

THE WINDY GENERAL ROUTING POLYHEDRON: A GLOBAL VIEW OF MANY KNOWN ARC ROUTING POLYHEDRA*

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Abstract. The windy postman problem consists of finding a minimum cost traversal of all of the edges of an undirected graph with two costs associated with each edge, representing the costs of traversing it in each direction. In this paper we deal with the windy general routing problem (WGRP), in which only a subset of edges must be traversed and a subset of vertices must be visited. This is also an *NP*-hard problem that generalizes many important arc routing problems (ARPs) and has some interesting real-life applications. Here we study the description of the WGRP polyhedron, for which some general properties and some large families of facet-inducing inequalities are presented. Moreover, since the WGRP contains many well-known routing problems as special cases, this paper also provides a global view of their associated polyhedra. Finally, for the first time, some polyhedral results for several ARPs defined on mixed graphs formulated by using two variables per edge are presented.

Key words. arc routing problems, windy general routing problem, polyhedra, facets

AMS subject classifications. 90C27, 90C57, 90B06

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1. Introduction. Arc routing problems (ARPs) have their origin in the celebrated Königsberg bridge problem solved by Euler. They basically consist of finding a traversal on a graph satisfying some conditions related to the links of the graph. ARPs have been studied in depth during the past 40 years due to the large number of real situations that can be modeled in this way: collection or delivery of goods, mail distribution, network maintenance (electrical lines or gas main inspection), snow removal, garbage collection, etc. The book edited by Dror [15] summarizes the state-of-the-art and real-life applications of the ARPs. Other interesting papers are those by Assad and Golden [1] and Eiselt, Gendreau, and Laporte [17, 18].

The first ARP presented in the literature is the well-known Chinese postman problem (CPP) [22]. It consists of finding a shortest tour (closed walk) traversing all of the *links* of a given graph G . Since then, this problem has been generalized in several aspects. Orloff [29] proposed the rural postman problem (RPP), in which not all of the links of the graph have to be traversed by the tour but only those in a given subset of “required” links. Orloff also proposed the general routing problem (GRP), in which the graph has a given set of required links to be traversed and a given set of required vertices to be visited. The GRP can also be considered as a generalization of the graphical traveling salesman problem (GTSP), studied by Cornuéjols, Fonlupt, and Naddef [14], Fleischmann [19], and Naddef and Rinaldi [27]. The GTSP consists of finding a shortest tour visiting all of the vertices of a given graph G at least once.

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All of these problems were defined on *undirected* graphs, where all of the links are *edges* that can be traversed in both directions with the same cost. These four problems are shown in Figure 1 (box labeled “undirected”), where an arc between two problems means that the first one is generalized by the second one.

Alternatively, the CPP can be defined on a *directed* graph, where all of the links are *arcs* that must be traversed in a given direction. This problem is known as the directed Chinese postman problem (DCPP) [16]. Obviously the RPP and the GRP can also be defined on a *directed* graph to obtain their “directed” versions, the DRPP [5, 33, 24] and the DGRP. The GTSP on a directed graph is known as the graphical asymmetric TSP (GATSP) [3]. These four problems are also represented in Figure 1 (box labeled “directed”).

A graph G is called a *mixed* graph if it has edges and arcs simultaneously. The CPP on a mixed graph, the mixed Chinese postman problem (MCP) [16, 4, 21, 28] is then a generalization of both the CPP and the DCPP. Note that any routing problem defined on a mixed graph generalizes the corresponding problems defined on undirected and directed graphs. This is the case for the mixed RPP (MRPP) [32, 11] and for the mixed GRP (MGRP) [6]. Note that since the GTSP is a pure node routing problem, the GTSP defined on a mixed graph is equivalent to the GATSP. These four “mixed” problems are also shown in Figure 1 (box labeled “mixed”). An arc between two “boxes” means that each problem in the first box is generalized by the corresponding problem in the second box.

Finally, we have the windy postman problem (WPP), proposed by Minieka [26] and also studied by Guan [23], Win [34], and Grötschel and Win [20, 21]. This problem consists of finding a minimum cost traversal of all of the edges of an undirected graph in which the cost of traversing an edge (i, j) in a given direction, c_{ij} , can be different from the cost of traversing it in the opposite direction, c_{ji} . Obviously, the WPP

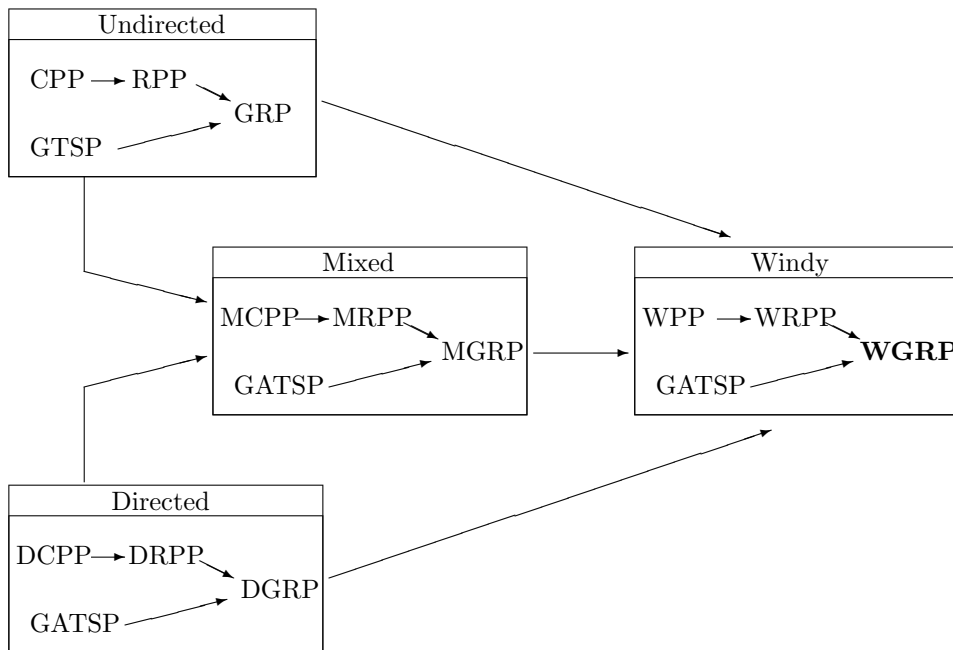


FIG. 1. Relationship among routing problems.

generalizes the undirected CPP (when $c_{ij} = c_{ji}$), but also the DCPP (each arc (i, j) with cost c can be modeled as an edge with costs $c_{ij} = c$ and $c_{ji} = \infty$) and therefore the MCPP. As before, windy routing problems are also represented in Figure 1 (box labeled “windy”). As for the mixed case, the windy version of the GTSP is equivalent to the GATSP.

In this paper we deal with the windy general routing problem (WGRP). This is the most general routing problem among those represented in Figure 1 and contains all of the other problems as special cases. Consequently, most of the theoretical results obtained for the WGRP and the algorithms designed for its approximate or exact resolution can be applied to many other ARPs.

Except for the CPP and the DCPP, which can be solved in polynomial time, all of the other problems in Figure 1 are *NP*-hard problems and therefore the WGRP is also *NP*-hard. The most successful approach to optimally solving these difficult routing problems is polyhedral combinatorics. It is based on the description of the polyhedron defined by the convex hull of all of the solutions associated with a formulation of the problem. This approach has recently been used by Corberán, Plana, and Sanchis [9] to optimally solve WGRP instances of large size, and its success is mostly due to the partial description of the WGRP polyhedron we present here. This description is the subject of this paper.

On the other hand, all of the formulations proposed in the literature for ARPs defined on undirected or directed graphs use only one variable per link. This variable represents, in the directed case, the number of times its associated arc is traversed (in the fixed direction), while, in the undirected case, it represents only the number of times the edge is traversed in either of the two possible directions. All windy problems are formulated by using two variables associated with each edge, representing the number of times the edge is traversed in each direction. However, ARPs defined on mixed graphs can be formulated by using either one variable per edge [28, 11, 6] or two variables [4, 31]. Although the polyhedra associated with ARPs defined on mixed graphs and formulated by using only one variable per edge have already been studied [11, 6], this is not the case when these problems are formulated by using two variables per edge. Then, as a by-product of the polyhedral study of the WGRP presented here, we also obtain some results on the polyhedra associated with ARPs defined on mixed graphs whose formulation uses two variables per edge.

Although the facet-inducing inequalities presented in this paper are associated with a formulation with two variables per edge, they can be easily transformed into facets of the polyhedra of the ARPs shown in Figure 1. Hence, a final contribution of this work is that it also gives a global view of the polyhedra associated with many well-known ARPs.

More specifically, the WGRP is formally defined and some notation is presented in section 2. Section 3 introduces the WGRP polyhedron and presents some results characterizing its facet-defining inequalities. In section 4 several families of valid inequalities and their facet-defining property are described. Section 5 discusses the application of these results to mixed ARPs formulated with two variables per edge. Finally, section 6 presents the conclusions.

2. Problem definition and notation. The WGRP can be defined as follows. Let $G = (V, E)$ be an undirected and connected graph with two nonnegative costs c_{ij} and c_{ji} associated with each edge $(i, j) \in E$, corresponding to the costs of traversing it from i to j and from j to i , respectively. Given a subset $E_R \subseteq E$ of required edges and a subset $V_R \subseteq V$ of required vertices, the problem is to find a minimum cost tour

(closed walk) on G traversing each required edge and visiting each required vertex at least once.

Note that the vertices incident with required edges can also be considered as required vertices and then included in V_R . Without loss of generality, we will assume that the original graph has been transformed to satisfy $V = V_R$. This is not a serious restriction as it is easy to transform WGRP instances which do not satisfy the assumption into equivalent instances which do. Hence, we will assume, in what follows, that we are working with a connected graph $G = (V, E) := (V_R, E_R \cup E_{NR})$.

If we remove all of the nonrequired edges from graph $G = (V, E)$, the resulting graph $G_R = (V, E_R)$ is, in general, not connected. Let p be the number of connected components of graph G_R , and let V_1, V_2, \dots, V_p denote the vertex sets of these connected components, which will be called R -sets. We will call the subgraphs $G(V_i)$ of G R -connected components. Note that each isolated required vertex itself defines an R -set.

