On ∞-Safety of Stochastic Differential Dynamics

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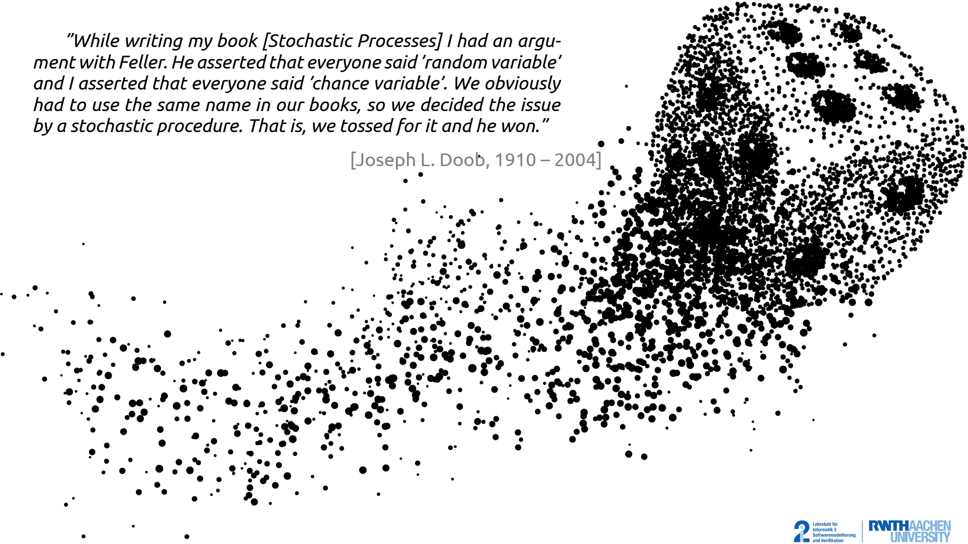
Stochasticity

"While writing my book [Stochastic Processes] I had an argument with Feller. He asserted that everyone said 'random variable' and I asserted that everyone said 'chance variable'. We obviously had to use the same name in our books, so we decided the issue by a stochastic procedure. That is, we tossed for it and he won."

[Joseph L. Doob, 1910 – 2004]



Stochasticity in Differential Dynamics

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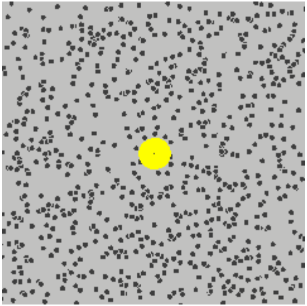
Louis Bachelier



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Brownian motion

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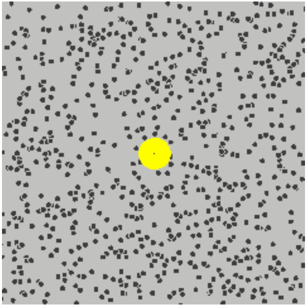

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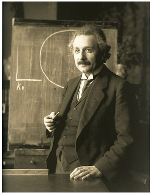
Brownian motion

"The mathematical expectation of the speculator is zero."

[L. Bachelier, Théorie de la spéculation, 1900]

Stochasticity in Differential Dynamics



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A. Einstein

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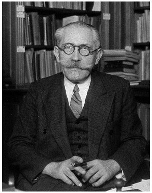
M. Smoluchowski



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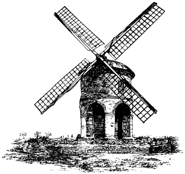
Applications of Stochastic Differential Dynamics



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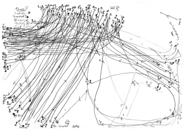
Wind forces



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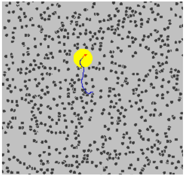
Pedestrian motion



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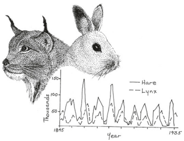
Brownian motion



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Population dynamics

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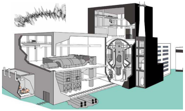
Stock options



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Robust control

Stochastic Differential Equations (SDEs)

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \sigma(X_t)\ \mathrm{d}W_t, \quad t \geq 0.$$

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$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \sigma(X_t)\ \mathrm{d}W_t, \quad t \geq 0.$$

The unique solution is the *stochastic process* $X_t(\omega) = X(t,\omega)\colon [0,\infty) \times \Omega \to \mathbb{R}^n$ s.t.

