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Theory and Methodology

The General Routing Problem polyhedron: Facets from the RPP and GTSP polyhedra [1]

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Abstract

In this paper we study the polyhedron associated with the General Routing Problem (GRP). This problem, first introduced by Orloff in 1974, is a generalization of both the Rural Postman Problem (RPP) and the Graphical Traveling Salesman Problem (GTSP) and, thus, is NP -hard. We describe a formulation of the problem such that from every non-trivial facet-inducing inequality for the RPP and GTSP polyhedra, we obtain facet-inducing inequalities for the GRP polyhedron. We describe a new family of facet-inducing inequalities for the GRP, the honeycomb constraints, which seem to be very useful for solving GRP and RPP instances. Finally, new classes of facets obtained by composition of facet-inducing inequalities are presented. © 1998 Elsevier Science B.V.

Keywords: General Routing Problem; Rural Postman Problem; Graphical Traveling Salesman Problem; Routing; Facets of polyhedra

1. Introduction

Let $G = (V, E)$ be a connected and non-directed graph and let c be a non-negative cost vector indexed by the set E of edges of G. We have two classical routing problems with a single vehicle defined on G, depending on whether the location of the service demand is on the nodes or on the edges of the graph:

(a) The *Graphical Traveling Salesman Problem* (GTSP) consists of finding the shortest tour going, at least once, through each vertex of G.

(b) The *Rural Postman Problem* (RPP) consists of, given a set of 'required' edges $E_R \in E$, finding the shortest tour on G such that each edge in E is traversed at least once.

The GTSP may be considered as a relaxation of the Traveling Salesman Problem (TSP) and has been studied, from a polyhedral point of view, by Fleischmann [7,8], Cornuejols, Fonlupt and Naddef [5] and Naddef and Rinaldi [12,13]. A polyhedral approach to the RPP is given in Corberán and Sanchis [4]. An extension of the above problems in which the service demand is located both on some of the nodes and on some of the edges of the graph provides a more general model that is quite more useful in solving real-life routing problems:

(c) Given a subset $E_R \in E$ of 'required' edges and given a subset $V_R \in V$ of 'required' vertices, the

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General Routing Problem (GRP) consists of finding the shortest tour on G such that each edge in E_R and each vertex in V_R are traversed at least once.

The GRP, first introduced in 1974 by Orloff [14] (see also Lenstra and Rinnooy Kan [11]), is then a generalization of both the Rural Postman Problem (RPP) (if $V_R = \emptyset$) and the Graphical Traveling Salesman Problem (GTSP) (if $E_R = \emptyset$ and $V_R = V$), and hence is NP-hard. Real-life applications in garbage collection, street sweeping, routing of sanitation vehicles, routing of electric meter readers, school bus transportation and others are discussed in the surveys by Eiselt, Gendreau and Laporte [6] and by Assad and Golden [1]. Also, a heuristic algorithm for the Capacitated GRP defined on a mixed graph is presented in Pandit and Muralidharan [15]; this procedure has been tested on a set of instances generated with respect to a curb-side waste-collection problem arising in residential areas. Finally, new results for the GRP (and for the RPP) polyhedron have been recently published by Letchford [9,10].

The aim of this paper is to contribute to the knowledge of the GRP (and RPP) polyhedron, thus obtaining a better linear programming relaxation of the problem that could be used to solve large size instances in the near future.

2. Problem definition and notation

Note that if $i \in V$ is a vertex incident with any required edge $e \in E_R$, the condition on the tour passing through edge e contains the condition of visiting vertex i. So, in the following, we will assume that V_R contains the set of vertices incident with the required edges.

In order to simplify the problem structure and formulation, we could transform the original graph G in the way proposed by Christofides et al. [3] for the RPP. Such a transformation, that eliminates the non-required vertices, makes easier both the formulation of the problem and our approach but sometimes this transformed graph could have more edges than the original one and some polynomial instances of the GRP could become then non-polynomial if $P \neq$ NP. Hence, we will work with the original graph $G = (V_R \cup V_S, E_R \cup E_S)$ where $V_S = V \setminus V_R$ and $E_S = E \setminus E_R$.

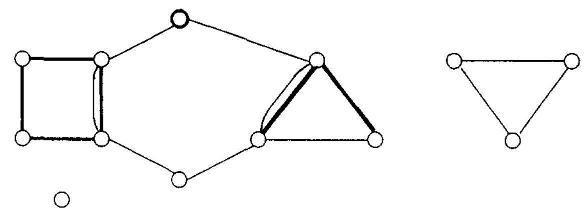


Fig. 1. Tour for the GRP.

In general, the graph $G^R = G(V_R, E_R)$ will be a non-connected graph. Let p be the number of connected components of G^R and let $V^1, V^2, \ldots, V^p$ be the sets of vertices corresponding to the p connected components of G^R. We will represent by $G(V^i)$ the subgraphs of G induced by the vertex sets $V^i, i = 1, 2, \ldots, p$, and they will be referred to as *R-connected components of G*. Notice that every isolated required vertex is a R-connected component of G. Given $v \in V$, $\delta(v)$ will denote the set of edges incident with v. Given two sets of vertices $S_1, S_2 \subset V$, $S_1 \cap S_2 = \emptyset$, $(S_1 : S_2)$ will represent the edge set of G with one extremal vertex in S_1 and the other in S_2 . Given $S \subset V$, we call the set of edges $\delta(S) = (S : V \setminus S)$ the *edge cutset* and we will denote by $\gamma(S)$ the set of edges with both endpoints in S. Finally, given $x \in \mathcal{R}^E$ and $T \subset E$, $x(T)$ denotes $\sum_{e \in T} x_e$.

A *tour for the GRP*, briefly *tour*, is a family $\mathcal{F}$ of edges such that the graph with set of vertices V and edge set $\mathcal{F}^*$, where $\mathcal{F}^*$ is obtained from $\mathcal{F}$ by considering every copy of a same edge as a different element, must be an even graph and all the edges in E_R belong to the same connected component. Note that this definition is more general than the usual one in the sense that tours as the one shown in Fig. 1 are allowed (required edges and isolated required vertices are represented in bold).

