Macroeconomics - Notes(models)

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RBC model

Solution Outline

- **Households**
  1. Solve the utility maximization problem by maximizing with respect to consumption, investments, labor and future capital (bonds might also appear in some settings)
- **Firms**
  1. Solve the profit maximization problem by maximizing with respect to current capital and labor. (The firm is not a monopoly so it does not get to set prices !)
- **Steady state**
  1. Write the equations for the non-stochastic steady state (no time index)
  2. Proceed to log linearize equilibrium equations using non-stochastic steady state equations [a]

[a] See Isabel and Gustavo's notes for the details

1.1 Set-up

1.1.1 Households

- Goal : Maximize the expected utility of the household over its lifetime by picking a path defining optimal consumption, leisure, investment and capital stock at each period depending on the state of the economy (technology shocks)
- Objective function :

$$\max_{C_t,N_t,I_t} E_0[\beta^t(\frac{C_t^{1-\gamma}}{1-\gamma} - \theta\frac{N_t^{1-\psi}}{1+\psi}] \tag{1}$$

- Constraints :

$$\begin{cases} \text{Budget} : C_t + I_t = W_tN_t + r_tK_t \\ \text{Capital Accumulation} : K_{t+1} = (1-\delta)K_t + I_t \\ \text{Time Endowment} : N_t = 1 - L_t \end{cases} \tag{2}$$

1.1.2 Firm

- Goal : Maximize the firm's profit through choosing the optimal combination/level of factor endowments at each period t
- Objective function : (no constraints here !)

$$\max_{K_t,N_t} \pi_t = Y_t - \tilde{R}_tK_t - W_tN_t \tag{3}$$

1.1.3 Steady state

Equilibrium Equations System

- **Euler Equation :** $C_t^{-\gamma} = E_t[\beta C_{t+1}^{-\gamma}(r_{t+1} + 1 - \delta)]$
- **Work/Leisure tradeoff :** $\theta N_t^{-\psi} = W_t C_t^{-\gamma}$
- **Time endowment constraint :** $L_t = 1 - N_t$
- **Flow budget constraint :** $C_t + I_t = Y_t$ [obtained from the firm's profit equation given that profit is null, combined with BC in equation (2)]
- **Capital accumulation constraint :** $K_{t+1} = I_t + (1-\delta)K_t$
- **Firm's MPK :** $\alpha \frac{Y_t}{K_t} = r_t$
- **Firm's MPN :** $(1-\alpha)\frac{Y_t}{N_t} = W_t$
- **Production function :** $Y_t = Z_t K_t^{\alpha} N_t^{1-\alpha}$
- **Law of motion of productivity/technology :** $ln(Z_t) = \rho ln(Z_{t-1}) + \epsilon_t$

Non-stochastic steady state

- Goal : We wish to find the point around wish the economy fluctuates to see how shocks create deviations from this equilibrium (how fast does the economy converge back to its steady state etc..)
- Method : see "Tricks & Results" part

Log-linearization

- Goal : Transform our system of non-linear equations in a system of linear ones. This helps greatly for our analysis (for instance it helps with moment calculation as we have a linear expectation term)
- Limitations : Log linearization is only valid if
  1. We only have small deviations from steady state
  2. The model does not have significant nonlinearities
- Method : See "Tricks & Results" part [1]

1.2 Tricks and results

1.2.1 Households

1. Simplify the constraint : Take the capital accumulation constraint and express I_t as a function of the other variables. Then substitute I_t in the budget constraint to get rid of investments.

$$\textbf{New constraint :}\ W_t N_t + r_t K_t + (1-\delta)K_t = C_t + K_{t+1} \tag{4}$$

2. Derive the FOC with respect to C_t to obtain the shadow value λ_t of increasing consumption.

$$\textbf{Expected result :}\ C_t^{-\gamma} = \lambda_t \tag{5}$$

[1] For additional details, please refer to Gustavo and Isabel's notes

3. Derive the FOC with respect to K_t to get the Euler Equation. Replace λ_t and λ_{t+1} using (1)

$$\textbf{Expected result : } \underbrace{C_t^{-\gamma}}_{=\lambda_t} = E_t[\beta \underbrace{C_{t+1}^{-\gamma}}_{=\lambda_{t+1}}(r_{t+1} + 1 - \delta)] \tag{6}$$

4. Derive the FOC with respect to N_t to get the leisure/work tradeoff equation.

$$\textbf{Expected result : } \theta N_t^{-\psi} = W_t C_t^{-\gamma} \tag{7}$$

1.2.2 Firm

1. Derive the FOC with respect to K_t to find the marginal product of capital (MPK)

$$\textbf{Expected result : } \alpha \frac{Y_t}{K_t} = \tilde{R}_t \tag{8}$$[2]

2. Derive the FOC with respect to N_t to find the marginal product of labor (MPN)

$$\textbf{Expected result : } (1-\alpha)\frac{Y_t}{N_t} = W_t \tag{9}$$

1.2.3 Steady state

Non-stochastic steady state

1. To obtain the system of non-stochastic steady state equation you need to remove all time/period indexes from the variables and rearrange each equation when needed
2. For the law of motion of productivity : The ϵ term disappears as there are no shocks in steady state
   - BEWARE : (assuming $\rho \neq 1$) you can't divide by $log(Z)$ as we can't tell whether it is equal to 0. Therefore move log on the same side and divide by the constant to end up with $Z = 1$

Non-Stochastic Steady State System

- **Euler Equation :** $1 = \beta(r + 1 - \delta) = \beta R$
- **Work/Leisure tradeoff :** $\theta N^{-\psi} = WC^{-\gamma}$
- **Time endowment constraint :** $L = 1 - N$
- **Flow budget constraint :** $C + I = Y$
- **Capital accumulation constraint :** $I = \delta K$
- **Firm's MPK :** $\alpha \frac{Y}{K} = r$
- **Firm's MPN :** $(1-\alpha)\frac{Y}{N} = W$
- **Production function :** $ZK^{\alpha}N^{1-\alpha}$
- **Law of motion of productivity/technology :** $Z = 1$

[2] α corresponds to the share of production due to capital (generally fixed to 2/3)

Log-Linearization

1. We replace the variables by their exponential forms ($X_t = Xe^{\hat{X}_t}$)
2. We cancel out constant terms using steady state equations
3. If we have multiplications of variables by one another we use Taylor

