Supplementary Material: Distributed and Differentiable Vector Field Control within a Curve Virtual Tube for a Robotic Swarm under Field-of-View Constraints

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I. Introduction

This supplementary material presents a detailed mathematical derivation of the vector field controller proposed in the paper "Distributed and Differentiable Vector Field Control within a Curve Virtual Tube for a Robotic Swarm under Field-of-View Constraints". To make a real robot track its velocity command, we present its first and second derivatives with respect to time.

II. Vector Field Controller

In the paper "Distributed and Differentiable Vector Field Control within a Curve Virtual Tube for a Robotic Swarm under Field-of-View Constraints", we propose a distributed swarm controller for guiding a robotic swarm to pass through a curve virtual tube under field-of-view constraints. The proposed controller can be viewed as a velocity command for the ith robot, which is shown as

$$\mathbf{v}_{\mathrm{c},i}=\mathbf{v}_2\left(\mathcal{T},\mathbf{p}_i,\overline{\mathbf{p}}_i\right)=\begin{cases}\left(1-\sigma_2\left(\cos\mu_i,0,\epsilon_2\right)\right)\mathbf{u}_{\mathrm{mc},i} & \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m1},i}\geq 0\\ \mathbf{0} & \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m1},i}<0\end{cases} \tag{1}$$

and

$$\begin{aligned}\cos\mu_i&=\frac{\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m1},i}}{\left\|\mathbf{u}_{\mathrm{mc},i}\right\|\left\|\mathbf{u}_{\mathrm{m1},i}\right\|},\\ \mathbf{u}_{\mathrm{mc},i}&=\mathrm{sat}\left(\mathbf{u}_{\mathrm{m1},i}+\mathbf{u}_{2,i}+\mathbf{u}_{\mathrm{m3},i},v_{\mathrm{m},i}\right)=\sigma_1\left(\left\|\mathbf{u}_i\right\|,\epsilon_1 v_{\mathrm{m},i},v_{\mathrm{m},i}\right)\frac{\mathbf{u}_i}{\left\|\mathbf{u}_i\right\|},\\ \mathbf{u}_i&=\mathbf{u}_{\mathrm{m1},i}+\mathbf{u}_{2,i}+\mathbf{u}_{\mathrm{m3},i},\\ \mathbf{u}_{\mathrm{m1},i}&=v_{\mathrm{m},i}\mathbf{t}\left(s_i\right),\\ \mathbf{u}_{2,i}&=-\sum_{j\in\mathcal{N}_{\mathrm{d},i}}\frac{\partial V_{\mathrm{m},ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|}\frac{\tilde{\mathbf{p}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|},\\ \mathbf{u}_{\mathrm{m3},i}&=\frac{\partial V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}}\frac{\mathbf{p}_i-\mathbf{m}\left(s_i\right)}{\left\|\mathbf{p}_i-\mathbf{m}\left(s_i\right)\right\|}.\end{aligned}$$

III. First Derivative of Vector Field Controller with respect to Time

In this section, we propose the first derivative of the vector field controller (1) with respect to time. The position and velocity of the ith robot are expressed as $\mathbf{p}_i$ and $\mathbf{v}_i$. Besides, we have $\tilde{\mathbf{p}}_{ij} = \mathbf{p}_i - \mathbf{p}_j$ and $\tilde{\mathbf{v}}_{ij} = \mathbf{v}_i - \mathbf{v}_j$. First, we have

$$\begin{aligned}
\dot{s}_i &= \frac{R_{\gamma}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)}\mathbf{t}\left(s_i\right)^{\mathrm{T}}\mathbf{v}_i,\\
\frac{\partial \mathbf{t}\left(s_i\right)}{\partial s_i} &= \kappa_{\mathrm{r}}\left(s_i\right)\mathbf{n}\left(s_i\right),\\
\frac{\partial \mathbf{n}\left(s_i\right)}{\partial s_i} &= -\kappa_{\mathrm{r}}\left(s_i\right)\mathbf{t}\left(s_i\right),\\
\dot{d}_{\mathrm{t},i} &= \left(\frac{\partial r_{\mathrm{t}}\left(s_i\right)}{\partial \mathbf{p}_i} - \frac{\left(\mathbf{p}_i - \mathbf{m}\left(s_i\right)\right)^{\mathrm{T}}}{\left\|\mathbf{p}_i - \mathbf{m}\left(s_i\right)\right\|}\left(\mathbf{I}_2 - \frac{\partial \mathbf{m}\left(s_i\right)}{\partial \mathbf{p}_i}\right)\right)\mathbf{v}_i,
\end{aligned}$$

