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On the existence of a unique class of yield and failure criteria comprising Tresca, von Mises, Drucker-Prager, Mohr-Coulomb, Galileo - Rankine, Matsuoka-Nakai and Lade-Duncan

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In this paper it is mathematically demonstrated that classical yield and failure criteria such as Tresca, von Mises, Drucker-Prager, Mohr-Coulomb, Matsuoka-Nakai and Lade-Duncan are all defined by the same equation. This can be seen as one of the three solutions of a cubic equation of the principal stresses, and suggests that all such criteria belong to a more general class of non-convex formulations which also comprises a recent generalization of the Galileo-Rankine criterion. The derived equation is always convex and can also provide a smooth approximation of continuity of at least class C^2 of the original Tresca and Mohr-Coulomb criteria. It is therefore free from all the limitations which restrain the use of some of them in numerical analyses. The mathematical structure of the presented equation is based on a separate definition of the meridional and deviatoric sections of the graphical representation of the criteria. This enables the use of an efficient implicit integration algorithm which results in a very short machine runtime even when demanding boundary value problems are analysed.

1. Introduction

The range of applications of classical yield and failure criteria has grown beyond the original purposes and their use is not any more restricted just to hand calculations such as e.g. those for the evaluation of the limit of elasticity of a beam or of the stability of an earth work. Classical yield and failure criteria nowadays play a fundamental role also in numerical analyses of boundary value problems as they are widely adopted in constitutive models for reproducing material behaviour. As discussed in what follows, this has unveiled that many of them are affected by limitations, some even of theoretical nature, which create numerical difficulties.

The arguments presented in this paper are restricted to criteria for isotropic materials, and the term *classical* will be used to indicate not only the Tresca, von Mises, Drucker-Prager and Mohr-Coulomb criteria but also those more recent but well established proposed by Matsuoka and Nakai [23] and by Lade and Duncan [13]. A recent no-tension criterion which represents a generalization of Galileo-Rankine, formulated by Lagioia et al. [16] and hereafter referred to as the I_3 tension cut-off, will be also included in the discussion. It should be noted however, that although this presentation is confined to these criteria only, its findings might be of a more general validity.

Classical yield and failure criteria for metals and soils adopt different mechanical quantities as an indicator of the proximity to the critical condition. Their mathematical definition varies very much from one another and so does their graphical representation in the Haigh-Westergaard principal stress space. It is no surprise then that the individual characteristics are usually emphasized whilst unifying aspects have seldom been explored. Investigating common features of classical criteria can be beneficial, and researchers have occasionally looked for equations defining a unifying class capable of comprising all individual criteria or yield functions (e.g. van Eekelen [35], Mortara [24], ...).

In this paper it will be mathematically demonstrated that the classical yield and failure criteria for metals and soils are all defined by one and only equation and depend on the first invariant of the stress tensor I_1, the second invariant J_{2D} of the deviatoric stress tensor and the Lode's angle, θ, [20]. At variance with the original defining equation of some of the classical criteria, the proposed formulation is convex and can also provide a rounded version of Tresca and Mohr-Coulomb, both circumscribing or inscribing their hexagons in the deviatoric section. The obtained equation is therefore free from all the problems which affect the original definition of some criteria and due to its structure proves extremely efficient in numerical analyses. Moreover it bears the fundamental advantage of allowing with one single implementation to reproduce all the above classical surfaces.

It should be noted that the formulation presented in this paper presents very close similarities with that proposed by Ottosen [25], who developed a phenomenological shape function to fit the failure of concrete in the deviatoric plane. It was obtained by means of a mechanical analogy of the deflection of a pressurized membrane supported along the edges of an equilateral triangle. As such, it was neither the result of theoretical developments nor was intended at reproducing the shape of any of the classical criteria. It is an amazing result that it happens to bear a very similar mathematical structure to that presented in this paper which is the outcome of rigorous mathematical elaborations of the definitions of the classical criteria. Podgórski [28], Bardet [3] and Bigoni and Piccolroaz [4], worked on similar formulations, however their origin is unknown and it is unclear whether they constitute an exact or an approximated reformulation of classical criteria. It is important to highlight that, contrary to the developments of this paper, both Ottosen and Bardet propose two separate equations to describe the shape of the criterion in the deviatoric plane depending on the value of the Lode's angle. Podgórski adopted Ottosen's shape function to formulate a failure criterion, which is stated without demonstration to coincide with Matsuoka-Nakai and Lade-Duncan, and can include other criteria. Contrary to what reported in the paper, his formulation does not correctly reproduce all listed criteria (e.g. the Lade-Duncan criterion). Bigoni and Piccolroaz studied the convexity of a similar formulation for different values of the defining parameters.

2. Notation

Throughout this manuscript the Soil Mechanics sign convention is adopted, whereby compressive stresses are considered positive. Moreover when referred to soils all stress quantities have to be considered effective, although for simplicity the 'dash' is omitted.

Unordered and ordered principal stresses will be indicated as $\sigma_1, \sigma_2, \sigma_3$ and $\sigma_{\rm I} \geq \sigma_{\rm II} \geq \sigma_{\rm III}$, respectively. Similarly $\tau_{12}, \tau_{23}, \tau_{31}$ refer to the three maximum shear stresses associated to the (σ_1, σ_2), (σ_2, σ_3) and (σ_3, σ_1) principal stress states.

The term 'stress obliquity' will be used to indicate the ratio $\frac{\tau}{\sigma_{\rm n}} = \tan\phi_{\rm m}$ between the tangential and normal stress acting on a general plane, $\phi_{\rm m}$ indicating the 'mobilized angle of shearing resistance' on that

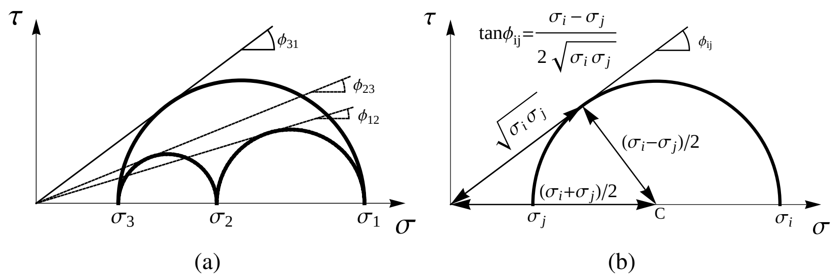


Figure 1. Mohr's circles and maximum stress obliquities (modified from Panteghini and Lagioia [27]).

plane. The three largest stress obliquities associated to the (σ_1, σ_2), (σ_2, σ_3) and (σ_3, σ_1) principal stress states will then be indicated as $\tan\phi_{12}$, $\tan\phi_{23}$ and $\tan\phi_{31}$ (figure 1).

The invariants of the characteristic equation of the effective stress tensor $\boldsymbol{\sigma}$ (e.g. Puzrin [30]) are

$$\begin{aligned} I_1 &= tr(\boldsymbol{\sigma}) = \sigma_{ii} \\ I_2 &= \frac{1}{2}(I_1^2 - \boldsymbol{\sigma} : \boldsymbol{\sigma}) = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \\ I_3 &= det\,\boldsymbol{\sigma} = \frac{1}{6}(\sigma_{ii}\sigma_{jj}\sigma_{kk} + 2\sigma_{ij}\sigma_{jk}\sigma_{ki} - 3\sigma_{ij}\sigma_{ij}\sigma_{kk}) \end{aligned} \tag{2.1}$$

and for principal stresses reduce to

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\,\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\,\sigma_2\,\sigma_3 \end{aligned} \tag{2.2}$$

The deviatoric stress tensor is defined as $\boldsymbol{s} = \boldsymbol{\sigma} - \frac{1}{3}tr(\boldsymbol{\sigma})\boldsymbol{I}$, and $\boldsymbol{I}$ indicates the second order identity tensor, and the associated invariants are

$$\begin{aligned} J_{2D} &= \frac{1}{2}\boldsymbol{s} : \boldsymbol{s} = \frac{1}{2}s_{ij}\,s_{ij} \\ J_{3D} &= det\boldsymbol{s} = \frac{1}{3}s_{ij}s_{jk}s_{ki} \end{aligned} \tag{2.3}$$

The mean stress and the square root of the second invariant of the deviatoric stress tensor, J_{2D}, are respectively indicated as

$$p = \frac{1}{3}tr(\boldsymbol{\sigma}) \tag{2.4}$$

and

$$J = \sqrt{J_{2D}} \tag{2.5}$$

In addition the invariants of the stress and deviatoric stress tensors are related by the following equations

$$\begin{aligned} I_2 &= \frac{1}{3}I_1^2 - J_{2D} \\ I_3 &= J_{3D} + \frac{1}{27}I_1^3 - \frac{1}{3}I_1 J_{2D} \end{aligned} \tag{2.6}$$

