_t(i) \, di \right]$$

2. Take the FOC's with respect to $C_t, N_t, N_t(i)$ and use costates to find back the following optimality conditions :

expected result :
$$\begin{cases} C_t(i) = C_t, \forall i \in [0,1] \\ N_t(i) = N_t \\ -U_{n,t} = A_t U_{c,t} \end{cases} \tag{28}$$

2.2.4 Steady state

A) Non-stochastic steady state

1. We start by proceeding in the same way as in the RBC model (remove time indexes)
2. For adjusted prices : move the constant terms (W, A, P, and C) outside of the sum. The sums can then be canceled out, along with the term in P and C.
3. For aggregate labor and prices : Notice that $P_s(i), X_s(i)$ and $N_s(i)$ do not depend on i (see equation 24,25). In other words, at equilibrium, all firms will price in the same way and demand the same quantity of labor. This entails that : $\forall i, N(i) = N$ and $P(i) = P$ in the non-stochastic steady state.

Non-Stochastic Steady State System

- Demand for a differentiated good : $C(i) = (\frac{P(i)}{P})^{-\theta} C$
- Euler Equation (inter-temporal) : $\beta \frac{R}{\Pi} = 1$
- Labor supply : $\phi N^{\psi} = \frac{C^{-\gamma}}{P} W$
- Labor demand (per firm) : $N(i) = \frac{C}{A} = \frac{Y}{A}$
- Labor market clearing : $N = \int_0^1 N(i) di$
- Price index : $P = (\int_0^1 P(i)^{1-\theta} di)^{\frac{1}{1-\theta}}$
- Adjusted price (sticky) : $X = X(i) = \frac{\theta}{(\theta-1)(1+\tau)} \frac{W}{A}$
- Flexible price : same as adjusted price ! $X = P = P(i)$
- Taylor rule (monetary policy) : $1 = \Pi$ [a]

[a] The gross inflation rate Π is equal to 1 so there is zero net inflation

B) Log-linearization

1. The overall method is the same as before, however some additional tricks are necessary (especially for prices).

2. Euler Equation :

   - Replace variables by their exponential form (Xe^{x_t}) and cancel out constant terms using steady state equations
   - For efficacy, it is preferable to group together exponents and THEN use Taylor

   Trick n°1 : Take $e^{r_t - \pi_t - \gamma c_t}$ which gives $(r_t - \pi_t) - \gamma c_t + 1$ (using Taylor)[a]

   [a] WARNING : here r_t is just the equivalent of $\hat{R}_t$ which is the notation we used for the RBC model

3. Adjusted price :

   - The first steps are the same as the Euler equation (substitute variables in exp form, cancel out constants, group exponentials in each sum and use Taylor)
   - We rearrange the sums and use Taylor a second time but for functions of the form $\frac{1}{1+x}$ (see details below)

Applying Taylor : Detailed method

We start by noting :

$$\Phi_s \equiv (1-\gamma)c_s + (\theta-1)p_s \tag{29}$$

This gives us the following equation :

$$1 + x_t(i) = \frac{\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(1+\Phi_s+w_s-a_s)}{\lambda\beta)^{s-t}(1+\Phi_s)} \tag{30}$$

For improved clarity, we now set the following :

$$\begin{cases} S \equiv \sum_{s=t}^{\infty}(\lambda\beta)^{s-t} \\ \epsilon_N \equiv \sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(\Phi_s+w_s-a_s) \\ \epsilon_D \equiv \sum_{s=t}^{\infty}(\lambda\beta)^{s-t}\Phi_s \end{cases} \tag{31}$$

This gives us this, which we will simplify using Taylor :

$$1 + x_t(i) = \frac{S+\epsilon_N}{S+\epsilon_D} \tag{32}$$

We can rewrite this to take the following form :

$$1 + x_t(i) = \frac{S(1+\epsilon_N/S)}{S(1+\epsilon_D/S)} = \frac{(1+\epsilon_N/S)}{(1+\epsilon_D/S)} \tag{33}$$

Now we can linearize the denominator using Taylor [a] on $\frac{1}{1+\epsilon_D/S}$ (we take $X \equiv \epsilon_D/S$)

$$\frac{1}{1+\epsilon_D/S} \sim (1-\epsilon_D/S) \tag{34}$$

Now, we multiply it with the numerator and get rid of higher order terms :

$$\frac{(1+\epsilon_N/S)}{(1+\epsilon_D/S)} = (1-\epsilon_D/S)(1+\epsilon_N/S) = 1 + \frac{\epsilon_N}{S} - \frac{\epsilon_D}{S} = 1 + \frac{\epsilon_N-\epsilon_D}{S} \tag{35}$$

Notice the following :

$$\epsilon_N - \epsilon_D = \sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(w_t-a_t) \tag{36}$$

[a] Please refer to the math appendix 4.1

- Following the second Taylor expansion we have this expression :

$$\textbf{Expected result: } 1 + x_t(i) = 1 + \frac{\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(w_t-a_t)}{\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}} \tag{37}$$

- We are now left with two little steps : cancel out the ones and rewrite the denominator. For the latter, notice that it is a geometric series and can be rewritten as $\frac{1}{1-\lambda\beta}$[14]

[14] See math appendix 4.2

- Just flip the denominator and you're done !

4. Price index :

- Right-most equality : Write variables in exponent form and get rid of constant terms based on non-stochastic steady state. Then, use the first order Taylor approximation and cancel out the ones.
- Left-most equality, we use the fact that, at equilibrium :

$$P_t^{1-\theta} = ((1-\lambda)X_t^{1-\theta} + \lambda P_{t-1}^{1-\theta})\,[15] \tag{38}$$

- So at the NS steady state we get : $P^{1-\theta} = (1-\lambda)X^{1-\theta} + \lambda P^{1-\theta}$. From there we just apply the usual method (exp + NS steady state cancellation + Taylor expansion)

5. Expected end result :

Log-Linearized System

- Demand for a differentiated good : $c_t(i) = -\theta(p_t(i) - p_t) + c_t$
- Euler equation : $c_t = E_t[-\frac{1}{\gamma}(r_t - \pi_{t+1}) + c_{t+1}]$
- Labour supply : $w_t - p_t = \psi n_t + \gamma c_t$
- Labour market clearing : $n_t = \int_0^1 n_t(i)di$
- Labour demand : $n_t(i) = c_t - a_t - \theta(p_t(i) - p_t)$
- Price index : $p_t = \int_0^1 p_t(i)di = \lambda p_{t-1} + (1-\lambda)x_t$
- Adjusted price (sticky)[a] : $x_t(i) = (1-\lambda\beta)E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(w_s - a_s)]$ and $x_t = x_t(i)$
- Flexible price : $p_t^f = p_t^f(i) = w_t^f - a_t$
- Taylor rule (monetary policy) : $r_t = \phi_\pi \pi_t + \epsilon_t^r$

[a] notice that neither adjusted prices nor flexible ones depend on the firm, all price the same way !

C) Phillips curve (normal version)

1. Start from x_t after log linearization
2. Split the sum with the term where s=t on one side, and all the others on the other side
3. Move one of the $\lambda\beta$ outside the sum and outside the expectation
4. Managing expectations :

- For the term where s=t you can remove the expectation (it does not depend on future periods)
- For the other term : we use the law of iterated expectation to make E_{t+1} appear in our expression so we'll be able to substitute it with x_{t+1} in the next step.