Taylor Expansion Theorem

Taylor's Theorem: For $f \in C^{n+1}(I)$ where I is an open interval containing a:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \text{ where } \xi \text{ lies between } a \text{ and } x.$$

Conditions:

- $f^{(n+1)}$ exists on I
- f is continuous on $[a, x]$

4. Otherwise, using the log is sufficient to find back what we need
5. Note : for the log linearization of the EE, it is useful to use the following expression which is also obtained through log-linearization $R\hat{R}_{t+1} = r\hat{r}_{t+1}$ [3]

Log-Linearized System

- **Euler Equation :** $\hat{C}_t = E_t[\hat{C}_{t+1} - \frac{1}{\gamma}\hat{R}_{t+1}]$
- **Work/Leisure tradeoff :** $\psi\hat{N}_t = \hat{W}_t - \gamma\hat{C}_t$
- **Time endowment constraint :** $\hat{N}_t = -\frac{L}{N}\hat{L}_t$
- **Flow budget constraint :** $\frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t = \hat{Y}_t$
- **Capital accumulation constraint :** $\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t$
- **Interest rate (firm's MPK):** $\hat{r}_t = \hat{Y}_t - \hat{K}_t$
- **wage (Firm's MPN) :** $\hat{W}_t = \hat{Y}_t - \hat{N}_t$
- **Production function :** $\hat{Y}_t = \hat{Z}_t + \alpha\hat{K}_t + (1-\alpha)\hat{N}_t$
- **Law of motion of productivity/technology :** $\hat{Z}_t = \rho\hat{Z}_{t-1} + \epsilon_t$ [4]
- **Gross interest rate** $\hat{R}_t = \frac{r}{R}\hat{r}_t$

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[3] See Gustavo's note, p.10

[4] be careful, we keep ϵ as is here, we do not log-linearize it

New-Keynesian model

Solution Outline

- **Households**
  1. Intratemporal : Solve the expenditure minimization problem for Household by minimizing with respect to consumption of a given good at a given period ($C_s(i)$)
  2. Intertemporal : Solve the utility maximization problem by maximizing expected utility with respect to aggregate consumption, labor and bonds
- **Firms**
  1. Flexible prices : Solve the profit maximization problem, assuming that prices can be adjusted at each period, by maximizing with respect to the price of a given good at a given period ($P_t(i)$)
  2. Sticky prices : Solve the profit maximization problem, assuming prices have a non null probability of carrying over to the next period, by maximizing with respect to the price of a given good at the period the price is set - right before the carryover($P_t(i)$)
- **Steady State**
  1. Log Linearization : same process as with RBC with a little twist for adjusted prices[a]
  2. Phillips curve : find the equation from the phillips curve based based on the log-linearized system. It shows how current inflation depends on expected future inflation and current economic conditions (marginal cost)

[a]See "Results & Methods" part for the details

2.1 Set-up

2.1.1 Households

A) The Expenditure Minimization Problem

- Goal : Minimize total expenditures across all goods while achieving a certain level of aggregate consumption
- Objective function :

$$\min_{C_s(j)} \int_0^1 P_s(i)C_s(i)di \tag{10}$$

- Constraint :

$$C_s = (\int_0^1 C_s(i)^{\frac{\theta-1}{\theta}})^{\frac{\theta}{\theta-1}} \text{ with } C_s \text{ the desired aggregate consumption level}^{5} \tag{11}$$

B) The Utility Maximization Problem

- Goal : maximize expected utility over the lifetime of the consumer
- objective function :

$$\max_{N_s,C_s,B_s} E_t[\sum_{s=t}^{\infty} \beta^{s-t}(\frac{C_s^{1-\gamma}-1}{1-\gamma} - \phi\frac{N_s^{1+\psi}}{1+\psi})]^{6} \tag{12}$$

- Constraints :

$$\begin{cases} \text{Budget : } R_{s-1}B_{s-1} + W_sN_s + D_s = T_s + B_s + P_sC_s \text{ with } T_s = \text{taxes} \\ \text{Initial condition : } B_{t-1} = 0 \\ \text{TVC : } \lim_{s\to\infty} B_s(\Pi_{i=0}^{s} R_{s-i}) = 0 \end{cases} \tag{13}$$

2.1.2 Firms

A) The Profit Maximization Problem : Flexible Prices

- Goal : set optimal prices at each period so as to maximize the companies profit (monopoly behavior : does not take prices as given)
- Objective function :

$$\max_{P_s(i),N_s(i)} \pi_s(i) = (1+\tau)P_s(i)Y_s(i) - W_sN_s(i) \tag{14}$$

- Constraints :

$$\begin{cases} \text{Production function : } Y_s(i) = A_sN_s(i)\ ^{7} \\ \text{Technology flow : } ln(A_s) = \rho_A ln(A_{s-1}) + \epsilon_s^a\ ^{8} \\ \text{Monopoly : } Y_s(i) = C_s(i) = (\dfrac{P_s}{P_s(i)})^{\theta} C_s\ ^{9} \end{cases} \tag{15}$$

B) The Profit Maximization Problem : Sticky Prices

- Goal : find the optimal price that maximizes profits over time given the fact that price can't be adjusted for a certain number of periods
- Problem :

$$X_t(i) = \arg\max_{P_t(i)} E_t[\sum_{s=t}^{\infty} (\lambda\beta)^{s-t} \frac{C_s^{-\gamma}/P_s}{C_t^{-\gamma}/P_t}((1+\tau)P_t(i) - \frac{W_s}{A_s})(\frac{P_t(i)}{P_s})^{-\theta} C_s]\ ^{10\ 11} \tag{16}$$

2.1.3 Planner

- objective : allocate consumption to the representative household and labor to the firms in a way that maximizes the well-being of the household. The allocation it chooses is the efficient allocation and can be compared to the allocation obtained in the decentralized case to identify inefficiencies.
- objective function :

$$\max_{C_t,N_t} U(C_t, N_t) \tag{17}$$

[6] θ is the elasticity of substitution across goods (the lower it is, the more market power firms will have

[6] γ is the inverse of IES (intertemporal elasticity of substitution), ψ the inverse of the Frisch elasticity of labor supply(the lower it is, the less responsive the household is to wage changes) and ϕ governs the degree of wage rigidity