Yield and failure criteria in the principal stress space can also be referred to cylindrical coordinates

$$\begin{aligned} \zeta &= \sqrt{3}p \\ \rho &= \sqrt{2}J \\ \theta &= \frac{1}{3}\arcsin\left(-\frac{\sqrt{27}}{2}\frac{J_{3D}}{J_{2D}^{\frac{3}{2}}}\right) \end{aligned} \tag{2.7}$$

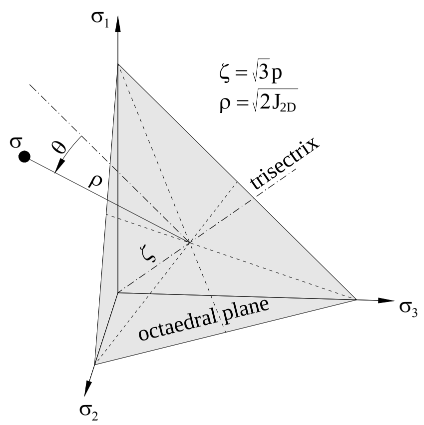


Figure 2. Cylindrical coordinates in the principal stress space.

where θ is the Lode's angle [20] and the geometrical interpretation of the cylindrical coordinates in shown in figure 2. A useful set of equations defines the ordered principal stresses as a function of the invariants

$$\begin{cases} \sigma_I = p + \frac{2}{\sqrt{3}} J \sin(\theta + \frac{2}{3}\pi) \\ \sigma_{II} = p + \frac{2}{\sqrt{3}} J \sin(\theta) \\ \sigma_{III} = p + \frac{2}{\sqrt{3}} J \sin(\theta - \frac{2}{3}\pi) \end{cases} \tag{2.8}$$

and triaxial compression ($\sigma_{\text{II}} = \sigma_{\text{III}}$) and extension ($\sigma_{\text{II}} = \sigma_{\text{I}}$) conditions are obtained for $\theta = -\pi/6$ and $\theta = \pi/6$, respectively.

3. Classical failure criteria

(a) ideal formulation

Earlier classical criteria, such as Tresca [34], Mohr-Coulomb [8] and von Mises [36], are usually defined in terms of stresses. For the purpose of implementation in programs for numerical analyses they can be easily reformulated in terms of invariants as shown in table 1. More recent criteria, such as Matsuoka-Nakai and Lade-Duncan, are on the other hand directly expressed as a combination of invariants (equations (3.7) and (3.12)). Although this tends to hide the underlying mechanical principle upon which they are based, an insight can be obtained by examining table 2. This shows that there is a close analogy between the criteria for soils and those for metals, when 'tangential stress' is substituted for 'stress obliquity'. It then appears that the Matsuoka-Nakai criterion, whose critical mechanical quantity is the root mean square of the three largest stress obliquities, is nothing other than the equivalent for soils of von Mises [1]. Similarly, the table also indicates that the Lade-Duncan criterion adopts as a critical mechanical quantity the product of the three principal stresses, each normalized by the mean pressure, p.

The use of isotropic yield and failure criteria in constitutive models is facilitated when the formulation in terms of invariants is adopted, however a considerably more efficient implementation can be achieved if its mathematical structure reflects the shape of the graphical representation of the criteria in the principal stress space. As this is an open surface, characterized by a linear meridional section and a closed curve in the deviatoric section the most favourable formulation is

$$f(\boldsymbol{\sigma}) = -F(p) + J\,\Gamma(\theta) = 0 \tag{3.1}$$

[1] It should be noted that whilst in the original paper of Matsuoka and Nakai [23] equation (3.7) is considered equivalent to the root mean square of equation (3.17) attaining a critical value, this is actually not the case and two different criteria are obtained, as pointed out by Panteghini and Lagioia [27]

where $F(p)$ defines the meridional section in *triaxial compression* associated to a Lode's angle $\theta = -\frac{\pi}{6}$, whilst $\Gamma(\theta) = \frac{1}{g(\theta)}$, $g(\theta)$ being the *shape function* which defines the deviatoric section curve.

For constitutive models based on perfect plasticity both the yield and the plastic potential functions are usually defined by the same yield/failure criterion, after which they are then named (e.g. the Tresca, the von Mises, the Mohr-Coulomb... *models*) . More complex models, such as those based on strain hardening/softening plasticity (e.g. Cam-Clay [31] for soils) adopt yield and plastic potential functions which plot as closed surfaces in the principal stress space. Also in this case, however, classical criteria can play a fundamental role if they can be reformulated according to equation (3.1). Their $\Gamma(\theta)$ can then be combined to a closed meridional curve to shape the surface around the mean pressure axis. Typical examples are the rotational ellipsoid of the Cam-Clay model [31] resulting from the combination of an elliptic meridional section and the von Mises deviatoric section or the surface proposed by Lagioia et al. [19] which combines a more versatile meridional curve with the Matsuoka-Nakai shape function.

Reformulating a criterion according to equation (3.1) not only provides greater versatility as new criteria and yield and plastic potential functions can be defined by combining new meridional curves with classical shape functions $g(\theta)$, but it also allows a very efficient implementation in programs for numerical analyses. Panteghini and Lagioia [26] have in fact shown that in this case an implicit algorithm can be formulated for the integration of the constitutive law which is based on a single scalar equation, and as such is extremely efficient and results in a very short machine runtime.

Finally, as the meridional section of classical criteria is linear, the problem of formulating criteria according to equation (3.1) comes down to the determination of the appropriate function $\Gamma(\theta) = g(\theta)^{-1}$.

(b) limitations of some original formulations

The use of classical criteria in their original formulation for defining yield and plastic potential surfaces is often problematic. The most notorious problem is probably that associated to the lack of smoothness of the Tresca and Mohr-Coulomb surfaces, which plot in the principal stress space as an hexagonal prism and an irregular hexagonal pyramid, respectively, their central axis lying on the space diagonal. In both cases the hexagonal deviatoric section is of class C^0 continuity resulting in severe numerical difficulties, which can be mitigated by adopting solutions such as those suggested by Sloan and Booker [33] and Abbo et al. [2]. However such remedies are expensive in terms of efficiency of the implementation. Abbo and Sloan [1] also removed the singularity at the apex of the Mohr-Coulomb pyramid by adopting an hyperbolic meridional section.

More subtle difficulties emerge when the Matsuoka-Nakai and the Lade-Duncan criteria are adopted to reproduce the behaviour of geomaterials. Admissible stress states for particulate media unable to resist tension should be confined within the first, compressive octant of the principal stress space, hence their failure criteria should not cross the boundaries of that octant. However, as shown in figure 3 both the Matsuoka-Nakai and the Lade-Duncan criteria plot also in tensile octants and each comprises two additional branches which define false elastic and failure states. A yield surface, more generally defined as

$$f(\boldsymbol{\sigma}) = 0 \tag{3.20}$$

when described by the Matsuoka-Nakai and Lade-Duncan criteria would therefore be non-convex, thus violating one of the basic requirements of the plasticity theory. As shown by Panteghini and Lagioia [27] for Matsuoka-Nakai, both criteria are non-convex not only when its graphical representation, $f = 0$, is considered but also when surfaces defined as

$$f(\boldsymbol{\sigma}) = k \tag{3.21}$$

are considered, with $k \neq 0$. This creates additional problems during the integration of constitutive laws, particularly when implicit integration schemes are adopted. The Mohr-Coulomb criterion too can present problems of lack of convexity when formulated referring to the largest stress obliquity $\tan\phi_{\mathrm{I,III}}$ as indicated in equation (3.15). The definition of stress obliquity, shown in figure 1, allows also the existence of a symmetric pyramid in the negative octant of the stress space, which is cancelled out if the formulation based on the sine function of equations (3.3) and (3.5) is adopted.