The family of edges obtained from any tour by deleting one copy of every edge in E_R will be called *semitour for the GRP*. As in the Rural Postman Problem (see Corberán and Sanchis [4]), we can formulate the GRP with respect to the semitours. We associate to each semitour an integer vector $x = (x_e : e \in E) \in \mathcal{R}^E$, where x_e represents the number of times edge $e \in E$ appears in the semitour. For notational convenience, we will use also the word semitour to designate this vector. Every vertex $v \in V$ incident with an even or zero (odd) number of edges in E_R, will be called *R-even* (*R-odd*). Note that every isolated required vertex is R-even. It is easy to see that the set of semitours for the GRP on G is the set of vectors $x \in \mathcal{R}^E$ satisfying

$$x \geqslant 0 \quad \text{and integer} \; \forall\, e \in E, \tag{2.1}$$

$$x(\delta(v)) \equiv 0 \pmod 2 \quad \forall\, v \in V_R : v \text{ is R-even, and } \forall\, v \in V_S, \tag{2.2}$$

$$x(\delta(v)) \equiv 1 \pmod 2 \quad \forall\, v \in V_R : v \text{ is R-odd}, \tag{2.3}$$

$$x(\delta(S)) \geqslant 2 \quad \forall\, S = (\bigcup_{i \in T} V^i) \cup W, \quad \emptyset \neq T \subset \{1,2,\dots,p\}, \quad W \subseteq V_S. \tag{2.4}$$

Then, the GRP can be formulated as:

$$\text{Minimize} \quad \{cx : x \text{ satisfies (2.1)–(2.4)}\}. \tag{2.5}$$

Let GRP(G) be the convex hull of all the semitours for the GRP on G. We will see that GRP(G) is an unbounded and full dimensional polyhedron and we will find a linear description of it as complete as possible. First, we define a *configuration* $\mathcal{C}$ *on* G, in the way proposed by Naddef and Rinaldi [12] for the GTSP, as any pair $(\mathcal{B}, c)$ where $\mathcal{B} = \{B_1, \dots, B_j, \dots, B_q\}$ is a partition of V and c is a real function defined on $\mathcal{B} \times \mathcal{B}$ satisfying:

(a) For every set B_j, the induced subgraph $G(B_j)$ is connected.

(b) For every pair $B_i \neq B_j \in \mathcal{B}$,

$$c(B_i, B_j) = \begin{cases} c(B_j, B_i) > 0 & \text{if } (B_i : B_j) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

We can associate to a configuration $\mathcal{C}$ of G, a weighted graph G_C with node set $\mathcal{B}$ and having an edge $[B_i, B_j]$ with weight $c(B_i, B_j)$ for every pair of nodes $B_i \neq B_j \in \mathcal{B}$ such that $(B_i : B_j) \neq \emptyset$. In the following, we will use a configuration to designate both $\mathcal{C}$ and G_C. The *configuration inequality* defined by $\mathcal{C}$ on G is the inequality

$$\sum_{e \in E} c_e x_e \geqslant c_0,$$

where:

(a) $c_e = 0$ for all $e \in \gamma(B_i), i = 1, \dots, q$.

(b) $c_e = c(B_i, B_j)$, where B_i and B_j are such that $e \in (B_i : B_j)$.

(c) c_0 is the length of a shortest c-length semitour of G_C.

G_C is the graph obtained from $G = (V, E)$ by shrinking every node set B_i to a single node and keeping a unique copy of the edges linking two same node sets if there exists one. As in the GTSP, one does not care what happens inside the nodes B_i, so it suffices to consider the case $|B_i| = 1$, i.e., we can work directly on the configuration graph G_C.

As in the GTSP (see Naddef and Rinaldi [12]) and in the RPP, all the linear inequalities needed to describe GRP(G), except the trivial ones, are configuration inequalities.

3. Facets obtained from the RPP and GTSP polyhedra

As was said before, RPP and GTSP are particular cases of, and strongly related to, the GRP. We are going to show that the polyhedral study of RPP and GTSP provides an important part of the facial description of the polyhedron GRP(G).

Corberán and Sanchis [4] obtain a formulation of the RPP in terms of semitours. They define an unbounded and full dimensional polyhedron associated to its solutions and describe several families of facet-inducing inequalities. The formulation (2.5) for the GRP is exactly the same as that obtained for the RPP; the RPP and the GRP formulated like this differ only in the structure defined on the graph G: in the GRP there are some R-connected components formed by only one vertex (that of the isolated required vertices). Notice that in the study done in that paper about the RPP polyhedron, the only property needed about the R-connected components was their connectivity, and every vertex is obviously connected. Hence:

GRP(G) *is an unbounded and full dimensional polyhedron in* $\mathcal{R}^E$ (*iff all vertices in* V_R *and all edges in* E_R *are in the same connected component of* G),

and the following inequalities induce facets of it:

1. *Trivial inequalities:*

$$x_e \geqslant 0 \tag{3.1}$$

(iff e is not a cut-edge $(S : V \setminus S)$ with $S \cap V_R \neq \emptyset$ and $(V \setminus S) \cap V_R \neq \emptyset$).

2. *Connectivity inequalities:*

$$x(\delta(S)) \geqslant 2 \quad \forall\, S = \Big(\bigcup_{i \in T} V^i\Big) \cup W,$$

$$\emptyset \neq T \subset \{1,2,\dots,p\}, \quad W \subseteq V_S \tag{3.2}$$

(iff $G(S)$ and $G(V \setminus S)$ are connected).

3. *R-odd inequalities:*

$$x(\delta(S)) \geqslant 1 \quad \forall\, S \textit{ such that } |\delta(S) \cap E_R| \textit{ is odd } \text{(R-odd edge cutset)} \tag{3.3}$$

(iff $G(S)$ and $G(V \setminus S)$ are connected).

4. *K-C inequalities:* A K-C configuration (Fig. 2) is defined by an integer $K \geqslant 3$ and a partition of V into $\{M_0, M_1, M_2, \dots, M_{K-1}, M_K\}$ such that:

(a) Each node set V^i, $1 \leqslant i \leqslant p$, (of the R-connected components) is contained in exactly one of the node sets $M_0 \cup M_K, M_1, M_2, \dots, M_{K-1}$ and each set $M_0, M_1, \dots, M_K$ contains, at least, one set V^i.

(b) The induced subgraphs $G(M_i)$, $i = 0,1,2,\dots,K$, are connected.

(c) $(M_0 : M_K)$ contains a positive and even number of required edges.

(d) The sets $(M_i : M_{i+1})$, $i = 0,1,\dots,K-1$, are nonempty.

The $K - C$ *inequality* corresponding to this configuration is defined by

$$(K-2)\,x\big((M_0 : M_K)\big) + \Big\{ \sum_{\substack{0 \leqslant i < j \leqslant K \\ (i,j) \neq (0,K)}} |i-j| \times x\big((M_i : M_j)\big) \Big\} \geqslant 2(K-1). \tag{3.4}$$

5. *Facets from the GTSP polyhedron:* Cornuejols, Fonlupt and Naddef [5] showed that the convex hull of all the tours for the GTSP in G, denoted by GTSP(G), is an unbounded and full dimensional polyhedron (iff G is connected) and described several families of facet-inducing inequalities.

If we consider any partition of V into K sets $M_1, M_2, \dots, M_K$, $2 \leqslant K \leqslant p$, in such a way each V^i is contained in one M_j, each M_j contains, at least, one set V^i and the induced subgraphs $G(M_j)$ are connected, and we call G' the graph obtained by shrinking every node set M_j, $j = 1,2,\dots,K$ into a single node, every semitour for the GRP on G provides a tour for the GTSP on G'. In [4], Corberán and Sanchis show that from every facet-inducing inequality for GTSP(G'), except the trivial ones, we obtain a facet-inducing inequality for RPP(G); using a similar proof, this is also true for GRP(G). In terms of configurations, this can be expressed as follows: a facet-inducing inequality of GTSP(G) corresponding to a configuration $\mathcal{C} = (\mathcal{B}, c)$ defined on G also induces a facet for GRP(G) if $\mathcal{B} = \{B_1, \dots, B_j, \dots, B_q\}$ satisfies that each node set V^i, $1 \leqslant i \leqslant p$, is completely contained in exactly one node set B_j and each B_j contains, at least, one set V^i.