[9] A_s is productivity/technology at period s

[9] ϵ represents technology shocks, ρ represents determines how much of the previous period's technology level carries over to the current period (if 0 shocks are short-lived

[9] The firm behaves monopolistically so it equates quantities produced with quantities demanded

[11] alternative notation : $\Lambda_s = \beta^{s-t}(\frac{C_s^{-\gamma}/P_s}{C_t^{-\gamma}/P_t})$ and $\forall s, \tilde{c_s}(i) = (\frac{P_t(i)}{P_s})^{-\theta} C_s$

[11] Λ_s is named the relevant stochastic discount factor for nominal payoff, $\tilde{c_s}(i)$ is the demand for good i conditional on the price of this good not varying over time

- constraints :

$$\begin{cases} \textbf{Aggregate Consumption Constraint :}\ C_t = \left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}} \\ \textbf{Production Function for each good i :}\ C_t(i) = A_t N_t(i), \forall i \in [0,1] \\ \textbf{Aggregate Labor Constraint :}\ N_t = \int_0^1 N_t(i) di \end{cases} \tag{18}$$

- results : at optimality households consume the same quantity of all goods, allocate the same amount of labor to all firms (why ? : 1) goods enter utility function in the same way, 2)concave utility) and the marginal disutility of labor equals its marginal benefit.
- potential inefficiencies in decentralized equilibrium :
  1. Price stickiness
  2. Market power (monopolies)
  3. Price dispersion (linked to price stickiness but may be problematic even in the period t-1)

2.1.4 Steady state

Equilibrium Equations System

- Demand for a differentiated good : $C_s(i) = (\frac{P_s}{P_s(i)})^{\theta} C_s$
- Euler Equation (inter-temporal) : $E_t\left[\beta \frac{R_s}{\Pi_{s+1}} C_{s+1}^{-\gamma}\right] = C_s^{-\gamma}$
- Labor supply : $\phi N_s^{\psi} = \frac{C_s^{-\gamma}}{P_s} W_s$
- Labor demand (per firm) : $N_t(i) = \frac{1}{A_t}(\frac{P_t(i)}{P_t})^{-\theta} C_t$
- Labor market clearing : $N_t = \int_0^1 N_t(i) di$
- Price index : $P_t = (\int_0^1 P_t(i)^{1-\theta} di)^{\frac{1}{1-\theta}}$
- Adjusted price (sticky) : $X_t(i) = \frac{\theta}{(\theta-1)(1+\tau)} \frac{E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}\frac{W_s}{A_s}P_s^{\theta}C_s^{1-\gamma}]}{E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}P_s^{\theta-1}C_s^{1-\gamma}]}$
- Flexible price : $P_t(i) = \frac{1}{1+\tau}\frac{\theta}{\theta-1}\frac{W_t}{A_t}$
- Taylor rule (Monetary policy) : $R_t = R\Pi_t^{\phi_\pi} e^{\epsilon_t^r}$

Non-stochastic steady state and Log-linearization

- Goal : same as with the RBC model
- Method : see "Tricks & Results" part

Phillips curve

- Goal : show current inflation depends on expected future inflation and current economic conditions (marginal cost)
- Method (how to get it) : see "Tricks & Results" part

2.2 Tricks and results

2.2.1 Households

A) The Expenditure Minimization Problem

1. Derive the FOC wrt. $C_s(i)$: be mindful that we are considering a single good in a single period so the integral in the left hand term can be safely ignored (it's as if we were to take a specific term in a sum

$$\textbf{Expected result :}\ C_s(i) = (\frac{\lambda_s}{P_s(i)})^{\theta} C_s \tag{19}$$

2. Substitute the result obtained for $C_s(i)$ in the constraint to find an expression for λ_s

$$\textbf{Expected result :}\ \lambda_s = P_s \tag{20}$$ [12]

3. Substitute λ_s in the expression for the optimal $C_s(i)$ that we found earlier
4. Substitute this new expression for $C_s(i)$ in the objective function

$$\textbf{Expected result :}\ TE = P_s C_s \tag{21}$$

B) The Utility Maximization Problem

1. FOC wrt C_s : define λ_s as a function of the rest

$$\textbf{Expected result :}\ \lambda_s = \frac{C_s^{-\gamma}}{P_s} \tag{22}$$

2. FOC wrt N_s (work/leisure tradeoff): equalize marginal cost (hours spent working) with marginal benefit of labor (consumption derived from labor income) → can't improve situation by working more/less

$$\textbf{Expected result :}\ \phi N_s^{\psi} = \frac{C_s^{-\gamma}}{P_s} W_s \tag{23}$$

3. FOC wrt B_s (Euler equation) : use (8) and equalize marginal expected utility of future consumption and marginal benefit of current consumption → can't improve situation by consuming more now or in the future

$$\textbf{Expected result :}\ E_t\left[\beta \frac{R_s}{\Pi_{s+1}} C_{s+1}^{-\gamma}\right] = C_s^{-\gamma} \tag{24}$$ [13]

[12] interpretation : the shadow value of increasing aggregate consumption is equal to the price index

[13] $\Pi_{s+1} = \frac{P_{s+1}}{P_s}$ and corresponds to inflation

2.2.2 Firms

A) The Profit Maximization Problem : Flexible Prices

1. Rearrange the problem to get rid of $N_s(i)$ and the constraint : to do that replace $C_s(i)$ by its constraint value and use the production function to express $N_s(i)$ as a function of other variables

$$\textbf{Expected result :}\ \max_{P_s(i)}[(P_s(i)(1+\tau)-\frac{W_s}{A_s}]P_s(i)^{-\theta}P_s^{\theta}C_s \tag{25}$$

2. Find the FOC wrt to $P_s(i)$: consumption and price aggregates cancel out, then just rearrange the terms

$$\textbf{Expected result :}\ P_s(i)=\frac{1}{1+\tau}\frac{\theta}{\theta-1}\frac{W_s}{A_s} \tag{26}$$

B) The Profit Maximization Problem : Sticky Prices

- Deriving the FOC with respect to $P_t(i)$:
  1. Start by canceling out aggregate terms with index t
  2. divide by $P_t(i)^{-\theta-1}$ on each side
  3. Move all terms that do not depend on s outside of the expectation
  4. Put the remaining $P_t(i)$ on one side and rearrange term to express it as a function of all other remaining terms

$$\textbf{Expected result :}\ X_t(i)=\frac{\theta}{(\theta-1)(1+\tau)}\frac{E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}\frac{W_s}{A_s}P_s^{\theta}C_s^{1-\gamma}]}{E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}P_s^{\theta-1}C_s^{1-\gamma}]} \tag{27}$$