[15] For additional details on how we get this expression, look at slide 19 in the lecture notes

Law of iterated expectation applied to E_t

Remember that E_t is a conditional expectation that can also be noted in the following way :

$$E_t \equiv E[.|z_t] \text{ with } z_t \text{ the history of } Z \text{ until period t included } ^a \tag{39}$$

Thus, it becomes possible to use the law of iterated expectation in the following way

$$E_t = E[E(.|z_{t+1})|z_t] = E_t[E_{t+1}] \tag{40}$$

[a] for more details, see Pset 1, Problem 3

5. Move one $\lambda\beta$ coefficient outside of $E_t[E_{t+1}]$ and notice that we get exactly $\lambda\beta E_t[x_{t+1}]$

$$\textbf{Expected result :}\ x_t = (1-\lambda\beta)(w_t - a_t) + (\lambda\beta)E_t[x_{t+1}] \tag{41}$$

6. Now we use the log-linearized price index equation : $p_t = \lambda p_{t-1} + (1-\lambda)x_t$ and substitute our new expression for x_t
7. We rearrange the term and express $\pi_t = p_t - p_{t-1}$ as a function of w_t, a_t, p_t and $E_t[\pi_{t+1}]$. Detailed steps :
   - Start from the price index equation and withdraw p_{t-1} on both sides to make π_t appear
   - Factor out $(1-\lambda)$ and substitute x_t from equation (38) to obtain this :

$$\textbf{Expected result :}\ \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}(w_t - a_t - p_t) + \beta E_t[\pi_{t+1}] \tag{42}$$

   - Now, take the price index equation we used in the beginning but in p_{t+1} and rewrite x_{t+1} as function of p_t, p_{t+1}

$$\textbf{Expected result :}\ x_{t+1} = \frac{p_{t+1} - \lambda p_t}{1-\lambda} = \frac{\pi_{t+1}}{1-\lambda} + p_t \tag{43}$$

   - We take the expectation of this new expression of x_{t+1} and substitute it in equation (39) to get the Phillips curve !

$$\textbf{Expected result :}\ \pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}(w_t - a_t - p_t) + \beta E_t[\pi_{t+1}] \tag{44}$$

D) Phillips curve (output gap version)

1. substitute the log-lin labour demand function into the log-lin labour market clearing condition to get a new expression of n (you also need to use the log lin price index to get rid of p_t)

$$n_t = c_t - a_t \tag{45}$$

2. combine (2) with the log lin labour supply equation to get :

$$(\psi + \gamma)c_t = w_t - p_t + \psi a_t \tag{46}$$

3. start from (3) and rewrite the equality for c_t^f (w_t and p_t take an f too). Then simplify the right hand side to express it as a function of a_t (remember that $p_t^f = w_t^f - a_t$ from the log linearized system)

$$(\psi + \gamma)c_t^f = (\psi + 1)a_t \Longleftrightarrow c_t^f = \frac{\psi + 1}{\psi + \gamma}a_t \tag{47}$$

4. we can now safely set ξ such that : $(w_t - a_t - p_t) = \xi(c_t - c_t^f)$
5. we substitute this in our previous expression for the phillips curve (see C) and get this new equation :

$$\pi_t = \kappa(c_t - c_t^f) + \beta E_t[\pi_{t+1}] \tag{48}$$

6. So with market clearing ($c_t = y_t$) we get :

$$\pi_t = \kappa(y_t - y_t^f) + \beta E_t[\pi_{t+1}] \tag{49}$$

3 Monetary policy

3.1 Optimal monetary policy

3.1.1 Efficient flex-price allocation

- Goal for the central bank : replicate the equilibrium allocation under flexible prices (assumed to be efficient here, $c_t^f = c_t^e$)
  1. limitations : need to be able to replicate flex-price allocation, flex price allocation need to be efficient(no externality, information friction..), need to know the natural IR)[16]
- Necessary conditions to achieve the equilibrium allocation :
  1. complete subsidy (compensates markup completely) : $\tau = \frac{\theta}{\theta - 1} - 1$
  2. no price dispersion at the onset : $P_{-1}(i) = P_{-1}, \forall i \in [0, 1]$
  3. set nominal IR (r_t) = real IR (r_t^n) [17]
  4. inflation is null (given since we consider flexible prices)
- multiplicity of equilibrias : the optimal equilibrium we seek satisfies $\forall t, c_t = c_t^f$ and $\pi_t = 0$
- Conditions for unicity of the equilibrium :
  1. $r_t = r_t^n + \phi_\pi \pi_t + \phi_c(c_t - c_t^f)$
  2. $\phi_\pi + \frac{(1-\beta)\phi_c}{\kappa} > 1$

[16] See extensions for inefficient flex price allocation case
[17] see methods and results for the derivation of the r_t^n under flexible prices

3.1.2 Inefficient flex-price allocation : The Quadratic loss function

- expresses the tradeoff between minimizing inflation (cross-sectional efficiency) and minimizing the output gap (aggregate efficiency)[18]

The loss minimization problem :

- objective function: with $x_t = c_t - c_t^e$ (the output gap)

  - We denote ν the weight put on output stabilization relative to inflation stabilization. Based on Woodford, $\nu = \frac{\kappa}{\theta}$

$$\min_{x_t,\pi_t} E_0[\sum_{t=0}^{\infty} \beta^t(\pi_t^2 + \nu x_t^2)] \tag{50}$$

- Constraint function: NKPC with tradeoffs (obtained from the regular NKPC by introducing c_t^e)

$$\pi_t = \kappa x_t + \beta E_t[\pi_{t+1}] + u_t \tag{51}$$

- variables :
  1. endogenous : $x_t = c_t - c_t^e$ (output gap)
  2. exogenous : $u_t = \kappa(c_t^e - c_t^f)$ (cost push shock) and π_{t+1} (inflation at the next period $\implies$ is impacted by previous decisions but is not a choice variable !)

3.1.3 Two types of monetary policy : discretionary vs commitment

1. discretionary policy : central bank takes the optimal policy at each period without being constrained by former commitments [19]
   - cannot commit so higher equilibrium inflation
   - depends only on current state
2. with commitment : central bank can commit to state-contingent policy plan with the risk that actions taken will be suboptimal.
   - can influence expectations directly which may lead to better policy trade offs due to anticipation (higher welfare)
   - depends on state history

3.2 Methods and results

3.2.1 deriving the natural rate of interest

1. Consider the relation between c_t^f and a_t derived in the decentralized new keynesian model.
2. Assume that a_t follows an $AR(1)$ process [20]
3. replace c_t^f and c_{t+1}^f in $r_t^n = \gamma E_t[c_{t+1}^f - c_t^f]$

Expected result : $r_t^n = \gamma \frac{\psi+1}{\psi+\gamma}(\rho_a - 1)a_t$

[18] 1) cross sectional efficiency deals with inefficiency across firms for a given period (inflation combined with sticky prices lead to varying consumption across goods), 2) aggregate efficiency stands for the taste that households have for a smoooth consumption path (concave utility function)

[19] see more details in extensions

[20] X follows an AR(1) process means that : $x_t = \rho_x x_{t-1} + \epsilon_t^x$