If S_1 and S_2 are two vertex sets, (S_1, S_2) represents the set of edges with an end point in S_1 and another end point in S_2 , while $(S_1, S_2)_R$ and $(S_1, S_2)_{NR}$ represent, respectively, the sets of required and nonrequired edges in (S_1, S_2) . Given a vertex subset S , $\delta(S)$ denotes the cutset $(S, V \setminus S)$, i.e., the set of edges with an end point in S and another end point not in S , while $E(S)$ denotes the set of edges with both end points in S . A vertex is called R -even (R -odd) if it is incident with an even (odd) number of required edges, and a subset of vertices $S \subseteq V$ is R -even (R -odd) if it contains an even (odd) number of R -odd vertices.

A tour for the WGRP is a closed walk on the edges of G traversing each required edge and visiting each required vertex at least once. Each tour is represented by an incidence vector $x \in \mathbb{R}^{2|E|}$ with two variables x_{ij}, x_{ji} associated with each edge $e = (i, j) \in E$ representing the number of times e is traversed in each direction. Given an edge set $F \subseteq E$, $x(F)$ denotes the sum of the variables in x corresponding to the edges in F , i.e., $x(F) := \sum_{(i,j) \in F} (x_{ij} + x_{ji})$. Moreover, a tour for the WGRP can be represented by a directed graph (V, A) , where A contains as many copies of arc (i, j) as the number of times the original edge (i, j) has been traversed from i to j . In a similar way to that given in [2] for the WRPP, the WGRP can be formulated as follows:

$$\begin{aligned}
 & \text{Minimize} && \sum_{(i,j) \in E} (c_{ij}x_{ij} + c_{ji}x_{ji}) \\
 & \text{subject to} \\
 (1) & && x_{ij} + x_{ji} \geq 1 \quad \forall (i, j) \in E_R, \\
 (2) & && \sum_{(i,j) \in \delta(i)} (x_{ij} - x_{ji}) = 0 \quad \forall i \in V, \\
 (3) & && \sum_{i \in S, j \notin S} x_{ij} \geq 1 \quad \forall S = \bigcup_{k \in Q} V_k, \quad Q \subset \{1, \dots, p\}, \\
 (4) & && x_{ij}, x_{ji} \geq 0 \quad \forall (i, j) \in E, \\
 (5) & && x_{ij}, x_{ji} \text{ integer} \quad \forall (i, j) \in E.
 \end{aligned}$$

Traversing inequalities (1) force the tour x to traverse each required edge at least once (in any direction). *Symmetry* equations (2) force the tour x to depart from each vertex as many times as it arrives at it. Given any edge cutset of G , $(S, V \setminus S)$, by

combining some of the equations (2),

$$\sum_{i \in S, j \notin S} x_{ij} = \sum_{i \in S, j \notin S} x_{ji}$$

is obtained. *Connectivity* inequalities (3) prevent the formation of subtours. Since inequalities (1) guarantee that x traverses each edge cutset containing required edges, inequalities (3) are needed only for edge cutsets joining different R -sets. Finally, (4) and (5) assure that all of the variables are nonnegative and integer.

3. The WGRP polyhedron. Let $\text{WGRP}(G)$ denote the convex hull of all of the tours for the WGRP on graph G , i.e., all of the integer vectors $x \in \mathbb{R}^{2|E|}$ satisfying (1)–(4). $\text{WGRP}(G)$ is an unbounded polyhedron which has the following properties.

THEOREM 1. *If G is a connected graph, the dimension of $\text{WGRP}(G)$ is $2|E| - |V| + 1$.*

Proof. The matrix defined by the coefficients of (2) has rank $|V| - 1$, and therefore it follows that $\dim(\text{WGRP}(G)) \leq 2|E| - (|V| - 1)$. On the other hand, it is not difficult to see that $\text{aff}(\text{WGRP}(G)) = \{x \in \mathbb{R}^{2|E|} : x \text{ satisfies (2)}\}$. Therefore, $\dim(\text{WGRP}(G)) \geq 2|E| - |V| + 1$, and the result follows. \square

Note that $\text{WGRP}(G)$ is not of full dimension, and different inequalities can induce the same facet of it. Such inequalities are called equivalent. In this section we show that inequalities (4), (1), and (3) from the formulation of the problem are facet-inducing for $\text{WGRP}(G)$.

THEOREM 2. *If edge $e = (i, j) \in E$ is not a bridge of G , then inequalities $x_{ij} \geq 0$ and $x_{ji} \geq 0$ are facet-inducing for $\text{WGRP}(G)$. If in addition $e = (i, j) \in E_R$, then inequality $x_{ij} + x_{ji} \geq 1$ is facet-inducing for $\text{WGRP}(G)$.*

Proof. We will study only the case when $e \in E_R$. If e is not a bridge of G , graph $G \setminus \{e\}$ is connected, and there are $K - 1$ affinely independent tours for the WGRP on $G \setminus \{e\}$, x^1, x^2, \dots, x^{K-1} , where $K = \dim(\text{WGRP}(G))$. All of these tours are not tours for the WGRP on G because they do not traverse the required edge e . Let C^1 be a cycle on G traversing e once from j to i . Then $x^1 + C^1, x^2 + C^1, \dots, x^{K-1} + C^1, x^1 + 2C^1$ are K affinely independent tours for the WGRP on G satisfying $x_{ij} = 0$. Hence, inequality $x_{ij} \geq 0$ is facet-inducing for $\text{WGRP}(G)$.

Consider now the inequality $x_{ij} + x_{ji} \geq 1$, and let C^2 be a cycle on G traversing e once from i to j . Then $x^1 + C^1, x^2 + C^1, \dots, x^{K-1} + C^1, x^1 + C^2$ are K affinely independent tours for the WGRP on G satisfying $x_{ij} + x_{ji} = 1$. \square

Before discussing the facet-defining property of connectivity and other families of inequalities, we will present some properties that are common to all of them in what follows.

3.1. Weak configuration inequalities. The polyhedra based on standard formulations using only one variable per edge corresponding to the routing problems in Figure 1 satisfy the following property. All of the facet-inducing inequalities (except the trivial ones) are *configuration inequalities* [27]. An inequality $F(x) \geq b_0$ is a configuration inequality if there is a partition $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_r\}$ of V such that the subgraphs $G(\mathcal{B}_k)$ are connected, the variables associated with links in the sets $E(\mathcal{B}_k)$ have coefficient zero in the inequality, and the variables associated with links from a given set \mathcal{B}_p to another given set \mathcal{B}_q have the same coefficient in the inequality.

It is easy to see that $\text{WGRP}(G)$ does not satisfy the above property. Connectivity inequalities (3), for example, are configuration inequalities with $\mathcal{B} = \{S, V \setminus S\}$, while traversing inequalities (1), and others that will be presented later, are not configuration inequalities. In what follows we will show that all of the facet-inducing

inequalities for $\text{WGRP}(G)$, except the trivial and the traversing ones, share some of the properties described above for the configuration inequalities. This will allow us to define a new wider class, the *weak configuration inequalities*.

LEMMA 3. Let $F(x) \geq b_0$ be a facet-inducing inequality for $\text{WGRP}(G)$ different from the trivial and the traversing ones.

- (a) For each variable x_{ij} there is a tour x satisfying $F(x) = b_0$ and $x_{ij} \geq 1$.
- (b) For each required edge $e = (i, j) \in E_R$ there is a tour x satisfying $F(x) = b_0$ and $x_{ij} + x_{ji} \geq 2$.

Proof. If (a) is not satisfied, every tour x such that $F(x) = b_0$ satisfies $x_{ij} = 0$. Then

$$\{x \text{ WGRP tour on } G : F(x) = b_0\} \subseteq \{x \text{ WGRP tour on } G : x_{ij} = 0\},$$

and, hence, either $F(x) \geq b_0$ is equivalent to a trivial inequality or it is not facet-inducing. In a similar way, if (b) is not satisfied, every tour x such that $F(x) = b_0$ satisfies $x_{ij} + x_{ji} = 1$, and either $F(x) \geq b_0$ is equivalent to a traversing inequality or it is not facet-inducing. \square

Part (a) in Lemma 3 implies that, for each variable x_{ij} associated with a *nonrequired* edge (i, j) , there is a tour x such that $F(x) = b_0$ and uses x_{ij} an *extra time*. This means that if a copy of the arc (i, j) is removed from the directed graph associated with x and is replaced by any other path from i to j , another WGRP tour is obtained. Part (a) also implies that, for each variable x_{ij} associated with a *required* edge (i, j) , there is a tour x such that $F(x) = b_0$ and uses x_{ij} at least once, although not necessarily an *extra time*. Finally, part (b) implies that, at least for one variable of each pair x_{ij}, x_{ji} associated with a *required* edge (i, j) , there is a tour x such that $F(x) = b_0$ and uses it an *extra time*.

An inequality $F(x) \geq b_0$ will be called a *weak configuration inequality* if there is a partition $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_r\}$ of V such that the subgraphs $G(\mathcal{B}_k)$ are connected, the variables associated with edges in the sets $E(\mathcal{B}_k)$ have coefficient zero in the inequality, and the variables associated with *nonrequired* edges from a given set \mathcal{B}_p to another given set \mathcal{B}_q have the same coefficient in the inequality. Note that if, in addition, the variables associated with *required* edges from \mathcal{B}_p to \mathcal{B}_q also have the same coefficient in the inequality, we have a (strong) configuration inequality.

THEOREM 4. All of the facet-inducing inequalities for the $\text{MGRP}(G)$, except those equivalent to the trivial or the traversing ones, are weak configuration inequalities.

Proof. Let $F(x) := \sum_{(i,j) \in E} (a_{ij}x_{ij} + a_{ji}x_{ji}) \geq b_0$ be such an inequality, and let $E^0 = \{e = (i, j) \in E : a_{ij} = a_{ji} = 0\}$. Let S be the node set of a connected component of $G^0 = (V, E^0)$.

