

# Cars 4 – Lightnight McQuation

Rhitt C

Feb 2026

## Abstract

Lightning McQueen, old and rusty, decides to officially retire from the racing scene, leaving Cruz to carry on his legacy.

In search of a new career, he thinks back to all the crazy stunts he'd pulled over the years and realises he has no idea how they were even possible physically. So he has a crazy idea: why not become a physicist and research stunt mechanics?

Who knows, maybe a strong understanding will lay the foundations for a future NASCAR training business ...

Note to self: learn simulink and experiment further with simulations

## 1 The Basics of Friction – Walking

## 2 First Paradigm – Skidding and Drifting

## 3 Second Paradigm – Banked Tracks

For ease of calculation, we deal with both friction down the slope (**Case 1**) and up the slope (**Case 2**) at once using  $(\pm)$  and  $(\mp)$ .

Throughout the calculations, the sign on top (i.e.  $(+)$  in  $(\pm)$  and  $(-)$  in  $(\mp)$ ) is for **Case 1** and the sign on the bottom is for **Case 2**.

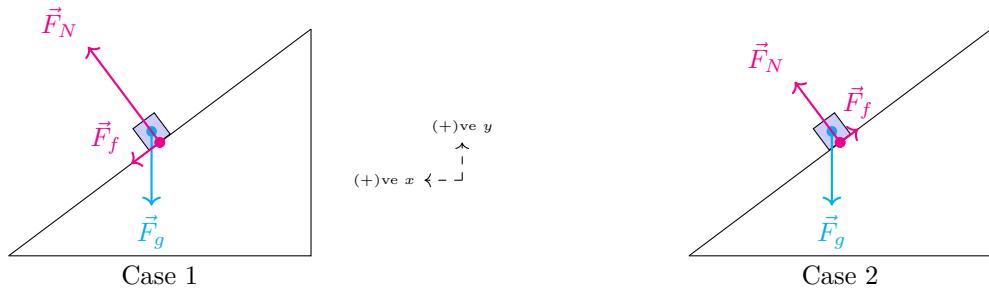
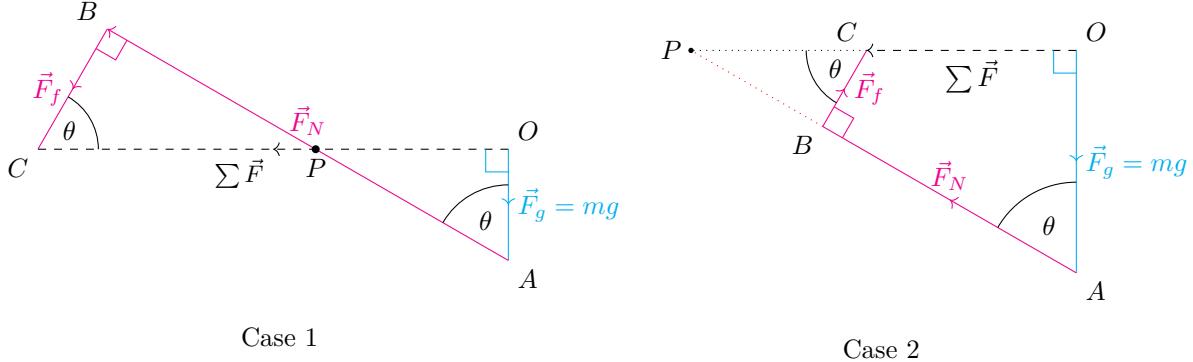


Figure 1: Toy Car on banked track with  $m = 0.1$ ,  $r = 2$ ,  $\theta = \arctan\left(\frac{3}{4}\right)$ ,  $\mu = 0.3$ ,  $g = 9.81$  SI units

### 3.1 Deriving a general formula for Friction and Normal forces

#### 3.1.1 Geometric Approach

We add the force vectors tip to tail as follows (using different parameters for better illustration):