3.2.2 deriving the inflation rate

1. just apply recursive reasoning to the expression for outputgap PC in the decentralized model

Expected result : $\pi_t = \kappa \sum_{s=t}^{\infty} \beta^{s-t} E_t[c_s - c_s^f]$

4

Fiscal Policy

4.1 Definitions and concepts

- Fiscal multiplier : effect of an increase in gvt expenditures on output (transitory/permanent = short-run/long-run multiplier)
- Fiscal policy shock : unexpected change in government spending or taxation that is not anticipated by economic agents. It alters the aggregate demand by affecting disposable income or direct government expenditures, thereby impacting overall economic activity
- Monetary policy shock : unexpected adjustment in the central bank's policy stance—for example, an unanticipated change in the interest rate or another policy instrument. It affects borrowing costs, credit conditions, and economic activity beyond what was forecasted by the markets.
- The ad-hoc consumption function[21] : with c_0 (autonomous consumption[22]), and c_1(marginal propensity to consume)

$$C_t = c_0 + c_1 Y_t \tag{52}$$

  - We have a positive feedback loop : $\nearrow g_t \Longrightarrow \nearrow Y_t \Longrightarrow C_t \Longrightarrow \nearrow\nearrow Y_t$

4.2 Methods and results

- Goal : understand effect of a given policy shock under certain conditions on prices (different from deriving optimal policy as we did with monetary policy!)

4.2.1 Derivation of the Fiscal multiplier : Flexible price case

- Goods Market clearing : $Y_t \neq C_t$ as the government purchases a share ν_t of the production. Gvt expenditures are financed with debt (bond issuance) or taxes (impacts household BC + gvt BC).

$$\begin{gathered} \textbf{standard : } Y_t = C_t + \nu_t Y_t \\ \textbf{LL : } y_t = c_t + g_t \text{ with } g_t = log(\frac{1-\nu}{1-\nu_t}) \end{gathered} \tag{53}$$

- Production function : UNCHANGED

$$\begin{gathered} \textbf{standard : } Y_t = A_t N_t \\ LL : y_t = a_t + n_t \end{gathered} \tag{54}$$

- Household maximization (work/leisure tradeoff: UNCHANGED

$$\textbf{LL : } w_t - p_t = \psi n_t + \gamma c_t \tag{55}$$

[21] WARNING : this equation is not derived, it is an assumption used to explain why the traditional Keynesian tends to be higher than 1

[22] Consumption occuring even with 0 income

- Optimal flexible price : to obtain the rewritten version 1) use [54] to introduce n_t, c_t, 2) use [52,53] to introduce y_t, g_t 3) reorganize terms and note $\Gamma \equiv \frac{\gamma}{\gamma+\psi}$

$$\begin{aligned} \textbf{LL : } p_t^f &= w_t^f - a_t \\ \textbf{rewritten LL : } y_t^f &= \frac{1+\psi}{\gamma+\psi} a_t + \Gamma g_t \end{aligned} \tag{56}$$

Recap of the system

Log-Linearize system :

1. GM clearing : $y_t = c_t + g_t$
2. Production : $y_t = a_t + n_t$
3. Work/leisure tradeoff : $w_t - p_t = \psi n_t + \gamma c_t$
4. Optimal flexible price : $p_t^f = w_t^f - a_t$
5. Optimal production : $y_t^f = \frac{1+\psi}{\gamma+\psi} a_t + \Gamma g_t$

- Fiscal multiplier (Γ) : links y_t^f to g_t
  1. $0 < \Gamma < 1$ since $\psi > 0$

4.2.2 Derivation of the Fiscal multiplier : Rigid price case

- Set-up :

Recap of the system

Log-linearized system :

1. GMC : $c_t = y_t - g_t$
2. EE : $c_t = \frac{-1}{\gamma}(r_t - E_t[\pi_{t+1}]) + E_t[c_{t+1}]$
3. output EE : $y_t - y_t^f = \frac{1}{\gamma}(r_t - r_t^n - E_t[\pi_{t+1}]) + E_t[y_{t+1} - y_{t+1}^f]$
4. Real IR (flexible) : $r_t^f = \gamma(E_t[c_{t+1}^f] - c_t^f)$[a]
5. NKPC : $\pi_t = \kappa(y_t - y_t^f) + \beta E_t[y_{t+1} - y_{t+1}^f]$
6. Monetary policy rule : $r_t = \phi_\pi \pi_t + \phi_y(y_t - y_t^f)$

[a] reminder : $r_t^f = r_t^n$ under flexible prices

- Assume a_t, g_t folow AR(1) processes with ρ^a, ρ^g and $\epsilon_t^a, \epsilon_t^g$ (fiscal policy shocks)
- Take the EE and withdraw the LHS from both side to have an expression that equals 0. Then add this expression and distribute the c_ts.
- Obtaining the output EE : Use the GMC for c_t, c_t^f and substitute it in the previous expression to introduce the output gap instead of the consumption gap. For convenience, lets note $\bar{y}_t \equiv y_t - y_t^f$

$$\text{Output EE : } \bar{y}_t = -\frac{1}{\gamma}(r_t - r_t^f - E_t[\pi_{t+1}]) + E_t[y_{t+1} - y_{t+1}^f] \tag{57}$$

- Obtaining a new expression for the flex-price IR : Take the expression for the r_t^f, substitute c_ts using the GMC, then substitute away y_ts using the expression for the flexible output level y_t^f we computed in the previous case (see equation [56])

$$\text{Standard : } r_t^f = \underbrace{-\gamma\frac{1+\psi}{\gamma+\psi}(1-\rho^a)a_t}_{\text{Productivty shock}} + \underbrace{\gamma(1-\Gamma)(1-\rho^g)g_t}_{\text{Fiscal policy shock}} \tag{58}$$

- We assume away productivity shocks for simplicity (productivity shock term becomes null)
- Apply the undetermined coefficients method to $\pi_t, \bar{y}_t$ (For relation coefficients, I use a notation that differs from the course for simplicity)

$$\begin{cases} \pi_t = \psi_\pi g_t \\ \bar{y}_t = \psi_{\bar{y}} g_t \end{cases} \tag{59}$$

- See PS6 for the value of $\psi_\pi, \psi_{\bar{y}}, \psi_y$ knowing $y_t^f = \Gamma g_t$
- We find the fiscal multiplier ψ_y remains < 1 but is superior to the one under flexible prices as $0 < \Gamma < 1$

4.3 Dynamics summary

4.3.1 Reaction to increase in g_t : flexible case

- Household work more and consume less (long-run fiscal multiplier < 1), dampening the positive feedback loop effect !
- Why ? : anticipate increase in taxes due to finance current g_t + consumption smoothing (will save more to maintain C level) [23]
- detailed dynamics :

$$\nearrow g_t \Longrightarrow \searrow c_t \Longrightarrow \nearrow MUC \Longrightarrow \nearrow n_t \Longrightarrow \nearrow y_t \tag{60}$$

4.3.2 Reaction to increase in g_t : Rigidity case

- Heightened importance of monetary policy as changes in nominal spending now have an impact on real output. We also have a lower than 1 fiscal multiplier however less so than with flexible prices
- observations : the multiplier **decreases** in ϕ_π and ϕ_y, it **increases** in λ (probability of price stickyness)
- why ? :
  1. impact of price stickiness : higher λ implies inflation is less responsive to change in the output gap. So when g_t goes up, firms responds by increasing production rather than raising prices (which they cant always do). Changes in real output are thus stronger than under flexible prices.

$$\nearrow \lambda | \nearrow g_t \Longrightarrow \searrow \text{increase in } \pi_t | \nearrow \text{increase in } y_t \tag{61}$$

  2. impact of ϕ_π and ϕ_y : remember that

$$r_t = r_t^n + \phi_\pi \pi_t + \phi_y(y_t - y_t^n) \tag{62}$$

     - when ϕ_π is high the ECB responds aggressively to inflation. Fiscal expansion raises aggregate demand (positive output gap) putting an upward pressure on prices (inflation). This means the monetary response (real interest rate) set by the ECB will be stronger, counteracting the fiscal stimulus by reducing private spending (Ripple effect of fiscal expansion is dampened).
     - when ϕ_y is high the ECB is very sensitive to the output gap (stronger response).