Assume that there is an edge $e = (i, j) \in E(S)$ with $a_{ij} \neq 0$ (necessarily $a_{ij} > 0$). Let x^* be a tour for the WGRP on G such that $F(x^*) = b_0$ and $x_{ij}^* \geq 1$ (it exists from Lemma 3). Let us replace in x^* arc (i, j) by a path from i to j in $E^0(S)$. If x^* uses x_{ij} an extra time, we obtain a tour x^{**} for the WGRP on G satisfying $F(x^{**}) < b_0$, which is a contradiction. From Lemma 3, this is true for all of the variables associated with nonrequired edges and for one variable of each pair associated with a required edge. Then, if x^* does not use x_{ij} an extra time, we add to x^* the arcs in a cycle formed by the arc from j to i (whose associated variable x_{ji} , which is used an extra time, has coefficient zero) and a path in E^0 from i to j , obtaining a tour for the WGRP on G violating the inequality. Therefore, if $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_r\}$ are the vertex sets of the connected components of G^0 , the induced graphs $G(\mathcal{B}_k)$ are connected and $a_{ij} = a_{ji} = 0$, for all $e = (i, j) \in E(\mathcal{B}_k)$, $k = 1, 2, \dots, r$.

Let us suppose now that there are two nonrequired edges $e = (i, j)$, $f = (u, v)$ with $i, u \in \mathcal{B}_p$ and $j, v \in \mathcal{B}_q$ such that $a_{ij} > a_{uv}$. Let x^* be a tour for the WGRP on G using x_{ij} an extra time and $F(x^*) = b_0$. Replace in x^* arc (i, j) by a path in $G(\mathcal{B}_p)$ from i to u , the edge f traversed from u to v , and a path in $G(\mathcal{B}_q)$ from v to j . This way, a tour violating the inequality is obtained, which is a contradiction. \square

Then, associated with each facet-inducing inequality for the WGRP(G), we have a *configuration graph* $G_C = (V^C, E^C)$. This is the graph resulting from shrinking vertex sets \mathcal{B}_k , $k = 1, \dots, r$, into a single vertex each and, after that, shrinking each set of nonrequired parallel edges into one single nonrequired edge but keeping all of the required edges. If two costs equal to the coefficients of the corresponding variables in the inequality are associated with each edge in G_C , this graph keeps all of the information about the inequality, since b_0 is the length of the shortest tour for the WGRP on graph G_C . Hence, each facet-inducing inequality for the WGRP(G) can be represented by means of its corresponding configuration graph G_C . The following theorem implies that, in order to show that a given inequality is facet-inducing for WGRP(G), it will suffice to prove it for the polyhedron associated with the configuration graph, which is simpler in general.

THEOREM 5. *If a weak configuration inequality is facet-inducing for WGRP(G_C), it is also facet-inducing for WGRP(G).*

Proof. It suffices to prove the result for a graph G obtained from G_C by replacing vertex \mathcal{B}_1 (for example) by a connected graph $G(\mathcal{B}_1) = (V^{\mathcal{B}_1}, E^{\mathcal{B}_1})$. For simplicity, $F(x) \geq b_0$ will denote both the inequality on G and the inequality on G_C . Note that, given a tour x for the WGRP on G , by shrinking $G(\mathcal{B}_1)$ into a single vertex we obtain a tour x^* for the WGRP on G_C satisfying $F(x) = F(x^*) \geq b_0$.

Let $K = \dim(\text{WGRP}(G_C))$, and let us suppose that $b_0 \neq 0$. Then there are K linearly independent tours for the WGRP on G_C satisfying $F(x) = b_0$, say, x^1, x^2, \dots, x^K . After replacing \mathcal{B}_1 by the graph $G(\mathcal{B}_1)$, some nonrequired edges may appear between the vertices of $G(\mathcal{B}_1)$ and nodes \mathcal{B}_i , $i \neq 1$. Let us call G'_C the graph obtained from G_C by adding Q nonrequired edges e'_1, e'_2, \dots, e'_Q , where each e'_i is parallel to a nonrequired edge (not necessarily distinct) e_j of G_C .

For each e'_i , $i = 1, 2, \dots, Q$, we will construct two tours in G'_C . Let $e_j = (\mathcal{B}_1, \mathcal{B}_q)$ be the original edge of G_C to which e'_i is parallel. From Lemma 3, there is a tour in G_C satisfying $F(x) = b_0$ and that traverses e_j from \mathcal{B}_1 to \mathcal{B}_q and another one that traverses it in the opposite direction. By replacing the edge e_j by the edge e'_i in these two tours, we obtain two new tours on G'_C satisfying $F(x) = b_0$. Then we have $K + 2Q$ linearly independent tours on G'_C satisfying $F(x) = b_0$. These tours can be completed with edges of $G(\mathcal{B}_1)$ to obtain $K + 2Q$ tours on G , $x^1, x^2, \dots, x^{K+2Q}$, also satisfying $F(x) = b_0$ because the variables associated with the edges of $G(\mathcal{B}_1)$ do not appear in the inequality.

Consider now the WGRP defined on graph $G(\mathcal{B}_1)$. Let $M = \dim(\text{WGRP}(G(\mathcal{B}_1)))$, and let y^1, y^2, \dots, y^M be M linearly independent tours for the WGRP on $G(\mathcal{B}_1)$. Then $x^1 + y^1, x^1 + y^2, \dots, x^1 + y^M$ are also tours for the WGRP on G satisfying $F(x) = b_0$. By expressing the $K + 2Q + M$ tours as rows of a matrix and subtracting the first row x^1 from the last M rows, we obtain a full rank matrix. Then, since $K = 2|E^C| - |V^C| + 1$ and $M = 2|E^{\mathcal{B}_1}| - |V^{\mathcal{B}_1}| + 1$, we have $K + 2Q + M = 2(|E^C| + |E^{\mathcal{B}_1}| + Q) - (|V^C| - 1 + |V^{\mathcal{B}_1}|) + 1 = \dim(\text{WGRP}(G))$ linearly independent tours satisfying $F(x) = b_0$. \square

As mentioned before, strong and weak configuration inequalities differ only in the fact that, in the first one, “parallel” variables (variables x_{ij} , x_{km} associated with edges in $(\mathcal{B}_p, \mathcal{B}_q)$, with $i, k \in \mathcal{B}_p$ and $j, m \in \mathcal{B}_q$) have equal coefficients, while the

last one can have parallel variables with different coefficients. By arguing as in the proof of Theorem 4 above, it can be shown that, if for each variable x_{ij} there is a tour satisfying $F(x) = b_0$ and using x_{ij} an extra time, then $F(x) \geq b_0$ is a strong configuration inequality, with all of the variables in $(\mathcal{B}_p, \mathcal{B}_q)$ having a coefficient equal to a_{pq} or a_{qp} . A variable with a different coefficient in a weak configuration inequality is then a variable x_{ij} associated with a required edge $(i, j) \in (\mathcal{B}_p, \mathcal{B}_q)$ such that there is no tour satisfying $F(x) = b_0$ which uses x_{ij} an extra time. It can be seen that, in this case, the coefficients of variables x_{ij} and x_{ji} are, respectively, $a_{pq} + k$ and a_{qp} , for some k satisfying $0 < k \leq a_{pq} + a_{qp}$.

Therefore, if a given facet-inducing inequality $F(x) \geq b_0$ for $\text{WGRP}(G)$ is not a strong configuration inequality, then there is at least one variable x_{ij} (associated with a required edge) such that every tour x satisfying $F(x) = b_0$ also satisfies either $x_{ij} = 0$ or $x_{ij} + x_{ji} = 1$. In other words, if a given facet \mathcal{F} of $\text{WGRP}(G)$ is not induced by a strong configuration inequality, then every tour for the WGRP on G which lies in \mathcal{F} also lies either in a trivial facet or in a traversing facet of $\text{WGRP}(G)$. Hence, we think that there must be relatively few such facets in $\text{WGRP}(G)$. In fact, the only inequalities of this type that we have found are a subset of the class of zigzag inequalities. All of the other inequalities presented in the following sections are strong configuration inequalities.

3.2. Connectivity inequalities. To finish this section, we will study the conditions under which connectivity inequalities (3) are facet-inducing for $\text{WGRP}(G)$. Furthermore, the proof of Theorem 6 shows the usefulness of the configuration issue and illustrates the use of Theorem 5 above.

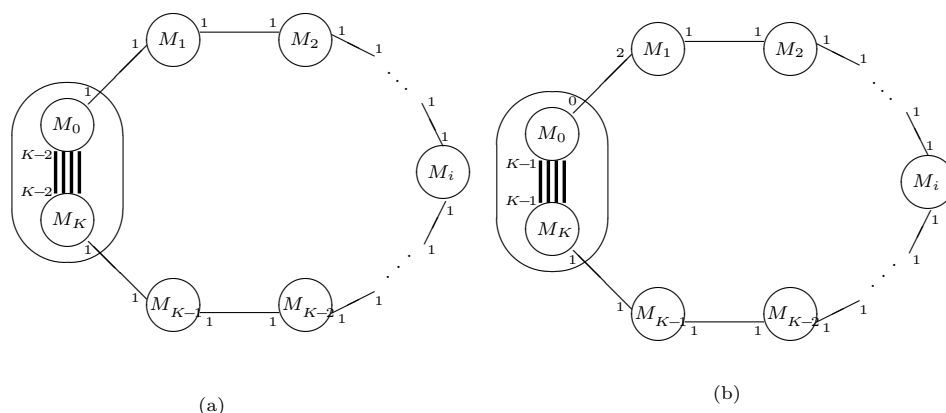
THEOREM 6. *Connectivity inequalities (3) are facet-inducing for $\text{WGRP}(G)$ if subgraphs $G(S)$ and $G(V \setminus S)$ are connected.*

Proof. The configuration graph G_C associated with connectivity inequalities has only two nodes, say, 1 and 2, corresponding to S and to $V \setminus S$, respectively, joined by a single nonrequired edge. Therefore, $\dim(\text{WGRP}(G_C)) = 1$, and, since the tour $x_{12} = x_{21} = 1$ satisfies $\sum_{i \in S, j \notin S} x_{ij} = x_{12} = 1$, the inequality is facet-inducing for $\text{WGRP}(G_C)$. By applying Theorem 5, connectivity inequalities (3) are facet-inducing for $\text{WGRP}(G)$. \square

4. Other facets of the WGRP polyhedron. In the presence of the integrality conditions (5), equations (2) and inequalities (4), (1), and (3) in the formulation are enough to describe all of the tours for the WGRP on G . Nevertheless, this is not the case when the integrality conditions are relaxed. In this section we study some other families of inequalities which are facet-inducing for $\text{WGRP}(G)$.