in which

$$\begin{aligned}
\frac{\partial r_{\mathrm{t}}\left(s_i\right)}{\partial \mathbf{p}_i} &= \frac{1}{2}\left(\frac{\partial \lambda\left(s_i,0\right)}{\partial s_i} + \frac{\partial \lambda\left(s_i,\pi\right)}{\partial s_i}\right)\frac{R_{\gamma}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)}\mathbf{t}\left(s_i\right)^{\mathrm{T}},\\
\frac{\partial \mathbf{m}\left(s_i\right)}{\partial \mathbf{p}_i} &= \left(\mathbf{t}\left(s_i\right) + \frac{1}{2}\left(\frac{\partial \lambda\left(s_i,0\right)}{\partial s_i} - \frac{\partial \lambda\left(s_i,\pi\right)}{\partial s_i}\right)\mathbf{n}\left(s_i\right) - \frac{1}{2}\left(\lambda\left(s_i,0\right) - \lambda\left(s_i,\pi\right)\right)\kappa_{\mathrm{r}}\left(s_i\right)\mathbf{t}\left(s_i\right)\right)\frac{R_{\gamma}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)}\mathbf{t}\left(s_i\right)^{\mathrm{T}}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\dot{\mathbf{u}}_{\mathrm{m1},i} &= v_{\mathrm{m},i}\frac{\partial \mathbf{t}\left(s_i\right)}{\partial s_i}\dot{s}_i = v_{\mathrm{m},i}\kappa_{\mathrm{r}}\left(s_i\right)\frac{R_{\gamma}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)}\mathbf{t}\left(s_i\right)^{\mathrm{T}}\mathbf{v}_i\mathbf{n}\left(s_i\right),\\
\dot{\mathbf{u}}_{2,1} &= -\sum_{j\in\mathcal{N}_{\mathrm{d},i}}\left(\frac{\partial^2 V_{\mathrm{m},ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|^2}\frac{\tilde{\mathbf{p}}_{ij}^{\mathrm{T}}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|}\tilde{\mathbf{v}}_{ij}\frac{\tilde{\mathbf{p}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|} + \frac{\partial V_{\mathrm{m},ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|}\frac{\tilde{\mathbf{v}}_{ij}\left\|\tilde{\mathbf{p}}_{ij}\right\|^2 - \tilde{\mathbf{p}}_{ij}\tilde{\mathbf{p}}_{ij}^{\mathrm{T}}\tilde{\mathbf{v}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|^3}\right),\\
\dot{\mathbf{u}}_{\mathrm{m3},i} &= \frac{\partial^2 V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}^2}\dot{d}_{\mathrm{t},i}\frac{\mathbf{p}_i - \mathbf{m}\left(s_i\right)}{\left\|\mathbf{p}_i - \mathbf{m}\left(s_i\right)\right\|} + \frac{\partial V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}}\frac{\mathbf{v}_i - \frac{\partial \mathbf{m}\left(s_i\right)}{\partial \mathbf{p}_i}\mathbf{v}_i}{\left\|\mathbf{p}_i - \mathbf{m}\left(s_i\right)\right\|} - \frac{\partial V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}}\frac{\left(\mathbf{p}_i - \mathbf{m}\left(s_i\right)\right)\left(\mathbf{p}_i - \mathbf{m}\left(s_i\right)\right)^{\mathrm{T}}\left(\mathbf{v}_i - \frac{\partial \mathbf{m}\left(s_i\right)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)}{\left\|\mathbf{p}_i - \mathbf{m}\left(s_i\right)\right\|^3}.
\end{aligned}$$

It has

$$\dot{\mathbf{u}}_i = \dot{\mathbf{u}}_{\mathrm{m1},i} + \dot{\mathbf{u}}_{2,i} + \dot{\mathbf{u}}_{\mathrm{m3},i},$$

and

$$\dot{\mathbf{u}}_{\mathrm{mc},i} = \frac{\partial \sigma_1}{\partial\left\|\mathbf{u}_i\right\|}\frac{\mathbf{u}_i^{\mathrm{T}}}{\left\|\mathbf{u}_i\right\|}\dot{\mathbf{u}}_i\frac{\mathbf{u}_i}{\left\|\mathbf{u}_i\right\|} + \sigma_1\frac{\dot{\mathbf{u}}_i\left\|\mathbf{u}_i\right\|^2 - \mathbf{u}_i\mathbf{u}_i^{\mathrm{T}}\dot{\mathbf{u}}_i}{\left\|\mathbf{u}_i\right\|^3},$$

where σ_1 represents $\sigma_1\left(\left\|\mathbf{u}_i\right\|, \epsilon_1 v_{\mathrm{m},i}, v_{\mathrm{m},i}\right)$ for short. Then the time derivative of the velocity command is shown as

$$\dot{\mathbf{v}}_{\mathrm{c},i} = \dot{\mathbf{v}}_2\left(\mathcal{T},\mathbf{p}_i,\overline{\mathbf{p}}_i\right) = \begin{cases}\left(1-\sigma_2\right)\dot{\mathbf{u}}_{\mathrm{mc},i} - \frac{\partial\sigma_2}{\partial\cos\mu_i}\frac{\partial\cos\mu_i}{\partial t}\mathbf{u}_{\mathrm{mc},i} & \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m1},i} \geq 0\\ \mathbf{0} & \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m1},i} < 0\end{cases},$$

where σ_2 represents $\sigma_2\left(\cos\mu_i, 0, \epsilon_2\right)$ for simplicity and

$$\frac{\partial\cos\mu_i}{\partial t} = \frac{\dot{\mathbf{u}}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m1},i} + \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m1},i}}{\left\|\mathbf{u}_{\mathrm{mc},i}\right\|\left\|\mathbf{u}_{\mathrm{m1},i}\right\|} - \frac{\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m1},i}\left(\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{mc},i}\left\|\mathbf{u}_{\mathrm{m1},i}\right\|^2 + \mathbf{u}_{\mathrm{m1},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m1},i}\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^2\right)}{\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^3\left\|\mathbf{u}_{\mathrm{m1},i}\right\|^3}.$$

