Introduction to exceptional points

- Proposed by Wiersig in 2014
- Degeneracy in non-hermitian Hamiltonian is called an exceptional point (EP)
- Typical systems like 2 coupled non-linear oscillators show EP's
- Hamiltonian (assume $\hbar = 1$)

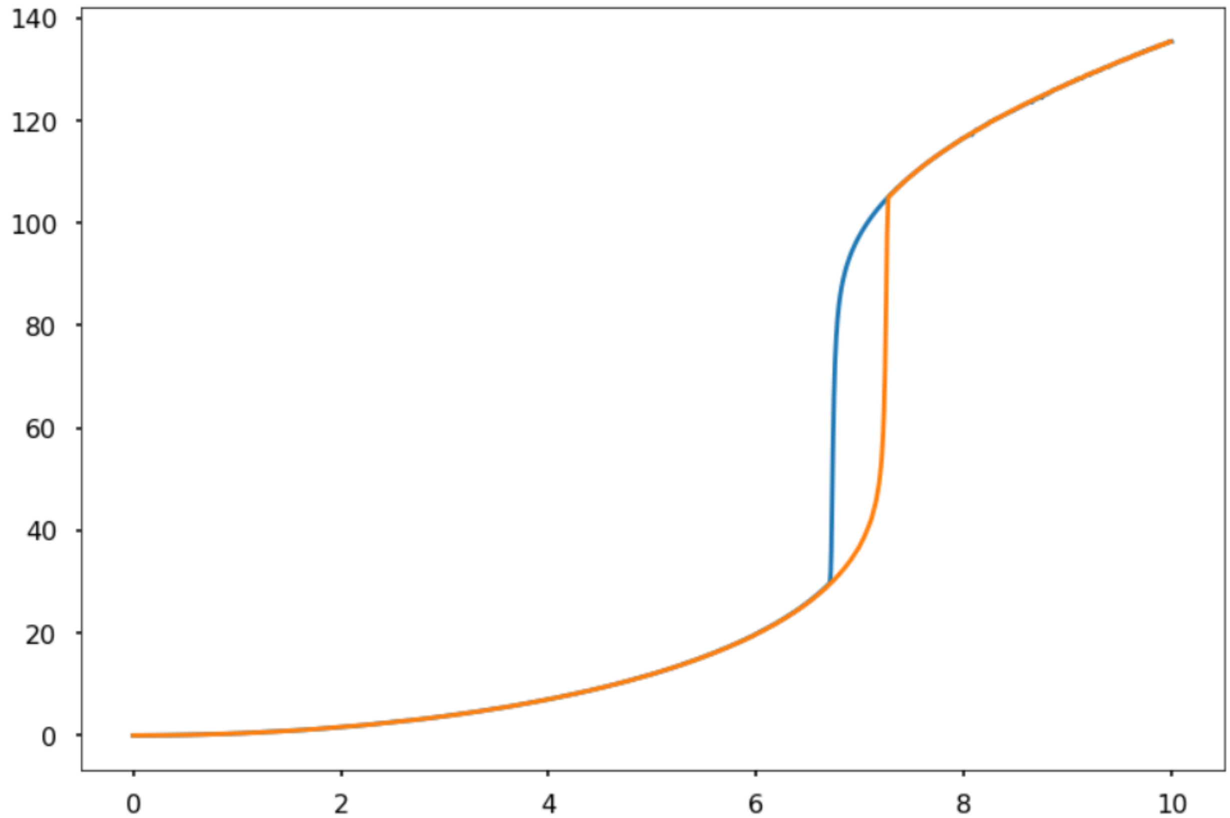
$$\begin{pmatrix} \omega_1 & g \\ g & \omega_2 \end{pmatrix}$$

- Eigenvalues: $\omega_{\pm} = \tilde{\omega}_{av} \pm \tilde{\Delta}/2\sqrt{1 + (\frac{2g}{\tilde{\Delta}})^2}$ where $w_{av} = (\tilde{\omega}_1 + \tilde{\omega}_2)/2$ and $\tilde{\Delta} = \tilde{\omega}_1 - \tilde{\omega}_2$
- Here, exceptional point occurs at $\frac{2g}{\Delta} = i$ (square root singularity)
- Around exceptional point perturbation of the form Re($\tilde{\omega}_i$)+ϵ results in a splitting Re($\omega_+ - \omega_-$) $\propto \sqrt{\epsilon}$ for small ϵ S
- Tradeoff: Highly sensitive to noise due to enhanced sensitivity

Alternative: Non-linear optical sensor

- Single mode coherently driven non-linear Kerr oscillator
- Measured signal: Splitting in transmitted intensities at 2 endpoints of hysteresis cycle

- N (y axis) vs $F/\sqrt{\Gamma}$ (x axis)



Non-Linear Optical sensor cont...