We first obtain valid inequalities for the WGRP from known valid inequalities for the undirected GRP. Let $G = (V, E)$ be an undirected graph, and let $\text{GRP}(G)$ be the polyhedron associated with the formulation of the undirected GRP using one variable y_e per edge $e = (i, j) \in E$, representing the number of times the edge is traversed in any direction. Consider now the same graph $G = (V, E)$ but with two nonnegative costs associated with each edge, and let $\text{WGRP}(G)$ be the polyhedron studied in this paper. In [2] it is shown that, if $\sum_{e \in E} \beta_e y_e \geq b_0$ is a valid inequality for $\text{GRP}(G)$, the inequality $\sum_{e=(i,j) \in E} \beta_e (x_{ij} + x_{ji}) \geq b_0$ is valid for $\text{WGRP}(G)$. Therefore, the R -odd cut, K-C, path-bridge, and honeycomb inequalities presented in what follows are valid inequalities for the $\text{WGRP}(G)$.

4.1. R -odd cut inequalities. An edge cutset $\delta(S)$ is called R -odd if it contains an odd number of required edges. Given that any tour must cross any given edge cutset

FIG. 2. K -C and K -C₀₂ configuration graphs.

an even number of times, the R -odd cut inequalities

$$(6) \quad x(\delta(S)) \geq |\delta_R(S)| + 1 \quad \forall S \subset V \quad \text{such that } \delta(S) \text{ is } R\text{-odd}$$

are valid for WGRP(G). It can be proved [30] that they are facet-inducing for WGRP(G) if subgraphs $G(S)$ and $G(V \setminus S)$ are connected.

4.2. K-C inequalities. K-C inequalities were introduced in [12] for the undirected RPP, and in [11] they were generalized to the mixed GRP in two families, the K-C and the K-C₀₂ inequalities. In this section we will study their generalization to the WGRP.

Consider a partition of V into $K+1$ subsets $\{M_0, M_1, M_2, \dots, M_{K-1}, M_K\}$, $K \geq 3$, such that each R -set V_i is contained by one of the node sets $M_0 \cup M_K, M_1, M_2, \dots, M_{K-1}$, the induced subgraphs $G(M_i)$, $i = 0, 1, 2, \dots, K$, are connected, and (M_0, M_K) contains a positive and even number of required edges. Furthermore, we will assume that sets (M_i, M_{i+1}) are nonempty, $i = 0, 1, \dots, K-1$. Such a partition defines the configuration graph shown in Figure 2(a), where the costs of its edges are $c(M_0, M_K) = c(M_K, M_0) = K-2$ and $c(M_i, M_j) = |i-j|$ for all i, j such that $\{i, j\} \neq \{0, K\}$. These costs, shown in Figure 2(a), correspond to the coefficients of the variables in the inequality. Each number represents the coefficient of the variable associated with the traversal of the edge from the nearest node to the farthest one. The K -C inequality is then

$$(7) \quad (K-2) x(M_0, M_K) + \sum_{\substack{0 \leq i < j \leq K \\ \{i, j\} \neq \{0, K\}}} |i-j| x(M_i, M_j) \geq 2(K-1) + (K-2)|(M_0, M_K)_R|.$$

Since K-C inequalities are a particular case of path-bridge inequalities, which will be presented later, the proof that K-C inequalities are facet-inducing is omitted here.

4.3. K-C₀₂ inequalities. With the same underlying configuration graph, K-C₀₂ inequalities differ from the standard ones in that integer K can now take value 2 and in that the coefficients of the variables and the right-hand side (RHS) change as follows:

- $c(M_0, M_K) = c(M_K, M_0) = K-1$;
- $c(M_0, M_1) = 0$, $c(M_1, M_0) = 2$;
- $c(M_i, M_{i+1}) = c(M_{i+1}, M_i) = 1$, $i = 1, 2, \dots, K-1$;

- otherwise, $c(M_i, M_j)$ is the shortest path cost from M_i to M_j in G_C , i.e.,
 - $c(M_0, M_j) = j - 1$, $1 \leq j \leq K - 1$,
 - $c(M_j, M_0) = j + 1$, $1 \leq j \leq K - 1$,
 - $c(M_i, M_j) = |i - j|$, $1 \leq i, j \leq K$.

The corresponding K -C₀₂ inequality is then

$$(8) \quad \sum_{(i,j) \in E} (a_{ij}x_{ij} + a_{ji}x_{ji}) \geq 2(K-1) + (K-1)|(M_0, M_K)_R|,$$

where $a_{ij} = c(M_p, M_q)$ if $i \in M_p$ and $j \in M_q$. These inequalities are also a special case of the path-bridge₀₂ inequalities presented next.

4.4. Path-bridge inequalities. Path-bridge (PB) inequalities were introduced by Letchford [25] for the undirected GRP and are based on the path inequalities for the GTSP proposed by Cornuéjols, Fonlupt, and Naddef [14]. In this section we study their extension to the WGRP. Let $P \geq 1$ and $B \geq 0$ be integers such that $P + B \geq 3$ is an odd number, and let $n_i \geq 2$, $i = 1, \dots, p$, be integer numbers. Let us define a partition $\{A, Z, \{V_j^i\}_{j=1, \dots, n_i}^{i=1, \dots, P}\}$ of V satisfying that each R -set is contained either in $A \cup Z$ or in a set V_j^i , the induced subgraphs $G(V_j^i)$ are connected, $i = 1, 2, \dots, P$, $j = 0, 1, 2, \dots, n_i + 1$ (where for the sake of simplicity we identify A with V_0^i and Z with $V_{n_i+1}^i$ for all i), and (A, Z) contains a number B of required edges. Furthermore, we will suppose that sets (V_j^i, V_{j+1}^i) are nonempty. Then the configuration graph has P paths from A to Z , each of them with $n_i + 2$ vertices and $n_i + 1$ edges. The edges joining the sets A and Z form the *bridge*. This configuration graph is shown in Figure 3, where only one number near an edge means that the coefficients of its two corresponding variables are equal. These coefficients are

- $c(A, Z) = c(Z, A) = 1$,
- $c(V_j^i, V_k^i) = \frac{|j-k|}{n_i-1}$ for all $j, k \in \{0, 1, \dots, n_i+1\}$, $i \in \{1, \dots, P\}$,
- $c(V_j^i, V_k^r) = \frac{1}{n_i-1} + \frac{1}{n_r-1} + \left| \frac{j-1}{n_i-1} - \frac{k-1}{n_r-1} \right|$ for all $i \neq r \in \{1, \dots, P\}$, $j \in \{1, \dots, n_i\}$, $k \in \{1, \dots, n_r\}$.

The PB inequality is then

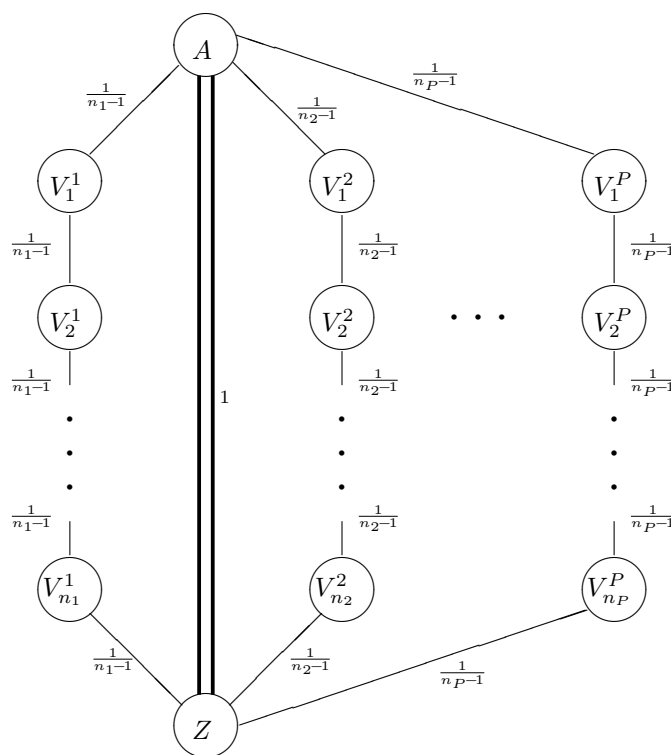
$$(9) \quad \sum_{(i,j) \in E} (a_{ij}x_{ij} + a_{ji}x_{ji}) \geq 1 + \sum_{i=1}^P \frac{n_i + 1}{n_i - 1} + |(A, Z)_R|,$$

where, again, $a_{ij} = c(V_p^k, V_q^l)$ if $i \in V_p^k$ and $j \in V_q^l$.

A PB inequality is called n -regular if all of the paths have the same length n . In this case, by multiplying it by $n-1$, the inequality (9) can be rewritten in an easier way. Note that a K-C inequality is a particular case of a PB inequality with $P = 1$ and that PB inequalities reduce to the s -path inequalities for the GATSP, introduced by Chopra and Rinaldi [3], when there are no required links crossing from A to Z .

THEOREM 7. *PB inequalities (9) are facet-inducing for WGRP(G).*

Proof. Let us first suppose that the number B of required edges joining A and Z is even. Then the number P of paths from A to Z is odd. If we consider in G_C all of the required edges as nonrequired and we replace each edge by a pair of opposite arcs, we obtain an instance G'_C of the GATSP. By subtracting $|(A, Z)_R|$ from the RHS of the PB inequality, an s -path inequality is obtained, which is facet-inducing for GATSP(G'_C) (see [3]). The dimension of this polyhedron is $2|E_C| - |V_C| + 1$, and there is this number of affinely independent tours for the GATSP on G'_C satisfying the inequality as an equality. Each one of these tours can be easily transformed into

FIG. 3. *PB configuration graph.*

tours for the WGRP on G_C (by adding to them one arc per each required edge in (A, Z) , half of them directed from A to Z and the other half from Z to A) satisfying the PB inequality (9) as an equality. Hence, PB inequalities are facet-inducing for $\text{WGRP}(G_C)$ and, from Theorem 5, for $\text{WGRP}(G)$.