IV. Second derivative of Vector Field Controller with Respect to Time

In this section, we propose the second derivative of the velocity command (1) with respect to time. The position, velocity and acceleration of the ith robot are expressed as $\mathbf{p}_i$, $\mathbf{v}_i$ and $\mathbf{a}_i$. Besides, we have $\tilde{\mathbf{p}}_{ij}=\mathbf{p}_i-\mathbf{p}_j$, $\tilde{\mathbf{v}}_{ij}=\mathbf{v}_i-\mathbf{v}_j$, $\tilde{\mathbf{a}}_{ij}=\mathbf{a}_i-\mathbf{a}_j$. First, we calculate $\frac{\partial \kappa(s)}{\partial s}$ and $\frac{\partial \kappa_{\mathrm{r}}(s)}{\partial s}$. We have

$$\begin{aligned}
&\frac{\partial \boldsymbol{\gamma}(s)}{\partial s}=\mathbf{t}(s),\\
&\frac{\partial^2 \boldsymbol{\gamma}(s)}{\partial s^2}=\kappa(s)\,\mathbf{n}_{\mathrm{p}}(s),\\
&\frac{\partial^3 \boldsymbol{\gamma}(s)}{\partial s^3}=\frac{\partial \kappa(s)}{\partial s}\mathbf{n}_{\mathrm{p}}(s)+\kappa(s)\frac{\partial \mathbf{n}_{\mathrm{p}}(s)}{\partial s}=\frac{\partial \kappa(s)}{\partial s}\mathbf{n}_{\mathrm{p}}(s)-\kappa(s)^2\,\mathbf{t}(s).
\end{aligned}$$

Hence we have

$$\frac{\partial \kappa(s)}{\partial s}=\mathbf{n}_{\mathrm{p}}(s)^{\mathrm{T}}\frac{\partial^3 \boldsymbol{\gamma}(s)}{\partial s^3}.$$

Similarly, we have

$$\frac{\partial \kappa_{\mathrm{r}}(s)}{\partial s}=\mathbf{n}(s)^{\mathrm{T}}\frac{\partial^3 \boldsymbol{\gamma}(s)}{\partial s^3}.$$

In the paper “Distributed and Differentiable Vector Field Control within a Curve Virtual Tube for a Robotic Swarm under Field-of-View Constraints”, we have proposed that

$$\begin{aligned}
R_{\boldsymbol{\gamma}}(s_i)&=\frac{1}{\kappa(s_i)},\\
R_{\mathcal{T}}(\mathbf{p}_i)&=R_{\boldsymbol{\gamma}}(s_i)-(\mathbf{p}_i-\boldsymbol{\gamma}(s_i))^{\mathrm{T}}\mathbf{n}_{\mathrm{p}}(s_i).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\dot{\kappa}(s_i)&=\frac{\partial \kappa(s_i)}{\partial s_i}\dot{s}_i=\mathbf{n}_{\mathrm{p}}(s_i)^{\mathrm{T}}\frac{\partial^3 \boldsymbol{\gamma}(s_i)}{\partial s_i^3}\dot{s}_i,\\
\dot{\kappa}_{\mathrm{r}}(s_i)&=\frac{\partial \kappa_{\mathrm{r}}(s_i)}{\partial s_i}\dot{s}_i=\mathbf{n}(s_i)^{\mathrm{T}}\frac{\partial^3 \boldsymbol{\gamma}(s_i)}{\partial s_i^3}\dot{s}_i,\\
\dot{R}_{\boldsymbol{\gamma}}(s_i)&=\frac{\partial R_{\boldsymbol{\gamma}}(s_i)}{\partial \kappa(s_i)}\dot{\kappa}(s_i)=-\frac{1}{\kappa(s_i)^2}\dot{\kappa}(s_i),\\
\dot{R}_{\mathcal{T}}(\mathbf{p}_i)&=\dot{R}_{\boldsymbol{\gamma}}(s_i)-(\mathbf{v}_i-\mathbf{t}(s_i)\,\dot{s}_i)^{\mathrm{T}}\mathbf{n}_{\mathrm{p}}(s_i)+(\mathbf{p}_i-\boldsymbol{\gamma}(s_i))^{\mathrm{T}}\kappa(s_i)\,\mathbf{t}(s_i)\,\dot{s}_i.
\end{aligned}$$