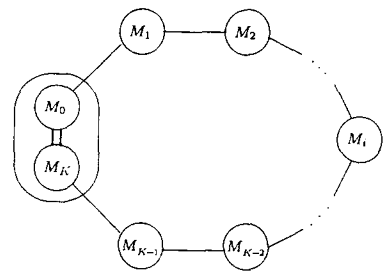


Fig. 2. K-C configuration.

Known facet-inducing inequalities for GTSP(G) include:

– $x(\delta(S)) \geqslant 2$ for every $S \subset V$ (iff the induced subgraphs $G(S)$ and $G(V \setminus S)$ are connected), which becomes the connectivity inequalities (3.2);

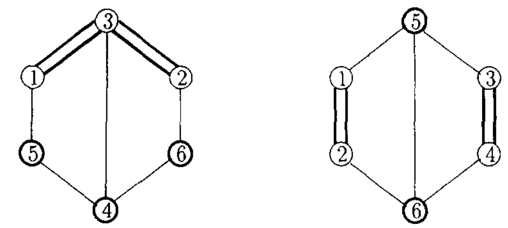
– *path* (bicycle and wheelbarrow) *inequalities* due to Cornuejols et al. [5];

– *star inequalities* due to Fleischmann [8];

– *path-tree inequalities* due to Naddef and Rinaldi [12].

6. Recently, Letchford [9,10] has proposed new families of facet-inducing inequalities for the GRP (and for the RPP). In particular, the *K-C inequalities* are shown to be precisely the *path-bridge inequalities* with a single path.

Note that all the facet-inducing inequalities for the GRP obtained from the GTSP have not active variables (those with non-zero coefficient) associated to edges in the set $E_{\mathrm{IN}} = \bigcup_{i=1}^{p} \gamma(V^i)$ (edges with the two endpoints in the same R-connected component).



(3.a) (3.b)

Fig. 3. Two ways to generalize K-C constraints.

It is not difficult to see that, when $V_R = V$, there are not facet-inducing inequalities for GRP(G) without active variables associated to edges in E_{IN} different from those obtained from the GTSP polyhedron: such an inequality, say $F(x) \geqslant b$, would correspond to a configuration $\mathcal{C} = (\mathcal{B}, c)$ on G, where each set V^i, $i = 1, \ldots, p$, would be contained in exactly one set B_j of $\mathcal{B}$ and then, each semitour x for the GRP on G_C would be also a tour for the GTSP on G_C and vice-versa, obviously with the same value for $F(x)$; so, $F(x) \geqslant b$ also induce a facet for GTSP(G).

4. Honeycomb inequalities

In a K-C configuration, an R-connected component (or a cluster of R-connected components) is divided into two parts. In this section we generalize this configuration simultaneously both in the number of parts an R-connected component is divided into and in the number of R-connected components we divide.

As an illustration, consider either of the graphs in Fig. 3, where the required edges and isolated required vertices are represented in bold. It can be seen that the vector x such that $x_e = 1$ for each $e \in E_S$ and $x_e = 0$ for each $e \in E_R$ is an extremal point of the polyhedron defined by all the inequalities described in Section 2, but x is not a semitour. The inequality that it is not satisfied by x is $\sum_{e \in E} x_e \geqslant 6$, which could be considered as a more general K-C inequality dividing an R-connected component into 3 parts (this would be the case of Fig. 3a) or dividing two R-connected components simultaneously (Fig. 3b). We show below that these (honeycomb) inequalities are facet-inducing of GRP(G).

Consider a partition of the set of vertices V into K vertex sets $\mathcal{A} = \{A_1, A_2, \ldots, A_L, A_{L+1}, \ldots, A_K\}$, $3 \leqslant K \leqslant p$, $1 \leqslant L \leqslant K$, in such a way that each V^j, $1 \leqslant j \leqslant p$, is contained in exactly one A_i, each A_i contains, at least, one V^j and the induced subgraphs $G(A_i)$, $i = 1, 2, \ldots, K$, are connected.

Suppose we can now partition each set A_i, $i = 1, 2, \ldots, L$, into $\gamma_i \geqslant 2$ subsets, $A_i = B_i^1 \cup B_i^2 \cup \cdots \cup B_i^{\gamma_i}$, satisfying:

(H.1) Each B_i^j contains an even number of R-odd nodes, $j = 1, 2, \ldots, \gamma_i$.
(H.2) The induced subgraphs $G(B_i^j)$, $j = 1, 2, \ldots, \gamma_i$ are connected.
(H.3) The graph with node set $B_i^1, B_i^2, \ldots, B_i^{\gamma_i}$ and having an edge $[B_i^j, B_i^k]$ for every pair of nodes $B_i^j \neq B_i^k$, such that $(B_i^j : B_i^k) \cap E_R \neq \emptyset$, is connected.

Condition H.3 is obviously satisfied when A_i is exactly a single R-connected component. When A_i consists of several R-connected components, condition H.3 implies that the partition of A_i into the B_i^j is made by cutting the R-connected components. For notational convenience, we denote $B_i^0 = A_i$, $i = L+1, \ldots, K$. We have therefore the following partition of V:

$$\mathcal{B} = \{B_1^1, B_1^2, \ldots, B_1^{\gamma_1}, B_2^1, B_2^2, \ldots, B_2^{\gamma_2}, \ldots, B_L^1, B_L^2, \ldots, B_L^{\gamma_L}, B_{L+1}^0, \ldots, B_K^0\}.$$

This partition $\mathcal{B}$ defines a graph $(\mathcal{B}, \mathcal{E})$ with a set of nodes $\mathcal{B}$ and a set of edges $\mathcal{E}$ formed by an edge $[B_r^i, B_q^j]$ between each couple of nodes B_r^i, B_q^j such that $(B_r^i : B_q^j) \neq \emptyset$. This graph $(\mathcal{B}, \mathcal{E})$ is obtained by shrinking each node set B_i^j of the original graph $G = (V, E)$ to a single node and, then, by shrinking each set of parallel edges to a single edge.

Let $T \subset \mathcal{E}$ be an edge set so that graph $(\mathcal{B}, T)$ is a spanning tree of $(\mathcal{B}, \mathcal{E})$. For every pair of nodes B_r^i, B_q^j we denote by $d(B_r^i, B_q^j)$ the number of edges in the unique path in $(\mathcal{B}, T)$ joining B_r^i and B_q^j. We assume T satisfies

(H.4) $d(B_q^i, B_q^j) \geqslant 3, \quad \forall\, q = 1, \ldots, L$ and $\forall\, i \neq j$.