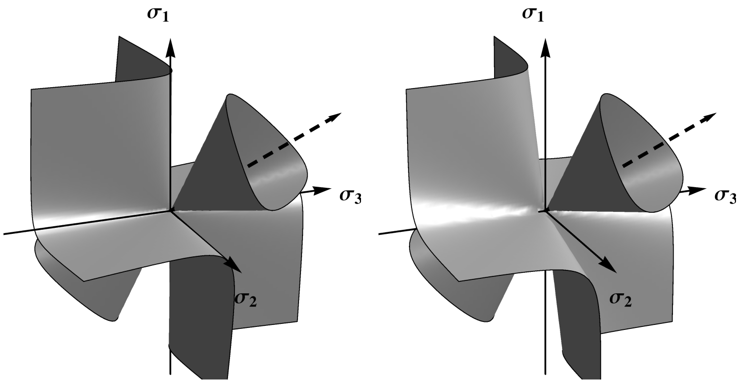
Table 1. Defining equations for classical failure criteria. Note that constants M, K_{MN} and K_{LD} are set so that Mohr-Coulomb, Drucker-Prager, Matsuoka-Nakai and Lade-Duncan predict the same strength for triaxial compression conditions $\sigma_{\text{II}} = \sigma_{\text{III}}$ or $\theta = -\pi/6$.

METALS		SOILS	
Tresca		Mohr-Coulomb	
$\sigma_{\text{I}} - \sigma_{\text{III}} = 2S_{\text{u}}$	(3.2)	$\sigma_{\text{I}} - \sigma_{\text{III}} = (\sigma_{\text{I}} + \sigma_{\text{III}})\sin\phi$	(3.3)
Tresca (in terms of invariants)		Mohr-Coulomb (in terms of invariants)	
$J\cos(\theta) - S_{\text{u}} = 0$	(3.4)	$J - p\,\dfrac{\sin\phi}{\cos\theta + \frac{\sin\theta\sin\phi}{\sqrt{3}}} = 0$	(3.5)
Von Mises		Matsuoka-Nakai	
$J = K_{\text{vM}}$	(3.6)	$\dfrac{I_1 I_2}{I_3} = K_{\text{MN}}$	(3.7)
		$K_{\text{MN}} = \dfrac{9 - \sin^2\phi}{1 - \sin^2\phi}$	(3.8)
		Drucker-Prager	
		$J - Mp = 0$	(3.9)
		$M = \dfrac{1}{\sqrt{3}} M_{\text{c}} = \dfrac{1}{\sqrt{3}}\dfrac{6\sin\phi}{3 - \sin\phi}$	(3.10)
I_3 tension criterion		Lade-Duncan	
$I_3 = K$	(3.11)	$\dfrac{I_1^3}{I_3} = K_{\text{LD}}$	(3.12)
		$K_{\text{LD}} = \dfrac{(3 - \sin\phi)^3}{(1 + \sin\phi)(1 - \sin\phi)^2}$	(3.13)

Table 2. Mechanical quantities adopted as critical indicators by different classical criteria

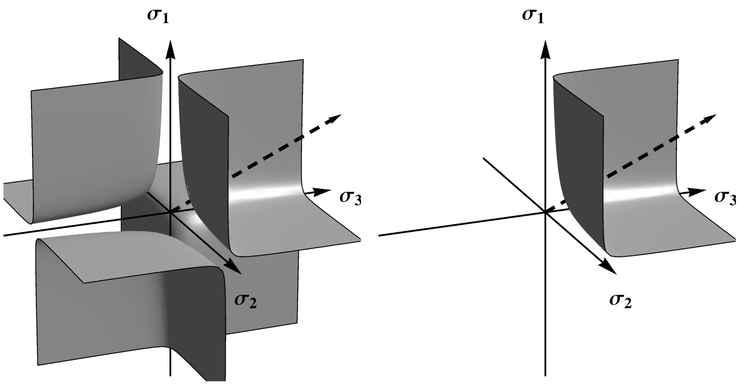
METALS		SOILS	
Tresca		Mohr-Coulomb	
$max\,\{\lvert\tau_{12}\rvert, \lvert\tau_{23}\rvert, \lvert\tau_{31}\rvert\}$	(3.14)	$max\,\{\tan\phi_{12}, \tan\phi_{23}, \tan\phi_{31}\}$	(3.15)
Von Mises		Matsuoka-Nakai	
$\sqrt{\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2}$	(3.16)	$\sqrt{\tan^2\phi_{12} + \tan^2\phi_{23} + \tan^2\phi_{31}}$	(3.17)
I_3 tension criterion		Lade-Duncan	
$\sigma_1\sigma_2\sigma_3$	(3.18)	$\dfrac{\sigma_1}{I_1}\dfrac{\sigma_2}{I_1}\dfrac{\sigma_3}{I_1}$	(3.19)

Despite the described limitations, the original formulation of these criteria has often been adopted in the last twenty years for defining yield and plastic potential functions (e.g. Lagioia and Nova [14], Lagioia et al. [19], di Prisco et al. [11], Bodas Freitas et al. [5]) and such models have been implemented and successfully



(a) Matsuoka-Nakai (b) Lade-Duncan

Figure 3. Plot of Matsuoka-Nakai and Lade-Duncan criteria as defined by the original formulations.



(a) I_3 tension cut-off (b) I_3 tension cut-off convexified

Figure 4. Plot of the I_3 tension cut-off (Lagioia et al. [16]).

employed for performing numerical analyses of boundary value problems (e.g. Fontana et al. [12], Lagioia and Potts [17], Lagioia and Potts [18], Shin [32], Bodas Freitas et al. [6]). It should be noted however that although skilled implementation can mitigate the effects of the described limitations, numerical problems cannot be completely eliminated and in some situations they cause early crushes of analyses, particularly if there are Gauss points with a very low stress state.

Problems of non-convexity were also encountered by Lagioia et al. [16] when they generalized the Galileo-Rankine criterion to a full three-dimensional smooth no-tension criterion. The resulting formulation plots as a non convex surface, shown in figure 4(a), which was however successfully convexified by means of an approach presented in the paper, providing the surface of figure 4(b).

4. Formulation of a single criterion

(a) the Matsuoka-Nakai and Lade-Duncan criteria

The limitations of the Matsuka-Nakai criterion have been discussed in detail by Panteghini and Lagioia [27] who have then convexified it in [26] using a technique similar to that presented by Lagioia et al. [16]. The resulting expression plots in the compressive octant only of the principal stress space and exactly reproduces that part of the multi-branch surfaces of the original criterion. That approach can also be used for the Lade-Duncan criterion, and as shown in what follows results in the same mathematical expression for both criteria. Equations (3.7) and (3.12) can be rearranged in a more convenient form for a numerical application as

$$f(\boldsymbol{\sigma}) = I_1\, I_2 - I_3 K_{MN} = 0 \tag{4.1}$$

and

$$f(\boldsymbol{\sigma}) = I_1^3 - I_3 K_{LD} = 0 \tag{4.2}$$

Replacing the invariants I_2 and I_3 with J_{2D} and J_{3D} using equation (2.6) results in

$$p^3 - A_1 J_{2D}\, p + A_2 J_{3D} = 0 \tag{4.3}$$

or equivalently

$$p^3 - A_1 J_{2D}\, p - \frac{2}{\sqrt{27}} A_2 J_{2D}^{3/2} \sin 3\theta = 0 \tag{4.4}$$

where the coefficients A_1 and A_2 are for Matsuoka-Nakai

$$A_1 = \frac{k_{MN} - 3}{k_{MN} - 9} \qquad A_2 = \frac{k_{MN}}{k_{MN} - 9} \tag{4.5}$$

and for Lade-Duncan

$$A_1 = \frac{k_{LD}}{k_{LD} - 27} \qquad A_2 = A_1 \tag{4.6}$$

Equation (4.4) is of third degree both in p and $\sqrt{J_{2D}}$. Although it is tempting to solve it for $J = \sqrt{J_{2D}}$, and indeed many authors have followed this path in the past, the convexification of the surfaces can be only achieved by solving for the mean pressure p. Equation (4.4) is then a depressed cubic equation of type

$$p^3 + u\, p + w = 0 \tag{4.7}$$

in p, with coefficients

$$\begin{aligned} u &= -A_1 J_{2D} \\ w &= -A_2 \frac{2}{\sqrt{27}} J_{2D}^{3/2} \sin 3\theta \end{aligned} \tag{4.8}$$

This equation has three ordered real roots

$$p_j = 2\sqrt{-\frac{u}{3}} \cos\left[\frac{1}{3} \arccos\left(\frac{3w}{2u}\sqrt{-\frac{3}{u}}\right) - \frac{2}{3}\pi j\right] \tag{4.9}$$

with $j = 0, 1, 2$ and $p_0 \geq p_1 \geq p_2$ if

$$\begin{aligned} &u < 0 \\ &u^3 + 27w^2 \leq 0 \end{aligned} \tag{4.10}$$

As shown in Appendix I, these conditions are always satisfied for both Matsuoka-Nakai and Lade-Duncan. Each of the solutions given by equation (4.9) for $j = 0, 1, 2$ reproduces one of the three surface branches, shown in figure 3, of the original formulations. Since the only significant branch is that lying in the positive octant of the stress space, which is always characterized by the largest and positive mean pressure, the equation of the convexified version of both criteria is associated to $j = 0$

$$p_0 = 2\sqrt{-\frac{u}{3}} \cos\left[\frac{1}{3} \arccos\left(\frac{3w}{2u}\sqrt{-\frac{3}{u}}\right)\right] \tag{4.11}$$

which after substituting the expression of the coefficients given by equations (4.8), multiplying by the slope of the meridional section for triaxial compression, M, (equation (3.10)) and some algebraic manipulation becomes

$$f(\boldsymbol{\sigma}) = -Mp + J\frac{2}{\sqrt{3}}\sqrt{A_1} M \cos\left[\frac{1}{3} \arccos\left(\frac{A_2}{A_1^{3/2}} \sin 3\theta\right)\right] = 0 \tag{4.12}$$

This equation bears the typical general structure of equation (3.1).