$$\begin{aligned}
\ddot{d}_{\mathrm{t},i}=&\left(\frac{\partial^2 r_{\mathrm{t}}(s_i)}{\partial \mathbf{p}_i\partial t}-\frac{\left(\mathbf{v}_i-\frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)^{\mathrm{T}}}{\|\mathbf{p}_i-\mathbf{m}(s_i)\|}\left(\mathbf{I}_2-\frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\right)\right.\\
&+\frac{(\mathbf{p}_i-\mathbf{m}(s_i))^{\mathrm{T}}(\mathbf{p}_i-\mathbf{m}(s_i))^{\mathrm{T}}\left(\mathbf{v}_i-\frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)}{\|\mathbf{p}_i-\mathbf{m}(s_i)\|^3}\left(\mathbf{I}_2-\frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\right)\\
&\left.+\frac{(\mathbf{p}_i-\mathbf{m}(s_i))^{\mathrm{T}}}{\|\mathbf{p}_i-\mathbf{m}(s_i)\|}\frac{\partial^2 \mathbf{m}(s_i)}{\partial \mathbf{p}_i\partial t}\right)\mathbf{v}_i+\left(\frac{\partial r_{\mathrm{t}}(s_i)}{\partial \mathbf{p}_i}-\frac{(\mathbf{p}_i-\mathbf{m}(s_i))^{\mathrm{T}}}{\|\mathbf{p}_i-\mathbf{m}(s_i)\|}\left(\mathbf{I}_2-\frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\right)\right)\mathbf{a}_i,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 r_{\mathrm{t}}\left(s_i\right)}{\partial \mathbf{p}_i \partial t}= & \frac{1}{2}\left(\frac{\partial^2 \lambda\left(s_i, 0\right)}{\partial s_i^2}+\frac{\partial^2 \lambda\left(s_i, \pi\right)}{\partial s_i^2}\right) \dot{s}_i \frac{R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)} \mathbf{t}\left(s_i\right)^{\mathrm{T}} \\
& +\frac{1}{2}\left(\frac{\partial \lambda\left(s_i, 0\right)}{\partial s_i}+\frac{\partial \lambda\left(s_i, \pi\right)}{\partial s_i}\right) \frac{R_{\mathcal{T}}\left(\mathbf{p}_i\right) \dot{R}_{\boldsymbol{\gamma}}\left(s_i\right)-\dot{R}_{\mathcal{T}}\left(\mathbf{p}_i\right) R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)^2} \mathbf{t}\left(s_i\right)^{\mathrm{T}} \\
& +\frac{1}{2}\left(\frac{\partial \lambda\left(s_i, 0\right)}{\partial s_i}+\frac{\partial \lambda\left(s_i, \pi\right)}{\partial s_i}\right) \frac{R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)} \kappa_{\mathrm{r}}\left(s_i\right) \mathbf{n}\left(s_i\right)^{\mathrm{T}} \dot{s}_i, \\
\frac{\partial^2 \mathbf{m}\left(s_i\right)}{\partial \mathbf{p}_i \partial t}= & \left(\kappa_{\mathrm{r}}\left(s_i\right) \mathbf{n}\left(s_i\right) \dot{s}_i+\frac{1}{2}\left(\frac{\partial^2 \lambda\left(s_i, 0\right)}{\partial s_i^2}-\frac{\partial^2 \lambda\left(s_i, \pi\right)}{\partial s_i^2}\right) \dot{s}_i \mathbf{n}\left(s_i\right)\right. \\
& -\frac{1}{2}\left(\frac{\partial \lambda\left(s_i, 0\right)}{\partial s_i}-\frac{\partial \lambda\left(s_i, \pi\right)}{\partial s_i}\right) \kappa_{\mathrm{r}}\left(s_i\right) \mathbf{t}\left(s_i\right) \dot{s}_i \\
& -\frac{1}{2}\left(\frac{\partial \lambda\left(s_i, 0\right)}{\partial s_i}-\frac{\partial \lambda\left(s_i, \pi\right)}{\partial s_i}\right) \dot{s}_i \kappa_{\mathrm{r}}\left(s_i\right) \mathbf{t}\left(s_i\right)-\frac{1}{2}\left(\lambda\left(s_i, 0\right)-\lambda\left(s_i, \pi\right)\right) \dot{\kappa}_{\mathrm{r}}\left(s_i\right) \mathbf{t}\left(s_i\right) \\
& \left.-\frac{1}{2}\left(\lambda\left(s_i, 0\right)-\lambda\left(s_i, \pi\right)\right) \kappa_{\mathrm{r}}\left(s_i\right)^2 \mathbf{n}\left(s_i\right) \dot{s}_i\right) \frac{R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)} \mathbf{t}\left(s_i\right)^{\mathrm{T}} \\
& +\left(\mathbf{t}\left(s_i\right)+\frac{1}{2}\left(\frac{\partial \lambda\left(s_i, 0\right)}{\partial s_i}-\frac{\partial \lambda\left(s_i, \pi\right)}{\partial s_i}\right) \mathbf{n}\left(s_i\right)\right. \\
& \left.-\frac{1}{2}\left(\lambda\left(s_i, 0\right)-\lambda\left(s_i, \pi\right)\right) \kappa_{\mathrm{r}}\left(s_i\right) \mathbf{t}\left(s_i\right)\right) \frac{R_{\mathcal{T}}\left(\mathbf{p}_i\right) \dot{R}_{\boldsymbol{\gamma}}\left(s_i\right)-\dot{R}_{\mathcal{T}}\left(\mathbf{p}_i\right) R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)^2} \mathbf{t}\left(s_i\right)^{\mathrm{T}} \\
& +\left(\mathbf{t}\left(s_i\right)+\frac{1}{2}\left(\frac{\partial \lambda\left(s_i, 0\right)}{\partial s_i}-\frac{\partial \lambda\left(s_i, \pi\right)}{\partial s_i}\right) \mathbf{n}\left(s_i\right)\right. \\
& \left.-\frac{1}{2}\left(\lambda\left(s_i, 0\right)-\lambda\left(s_i, \pi\right)\right) \kappa_{\mathrm{r}}\left(s_i\right) \mathbf{t}\left(s_i\right)\right) \frac{R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)} \kappa_{\mathrm{r}}\left(s_i\right) \mathbf{n}\left(s_i\right)^{\mathrm{T}} \dot{s}_i .
\end{aligned}
$$

