Analyzing volatility

GARCH MODELS IN R

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About the instructor

- Kris Boudt
  - PhD in financial risk forecasting
  - Use GARCH models to win by not losing (much)
- R package rugarch of Alexios Ghalanos.

Calculating returns

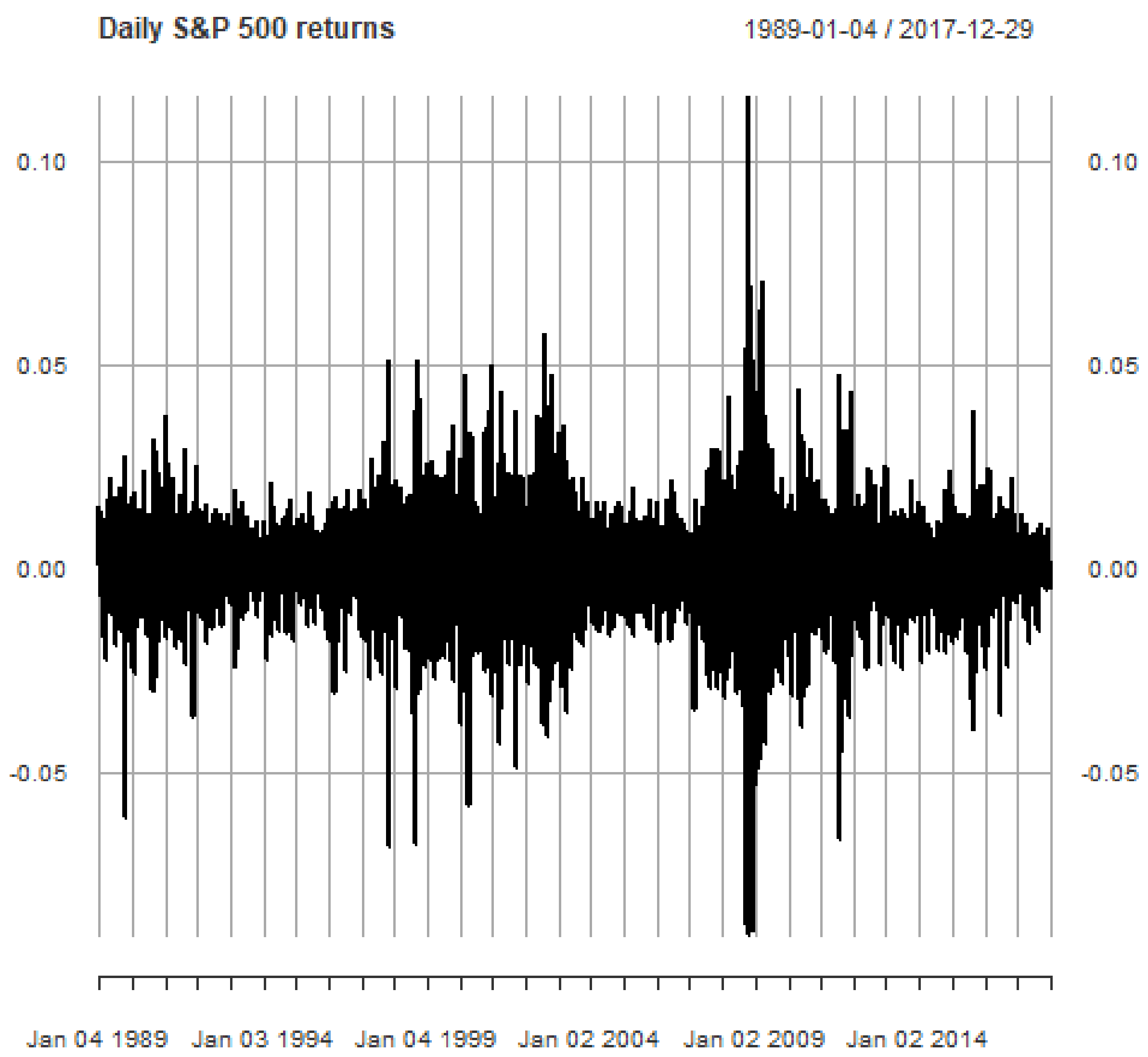
- Relative financial gains and losses, expressed in terms of returns

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

- Function `CalculateReturns` in `PerformanceAnalytics`

```
# Example in R for daily S&P 500 prices (xts object)
library(PerformanceAnalytics)
SP500returns <- CalculateReturns(SP500prices)
```