Let us now suppose that B is odd and P is even. Let \bar{e} be a given required edge in (A, Z) . We construct the tour x^1 as the one traversing edge \bar{e} once in each direction, half of the edges in $(A, Z)_R \setminus \{\bar{e}\}$ from A to Z and the other half from Z to A . Tour x^1 traverses half of the paths from A to Z and the other half from Z to A . From x^1 we construct the following tours:

- For each edge $e \in (A, Z) \setminus \{\bar{e}\}$ we define a tour traversing e from A to Z once more than tour x^1 does and traversing \bar{e} only from Z to A . All of the other edges are traversed as in x^1 . Another similar tour traversing e from Z to A and \bar{e} from A to Z is considered. Then we have $2|(A, Z)| - 2$ tours.
- For each edge $e = (V_i^k, V_{i+1}^k)$, we construct a tour x that traverses all of the edges in the path k , except e , in both directions. Edge \bar{e} is traversed by x in the same direction as path k is traversed by x^1 . All of the other edges are traversed by x as they are traversed by x^1 . Then we have $\sum_{i=1}^P (n_i + 1)$ tours.
- For each path k , it is possible to build $n_i + 1$ different tours by using the opposite orientations to those used in (b) for the other $P - 1$ paths. From among them we select, for each path k , only the tour that does not traverse the edge (V_0^k, V_1^k) , and we then have P tours.
- For each of the remaining edges (if any) there are always two tours for the

	\bar{e}	$(A, Z) \setminus \{\bar{e}\}$	Paths	E'
(a)	*	I	0	0
(b)+(c)	*	0	B	0
(d)	*	*	*	I

FIG. 4. Matrix appearing in the proof of Theorem 7.

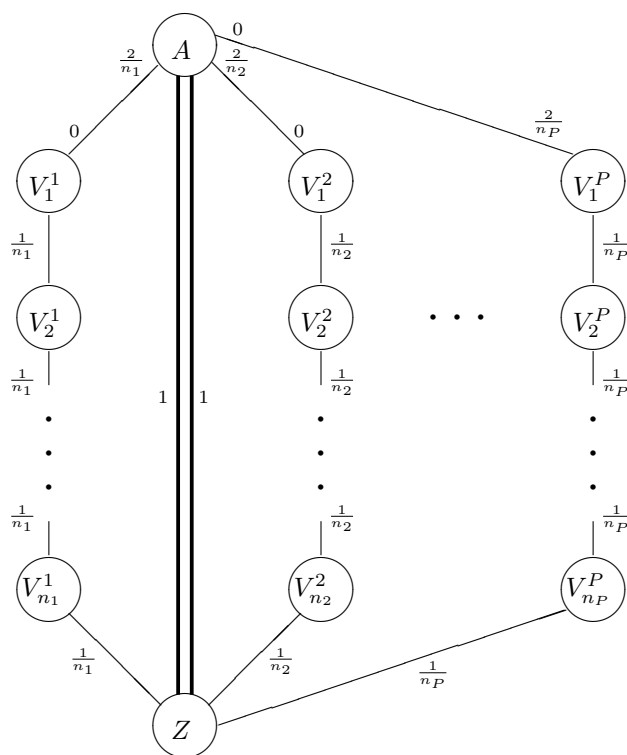
WGRP in G_C satisfying (9) as an equality, traversing this edge exactly once in one direction each, edges in (A, Z) and edges in the P paths. Then, if E' is the set of such remaining edges, we have obtained $2|E'|$ tours.

By including x^1 we have $1+2|(A, Z)|-2+\sum_{i=1}^P(n_i+1)+P+2|E'|$ tours, all of them satisfying the PB inequality as an equality. Hence we have $2|E_C| - \sum_{i=1}^P n_i - 1 = \dim(\text{WGRP}(G_C))$ tours. If we subtract x^1 from all of the other tours and express them as the rows of a matrix (sorted as (a), (b)+(c), and (d) above) and the variables associated with the edges as columns (sorted as \bar{e} , $(A, Z) \setminus \{\bar{e}\}$, (V_i^k, V_{i+1}^k) for all i, k , and E'), we obtain the matrix shown in Figure 4. Submatrix B is a full rank matrix (see [30] for details) and, hence, also the complete matrix in Figure 4. Therefore, the PB inequality is facet-inducing for $\text{WGRP}(G_C)$, and then it is also facet-inducing for $\text{WGRP}(G)$. \square

4.5. Path-bridge₀₂ inequalities. Path-bridge₀₂ inequalities were presented in [6] for the MGRP. The configuration graph corresponding to a path-bridge₀₂ inequality is similar to the previous one except that integers n_i can now take value 1 and the paths are classified into two types: AZ paths and ZA paths. The associated configuration graph is shown in Figure 5, where the two coefficients associated with an edge are represented only if they are different.

The coefficients are

- $c(A, Z) = c(Z, A) = 1$,
- $c(V_0^i, V_1^i) = 0$ and $c(V_1^i, V_0^i) = \frac{2}{n_i}$ for each AZ path i ,
- $c(V_0^i, V_1^i) = \frac{2}{n_i}$ and $c(V_1^i, V_0^i) = 0$ for each ZA path i ,
- $c(V_j^i, V_{j+1}^i) = c(V_{j+1}^i, V_j^i) = \frac{1}{n_i}$,
- $c(V_j^i, V_k^i)$, with $|j-k| > 1$, is the shortest path cost from V_j^i to V_k^i using edges of the path i , i.e.,
 - $c(V_j^i, V_k^i) = \frac{|j-k|}{n_i}$ for all $j \neq k \in \{1, 2, \dots, n_i+1\}$,
 - $c(V_0^i, V_k^i) = \frac{k-1}{n_i}$ and $c(V_k^i, V_0^i) = \frac{k+1}{n_i}$ if path i is of type AZ , $1 \leq k \leq n_i$,
 - $c(V_0^i, V_k^i) = \frac{k+1}{n_i}$ and $c(V_k^i, V_0^i) = \frac{k-1}{n_i}$ if path i is of type ZA , $1 \leq k \leq n_i$.
- Order the remaining edges (edges in (V_j^i, V_l^k) , $i \neq k$, if any) in an arbitrary way e_1, e_2, \dots, e_h . For $i=1$ to h , if $e_i = (u, v)$, we assign to c_{uv} the maximum value such that there is a WGRP tour on G_C with cost $P+1+|(A, Z)_R|$ traversing e_i just from u to v but not traversing the edges $\{e_{i+1}, \dots, e_h\}$. Similarly, we assign to c_{vu} the maximum value such that there is a tour of c -length $P+1+|(A, Z)_R|$ traversing e_i from v to u (perhaps also from u to v) but not traversing the edges $\{e_{i+1}, \dots, e_h\}$ (sequential lifting).

FIG. 5. PB_{02} configuration graph.

The path-bridge $_{02}$ inequality PB_{02} is then

$$(10) \quad \sum_{(i,j) \in E} (a_{ij}x_{ij} + a_{ji}x_{ji}) \geq P + 1 + |(A, Z)_R|.$$

Note that the main difference between PB_{02} and PB inequalities is that variables in sets (A, V_1^i) have different coefficients for its two associated variables. One of these coefficients is $\frac{2}{n_i}$, twice the cost of the other edges in path i , while the other coefficient is 0 (see Figure 5). The direction represented by the variable with coefficient 0 in each path determines whether the path is of type AZ or ZA . In a path-bridge $_{02}$ configuration graph, the pair of asymmetric coefficients $(\frac{2}{n_i}, 0)$ for each path i can be associated with any edge (V_j^i, V_{j+1}^i) in the path, thus obtaining different but equivalent inequalities (given the same ordering in the sequential lifting process). Finally, when $P=1$, the path-bridge $_{02}$ inequality (10) reduces to a $K-C_{02}$ inequality.

THEOREM 8. *Path-bridge $_{02}$ inequalities (10) are valid for $WGRP(G)$.*

Proof. The proof is similar to that in [6] for the MGRP and is omitted here for the sake of brevity. \square

THEOREM 9. *Let $F(x) \geq b_0$ be a path-bridge $_{02}$ inequality (10), and let P_{AZ} and P_{ZA} be the number of paths of types AZ and ZA , respectively. Then $F(x) \geq b_0$ is facet-inducing of $WGRP(G)$ if $|(A, Z)_R| \geq |P_{AZ} - P_{ZA}| + 1$.*

Proof. The proof is similar to that of Theorem 7 and is omitted here. Note only that condition $|(A, Z)_R| \geq |P_{AZ} - P_{ZA}| + 1$ is needed in order to construct $WGRP$ tours traversing the paths of type AZ (type ZA) from A to Z (from Z to A) and

satisfying $F(x) = b_0$. \square

As for the (standard) PB inequalities, when all of the paths are of the same length n , we obtain the n -regular PB₀₂ inequalities, which can be written in an easier way.

4.6. Honeycomb inequalities. Honeycomb inequalities were first proposed by Corberán and Sanchis [13] for the undirected GRP and by Corberán, Mejía, and Sanchis [6] for the MGRP. These inequalities also generalize the K-C inequalities. Consider a partition $\{A_1, A_2, \dots, A_L, A_{L+1}, \dots, A_K\}$ of V , $3 \leq K \leq p$, $1 \leq L \leq K$, such that each R -set is contained by one A_i and the induced subgraphs $G(A_i)$ are connected. Each set A_i , $i = 1, 2, \dots, L$, is divided into $\gamma_i \geq 2$ subsets, $A_i = B_i^1 \cup B_i^2 \cup \dots \cup B_i^{\gamma_i}$, satisfying that each B_i^j contains an even number of R -odd nodes and the induced subgraphs $G(B_i^j)$ are connected, $j = 1, 2, \dots, \gamma_i$, and the graph defined by nodes $B_i^1, B_i^2, \dots, B_i^{\gamma_i}$ and the required edges (B_i^j, B_i^k) is connected (and even).

