Supplemental Materials for

Nonlinear Regression Models and Applications in Agricultural Research

Sotirios V. Archontoulis and Fernando E. Miguez

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Supplemental tables with nonlinear equations and the associate parameter meaning

Table S.1 – Equations derived from the exponential equation. Numbering continues from Group I of the Table 1. Placing the parameters and variables together always indicates multiplication (e.g. *kt* = *k* × *t*). e = exp.

(#)	Name and/or Reference	Form	Parameter meaning
1.3	Parallel or double first order	$Y = fY_o e^{-k_1 t} + (1-f)Y_o e^{-k_2 t}$	*Y* is the response variable, *t* is the explanatory variable (e.g. time), Y_o is the initial or maximum *Y* value, k, k_1 and k_2 are rate constants that determine the steepness of the curve (units: time^{-1}) f, f_1 and f_2 are fractions (range: 0–1) that determine the size of each pool *b* is a rate constant for the linear component t_i is the infection point for the logistic component (units: time), t_d is the distance from infection point to ¾ maximum (units: time)
1.4	Triple first order	$Y = f_1 Y_o e^{-k_1 t} + f_2 Y_o e^{-k_2 t} + (1-f_1-f_2)Y_o e^{-k_3 t}$	
1.5	Consecutive first order (Andren and Paustian 1987)	$Y = fY_o e^{-k_1 t} + (1-f)Y_o\left[e^{-k_2 t} + \frac{k_2}{k_1-k_2}(e^{-k_2 t} - e^{-k_1 t})\right]$	
1.6	First order plus linear (Gills and Price 2011)	$Y = Y_o(1-e^{-kt}) + bt$	
1.7	First order plus logistic (Gills and Price 2011)	$Y = Y_o f(1-e^{-kt}) + \frac{Y_o(1-f)}{1+e^{-\frac{t-t_i}{t_d}}}$	
1.8	Leaf area index nitrogen limited equation Yin et al (2000, 2003a)	$Y_{LAI} = \frac{1}{k_n} ln\left(1+\frac{k_n X_N}{n_b}\right)$	Y_{LAI} *and* Y_N are the response variables (leaf area index and nitrogen content, respectively), X_N and X_{LAI} are the explanatory variables (canopy nitrogen content and leaf area index, respectively), k_n is the rate constant determines the steepness of the curve (also called nitrogen extinction coefficient), n_b is the minimum value of nitrogen at or below which leaf photosynthesis under saturated light conditions is zero (usually n_b ranges from 0.25 to 0.80; see review by Archontoulis et al., 2012), Y_o is the initial or maximum *Y* value, *c* is a dimensionless coefficient
1.9	Protein or nitrogen distribution within plant canopies (Johnson et al., 2010)	$Y_N = Y_o - (Y_o - n_b)(1-e^{-k_n X_{LAI}})^c$	
1.10	Photoperiodic sensitivity (Wang and Engel, 1998)	$Y = e^{-P_{sen}(X-P_c)}$	*Y* is the response variable, *X* is the explanatory variable (photoperiod), P_{sen} is the photoperiod sensitivity coefficient being positive for short-days plants and negative for long days plants, P_c is the critical day-length, around 11 h for short days and 18 hours for long days
1.11	Horton (1940) model for water infiltration	$Y = Y_c + (Y_o - Y_c)e^{-kt}$	*Y* is the response variable (infiltration capacity, mm/h), *t* is the explanatory variable (time, hours), *k* is the rate constant representing the rate of decrease,

			Y_c indicates a final or equilibrium capacity (mm/h) Y_o indicates the initial infiltration capacity (mm/h)

Table S.2 – Additional sigmoid type functions. Numbering continues from Group II of the Table 1. Placing the parameters and variables together always indicates multiplication (e.g. $kt = k \times t$). e = exp.