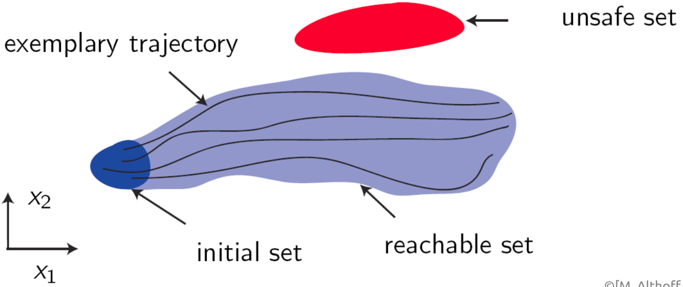
$$X_t = X_0 + \int_0^t b(X_s)\ \mathrm{d}s + \int_0^t \sigma(X_s)\ \mathrm{d}W_s.$$

The solution $\{X_t\}$ is also referred to as an *(Itô) diffusion process.*

Safety Verification of ODEs

Given $T \in \mathbb{R}$, $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{X}_0 \subset \mathcal{X}$, $\mathcal{X}_u \subset \mathcal{X}$, weather

$$\forall \mathbf{x}_0 \in \mathcal{X}_0 : \quad \left(\bigcup_{t \leq T} \mathbf{x}_{t,\mathbf{x}_0}\right) \cap \mathcal{X}_u = \emptyset \quad ?$$



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- System is *T*-safe, if no trajectory enters $\mathcal{X}_u$ over $[0, T]$; Unbounded : $T = \infty$.

∞-Safety of SDEs

Bound the failure probability

$$P\left(\exists t \in [0,\infty) \colon \tilde{X}_t \in \mathcal{X}_u\right), \quad \forall X_0 \in \{X \mid \mathsf{supp}(X) \subseteq \mathcal{X}_0\},$$

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where $\tilde{X}_t$ is the process that will stop at the boundary of $\mathcal{X}$:

$$\tilde{X}_t \mathrel{\widehat{=}} X_{t \wedge \tau_{\mathcal{X}}} = \begin{cases} X(t, \omega) & \text{if } t \leq \tau(\omega), \\ X(\tau(\omega), \omega) & \text{otherwise}, \end{cases}$$

with $\tau_{\mathcal{X}} \mathrel{\widehat{=}} \inf\{t \mid X_t \notin \mathcal{X}\}$.

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with $\tau_{\mathcal{X}} \mathrel{\widehat{=}} \inf\{t \mid X_t \notin \mathcal{X}\}$.

$$\phi \mathrel{\widehat{=}} \text{“}\tilde{X}_t \text{ evolves within } \mathcal{X}\text{”}, \quad \psi \mathrel{\widehat{=}} \text{“}\tilde{X}_t \text{ evolves into } \mathcal{X}_u\text{”}$$

$$\Downarrow$$

∞-safety asks for a bound on $P(\phi\, \mathcal{U}\, \psi)$.

Overview of the Idea

Observe that for any $0 \leq T < \infty$,

$$P(\exists t \geq 0 \colon \tilde{X}_t \in \mathcal{X}_u) \leq P(\exists t \in [0, T] \colon \tilde{X}_t \in \mathcal{X}_u) + P(\exists t \geq T \colon \tilde{X}_t \in \mathcal{X}_u).$$

⇒ S. Feng, M. Chen, B. Xue, S. Sankaranarayanan, N. Zhan : *Unbounded-time safety verification of stochastic differential dynamics*. CAV'20.