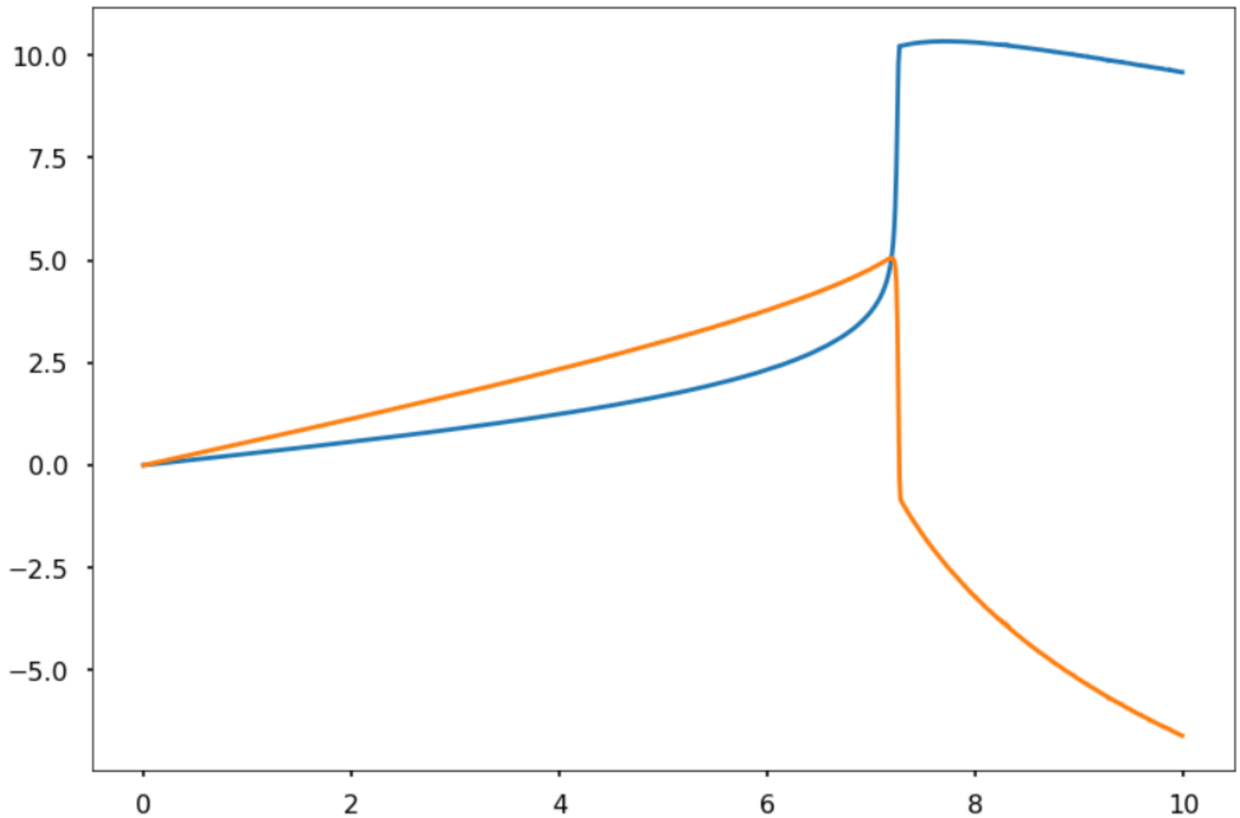
- Realised using a Fabry-Perot setup or any other architecture (whispering gallery, ring, photonic crystal etc.) having one mode spectrally distant from other modes and probes an intensity dependent refractive index
- In a frame rotating at driving frequency ω intracavity field α obeys following equation $i\dot{\alpha}(t) = -\Delta - \frac{i\Gamma}{2} + U(|\alpha(t)|^2 - 1)\alpha(t) + i\sqrt{\kappa_L}F + D\xi(t)$
  - α: Intracavity Field
  - Δ: Detuning $(\omega - \omega_0)$ where ω_0 is the resonance frequency
  - Γ: Total Loss=$\gamma + \kappa_L + \kappa_R$ where γ is intrinsic loss, κ_L, κ_R is loss through left and right ports respectively
  - U: Kerr non-linearity strength
  - F: Amplitude of input laser
  - D$\xi(t)$: Gaussian white noise with variance D^2
  - Assume D=0 for now

