

Tao Analysis - My Visualisations and Proofs

Rhitt C

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1 Abstract

This text is heavily inspired by Terrence Tao's Analysis Vol. I and II. However, I have taken many liberties in altering formulations to my liking and providing visuals. I'm also including my solutions to the exercises for future reference.

2 Construction of \mathbb{N}

2.1 Peano Axioms

We lay our assumed axioms under the Peano construction of $\mathbb{N} = \{0, 1, 2, \dots\}$ as follows, motivated by the physical intuition of “counting numbers”. In doing so, we implicitly rely on some set theory, namely some of the ZFC axioms of sets¹. The Peano and ZFC Axioms are our *only* assumptions in all of Analysis. We shall see that this is sufficient for a very rich set of results about $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and some even more general statements² to follow directly by formal logic from these foundational axioms, given some definitions. In fact, almost all of modern maths relies on no more than the 8-9 ZFC and 5 Peano axioms. As I’m sure you’ll agree after seeing how intuitive the axioms really are, this is quite a remarkable development in modern maths.

We start with the bare minimum by demanding that some fixed object, namely 0, be well defined. 0 will be crucial to defining a few later axioms, as it gives us a sort of starting point.

Axiom P.1: Existence of 0

There exists an object 0 that is natural and can be used in later axioms.

$$\exists 0 \in \mathbb{N} \quad \text{s.t.} \quad \text{P.3 and P.5 are met.}$$

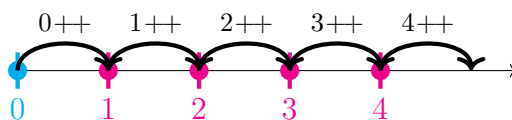


Now to populate \mathbb{N} , we introduce a new primitive operation (that will later be superseded by addition) called *succession*, with the property of closure. Informally, we hope that it behaves like our intuition for incrementing a number by 1, and thus forces the infinitely many “counting numbers” to now be well defined and in \mathbb{N} . Denote the succession of all $n \in \mathbb{N}$ as $n++$, and label $(1 := 0++)$, $(2 := 1++)$, $(3 := 2++)$, ...

Axiom P.2: Closure of \mathbb{N} under $(++)$

All naturals have natural successions.

$$\forall n \in \mathbb{N} \quad n \in \mathbb{N} \implies (n++) \in \mathbb{N}$$



¹ZFC is defined in the next chapter, so that the formulation of the Axiom of Infinity can be in terms of \mathbb{N} and is thus much more intuitive. Rest assured, while the Peano axioms refer to \mathbb{N} , they do not themselves make reference to the Axiom of Infinity, so there’s no circularity.

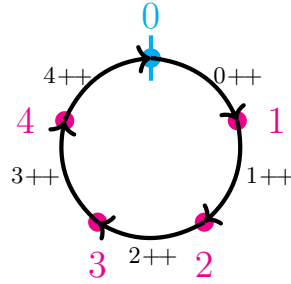
²All structures isomorphic to the ones mentioned, and also much later on, all metric spaces

However, P.1 and P.2 alone are not strong enough to force \mathbb{N} to populate with our infinitely many elements. In particular, they do not prevent existence of an $n \in \mathbb{N}$ with $n++ = 0$. And yet, informally speaking, this leads to a modulo cycle in which only $n + 1$ elements exist in \mathbb{N} , and everything beyond that overflows back to 0, like in a computer. Modulo sets have many applications, but they are not what we desire as a full set of “counting numbers”, particularly if that n turns out to be 0 itself, leading to a trivial $\mathbb{N} = \{0\}$. So we explicitly force \mathbb{N} to not be such a modulo cycle by demanding that a natural’s succession never overflows back to 0.

Axiom P.3: \mathbb{N} is not a modulo cycle overflowing to 0

No natural’s succession can be 0.

$$\forall n \in \mathbb{N} \quad (n++) \neq 0$$

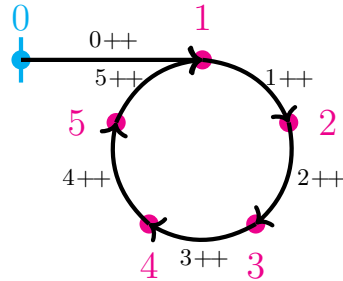


As it stands, the issue hasn’t been fully resolved yet. There is still the pathological case of an *offset* modulo cycle, in which an $n \in \mathbb{N}$ overflows back so that $n \mapsto m++$ even though we already have a previous $m \mapsto m++$. We now complete our requirements on $(++)$ by demanding no such dual mappings to the same succession i.e. we make $(++)$ *injective*.

Axiom P.4: \mathbb{N} has no otherwise offset modulo cycles either

Succession is injective (one-to-one). That is, distinct naturals have distinct successions.

$$\forall n, m \in \mathbb{N} \quad n \neq m \implies (n++) \neq (m++)$$

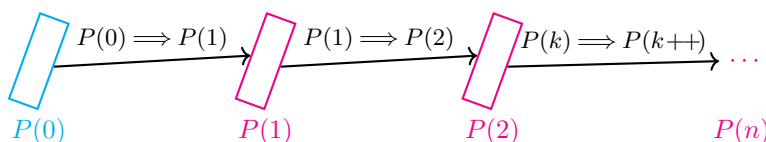


We now lay out by far the most useful axiom of \mathbb{N} to handle the sheer infinite size of \mathbb{N} recursively. It is induction.

Axiom P.5: (PMI) Principle of Mathematical Induction

If a proposition of naturals is met for 0 (Base Case), and it being satisfied for some natural implies it's met for the succession as well (Inductive Case), then as a domino effect, then the proposition is met for all naturals.

$$\forall P : \mathbb{N} \rightarrow \{\text{True}, \text{False}\} \quad (P(0) \ \& \ (\forall k \in \mathbb{N}) P(k) \implies P(k++)) \implies \forall n \in \mathbb{N} \quad P(n)$$



Abstractly speaking, all sets X equipped with an analogous operation $S : X \rightarrow X$ (referred to together as an algebraic structure (X, S)), that obeys the 5 Peano Axioms is said to be *isomorphic* with $(\mathbb{N}, (++)$), denoted $(X, S) \cong (\mathbb{N}, (++)$). Note that due to our later discussions only relying on logical implications from the axioms, which are met by (X, S) , the results we derive must apply to every such (X, S) , given we rewrite our definitions accordingly. As such, we do not strictly define a particular $(\mathbb{N}, (++)$), but rather refer abstractly to any such solution (X, S) to Peano. So in a sense, the Peano Axioms lay out a kind of citizenship test, in which (X, S) is considered a natural structure if and only if it passes, and it gets all the perks of being a natural citizen that we will now derive.

There is one final assumption that must be

2.2 Addition

2.3 Multiplication