For horizontal circular motion we require  $\sum \vec{F} = F_{cp} = m \left( \frac{v^2}{r} \right)$  horizontally in towards center of velodrome. Hence, from the diagrams:

$$\begin{aligned}
 |\sum \vec{F}| &= OP \pm PC \\
 &= (OA \tan \theta) \pm PC \quad \because \triangle OAP \\
 \implies m \frac{v^2}{r} &= mg \tan \theta \pm PC \\
 \therefore PC &= \pm m \left( \frac{v^2}{r} - g \tan \theta \right)
 \end{aligned} \tag{E1.1.1}$$

This  $PC$  in  $\triangle PBC$  now gives us the vital constraint on how far  $\vec{F}_N$  and  $\vec{F}_f$  can extend, and so we can now solve for them explicitly:

$$\begin{aligned}
 |F_N| &= AP \pm PB & |F_f| &= BC \\
 &= \left( \frac{OA}{\cos \theta} \right) \pm PC \sin \theta & \because \triangle OAP, \triangle PBC &= PC \cos \theta \quad \because \triangle PBC \\
 &= \frac{mg}{\cos \theta} + m \left( \frac{v^2}{r} \sin \theta - g \frac{\sin^2 \theta}{\cos \theta} \right) & \because (E1.1.1) &= \pm m \left( \frac{v^2}{r} \cos \theta - g \sin \theta \right) \quad \because E1.1.1 \\
 &= m \left( \frac{v^2}{r} \sin \theta + g \frac{1 - \sin^2 \theta}{\cos \theta} \right) \\
 &= m \left( \frac{v^2}{r} \sin \theta + g \cos \theta \right) \quad \because \sin^2 \theta + \cos^2 \theta = 1
 \end{aligned}$$

Thus, we conclude:

$$\vec{F}_N = m \left( \frac{v^2}{r} \sin \theta + g \cos \theta \right) \text{ up perpendicular to slope}$$

and splitting the answer for friction into the two cases:

$$\vec{F}_{f(1)} = m \left( \frac{v^2}{r} \cos \theta - g \sin \theta \right) \text{ down the slope} \quad \text{or} \quad \vec{F}_{f(2)} = m \left( g \sin \theta - \frac{v^2}{r} \cos \theta \right) \text{ up the slope}$$

### 3.1.2 Longer Algebraic Approach

The first constraint is that the car travels in a purely horizontal plane i.e. there is no vertical movement and so net vertical force on the car is  $\vec{0}$ .

Setting the upwards direction as positive, we have the following:

$$\begin{aligned} \sum \vec{F}_y &= \vec{0} \\ \vec{0} &= \vec{F}_g + \vec{F}_{fy} + \vec{F}_{Ny} \\ \vec{0} &= -mg \mp |F_f| \sin \theta + |F_N| \cos \theta \end{aligned} \quad (\text{E1})$$

$$(\text{E1}) \cdot \cos \theta \implies \vec{0} = -mg \cos \theta \mp |F_f| \sin \theta \cos \theta + |F_N| \cos^2 \theta \quad (\text{E2})$$

The second constraint is that the car undergoes uniform circular motion.

Setting the direction in towards the center of the velodrome to be positive, we must have positive centripetal force and hence:

$$\begin{aligned} \sum \vec{F}_x &= +m \frac{v^2}{r} \\ \vec{F}_{fx} + \vec{F}_{Nx} &= m \frac{v^2}{r} \\ \vec{0} &= m \frac{v^2}{r} - \vec{F}_{fx} - \vec{F}_{Nx} \\ \vec{0} &= m \frac{v^2}{r} \mp |F_f| \cos \theta - |F_N| \sin \theta \end{aligned} \quad (\text{E3})$$

$$(\text{E3}) \cdot \sin \theta \implies \vec{0} = m \frac{v^2}{r} \sin \theta \mp |F_f| \sin \theta \cos \theta - |F_N| \sin^2 \theta \quad (\text{E4})$$

Now we combine to obtain an equation involving  $F_f$  but not  $F_N$

$$\begin{aligned} (\text{E4}) - (\text{E2}) \implies \vec{0} - \vec{0} &= m \frac{v^2}{r} \sin \theta + mg \cos \theta + (\vec{0}) + |F_N| \cdot (-\sin^2 \theta - \cos^2 \theta) \\ \vec{0} &= m \frac{v^2}{r} \sin \theta + mg \cos \theta + |F_N| \cdot (-1) \\ |F_N| &= m \left( \frac{v^2}{r} \sin \theta + g \cos \theta \right) \end{aligned} \quad (\text{E5})$$