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$$P(\exists t \geq 0\colon \tilde{X}_t \in \mathcal{X}_u) \leq \underbrace{P(\exists t \in [0, T]\colon \tilde{X}_t \in \mathcal{X}_u)}_{\text{Bounded by a \textit{time-dependent barrier certificate}}} + \underbrace{P(\exists t \geq T\colon \tilde{X}_t \in \mathcal{X}_u)}_{\text{Bounded by an \textit{exponential barrier certificate}}}.$$

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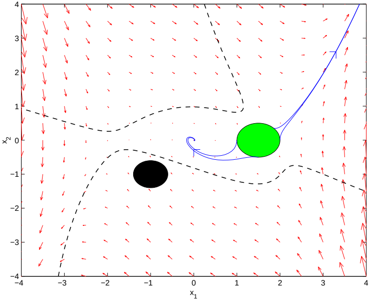
Recap : Barrier Certificate Witnesses ∞-Safety

$$B(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \mathcal{X}_u,$$
$$B(\mathbf{x}) \leq 0 \quad \forall \mathbf{x} \in \mathcal{X}_0,$$
$$\frac{\partial B}{\partial \mathbf{x}}(\mathbf{x}) b(\mathbf{x}) < 0 \quad \forall \mathbf{x} \in \partial B.$$



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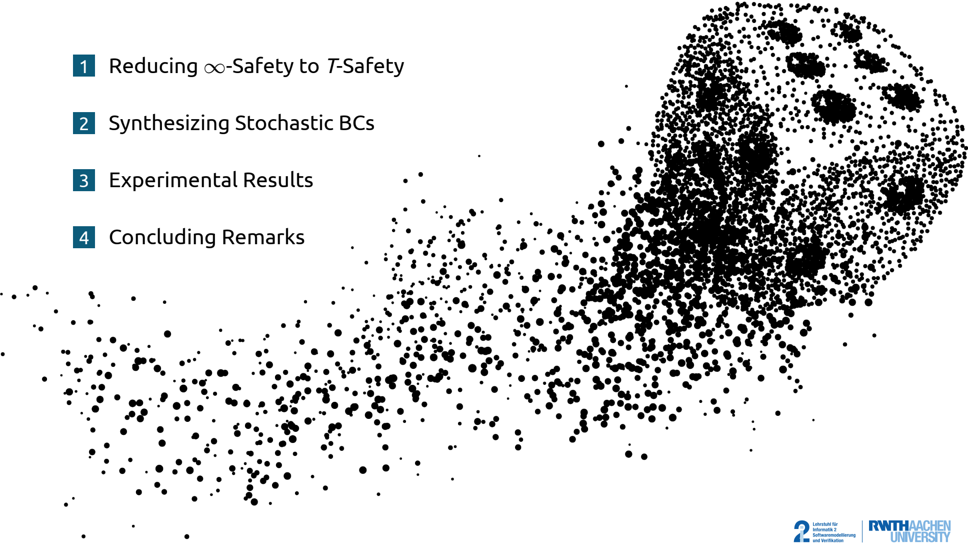
Outline

1 Reducing ∞-Safety to T-Safety

2 Synthesizing Stochastic BCs

3 Experimental Results

4 Concluding Remarks



Infinitesimal Generator

Definition (Infinitesimal generator [Øksendal, 2013])

Let $\{X_t\}$ be a diffusion process in $\mathbb{R}^n$. The *infinitesimal generator* $\mathcal{A}$ *of* X_t is defined by

$$\mathcal{A}f(s, \mathbf{x}) = \lim_{t \downarrow 0} \frac{E^{s,\mathbf{x}}\left[f(s+t, X_t)\right] - f(s, \mathbf{x})}{t}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Let $\mathcal{D}_{\mathcal{A}}$ denote the set of functions for which the limit exists for all $(s, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$.

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Let $\mathcal{D}_\mathcal{A}$ denote the set of functions for which the limit exists for all $(s,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$.

Lemma ([Øksendal, 2013])

Let $\{X_t\}$ be a diffusion process defined by an SDE. If $f \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ with compact support, then $f \in \mathcal{D}_\mathcal{A}$ and

$$\mathcal{A}f(t,\mathbf{x}) = \frac{\partial f}{\partial t} + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j} (\sigma\sigma^\mathsf{T})_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

$\mathcal{A}f(t,\mathbf{x})$ generalizes the *Lie derivative* that captures the evolution of $f(t,\mathbf{x})$ along X_t.