2.2.3 Planner

1. Solving the planners problem : set up the lagrangian with a costate variable for each of the three constraints. Be careful for the production constraint as you need to introduce a constraint for each good.

expected result :

$$\mathcal{L}=U(C_t,N_t)+\lambda_t\left[C_t-\left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}}\,di\right)^{\frac{\theta}{\theta-1}}\right]$$
$$+\int_0^1 \mu_t(i)\,[C_t(i)-A_t\,N_t(i)]\,di+\nu_t\left[N_t-\int_0^1 N_t(i)\,di\right]$$

2. Take the FOC's with respect to $C_t, N_t, N_t(i)$ and use costates to find back the following optimality conditions :

expected result :

$$\begin{cases} C_t(i)=C_t, \forall i\in[0,1] \\ N_t(i)=N_t \\ -U_{n,t}=A_tU_{c,t} \end{cases} \tag{28}$$

2.2.4 Steady state

A) Non-stochastic steady state

1. We start by proceeding in the same way as in the RBC model (remove time indexes)

2. For adjusted prices : move the constant terms (W, A, P, and C) outside of the sum. The sums can then be canceled out, along with the term in P and C.

3. For aggregate labor and prices : Notice that $P_s(i), X_s(i)$ and $N_s(i)$ do not depend on i (see equation 24,25). In other words, at equilibrium, all firms will price in the same way and demand the same quantity of labor. This entails that : $\forall i, N(i) = N$ and $P(i) = P$ in the non-stochastic steady state.

Non-Stochastic Steady State System

- Demand for a differentiated good : $C(i) = (\frac{P(i)}{P})^{-\theta}C$
- Euler Equation (inter-temporal) : $\beta\frac{R}{\Pi} = 1$
- Labor supply : $\phi N^{\psi} = \frac{C^{-\gamma}}{P}W$
- Labor demand (per firm) : $N(i) = \frac{C}{A} = \frac{Y}{A}$
- Labor market clearing : $N = \int_0^1 N(i)di$
- Price index : $P = (\int_0^1 P(i)^{1-\theta}di)^{\frac{1}{1-\theta}}$
- Adjusted price (sticky) : $X = X(i) = \frac{\theta}{(\theta-1)(1+\tau)}\frac{W}{A}$
- Flexible price : same as adjusted price ! $X = P = P(i)$
- Taylor rule (monetary policy) : $1 = \Pi$ [a]

[a] The gross inflation rate Π is equal to 1 so there is zero net inflation

B) Log-linearization

1. The overall method is the same as before, however some additional tricks are necessary (especially for prices).

2. Euler Equation :

   - Replace variables by their exponential form (Xe^{x_t}) and cancel out constant terms using steady state equations
   - For efficacy, it is preferable to group together exponents and THEN use Taylor

   Trick n°1 : Take $e^{r_t-\pi_t-\gamma c_t}$ which gives $(r_t - \pi_t) - \gamma c_t + 1$ (using Taylor)[a]

   [a] WARNING : here r_t is just the equivalent of $\hat{R}_t$ which is the notation we used for the RBC model

3. Adjusted price :

   - The first steps are the same as the Euler equation (substitute variables in exp form, cancel out constants, group exponentials in each sum and use Taylor)
   - We rearrange the sums and use Taylor a second time but for functions of the form $\frac{1}{1+x}$ (see details below)

Applying Taylor : Detailed method

We start by noting :

$$\Phi_s \equiv (1-\gamma)c_s + (\theta - 1)p_s \tag{29}$$

This gives us the following equation :

$$1 + x_t(i) = \frac{\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(1 + \Phi_s + w_s - a_s)}{\lambda\beta)^{s-t}(1 + \Phi_s)} \tag{30}$$

For improved clarity, we now set the following :

$$\begin{cases} S \equiv \sum_{s=t}^{\infty}(\lambda\beta)^{s-t} \\ \epsilon_N \equiv \sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(\Phi_s + w_s - a_s) \\ \epsilon_D \equiv \sum_{s=t}^{\infty}(\lambda\beta)^{s-t}\Phi_s \end{cases} \tag{31}$$

This gives us this, which we will simplify using Taylor :

$$1 + x_t(i) = \frac{S + \epsilon_N}{S + \epsilon_D} \tag{32}$$

We can rewrite this to take the following form :

$$1 + x_t(i) = \frac{S(1 + \epsilon_N/S)}{S(1 + \epsilon_D/S)} = \frac{(1 + \epsilon_N/S)}{(1 + \epsilon_D/S)} \tag{33}$$

Now we can linearize the denominator using Taylor [a] on $\frac{1}{1+\epsilon_D/S}$ (we take $X \equiv \epsilon_D/S$)

$$\frac{1}{1 + \epsilon_D/S} \sim (1 - \epsilon_D/S) \tag{34}$$

Now, we multiply it with the numerator and get rid of higher order terms :

$$\frac{(1 + \epsilon_N/S)}{(1 + \epsilon_D/S)} = (1 - \epsilon_D/S)(1 + \epsilon_N/S) = 1 + \frac{\epsilon_N}{S} - \frac{\epsilon_D}{S} = 1 + \frac{\epsilon_N - \epsilon_D}{S} \tag{35}$$

Notice the following :

$$\epsilon_N - \epsilon_D = \sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(w_t - a_t) \tag{36}$$

[a] Please refer to the math appendix 4.1

- Following the second Taylor expansion we have this expression :

$$\textbf{Expected result: } 1 + x_t(i) = 1 + \frac{\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(w_t - a_t)}{\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}} \tag{37}$$

- We are now left with two little steps : cancel out the ones and rewrite the denominator. For the latter, notice that it is a geometric series and can be rewritten as $\frac{1}{1-\lambda\beta}$ [14]

[14] See math appendix 4.2

- Just flip the denominator and you're done !