[23] See "methods" part for the details

5

Zero Lower Bound

5.1 Definitions and Concepts

- Markovian process : current state is sufficient to predict probability to move to another state

5.2 Set-up

- About ϵ^d :
  - materializes the impact of a negative preference (discount rate) shock which makes people more patient (want to save more) [24]
  - null in **normal state** (SS) and < 0 in **crisis state**. It is constant in each period separately.
  - the economy returns to the normal (exits the shock) with probability $1 - \mu$ at each period
- About g_t : government expenditure is discrete and depends on the state, taking either the value g^H when in the crisis state and g^L when in the normal state
- Normal(H) vs crisis state(L) : $\bar{y}^H, \pi^H, r^H, g^H, e_d^H$ are null in the normal state and constant in each state separately (crisis value remains the same as long as the crisis lasts)

Recap of the system

1. Household objective function: $E_t \left[\sum_{s=t}^{\infty} \beta^{s-t} e^{\epsilon_s^d} \left(\frac{C_s^{1-\gamma} - 1}{1-\gamma} - \Phi \frac{N_s^{1+\psi}}{1+\psi} \right) \right]$
2. Modified EE (output): $\bar{y}_t = \frac{-1}{\gamma} \Big(\epsilon_{t+1}^d - \epsilon_t^d + r_t - r_t^n - E_t[\pi_{t+1}] \Big) + E_t[\bar{y}_{t+1}]$
3. New Keynesian Phillips Curve (NKPC): $\pi_t = \kappa \bar{y}_t + \beta E_t[\pi_{t+1}]$
4. Monetary policy rule: $r_t = \phi_\pi \pi_t + \phi_y \bar{y}_t$ (non-LL: $R_t = \Pi_t^{\phi_\pi} \left(\frac{Y_t}{Y_t^f} \right)^{\phi_y}$)
5. Flexible output: $y_t^f = \Gamma g_t$
6. Flexible interest rate: $r_t^f = \gamma(1 - \Gamma) \Big(g_t - E_t[g_{t+1}] \Big)$
7. Zero lower bound (ZLB) on the interest rate set by the ECB: $r_t = \max \Big\{ 0,\ \phi_\pi \pi_t + \phi_y \bar{y}_t \Big\}$

[24] to push people to save more you can also take an upward shock in the interest rate spread : borrowing money to fund consumption becomes more expensive relative to lending, which makes saving more attractive

5.3 A) The crisis state

5.3.1 1) The system

Recap of the system

1. inflation : $\pi^L = \frac{\kappa}{1-\beta\mu}\bar{y}^L$
2. Flex-price IR : $(r^f)^L = \gamma(1-\Gamma)(1-\mu)g^L$
3. Flex-price output : $(y^f)^L = \Gamma g^L$
4. EE (output) : $\bar{y}^L = -\frac{1}{\gamma}E_t[-\Delta_0 + r^L - (r^f)^L - \mu\pi^L] + \mu\bar{y}^L \Longleftrightarrow \bar{y}^L = \vartheta_r(\Delta_0 - r^L) + \vartheta_g g^L$
5. Policy rule : $r^L = \max\left\{0, (\phi_\pi \frac{\kappa}{1-\beta\mu} + \phi_y)(\vartheta_r(\Delta_0 - r^L) + \vartheta_g g^L\right\}$

- Inflation : use the NKPC (rewrite expectation)

$$\pi^L = \kappa\bar{y}^L + \beta\underbrace{(\mu\pi^L + (1-\mu)\pi^H)}_{=E_t[\pi_{t+1}]} \Longleftrightarrow \pi^L = \frac{\kappa}{1-\beta\mu}\bar{y} \tag{63}$$

- EE (output) : 1) get rid of expectations (including for ϵ^d_{t+1})[25], 2) substitute $\pi^L, (r^f)^L = (r^n)^L$, 3) note $\vartheta_r \equiv \frac{(1-\beta\mu)}{\gamma(1-\mu)(1-\beta\mu)-\mu\kappa}$ and $\vartheta_g = \vartheta_r(1-\Gamma)\gamma(1-\mu)$ and reorganize the terms

$$\begin{aligned}
&1)\ \bar{y}^L = -\frac{1}{\gamma}E_t[-\underbrace{(1-\mu)\epsilon^d_0}_{\equiv\Delta_0} + r^L - (r^f)^L - \mu\pi^L] + \mu\bar{y}^L \\
&2)\ \frac{\gamma(1-\mu)(1-\beta\mu)-\mu\kappa}{\gamma(1-\beta\mu)}\bar{y}^L = \frac{1}{\gamma}(\Delta_0 - r^L) - (1-\Gamma)(1-\mu)g^L \\
&3)\ \bar{y}^L = \vartheta_r(\Delta_0 - r^L) + \vartheta_g g^L
\end{aligned} \tag{64}$$

- policy rule: use the expression for the inflation and the EE (to replace $\bar{y}^L$

5.3.2 2) The dynamics

- output gap : We note that $\vartheta_g > (1-\Gamma) > 0$, so the output gap depends positively on government spending !
- IR : depends positively on government spending through ϑ_r as well.

5.3.3 3) To bind or not to bind

[26] **Finding the conditions for ZBL to be binding**

Recap of the conditions for the ZLB to Bind

1. condition on Δ_0 (holding g^L fixed) : $\Delta_0 < \frac{-\vartheta_g}{\vartheta_r}g^L < 0$ and $\epsilon^d_0 < \epsilon^d_{critical}(g^L) < 0$
2. condition on g^L (holding Δ_0 fixed) : $g^L < -\frac{\vartheta_r}{\vartheta_g}\Delta_0$ and $\epsilon^d_0 < \epsilon^d_{critical}(\Delta_0)[> 0]$

- Let's find conditions on Δ_0 and on g^L to see when the ZLB binds (ceteris et paribus)

[25] Note that Δ_0 is the value of the gvt spending shock
[26] that is the question...