Given $\tilde{\mathbf{a}}_{ij}=\mathbf{a}_i-\mathbf{a}_j$, we have

$$
\begin{aligned}
\ddot{\mathbf{u}}_{\mathrm{m}1, i}= & v_{\mathrm{m}, i} \dot{\kappa}_{\mathrm{r}}\left(s_i\right) \frac{R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)} \mathbf{t}\left(s_i\right)^{\mathrm{T}} \mathbf{v}_i \mathbf{n}\left(s_i\right)+v_{\mathrm{m}, i} \kappa_{\mathrm{r}}\left(s_i\right) \frac{R_{\mathcal{T}}\left(\mathbf{p}_i\right) \dot{R}_{\boldsymbol{\gamma}}\left(s_i\right)-\dot{R}_{\mathcal{T}}\left(\mathbf{p}_i\right) R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)^2} \mathbf{t}\left(s_i\right)^{\mathrm{T}} \mathbf{v}_i \mathbf{n}\left(s_i\right) \\
& +v_{\mathrm{m}, i} \kappa_{\mathrm{r}}\left(s_i\right)^2 \frac{R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)} \dot{s}_i \mathbf{n}\left(s_i\right)^{\mathrm{T}} \mathbf{v}_i \mathbf{n}\left(s_i\right)+v_{\mathrm{m}, i} \kappa_{\mathrm{r}}\left(s_i\right) \frac{R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)} \mathbf{t}\left(s_i\right)^{\mathrm{T}} \mathbf{a}_i \mathbf{n}\left(s_i\right) \\
& -v_{\mathrm{m}, i} \kappa_{\mathrm{r}}\left(s_i\right)^2 \frac{R_{\boldsymbol{\gamma}}\left(s_i\right)}{R_{\mathcal{T}}\left(\mathbf{p}_i\right)} \dot{s}_i \mathbf{t}\left(s_i\right)^{\mathrm{T}} \mathbf{v}_i \mathbf{t}\left(s_i\right), \\
\ddot{\mathbf{u}}_{2,1}= & -\sum_{j \in \mathcal{N}_{\mathrm{d}, i}}\left(\frac{\partial^3 V_{\mathrm{m}, ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|^3}\left(\frac{\tilde{\mathbf{p}}_{ij}^{\mathrm{T}} \tilde{\mathbf{v}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|}\right)^2 \frac{\tilde{\mathbf{p}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|}+\frac{\partial^2 V_{\mathrm{m}, ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|^2} \frac{\tilde{\mathbf{v}}_{ij}{ }^{\mathrm{T}}\left\|\tilde{\mathbf{p}}_{ij}\right\|^2-\tilde{\mathbf{p}}_{ij}^{\mathrm{T}} \tilde{\mathbf{p}}_{ij}^{\mathrm{T}} \tilde{\mathbf{v}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|^3} \tilde{\mathbf{v}}_{ij} \frac{\tilde{\mathbf{p}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|}\right. \\
& +\frac{\partial^2 V_{\mathrm{m}, ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|^2} \frac{\tilde{\mathbf{p}}_{ij}^{\mathrm{T}}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|} \tilde{\mathbf{a}}_{ij} \frac{\tilde{\mathbf{p}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|}+2 \frac{\partial^2 V_{\mathrm{m}, ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|^2} \frac{\tilde{\mathbf{p}}_{ij}^{\mathrm{T}}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|} \tilde{\mathbf{v}}_{ij} \frac{\tilde{\mathbf{v}}_{ij}\left\|\tilde{\mathbf{p}}_{ij}\right\|^2-\tilde{\mathbf{p}}_{ij} \tilde{\mathbf{p}}_{ij}^{\mathrm{T}} \tilde{\mathbf{v}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|^3} \\
& \left.+\frac{\partial V_{\mathrm{m}, ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|} \frac{\tilde{\mathbf{a}}_{ij}\left\|\tilde{\mathbf{p}}_{ij}\right\|^2+\tilde{\mathbf{v}}_{ij} \tilde{\mathbf{p}}_{ij}^{\mathrm{T}} \tilde{\mathbf{v}}_{ij}-\tilde{\mathbf{p}}_{ij} \tilde{\mathbf{v}}_{ij}^{\mathrm{T}} \tilde{\mathbf{v}}_{ij}-\tilde{\mathbf{p}}_{ij} \tilde{\mathbf{p}}_{ij}^{\mathrm{T}} \tilde{\mathbf{a}}_{ij}}{\left\|\tilde{\mathbf{p}}_{ij}\right\|^3}-\frac{\partial V_{\mathrm{m}, ij}}{\partial\left\|\tilde{\mathbf{p}}_{ij}\right\|} \frac{3 \tilde{\mathbf{p}}_{ij}^{\mathrm{T}} \tilde{\mathbf{v}}_{ij}\left(\tilde{\mathbf{v}}_{ij}\left\|\tilde{\mathbf{p}}_{ij}\right\|^2-\tilde{\mathbf{p}}_{ij} \tilde{\mathbf{p}}_{ij}^{\mathrm{T}} \tilde{\mathbf{v}}_{ij}\right)}{\left\|\tilde{\mathbf{p}}_{ij}\right\|^5}\right),
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\mathbf{u}}_{\mathrm{m}3,i} =& \frac{\partial^3 V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}^3}\dot{d}_{\mathrm{t},i}^2 \frac{\mathbf{p}_i - \mathbf{m}(s_i)}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|} + \frac{\partial^2 V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}^2}\ddot{d}_{\mathrm{t},i} \frac{\mathbf{p}_i - \mathbf{m}(s_i)}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|} \\
&+ \frac{\partial^2 V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}^2}\dot{d}_{\mathrm{t},i} \left(\frac{\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|} - \frac{(\mathbf{p}_i - \mathbf{m}(s_i))(\mathbf{p}_i - \mathbf{m}(s_i))^{\mathrm{T}} \left(\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|^3} \right) \\
&+ \frac{\partial^2 V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}^2}\dot{d}_{\mathrm{t},i} \frac{\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|} + \frac{\partial V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}} \frac{\mathbf{a}_i - \frac{\partial^2 \mathbf{m}(s_i)}{\partial \mathbf{p}_i \partial t}\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{a}_i}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|} \\
&- 2\frac{\partial V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}} \frac{\left(\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)(\mathbf{p}_i - \mathbf{m}(s_i))^{\mathrm{T}} \left(\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|^3} \\
&- \frac{\partial^2 V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}^2}\dot{d}_{\mathrm{t},i} \frac{(\mathbf{p}_i - \mathbf{m}(s_i))(\mathbf{p}_i - \mathbf{m}(s_i))^{\mathrm{T}} \left(\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|^3} \\
&- \frac{\partial V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}} \frac{(\mathbf{p}_i - \mathbf{m}(s_i))\left(\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)^{\mathrm{T}} \left(\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|^3} \\
&- \frac{\partial V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}} \frac{(\mathbf{p}_i - \mathbf{m}(s_i))(\mathbf{p}_i - \mathbf{m}(s_i))^{\mathrm{T}} \left(\mathbf{a}_i - \frac{\partial^2 \mathbf{m}(s_i)}{\partial \mathbf{p}_i \partial t}\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{a}_i\right)}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|^3} \\
&+ \frac{\partial V_{\mathrm{t},i}}{\partial d_{\mathrm{t},i}} \frac{3(\mathbf{p}_i - \mathbf{m}(s_i))\left((\mathbf{p}_i - \mathbf{m}(s_i))^{\mathrm{T}} \left(\mathbf{v}_i - \frac{\partial \mathbf{m}(s_i)}{\partial \mathbf{p}_i}\mathbf{v}_i\right)\right)^2}{\|\mathbf{p}_i - \mathbf{m}(s_i)\|^5}.
\end{aligned}
$$