(b) The Tresca and Mohr-Coulomb criteria

The Tresca and Mohr-Coulomb failure criteria can be easily formulated according to the structure given by equation (3.1), deriving $\Gamma(\theta)$ directly from trigonometrical considerations. Whilst for Tresca the radial path perpendicular to the hexagon occurs for a Lode's angle $\theta = 0$ (figure 5), in the case of Mohr-Coulomb this path is shifted by a positive value of Lode's angle $\bar{\theta}$, which depends on the angle of shearing resistance ϕ. This angle can be determined imposing that the cylindrical coordinate ρ_n associated to the orthogonal path is equally given by

$$\rho_n = \rho_{-\frac{\pi}{6}} \cos(\frac{\pi}{6} + \bar{\theta}) = \rho_{\frac{\pi}{6}} \cos(\frac{\pi}{6} - \bar{\theta}) \tag{4.13}$$

where $\rho_{-\frac{\pi}{6}}$ and $\rho_{\frac{\pi}{6}}$ represent the cylindrical coordinates of the vertices of the hexagon associated to triaxial compression and triaxial extension, respectively. This results in a relationship between $\bar{\theta}$ and ϕ

$$\frac{\cos(\frac{\pi}{6} + \bar{\theta})}{\cos(\frac{\pi}{6} - \bar{\theta})} = \frac{\rho_{-\frac{\pi}{6}}}{\rho_{\frac{\pi}{6}}} = \frac{3 + \sin\phi}{3 - \sin\phi} \tag{4.14}$$

which after trigonometric expansion leads to

$$\bar{\theta} = \arctan\frac{\sin\phi}{\sqrt{3}} \tag{4.15}$$

The cylindrical coordinate $\rho(\theta) = \rho_\theta$ which describes the irregular hexagon of the Morh-Coulomb criterion as a function of θ is then given by

$$\rho_\theta = \rho_n \frac{1}{\cos(\theta - \bar{\theta})} = \rho_{-\frac{\pi}{6}} \frac{\cos(\frac{\pi}{6} + \bar{\theta})}{\cos(\theta - \bar{\theta})} \tag{4.16}$$

which can be rewritten in terms of J, using the second of equations (2.7). Moreover for frictional materials the value of $J_{-\frac{\pi}{6}} = \rho_{-\frac{\pi}{6}}/\sqrt{2}$ at triaxial compression grows linearly with the mean pressure as $J_{-\frac{\pi}{6}} = K + Mp$, where M is given by equation (3.10) and the constant parameter $K = c' \cot\phi$ accounts for an optional cohesive term c' in the Mohr-Coulomb criteria. Equation (4.16) can then be rewritten, leading to a formulation of the Mohr-Coulomb criterion which reflects equation (3.1)

$$-(K + Mp) + J\,\frac{\cos(\theta - \bar{\theta})}{\cos(\frac{\pi}{6} + \bar{\theta})} = 0 \tag{4.17}$$

where the inverse of the shape function is

$$\Gamma(\theta) = \frac{1}{g(\theta)} = \frac{\cos(\theta - \bar{\theta})}{\cos(\frac{\pi}{6} + \bar{\theta})} \tag{4.18}$$

This equation for a non frictional material characterized by $\phi = 0$ reduces to the Tresca criterion, as both $\bar{\theta}$ and M are nil. It should be noted that substituting the value of $\bar{\theta}$ given by equation (4.15) into equation (4.17) and expanding both cosines, one obtains

$$-(K + Mp) + \left(2\frac{\sqrt{3}\cos\theta + \sin\theta\sin\phi}{3 - \sin\phi}\right) J = 0 \tag{4.19}$$

which is indeed equivalent to the equation (3.5) usually found in the literature (e.g. Potts and Zdravkovic [29], de Souza Neto et al. [10], ...) when both terms of that equation are multiplied for M and the cohesive term is set to zero.

The angle $\bar{\theta}$ can be rewritten as

$$\bar{\theta} = \bar{\gamma}\frac{\pi}{6} \tag{4.20}$$

where

$$\bar{\gamma} = \frac{6}{\pi}\arctan\frac{\sin\phi}{\sqrt{3}} \quad \in [0, 1] \tag{4.21}$$

so that equation (4.18) can then be rewritten as

$$\Gamma(\theta) = \frac{\cos\left[\theta - \bar{\gamma}\frac{\pi}{6}\right]}{\cos\left[(\bar{\gamma} + 1)\frac{\pi}{6}\right]} \tag{4.22}$$

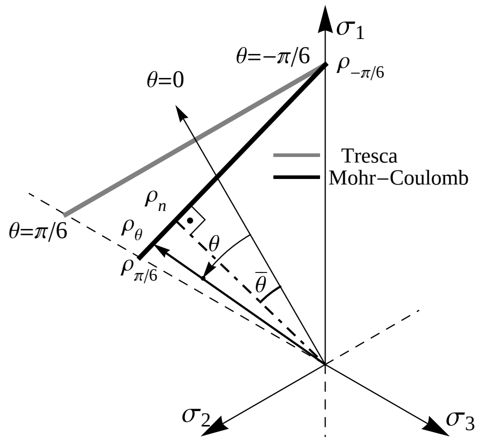


Figure 5. Deviatoric plane section of the Morh-Coulomb and Tresca criteria (due to symmetry only the $\sigma_1 \geq \sigma_2 \geq \sigma_3$ portion is drawn.

As the cosine is a even function,

$$\cos\left[\theta - \bar{\gamma}\frac{\pi}{6}\right] = \cos\left[\bar{\gamma}\frac{\pi}{6} - \theta\right] \tag{4.23}$$

and in the domain $[-\frac{\pi}{6}, \frac{\pi}{6}]$ θ can be rewritten as

$$\theta = \frac{1}{3}\arcsin(\sin 3\theta) = \frac{1}{3}\left[\frac{\pi}{2} - \arccos(\sin 3\theta)\right] \tag{4.24}$$

equation (4.22) becomes

$$\Gamma(\theta) = \frac{1}{\cos\left[(\bar{\gamma}+1)\frac{\pi}{6}\right]}\cos\left[\frac{1}{3}\arccos(\sin 3\theta) - (1-\bar{\gamma})\frac{\pi}{6}\right] \tag{4.25}$$

The Tresca and Mohr-Coulomb criteria expressed by equation (4.17) can then be rewritten as

$$f(\boldsymbol{\sigma}) = -(K + Mp) + J\,\frac{1}{\cos\left[(\bar{\gamma}+1)\frac{\pi}{6}\right]}\cos\left[\frac{1}{3}\arccos(\sin 3\theta) - (1-\bar{\gamma})\frac{\pi}{6}\right] = 0 \tag{4.26}$$

which is indeed similar to equation (4.12) obtained for Matsuoka-Nakai and Lade-Duncan.

5. The general equation of the criteria

A general defining equation for describing all the classical criteria considered in this paper can be now formulated. Equations (4.12) and (4.26) which have been retrieved for the Matsuoka-Nakai, Lade-Duncan, Tresca and Mohr-Coulomb criteria, can be collected in a general expression

$$f(\boldsymbol{\sigma}) = -(K + Mp) + J\,\alpha\cos\left[\frac{\arccos(\beta\sin 3\theta)}{3} - \gamma\frac{\pi}{6}\right] = 0 \tag{5.1}$$

which reflects the general structure given by equation (3.1) as the meridional section is given by $F(p) = K + Mp$ whilst the $\Gamma(\theta)$ function is

$$\Gamma(\theta) = \alpha\cos\left[\frac{\arccos(\beta\sin 3\theta)}{3} - \gamma\frac{\pi}{6}\right] \tag{5.2}$$

Parameters M and K define the linear meridional section. For particulate materials, M sets the slope in triaxial compression conditions and is a function of the angle of shearing resistance ϕ (equation (3.10)), whilst K is a *cohesive* term which depends on the two classical Mohr-Coulomb quantities c' and ϕ via $K = c'\cot\phi$. Non-frictional materials such as metals are characterized by $\phi = 0$. Consistently for both the

Tresca and von Mises criteria equation (3.10) results in $M=0$, whilst parameter K is a function of the yield stress in mono-axial tension tests. Parameters α, β and γ which define the deviatoric shape function are given in table 3 for all classical criteria. The parameters which provide the circular deviatoric section of the von Mises and Drucker-Prager criteria can be immediately determined by inspection of equation (5.2) and are also listed in the table.