- **Condition on Δ_0 :** we know we need to have $r^L = 0$ when ZLB binds, in other words the RHS of the max function must be inferior to 0 for it to be valid. So we get :

$$RHS(r^L = 0) < 0 \iff \vartheta_r \Delta_0 + \vartheta_g g^L < 0 \iff \Delta_0 < \underbrace{\frac{-\vartheta_g}{\vartheta_r} g^L}_{\equiv \epsilon^d_{critical}} < 0 \tag{65}$$

Main Takeaways

- We need a large enough negative shock ($\epsilon^d_0 < \epsilon^d_{critical}$) that will increase the desire to save sufficiently for the ZLB to bind
- Based on the sign of the critical value, if $g^L = 0$ any strictly negative shock will suffice for ZLB to bind.[a]

[a] if the economy has an intercept, $r_t = \alpha + \phi_\pi \pi_t + \phi_y \bar{y}$, this result no longer applies as we now need the critical value to be lower than α/ϑ_r

- **Condition on g^L :** we follow the same reasoning which yields :

$$g^L < -\frac{\vartheta_r}{\vartheta_g} \Delta_0 \tag{66}$$

Main Takeaways

- since $\Delta_0 < 0$ we need $g^L_{critical} > 0$ so below a certain amount of gvt expenditure increase (positive) in crisis state the ZLB is actually binding
- too great an increase in g_t prevents the ZLB from being binding

Finding the fiscal multiplier when the ZLB is binding

- Getting the output gap multiplier : we start back from the first inequality we have using the RHS of the policy rule, however this time we write the RHS as a function of the output gap in the crisis state $\bar{y}^L$.

$$\bar{y}^L < 0 \text{ and } \frac{\delta \bar{y}^L(r^L = 0)}{\delta g^L} = \vartheta_g \tag{67}$$

- ϑ_g is thus the partial effect (multiplier) on the output gap of a small enough increase in g^L for the ZLB to keep binding
- Getting the fiscal multiplier : we know $\frac{\delta (y^f)^L}{\delta g^L} = \Gamma$ (trivial), now let's combine this multiplier with the previous one to find the fiscal multiplier :

$$\text{Fiscal multiplier : } \frac{\delta y^L}{\delta g^L} = \frac{\delta \bar{y}^L}{\delta g^L} + \frac{\delta (y^f)^L}{\delta g^L} = \vartheta_g + \Gamma \tag{68}$$

Main Takeaways

- $\vartheta_g > 1$ so the fiscal multiplier is more than 1 !
- comparative statics :
  1. $\nearrow \mu \Longrightarrow \nearrow$ *multiplier* (makes sense since we stay longer in a recession and we know the fiscal multiplier is stronger in recession periods)
  2. $\nearrow \kappa \Longrightarrow \nearrow$ *multiplier*
  3. $\nearrow (1-\lambda) \Longrightarrow \nearrow$ *multiplier* [a]
- Taking the expression for π^L and differentiating it in g^L we also find that inflation depends positively on government spending. This helps avoid the deflationary spiral we document in the extension.

[a] see below for a detailed explanation on the impact of price flexibility on the multiplier

Impact of price flexiblity on the fiscal multiplier

- When prices are more flexible κ is larger (the slope of the NKPC is steeper) so, when faced with a negative expenditure shock, the inflation response is stronger.
- because at the ZLB monetary policy does not work, having a stronger inflation response to changes in output reinforces the power of fiscal policies. (a fiscal stimulus that raises output creates a larger increase in inflation which helps lower the real IR even more)
-

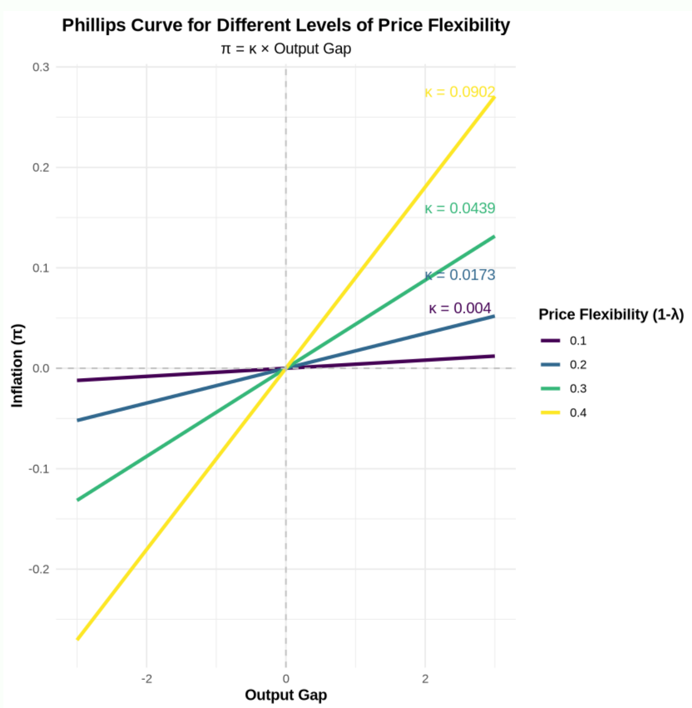


Figure 1: Comparison of inflation response under different levels of price flexibility

Extensions

6.1 RBC

6.1.1 Alternative formulation conditional on technology shocks

Why this formulation ? : This alternative formulation consists in making visible the fact that the planner chooses a plan for C,S and K from now to infinity **based on what happens in the economy (the history of Z)** Indeed, Z can take an infinity of different values at each period, so there exists an infinity of possible history of Z and the planner must decide on a path for each of them. In the following notation, we note Z_i the value taken by Z in period i and we note $z^{t+i} = (Z_0, ...Z_{t+i})$ the history of Z from period 0 to period $t+i$. **What does it look like ? :**

The planner problem in conditional form

$$\max_{C^{t+i}(z^{t+i}), S_{t+i}(.), K_{t+i+1}(.)} \sum_{i=0}^{\infty} \sum_{z_{t+i}} \beta^i \Pi^{t+i}(z^{t+i}) U(C_{t+i}(z^{t+i}))^a \tag{69}$$

constraints : $\forall i, z^{t+i}$

$$\begin{cases} \text{Budget} : C_{t+i}(z^{t+i}) + S_{t+i}(z^{t+i}) \leq Z_{t+i}(z^{t+i}) F(K_{t+i}(z^{t+i+1})) \\ \text{Capital accumulation} : K_{t+i+1}(z^{t+i}) = (1-\delta) K_{t+i}(z^{t+i-1}) + S_{t+i}(z^{t+i}) \\ \text{Non negativity} : K_{t+i+1}(z^{t+i}) \geq 0 \end{cases} \tag{70}$$

[a] NOTE : We take z_{t+i} and not z_{t+i+1} for K_{t+i+1} since we choose K for the next period based on the known history of Z which goes all the way to the current period. For K_{t+i} however, we take z_{t+i-1} since we take it as granted as it has been determined in the previous period given the known history at the time.

Deriving the FOCs :

- tip : We can get rid of the sum of probability terms by just substituting them by E_t of the remaining terms. Then use linearity of expectation to put E_t on the left side making computations easier.
- The mecanism is the same as the regular approach : We derive the objective function in C, K with respect to a certain period $t+i$ and a certain history z_{t+i}.
- We have the associated probability terms in the FOC's. When we have terms conditional on z_{t+i+1}, we must not forget to sum those terms over all possible z^{t+i+1} (This makes sense as Z_{t+i+1} is not yet known so there is an infinity of possible z^{t+i+1} to consider)
- We use Bayes law to rewrite $\Pi^{t+i+1}(z^{t+i+1})$ as $\Pi^{t+i}(z^{t+i})\Pi(z^{t+i+1}|z^{t+i})$ so we can cancel out the $\Pi^{t+i}(z^{t+i})$
- Finally we simplify the resulting expression by replacing the sum of probabilities in z^{t+i+1} by E_{t+i}

Expected result : $\lambda_{t+i}(z^{t+i}) = \beta E_{t+i}(\lambda_{t+i+1}(z^{t+i+1}) R_{t+i+1}(z^{t+i+1})|z^{t+i})$

6.1.2 RBC with capital utilization

What does this model introduce ? Why is it useful ?