Daily S&P 500 returns



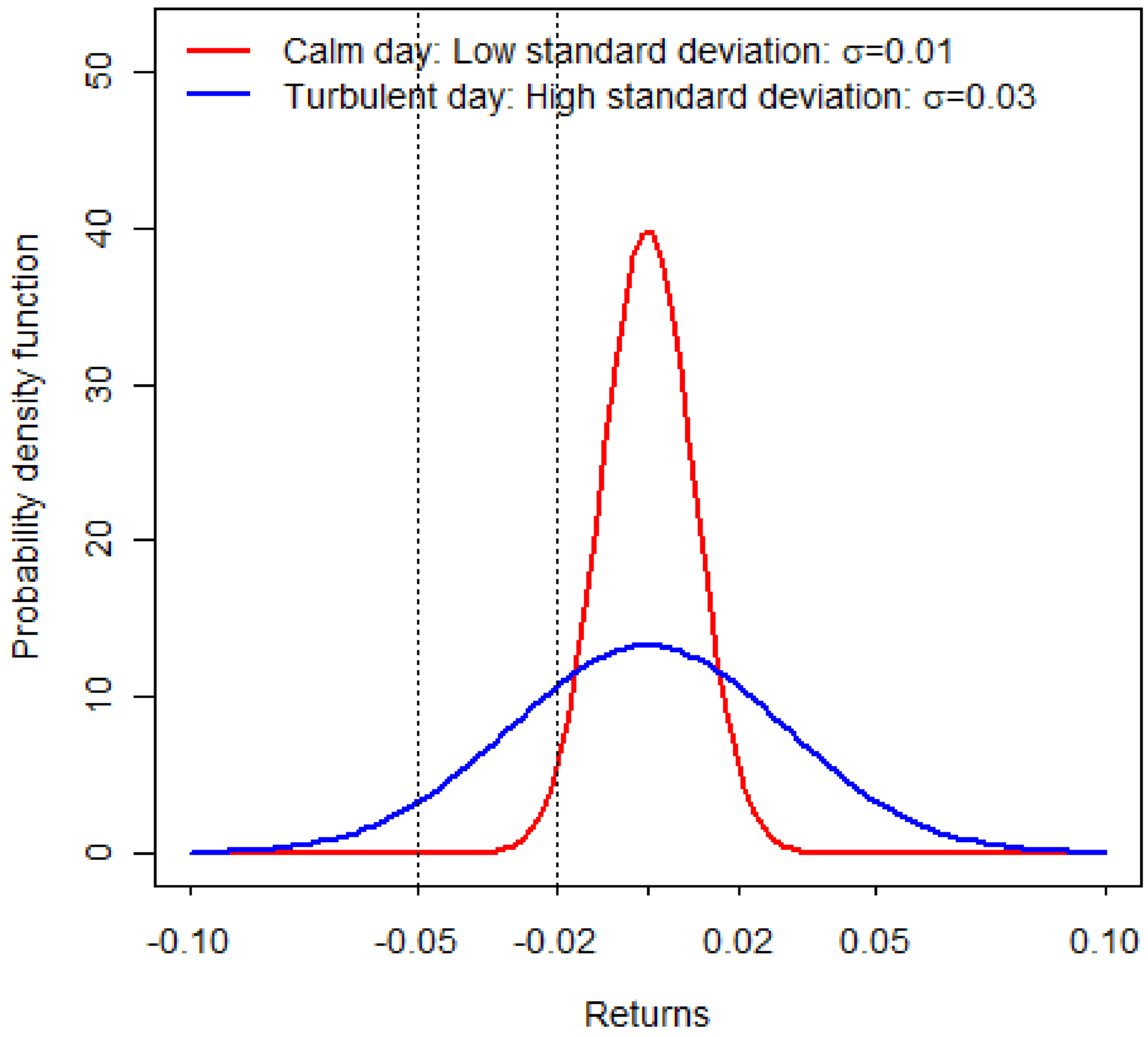
Properties of daily returns:

- The average return is zero
- Return variability changes through time

Standard deviation = measure of return variability.

Synonym: Return **volatility**.

Greek letter σ_t.


Calm day: Low standard deviation: σ=0.01
Turbulent day: High standard deviation: σ=0.03
Probability density function
0
10
20
30
40
50
-0.10
-0.05
-0.02
0.02
0.05
0.10
Returns

How to estimate return volatility

- Function `sd()` computes the (daily) standard deviation:

```
sd(sp500ret)
```

```
0.01099357
```

- Corresponding formula for $\hat{\sigma}$ computed using T daily returns:

$$\hat{\sigma} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(R_t - \hat{\mu})^2},$$

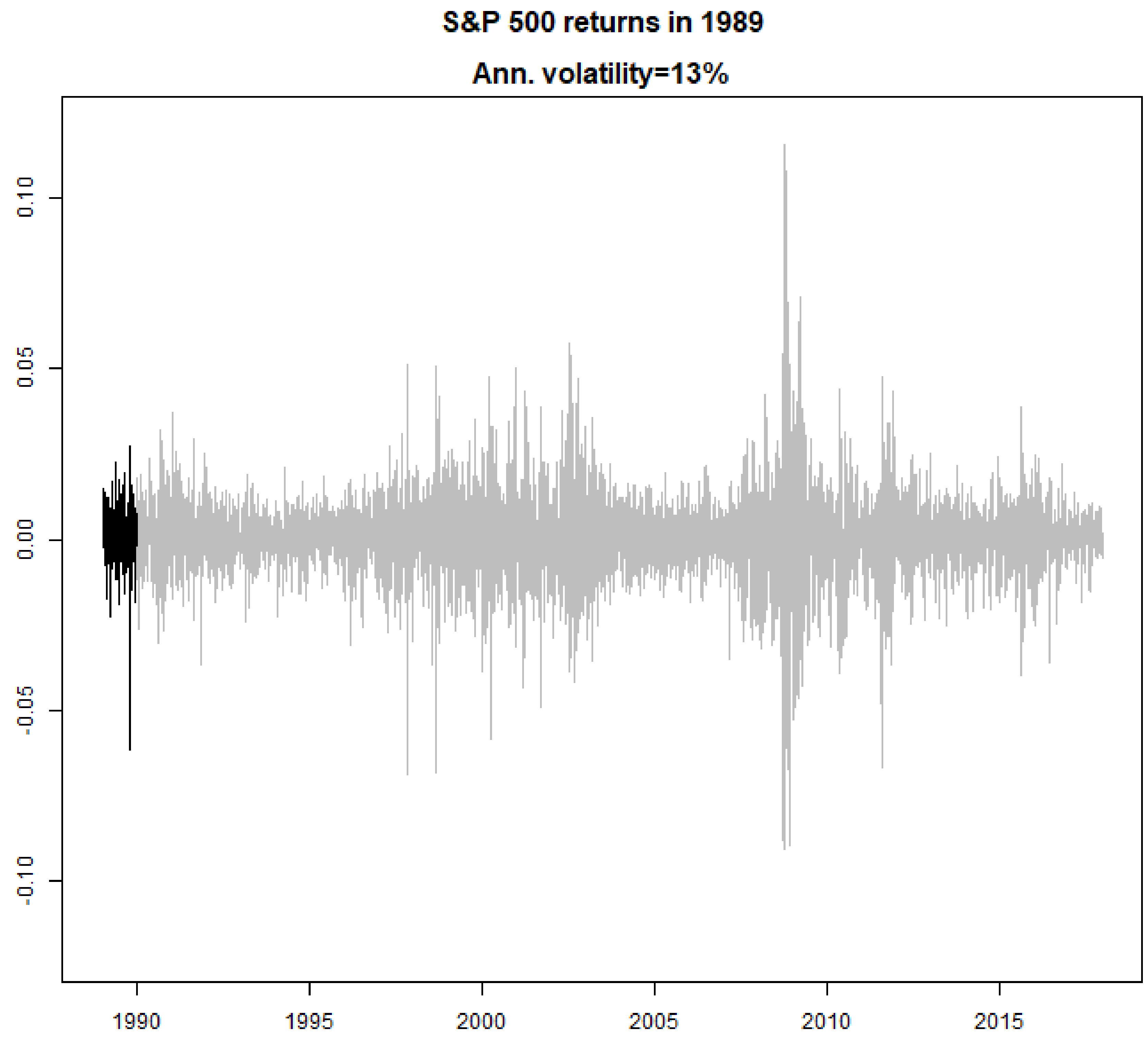
- where $\hat{\mu}$ is the mean return.