This spanning tree $(\mathcal{B}, T)$ defines the 'skeleton' of the configuration (see Figs. 4a, 5a and 6a) and looks like a honeycomb where each cell is a K-C configuration defined by a pair of nodes B_q^i, B_q^j (related to the same A_q) and by the unique path in T joining B_q^i

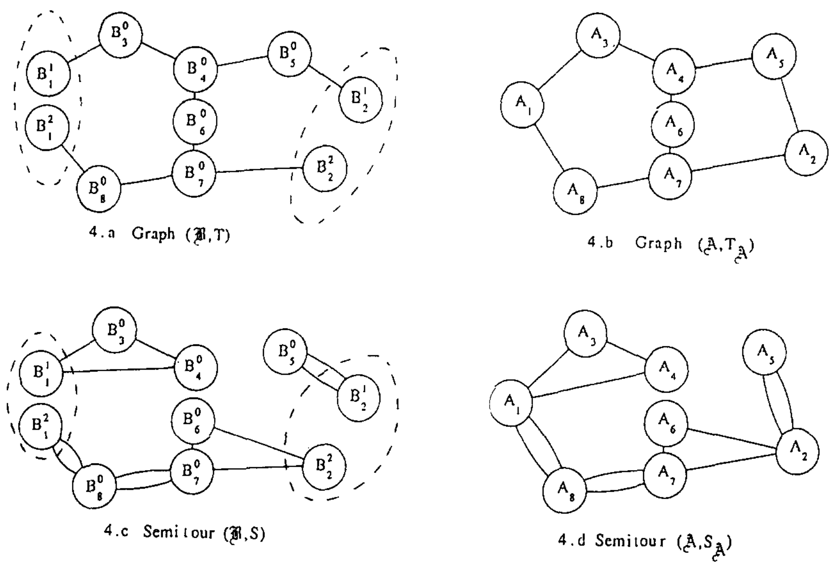


Fig. 4.

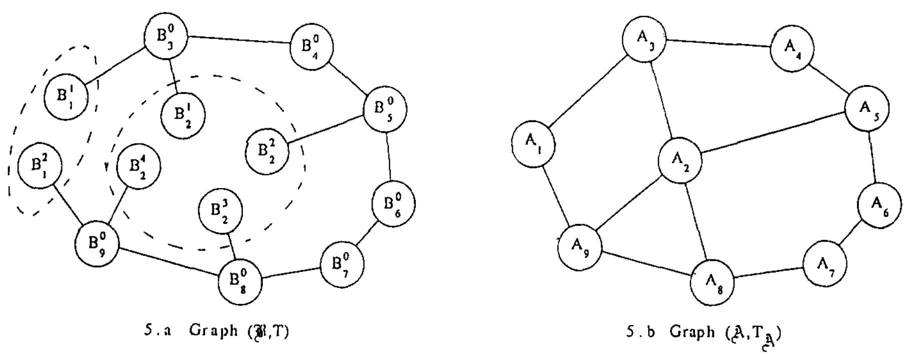


Fig. 5.

and B_q^j ($K \geqslant 3$ since H.4 is satisfied). This resemblance with a honeycomb may appear less clear in some cases, e.g. see Fig. 7a.

We define the configuration cost function $c : \mathcal{B} \times \mathcal{B} \to \mathcal{R}^+$ on the edges of $(\mathcal{B}, \mathcal{E})$ as follows:

(I) For the edges $[B_r^i, B_r^j]$ such that the path in $(\mathcal{B}, T)$ joining B_r^i and B_r^j does not contain more than one node related to the same A_s except the nodes B_r^i and B_r^j:

$$c(B_r^i, B_r^j) = d(B_r^i, B_r^j) - 2.$$

(Note that, since H.4 is satisfied, $c(B_r^i, B_r^j) \geqslant 1$.)

(II) For the edges $[B_r^i, B_q^j]$ with $r \neq q$ such that the path in $(\mathcal{B}, T)$ joining B_r^i and B_q^j does not contain more than one node related to the same A_s:

$$c(B_r^i, B_q^j) = d(B_r^i, B_q^j).$$

(III) The remaining edges (if any), i.e., the edges $[B_r^i, B_q^j]$ such that in the path in $(\mathcal{B}, T)$ joining B_r^i and B_q^j there is some pair of nodes related to the same A_s (distinct from pair B_r^i, B_q^j, when $r = q$), are ordered in an arbitrary way $e_1, e_2, \ldots, e_k$. For $h = 1, \ldots, k$, if $e = [B_r^i, B_q^j]$, let $c(B_r^i, B_q^j)$ be the maximum value

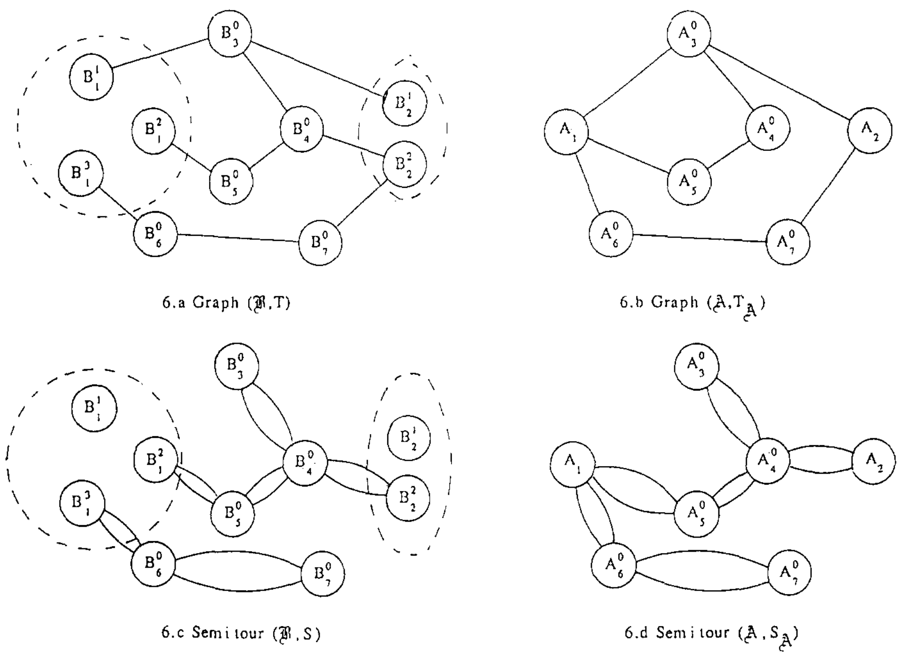


Fig. 6.

such that e belongs to a semitour of c-cost $2(K-1)$ only using edges from $\mathcal{E} \setminus \{e_{h+1}, \dots, e_k\}$ (sequential lifting).