Solving for α

Setting α as a+ib where a and b are real, we transform the equation for $\dot{\alpha}$ to the following coupled non-linear differential equations.

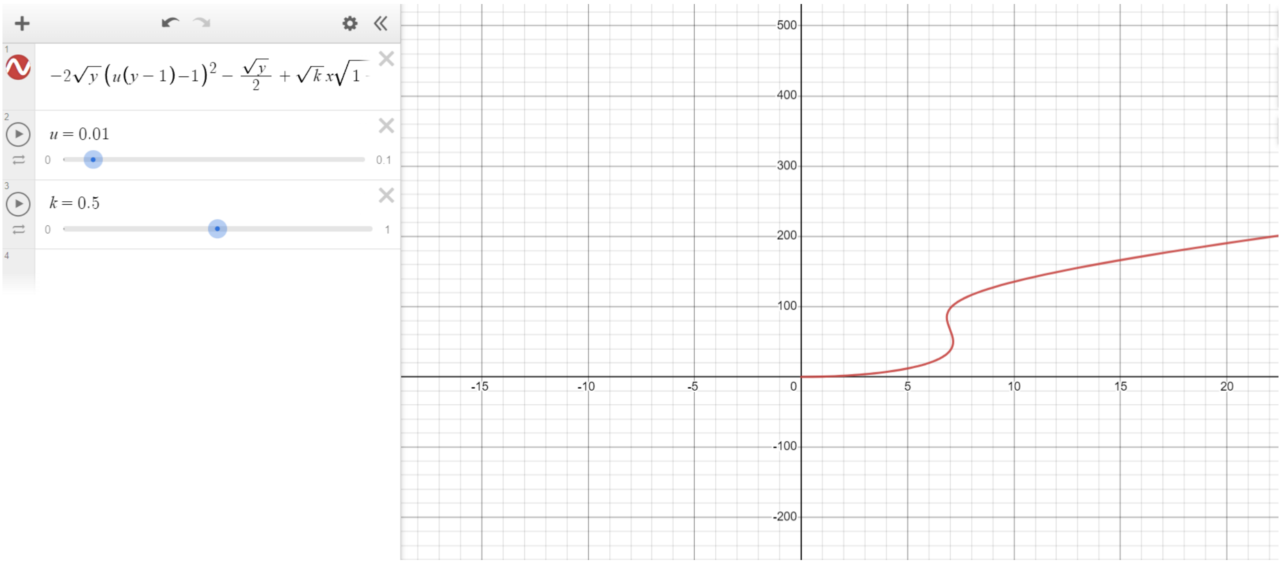
$$\dot{a} = (-\Delta + U(a^2 + b^2 - 1))b - \frac{a\Gamma}{2} + \sqrt{\kappa_L}F$$
$$\dot{b} = \frac{-b\Gamma}{2} - a(-\Delta + U(a^2 + b^2 - 1))$$

Hence we can see U represents the strength of non-linearity in this equation
(blue- variation of a (y axis) with time (x axis))
(orange- variation of b (y axis) with time (x axis))