Annualized volatility

- `sd(sp500ret)` is daily volatility
- Annualized volatility = $\sqrt{252} \times$ daily volatility

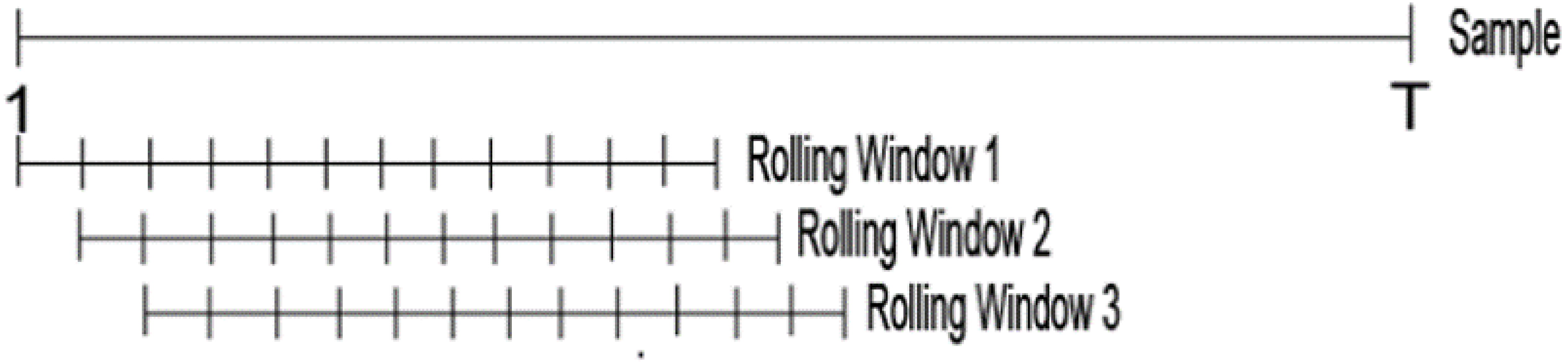
```
# Compute annualized standard deviation
sqrt(252) * sd(sp500ret)
```

```
0.1745175
```


S&P 500 returns in 1989
Ann. volatility=13%
0.10
0.05
0.00
-0.05
-0.10
1990
1995
2000
2005
2010
2015

Rolling volatility estimation

- Rolling estimation windows :