Equation (5.1) provides an exact definition of all classical criteria. It is, however free from all the limitations discussed in earlier sections. It is convex for all criteria. The two additional branches which bear no mechanical significance in the original formulations of both the Matsuoka-Nakai and Lade-Duncan criteria have been effectively eliminated. This also applies to the Mohr-Coulomb criterion, which is also non-convex if formulated in terms of stress obliquity. A very significant result for numerical applications is that the defining equation (5.1) is convex both when describing the criterion surface for $f(\boldsymbol{\sigma})=0$ and also for $f(\boldsymbol{\sigma})\neq 0$. In addition as the distance of a generic stress state $\boldsymbol{\sigma}^*$ from the criterion surface $f(\boldsymbol{\sigma})=0$ increases, the value of f evaluated at that stress state, $f(\boldsymbol{\sigma}^*)$, linearly increases, which is particularly convenient for the efficiency of the implementation. A distinct formulation for $\Gamma(\theta)$ also allows, if required, to assign a particular deviatoric plane shape independently of the inclination of the meridional section.

Finally the general expression of the $\Gamma(\theta)$ function can also provide a rounded hexagon in the deviatoric plane for both Tresca and Mohr-Coulomb. In fact it can be noted that whilst the original hexagons are obtained for parameter $\beta=1$, both Matsuoka-Nakai and Lade-Duncan which are of continuity class of at least C^2 are characterized by a value of parameter $\beta<1$. The effect of parameter β in the function $\arccos[\beta\sin(x)]$ is shown in figure 6. For values of $\beta<1$ the argument of the cosine function in equation (5.2) becomes a continuous function of the Lode's angle θ, which in turn results in $\Gamma(\theta)$ being stationary for both $\theta=-\dfrac{\pi}{6}$ and $\theta=\dfrac{\pi}{6}$, as shown in figure 7.

Setting parameter β to a value less than unity can therefore be used to smoothen both Tresca and Mohr-Coulomb criteria as shown in figure 8. If the same parameters α and γ of the original discontinuous criterion are retained (table 3), a rounded hexagon is obtained which is 'inscribed' in the original one. However if circumscribed hexagons are required, table 3 also provides the necessary parameters α and γ. By selecting

Table 3. Shape function parameters for different failure criteria (Coefficients A_1 and A_2 for the Matsuoka-Nakai and Lade-Duncan criteria are given by equations (4.5) and (4.6), respectively) .

Model	α	β	γ
von Mises and Drucker-Prager	1	0	1
Tresca	$\sec\left[\frac{\pi}{6}\right]$	1	1
Rounded Tresca	$\sec\left[\frac{\pi}{6}\right]$	<1	1
Mohr-Coulomb	$\sec\left[(\bar{\gamma}+1)\frac{\pi}{6}\right]$	1	$(1-\bar{\gamma})$
Inner Mohr-Coulomb	$\sec\left[(\bar{\gamma}+1)\frac{\pi}{6}\right]$	<1	$(1-\bar{\gamma})$
Outer Mohr-Coulomb	$\csc\left[(1+\gamma)\frac{\pi}{6}+\frac{\arccos\beta}{3}\right]$	<1	$\frac{2}{\pi}\left[-3\arctan\left(\frac{\sin\phi\cot\frac{\arcsin\beta}{3}}{3+\sin\phi}-\frac{3\tan\frac{\arcsin\beta}{3}}{3+\sin\phi}\right)+\arccos\beta\right]$
Matsuoka-Nakai	$\frac{2}{\sqrt{3}}\sqrt{A_1}M$	$\frac{A_2}{A_1^{3/2}}$	0
Lade-Duncan	$\frac{2}{\sqrt{3}}\sqrt{A_1}M$	$\frac{A_2}{A_1^{3/2}}$	0

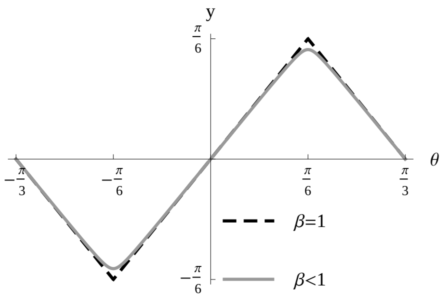


Figure 6. Effect of parameter β in a function $\frac{\pi}{6} - \arccos[\beta \sin(x)]$

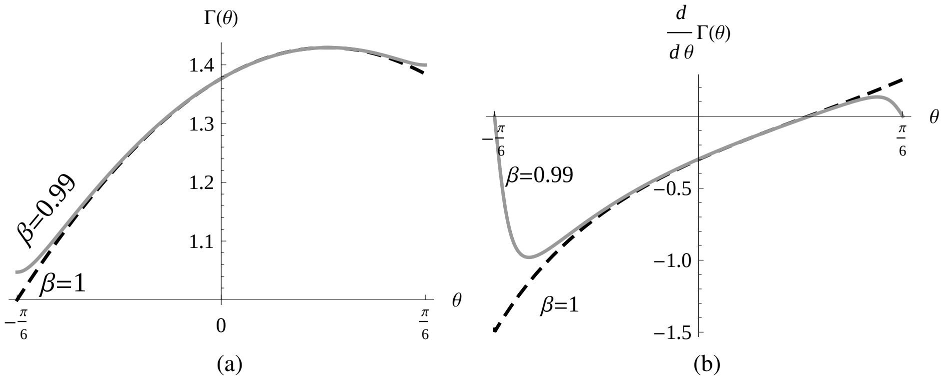


Figure 7. Effect of parameter β on the shape function $\Gamma(\theta)$ (a) and its derivative (b) of the inner Mohr-Coulomb criterion plotted for $\phi = 30°$.

a β lower than but very close to unity the smooth versions of the criteria can match very closely the original ones. It should be noted, however, that particularly for the Mohr-Coulomb criterion, due to the lack of symmetry with respect to $\theta = 0$, failure in plane strain will be attained for a value of Lode's angle θ different from that obtained with the discontinuous version (see Lagioia and Panteghini [15]). An indication of the percentage error associated to the adoption of the circumscribed and 'inscribed' hexagons with different β for the Mohr-Coulomb criterion are shown in figure 9 for three values of the angle of shearing resistance $\phi = 15°$, $\phi = 30°$ and $\phi = 45°$. The error e is defined as

$$e = \frac{g_{\text{smooth}}(\theta) - g_{\text{exact}}(\theta)}{g_{\text{exact}}(\theta)} 100 \tag{5.3}$$

where $g_{\text{smooth}}(\theta)$ and $g_{\text{exact}}(\theta)$ are the shape functions of the rounded and exact hexagons, respectively. Consistently the error shown in figure 9 is positive for the circumscribed and negative for the 'inscribed' criterion. Due to the rounding of the edges, the largest error occurs for a Lode's angle close to triaxial compression ($\theta = -\frac{\pi}{6}$) no matter the angle of shearing resistance. Moreover for any value of the rounding parameter β the circumscribed hexagons is also affected by a smaller error as compared to the 'inscribed' one. It should also be noticed that the inner hexagon is not strictly speaking an 'inscribed' hexagon as for values of θ which depend on the value of the angle of shearing resistance ϕ it also crosses the true Morh-Coulomb hexagon, as shown in figure 9(c) for $\beta = 0.95$ (continuous black line).