Note that, when A_i consists of several R -sets, the last condition implies that the partition of A_i into the B_i^j is made by cutting the R -sets. For notational convenience, we denote $B_i^0 = A_i$, $i = L+1, \dots, K$. Therefore we have the following partition of V :

$$\mathcal{B} = \{B_1^1, B_1^2, \dots, B_1^{\gamma_1}, B_2^1, B_2^2, \dots, B_2^{\gamma_2}, \dots, B_L^1, B_L^2, \dots, B_L^{\gamma_L}, B_{L+1}^0, \dots, B_K^0\}.$$

This partition \mathcal{B} defines the configuration graph $G_C = (\mathcal{B}, \mathcal{E})$. Let T be a set of nonrequired edges in G_C joining nodes corresponding to different A_j such that (\mathcal{B}, T) is a spanning tree (see Figure 6, where the arcs in T are represented by thin lines and the required links by bold lines). Then, for each pair of nodes B_i^j, B_p^q in \mathcal{B} , let $d(B_i^j, B_p^q)$ denote the number of edges in the unique path in (\mathcal{B}, T) joining B_i^j to B_p^q . We will also assume that $d(B_i^j, B_i^q) \geq 3$ for all $i = 1, \dots, L$ and for all $j \neq q$.

We divide the set \mathcal{E} into 3 subsets: the set \mathcal{C} formed by the edges joining nodes B_i^p, B_i^q , with $p, q \neq 0$ (the nodes obtained by “cutting” the sets A_i , $i = 1, \dots, L$), the set of edges in T , and the set formed by the remaining edges, which will be called $\mathcal{I}n$. The costs of the edges of graph $G_C = (\mathcal{B}, \mathcal{E})$ are (see Figure 6) the following:

- (I) For the edges $(B_i^p, B_i^q) \in \mathcal{C}$ such that the path in (\mathcal{B}, T) joining B_i^p and B_i^q does not contain more than one node related to the same A_s (except the nodes B_i^p and B_i^q), $c(B_i^p, B_i^q) = c(B_i^q, B_i^p) = d(B_i^p, B_i^q) - 2$.

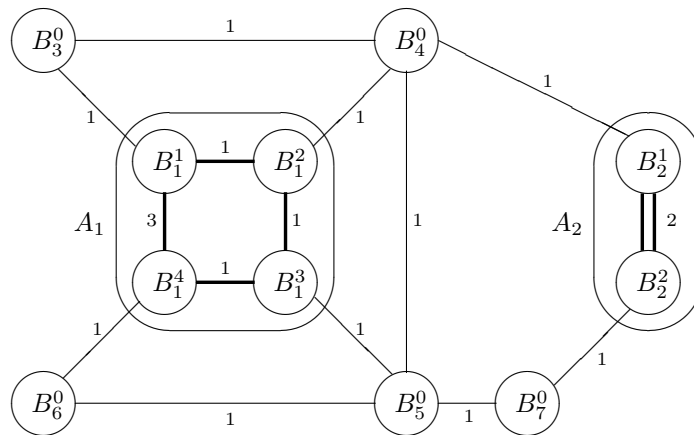


FIG. 6. Honeycomb configuration graph.

- (II) For the edges $(B_r^i, B_q^j) \in T \cup \mathcal{I}n$, $r \neq q$, such that the path in (\mathcal{B}, T) joining B_r^i and B_q^j does not contain more than one node related to the same A_s , $c(B_r^i, B_q^j) = c(B_q^j, B_r^i) = d(B_r^i, B_q^j)$.
- (III) For the remaining edges (if any), we compute their two costs with a process of sequential lifting such as the one described for PB_{02} inequalities.

The corresponding *honeycomb inequality* is then

$$(11) \quad \sum_{(i,j) \in E} (a_{ij}x_{ij} + a_{ji}x_{ji}) \geq 2(K-1) + \sum_{(B_q^i, B_q^j) \in \mathcal{E}_R} c(B_q^i, B_q^j),$$

where $a_{ij} = c(B_r^i, B_q^j)$ if $i \in B_r^i$ and $j \in B_q^j$.

Note that, when the configuration graph has no edges of type III, all of the coefficients in the honeycomb inequality can be computed in terms of the shortest distances in the graph (\mathcal{B}, T) . This occurs, for example, when every node B_q^i , $i \neq 0$, has degree 1 in (\mathcal{B}, T) . Given that the sequential lifting process for a set of edges guarantees the validity of an inequality if it is valid without this set of edges, and that this is also true for the facet-inducing property, in what follows we will assume that the honeycomb configuration has no edges of type III.

Before proving that the honeycomb inequalities are facet-inducing for $WGRP(G)$, let us show how to build tours for the WGRP on $G_C = (\mathcal{B}, \mathcal{E})$ satisfying the honeycomb inequality $F(x) \geq b_0$ as an equality. Let $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ denote the graph with node set $\bar{\mathcal{A}} = \{A_1, A_2, \dots, A_L, A_{L+1}, \dots, A_K\}$ and having an edge (A_i, A_j) for each edge (B_i^p, B_j^q) in T (Figure 7(a)). Then, by traversing all of the $K-1$ edges of any spanning tree of $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ once in each direction, and each required edge in G_C once, we obtain a tour of cost b_0 (Figure 7(b)).

THEOREM 10. *Honeycomb inequalities (11) are facet-inducing of $WGRP(G)$ if the shrunk graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ is 2-connected.*

Proof. The dimension of $WGRP(G_C)$ is $2|\mathcal{E}| - |\mathcal{B}| + 1 = 2|\mathcal{E}| - |T| = 2|\mathcal{C}| + 2|\mathcal{I}n| + |T|$, and this is the number of linear independent WGRP tours x satisfying the inequality as an equality that we build in what follows. Let x^d be the incidence vector of a set of L cycles traversing each required edge exactly once (such cycles exist since the required edges in G_C form even graphs).

For each variable x_{uv} associated with an edge in $\mathcal{C} \cup \mathcal{I}n$, consider the tour that uses x_{uv} exactly once, the path in T from v to u , and the required edges as in x^d . The nodes still not visited are connected with edges in T used once in each direction. Then we have $2|\mathcal{C}| + 2|\mathcal{I}n|$ tours.

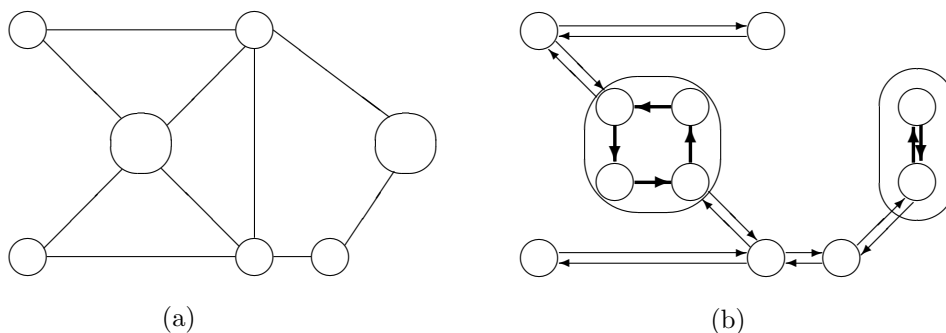


FIG. 7. Graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ and a WGRP tour on G_C satisfying (11) as an equality.

Since the graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ is 2-connected, it can be proved that the only equation satisfied by the incidence vectors of all its spanning trees is $\sum_{e \in T_{\bar{\mathcal{A}}}} x_e = |\bar{\mathcal{A}}| - 1$. Hence, $|T_{\bar{\mathcal{A}}}| = |T|$ different spanning trees can be selected in such a way that their incidence vectors are linearly independent. For each spanning tree of graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ we can define a WGRP tour, such as the one shown in Figure 7(b), which uses each edge in the tree once in each direction and the required edges as in x^d .

If we subtract x^d from the $2|\mathcal{C}| + 2|\mathcal{I}n| + |T|$ tours above and we express them as rows of a matrix, then we obtain a full rank matrix. Hence, the honeycomb inequality is facet-inducing for $\text{WGRP}(G_C)$, and, by applying Theorem 5, it is facet-inducing for $\text{WGRP}(G)$. \square

4.7. Honeycomb₀₂ inequalities. As for K-C and PB inequalities, there is another version of the honeycomb inequalities in which some edges have coefficients 0 and 2. Honeycomb₀₂ inequalities were presented in [6] for the MGRP. As in that paper, we consider here only the case $L=1$, i.e., when only one R -set (or cluster of R -sets) is divided into γ_1 parts. We have not studied the general case because it is quite a bit more complicated. The reason is that, after replacing some pairs of coefficients (1, 1) in a honeycomb configuration with $L > 1$ by coefficients (0, 2), a nonvalid inequality can be obtained.

With a similar configuration graph $G_C = (\mathcal{B}, \mathcal{E})$ and tree T (see Figure 8), the nodes in set A_1 are classified into two types, nodes of type \mathcal{O} (those that will be incident with edges in T with coefficients 0 and 2) and nodes of type \mathcal{I} , in such a way that there is at least one node of each type. We will assume that $d(B_1^i, B_1^j) \geq 3$ if B_1^i, B_1^j are nodes of the same type, while $d(B_1^i, B_1^j) \geq 2$ if B_1^i, B_1^j are of different types. The coefficients are defined as

- $c(B_1^i, B_1^j) = d(B_1^i, B_1^j) - 2$ if B_1^i, B_1^j are nodes of the same type;
- $c(B_1^i, B_1^j) = d(B_1^i, B_1^j) - 1$ if B_1^i, B_1^j are nodes of different types;
- $c(B_1^i, B_q^0) = 0$ and $c(B_q^0, B_1^i) = 2$ if B_1^i is of type \mathcal{O} and $(B_1^i, B_q^0) \in T$;
- for the remaining edges in T , $c(B_q^i, B_r^j) = c(B_r^j, B_q^i) = 1$;
- otherwise, $c(B_q^i, B_r^j)$ is the shortest path cost in T from B_q^i to B_r^j .

With these coefficients, the honeycomb₀₂ inequality is expressed as in (11).