We call $(\mathcal{B}, c)$ a *honeycomb configuration*. The associated *honeycomb inequality* is

$$\sum_{e \in E} c_e x_e \geqslant 2(K-1), \tag{4.1}$$

where

$$c_e = c(B_r^i, B_q^j) \quad \forall\, e \in (B_r^i, B_q^j),$$
$$c_e = 0 \quad \forall e \in \gamma(B_r^i).$$

It is interesting to note that, when the degree of every node B_q^i, $i \neq 0$, in $(\mathcal{B}, T)$ is equal to 1 (the nodes from parts of R-connected components are leaves of the tree (Figs. 4a and 5a)), the configuration graph $(\mathcal{B}, \mathcal{E})$ *has no edges of type II*. Then, there is no need for the sequential lifting and all the coefficients in the honeycomb inequality can be computed in terms of shortest distances in graph $(\mathcal{B}, T)$. By contrast, when any node B_q^i, $i \neq 0$, has degree in $(\mathcal{B}, T)$ greater than 1, sequential lifting may be required with some edges (node B_2^2 and edges $[B_2^1, B_7^0]$, $[B_2^1, B_6^0]$ and $[B_2^1, B_1^3]$ in Fig. 6a; see also Fig. 7a).

Before showing that the honeycomb inequalities are valid for GRP(G) and that, under certain conditions, they are facet-inducing, let us see how to build semitours on $(\mathcal{B}, \mathcal{E})$. To do that, we use the following notation: given an edge family $F \subseteq \mathcal{E}$, we denote by $F_{\mathcal{A}}$ the edge family formed by an edge $[A_i, A_j]$ for each edge $[B_i^m, B_j^s]$ of F. So, graph $(\mathcal{A}, F_{\mathcal{A}})$ is the result of shrinking in graph $(\mathcal{B}, F)$ every node set $\{B_i^1, B_i^2, \dots, B_i^{\gamma_i}\}$ to a single node A_i, and may have parallel edges (see Figs. 4b, 5b, 6b and 7b). With this notation, a semitour in $(\mathcal{B}, \mathcal{E})$ is any edge family $S \subseteq \mathcal{E}$ satisfying the following conditions:

(1) Graph $(\mathcal{B}, S)$ is even.
(2) Graph $(\mathcal{A}, S_{\mathcal{A}})$ is connected.

Condition (1) means that every node B_i^j is incident with an even (or zero) number of edges in S, and it is due to the fact that every node B_i^j contains an even number of R-odd nodes. With respect to condition (2), it is interesting to note that graph $(\mathcal{B}, S)$ may be connected or disconnected (Figs. 4c and 6c), since connectivity among different B_i^j, $j = 1, 2, \dots, \gamma_i$ (within the same A_i) is assured by H.3 and within B_i^j by H.2. Nevertheless, S must connect, at least, one node B_i^j

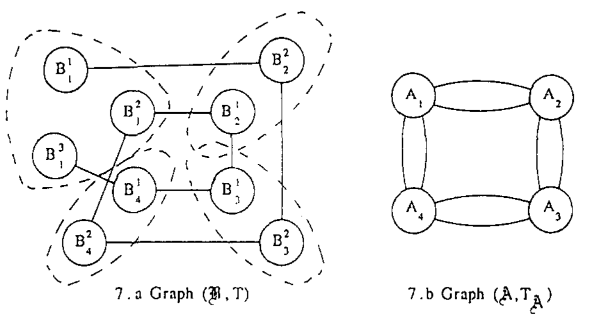


Fig. 7.

of each A_i, $i = 1, \ldots, K$, so that after shrinking each node set $\{B_i^1, B_i^2, \ldots, B_i^{\gamma_i}\}$ into a single node A_i, we obtain a connected graph (see Figs. 4d and 6d).

As an illustration, by taking two copies of each one of the $K-1$ edges of any spanning tree of $(\mathcal{A}, T_{\mathcal{A}})$ we obtain a semitour of c-cost $2(K-1)$ (see Fig. 6c). These semitours play a crucial role on the proof that the honeycomb inequalities are facet-inducing of GRP(G).

Theorem 4.1. *Honeycomb inequalities* (4.1) *are valid with respect to* GRP(G).

Proof. We only have to prove validity for the semitours that do not use edges of type II (sequential lifting), since for the others validity is guaranteed by the choice of their coefficients.

Let S be a semitour that does not use edges of type II. If S contains any edge $[B_r^i, B_q^j]$, $r \neq q$, of type I.b, we can replace it by the edges of a path linking B_r^i to B_q^j in $(\mathcal{B}, T)$ at the same cost. In the same way, we can delete the edges of any cycle in S formed only by edges of type I.c (obviously within the same A_q), since we obtain a lower cost semitour.

Therefore, we can assume that S only uses edges in T and a number J of edges of type I.c not forming cycles, say $e_1, e_2, \ldots, e_J$. The subgraph of $(\mathcal{B}, \mathcal{E})$ induced by these J edges is, then, a forest: let h be the number of its connected components and let $\{u_i^1, \ldots, u_i^{\alpha_i}\}$ be the node set of the i-th connected component, $i = 1, \ldots, h$ (each u_i^j is a node of type B_q^r, $r \neq 0$). As every one of these h connected components is a tree, it has exactly $\alpha_i - 1$ edges, $i = 1, \ldots, h$, and we have

$$(\alpha_1 - 1) + (\alpha_2 - 1) + \cdots + (\alpha_h - 1) = J.$$

From S being a semitour we know that the graph obtained by shrinking in graph $(\mathcal{B}, S)$ each node set $\{B_j^1, B_j^2, \ldots, B_j^{\gamma_j}\}$ into a single node A_j is connected. In this case, however, since S uses the edges $e_1, e_2, \ldots, e_J$, if in this shrinking process we do not shrink nodes $u_i^1, \ldots, u_i^{\alpha_i}$, $i = 1, 2, \ldots, h$ (i.e., we shrink only one node u_i^1 in every set $\{u_i^1, \ldots, u_i^{\alpha_i}\}$, $i = 1, 2, \ldots, h$, into its corresponding A_j), we will also obtain a connected graph. Notice that every node not shrunk above u_i^s, $s \neq 1$ is connected to its corresponding A_i. This graph will have

$$K + (\alpha_1 - 1) + (\alpha_2 - 1) + \cdots + (\alpha_h - 1) = K + J$$

nodes.

We replace in $(\mathcal{B}, S)$ every edge e_i, $i = 1, 2, \ldots, J$, by the edges of the unique path in $(\mathcal{B}, T)$ linking their extremal nodes, say B_q^k, B_q^j. As for these edges $c(e_i) = c(B_q^k, B_q^j) = d(B_q^k, B_q^j) - 2$, we then obtain a semitour S' satisfying $c(S') = c(S) + 2J$ and *only using edges in* T. By construction of S', the nodes of every set $\{u_i^1, \ldots, u_i^{\alpha_i}\}$, $i = 1, 2, \ldots, h$, are still connected among themselves, and so, the resulting graph of the 'partial shrinking' described above applied to graph $(\mathcal{B}, S')$ is a connected graph with $K + J$ nodes. Therefore S' uses, at least, $K + J - 1$ edges (all of them in T). Since the edges of T form a spanning tree of $(\mathcal{B}, \mathcal{E})$, for $(\mathcal{B}, S')$ being an even graph only using edges in T, it must contain an even (non-zero) number of copies of every edge used by S'. Hence, $c(S') \geqslant 2(K + J - 1)$ and so $c(S) \geqslant 2(K-1)$. □

Theorem 4.2. *Given a honeycomb configuration* $(\mathcal{B}, c)$ *on graph* G, *their corresponding honeycomb inequality is facet-inducing of* GRP(G) *if the 'shrunk' graph* $(\mathcal{A}, T_{\mathcal{A}})$ *is* 2*-connected.*

Proof. If the honeycomb inequality $cx \geqslant 2(K-1)$ is not facet-inducing, there exists an inequality, say $dx \geqslant 2(K-1)$, which is facet-inducing, and which is satisfied with equality for all semitours which satisfy $cx = 2(K-1)$, i.e.,

$$\{x \in \text{GRP}(G) : cx = 2(K-1)\}$$
$$\subseteq \{x \in GRP(G) : dx = 2(K-1)\}.$$

We will show that the coefficients d_e for edges of types (I), (II) and (III) (i.e., $\forall\ e \in \mathcal{E}$), are equal to the coefficients c_e of the honeycomb inequality.