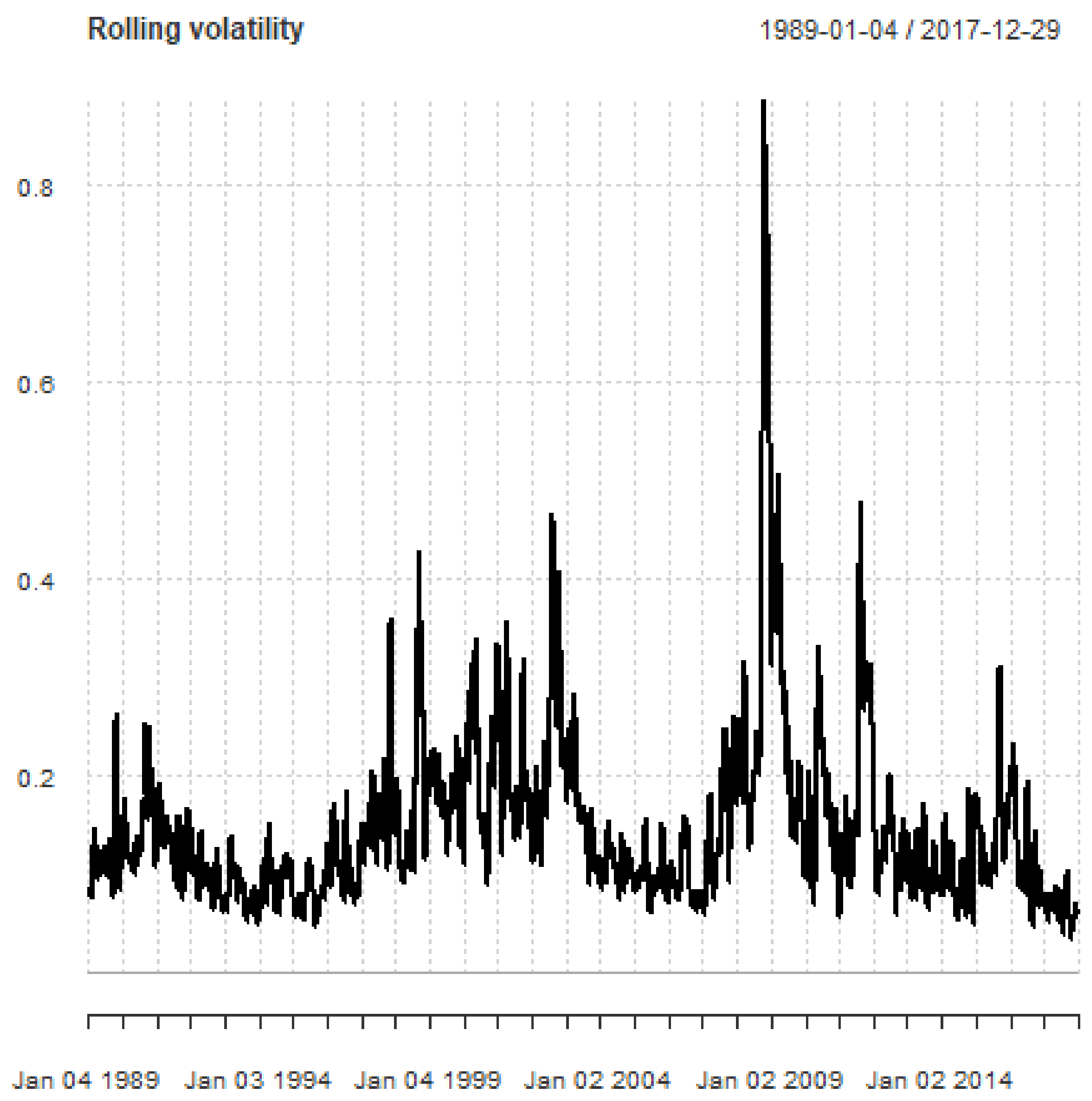


- Window width? Multiple of 22 (trading days).

Function chart.RollingPerformance()

```
library(PerformanceAnalytics)

chart.RollingPerformance(R = sp500ret,
                         width = 22,
                         FUN = "sd.annualized",
                         scale = 252,
                         main = "Rolling 1 month volatility")
```


Rolling volatility
1989-01-04 / 2017-12-29
0.8
0.6
0.4
0.2
Jan 04 1989
Jan 03 1994
Jan 04 1999
Jan 02 2004
Jan 02 2009
Jan 02 2014

About GARCH models in R

- Estimation of σ_t requires time series models, like GARCH.

Let's refresh the basics of computing rolling standard deviations in R

GARCH MODELS IN R

GARCH models: The way forward

GARCH MODELS IN R

Kris Boudt
Professor of finance and econometrics

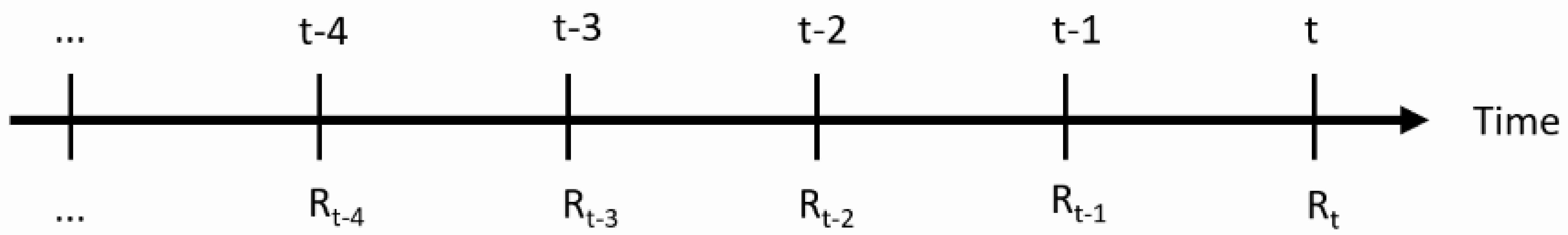
Inventors of GARCH models

Robert Engle

Tim Bollerslev

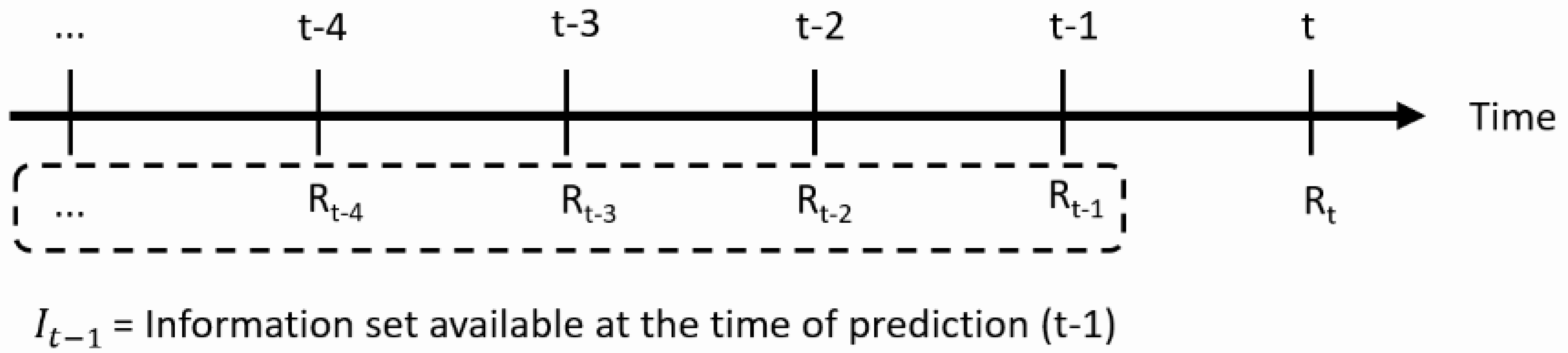
Notation (i)