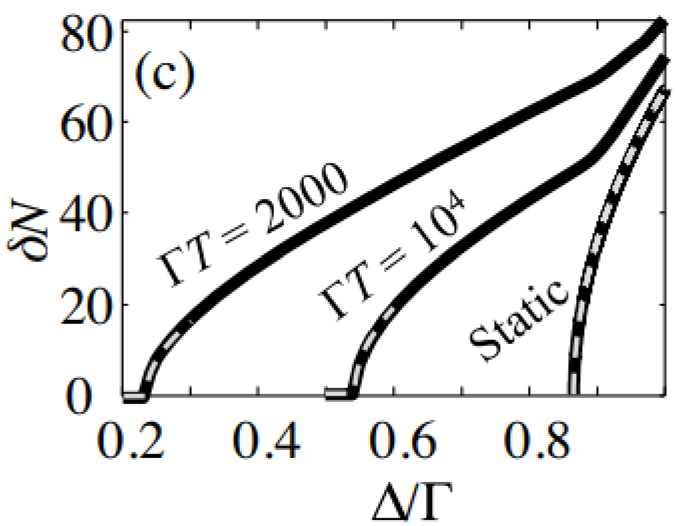
Steady state solutions

- Set $\dot{\alpha} = 0$ (steady state approximation)
- Also set $\frac{d|F|^2}{dN} = 0$ (turning points)
- Solutions are $\tilde{N}_{\pm} = \frac{2\Delta}{3U} \pm \frac{2\Delta}{6U}\sqrt{1 - (\frac{\sqrt{3}\Gamma}{2\Delta})^2}$ (notice similarity with EP sensors)
- https://www.desmos.com/calculator/zhxszbbxrs

- 

Properties

- Notice that the transmitted signal ($\delta N = N_+ - N-$) depends on Δ and measurement time.
- The dependence (as plotted by the authors) has been shown below

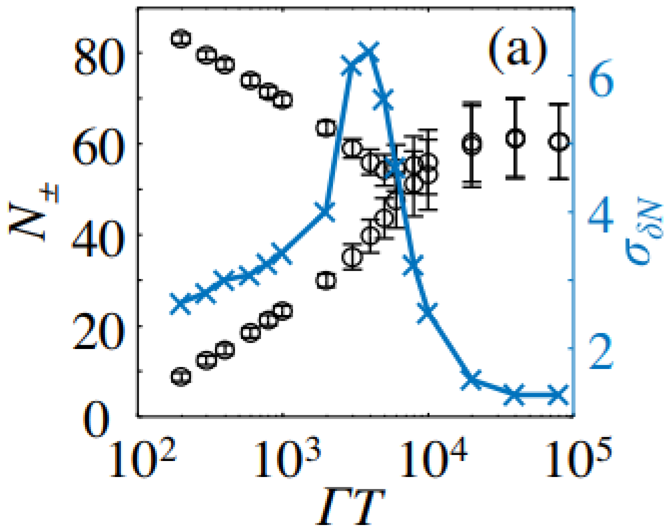


- Observe: For faster measurement time, less detuning needed for signal $\Rightarrow$ better sensitivity for faster measurement times.
- Dashed Lines: Square root fit near each singularity

Effects of Noise

- TIll now analysis was based on assumption D=0

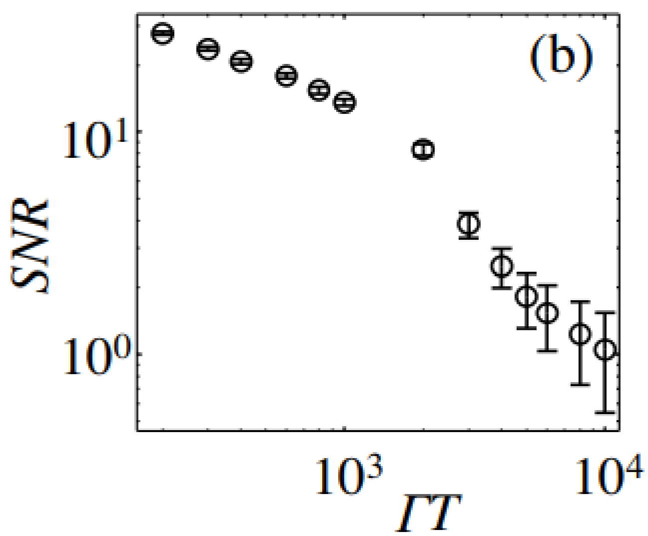
- WIth noise, solve numerically with different random seeds



