Model Moving Average, MA(q)

- Model MA mempunyai ordo yang besarnya dinotasikan dengan huru *"q"*, sehingga dinotasikan dengan *MA(q)*
- Model MA mengasumsikan tiap observasi dibentuk oleh rata-rata periode ke belakang
- Model umum *MA(q)* ditulis :

$$Y_t = \theta_0 + \varepsilon_t - \theta_1 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}$$

Model Autoregressive, AR(p)

- Model AR mempunyai ordo yang besarnya dinotasikan dengan huru *"p"*, sehingga dinotasikan dengan *AR(p)*
- Nilai Y_t pada periode saat ini (t) adalah kombinasi linear dari p nila periode sebelumnya ditambah komponen e_t
- Model umum *AR(p)* ditulis :

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

Asumsi: e_t saling bebas thdp $Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p}$

Model Moving Average, MA(1)

MA(1) : Nilai deret waktu saat ini bergantung pada nilai eror terakhir, satu unit waktu di masa lalu.

$$Y_t = \theta_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$Cov(Y_t, Y_{t-1}) = Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}) = Cov(-\theta e_{t-1}, e_{t-1}) = -\theta\sigma_e^2$$

$$Cov(Y_t, Y_{t-2}) = Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) = 0$$

$$E(Y_t) = 0$$

$$\gamma_0 = Var(Y_t) = \sigma_e^2(1+\theta^2)$$

$$\gamma_1 = -\theta\sigma_e^2$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} \Rightarrow \rho_1 = (-\theta)/(1+\theta^2), \quad \gamma_k = \rho_k = 0 \quad \text{for } k \geq$$

Model Autoregressive, AR(1)

Model: $Y_t = \phi Y_{t-1} + e_t$

untuk $k = 1,2,\ldots$ $\quad E(Y_{t-k}Y_t) = \phi E(Y_{t-k}Y_{t-1}) + E(e_t Y_{t-k})$

$$\gamma_k = \phi\gamma_{k-1} + E(e_t Y_{t-k})$$

untuk $k = 1$, $\gamma_1 = \phi\gamma_0 = \frac{\phi\sigma_e^2}{1-\phi^2}$

untuk $k = 2$, $\gamma_2 = \phi\gamma_1 = \frac{\phi^2\sigma_e^2}{1-\phi^2}$

$$\Rightarrow \gamma_k = \frac{\phi^k\sigma_e^2}{1-\phi^2}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k \quad \text{for } k = 1, 2, 3, \ldots$$

Model Moving Average, MA(2)

MA(2) : Nilai deret waktu saat ini bergantung pada dua nilai eror terakhir, satu unit waktu dan dua unit waktu di masa lalu.

$$Y_t = \theta_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

$$\gamma_0 = Var(Y_t) = Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) = (1+\theta_1^2+\theta_2^2)\sigma_e^2$$

$$\begin{aligned}
\gamma_1 &= Cov(Y_t, Y_{t-1}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\
&= Cov(-\theta_1 e_{t-1}, e_{t-1}) + Cov(-\theta_1 e_{t-2}, -\theta_2 e_{t-2}) \\
&= [-\theta_1 + (-\theta_1)(-\theta_2)]\sigma_e^2 \\
&= (-\theta_1 + \theta_1\theta_2)\sigma_e^2 \\
\gamma_2 &= Cov(Y_t, Y_{t-2}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) \\
&= Cov(-\theta_2 e_{t-2}, e_{t-2}) \\
&= -\theta_2\sigma_e^2
\end{aligned}$$

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$\rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$\rho_k = 0 \quad \text{for } k = 3, 4, \ldots$$

ACF for AR (2)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

$$E(Y_t) = 0,$$

$$Var(Y_t) = \gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma_e^2$$

$$Cov(Y_t, Y_{t-k}) = \gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} \; for \; k = 1,2,3,\ldots$$

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} for \; k = 1,2,3,\ldots$$

The General MA(q) Process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

$$\gamma_0 = (1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2)\sigma_e^2$$

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2} & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases}$$

We normally restrict autoregressive models to stationary data, in which case some constrai on the values of the parameters are required.

- For an AR(1) model: $-1 < \phi_1 < 1$.
- For an AR(2) model: $-1 < \phi_2 < 1$, $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$.