THEOREM 11. *Honeycomb₀₂ inequalities are valid for $\text{WGRP}(G)$.*

Proof. The proof is similar to that in [6] for the MGRP and is omitted here for the sake of brevity. \square

THEOREM 12. *Honeycomb₀₂ inequalities are facet-inducing for $\text{WGRP}(G)$ if the shrunk graph $(\bar{\mathcal{A}}, T_{\bar{\mathcal{A}}})$ is 2-connected.*

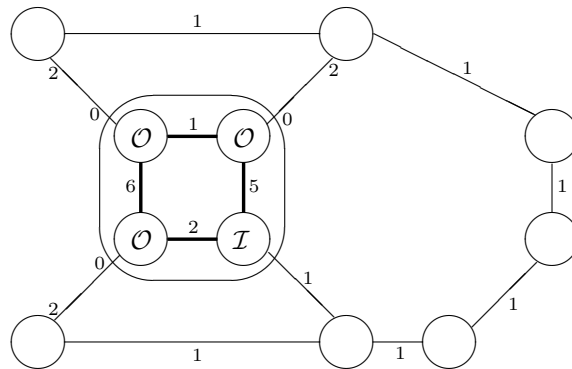


FIG. 8. Honeycomb₀₂ configuration graph with $L=1$.

Proof. The proof is similar to that of Theorem 10, and the details can be found in [30]. The main difference is the way the tours associated with variables that correspond to traversing an edge $e = (u, v)$ in \mathcal{C} from a node u of type \mathcal{O} to a node v of type \mathcal{I} are defined. If (u, v) is required, we construct a tour traversing each edge in the path in T from u to v twice and some edges in T once in each direction to connect those nodes B_i^0 that have still not been visited. In this case not all of the required edges can be traversed as in x^d , and some of them are redirected to obtain a symmetric graph. If e is not required, the tour is constructed in a similar way. \square

4.8. Zigzag inequalities. Zigzag inequalities were introduced in [8]. Here we first describe two families of zigzag inequalities: the even and odd zigzag inequalities. Consider a partition of set V into 4 subsets: M^1 , M^2 , M^3 , and M^4 . Let α_{ij} denote the number of required edges in (M^i, M^j) , and suppose that one of the following conditions is satisfied:

- *Even case:* M^1 , M^2 , M^3 , and M^4 are R -even and $\alpha_{12} = \alpha_{34} = \alpha_{14} = \alpha_{23} = 0$.
- *Odd (simple) case:* M^1 , M^2 , M^3 , and M^4 are R -odd and $\alpha_{12} + \alpha_{34} = \alpha_{14} + \alpha_{23}$.

The configuration graphs $G_{\mathcal{C}}$ associated with even and odd (simple) zigzag inequalities are shown in Figures 9(a) and 9(b) (where the required edges are represented in bold lines) and are defined by the partition of V above and by the following coefficients:

$$c(M^1, M^2) = c(M^3, M^4) = 0, \quad c(M^2, M^1) = c(M^4, M^3) = 2, \quad c(M^i, M^j) = 1 \text{ otherwise.}$$

Then the even and odd (simple) zigzag inequalities can be expressed as

$$(12) \quad x(\delta(M^1 \cup M^2)) + 2x(M^2 : M^1) + 2x(M^4 : M^3) \geq \alpha_{13} + \alpha_{24} + \alpha_{14} + \alpha_{23} + 2,$$

where, given any set of edges (S_1, S_2) , $x(S_1 : S_2) = \sum_{(i,j): i \in S_1, j \in S_2} x_{ij}$.

In the above-mentioned paper it is shown that even and odd (simple) zigzag inequalities (12) are valid and facet-inducing for $\text{WGRP}(G)$ if the following conditions hold: If $G_{\mathcal{C}}$ is a complete graph and $\alpha_{13}, \alpha_{24} \geq 2$ (even case) or $\alpha_{13} \geq |\alpha_{12} - \alpha_{14}| + 1$ and $\alpha_{24} \geq |\alpha_{12} - \alpha_{23}| + 1$ (odd case). All of these inequalities are strong configuration inequalities. Furthermore, it is also proved that odd (simple) zigzag inequalities are equivalent to 3-wheel inequalities for the WPP [34].

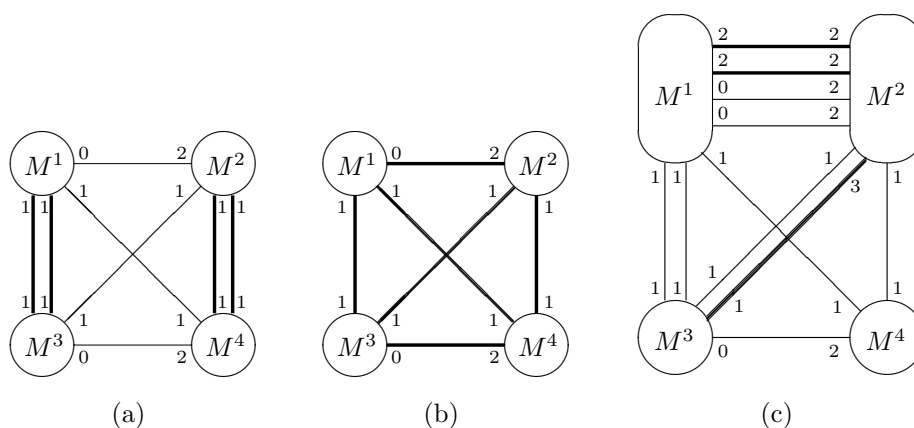


FIG. 9. Zigzag configuration graphs.

Condition $\alpha_{12} + \alpha_{34} = \alpha_{14} + \alpha_{23}$ for the odd (simple) case above is very strong and can be relaxed as follows to obtain a more general version of odd zigzag inequalities. Let M^1, M^2, M^3 , and M^4 be a partition of V into R -odd sets. Let us represent the horizontal edges by $\mathcal{H} = (M^1, M^2) \cup (M^3, M^4)$ and the diagonal ones by $\mathcal{D} = (M^2, M^3) \cup (M^1, M^4)$, and note that $\mathcal{H} \cup \mathcal{D} = \delta(M^1 \cup M^3)$. Let $\mathcal{F} \subset (\mathcal{H} \cup \mathcal{D})_R$ be a subset of required edges (shown as bold lines in Figure 9(c), where all of the edges are required) satisfying

$$(13) \quad |\mathcal{H}_R \setminus \mathcal{F}| + |\mathcal{D}_R \cap \mathcal{F}| = |\mathcal{D}_R \setminus \mathcal{F}| + |\mathcal{H}_R \cap \mathcal{F}|.$$

The more general odd zigzag inequality is

$$(14) \quad \begin{aligned} & x(\delta(M^1 \cup M^2)) + 2x(M^2 : M^1) + 2x(M^4 : M^3) + 2x(F_{zz}) \\ & \geq \alpha_{13} + \alpha_{24} + \alpha_{14} + \alpha_{23} + 2|\mathcal{H} \cap \mathcal{F}| + 2, \end{aligned}$$

where $x(F_{zz})$ denotes the variables associated with the edges in \mathcal{F} in the direction given by the zigzag, i.e., $M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow M^4 \rightarrow M^1$. The pairs of coefficients associated with each edge are shown in Figure 9(c). Odd zigzag inequalities (14) are valid and facet-inducing for $\text{WGRP}(G)$ if $G_C \setminus \mathcal{F}$ is a complete graph, there are two required edges $e_{13} \in (M^1, M^3)$ and $e_{24} \in (M^2, M^4)$, and the remaining required edges in G_C can be oriented to induce a symmetric graph satisfying that all of the edges in $(\mathcal{H} \cup \mathcal{D})_R \setminus \mathcal{F}$ are oriented in the direction of the zigzag and all of the edges in \mathcal{F} are oriented in the opposite direction.

Odd zigzag inequalities (14) are weak configuration inequalities when $\mathcal{F} \neq \emptyset$. If $\mathcal{F} = \emptyset$, they reduce to inequalities (12), and condition (13) becomes $\alpha_{12} + \alpha_{34} = \alpha_{14} + \alpha_{23}$.

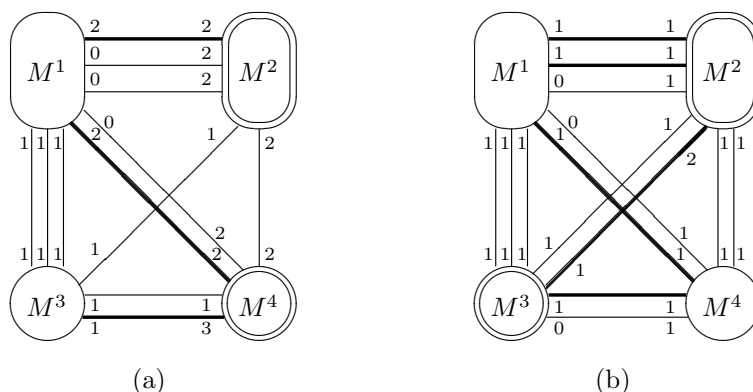
In the odd zigzag inequalities, the direction of the zigzag is $M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow M^4 \rightarrow M^1$ and is determined by the orientation of some edges whose associated variables take fractional values in the linear program (LP) solution. More recently, in [10], two more versions of zigzag inequalities have been described for the WPP, the even-even, and the odd-odd zigzag inequalities. These two kinds of inequalities are designed to cut fractional LP solutions in which some edges in the zigzag have a different orientation, $M^1 \rightarrow M^2 \rightarrow M^3$ and $M^1 \rightarrow M^4 \rightarrow M^3$ for the even-even zigzag inequalities and $M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow M^4$ and $M^1 \rightarrow M^4$ for the odd-odd zigzag inequalities. The name of the inequalities also refers to the degree of the two shores of the edge cutset $\delta(M^1 \cup M^3)$ (see Figures 10(a) and 10(b), where all of the edges are required and a set M^i represented with a double circle is R -odd). These two last types of inequalities are proved to be facet-inducing for the WPP polyhedron under mild conditions. In what follows we describe the generalization of these inequalities to the WGRP.