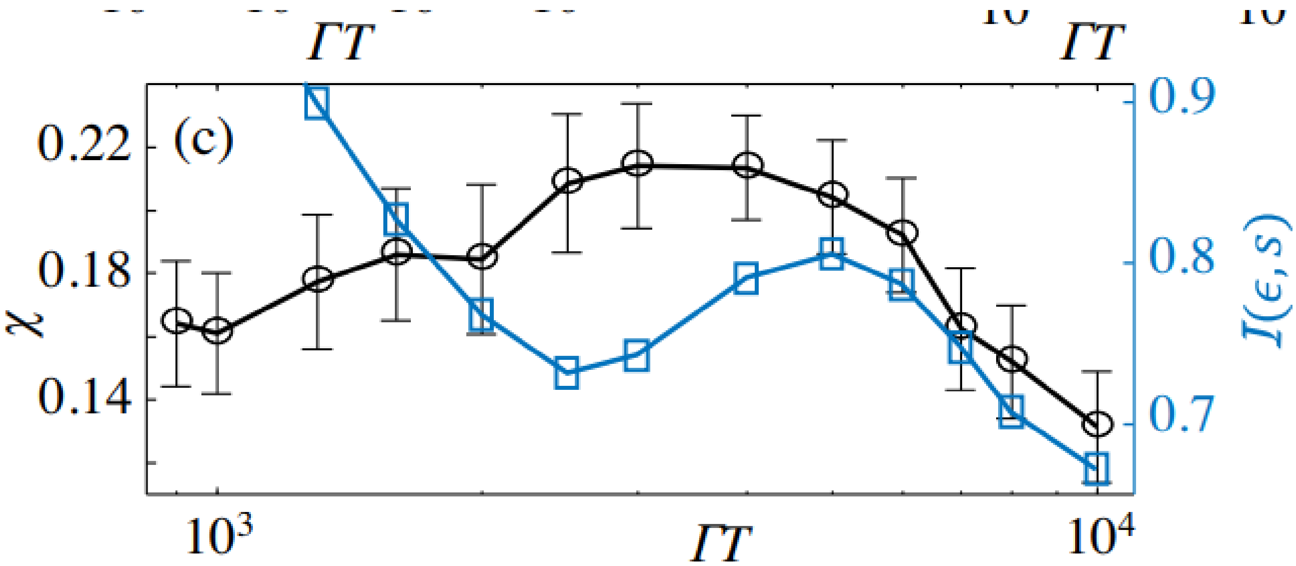
(plot from reference)

Parameters: $\Delta = 0.7\Gamma$

Advantages cont...



(plots from reference)



SNR: Signal to noise ratio defined as $\frac{\delta N}{\sigma_{\delta N}}$

χ : Precision figure of merit: $\frac{\delta \bar{N}_\epsilon - \delta \bar{N}_0}{\sigma_0 + \sigma_\epsilon}$; Quantifies the mean change in signal relative to measurement uncertainty

Alternative viewpoint: Information theory

- Mutual information between 2 random variables: Measure of certainty of a random variable given knowledge of the other variable
- $I(\epsilon; S) = \sum_{s \in S} \sum_{\epsilon \in E} p_{\epsilon, S}(\epsilon, s) log \frac{p_{(\epsilon, S)}(\epsilon, s)}{p_\epsilon(\epsilon) p_S(s)}$ (from KL divergence between $p_{(\epsilon, S)}(\epsilon, s)$ and $p_\epsilon p_S(s)$)
- KL divergence: Measure of similarity between to probability distributions P and Q
- Set $p_\epsilon(\epsilon)$ gaussian with mean $\Gamma/100$ and $\sigma = \Gamma/1000$ and calculate $p_s(s)$ numerically using detuning $\Delta_0 = 0.7\Gamma$. Then based on value of s we get the joint probability distribution
- Observe the enhancement of information content near T_{ss}
- Correlation between square root singularity, mutual information and precision.
- Results are general for $\epsilon << \Gamma$

Effects of averaging

- Smoothens effects of noise thus improving precision
- For same protocol F, it enhances sensitivity
- Tradeoff: Measurement time vs precision: averaging detrimental to fast sensing
- Non-trivial dependence of χ on number of cycles n.
- Peak followed by fall (unexpected) followed by gradual rise (expected)
  $\Rightarrow$ Possibility of model parameters having best of both worlds (speed and precision)