(#)	Name and/or Reference	Form	Parameter meaning & application
Different version of the logistic equation[a]			
2.7	5-parameter	$Y = Y_{low} + \frac{Y_{asym}}{[1+e^{-k(t-t_m)}]^c}$	Y is the response variable (e.g. biomass), t is the explanatory variable (e.g. time), Y_{asym} is the asymptotic Y value, Y_{low} is the initial Y value, t_m is the infection point at which the growth rate is maximum, k determines the steepness of the curve, c is a shape parameter
2.8	4-parameter	$Y = Y_{low} + \frac{Y_{asym}}{1+e^{-k(t-t_m)}}$	
2.9	2-parameter	$Y = \frac{1}{1+e^{-k(t-t_m)}}$	
2.10	1-parameter	$Y = \frac{1}{1+e^{-(t-t_m)}}$	
Different versions of the beta function[b]			
2.11	5-parameter; for cases when initial phase is important; Yin et al., 2003	$Y = Y_b + (Y_{max} - Y_b)\left(1+\frac{t_e-t}{t_e-t_m}\right)\left(\frac{t-t_b}{t_e-t_b}\right)^{\frac{t_e-t_b}{t_e-t_m}}$	Y is the response variable, t is the explanatory variable (e.g. time), Y_{max} is the maximum Y value, t_m is the infection point at which growth rate is maximized, t_e is the t value (time) when $Y = Y_{max}$, t_b is the t value when growth starts, Y_b is the Y value at time t_b
2.12	2-parameter; for cases when maximum growth rate achieved at the beginning of growth; Yin et al., 2003	$Y = Y_{max}(2t_e - t)\frac{t}{t_e^2}$	
2.13	2-parameter; for cases when maximum growth achieved towards the end of growth period; Yin et al., 2003	$Y = Y_{max}(3t_e - 2t)\frac{t^2}{t_e^3}$	
2.14	3-parameter; Applied to describe leaf senescence; Yin et al., 2009	$Y = Y_{max}\left[1-\left(1+\frac{t_e-t}{t_e-t_m}\right)\left(\frac{t}{t_e}\right)^{\frac{t_e}{t_e-t_m}}\right]$	

Expolinear functions for plant growth analysis			
2.15	Expolinear (Goudriaan and Monteith 1990)	$Y = \frac{c_m}{r_m} ln\left[1 + e^{r_m(t-t_o)}\right]$	*Y* is the response variable (e.g. biomass), *t* is the explanatory variable (time), c_m is the maximum growth rate in the linear phase, r_m is the maximum relative growth rate in the exponential phase, t_o is the time when growth begins, Y_{max} is the maximum *Y* value
2.16	Symmetrical Expolinear (Goudriaan 1994)	$Y = \frac{c_m}{r_m} ln\left[\frac{1+e^{r_m(t-t_o)}}{1+e^{r_m\left(t-t_o-\frac{Y_{max}}{c_m}\right)}}\right]$	
Additional generic sigmoid functions			
2.17	Levakovic (cited in Zeide, 1993)	$Y = Y_{asym}\left(\frac{t^d}{\beta + t^d}\right)^c$	*Y* is the response variable (e.g. biomass), *t* is the explanatory variable (time), Y_{asym} is the asymptotic *Y* value, *β and c* determine the start of growth and the inflection point, *d* controls the shape of the curve,
2.18	Korf (cited in Zeide, 1993)	$Y = Y_{asym}e^{-\beta t^{-c}}$	
2.19	Special Von Bertalanffly, 1938[c]	$Y = Y_{asym}\left(1 - e^{-k(t-t_o)}\right)$	*Y* is the response variable (e.g. biomass), *t* is the explanatory variable (time), Y_{asym} is the asymptotic maximum *Y* value t_o is the time when growth begins, t_{50} is the time when Y = ½ of Y_{asym}, *k* controls the steepness of the curve *v* controls the asymmetry and the shape of the curve (for *v*=1, 2.20 becomes 2.19)
2.20	Generalized Von Bertalanffly, 1938[c]	$Y = Y_{asym}\left(1 - e^{-k(t-t_o)}\right)^v$	
2.21	Log-logistic distribution (Ritz et al., 2013)	$Y = \frac{Y_{asym}}{1+exp[k\{\log(t)-\log(t_{50})\}]} = \frac{Y_{asym}}{1+\left(\frac{t}{t_{50}}\right)^k}$	
Inverse sigmoid functions used to quantify water retention curves[d]			
2.22	Van Genuchten (1980)	$\theta_h = \theta_r + \frac{\theta_s - \theta_r}{[1 + (ah)^n]^{1-\frac{1}{n}}}$	θ_ψ is the volumetric water content (mm/mm), *h* is the matric potential (kPa) θ_r is the residual water content (mm/mm), θ_s is the saturated water content (mm/mm), *a* is the inverse of the air entry suction (*a*>0), *n* is a measure of the poor-size distribution (n>1)
2.23	Gardner (1958)	$\theta_h = \theta_r + \frac{\theta_s - \theta_r}{1 + (ah)^n}$	
2.24	Brooks and Corey (1964)	$\theta_h = \theta_r + \frac{\theta_s - \theta_r}{(ah)^n}$	
2.25	Campbell (1974)	$\theta_h = \theta_s(ah)^n$	