4. Price index :

- Right-most equality : Write variables in exponent form and get rid of constant terms based on non-stochastic steady state. Then, use the first order Taylor approximation and cancel out the ones.
- Left-most equality, we use the fact that, at equilibrium :

$$P_t^{1-\theta} = ((1-\lambda)X_t^{1-\theta} + \lambda P_{t-1}^{1-\theta}) \tag{38}$$ [15]

- So at the NS steady state we get : $P^{1-\theta} = (1-\lambda)X^{1-\theta} + \lambda P^{1-\theta}$. From there we just apply the usual method (exp + NS steady state cancellation + Taylor expansion)

5. Expected end result :

Log-Linearized System

- Demand for a differentiated good : $c_t(i) = -\theta(p_t(i) - p_t) + c_t$
- Euler equation : $c_t = E_t[-\frac{1}{\gamma}(r_t - \pi_{t+1}) + c_{t+1}]$
- Labour supply : $w_t - p_t = \psi n_t + \gamma c_t$
- Labour market clearing : $n_t = \int_0^1 n_t(i)di$
- Labour demand : $n_t(i) = c_t - a_t - \theta(p_t(i) - p_t)$
- Price index : $p_t = \int_0^1 p_t(i)di = \lambda p_{t-1} + (1-\lambda)x_t$
- Adjusted price (sticky)[a] : $x_t(i) = (1-\lambda\beta)E_t[\sum_{s=t}^{\infty}(\lambda\beta)^{s-t}(w_s - a_s)]$ and $x_t = x_t(i)$
- Flexible price : $p_t^f = p_t^f(i) = w_t^f - a_t$
- Taylor rule (monetary policy) : $r_t = \phi_\pi \pi_t + \epsilon_t^r$

[a] notice that neither adjusted prices nor flexible ones depend on the firm, all price the same way !

C) Phillips curve (normal version)

1. Start from x_t after log linearization
2. Split the sum with the term where s=t on one side, and all the others on the other side
3. Move one of the $\lambda\beta$ outside the sum and outside the expectation
4. Managing expectations :

- For the term where s=t you can remove the expectation (it does not depend on future periods)
- For the other term : we use the law of iterated expectation to make E_{t+1} appear in our expression so we'll be able to substitute it with x_{t+1} in the next step.

[15] For additional details on how we get this expression, look at slide 19 in the lecture notes

Law of iterated expectation applied to E_t

Remember that E_t is a conditional expectation that can also be noted in the following way :

$$E_t \equiv E[.|z_t] \text{ with } z_t \text{ the history of } Z \text{ until period t included}^a \tag{39}$$

Thus, it becomes possible to use the law of iterated expectation in the following way

$$E_t = E[E(.|z_{t+1})|z_t] = E_t[E_{t+1}] \tag{40}$$

[a] for more details, see Pset 1, Problem 3

5. Move one $\lambda\beta$ coefficient outside of $E_t[E_{t+1}]$ and notice that we get exactly $\lambda\beta E_t[x_{t+1}]$

$$\textbf{Expected result :}\ x_t = (1-\lambda\beta)(w_t - a_t) + (\lambda\beta)E_t[x_{t+1}] \tag{41}$$

6. Now we use the log-linearized price index equation : $p_t = \lambda p_{t-1} + (1-\lambda)x_t$ and substitute our new expression for x_t
7. We rearrange the term and express $\pi_t = p_t - p_{t-1}$ as a function of w_t, a_t, p_t and $E_t[\pi_{t+1}]$. Detailed steps :
   - Start from the price index equation and withdraw p_{t-1} on both sides to make π_t appear
   - Factor out $(1-\lambda)$ and substitute x_t from equation (38) to obtain this :

$$\textbf{Expected result :}\ \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}(w_t - a_t - p_t) + \beta E_t[\pi_{t+1}] \tag{42}$$

   - Now, take the price index equation we used in the beginning but in p_{t+1} and rewrite x_{t+1} as function of p_t, p_{t+1}

$$\textbf{Expected result :}\ x_{t+1} = \frac{p_{t+1} - \lambda p_t}{1-\lambda} = \frac{\pi_{t+1}}{1-\lambda} + p_t \tag{43}$$

   - We take the expectation of this new expression of x_{t+1} and substitute it in equation (39) to get the Phillips curve !

$$\textbf{Expected result :}\ \pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}(w_t - a_t - p_t) + \beta E_t[\pi_{t+1}] \tag{44}$$

D) Phillips curve (output gap version)

1. substitute the log-lin labour demand function into the log-lin labour market clearing condition to get a new expression of n (you also need to use the log lin price index to get rid of p_t)

$$n_t = c_t - a_t \tag{45}$$

2. combine (2) with the log lin labour supply equation to get :

$$(\psi + \gamma)c_t = w_t - p_t + \psi a_t \tag{46}$$

3. start from (3) and rewrite the equality for c_t^f (w_t and p_t take an f too). Then simplify the right hand side to express it as a function of a_t (remember that $p_t^f = w_t^f - a_t$ from the log linearized system)

$$(\psi + \gamma)c_t^f = (\psi + 1)a_t \iff c_t^f = \frac{\psi + 1}{\psi + \gamma}a_t \tag{47}$$

4. we can now safely set ξ such that : $(w_t - a_t - p_t) = \xi(c_t - c_t^f)$
5. we substitute this in our previous expression for the phillips curve (see C) and get this new equation :

$$\pi_t = \kappa(c_t - c_t^f) + \beta E_t[\pi_{t+1}] \tag{48}$$

3

Monetary and Fiscal policy

3.1 Optimal monetary policy

3.1.1 Efficient flex-price allocation

- Goal for the central bank : replicate the equilibrium allocation under flexible prices (assumed to be efficient here, $c_t^f = c_t^e$)
    1. limitations : need to be able to replicate flex-price allocation, flex price allocation need to be efficient(no externality, information friction..), need to know the natural IR)[16]
- Necessary conditions to achieve the equilibrium allocation :
    1. complete subsidy (compensates markup completely) : $\tau = \frac{\theta}{\theta-1} - 1$
    2. no price dispersion at the onset : $P_{-1}(i) = P_{-1}$, $\forall i \in [0, 1]$
    3. set nominal IR (r_t) = real IR (r_t^n) [17]
    4. inflation is null (given since we consider flexible prices)
- multiplicity of equilibrias : the optimal equilibrium we seek satisfies $\forall t, c_t = c_t^f$ and $\pi_t = 0$
- Conditions for unicity of the equilibrium :
    1. $r_t = r_t^n + \phi_\pi \pi_t + \phi_c(c_t - c_t^f)$
    2. $\phi_\pi + \frac{(1-\beta)\phi_c}{\kappa} > 1$