We have

$$
\ddot{\mathbf{u}}_i = \ddot{\mathbf{u}}_{\mathrm{m}1,i} + \ddot{\mathbf{u}}_{2,i} + \ddot{\mathbf{u}}_{\mathrm{m}3,i}
$$

and

$$
\begin{aligned}
\ddot{\mathbf{u}}_{\mathrm{mc},i} =& \frac{\partial^2 \sigma_1}{\partial \|\mathbf{u}_i\|^2} \left(\frac{\mathbf{u}_i^{\mathrm{T}}}{\|\mathbf{u}_i\|}\dot{\mathbf{u}}_i \right)^2 \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|} + \frac{\partial \sigma_1}{\partial \|\mathbf{u}_i\|} \left(\frac{\dot{\mathbf{u}}_i \|\mathbf{u}_i\|^2 - \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}} \dot{\mathbf{u}}_i}{\|\mathbf{u}_i\|^3} \right)^{\mathrm{T}} \dot{\mathbf{u}}_i \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|} \\
&+ \frac{\partial \sigma_1}{\partial \|\mathbf{u}_i\|} \frac{\mathbf{u}_i^{\mathrm{T}}}{\|\mathbf{u}_i\|} \ddot{\mathbf{u}}_i \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|} + 2\frac{\partial \sigma_1}{\partial \|\mathbf{u}_i\|} \frac{\mathbf{u}_i^{\mathrm{T}}}{\|\mathbf{u}_i\|} \dot{\mathbf{u}}_i \frac{\dot{\mathbf{u}}_i \|\mathbf{u}_i\|^2 - \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}} \dot{\mathbf{u}}_i}{\|\mathbf{u}_i\|^3} \\
&+ \sigma_1 \frac{\|\mathbf{u}_i\|^2 \left(\ddot{\mathbf{u}}_i \|\mathbf{u}_i\|^2 + \dot{\mathbf{u}}_i \mathbf{u}_i^{\mathrm{T}} \dot{\mathbf{u}}_i - \mathbf{u}_i \dot{\mathbf{u}}_i^{\mathrm{T}} \dot{\mathbf{u}}_i - \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}} \ddot{\mathbf{u}}_i \right) - 3\mathbf{u}_i^{\mathrm{T}} \dot{\mathbf{u}}_i \left(\dot{\mathbf{u}}_i \|\mathbf{u}_i\|^2 - \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}} \dot{\mathbf{u}}_i \right)}{\|\mathbf{u}_i\|^5},
\end{aligned}
$$

where σ_1 represents $\sigma_1(\|\mathbf{u}_i\|, \epsilon_1 v_{\mathrm{m},i}, v_{\mathrm{m},i})$ for short. Then $\ddot{\mathbf{v}}_{\mathrm{c},i}$ is shown as

$$
\ddot{\mathbf{v}}_{\mathrm{c},i} = \ddot{\mathbf{v}}_2(\mathcal{T}, \mathbf{p}_i, \overline{\mathbf{p}}_i) = \begin{cases} \ddot{\mathbf{v}}_2'(\mathcal{T}, \mathbf{p}_i, \overline{\mathbf{p}}_i) & \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}} \mathbf{u}_{\mathrm{m}1,i} \geq 0 \\ \mathbf{0} & \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}} \mathbf{u}_{\mathrm{m}1,i} < 0 \end{cases}
$$

$$
\begin{aligned}
\ddot{\mathbf{v}}_2'(\mathcal{T}, \mathbf{p}_i, \overline{\mathbf{p}}_i) =& -\frac{\partial \sigma_2}{\partial \cos \mu_i} \frac{\partial \cos \mu_i}{\partial t} \dot{\mathbf{u}}_{\mathrm{mc},i} + (1 - \sigma_2) \ddot{\mathbf{u}}_{\mathrm{mc},i} - \frac{\partial^2 \sigma_2}{\partial \cos \mu_i^2} \left(\frac{\partial \cos \mu_i}{\partial t} \right)^2 \mathbf{u}_{\mathrm{mc},i} \\
&- \frac{\partial \sigma_2}{\partial \cos \mu_i} \frac{\partial^2 \cos \mu_i}{\partial t^2} \mathbf{u}_{\mathrm{mc},i} - \frac{\partial \sigma_2}{\partial \cos \mu_i} \frac{\partial \cos \mu_i}{\partial t} \dot{\mathbf{u}}_{\mathrm{mc},i},
\end{aligned}
$$

where σ_2 represents $\sigma_2\left(\cos\mu_i, 0, \epsilon_2\right)$ for short and