[a]: the 3-parameter logistic equation is given in Table 1 (see equation 2.1)
[b]: For additional versions of the beta growth function see Yin et al (2003b)
[c]: for a comparison of the two Von Bertalanffly (1938) models see Urban, 2002
[d]: The R-package ‘HydroMe’ includes the listed water retention equations (see Omuto and Gumbe, 2009)

Table S.3 – Additional equations for photosynthesis. Numbering continues from Group III of the Table 1. Placing the parameters and variables together always indicates multiplication (e.g. *kt* = *k* × *t*). e = exp.			
(#)	Name and/or Reference	Form	Parameter meaning & application
3.7	Modified rectangular hyperbola proposed to describe photosynthetic reduction at over-saturated irradiance levels (e.g. photorespiration) (Ye and Zhao, 2010)	$Y = aI\frac{1-\beta I}{1+\gamma I} - R_d$	Y is the response variable (net photosynthesis), I is the explanatory variable (irradiance), a is the slope of the curve when I equals zero, β and γ are coefficients with no clear (biological) meaning
3.8	Modified hyperbola used to describe light saturated photosynthesis response to nitrogen (del Pozo & Dennett 1999)	$Y = Y_{asym}\frac{N-N_{min}}{N+k}$	Y is the response variable (light saturated net photosynthesis), N is the explanatory variable (leaf nitrogen), Y_{asym} is the asymptotic maximum Y value, k determines the curvature of the curve, N_{min} is the N value at or below which Y=0
3.9	Weibull type sigmoid function used to assess water stress effects on photosynthesis (Vico and Porporato, 2008)	$Y = Y_{asym}e^{\left[-\left(\frac{-\Psi_L}{d}\right)^b\right]}$	Y is the response variable (e.g. carboxylation rate), Ψ_L is the explanatory variable (water potential), Y_{asym} is the asymptotic Y value, b and d determine the shape of the curve,

Table S.4 – Additional equations for temperature dependencies. Numbering continues from Group IV of the Table 1. Placing the parameters and variables together always indicates multiplication (e.g. $kt = k \times t$). e = exp.

(#)	Name and/or Reference	Form	Parameter meaning & application
4.6	Power (e.g. Shibu et al., 2006)	$Y = \left(\frac{T}{T_o}\right)^c$	Y is the response variable, T is the explanatory variable (temperature), T_o is the optimum temperature for maximizing Y, c is a shape factor (usually c=2), a determines the magnitude of the Y value, b determines the shape of the curve
4.7	Logarithmic (e.g. Foereid et al., 2011)	$Y = a + bln(T)$	
4.8	Exponential (e.g. Shibu et al., 2006)	$Y = ae^{bT}$	
4.9[a]	Modified exponential (O'Connell 1990)	$Y = e^{\left[a+bT\left(1-0.5\frac{T}{T_o}\right)\right]}$	
4.10	Bell-shaped 3-parameter; Used to describe photosynthesis response to temperature (Johnson et al., 2010)	$Y = \left(\frac{(1+c)T_o - T_{min} - cT}{(1+c)T_o - T_{min} - cT_{ref}}\right)\left(\frac{T - T_{min}}{T_{ref} - T_{min}}\right)^c$	Y is the response variable (e.g. photosynthesis), T is the explanatory variable (temperature), T_{ref} is a reference temperature at which Y=1, T_{min} is the minimum temperature for Y=0, T_o is the optimum temperature for maximum Y, c is a shape factor