3.1.2 Inefficient flex-price allocation : The Quadratic loss function

- expresses the tradeoff between minimizing inflation (cross-sectional efficiency) and minimizing the output gap (aggregate efficiency)[18]

The loss minimization problem :

- objective function: with $x_t = c_t - c_t^e$ (the output gap)

  - We denote ν the weight put on output stabilization relative to inflation stabilization. Based on Woodford, $\nu = \frac{\kappa}{\theta}$

$$\min_{x_t, \pi_t} E_0[\sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \nu x_t^2)] \tag{49}$$

- Constraint function: NKPC with tradeoffs (obtained from the regular NKPC by introducing c_t^e)

$$\pi_t = \kappa x_t + \beta E_t[\pi_{t+1}] + u_t \tag{50}$$

[16] See extensions for inefficient flex price allocation case

[17] see methods and results for the derivation of the r_t^n under flexible prices

[18] 1) cross sectional efficiency deals with inefficiency across firms for a given period (inflation combined with sticky prices lead to varying consumption across goods), 2) aggregate efficiency stands for the taste that households have for a smoooth consumption path (concave utility function)

- variables :
  1. endogenous : $x_t = c_t - c_t^e$ (output gap)
  2. exogenous : $u_t = \kappa(c_t^e - c_t^f)$ (cost push shock) and π_{t+1} (inflation at the next period $\Longrightarrow$ is impacted by previous decisions but is not a choice variable !)

3.1.3 Two types of monetary policy : discretionary vs commitment

1. discretionary policy : central bank takes the optimal policy at each period without being constrained by former commitments [19]
   - cannot commit so higher equilibrium inflation
   - depends only on current state
2. with commitment : central bank can commit to state-contingent policy plan with the risk that actions taken will be suboptimal.
   - can influence expectations directly which may lead to better policy trade offs due to anticipation (higher welfare)
   - depends on state history

3.2 Methods and results

3.2.1 deriving the natural rate of interest

1. Consider the relation between c_t^f and a_t derived in the decentralized new keynesian model.
2. Assume that a_t follows an $AR(1)$ process [20]
3. replace c_t^f and c_{t+1}^f in $r_t^n = \gamma E_t[c_{t+1}^f - c_t^f]$

Expected result : $r_t^n = \gamma \frac{\psi+1}{\psi+\gamma}(\rho_a - 1)a_t$

3.2.2 deriving the inflation rate

1. just apply recursive reasoning to the expression for outputgap PC in the decentralized model

Expected result : $\pi_t = \kappa \sum_{s=t}^{\infty} \beta^{s-t} E_t[c_s - c_s^f]$

4

Extensions

4.1 RBC

4.1.1 Alternative formulation conditional on technology shocks

Why this formulation ? : This alternative formulation consists in making visible the fact that the planner chooses a plan for C,S and K from now to infinity **based on what happens in the economy (the history of Z)** Indeed, Z can take an infinity of different values at each period, so there exists an infinity of possible history of Z and the planner must decide on a path for each of them. In the following notation, we note Z_i

[19] see more details in extensions

[20] X follows an AR(1) process means that : $x_t = \rho_x x_{t-1} + \epsilon_t^x$

the value taken by Z in period i and we note $z^{t+i} = (Z_0, ...Z_{t+i})$ the history of Z from period 0 to period $t+i$. **What does it look like ? :**

The planner problem in conditional form

$$\max_{C^{t+i}(z^{t+i}),S_{t+i}(.),K_{t+i+1}(.)} \sum_{i=0}^{\infty} \sum_{z_{t+i}} \beta^i \Pi^{t+i}(z^{t+i}) U(C_{t+i}(z^{t+i}))^a \tag{51}$$

constraints : $\forall i, z^{t+i}$

$$\begin{cases} \text{Budget} : C_{t+i}(z^{t+i}) + S_{t+i}(z^{t+i}) \leq Z_{t+i}(z^{t+i}) F(K_{t+i}(z^{t+i+1})) \\ \text{Capital accumulation} : K_{t+i+1}(z^{t+i}) = (1-\delta) K_{t+i}(z^{t+i-1}) + S_{t+i}(z^{t+i}) \\ \text{Non negativity} : K_{t+i+1}(z^{t+i}) \geq 0 \end{cases} \tag{52}$$

[a] NOTE : We take z_{t+i} and not z_{t+i+1} for K_{t+i+1} since we choose K for the next period based on the known history of Z which goes all the way to the current period. For K_{t+i} however, we take z_{t+i-1} since we take it as granted as it has been determined in the previous period given the known history at the time.

Deriving the FOCs :

- tip : We can get rid of the sum of probability terms by just substituting them by E_t of the remaining terms. Then use linearity of expectation to put E_t on the left side making computations easier.
- The mecanism is the same as the regular approach : We derive the objective function in C, K with respect to a certain period $t+i$ and a certain history z_{t+i}.
- We have the associated probability terms in the FOC's. When we have terms conditional on z_{t+i+1}, we must not forget to sum those terms over all possible z^{t+i+1} (This makes sense as Z_{t+i+1} is not yet known so there is an infinity of possible z^{t+i+1} to consider)
- We use Bayes law to rewrite $\Pi^{t+i+1}(z^{t+i+1})$ as $\Pi^{t+i}(z^{t+i})\Pi(z^{t+i+1}|z^{t+i})$ so we can cancel out the $\Pi^{t+i}(z^{t+i})$
- Finally we simplify the resulting expression by replacing the sum of probabilities in z^{t+i+1} by E_{t+i}

Expected result : $\lambda_{t+i}(z^{t+i}) = \beta E_{t+i}(\lambda_{t+i+1}(z^{t+i+1}) R_{t+i+1}(z^{t+i+1})|z^{t+i})$

4.1.2 RBC with capital utilization

What does this model introduce ? Why is it useful ?

- Adjusting the capital level below or above its renewal level (where K_t is only increased by the amount δK_t required to compensate for intertemporal depreciation).
- It is closer to reality as it smooths the response to shocks : in reality capital cannot be adjusted instantaneously so we should expect more of a hump-shaped impulse response rather than a spike.