- Input: Time series of returns



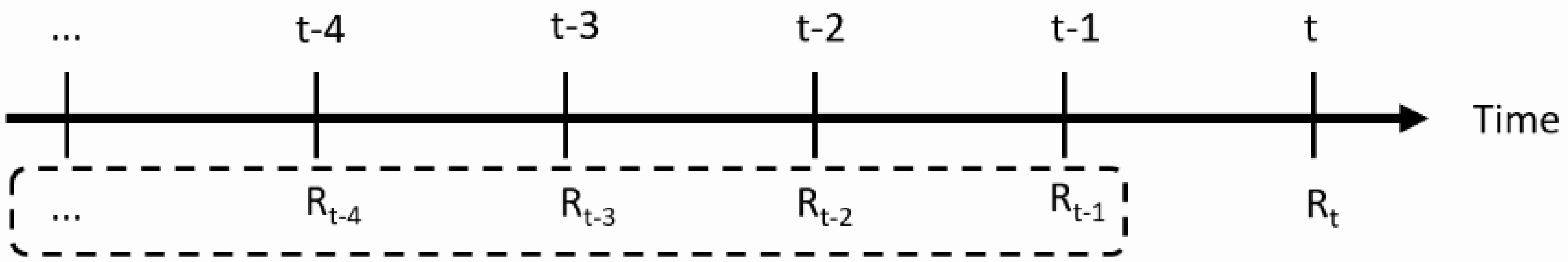
Notation (ii)

- At time $t-1$, you make the prediction about the the future return R_t, using the *information set* available at time $t-1$:



Notation (iii)

- Predicting the **mean return:** what is the best possible prediction of the actual return?



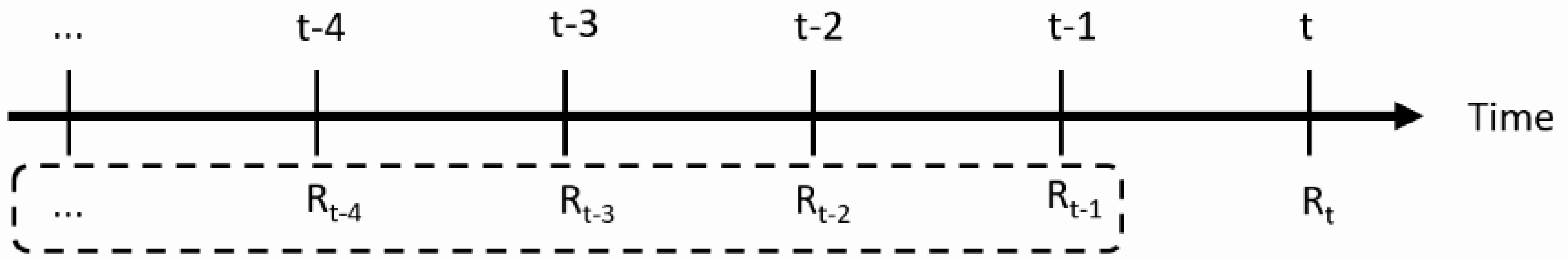
I_{t-1} = Information set available at the time of prediction (t-1)

$$\mu_t = E[R_t \mid I_{t-1}]$$

Prediction error: $e_t = R_t - \mu_t$

Notation (iv)

- We then predict the **variance:** how far off the return can be from its mean?

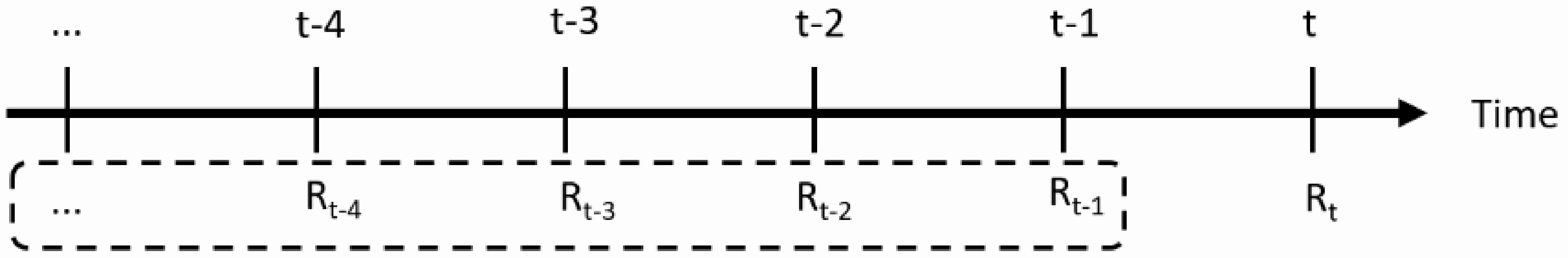


I_{t-1} = Information set available at the time of prediction (t-1)

$$\sigma_t^2 = var(R_t \mid I_{t-1})$$
$$= E[(R_t - \mu_t)^2 \mid I_{t-1}]$$
$$= E[e_t^2 \mid I_{t-1}]$$
$$\sigma_t = \sqrt{\sigma_t^2}$$

From theory to practice: Models for the mean

- We need an equation that maps the past returns into a prediction of the mean

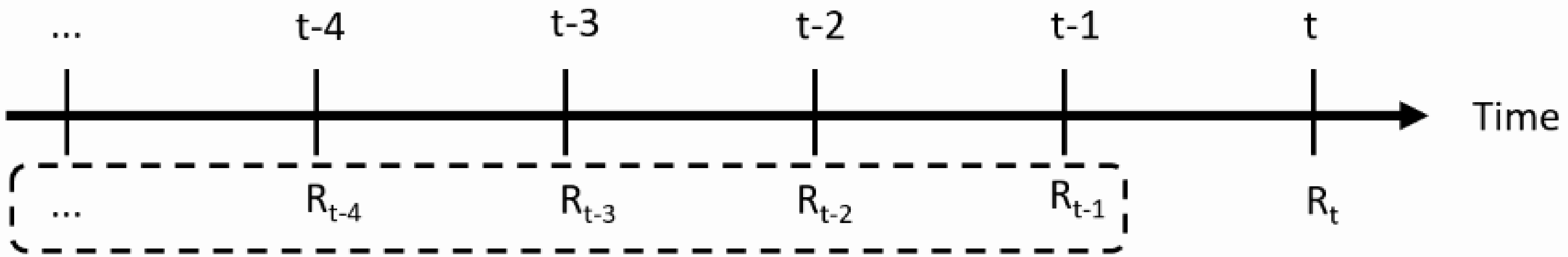


Rolling mean model: $\mu_t = \frac{1}{M}\sum_{i=1}^{M} R_{t-i}$

For AR(MA) models for the mean, see Datacamp course on time series analysis.