[a]: Eq. 4.9 follows an increasing and decreasing pattern like the modified Arrhenius equation (see Eq. 4.3 in Table 1, Fig. 2). It was used by Kirschbaum (1995) to describe soil organic matter decomposition as a function of temperature.

Table S.5 – Additional peak or bell shape equations. Numbering continues from Group V of the Table 1. Placing the parameters and variables together always indicates multiplication (e.g. *kt* = *k* × *t*). e = exp.

(#)	Name and/or Reference	Form	Parameter meaning & application
5.4[a]	Lorentzian 3-parameter	$Y = \dfrac{a}{1 + \left(\dfrac{X - X_o}{b}\right)^2}$	*Y* is the response variable, *X* is the explanatory variable, Y_o is the initial Y value, X_o is the position of the center of the peak, *a* and *b* are coefficients controlling the height and width of the peak *c* determines the number of peaks (0 < c < 1 for one peak)
5.5[a]	Lorentzian 4-parameter	$Y = Y_o + \dfrac{a}{1 + \left(\dfrac{X - X_o}{b}\right)^2}$	
5.6[b]	Log Normal 3-parameter	$Y = ae^{-0.5\left(\frac{\ln\left(\frac{X}{X_o}\right)}{b}\right)^2}$	
5.7	Pseudo-Voigt 4-parameters	$Y = a\left[\dfrac{c}{1 + \left(\dfrac{X - X_o}{b}\right)^2} + (1 - c)e^{-0.5\left(\frac{X - X_o}{b}\right)^2}\right]$	

[a]: condition b > 1
[b]: condition 0 < b < 1

Table S.6 – Additional equations. Numbering continues from Group VI of the Table 1. Placing the parameters and variables together always indicates multiplication (e.g. $kt = k \times t$).

(#)	Name and/or Reference	Form	Parameter meaning & application
6.6[a]	Holling type IV (e.g. Bolker 2008)	$Y = \frac{Y_{asym}X^2}{b + cX + X^2}$	Y is the response variable, X is the explanatory variable, Y_{asym} is the asymptotic Y value, b and c define the position of the peak and the shape of the curve, Y_o is the initial or maximum Y value
6.7[b]	Shepherd (e.g. Bolker 2008)	$Y = \frac{Y_{asym}X}{b + X^c}$	
6.8[b]	Hassell (e.g. Bolker 2008)	$Y = \frac{Y_{asym}X}{(b + X)^c}$	
6.9[c]	2 parameter exponent (e.g. Ratkowsky 1993)	$Y = Y_o b^X$	
6.10	Monod kinetics	$Y = \frac{\mu X}{X + C_{sat}} \mathrm{B}$	Y is the response variable (e.g. mineralization rate), X is the explanatory variable, the substrate (e.g. OC), μ is the rate constant, C_{sat} is the half-saturation constant, B is the rate of microorganisms (e.g. size of microbial biomass), If B=1 then Monod becomes M-M equation

[a]: if $c < 0$ the equation takes a peak form; the peak occurs at $X = -2b/c$. This function is called rational function because expresses the ratio of two polynomial functions. It can also take a S-shape (see Bolker, 2008)
[b]: if $c > 0$ then equation takes a peak form
[c]: if $b > 1$ then Y increases; if $b < 1$ then Y decreases.

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