(a) Coefficients for edges in T. Let be $a = [B^i_r, B^j_q] \in T$. As graph $(\mathcal{A}, T_\mathcal{A})$ is 2-connected (it has no cutting nodes), by dropping node A_r we obtain a connected graph which would have a spanning tree $T'_\mathcal{A}$ with $K-2$ edges. Let $T' \subset T$ be the set of $K-2$ edges in T associated to $T'_\mathcal{A}$. As the edges in $(T' \cup \{a\})_\mathcal{A}$ form a spanning tree of $(\mathcal{A}, T_\mathcal{A})$, the vector x^1, defined as

$$x^1_a = 2, \qquad x^1_e = \begin{cases} 2 & \forall\, e \in T', \\ 0 & \forall\, e \in \mathcal{E} \setminus (T' \cup \{a\}), \end{cases}$$

is a semitour satisfying $cx^1 = 2(K-1)$ and, therefore, $dx^1 = 2(K-1)$.

Consider now any other edge $b = [B^k_r, B^m_s] \in T$, $s \neq q$ (b exists because $(\mathcal{A}, T_\mathcal{A})$ has no cutting edges). The edges in $(T' \cup \{b\})_\mathcal{A}$ also form a spanning tree of $(\mathcal{A}, T_\mathcal{A})$ and the vector x^2, defined as

$$x^2_b = 2, \qquad x^2_e = \begin{cases} 2 & \forall\, e \in T', \\ 0 & \forall\, e \in \mathcal{E} \setminus (T' \cup \{b\}), \end{cases}$$

is also a semitour satisfying $cx^2 = 2(K-1)$ and, therefore, $dx^2 = 2(K-1)$. As $d(x^1 - x^2) = 0$ we obtain $d_a = d_b$ and, therefore, all the edges in T incidents with any node B^i_r associated to A_r have the same coefficient. By iterating this argument we can conclude that all the edges in T have the same coefficient, σ, and from $dx^1 = 2\sigma(K-1) = 2(K-1)$ we obtain $\sigma = 1$.

(b) Coefficients for edges not in T of type (II). Let $a = [B^i_r, B^j_q]$, $r \neq q$, be an edge of such a type, where i and j may be both zero. In order to simplify the notation, we denote $[B^i_r, B^j_q] = [u, v]$. In $(\mathcal{B}, T)$ there exists an unique path linking u and v, say $u, e_1, w_1, e_2, w_2, \ldots, e_{H-1}, w_{H-1}, e_H, v$, with $H = c_a$. By definition of edge of type (II) every node $u, w_1, w_2, \ldots, w_{H-1}, v$ is related to a different A_i, and therefore, the edges in $\{e_1, e_2, \ldots, e_H\}_\mathcal{A}$ do not form a cycle in graph $(\mathcal{A}, T_\mathcal{A})$ and we can add to them $K-1-H$ edges in T to obtain a spanning tree of $(\mathcal{A}, T_\mathcal{A})$. We denote by T' the corresponding $K-1-H$ edges in T. Consider the semitour x^1 defined as

$$x^1_a = 1,$$
$$x^1_{e_i} = 1, \quad i = 1, 2, \ldots, H,$$
$$x^1_e = \begin{cases} 2 & \forall e \in T', \\ 0, & \text{otherwise}, \end{cases}$$

which satisfies

$$cx^1 = H + H + 2(K-1-H) = 2(K-1),$$

and the semitour x^2 defined as

$$x^2_a = 0,$$
$$x^2_{e_i} = 2, \quad i = 1, 2, \ldots, H,$$
$$x^2_e = \begin{cases} 2 & \forall e \in T', \\ 0, & \text{otherwise}, \end{cases}$$

which also satisfies

$$cx^2 = 2H + 2(K-1-H) = 2(K-1).$$

From $dx^1 = 2(K-1) = dx^2$, we have

$$d_a + \sum_{i=1}^{H} d_{e_i} = 2\sum_{i=1}^{H} d_{e_i},$$

where, as we have shown above, each $d_{e_i} = 1$ and so, we obtain $d_a = H = c_a$.

(c) Coefficients for edges of type (I). Let $a = [B^i_q, B^j_q]$ be an edge of such a type denoted by $a = [u, v]$. In $(\mathcal{B}, T)$ there exists an unique path linking u and v, say $u, e_1, w_1, e_2, w_2, \ldots, e_{H-1}, w_{H-1}, e_H, v$, where $H - 2 = c_a$. By the definition of edges of type (I), every node $u, w_1, w_2, \ldots, w_{H-1}$ is related to a different A_i , and therefore, the edges in $\{e_1, e_2, \ldots, e_H\}_\mathcal{A}$ form an elementary cycle in graph $(\mathcal{A}, T_\mathcal{A})$. Nevertheless, if we delete for example edge e_H, the remaining edges $\{e_1, e_2, \ldots, e_{H-1}\}_\mathcal{A}$ do not form a cycle in $(\mathcal{A}, T_\mathcal{A})$ and we can add to them $K-H$ edges in $T_\mathcal{A}$ in order to obtain a spanning tree of $(\mathcal{A}, T_\mathcal{A})$. We denote by T' their corresponding $K-H$ edges in T. Consider the semitour x^1 defined as

$$x^1_a = 1,$$
$$x^1_{e_i} = 1, \quad i = 1, 2, \ldots, H,$$
$$x^1_e = \begin{cases} 2 & \forall e \in T', \\ 0, & \text{otherwise}, \end{cases}$$

which satisfies

$$cx^1 = (H-2) + H + 2(K-H) = 2(K-1),$$

and the semitour x^2 defined as

$x_a^2 = 0,$

$x_{e_i}^2 = 2, \quad i = 1, 2, \ldots, H-1, \qquad x_{e_H}^2 = 0,$

$$x_e^2 = \begin{cases} 2 & \forall e \in T', \\ 0, & \text{otherwise,} \end{cases}$$

which also satisfies

$cx^2 = 2(H-1) + 2(K-H) = 2(K-1).$

From $dx^1 = 2(K-1) = dx^2$, we have

$$d_a + \sum_{i=1}^{H} d_{e_i} = 2\sum_{i=1}^{H-1} d_{e_i},$$

where each $d_{e_i} = 1$ and so, we obtain $d_a + H = 2(H-1)$, and hence, $d_a = H - 2 = c_a$.