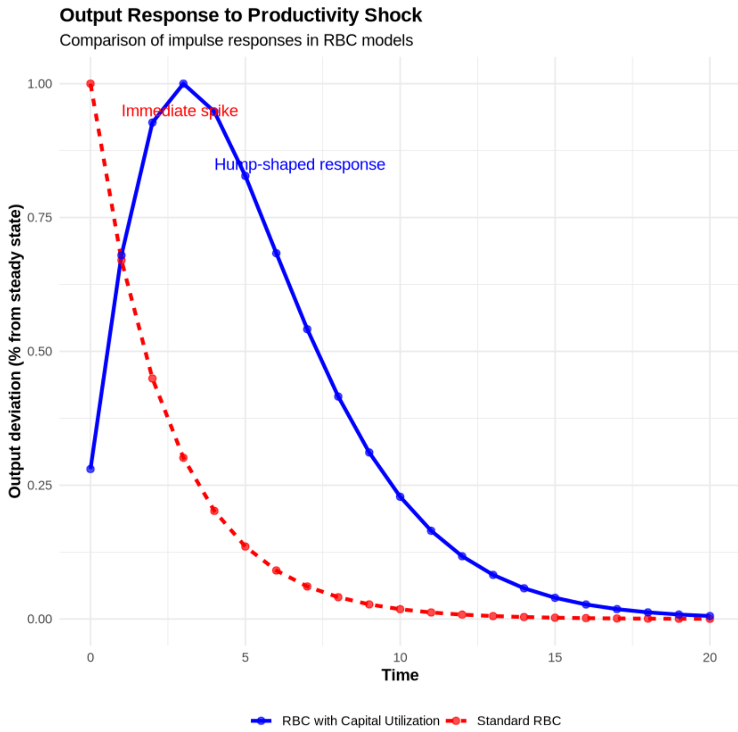


Figure 1: Comparison of impulse responses to productivity shock (Standard vs Cap. Util.

The new capital accumulation constraint :

$$K_{t+1} = I_t - \frac{\phi}{2}(\frac{I_t}{K_t} - \delta)^2 K_t + (1-\delta)K_t \tag{53}$$

ϕ is the **adjustment cost parameter**, it quantifies the cost incurred when the investment to capital ratio deviates from the steady state renewal rate δ. The higher it is, the steeper the price for adjusting capital over its normal level.

About Tobin's q

- definition : $\frac{\mu_t}{\lambda_t}$
- μ_t is the marginal utility of having extra installed capital (K_{t+1}, λ_t is the marginal utility of having some extra consumption.
- tobin's q is the ratio of how much consumption one would be willing to give up on to have spome extra ufture capital ($\iff$ relative price of capital in terms of consumption)

4.1.3 RBC with government spending shocks

What does this model introduce ?

1. Government now tax households through a lump-sum tax T and funds its spendings exclusively through tax income ($T_t = G_t$).
2. Utility is additively separable in consumption, leasure and government spending : a change in one component does not directly affect the marginal utility of the others. $\rightarrow$ the FOC for consumption and leisure deicsions are derived independently of the government spending term

3. Government spending has no effect on TFP and capital stock : the government has no impact on firms as the tax or spendings do not have an impact positively or negatively the productivity of the economy, nor do they affect the available capital for production (capital accumulation equation is untouched)

4. Government spending is not reinvested back in the economy ! Money can be assumed to be thrown away through government spendings after it has been funded by the new tax.

Change in set up and results :
A) Household flow budget constraint

$$\textbf{new flow BC :}\ C_t + I_t = w_t N_t + r_t K_t - T_t = w_t N_t + r_t K_t - G_t \tag{54}$$

B) Steady state flow budget constraint :

$$\textbf{new flow SS BC :}\ C_t + I_t = w_t N_t + r_t K_t - G_t = Y_t{}^{21} - G_t \tag{55}$$

C) Law of motion for government spending

$$ln(G_t/G) = \rho_G ln(G_{t-1}/G) + \epsilon_t^G \tag{56}$$

Variables: (1) G is the steady-state level of government spending. (2) $\rho_G \in [0, 1]$ is the persistence parameter; the higher it is, the more persistent the government spending shocks are over time. When it is equal to 0, government spending is entirely driven by the current shock and does not depend on past values. (3) ϵ_t^G is a random government spending shock at period t.

Outcome of the model :

- No capital case ($\alpha = 0$ in the production function) : positive gvt spending shocks increase output (based on equation 46) and decreases consumption initially (makes sense since we increase taxes, see 45)
  - economic rational : A rise in government spending, $G_t$financed by lump-sum taxes, reduces resources available for private consumption. To maintain their consumption, households respond by increasing labor supply N_t which increases output.

Detailed response process

1. Positive government spending shock (ϵ_t^G) increases gvt spending (G_t) and thus taxes (T_t)
2. Household consumption (C_t) drops since the disposable income available is reduced by increased taxes.
3. The marginal utility of consumption (C_t^λ)rises in response (concave utility function)
4. Household supply more labor (N_t) to offset the loss in disposable income available for consumption and maintain their former consumption level.
5. The reservation wage (w_t) goes down because household are now willing to work for a lower price
6. Output rises as more labor is supplied

[21] the expression $Y_t = w_t N_t + r_t K_t$ remains unchanged as it only depends on firms