From theory to practice: Models for the variance

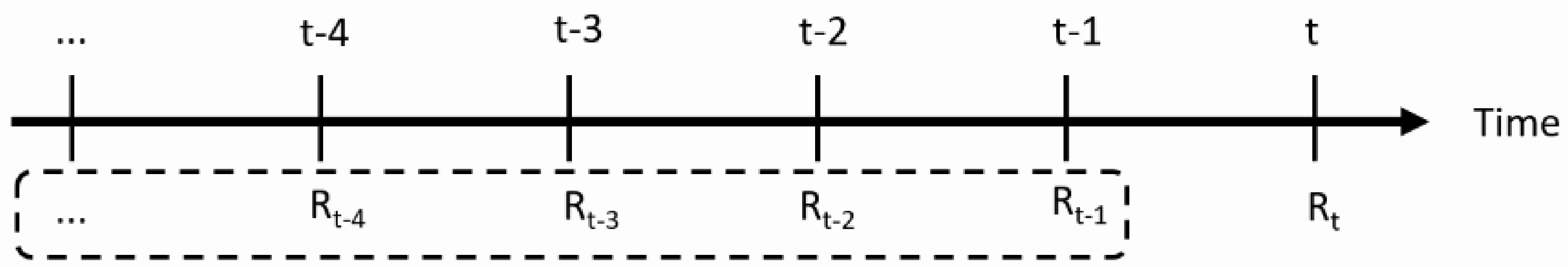
- We need an equation that maps the past returns into predictions of the variance



Rolling variance model: $\sigma_t^2 = \frac{1}{M}\sum_{i=1}^{M} e_{t-i}^2$

ARCH(p) model: Autoregressive Conditional Heteroscedasticity

- We need an equation that maps the past returns into predictions of the variance

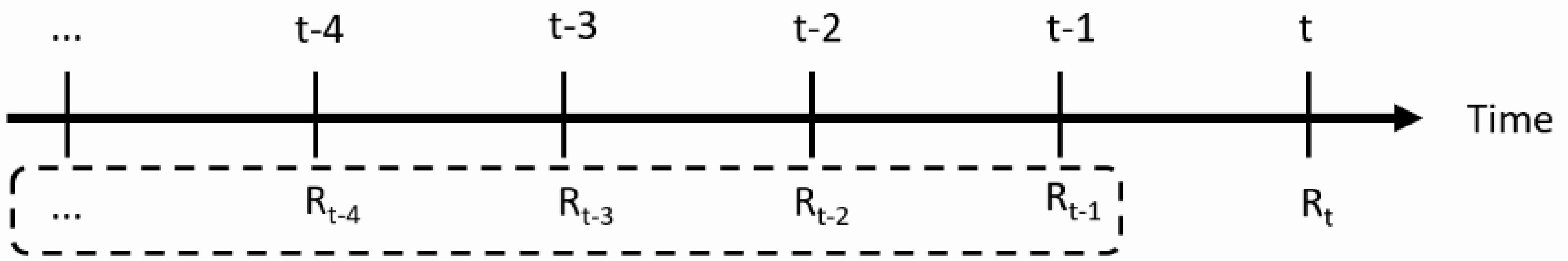


Rolling variance model: $\sigma_t^2 = \frac{1}{M}\sum_{i=1}^{M} e_{t-i}^2$

ARCH(p) model: $\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i e_{t-i}^2$

GARCH(1,1) model: Generalized ARCH

- We need an equation that maps the past returns into predictions of the variance



I_{t-1} = Information set available at the time of prediction (t-1)

$$\text{ARCH(p) model: } \sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i e_{t-i}^2$$

$$\text{GARCH(1,1) model: } \sigma_t^2 = \omega + \alpha e_{t-1}^2 + \beta \sigma_{t-1}^2$$

Parameter restrictions

To make the GARCH process realistic, we need that:

1. ω, α and β are > 0: this ensures that $\sigma_t^2 > 0$ at all times.
2. $\alpha + \beta < 1$: this ensures that the predicted variance σ_t^2 always returns to the long run variance:
   - The variance is therefore "mean-reverting"
   - The long run variance equals $\frac{\omega}{1-\alpha-\beta}$

R implementation - Specify the inputs

- Let's familiarize ourselves with the GARCH equations using R code:

$$\sigma_t^2 = \omega + \alpha e_{t-1}^2 \beta \sigma_{t-1}^2$$

```
# Set parameter values
alpha <- 0.1
beta <- 0.8
omega <- var(sp500ret) * (1 - alpha - beta)
# Then: var(sp500ret) = omega / (1 - alpha - beta)
```

```
# Set series of prediction error
e <- sp500ret - mean(sp500ret) # Constant mean
e2 <- e ^ 2
```

R implementation - compute predicted variances

```
# We predict for each observation its variance.
nobs <- length(sp500ret)
predvar <- rep(NA, nobs)
```

```
# Initialize the process at the sample variance
predvar[1] <- var(sp500ret)
```