- Adjusting the capital level below or above its renewal level (where K_t is only increased by the amount δK_t required to compensate for intertemporal depreciation).
- It is closer to reality as it smooths the response to shocks : in reality capital cannot be adjusted instantaneously so we should expect more of a hump-shaped impulse response rather than a spike.

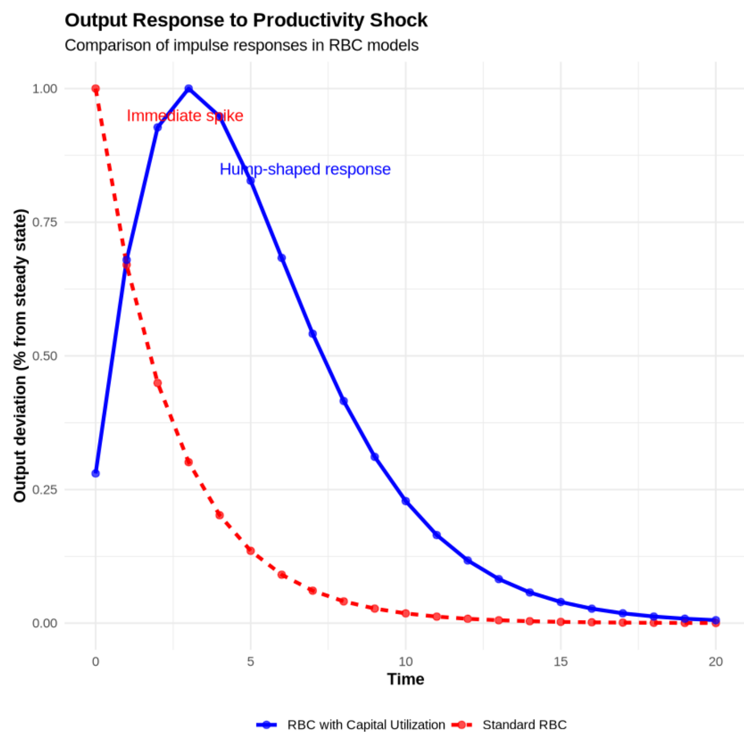


Figure 2: Comparison of impulse responses to productivity shock (Standard vs Cap. Util.

The new capital accumulation constraint :

$$K_{t+1} = I_t - \frac{\phi}{2}(\frac{I_t}{K_t} - \delta)^2 K_t + (1-\delta)K_t \tag{71}$$

ϕ is the **adjustment cost parameter**, it quantifies the cost incurred when the investment to capital ratio deviates from the steady state renewal rate δ. The higher it is, the steeper the price for adjusting capital over its normal level.

About Tobin's q

- definition : $\frac{\mu_t}{\lambda_t}$
- μ_t is the marginal utility of having extra installed capital (K_{t+1}, λ_t is the marginal utility of having some extra consumption.
- tobin's q is the ratio of how much consumption one would be willing to give up on to have spome extra ufture capital ($\iff$ relative price of capital in terms of consumption)

6.1.3 RBC with government spending shocks

What does this model introduce ?

1. Government now tax households through a lump-sum tax T and funds its spendings exclusively through tax income ($T_t = G_t$).
2. Utility is additively separable in consumption, leasure and government spending : a change in one component does not directly affect the marginal utility of the others. $\rightarrow$ the FOC for consumption and leisure deicsions are derived independently of the government spending term

3. Government spending has no effect on TFP and capital stock : the government has no impact on firms as the tax or spendings do not have an impact positively or negatively the productivity of the economy, nor do they affect the available capital for production (capital accumulation equation is untouched)

4. Government spending is not reinvested back in the economy ! Money can be assumed to be thrown away through government spendings after it has been funded by the new tax.

Change in set up and results :
A) Household flow budget constraint

$$\textbf{new flow BC :} \; C_t + I_t = w_t N_t + r_t K_t - T_t = w_t N_t + r_t K_t - G_t \tag{72}$$

B) Steady state flow budget constraint :

$$\textbf{new flow SS BC :} \; C_t + I_t = w_t N_t + r_t K_t - G_t = Y_t{}^{27} - G_t \tag{73}$$

C) Law of motion for government spending

$$ln(G_t/G) = \rho_G ln(G_{t-1}/G) + \epsilon_t^G \tag{74}$$

Variables: (1) G is the steady-state level of government spending. (2) $\rho_G \in [0,1]$ is the persistence parameter; the higher it is, the more persistent the government spending shocks are over time. When it is equal to 0, government spending is entirely driven by the current shock and does not depend on past values. (3) ϵ_t^G is a random government spending shock at period t.

Outcome of the model :

- No capital case ($\alpha = 0$ in the production function) : positive gvt spending shocks increase output (based on equation 46) and decreases consumption initially (makes sense since we increase taxes, see 45)
  - economic rational : A rise in government spending, $G_t$financed by lump-sum taxes, reduces resources available for private consumption. To maintain their consumption, households respond by increasing labor supply N_t which increases output.

Detailed response process

1. Positive government spending shock (ϵ_t^G) increases gvt spending (G_t) and thus taxes (T_t)
2. Household consumption (C_t) drops since the disposable income available is reduced by increased taxes.
3. The marginal utility of consumption (C_t^λ)rises in response (concave utility function)
4. Household supply more labor (N_t) to offset the loss in disposable income available for consumption and maintain their former consumption level.
5. The reservation wage (w_t) goes down because household are now willing to work for a lower price
6. Output rises as more labor is supplied

[27] the expression $Y_t = w_t N_t + r_t K_t$ remains unchanged as it only depends on firms