This gives us the normal force! Now, plugging into (E1), we find the friction force

$$\begin{aligned} (\text{E5}) \rightarrow (\text{E1}) \implies \vec{0} &= -mg \mp |F_f| \sin \theta + m \left( g \cos \theta + \frac{v^2}{r} \sin \theta \right) \cos \theta \\ \pm |F_f| \sin \theta &= m \left( \frac{v^2}{r} \sin \theta \cos \theta + g(-1 + \cos^2 \theta) \right) \\ &= m \left( \frac{v^2}{r} \sin \theta \cos \theta + g(-\sin^2 \theta) \right) \\ |F_f| &= \pm m \left( \frac{v^2}{r} \cos \theta - g \sin \theta \right) \end{aligned}$$

Summarising our results thus far, we have,

$$\vec{F}_N = m \left( \frac{v^2}{r} \sin \theta + g \cos \theta \right) \text{ up perpendicular to slope}$$

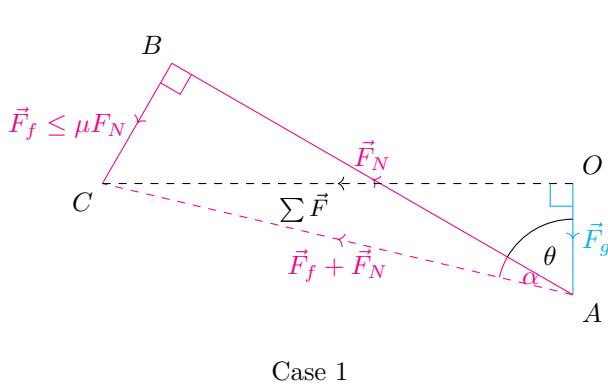
and splitting the answer for friction into the two cases:

$$\vec{F}_{f(1)} = m \left( \frac{v^2}{r} \cos \theta - g \sin \theta \right) \text{ down the slope} \quad \text{or} \quad \vec{F}_{f(2)} = m \left( g \sin \theta - \frac{v^2}{r} \cos \theta \right) \text{ up the slope}$$

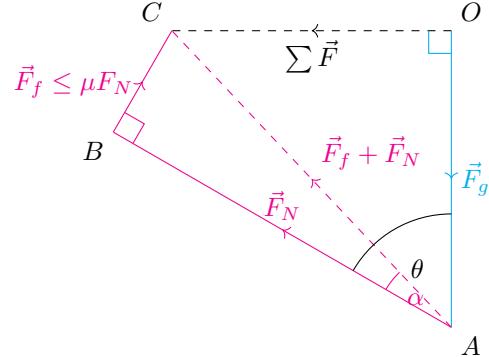
## 3.2 Deriving Range of Speeds

### 3.2.1 Geometric Approach

By noting that maximal static friction is proportional to the normal force, we can obtain the range of speeds explicitly.



Case 1



Case 2

As before, we combine the two cases in our calculations using  $(\pm)$ . From the diagrams, we have:

$$\begin{aligned} \tan(\theta \pm \alpha) &= \frac{|\sum \vec{F}|}{|F_g|} \quad \because \triangle OAC \\ |\sum \vec{F}| &= |F_g| \tan(\theta \pm \alpha) \\ m \frac{v^2}{r} &= mg \cdot \tan(\theta \pm \alpha) \\ \therefore |v| &= \sqrt{rg \cdot \tan(\theta \pm \alpha)} \end{aligned}$$

We now find the range of values for  $\alpha$ , which doesn't depend on the case so there is no  $(\pm)$ :

$$\begin{aligned} \tan \alpha &= \frac{|F|_f}{|F_N|} \\ &\leq \frac{\mu |F_N|}{|F_N|} \\ &\leq \mu \\ \therefore 0 \leq \alpha &\leq \arctan \mu \end{aligned}$$

When  $\alpha$  is at max,  $\tan(\theta + \alpha)$  is at max and  $\tan(\theta - \alpha)$  is at min. Thus, we conclude:

$$\sqrt{rg \cdot \tan(\theta - \arctan \mu)} \leq |v| \leq \sqrt{rg \cdot \tan(\theta + \arctan \mu)}$$

### 3.2.2 Algebraic Approach

This is significantly longer than the geometric approach and requires some more advanced knowledge to simplify the answer fully.