$$\begin{aligned}\frac{\partial^2\cos\mu_i}{\partial t^2} = & \frac{\ddot{\mathbf{u}}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m}1,i} + 2\dot{\mathbf{u}}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m}1,i} + \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\ddot{\mathbf{u}}_{\mathrm{m}1,i}}{\left\|\mathbf{u}_{\mathrm{mc},i}\right\|\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|} \\ & - \frac{2\left(\dot{\mathbf{u}}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m}1,i} + \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m}1,i}\right)\left(\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{mc},i}\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^2 + \mathbf{u}_{\mathrm{m}1,i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m}1,i}\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^2\right)}{\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^3\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^3} \\ & - \frac{\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m}1,i}\left(\dot{\mathbf{u}}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{mc},i}\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^2 + \mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\ddot{\mathbf{u}}_{\mathrm{mc},i}\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^2 + 2\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{mc},i}\mathbf{u}_{\mathrm{m}1,i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m}1,i}\right)}{\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^3\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^3} \\ & - \frac{\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m}1,i}\left(\dot{\mathbf{u}}_{\mathrm{m}1,i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m}1,i}\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^2 + \mathbf{u}_{\mathrm{m}1,i}^{\mathrm{T}}\ddot{\mathbf{u}}_{\mathrm{m}1,i}\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^2 + 2\mathbf{u}_{\mathrm{m}1,i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m}1,i}\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{mc},i}\right)}{\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^3\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^3} \\ & + \frac{\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\mathbf{u}_{\mathrm{m}1,i}\left(\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{mc},i}\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^2 + \mathbf{u}_{\mathrm{m}1,i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m}1,i}\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^2\right)\left(3\mathbf{u}_{\mathrm{mc},i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{mc},i}\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^2 + 3\mathbf{u}_{\mathrm{m}1,i}^{\mathrm{T}}\dot{\mathbf{u}}_{\mathrm{m}1,i}\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^2\right)}{\left\|\mathbf{u}_{\mathrm{mc},i}\right\|^5\left\|\mathbf{u}_{\mathrm{m}1,i}\right\|^5}.\end{aligned}$$

V. Simulation Result

In this section, the correctness of the presented mathematical derivation is verified in a numerical simulation. Consider a scenario that one robotic swarm made up of $M = 20$ robots are asked to pass through a prescribed curve virtual tube. The distributed swarm controller in (1) is used to control these robots. All robots satisfy the three integrator model as

$$\begin{cases}\dot{\mathbf{p}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = \mathbf{a}_i \\ \dot{\mathbf{a}}_i = \mathbf{j}_{\mathrm{c},i}\end{cases}, \tag{2}$$

where $\mathbf{v}_i \in \mathbb{R}^2, \mathbf{a}_i \in \mathbb{R}^2$ represent the ith robot's velocity and acceleration, $\mathbf{j}_{\mathrm{c},i} \in \mathbb{R}^2$ represents the desired jerk. Then, a control law for (2) is designed as

$$\mathbf{j}_{\mathrm{c},i} = \ddot{\mathbf{v}}_{\mathrm{c},i} + k_{\mathbf{v}}\left(\mathbf{v}_{\mathrm{c},i} - \mathbf{v}_i\right) + k_{\mathbf{a}}\left(\dot{\mathbf{v}}_{\mathrm{c},i} - \mathbf{a}_i\right), \tag{3}$$

where $k_{\mathbf{v}}, k_{\mathbf{a}} > 0$. It is easy to obtain that $\lim_{t\to\infty}\mathbf{v}_i = \mathbf{v}_{\mathrm{c},i}$. Hence, according to (3), $\dot{\mathbf{v}}_{\mathrm{c},i}$ and $\ddot{\mathbf{v}}_{\mathrm{c},i}$ are necessary for the simulation.

Different from the simulation result presented in the paper "Distributed and Differentiable Vector Field Control within a Curve Virtual Tube for a Robotic Swarm under Field-of-View Constraints", how robots move inside the curve virtual tube is not the focus in this supplementary material. Here we only concern about whether the calculation results of $\dot{\mathbf{v}}_{\mathrm{c},i}$ and $\ddot{\mathbf{v}}_{\mathrm{c},i}$ are correct or not. Let $\mathbf{v}_{\mathrm{c},i} = \left[v_{x,i}\ v_{y,i}\right]^{\mathrm{T}}$. Without loss of generality, we only talk about $v_{x,10}$, $\dot{v}_{x,10}$ and $\ddot{v}_{x,10}$ of the 10th robot. The simulation lasts 10 seconds. Figure 1 shows the changes of $v_{x,10}$, $\dot{v}_{x,10}$ and $\ddot{v}_{x,10}$. To verify the correctness of $\dot{v}_{x,10}$, the comparison of $v_{x,10}$ and $\int_0^t \dot{v}_{x,10}\mathrm{d}t$ is shown in Figure 2. It can be seen that these two curves has an initial error due to $v_{x,10}\left(0\right) \neq 0$. Besides, at $t = 3.013\ s$, $v_{x,10}$ has a jump, which can be seen in the right plot of Figure 2. This is because another robot enters the avoidance area and detection area of the 10th robot at the same time. To verify the correctness of $\ddot{v}_{x,10}$, the comparison of $\dot{v}_{x,10}$ and $\int_0^t \ddot{v}_{x,10}\mathrm{d}t$ is shown in Figure 3. In Figure 3, we can also observe the jump at $t = 3.013\ s$. In a word, Figure 2 and 3 show the correctness of the calculation result of $\dot{\mathbf{v}}_{\mathrm{c},i}$ and $\ddot{\mathbf{v}}_{\mathrm{c},i}$.