Exponential Stochastic Barrier Certificate

Theorem

Suppose there exists an essentially non-negative matrix $\Lambda \in \mathbb{R}^{m\times m}$*, together with an m-dimensional polynomial function (termed* exponential stochastic barrier certificate*)* $V(\mathbf{x}) = (V_1(\mathbf{x}), V_2(\mathbf{x}), \ldots, V_m(\mathbf{x}))^{\mathsf{T}}$*, with* $V_i\colon \mathbb{R}^n \to \mathbb{R}$ *for* $1 \leq i \leq m$*, satisfying*

$$V(\mathbf{x}) \geq \mathbf{0} \quad \textit{for}\ \mathbf{x} \in \mathcal{X}, \tag{1}$$

$$\mathcal{A}V(\mathbf{x}) \leq -\Lambda V(\mathbf{x}) \quad \textit{for}\ \mathbf{x} \in \mathcal{X}, \tag{2}$$

$$\Lambda V(\mathbf{x}) \leq \mathbf{0} \quad \textit{for}\ \mathbf{x} \in \partial\mathcal{X}. \tag{3}$$

Define a function

$$F(t, \mathbf{x}) \mathrel{\widehat{=}} \mathrm{e}^{\Lambda t} V(\mathbf{x}),$$

then every component of $F(t, \tilde{X}_t)$ *is a supermartingale.*

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Proof

Based on Dynkin's formula [Dynkin, 1965] and Fatou's lemma.

Doob's Supermartingale Inequality

Lemma (Doob's supermartingale inequality [Karatzas and Shreve, 2014]**)**

Let $\{X_t\}_{t>0}$ *be a right continuous non-negative supermartingale adapted to a filtration* $\{\mathcal{F}_t \mid t > 0\}$. *Then for any* $\lambda > 0$,

$$\lambda P\left(\sup_{t\geq 0} X_t \geq \lambda\right) \leq E[X_0].$$

A bound on the probability that a non-negative supermartingale exceeds some given value over a given time interval.

Exponentially Decreasing Bound on the Tail Failure Probability

For cases where $V(\mathbf{x})$ is a scalar function [1]:

Proposition

Suppose there exists a positive constant $\Lambda \in \mathbb{R}$ and a scalar exponential stochastic barrier certificate $V\colon \mathbb{R}^n \to \mathbb{R}$. Then,

$$P\left(\sup_{t \geq T} V\left(\tilde{X}_t\right) \geq \gamma\right) \leq \frac{E\left[V(X_0)\right]}{\mathrm{e}^{\Lambda T}\gamma}$$

holds for any $\gamma > 0$ and $T \geq 0$. Moreover, if there exists $l > 0$ s.t.

$$V(\mathbf{x}) \geq l \quad \textit{for all } \mathbf{x} \in \mathcal{X}_u,$$

then

$$P\left(\exists t \geq T\colon \tilde{X}_t \in \mathcal{X}_u\right) \leq \frac{E[V(X_0)]}{\mathrm{e}^{\Lambda T} l}$$

holds for any $T \geq 0$.

[1] The result generalizes to the slightly more involved case where $V(\mathbf{x})$ is a vector function.

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holds for any $T \geq 0$.

Proof

Based on Doob's supermartingale inequality.

[1] The result generalizes to the slightly more involved case where $V(\mathbf{x})$ is a vector function.

Exponentially Decreasing Bound on the Tail Failure Probability

$\forall \epsilon > 0. \exists \tilde{T} \geq 0\colon$ the truncated $\tilde{T}$-tail failure probability is bounded by ϵ :

Theorem

If there exists $\alpha > 0$, s.t. $\forall \mathbf{x} \in \mathcal{X}_0\colon V_i(\mathbf{x}) \leq \alpha$ holds for some $i \in \{1, \ldots, m\}$. Then for any $\epsilon > 0$, there exists $\tilde{T} \geq 0$ s.t.

$$P\left(\exists t \geq \tilde{T}\colon \tilde{X}_t \in \mathcal{X}_u\right) \leq \epsilon.$$