We now wish to find the maximum value of  $|v|$  that meets our constraints for horizontal circular motion. This occurs in **Case 1** with maximal  $|F_f|$ , as  $F_f$  has a maximal centre-seeking component, and so does  $F_N$  as it increases to balance out both friction and gravity. Thus, they both contribute as much as possible to the centripetal force. Hence, we have:

$$\begin{aligned}
 |F_{f(1)}| &\leq \mu|F_N| \\
 m \left( \frac{v^2}{r} \cos \theta - g \sin \theta \right) &\leq \mu m \left( \frac{v^2}{r} \sin \theta + g \cos \theta \right) \\
 \frac{v^2}{r} \cos \theta - g \sin \theta &\leq \mu \left( \frac{v^2}{r} \sin \theta + g \cos \theta \right) \\
 \frac{v^2}{r} (\cos \theta - \mu \sin \theta) &\leq g(\sin \theta + \mu \cos \theta) \\
 \frac{v^2}{r} &\leq g \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} \\
 |v| &\leq \sqrt{rg \left( \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} \right)}
 \end{aligned}$$

Similarly, to minimise  $|v|$ , we want  $F_f$  to be maximally directed up the slope (i.e. replace  $(\pm)$  with  $(-)$ ). The primary reason is that this leads to a maximal component away from the center of the velodrome. The vertical component also “lessens the burden” of  $F_N$  in opposing the constant  $F_g$  downwards, and thereby weakens  $F_N$  alongside its center-seeking component to a min. Both consequences minimise centripetal force and thus speed as required.

Thus,

$$\begin{aligned}
 |F_{f(2)}| &\leq \mu|F_N| \\
 m \left( g \sin \theta - \frac{v^2}{r} \cos \theta \right) &\leq \mu m \left( \frac{v^2}{r} \sin \theta + g \cos \theta \right) \\
 g(\sin \theta - \mu \cos \theta) &\leq \frac{v^2}{r} (\cos \theta + \mu \sin \theta) \\
 |v| &\geq \sqrt{rg \left( \frac{\sin \theta - \mu \cos \theta}{\cos \theta + \mu \sin \theta} \right)}
 \end{aligned}$$

Thus we have the full range of speeds for which static friction and the normal force can work to ensure there is horizontal circular motion:

$$\boxed{\sqrt{rg \left( \frac{\sin \theta - \mu \cos \theta}{\cos \theta + \mu \sin \theta} \right)} \leq |v| \leq \sqrt{rg \left( \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} \right)}}$$

Simplification requires knowledge of sum of sinusoids with same frequency:

$$\sqrt{rg \left( \frac{\sin \theta - \mu \cos \theta}{\cos \theta + \mu \sin \theta} \right)} \leq |v| \leq \sqrt{rg \left( \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} \right)}$$

$$\sqrt{rg \left( \frac{\left(\sqrt{1+\mu^2}\right) \sin(\theta - \arctan \mu)}{\left(\sqrt{1+\mu^2}\right) \cos(\theta - \arctan \mu)} \right)} \leq |v| \leq \sqrt{rg \left( \frac{\left(\sqrt{1+\mu^2}\right) \sin(\theta + \arctan \mu)}{\left(\sqrt{1+\mu^2}\right) \cos(\theta + \arctan \mu)} \right)}$$

$$\therefore \boxed{\sqrt{rg \cdot \tan(\theta - \arctan \mu)} \leq |v| \leq \sqrt{rg \cdot \tan(\theta + \arctan \mu)}}$$

## 4 New Business – Lightnight McQuation