6. A general class of criteria

Following the work of Panteghini and Lagioia [27], it has been argued that not only the Matsuoka-Nakai but also the Lade-Duncan criterion is non-convex and comprises three branches. Indeed both criteria are

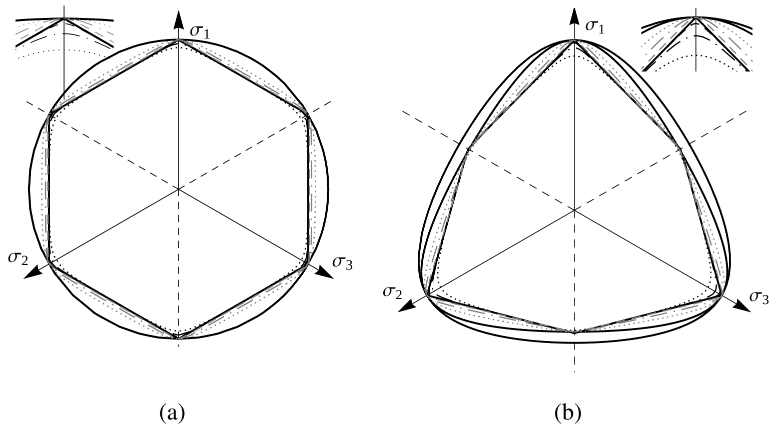


Figure 8. Original and rounded Tresca (a) and Mohr-Coulomb ($\phi = 30°$) (b) criteria, given by equation (5.2) with rounding parameter $\beta = 0.95, 0.99, 0.999$. Outer smooth criteria for both Tresca and Mohr-Coulomb are associated to the parameters α and γ in table 3 labeled for the latter criterion. von Mises (a) and Matsuoka-Nakai and Lade-Duncan (b) deviatoric sections are also shown.

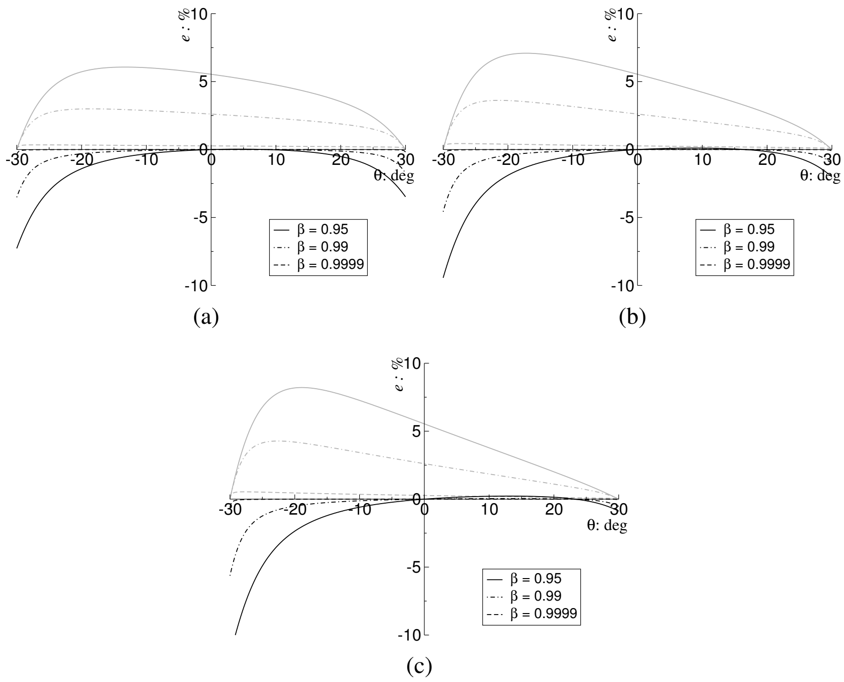


Figure 9. Percentage strength error of circumscribed (gray lines) and 'inscribed' (black lines) rounded hexagons as compared to true Mohr-Coulomb hexagon for different values of parameter β and angle of shearing resistance $\phi = 15°$ (a), $\phi = 30°$ (b) , $\phi = 45°$ (c) .

originally defined by a non-convex cubic function of the principal stresses, which was rewritten in terms of p, J and θ to provide a depressed cubic equation in the first two invariants. The three ordered solutions (equation (4.9)) of the latter cubic equation provide the expressions which define each of the three branches of both criteria.

Capurso [7] pointed out that a definition of the Tresca criterion alternative to equation (3.2) can be obtained by multiplying by each other the six relationships

$$\begin{aligned} \pm(\sigma_1 - \sigma_2) - 2S_u = 0 \\ \pm(\sigma_2 - \sigma_3) - 2S_u = 0 \\ \pm(\sigma_3 - \sigma_1) - 2S_u = 0 \end{aligned} \tag{6.1}$$

resulting in an equation of the sixth degree of the unordered principal stresses

$$\begin{aligned} &[(\sigma_1 - \sigma_2) - 2S_u]\,[-(\sigma_1 - \sigma_2) - 2S_u]\,[(\sigma_2 - \sigma_3) - 2S_u]\,[-(\sigma_2 - \sigma_3) - 2S_u] \\ &[(\sigma_3 - \sigma_1) - 2S_u]\,[-(\sigma_3 - \sigma_1) - 2S_u] = 0 \end{aligned} \tag{6.2}$$

However it should be noted that this equation is not completely equivalent to the Tresca criterion. In fact stress states which violate the true Tresca criterion can satisfy such equation, provided that at least one of the above τ is equal to the limit value S_u, whilst any of others simultaneously exceeds it. A better suited formulation than that proposed by Capurso can be obtained if the product of the square roots of each of the six equations (6.1) is considered, resulting in

$$\sqrt{4S_u^2 - (\sigma_1 - \sigma_2)^2}\,\sqrt{4S_u^2 - (\sigma_2 - \sigma_3)^2}\,\sqrt{4S_u^2 - (\sigma_3 - \sigma_1)^2} = 0 \tag{6.3}$$

This effectively defines the hexagonal prism of the Tresca criterion, however it is not suitable for numerical applications as the square roots constrain the domain of definition within the prism only.

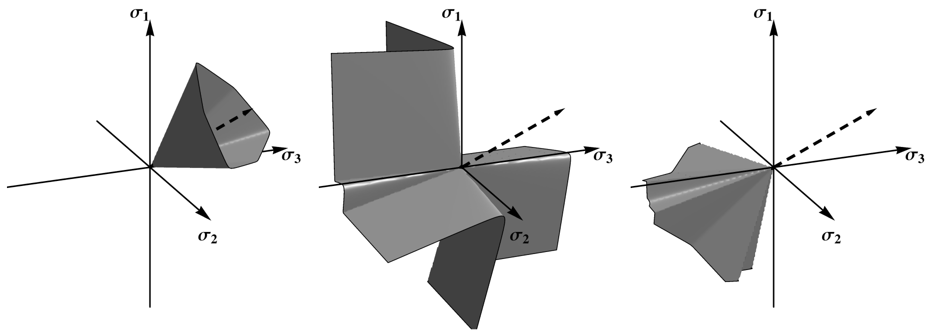
All this can also be directly extended to the Mohr-Coulomb criterion, provided that the extreme tangential stresses τ_{ij} associated to the unordered principal stresses (σ_i, σ_j) are replaced with the extreme obliquities $\tan\phi_{ij}$.

Despite not being a completely equivalent definition of the Tresca criterion, Capurso's arguments [7] suggest that, similarly to Matsuoka-Nakai and Lade-Duncan, also the Tresca and the Mohr-Coulomb criteria can be defined by a cubic equation of the principal stresses. This is coherent with the findings of this work. In fact both equations (4.25) and (5.2) which provide the $\Gamma(\theta)$ functions for Tresca and Mohr-Coulomb and for all classical criteria, respectively, can be rewritten to remove the term outside the arc-cosine function, as shown in appendix B. Equations (B.4) or (B.5) and (B.7) provide then a reformulation of the $\Gamma(\theta)$ for Tresca and Mohr-Coulomb and for all classical criteria, respectively, which reflect the structure of the largest of the three solutions of a cubic equation given by equation (4.9) for $j = 0$. It is interesting to observe that the remaining two solutions for $j = 1$ and $j = 2$ plot for the Mohr-Coulomb criterion as shown in figure 10 and this are associated to the obliquities $\tan\phi_{II,III}$ and $\tan\phi_{I,II}$ of the negative ordered principal stresses $(-\sigma_I, -\sigma_{II})$ and $(-\sigma_{II}, -\sigma_{III})$ attaining a limit value. It should be noted that the I_3 tension cut-off recently proposed by Lagioia et al. [16] as a three-dimensional and smooth generalization of the Galileo-Rankine criterion, is also described by a similar cubic equation, and as such belongs to the same class of non-convex criteria. As the coefficients of the equation do not satisfy the conditions given by equation (4.10) for all stress states, in that case convexification results in a piecewise solution.

7. Evaluation of the performance of the proposed general criterion

The general equation (5.1) was used to define the yield and the plastic potential surfaces of an elastic-perfect plastic constitutive model which was then implemented in Abaqus, a commercial finite element program.