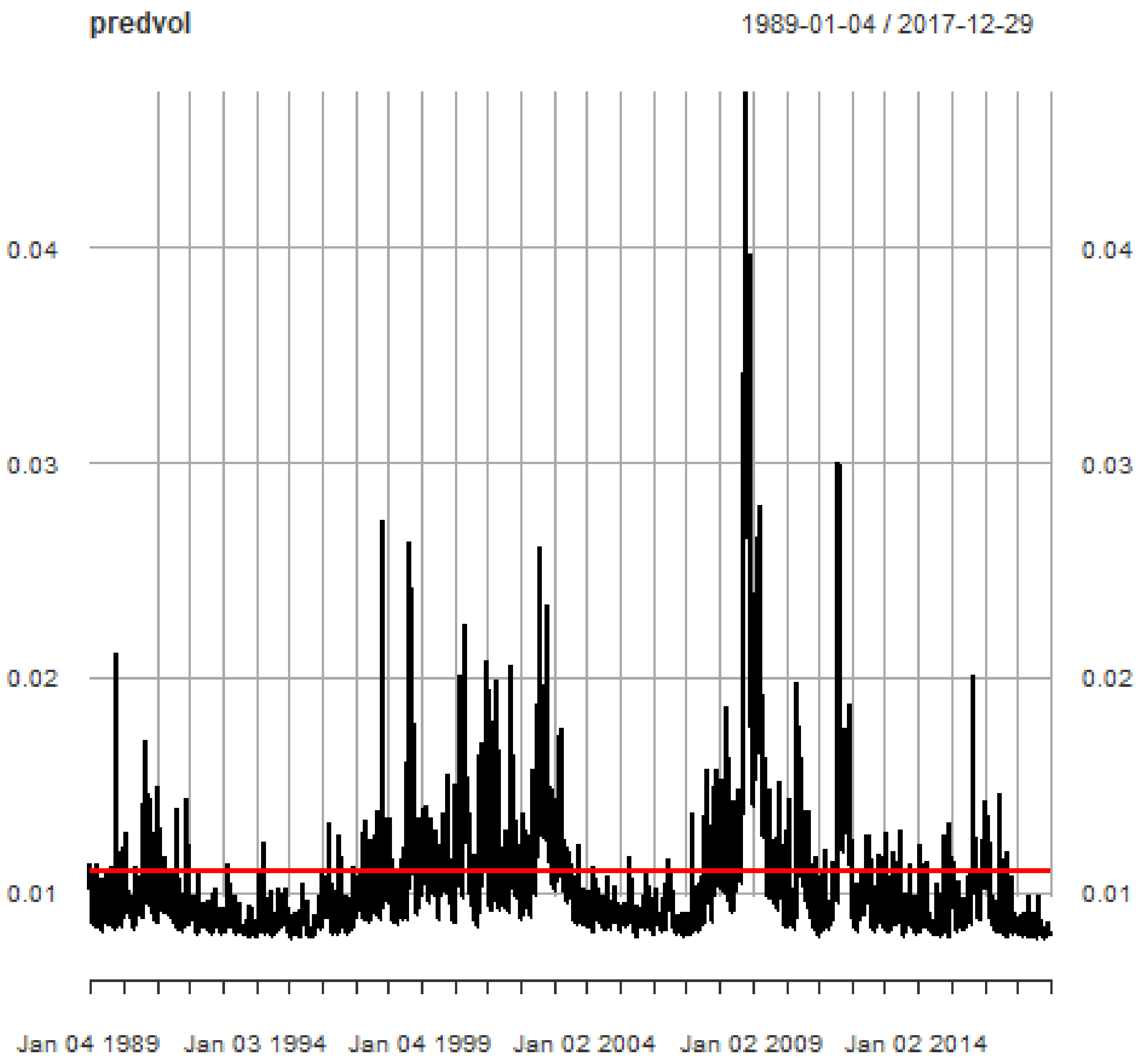
```
# Loop starting at 2 because of the lagged predictor
for (t in 2:nobs){
   predvar[t] <- omega + alpha * e2[t - 1] + beta * predvar[t-1]
}
```

R implementation - Plot of GARCH volatilities

```
# Volatility is sqrt of predicted variance
predvol <- sqrt(predvar)
predvol <- xts(predvol, order.by = time(sp500ret))
```

```
# We compare with the unconditional volatility
uncvol <- sqrt(omega / (1 - alpha-beta))
uncvol <- xts(rep(uncvol, nobs), order.by = time(sp500ret))
```

```
# Plot
plot(predvol)
lines(uncvol, col = "red", lwd = 2)
```


predvol
1989-01-04 / 2017-12-29
0.04
0.03
0.02
0.01
Jan 04 1989
Jan 03 1994
Jan 04 1999
Jan 02 2004
Jan 02 2009
Jan 02 2014

Let's practice!

GARCH MODELS IN R

The rugarch package

GARCH MODELS IN R

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The normal GARCH(1,1) model with constant mean

The normal GARCH model

$$R_t = \mu + e_t$$

$$e_t \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \omega + \alpha e_{t-1}^2 + \beta \sigma_{t-1}^2$$

- Four parameters: $\mu, \omega, \alpha, \beta$.
- Estimation by **maximum likelihood:** find the parameter values for which the GARCH model is most likely to have generated the observed return series.

Alexios Ghalanos

```
library(rugarch)
citation("rugarch")
To cite the rugarch package, please use:


Alexios Ghalanos (2018). rugarch: Univariate GARCH models.R package version 1.4-0.
```

Workflow

- Three steps:
  - `ugarchspec()` : Specify which GARCH model you want to use (mean μ_t, variance σ_t^2, distribution of e_t)
    - `ugarchfit()` : Estimate the GARCH model on your time series with returns $R_1, ..., R_T$.
    - `ugarchforecast()` : Use the estimated GARCH model to make volatility predictions for R_{T+1},...