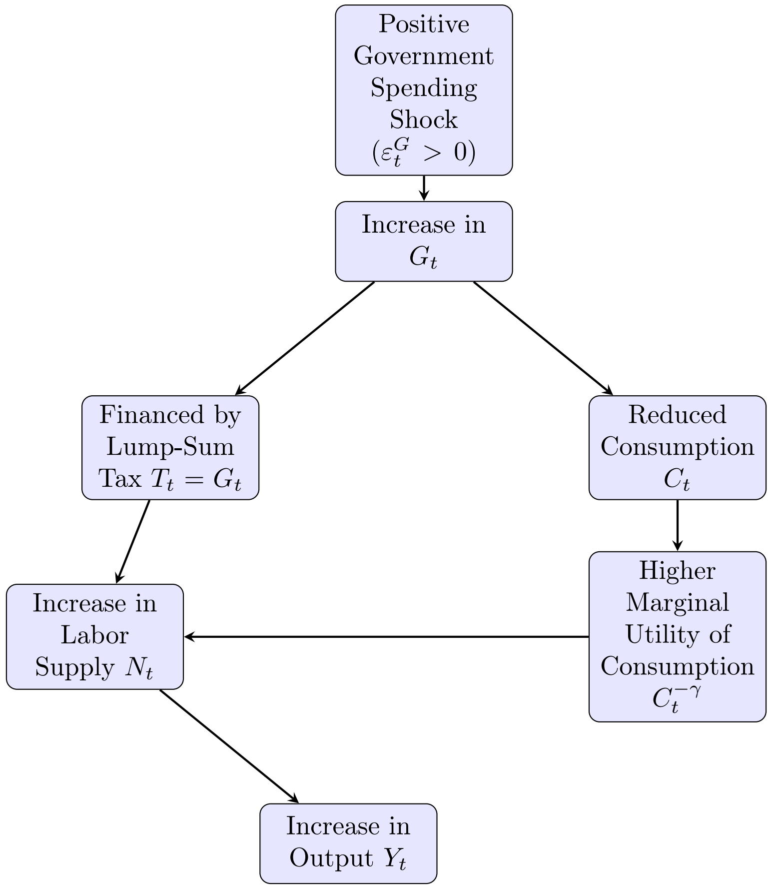


Figure 2: Transmission mechanism of a positive government spending shock (no capital case)

- Capital case ($\alpha \neq 0$) : same dynamics as in the no capital case but the impact on output is stronger. We indeed have both a short term effect (adjust level of labor provided) and a long term effect (adjust capital stock lent to firms) to compensate for drop in disposable income for consumption.

4.2 NK model

4.3 Optimal monetary policy

4.3.1 Inefficient flex price allocation case : time varying θ_t

- modification : θ is time-dependent so market power of firms can change from one period to the other and influence how high some firms will be able to price their goods. As the markup is updated at each period, it is impossible to compensate this inefficiency with a targeted subsidy.
- modified log-linearized system :
  1. static price (flexible) : $p_t^{\diamond} = \mu_t^n + w_t - a_t$ with $\mu_t^n = ln(\frac{\theta_t(\theta-1)}{\theta(\theta_t-1)})$
  2. labor supply : $w_t - p_t - \gamma c_t = \psi n_t$
  3. labor demand (under flexible prices) : $n_t = c_t - a_t$ (does not depend on the firm!)
- flexible consumption c_t^f is obtained from subsituting n_t and p_t in the labor supply equation

$$c_t^f = \frac{\psi + 1}{\psi + \gamma} a_t - \frac{1}{\psi + \gamma} \mu_t^n \tag{57}$$

- dynamics : when θ_t goes down, the markup μ_t^n goes up and c_t^f goes down
- modified NKPC :
  1. start from this :
$$\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}(p_t^\diamond - p_t) + \beta E_t[\pi_{t+1}] \tag{58}$$
  2. replace $p_t^\diamond$ by its expression then replace p_t using the labor supply equation
  3. replace n_t using the labor demand equation and pull together terms to recover c_t^f

expected result : $\pi_t = \frac{(1-\lambda)(1-\lambda\beta)}{\lambda}(\psi+\gamma)(c_t - c_t^f) + \beta E_t[\pi_{t+1}]$

- we find the Philipps curve remains unchanged !

In short

1. flexible price allocation is inefficient ($c_t^f \neq c_t^e$)
2. the static profit maximizing price changes and so does the equilibrium consumption under flexible prices
3. the NKPC remains unchanged !

4.3.2 Optimal discretionary policy

Solving method :

- start from regular loss minimization set-up and note $\nu_t = \beta E_t[\pi_{t+1}] + u_t$
- derive π_t with respect to output gap x_t and find the expression for the optimal output gap (knowing $\nu = \frac{\kappa}{\theta}$)
- substitute the expression for optimal x_t in the NKPC formula and find a new expression of π_t that depends only on $u_{t+k,k\in[0,\infty[}$ through reccursive reasoning

$$\pi_t = \frac{1}{1+\kappa\theta}\sum_{k=0}^{\infty}\left(\frac{\beta}{1+\kappa\theta}\right)^k E_t[u_{t+k}]. \tag{59}$$

- use the fact that u_t follows an AR(1) process so we can rewrite π_t as a function of u_t + notice we have a geometric sum
$$\pi_t = \frac{u_t}{1+\theta\kappa - \beta\rho} \tag{60}$$
- we guess that x_t and π_t also follow a AR(1) process : $E_t[x_{t+1}] = \rho x_t$ and $E_t[\pi_{t+1}] = \rho\pi_t$
- let's get a new expression for x_t : we take the Euler equation and rewrite $r_t = r_t + r_t^n - r_t^n$ with $r_t^n = \gamma E_t[c_{t+1}^e - c_t^e]$ (the natural IR at the efficient allocation)

Expected result :

5

Math Appendix

5.1 Usual Taylor Expansions

Function $f(x)$	Taylor Expansion form	...with sums
e^x	$1 + x + x^2 + \cdots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\frac{1}{1-x}$	$1 - x + x^2 - x^3 + \cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots$	$\sum_{n=0}^{\infty} x^n$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

5.2 Geometric Series

Geometric series take either of these 3 forms and give the following results (all can be admitted):

$$\begin{aligned} &1) \sum_{k=0}^{\infty} q^k = \sum_{k=t}^{\infty} q^{k-t} = \frac{1}{1-q} \\ &2) \sum_{k=0}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2} \\ &3) \sum_{k=0}^{\infty} k(k-1) q^{k-1} = \frac{2}{(1-q)^3} \end{aligned} \tag{61}$$

5.3 Expectation indices and period notations

The t+i notation : begins at time t

form of the sum :

$$E_t[\sum_{i=0}^{\infty} \beta^i X_{t+i}] \tag{62}$$

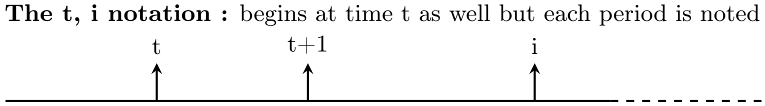
Note : (1) β is indexed by i and not t+i because between period t and t+i, only i periods have passed so we discount i times.

The t notation : begins at time 0

form of the sum :

$$E_0[\sum_{t=0}^{\infty} \beta^t X_t] \tag{63}$$

The t, i notation : begins at time t as well but each period is noted i instead of t+i



form of the sum :

$$E_t[\sum_{i=t}^{\infty} \beta^{i-t} X_i] \tag{64}$$