Time-Dependent Stochastic Barrier Certificate

Theorem

Suppose there exists a constant $\eta > 0$ *and a polynomial function (termed* time-dependent stochastic barrier certificate*)* $H(t, \mathbf{x}) \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, *satisfying*

$$H(t, \mathbf{x}) \geq 0 \quad \textit{for } (t, \mathbf{x}) \in [0, T] \times \mathcal{X}, \tag{4}$$

$$\mathcal{A}H(t, \mathbf{x}) \leq 0 \quad \textit{for } (t, \mathbf{x}) \in [0, T] \times (\mathcal{X} \setminus \mathcal{X}_u), \tag{5}$$

$$\frac{\partial H}{\partial t} \leq 0 \quad \textit{for } (t, \mathbf{x}) \in [0, T] \times \partial\mathcal{X}, \tag{6}$$

$$H(t, \mathbf{x}) \geq \eta \quad \textit{for } (t, \mathbf{x}) \in [0, T] \times \mathcal{X}_u. \tag{7}$$

Then,

$$P\left(\exists t \in [0, T] \colon \tilde{X}_t \in \mathcal{X}_u\right) \leq \frac{E[H(0, X_0)]}{\eta}.$$

Time-Dependent Stochastic Barrier Certificate

Theorem

Suppose there exists a constant $\eta > 0$ and a polynomial function (termed time-dependent stochastic barrier certificate*) $H(t, \mathbf{x}) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, satisfying*

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$$H(t, \mathbf{x}) \geq \eta \quad \textit{for } (t, \mathbf{x}) \in [0, T] \times \mathcal{X}_u. \tag{7}$$

Then,

$$P\left(\exists t \in [0, T] : \tilde{X}_t \in \mathcal{X}_u\right) \leq \frac{E[H(0, X_0)]}{\eta}.$$

Proof

Based on Dynkin's formula and Doob's supermartingale inequality.

Time-Dependent Stochastic Barrier Certificate

Corollary

Suppose there exists $\beta > 0$*, s.t.* $H(0, \mathbf{x}) \leq \beta$ *for* $\mathbf{x} \in \mathcal{X}_0$*. Then,*

$$P\left(\exists t \in [0, T] \colon \tilde{X}_t \in \mathcal{X}_u\right) \leq \frac{\beta}{\eta}.$$

SDP Encoding for Synthesizing $V(\mathbf{x})$

$$\underset{a,\alpha}{\texttt{minimize}} \quad \alpha \tag{8}$$

$$\texttt{subject to} \quad V^a(\mathbf{x}) \geq \mathbf{0} \quad \text{for } \mathbf{x} \in \mathcal{X} \tag{9}$$

$$\mathcal{A}V^a(\mathbf{x}) \leq -\Lambda V^a(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathcal{X} \tag{10}$$

$$\Lambda V^a(\mathbf{x}) \leq \mathbf{0} \quad \text{for } \mathbf{x} \in \partial\mathcal{X} \tag{11}$$

$$V^a(\mathbf{x}) \geq \mathbf{1} \quad \text{for } \mathbf{x} \in \mathcal{X}_u \tag{12}$$

$$V^a(\mathbf{x}) \leq \alpha\mathbf{1} \quad \text{for } \mathbf{x} \in \mathcal{X}_0 \tag{13}$$

SDP Encoding for Synthesizing $H(t, \mathbf{x})$

$$\begin{aligned}
\underset{b,\beta}{\texttt{minimize}} \quad & \beta && (14)\\
\texttt{subject to} \quad & H^b(t,\mathbf{x}) \geq 0 \quad \text{for } (t,\mathbf{x}) \in [0,T] \times \mathcal{X} && (15)\\
& \mathcal{A}H^b(t,\mathbf{x}) \leq 0 \quad \text{for } (t,\mathbf{x}) \in [0,T] \times (\mathcal{X} \setminus \mathcal{X}_u) && (16)\\
& \frac{\partial H^b}{\partial t} \leq 0 \quad \text{for } (t,\mathbf{x}) \in [0,T] \times \partial\mathcal{X} && (17)\\
& H^b(t,\mathbf{x}) \geq 1 \quad \text{for } (t,\mathbf{x}) \in [0,T] \times \mathcal{X}_u && (18)\\
& H^b(0,\mathbf{x}) \leq \beta \quad \text{for } \mathbf{x} \in \mathcal{X}_0 && (19)
\end{aligned}$$