The constitutive law was numerically integrated by means of an implicit scheme, following a procedure recently proposed by Panteghini and Lagioia [26]. This is an extremely efficient algorithm valid when the yield and plastic potential functions have the structure given by equation (3.1). Iterations at the constitutive level are then performed on a single nonlinear scalar equation, rather than on a system of



(a) $j=0$ (b) $j=1$ (c) $j=2$

Figure 10. 3D plot of the non-convex class, with parameters α, β and γ set for Mohr-Coulomb and the $\Gamma(\theta)$ function expressed by equation (B.7). Index j as in equation (4.9).

four or seven (depending on whether principal or general stresses are considered) as in standard implicit schemes involving isotropic materials (e.g. de Souza Neto et al. [10]) [2].

Numerical analyses were then performed to analyse a shallow circular footing (Figure 11) resting on a purely frictional material, with no lateral surcharge. It should be noted that a *true* N_γ problem was considered, in which both the lateral surcharge and the effective cohesion component were actually set to nil, i.e. $q_0 = 0$ and $c' = 0$. It is a common knowledge that this constitutes an extremely demanding boundary value problem (e.g. De Borst and Vermeer [9]) which causes program crushes at a very early stage of the analysis.

A $2m$ diameter circular footing, with a rough interface at the contact with the soil, resting on a 50m wide by 50m deep domain was considered. The actual footing was not considered in the analysis and the loading process was simulated by applying vertical displacements at the footing-soil contact nodes, whilst horizontal displacements were prevented to simulate the rough interface. The other boundary conditions are indicated in Figure 11. All the analyses were carried out in displacement control and assuming associated plasticity, while drained conditions were considered. A k_0 initial stress state was assumed, whereby the effective vertical stress increases linearly with depth as $\sigma_z = \gamma z$, whilst the horizontal stress is given by $\sigma_h = k_0 \sigma_v$ and $k_0 = 1 - \sin\phi$.

The whole soil domain was discretized in $3,272$ eight-nodes isoparametric and axis-symmetric quadratic elements with reduced Gaussian integration named CAX8R in Abaqus, which resulted in $10,009$ nodes. Material parameters for the soil were: angle of shearing resistance $\phi = 25°$, Young's modulus $E = 100,000$ kPa, Poisson's ratio $\nu = 0.3$ and unit weight $\gamma_d = 18$ kN$/$m^3.

Altogether ten different analyses were carried out. For one of them, the material behaviour was reproduced by means of the Mohr-Coulomb model, with the irregular hexagon characterized by the gradient discontinuities. This model was separately implemented in the finite element program, adopting the procedure described by de Souza Neto et al. [10] to deal with the cusps at the implementation level. Three more analyses were conducted setting the parameters α, β and γ in equation (5.2) to provide the Druker-Prager, Lade-Duncan and Matsuoka-Nakai surfaces. Finally six analyses were conducted with rounded Mohr-Coulomb pyramids, both circumscribed and 'inscribed' to the true failure criterion.

Results of all analyses are shown in Figure 12 together with the exact solution of the characteristic lines method obtained using the ABC program written by Martin ([21], [22]). For the sake of generality, the load - deflection curves are plotted in terms of bearing capacity factor for self-weight, N_γ against normalized settlements δ/B, where B is the breath of the footing.

[2] It should be noted that as in the case of the Drucker-Prager and von Mises criteria the deviatoric section is circular, the Lode's angle at the end of the step always coincides with that of the predictor. At the implementation level this needs to be treated as the exceptions provided in *Box 3* of the original paper of Panteghini and Lagioia [26].

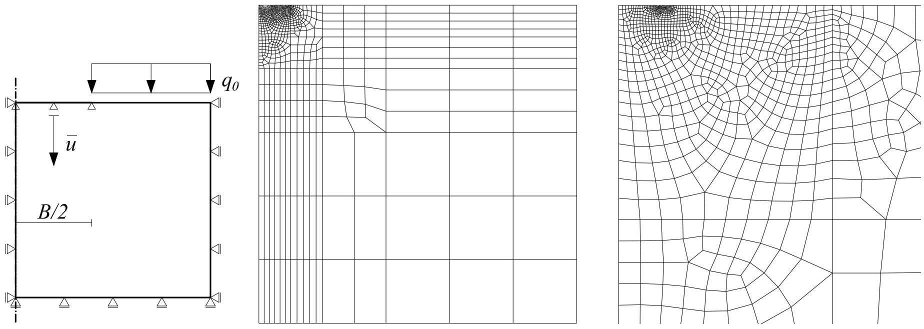


Figure 11. Model of the footing problem and mesh discretization.

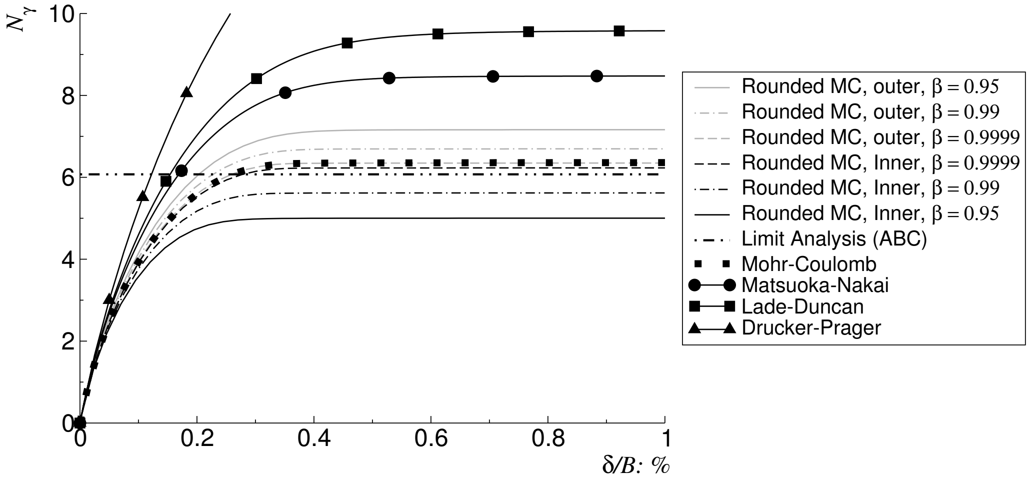


Figure 12. Bearing capacity factor N_γ versus normalized settlement for a circular shallow footing resting on elastic-perfect plastic Drucker-Prager (only part of the curve is shown), Matsuoka-Nakai, Lade-Duncan, Mohr-Coulomb, rounded circumscribed (gray curves) and 'inscribed' (black curves) Mohr-Coulomb soils. The Limit Analysis N_γ evaluated with the ABC program ([21] and [22]) is also shown for comparison.

Whilst the exact N_γ value was 6.07, numerical analysis of the true Mohr-Coulomb material (with gradient discontinuities) gave $N_\gamma = 6.36$, thus exceeding by 4.7% the correct value. As expected, the load-deflection curve of the Drucker-Prager material provides a largely overestimated bearing capacity ($N_\gamma = 16.65$), and it is only partly shown in Figure 12 for the sake of clearness. Bearing capacity factors for the Lade-Duncan and Matsuoka-Nakai soils were $N_\gamma = 9.59$ and $N_\gamma = 8.48$, respectively. In both cases a larger capacity than the Mohr-Coulomb soil was obtained, which accounts for the effect of the intermediate principal stress on shear strength.

Rounded Mohr-Coulomb soils overestimate or underestimate the true Mohr-Coulomb bearing capacity depending on whether a circumscribed or an inscribed hexagon, respectively, is adopted. However by adopting a rounding parameter β very close to , but less than, unity, the difference can be made insignificant for all practical purposes. As an example, out of the six analyses performed with a rounded hexagon, the best approximation of the true Mohr-Coulomb curve was $N_\gamma = 6.38$ associated to $\beta = 0.9999$. As the analyses lasted always less than ten minutes, despite the very fine mesh, better approximations can be easily obtained without appreciably increasing the run-time. It should be also noticed that for any given value of β the 'outer' hexagon converges faster to the true Mohr-Coulomb.

8. Conclusions

Classical yield and failure criteria, Tresca, von Mises, Drucker-Prager and Mohr-Coulomb and the more recent ones, Matsuoka-Nakai and Lade-Duncan, adopt different mechanical quantities as an indicator of the proximity to yield and/or failure. They considerably differ from one another both in mathematical definition and in graphical representation in the principal stress space. As the range of applications of these criteria has expanded to include constitutive modelling of building materials and geomaterials, numerous limitations have emerged in the original formulations which make their use in defining yield and plastic potential surfaces problematic. These include the lack of smoothness of criteria such as Tresca and Mohr-Coulomb, associated to the cusp at the vertices in the hexagonal deviatoric section and the lack of convexity of Matsuoka-Nakai and Lade-Duncan which plot as multi-branch surfaces in the principal stress space. The Mohr-Coulomb criterion too, if defined in terms of stress obliquities, comprises two symmetrical pyramids.