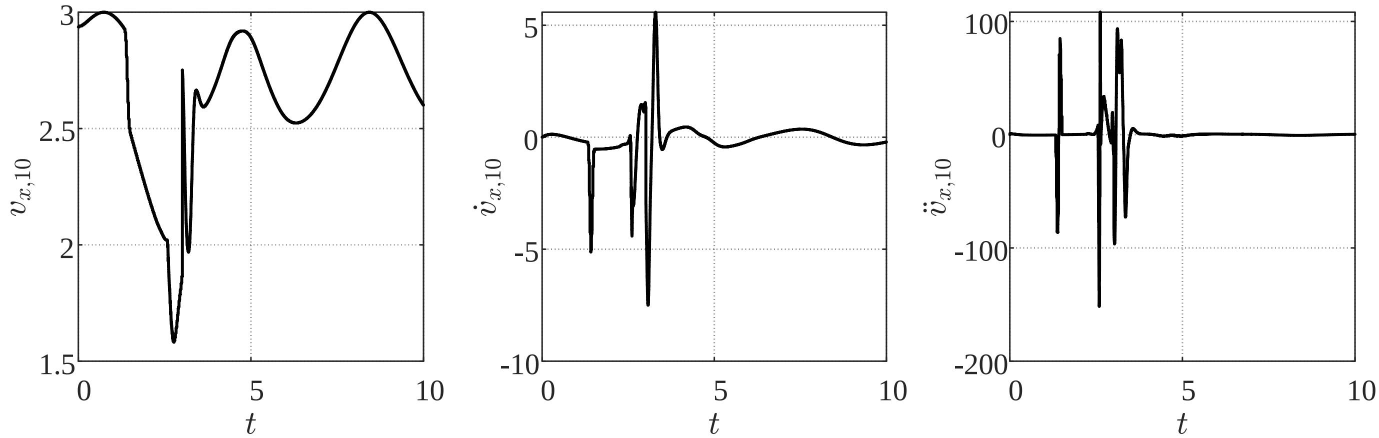

Fig. 1: $v_{x,10}$, $\dot{v}_{x,10}$ and $\ddot{v}_{x,10}$ in the simulation.

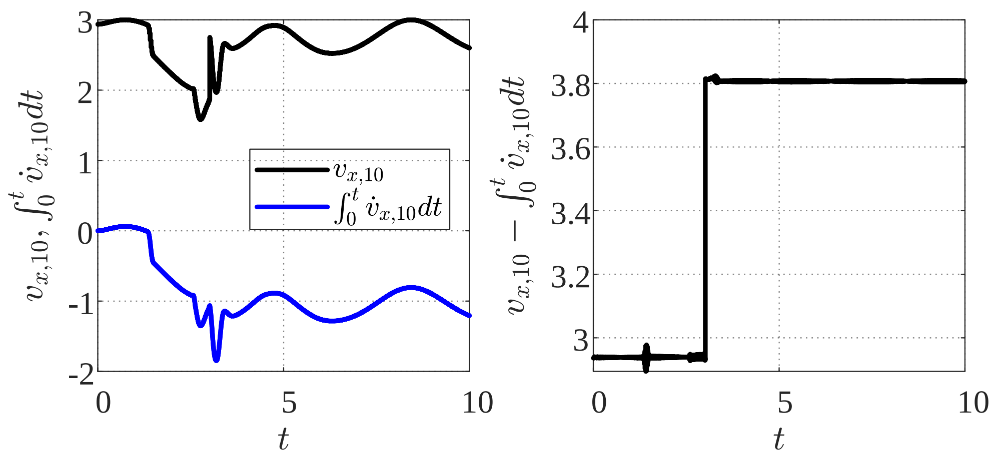

Fig. 2: The comparison of $v_{x,10}$ and $\int_0^t \dot{v}_{x,10}\mathrm{d}t$.

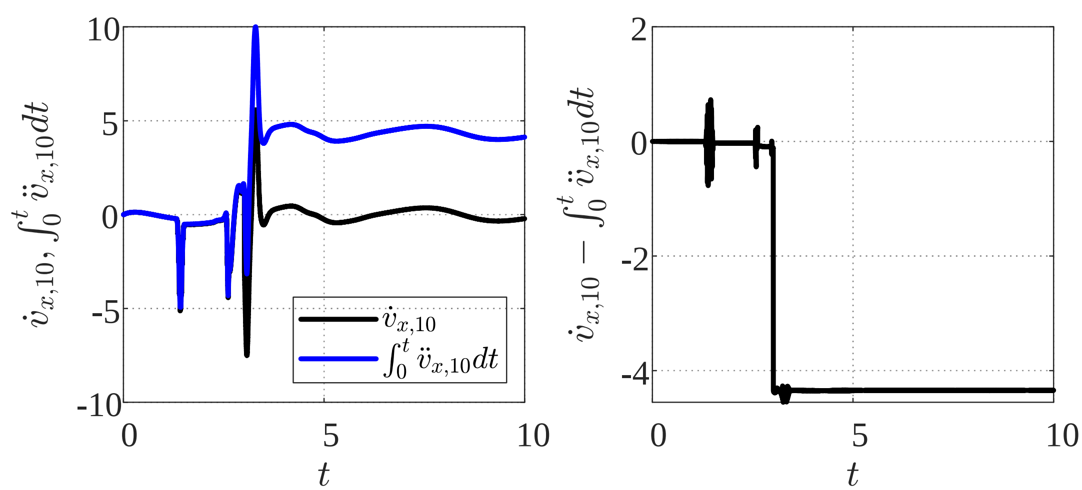

Fig. 3: The comparison of $\dot{v}_{x,10}$ and $\int_0^t \ddot{v}_{x,10}\mathrm{d}t$.