Example : Population Dynamics

Example (Population growth [Panik, 2017])

$$\mathrm{d}X_t = -X_t \,\mathrm{d}t + \sqrt{2}/2 X_t \,\mathrm{d}W_t.$$

∞-safety setting : $\mathcal{X} = \{\mathbf{x} \mid \mathbf{x} \geq 0\}, \mathcal{X}_0 = \{\mathbf{x} \mid \mathbf{x} = 1\}, \mathcal{X}_u = \{\mathbf{x} \mid \mathbf{x} \geq 2\}$.

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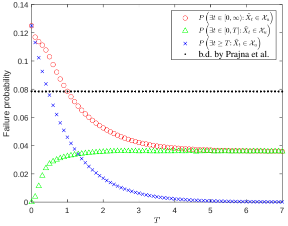
∞-safety setting : $\mathcal{X} = \{\mathbf{x} \mid \mathbf{x} \geq 0\}, \mathcal{X}_0 = \{\mathbf{x} \mid \mathbf{x} = 1\}, \mathcal{X}_u = \{\mathbf{x} \mid \mathbf{x} \geq 2\}.$

$$\begin{aligned} V(\mathbf{x}) &= 0.000001630047868 - 0.000048762786972\mathbf{x} \\ &\quad + 0.125025533525219\mathbf{x}^2 + 0.000000001603294\mathbf{x}^3. \end{aligned}$$

$$P\left(\exists t \geq T\colon \tilde{X}_t \in \mathcal{X}_u\right) \leq \frac{0.12498}{\mathrm{e}^T} \quad \forall T > 0.$$



Example : Harmonic Oscillator

Example (Harmonic oscillator [Hafstein et al., 2018])

$$\mathrm{d}X_t = \begin{pmatrix} 0 & \omega \\ -\omega & -k \end{pmatrix} X_t \,\mathrm{d}t + \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix} X_t \,\mathrm{d}W_t.$$

Constants : $\omega = 1, k = 7, \sigma = 2$.

∞-safety setting : $\mathcal{X} = \mathbb{R}^n$, $\mathcal{X}_0 = \{(x_1, x_2) \mid -1.2 \leq x_1 \leq 0.8, -0.6 \leq x_2 \leq 0.4\}$, $\mathcal{X}_u = \{(x_1, x_2) \mid |x_1| \geq 2\}$.

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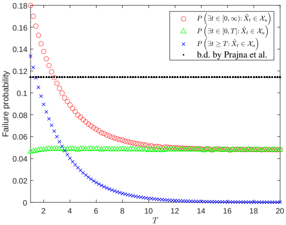
∞-safety setting : $\mathcal{X} = \mathbb{R}^n, \mathcal{X}_0 = \{(x_1, x_2) \mid -1.2 \leq x_1 \leq 0.8, -0.6 \leq x_2 \leq 0.4\},$
$\mathcal{X}_u = \{(x_1, x_2) \mid |x_1| \geq 2\}.$

$\forall T \geq 1:$

$$P\left(\exists t \geq T \colon \tilde{X}_t \in \mathcal{X}_u\right) \leq \frac{0.19927}{0.00005\mathrm{e}^{0.2T} + 1.00025\mathrm{e}^{0.4T}}.$$



Summary

For any $0 \leq T < \infty$,

$$P(\exists t \geq 0\colon \tilde{X}_t \in \mathcal{X}_u) \leq \underbrace{P(\exists t \in [0, T]\colon \tilde{X}_t \in \mathcal{X}_u)}_{\text{Bounded by a \textit{time-dependent barrier certificate}}} + \underbrace{P(\exists t \geq T\colon \tilde{X}_t \in \mathcal{X}_u)}_{\text{Bounded by an \textit{exponential barrier certificate}}}.$$

⇒ S. Feng, M. Chen, B. Xue, S. Sankaranarayanan, N. Zhan : *Unbounded-time safety verification of stochastic differential dynamics.* CAV '20.

SDEs with control inputs?

∞-Safety of Probabilistic Programs?

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