(d) Coefficients for edges of type (III). Let $\{e_1, e_2, \ldots, e_k\}$ be the set of such edges. By construction, for every e_i, $i = 1, 2, \ldots, k$, there exists a semitour with c-cost $2(K-1)$ only using e_i and edges in $\mathcal{E} \setminus \{e_{i+1}, \ldots, e_k\}$, so we obtain iteratively $d_{e_i} = c_{e_i}$. □

Note that as the instances on Figs. 4, 5, 6 and 7 satisfy that $(\mathcal{A}, T_\mathcal{A})$ is a 2-connected graph, their corresponding honeycomb inequalities are facet-inducing. This condition excludes situations like the one shown in Fig. 8, which can be considered as a composition of two K-C configurations (honeycomb configurations) with only a common node. Its corresponding inequality is not facet-inducing because it is dominated by the sum of the two inequalities corresponding to the configurations considerated separately. This is according to the sum process of configurations described by Naddef and Rinaldi for the GTSP where, at least, two common nodes are required in order to sum two configurations (2-sum), which will be considered in the next section.

The 2-connectivity condition on graph $(\mathcal{A}, T_\mathcal{A})$ also implies that when the degree of any node B_i^0 in $(\mathcal{B}, T)$ is 1, the corresponding honeycomb inequality, though valid, is not facet-inducing. Note that in the example of Fig. 9, we could shrink node B_5^0 and its adjacent node in $(\mathcal{B}, T)$, B_3^0, in order to obtain another honeycomb inequality that dominates the original one and that is facet-inducing. This idea is also true in general and provides the way in which the heuristic identification procedures for separation are being developed.

5. Composition of inequalities

In this section, we generalize the results obtained by Naddef and Rinaldi [12] related to the composition of inequalities for the GTSP.

Let $\mathcal{C}_1 = (\mathcal{B}_1, c^1)$ and $\mathcal{C}_2 = (\mathcal{B}_2, c^2)$ be two configurations defined on graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Let $B^{(1)}, B^{(2)}, \ldots, B^{(i)}, \ldots, B^{(s)}$ be nodes of $\mathcal{B}_1$ and let $D^{(1)}, D^{(2)}, \ldots, D^{(i)}, \ldots, D^{(s)}$ be nodes of $\mathcal{B}_2$ such that all necessary isomorphisms and equalities of edge coefficients are satisfied in order to identify set $B^{(i)}$ and $D^{(i)}$, $i = 1, \ldots, s$. Assume that the following conditions are satisfied:

$$\text{(i)} \quad c^1\big(B^{(i)}, B^{(i+1)}\big) = \alpha\, c^2\big(B^{(i)}, B^{(i+1)}\big) \quad \text{for some } \alpha > 0;$$

$$\text{(ii)} \quad c^1\big(B^{(i)}, B^{(i+j)}\big) = \sum_{p=i}^{j-1} c^1\big(B^{(p)}, B^{(p+1)}\big), \quad i = 1, \ldots, s-2, \quad j = 2, \ldots, s-i;$$

$$\text{(iii)} \quad c^2\big(D^{(i)}, D^{(i+j)}\big) = \sum_{p=i}^{j-1} c^2\big(D^{(p)}, D^{(p+1)}\big), \quad i = 1, \ldots, s-2, \quad j = 2, \ldots, s-i;$$

i.e., the sets can be ordered linearly on a path, the coefficient of any chord being equal to the sum of the coefficients of the portion of that path between its extremities. Let

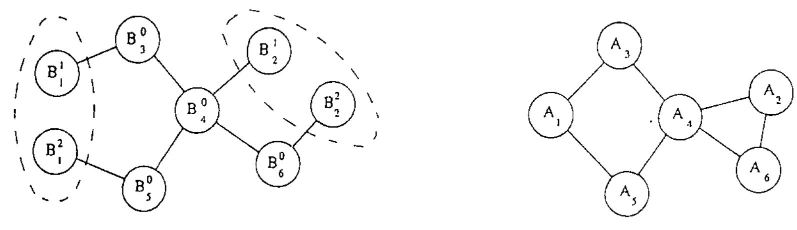
$$\varepsilon(s) = \sum_{p=i}^{s-1} c^1\big(B^{(p)}, B^{(p+1)}\big) = \sum_{p=i}^{s-1} c^2\big(D^{(p)}, D^{(p+1)}\big), \quad \alpha = 1.$$

Let $c^1x \geqslant c_0^1$ and $c^2x \geqslant c_0^2$ be the two configuration inequalities corresponding to $\mathcal{C}_1$ and $\mathcal{C}_2$ and define

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \setminus \{D^{(1)}, D^{(2)}, \ldots, D^{(i)}, \ldots, D^{(s)}\}$$

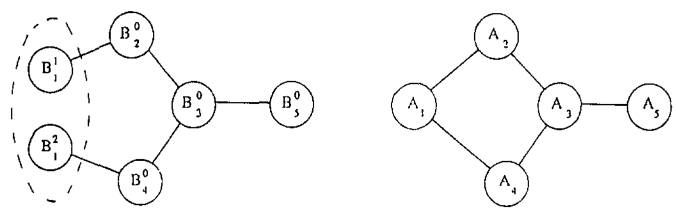
($B^{(i)}$ and $D^{(i)}$ are identified and called $B^{(i)}$) and c is defined by:

$$\text{(c1)} \quad c_0 = c_0^1 + c_0^2 - 2\varepsilon(s).$$



8.a Graph ($\mathcal{B}$,T) 8.b Graph ($\mathcal{A}$,$T_{\mathcal{A}}$)

Fig. 8. A non-facet-inducing honeycomb configuration.



9.a Graph ($\mathcal{B}$,T) 9.b Graph ($\mathcal{A}$,$T_{\mathcal{A}}$)

Fig. 9. Another non-facet-inducing honeycomb configuration.

(c2) $c_e = c_e^i$ if $e \in \gamma(V_i)$, $i \in \{1,2\}$.
(c3) coefficient for crossing edges (if any): from a sequential lifting.

The configuration $\mathcal{C} = (\mathcal{B}, c)$ is called a *linear s-sum of* $\mathcal{C}_1$ *and* $\mathcal{C}_2$. When $s = 2$ we have the 2-sum and the conditions of linearity, (i)–(iii) above, are obviously satisfied (see Naddef and Rinaldi [12]). A similar proof to that included in that paper provides the following results:

Theorem 5.1. *If $c^1x \geqslant c_0^1$ and $c^2x \geqslant c_0^2$ are valid inequalities for* GRP(G_1) *and* GRP(G_2), *respectively, then inequality $cx \geqslant c_0$ obtained by linear s-sum is valid for* GRP(G).

