Tao Analysis - My Visualisations and Proofs

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Contents

1 Abstract

This text is heavily inspired by Terrence Tao's Analysis Vol. I and II. However, I have taken many liberties in altering formulations to my liking and providing visuals. I'm also including my solutions to the exercises for future reference.

2 Construction of $\mathbb{N}$

2.1 Peano Axioms

We lay our assumed axioms under the Peano construction of $\mathbb{N} = \{0, 1, 2, \dots\}$ as follows, motivated by the physical intuition of "counting numbers". All later results, apart from definitions, must strictly follow logically from the Peano Axioms and ZFC Axioms (defined strictly in next chapter):

We start with the bare minimum by demanding that some fixed object, namely 0, be well defined:

Axiom P.1: Existence of 0

There exists an object 0 that is natural.

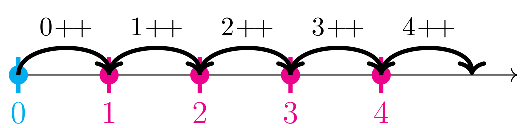
$$\exists 0 \quad s.t. \quad 0 \in \mathbb{N}$$

Now to populate $\mathbb{N}$, we introduce a new primitive operation (that will later by superseded by addition) called *succession*, with the property of closure. Informally, we hope that it behaves like our intuition for incrementing a number by 1, and thus forces the infinitely many "counting numbers" to now be well defined and in $\mathbb{N}$. Denote the succession of all $n \in \mathbb{N}$ as $n{+}{+}$, and label $(1 := 0{+}{+})$, $(2 := 1{+}{+})$, $(3 := 2{+}{+})$, $\dots$

Axiom P.2: Closure of $\mathbb{N}$ under $(++)$

All naturals have natural successions.

$$\forall n \in \mathbb{N} \quad n \in \mathbb{N} \implies (n{+}{+}) \in \mathbb{N}$$

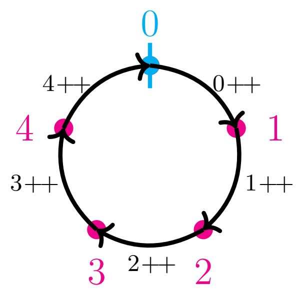


However, P.1 and P.2 alone are not strong enough to force $\mathbb{N}$ to populate with our infinitely many elements. In particular, they do not prevent existence of an $n \in \mathbb{N}$ with $n{+}{+} = 0$. And yet, informally speaking, this leads to a modulo cycle in which only $n+1$ elements exist in $\mathbb{N}$, and everything beyond that overflows back to 0, like in a computer. Modulo sets have many applications, but they are not what we desire as a full set of "counting numbers", particularly if that n turns out to be 0 itself, leading to a trivial $\mathbb{N} = \{0\}$. So we explicitly force $\mathbb{N}$ to not be such a modulo cycle by demanding that a natural's succession never overflows back to 0.

Axiom P.3: $\mathbb{N}$ is not a modulo cycle overflowing to 0

No natural's succession can be 0.

$$\forall n \in \mathbb{N} \quad (n{+}{+}) \neq 0$$

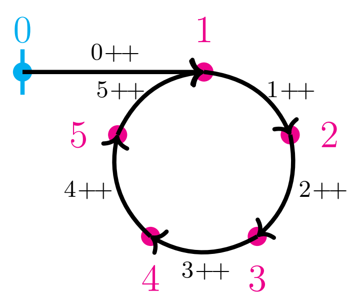


As it stands, the issue hasn't been fully resolved yet. There is still the pathological case of an *offset* modulo cycle, in which an $n \in \mathbb{N}$ overflows back so that $n \mapsto m{+}{+}$ even though we already have a previous $m \mapsto m{+}{+}$. We now complete our requirements on $({+}{+})$ by demanding no such dual mappings to the same succession i.e. we make $({+}{+})$ *injective*.

Axiom P.4: $\mathbb{N}$ has no otherwise offset modulo cycles either

Succession is injective (one-to-one). That is, distinct naturals have distinct successions.

$$\forall n, m \in \mathbb{N} \quad n \neq m \implies (n{+}{+}) \neq (m{+}{+})$$

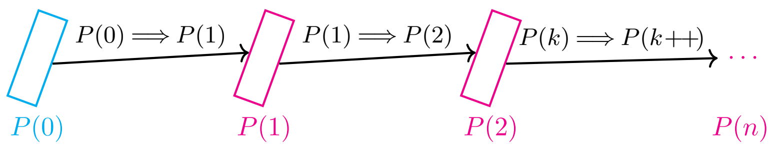


We now lay out by far the most useful axiom of $\mathbb{N}$ to handle the sheer infinite size of $\mathbb{N}$ recursively. It is induction.

Axiom P.5: (PMI) Principle of Mathematical Induction

If a proposition of naturals is met for 0 (Base Case), and it being satisfied for some natural implies it's met for the succession as well (Inductive Case), then as a domino effect, then the proposition is met for all naturals.

$$\forall P : \mathbb{N} \to \{\text{True}, \text{False}\} \qquad (P(0) \ \& \ (\forall k \in \mathbb{N}) P(k) \Longrightarrow P(k{+}{+})) \quad \Longrightarrow \quad \forall n \in \mathbb{N} \quad P(n)$$



Abstractly speaking, any other set X equipped with an analogous operation $S : X \to X$ that obeys the 5 Peano Axioms is said to be *isomorphic* with $\mathbb{N}$, denoted $X \cong \mathbb{N}$. Note that due to our later discussions only relying on logical implications from the axioms, which are met by X, the results we derive must also apply to X.

2.2 Addition

2.3 Multiplication