The invertibility constraints for other models are similar to the stationarity constrain

- For an MA(1) model: $-1 < \theta_1 < 1$.
- For an MA(2) model: $-1 < \theta_2 < 1$, $\theta_2 + \theta_1 > -1$, $\theta_1 - \theta_2 < 1$.

Invertibility

An MA(1) model: $Y_t = e_t - \theta e_{t-1}$

First rewriting this as $e_t = Y_t + \theta e_{t-1}$ and then replacing t by $t-1$ and substituting for e_{t-1} above, we get

$$\begin{aligned} e_t &= Y_t + \theta(Y_{t-1} + \theta e_{t-2}) \\ &= Y_t + \theta Y_{t-1} + \theta^2 e_{t-2} \end{aligned}$$

If $|\theta| < 1$, we may continue this substitution "infinitely" into the past and obtain the expression

$$e_t = Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \cdots$$

or

$$Y_t = (-\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \cdots) + e_t$$

- If $|\theta| < 1$, we see that the MA(1) model can be inverted into an infinite-order autoregressive model.
- We say that the MA(1) model is invertible if and only if $|\theta| < 1$.

ARMA(1,1) $\quad Y_t = \phi_1 y_{t-1} + e_t - \theta_1 e_{t-1}$

Asumsi :

e_t bebas terhadap $y_{t-1}, y_{t-2}, \ldots$

e_{t-1} bebas terhadap $y_{t-2}, y_{t-3}, \ldots$

Namun e_t dan e_{t-1} tidak bebas terhadap y_t

$$E(e_t Y_t) = E[e_t(\phi Y_{t-1} + e_t - \theta e_{t-1})] = \sigma_e^2$$

$$E(e_{t-1}Y_t) = E[e_{t-1}(\phi Y_{t-1} + e_t - \theta e_{t-1})] = \phi\sigma_e^2 - \theta\sigma_e^2 = (\phi - \theta)\sigma_e^2$$

$$\left.\begin{aligned} \gamma_0 &= \phi\gamma_1 + [1 - \theta(\phi-\theta)]\sigma_e^2 \\ \gamma_1 &= \phi\gamma_0 - \theta\sigma_e^2 \\ \gamma_k &= \phi\gamma_{k-1} \quad \text{for } k \geq 2 \end{aligned}\right\}$$

$$\gamma_0 = \frac{(1 - 2\phi\theta + \theta^2)}{1-\phi^2}\sigma_e^2$$

$$\rho_k = \frac{(1-\theta\phi)(\phi-\theta)}{1-2\theta\phi+\theta^2}\phi^{k-1} \quad \text{for } k \geq 1$$

General Behavior of the ACF and PACF for ARMA Models

	AR(*p*)	MA(*q*)	ARMA(*p*,*q*), *p*>0, and *q*>0
ACF	Tails off	Cuts off after lag *q*	Tails off
PACF	Cuts off after lag *p*	Tails off	Tails off

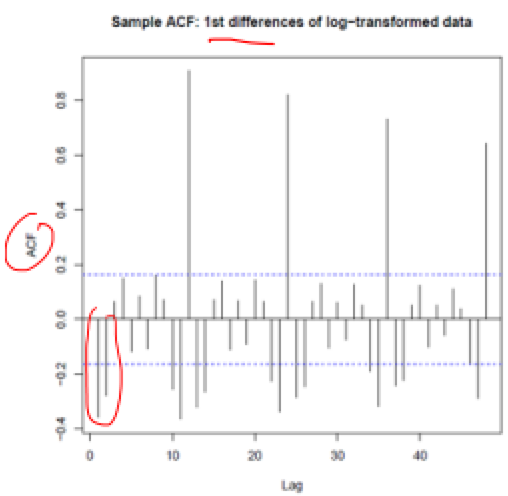
Model SARIMA$(p,d,q)(P,D,Q)^s$

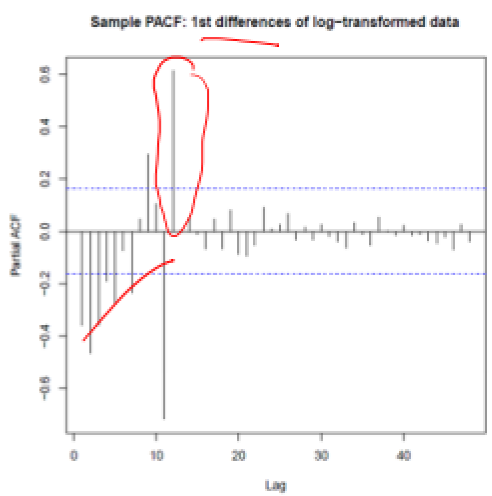
$$\boxed{\Phi_P(B^s)\phi_p(B)(1-B)^d(1-B^s)^D Y_t = \theta_q(B)\Theta_Q(B^s)e_t}$$