Workflow in R

`ugarchspec()` specifies which GARCH model you want to use.

```
garchspec <- ugarchspec(mean.model = list(armaOrder = c(0,0)),
                         variance.model = list(model = "sGARCH"),
                         distribution.model = "norm")
```

`ugarchfit()` estimates the GARCH model.

```
garchfit <- ugarchfit(data = sp500ret, spec = garchspec)
```

`ugarchforecast()` forecasts the volatility of the future returns.

```
garchforecast <- ugarchforecast(fitORspec = garchfit, n.ahead = 5)
```

ugarchfit object

- The `ugarchfit` yields an object that contains all the results related to the estimation of the garch model.
- Methods `coef`, `uncvar`, `fitted` and `sigma`:

```
# Coefficients
garchcoef <- coef(garchfit)
# Unconditional variance
garchuncvar <- uncvariance(garchfit)
# Predicted mean
garchmean <- fitted(garchfit)
# Predicted volatilities
garchvol <- sigma(garchfit)
```

GARCH coefficients for daily S&P 500 returns

```
print(garchcoef)
```

```
          mu        omega       alpha1        beta1
5.728020e-04 1.220515e-06 7.792031e-02 9.111455e-01
```

$$R_t = 5.7 \times 10^{-4} + e_t$$

$$e_t \sim N(0, \hat{\sigma}_t^2)$$

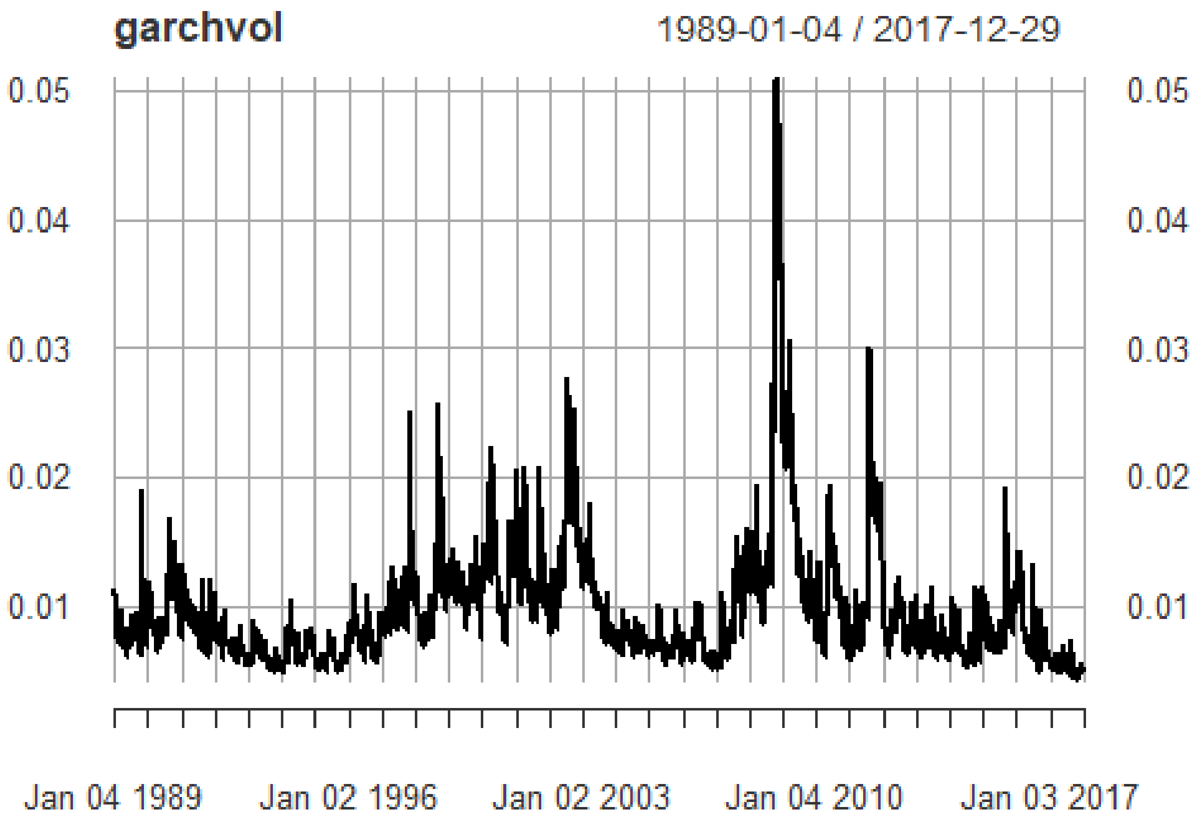
$$\hat{\sigma}_t^2 = 1.2 \times 10^{-6} + 0.08 e_{t-1}^2 + 0.91 \hat{\sigma}_{t-1}^2$$

```
sqrt(garchuncvar)
```

```
0.01056519
```

Estimated volatilities

```
garchvol <- sigma(garchfit)
plot(garchvol)
```



What about future volatility?

```
tail(garchvol, 1)
```

```
2017-12-29 0.004862908
```

What about the volatility for the days following the end of the time series?

Forecasting h-day ahead volatilities

- Applying the `sigma()` method to the `ugarchforecast` object gives the volatility forecasts:

```
sigma(garchforecast)
```

```
      2017-12-29
T+1  0.005034754
T+2  0.005127582
T+3  0.005217770
T+4  0.005305465
T+5  0.005390797
```

Forecasting h-day ahead volatilities

Applying the `fitted()` method to the `ugarchforecast` object gives the mean forecasts:

```
fitted(garchforecast)
```

```
      2017-12-29
T+1  0.000572802
T+2  0.000572802
T+3  0.000572802
T+4  0.000572802
T+5  0.000572802
```

Application to tactical asset allocation

A portfolio that invests a percentage w in a risky asset (with volatility σ_t) and keeps $1 - w$ on a risk-free bank deposit account has volatility equal to

$$\sigma_p = w\sigma_t$$

How to set w? One approach is **volatility targeting:** w is such that the predicted annualized portfolio volatility equals a target level, say 5%. Then:

$$w^* = 0.05/\sigma_t$$

Since GARCH volatilities change, the optimal weight changes as well.

Let's play with rugarch!

GARCH MODELS IN R