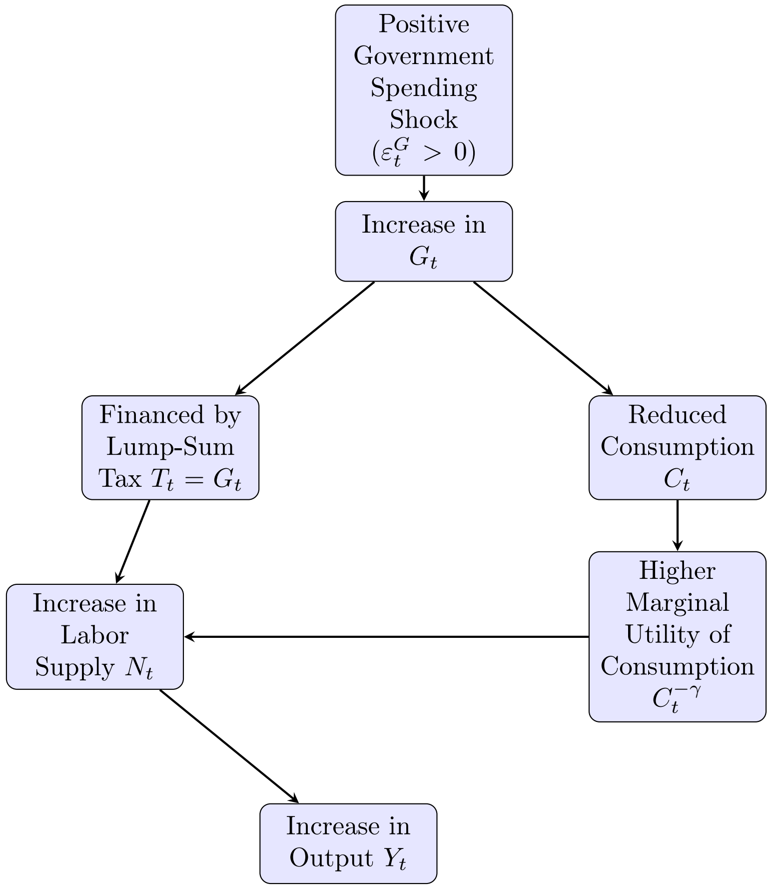


Figure 3: Transmission mechanism of a positive government spending shock (no capital case)

- Capital case ($\alpha \neq 0$) : same dynamics as in the no capital case but the impact on output is stronger. We indeed have both a short term effect (adjust level of labor provided) and a long term effect (adjust capital stock lent to firms) to compensate for drop in disposable income for consumption.

6.2 NK - Optimal monetary policy

6.2.1 Inefficient flex price allocation case : time varying θ_t

- modification : θ is time-dependent so market power of firms can change from one period to the other and influence how high some firms will be able to price their goods. As the markup is updated at each period, it is impossible to compensate this inefficiency with a targeted subsidy.
- modified log-linearized system :
  1. static price (flexible) : $p_t^{\diamond} = \mu_t^n + w_t - a_t$ with $\mu_t^n = ln(\frac{\theta_t(\theta-1)}{\theta(\theta_t-1)})$
  2. labor supply : $w_t - p_t - \gamma c_t = \psi n_t$
  3. labor demand (under flexible prices) : $n_t = c_t - a_t$ (does not depend on the firm!)
- flexible consumption c_t^f is obtained from subsituting n_t and p_t in the labor supply equation

$$c_t^f = \frac{\psi + 1}{\psi + \gamma} a_t - \frac{1}{\psi + \gamma} \mu_t^n \tag{75}$$

- dynamics : when θ_t goes down, the markup μ_t^n goes up and c_t^f goes down

- modified NKPC :
  1. start from this :
  $$\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}(p_t^\diamond - p_t) + \beta E_t[\pi_{t+1}] \tag{76}$$
  2. replace $p_t^\diamond$ by its expression then replace p_t using the labor supply equation
  3. replace n_t using the labor demand equation and pull together terms to recover c_t^f

expected result : $\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}(\psi+\gamma)(c_t - c_t^f) + \beta E_t[\pi_{t+1}]$

- we find the Philipps curve remains unchanged !

In short

1. flexible price allocation is inefficient ($c_t^f \neq c_t^e$)
2. the static profit maximizing price changes and so does the equilibrium consumption under flexible prices
3. the NKPC remains unchanged !

6.2.2 Discretionary Monetary Policy

Principle : The central bank picks the optimal output gap and inflation rate at each period t without committing to anything for periods to come. This ensures the optimal monetary policy is picked at its stage, but this may go against total welfare in some situations due to lack of credible commitment (impossible to commit to keep a low inflation over a predetermined period for instance).

Set-up : with u_t $(= c_t^e - c_t^f)$, $x_t(= c_t - c_t^e)$ and $\nu_t = \beta E_t[\pi_{t+1}] + u_t$

$$\begin{gathered}\min_{x_t,\pi_t}(\pi_t^2 + \vartheta x_t^2) \\ \text{subject to (NKPC)} : \pi_t = \kappa x_t + \beta E_t[\pi_{t+1}] + u_t\end{gathered} \tag{77}$$

Solving method :

- Start from regular loss minimization set-up and note $\nu_t = \beta E_t[\pi_{t+1}] + u_t$
- derive the FOC with respect to x_t and find the expression for the optimal output gap (with $\vartheta = \frac{\kappa}{\theta}$)

Expected result : $$x_t = -\frac{\kappa\pi_t}{\vartheta} = -\theta\pi_t \tag{78}$$

- The ECB leans against the wind : when output exceeds its flexible-price level ($x_t > 0$) then the ECB will seek to lower inflation and the other way around
- Replace ν_t by its expression, then substitute the expression for optimal x_t in the NKPC formula.
- Find a new expression of π_t that depends only on $u_{t+k,k\in[0,\infty[}$ through reccursive reasoning on the NKPC.

Expected result : $$\pi_t = \frac{1}{1+\kappa\theta}\sum_{k=0}^{\infty}\left(\frac{\beta}{1+\kappa\theta}\right)^k E_t[u_{t+k}]. \tag{79}$$

  - we notice that the optimal inflation is decreasing in θ (elasticity of substitution across goods), and in κ (sensitivity of inflation to output gap).

- Use the fact that u_t follows an AR(1) process ($u_{t+k} = \rho^k u_t$) so we can rewrite π_t as a function of u_t (NOTE : notice we have a geometric sum)

$$\text{Expected result : } \pi_t = \frac{u_t}{1 + \theta\kappa - \beta\rho} \tag{80}$$

- based on this result we can state that x_t and π_t also follow an AR(1) process : $E_t[x_{t+1}] = \rho x_t$ and $E_t[\pi_{t+1}] = \rho\pi_t$
- Let's get a new expression for x_t : we take the Euler Equation and rewrite $r_t = r_t + r_t^n - r_t^n$ with $r_t^n = \gamma E_t[c_{t+1}^e - c_t^e]$ (the natural IR at the efficient allocation)[28]

$$\text{Expected result : } x_t = \frac{-1}{\gamma}(r_t - r_t^n - E_t[\pi_{t+1}]) + E_t[x_{t+1}] \tag{81}$$

- Removing the expectation : Use the expression you got earlier for optimal x_t and the fact that π_t, x_t follow AR(1) processes. You get a new expression of r_t as a function of the nominal IR and the inflation rate

$$\text{Expected result : } r_t = r_t^n + (\rho + \theta\gamma(1-\rho))\pi_t \tag{82}$$

6.2.3 Monetary Policy with Commitment

Principle : The Central bank can credibly commit to a policy for periods to come. Compared to the discretionary model, it cares about the expected loss over future periods.

Set-up :

$$\min_{\{\pi_t, x_t\}_{t=0}^{\infty}} \frac{1}{2} E_0[\sum_{t=0}^{\infty} \beta^t(\pi_t^2 + \alpha x_t^2)] \tag{83}$$
$$\text{Subject to : } \forall t, \pi_t = \nu_t + \kappa x_t$$

Solving for optimal x_t

- write down the lagrangian and derive the FOC with respect to π_t
- get rid of the expectation term with λ_{t-1} and express λ_{t-1} as a function of λ_t, π_t

$$\text{Expected result : } \lambda_{t-1} = E_{t-1}[\lambda_t - \pi_t] \tag{84}$$

- derive the FOC with respect to x_t and express x_t as a function of λ_t

$$\text{Expected result : } x_t = \frac{-\kappa}{\alpha_x}\lambda_t \iff \lambda_t = \frac{-\alpha_x}{\kappa}x_t \tag{85}$$

- take [83] and substitute the lambdas using [84] to get the final expression for x_t

$$\text{Expected result : } x_t = E_t[x_{t+1} + \frac{\kappa}{\alpha_x}\pi_{t+1}] \tag{86}$$

[28] remember that $x_t = c_t - c_t^e$ and rearrange the c_ts !