Keterangan:

$\phi_p(B)$	: $(1-\phi_1 B - \cdots - \phi_p B^p)$, operator AR*(p)*
$\theta_q(B)$	: $(1-\theta_1 B - \cdots - \theta_q B^q)$, operator MA*(q)*
$\Phi_P(B^s)$	: $(1-\Phi_1 B^s - \cdots - \Phi_P B^{Ps})$, operator *Seasonal* AR*(P)*
$\Theta_Q(B^s)$	: $(1-\Theta_1 B^s - \cdots - \Theta_Q B^{Qs})$, operator *Seasonal* MA*(Q)*
$(1-B)^d$	: pembedaan nonmusiman
$(1-B^s)^D$	: pembedaan musiman
Y_t	: data pada waktu ke-*t*
(p, d, q)	: ordo bagian nonmusiman dari model
(P, D, Q)	: ordo bagian musiman dari model
s	: jumlah periode per musim
e_t	: sisaan pada waktu ke-t
B	: operator penggeser mundur *(backshift)*

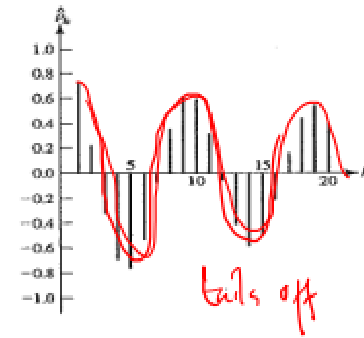
Berdasarkan plot ACF dan PACF berikut ini, maka model yang tepat adalah

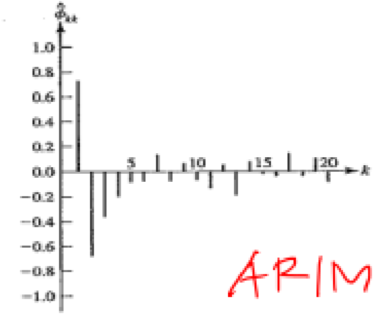




a. ARIMA(0,1,2)(1,0,0)$_{12}$
b. ARIMA(2,1,0) (1,0,0)$_{12}$
c. ARIMA(0,1,2) (0,1,1)$_{12}$
d. ARIMA(2,1,0) (0,1,1)$_{12}$

Berdasarkan plot ACF, PACF serta dugaan nilai ACF dan PACF berikut ini, maka model yang tepat adalah





ARIMA (0,0,1)

(a) ACF, $\hat{\rho}_k$

1–10	.73	.22	−.32	−.69	−.76	−.53	−.08	.35	.61	.59
St.E.	.13	.19	.20	.21	.25	.29	.30	.30	.31	.33
11–20	.31	−.06	−.41	−.58	−.49	−.21	.16	.44	.54	.40
St.E.	.35	.36	.37	.37	.39	.40	.40	.41	.41	.43

(b) PACF, $\hat{\phi}_{kk}$

1–10	.73	−.68	−.36	−.20	−.09	−.08	.13	−.08	.06	−.07
St.E.	.13	.13	.13	.13	.13	.13	.13	.13	.13	.13
11–20	−.13	.05	−.19	.07	−.02	−.04	.14	−.04	.10	−.09
St.E.	.13	.13	.13	.13	.13	.13	.13	.13	.13	.13

a. ARIMA (3,0,0)
b. ARIMA (0,0,3)
c. ARIMA(3,1,0) X
d. ARIMA (0,1,3) X

t hit..

Dalam suatu analisis, disimpulkan bahwa

- ACF untuk deret waktu {Yt} menurun sangat lambat sekali
- PACF untuk deret waktu $\{\nabla^2 Yt\}$ signifikan pada lags 1 dan 2 (pada lag lainnya tidak nyata) $(1-B)^2 Y_t$
- Uji Dickey-Fuller unit root test untuk deret waktu {Yt} tidak tolak H_0

Model manakah yang paling konsisten dengan kesimpulan di atas?

a. IMA(1,1)
b. ARI(2,2)
c. ARIMA(2,2,2)
d. IMA(2,2)

AR