Theorem 5.2. *If $c^1x \geqslant c_0^1$ and $c^2x \geqslant c_0^2$ are facet-inducing inequalities for* GRP(G_1) *and* GRP(G_2), *respectively, then inequality $cx \geqslant c_0$ obtained by linear s-sum is facet-inducing for* GRP(G) *if there exists a minimum c^1-cost semitour of G_1 using 2 edges in each $\left(B^{(i)} : B^{(i+1)}\right)$, $i = 1, ..., s-1$, and there exists a minimum c^2-cost semitour of G_2 using 2 edges in each $\left(D^{(i)} : D^{(i+1)}\right)$, $i = 1, ..., s-1$.*

As a consequence of the above theorems we obtain the following result:

Corollary 5.1. *Inequalities from the linear s-sum of two honeycomb facet-inducing configurations $(\mathcal{B}_1, c^1)$ and $(\mathcal{B}_2, c^2)$, identifying paths on its corresponding spanning trees T_1 and T_2 not having two nodes corresponding to the same A_i, are also facet-inducing of* GRP(G). *In fact, they are also honeycomb inequalities.*

Proof. As seen in the proof of Theorem 4.2, in a honeycomb configuration $(\mathcal{B}, c)$, any path on graph $(\mathcal{B}, T)$ not having two nodes corresponding to the same A_i (i.e., the 'projection' of this path on the graph $(\mathcal{A}, T_{\mathcal{A}})$ does not form a cycle), satisfies linearity conditions (i)–(iii) above and, furthermore, there exists a semitour x of c-cost $2(K-1)$ using 2 edges in each set $\left(B_r^i : B_q^j\right)$ of the path. □

As a particular case, the 2-sum of two honeycomb facet-inducing inequalities, identifying edges in T, is also facet-inducing of GRP(G).

Note that the successive linear sums of K-C configurations, identifying always paths from the trees such that their 'projection' on graph $(\mathcal{A}, T_{\mathcal{A}})$ does not form a cycle, are honeycomb configurations.

Furthermore, as in any K-C configuration the degree of nodes of type B_q^i, $i \neq 0$, in graph $(\mathcal{B}, T)$ is 1, if in the sum process we only allow to nodes of such a type to be identified with nodes of the same kind, their degree with respect to the resulting tree still remains equal to 1. Then, the honeycomb configuration obtained by this sum has no edges of type (III) and all the coefficients can be computed as shortest paths in $(\mathcal{B},T)$ (see Figs. 4a and 5a). So, honeycomb configurations with the degree of all the nodes of type B_q^i, $i \neq 0$, in graph $(\mathcal{B},T)$ equal to 1, provide an example of facet-inducing inequalities obtained by successive linear s-sum of facet-inducing inequalities where all the coefficients can be computed as shortest paths (and no sequential lifting is needed).

It is important to note that not all the honeycomb configurations can be obtained by successive linear s-sums of K-C configurations. This can be seen in the graph of Fig. 10. This graph shows the 'skeleton' of a honeycomb configuration that is facet-inducing for GRP(G), where sequential lifting was needed to compute some costs. This honeycomb configuration could be seen as a 'composition' of the 3 K-C configurations defined by the unique path from T linking nodes 1 and 8 $(1,2,3,4,5,6,7,8)$, 2 and 6 $(2,3,4,5,6)$ and 9 and 10 $(9,7,6,5,10)$, respectively. Nevertheless, it should be noted that this honeycomb configuration cannot be obtained by any 'combination' of two linear s-sums of the above K-C configurations, because we need to identify nodes 2, 3, 4, 5 and 6 whose edge coefficients do not satisfy one of the conditions (ii) or (iii) of linearity:

$$c_{26} = 2 \neq c_{23} + c_{34} + c_{45} + c_{56} = 4.$$

Neither can it be obtained by linear s-sums of the following K-C configurations: $(1,2,6,7,8)$, $(2,3,4,5,6)$ and $(9,7,6,5,10)$.

Hence, the honeycomb inequalities presented in Section 4, besides being more explicit (to which their coefficients refer), are a more general class of facet-inducing inequalities for the GRP(G) than those obtained by linear s-sums of K-C inequalities.

Finally, more general families of facet inducing inequalities for GRP(G) can be obtained. Note in the proof of Theorem 4.2 that in the honeycomb configuration, given any edge e in the tree T, there always exists a semitour of c-cost $2(K-1)$ using edge e twice. Any other family of facet-inducing inequalities satisfying the above property can be composed with the honeycomb configurations to get a new family of facet-inducing inequalities for GRP(G). For example in Naddef and Rinaldi [12], conditions under which

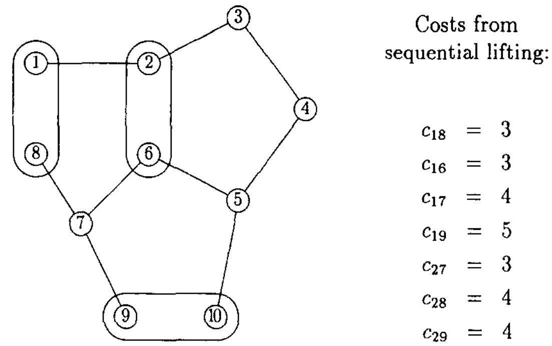


Fig. 10. A honeycomb configuration not obtained by s-sum of K-C's.

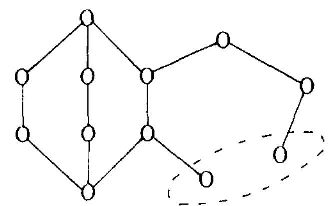


Fig. 11. 2-sum of path and K-C configurations.

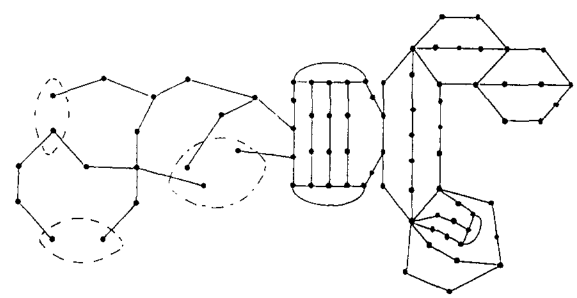


Fig. 12. 2-sum of path-tree and honeycomb configurations.

this property is satisfied by a new family of facet-inducing inequalities for the GTSP (called *path tree*) are given. Hence, configurations like those showed in Figs. 11 and 12 also provide facet-inducing inequalities for GRP(G).

6. Conclusions

In this paper we have presented a wide description of the GRP polyhedron, including a new class of explicit facet-inducing inequalities called honeycomb constraints, that generalize the K-C constraints and seem to be very useful in the resolution of GRP, and RPP, instances. At this moment, we are designing separation procedures to develop a cutting plane algorithm for the GRP. The separation problems for connectivity and R-odd constraints are known to be polynomially solvable. A similar result for the separation of the K-C constraints is not known for the moment. For these constraints we have developed several heuristics that seem to work very well in some preliminary computational testing. In fact, all 26 instances reported in Corberán and Sanchis [4] have been automatically solved to optimality, except instance $I21$ where, although we obtain the optimal cost, a honeycomb inequality is needed in order to get the optimal semitour. This is the reason for believing that the identification of violated honeycomb inequalities is a necessary tool in order to solve GRP and RPP instances of larger sizes. We are trying to devise heuristics for the separation of the most simple honeycomb inequalities based on that for the K-C constraints.

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