6.3 NK - Zero Lower Bound

6.3.1 ZLB with preference (discount rate) shock and R≠1

A) Set up

- The economy can enter a crisis state (via a shock $\epsilon_t^d < 0$ at period t which maintains itself from one period to the other with probability μ (otherwise returns to SS)

  - preference shock : sudden change in consumers' tastes or utility, which alters their consumption decisions independently of changes in income or relative prices. (affects the intertemporal tradeoff)

- Steady state : $c_t = 0, \epsilon_{t-1}^d = 0, \pi_t = 0$

- Monetary policy rule :

$$r_t = max\{-ln(R), \phi_\pi \pi_t\} \text{ with } R=\frac{1}{\beta} \tag{87}$$

- New keynesian equation :

$$c_t = E_t[-\frac{1}{\gamma}(\epsilon_{t+1}^d - \epsilon_t^d - r_t - \pi_{t+1} + c_{t+1}] \tag{88}$$

- The standard form of the NKPC remains unchanged !

B) General solving method

- step 1 : write down the expectation of ϵ_t, c_t and π_t depending on μ to get rid of expectations in the Philipps curve and the EE

- step 2 : use the rewritten PC and EE to get new expressions for c_t, π_t

- step 3 : guess and verify on r_t for 1)binding vs 2)non binding case and find c_t, π_t for each case (detailed below)

C) Understand how c_t, π_t depend on the shock using the method of undetermined coefficient

- case 1 : lower bound is not binding ($r_t = \phi_\pi \pi_t$)

  1. Guess : replace guessed r_t in the expression for c_t, π_t obtained in step 2, and express the latter as a function of the shock ϵ_t^d
  2. Verify : give a condition on the shock value ϵ such that the ZLB is indeed not binding ($\phi_\pi \pi_t > ln(R)$). This gives us the minimal value of the shock for which we know our guess that the lower bound is not binding will be verified.
  3. Find and Interpret how c_t, π_t depend on ϵ_t^d (sign, magnitude)

Economic interpretation : dynamics

– Both c_t, π_t depend positively on the shock :

$$\epsilon_t^d < 0 \Longrightarrow \searrow c_t, \pi_t \tag{89}$$

– Mediating role of monetary policy reaction (ϕ_π) : when the negative shock strikes, household face an increased desire to save (consume later). Thus they reduce consumption for the current period. To counteract this, the ECB raises the nominal interest rate(r_t^n). The stronger the response of the ECB, the stronger the compensating effect of r_t^n. $\Longrightarrow$ NO DEFLATIONARY SPIRAL

$$\nearrow \phi_p i \Longrightarrow \nearrow \text{drop in } r_t^n \Longrightarrow \searrow \text{drop in } c_t \tag{90}$$

- case 2 : lower bound is binding ($r_t = -ln(R)$

1. Guess : same approach as case 1
2. Verify : this time the condition will be $\phi_\pi \pi_t > ln(R)$ but we proceed in the same way
3. interpretation :

Economic interpretation : dynamics

– As before, households wish to save more, but because the lower bound is binding the ECB cannot further adjust the nominal interest rate. As consumption drops, the output gap widens (less demand at times t, means less production).

– Because, there is less production, firms push down salaries as they need to hire fewer workers (the marginal cost of production goes down). This means firms are able to reduce price. However, due to price stickiness (Calvo pricing), not all firms can adjust immediately, creating a gradual adjustment process. Household thus expect even lower prices in the future which further boosts their desire to save.

– The real interest rate at time t ($r_t = r_t^n - E_t[\pi_{t+1}]$) increases as π_{t+1} becomes more negative further dis-incentivizing consumption now. $\Longrightarrow$ DEFLATIONARY SPIRAL

$$\epsilon_t^d < 0 \Longrightarrow \searrow c_t \Longrightarrow \searrow y_t \Longrightarrow \searrow n_t \Longrightarrow \searrow w_t \Longrightarrow \searrow \pi_{t+1} \Longrightarrow \nearrow r_t \Longrightarrow \searrow\searrow c_t \tag{91}$$

Further implications of ZLB being binding :

- Odyssean forward guidance : credible commitment by the ECB to raise future inflation can serve to raise current consumption (since r_t^n is constant, the ECB can commit to lower the real interest rate by acting on expected future inflation)
- government spending multiplier can be very large. With the central bank unable to lower rates further, fiscal expansion leads directly to increased aggregate demand. Additionally, if the fiscal intervention modifies expectations (for instance, by signaling future policy changes or affecting inflation expectations), it further boosts demand.

7

Math Appendix

7.1 Usual Taylor Expansions

Function $f(x)$	Taylor Expansion form	...with sums
e^x	$1 + x + x^2 + \cdots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\frac{1}{1-x}$	$1 - x + x^2 - x^3 + \cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots$	$\sum_{n=0}^{\infty} x^n$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

7.2 Geometric Series

Geometric series take either of these 3 forms and give the following results (all can be admitted):

$$\begin{aligned} &1) \sum_{k=0}^{\infty} q^k = \sum_{k=t}^{\infty} q^{k-t} = \frac{1}{1-q} \\ &2) \sum_{k=0}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2} \\ &3) \sum_{k=0}^{\infty} k(k-1) q^{k-1} = \frac{2}{(1-q)^3} \end{aligned} \tag{92}$$

7.3 Expectation indices and period notations

The t+i notation : begins at time t

<u>form of the sum :</u>

$$E_t[\sum_{i=0}^{\infty} \beta^i X_{t+i}] \tag{93}$$

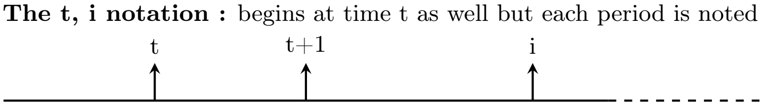
Note : (1) β is indexed by i and not t+i because between period t and t+i, only i periods have passed so we discount i times.

The t notation : begins at time 0

<u>form of the sum :</u>

$$E_0[\sum_{t=0}^{\infty} \beta^t X_t] \tag{94}$$

The t, i notation : begins at time t as well but each period is noted i instead of t+i



<u>form of the sum :</u>

$$E_t[\sum_{i=t}^{\infty} \beta^{i-t} X_i] \tag{95}$$

7.4 Undetermined Coefficient Method

- **Goal** : we wish to investigate how y_t depends on x_t
- **Step 1 : Guess**
  1. We assume y_t depends on x_t thourhg the following function : $y_t = \psi_x x_t$ with ψ_x unknown
- **Step 2 : Verify**
  1. Assuming the previous conjecture is true, we use the formulas we know to retrieve an expression of ψ_x based on existing parameters (not variables !).
  2. We investigate the sign and magnitude of ψ_x using the expression we just determined and infer how y_t will vary as a function of x_t depending on known parameters.

7.5 AR and VAR processes

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