Following the steps of Panteghini and Lagioia [26] and Lagioia et al. [16] who have recently convexified the Matsuoka-Nakai and a generalisation of the Galileo-Rankine criteria, it has been shown that a single equation exists which defines the six criteria and is free from all the limitations which affect the original formulations. Not only is this general criterion always convex, both for $f(\boldsymbol{\sigma})=0$ and for $f(\boldsymbol{\sigma})\neq 0$, but, if required, it can also provide a smooth Tresca and Mohr-Coulomb of continuity of at least class C^2. The mathematical structure of the expression allows for a very efficient implementation based on an implicit integration scheme formulated by Panteghini and Lagioia [26]. Finally, as it comprises a separate definition of the meridional and deviatoric sections, new yield and plastic potential surfaces can be easily formulated by combining new meridional sections with the deviatoric sections of classical criteria.

I has then been shown that, after some trigonometrical manipulations, the expression of the general criterion can be rewritten as the largest of the three ordered solutions of a cubic equation. For Matsuoka-Nakai and Lade-Duncan, the other two solutions define the two additional surface branches which make these criteria non-convex, whilst the product of all three solutions results in the original defining equation of both criteria. Following the steps of Capurso [7], it was argued that both Tresca and Mohr-Coulomb can also be defined by a cubic equation. It can then be postulated that a more general class on non-convex criteria exists, which is defined by a cubic equation and comprises the six criteria considered in this paper. It is interesting to observe that in the case of the Mohr-Coulomb criterion, the two additional surface branches defined by the intermediate and lowest solutions of the cubic equation, are associated to the intermediate and minor stress obliquities $\tan\phi_{\mathrm{I,II}}$ and $\tan\phi_{\mathrm{II,III}}$ attaining a critical value for tensile stresses. Such class also includes the recent generalization of the Galileo-Rankine criterion proposed by Lagioia et al. [16].

The general defining equation has been adopted to define the yield and the plastic potential surfaces of an elastic-perfect plastic model which has been implemented in a finite element program. The efficiency of the general criterion has then been evaluated by performing numerical analyses of a *true* N_γ problem for a shallow circular footing, where both material cohesion and lateral surcharge were set to zero.

Analyses were performed for Drucker-Prager, Matsuoka-Nakai, Lade-Duncan, original and a rounded Mohr-Coulomb soils. The bearing capacity for the original Mohr-Coulomb material is very close to, but larger than that obtained from the method of the characteristic lines. This bearing capacity can also be closely approximated by the rounded version of that failure criterion, if the rounding parameter β is set to a value very close to, but less than, unity. The effect of the intermediate principal stress at failure on the bearing capacity is shown by the results of the analyses of the Matsuoka-Nakai and Lade-Duncan materials, which indicate that they both largely exceed that of the Mohr-Coulomb material, the Lade-Duncan N_γ being the largest. These findings indicate that the general equation can be effectively adopted in substitution of the original formulations. This provides great versatility as numerical analyses can be readily repeated, by changing the value of the three defining parameters, to simulated different classical failure criteria.

9. Statements

Ethics Statement. This work does not involve humans Data accessibility statement. This work does not contain data. Competing interests statement. We have no competing interests. Author's contribution. This work is the result of an equal contribution of both authors. Funding. This work has received no funding

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Appendices

A. Coefficients of the depressed cubic equation of the Matsuoka-Nakai and Lade-Duncan criteria

Equation (4.4) is a depressed cubic equation of type (4.7) which has three real roots for both Matsuoka-Nakai and Lade-Duncan. In fact the requirements for this to happen are

$$u < 0 \tag{A.1}$$

and

$$4u^3 + 27w^2 \leq 0 \tag{A.2}$$

The first conditions given by equation (A.1) is always satisfied for both Matsuoka-Nakai and Lad-Duncan. In fact, since $k_{MN} \geq 9$, $k_{LD} \geq 27$ and J_{2D} is always positive, combining equations (4.8), (4.5) and (4.6) results in

$$\begin{aligned} u &= -A_1 J_{2D} = -\frac{k_{MN}-3}{k_{MN}-9} J_{2D} \leq 0 \\ u &= -A_1 J_{2D} = -\frac{k_{LD}}{k_{LD}-27} J_{2D} \leq 0 \end{aligned} \tag{A.3}$$

The secondon condition, given by equation (A.2), after some algebraic manipulations becomes for Matsuoka-Nakai

$$(k_{MN}-3)^3 \geq k_{MN}^2 (k_{MN}-9) \sin^2 3\theta \tag{A.4}$$

Since $\sin^2 3\theta \in [0,1]$ a majorant of the RHS is $k_{MN}^2\,(k_{MN}-9)$ and equation (A.4) becomes

$$(k_{MN}-3)^3 \geq k_{MN}^2\,(k_{MN}-9) \tag{A.5}$$

or equivalently

$$27(k_{MN}-1) \geq 0 \tag{A.6}$$

which is clearly always satisfied.

Similarly equation (A.2) for Lade-Duncan leads to

$$k_{LD}^3 \geq k_{LD}^2\,(k_{LD}-27)\sin^2 3\theta \tag{A.7}$$

and using again a majorant of the RHS becomes

$$k_{LD}^3 \geq k_{LD}^2\,(k_{LD}-27) \tag{A.8}$$

or equivalently

$$27k_{LD}^2 \geq 0 \tag{A.9}$$

which is clearly satisfied.

B. Reformulation of the general criteria to reflect the structure of ordered solutions of a cubic equation

Equation (5.1) does not reflect the structure of the three ordered solutions (4.9) of the depressed cubic of equation due to the additional term $-\gamma\dfrac{\pi}{6}$ which appears in $\Gamma(\theta)$. This also applies to the $\Gamma(\theta)$ of the Tresca and Mohr-Coulomb criteria, given by equation (4.25), for the presence of the term $-(1-\bar{\gamma})\dfrac{\pi}{6}$. However both $\Gamma(\theta)$ functions can be manipulated to remove the term outside the arc-cosine function.

In fact, as the cosine is an even function, equation (4.18) retrieved for the Tresca and Mohr-Coulomb criteria can be rewritten as

$$\Gamma(\theta) = \frac{\cos(\theta-\bar{\theta})}{\cos(\dfrac{\pi}{6}+\bar{\theta})} = \frac{\cos|\theta-\bar{\theta}|}{\cos(\dfrac{\pi}{6}+\bar{\theta})} \tag{B.1}$$

Moreover as

$$|\theta-\bar{\theta}| = \frac{1}{3}\arccos\left[\cos 3(\theta-\bar{\theta})\right] \tag{B.2}$$

and

$$\cos 3(\theta-\bar{\theta}) = \sin\left[\frac{\pi}{2}-3(\theta-\bar{\theta})\right] \tag{B.3}$$

equations (4.18) and (B.1) can be rewritten as

$$\Gamma(\theta) = \frac{\cos(\theta-\bar{\theta})}{\cos(\dfrac{\pi}{6}+\bar{\theta})} = \frac{1}{\cos(\dfrac{\pi}{6}+\bar{\theta})}\cos\frac{\arccos\left\{\sin\left[\frac{\pi}{2}-3(\theta-\bar{\theta})\right]\right\}}{3} \tag{B.4}$$

or equivalently as

$$\Gamma(\theta) = \frac{\cos(\theta-\bar{\theta})}{\cos(\dfrac{\pi}{6}+\bar{\theta})} = \frac{1}{\cos(\dfrac{\pi}{6}+\bar{\theta})}\cos\frac{\arccos\left\{\sin\left[(1+\bar{\gamma})\frac{\pi}{2}-3\theta\right]\right\}}{3} \tag{B.5}$$

in which indeed the term outside the arc-cosine function has been removed.

Similarly, for the general case of $\beta \neq 1$, equation (5.2) can be rewritten by considering that $\gamma\pi/6$ is equivalent to $1/3\arccos\left[\cos(\gamma\pi/2)\right]$ and that

$$|\arccos x - \arccos y| = \arccos\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right] \tag{B.6}$$

which results in

$$\Gamma(\theta) = \alpha\cos\frac{\arccos\left[\beta\sin 3\theta\cos\frac{\gamma\pi}{2} + \sqrt{1-\beta^2\sin^2 3\theta}\sin\frac{\gamma\pi}{2}\right]}{3} \tag{B.7}$$