For the even-even zigzag inequalities, sets M^2 and M^4 are R -odd, sets M^1 and M^3 are R -even, and \mathcal{F} is a subset of required edges $\mathcal{F} \subset \delta_R(M^1 \cup M^3)$ (shown as bold lines in Figure 10(a)), satisfying

$$(15) \quad \begin{aligned} & |(M^1, M^2 \cup M^4)_R \setminus \mathcal{F}| + |(M^2 \cup M^4, M^3)_R \cap \mathcal{F}| \\ & = |(M^1, M^2 \cup M^4)_R \cap \mathcal{F}| + |(M^2 \cup M^4, M^3)_R \setminus \mathcal{F}|. \end{aligned}$$

The corresponding even-even zigzag inequality is

$$(16) \quad \begin{aligned} & x(\delta(M^3)) + 2x(M^2, M^4) + 2x(M^2 : M^1) + 2x(M^4 : M^1) + 2x(F_{zz}) \\ & \geq \alpha_{13} + 2\alpha_{24} + \alpha_{23} + \alpha_{34} + 2|\delta(M^1) \cap \mathcal{F}| + 2, \end{aligned}$$

FIG. 10. Even-even and odd-odd zigzag configurations, with sets \mathcal{F} in bold lines.

where $x(F_{zz})$ denotes the variables associated with the edges in \mathcal{F} in the direction of the zigzag, i.e., $M^1 \rightarrow M^2 \rightarrow M^3$ and $M^1 \rightarrow M^4 \rightarrow M^3$. The coefficients of the variables are also shown in Figure 10(a).

For the odd-odd zigzag inequalities (see Figure 10(b)), sets M^2 and M^3 are R -odd, sets M^1 and M^4 are R -even, set $\mathcal{F} \subset \delta_R(M^1 \cup M^3)$ satisfies the condition

$$(17) \quad \begin{aligned} & |(M^1, M^2 \cup M^4)_R \setminus \mathcal{F}| + |(M^3, M^4)_R \setminus \mathcal{F}| + |(M^2, M^3)_R \cap \mathcal{F}| + 1 \\ & = |(M^1, M^2 \cup M^4)_R \cap \mathcal{F}| + |(M^3, M^4)_R \cap \mathcal{F}| + |(M^2, M^3)_R \setminus \mathcal{F}|, \end{aligned}$$

and the corresponding odd-odd zigzag inequality is

$$(18) \quad \begin{aligned} & x(M^2 \cup M^4 : M^1 \cup M^3) + x(M^1, M^3) + x(M^2, M^4) + x(M^3 : M^2) + x(F_{zz}) \\ & \geq \alpha_{13} + \alpha_{24} + |(M^2, M^3)_R \setminus \mathcal{F}| + |\mathcal{F}| + 1, \end{aligned}$$

where, again, $x(F_{zz})$ denotes the variables associated with the edges in \mathcal{F} in the direction of the zigzag, i.e., $M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow M^4$ and $M^1 \rightarrow M^4$.

The proofs of the following two theorems are similar to those for the WPP in [10] and are omitted here.

THEOREM 13. *Even-even zigzag inequalities (16) are valid for $WGRP(G)$. They are facet-inducing for $WGRP(G)$ if $G_C \setminus \mathcal{F}$ is a complete graph, there are three edges $e_{13}, f_{13} \in (M^1, M^3)_R$ and $e_{24} \in (M^2, M^4)_R$, and the remaining required edges in G_C can be oriented to induce a (directed) symmetric graph where all of the edges in $\delta_R(M^1 \cup M^3) \setminus \mathcal{F}$ are oriented in the direction of the zigzag and all of the edges in \mathcal{F} are oriented in the opposite direction.*

THEOREM 14. *Odd-odd zigzag inequalities (18) are valid for $WGRP(G)$. They are facet-inducing for $WGRP(G)$ if $G_C \setminus \mathcal{F}$ is a complete graph, there are three edges $e_{13} \in (M^1, M^3)_R$, $e_{24} \in (M^2, M^4)_R$, and $e_{14} \in (M^1, M^4)_R \cap \mathcal{F}$, and the remaining required edges in G_C can be oriented to induce a (directed) symmetric graph where all of the edges in $\delta_R(M^1 \cup M^3) \setminus \mathcal{F}$ are oriented in the direction of the zigzag and all of the edges in \mathcal{F} are oriented in the opposite direction.*

5. The mixed GRP polyhedron. As mentioned in the introduction, the MGRP is the general routing problem defined on a mixed graph $G = (V, E, A)$, where V is the set of vertices, E is the set of edges, and A is the set of arcs, and the MCP is a special case of it.

Nobert and Picard [28] proposed a formulation for the MCPP using only one variable per edge expressing the number of times a given edge is traversed in any direction. Such an approach has been also used for the MGRP in [11] and [6], where a wide polyhedral study is carried out. We call this kind of formulation F1. On the other hand, different formulations using two variables per edge were proposed for the MCPP by Christofides et al. [4] and Ralphs [31]. They will be called F2 formulations. A theoretical and computational comparison of both formulations for ARPs on mixed graphs can be found in [7]. Although formulation F2 is perhaps a more intuitive approach to a problem defined on a mixed graph, polyhedral investigations on it have not been presented up to now. That is the subject of this section, where we will take advantage of the previous WGRP study.

As for the WGRP, we will assume again that the original graph has been transformed to satisfy $V = V_R$. In addition, it can also be assumed that $E \setminus E_R = \emptyset$, because each nonrequired edge can be replaced by a pair of opposite nonrequired arcs [11]. Although this last transformation is not as important for formulation F2 as it is for F1, in what follows we suppose that we are working on a strongly connected graph $G = (V, E, A) := (V_R, E_R, A_R \cup A_{NR})$.

An MGRP instance can be transformed into a WGRP instance in which the cost of traversing the edges in the forbidden directions are set to infinity. Then the variables associated with the forbidden directions can be removed from the WGRP formulation, thus obtaining an MGRP formulation with just one variable associated with each arc and two variables associated with each edge. Hence, formulation F2 for the MGRP is exactly the one presented for the WGRP in section 2 except that some variables do not exist (all of the variables x_{ji} associated with arcs (i, j)). Therefore, constraints (1) and (4) are replaced by

$$(19) \quad x_{ij} + x_{ji} \geq 1 \quad \forall (i, j) \in E (= E_R),$$

$$(20) \quad x_{ij} \geq 1 \quad \forall (i, j) \in A_R,$$

$$(21) \quad x_{ij} \geq 0 \quad \text{otherwise.}$$

Let $\text{MGRP}(G)$ represent the convex hull of all of the feasible solutions of the F2 formulation for the MGRP on $G = (V, E, A)$ (MGRP tours). From the results obtained for the WGRP polyhedron, it can be shown that, if G is a strongly connected graph, the dimension of $\text{MGRP}(G)$ is $2|E| + |A| - |V| + 1$. Furthermore, *traversing* and *trivial* inequalities are facet-inducing for $\text{MGRP}(G)$ if the corresponding link (edge or arc) $e = (i, j)$ satisfies that $G \setminus \{e\}$ is a strongly connected graph. All other facet-inducing inequalities for the $\text{MGRP}(G)$ are *weak configuration inequalities*, and their associated configuration graphs can have parallel variables with different coefficients associated with their edges.

Obviously any valid inequality for the WGRP is also a valid inequality for the MGRP. Furthermore, the facet-inducing inequalities for $\text{WGRP}(G)$ described in this paper are also facet-inducing for $\text{MGRP}(G)$ if the arcs in G have the “appropriate direction,” i.e., if the removed variables do not avoid the existence of all of the tours needed to define a facet. Before presenting the following inequalities, we need some more notation. If $G = (V_R, E_R, A_R \cup A_{NR})$ is a mixed graph and S_1, S_2 are two vertex sets, we denote

$$E(S_1, S_2) = \{e = (i, j) \in E : i \in S_1, j \in S_2 \text{ or } j \in S_1, i \in S_2\},$$

$$A(S_1, S_2) = \{a = (i, j) \in A : i \in S_1, j \in S_2\},$$

$$A_R(S_1, S_2) = \{a = (i, j) \in A_R : i \in S_1, j \in S_2\}.$$

A mixed graph is called *balanced* if, for every $S \subset V$, the difference between the number of arcs leaving S and the number of arcs entering S is less than or equal to the number of edges in $E(S, V \setminus S)$.

It can be shown that the following inequalities are facet-inducing for $\text{MGRP}(G)$:

- Connectivity inequalities (3), if graphs $G(S)$ and $G(V \setminus S)$ are strongly connected.
- R -odd cut inequalities (6), if $G(S)$ and $G(V \setminus S)$ are strongly connected and $\epsilon > |\alpha - \beta|$, where $\epsilon = |E(S, V \setminus S)|$, $\alpha = |A_R(S, V \setminus S)|$, and $\beta = |A_R(V \setminus S, S)|$.
- K-C inequalities (7), if $\epsilon \geq |\alpha - \beta|$, and K-C₀₂ inequalities (8), if $\epsilon \geq |\alpha + 1 - \beta| + 1$, where $\epsilon = |E(M_0, M_K)|$, $\alpha = |A_R(M_0, M_K)|$, and $\beta = |A_R(M_K, M_0)|$. Here we assume that all of the sets $A(M_i, M_{i+1})$, $A(M_{i+1}, M_i)$ are nonempty.
- PB inequalities (9), if $\epsilon \geq |\alpha - \beta|$ for P odd and $\epsilon \geq |\alpha - \beta| + 1$ for P even, and path-bridge₀₂ inequalities (10), if $\epsilon \geq |\alpha + P_{AZ} - \beta - P_{ZA}| + 1$, where P_{AZ} and P_{ZA} are the number of paths of types AZ and ZA , respectively, $\epsilon = |E(A, Z)|$, $\alpha = |A_R(A, Z)|$, and $\beta = |A_R(Z, A)|$. The existence of all of the pairs of opposite (nonrequired) arcs forming the P paths is assumed here.
- Honeycomb inequalities (11), if the subgraph of G_C induced by the required links $E \cup A_R$ is balanced and the graph $(\bar{A}, T_{\bar{A}})$ is 2-connected. Honeycomb₀₂ inequalities if, in addition, for each edge or arc (u, v) , with node u of type \mathcal{O} and node v of type \mathcal{I} , the subgraph of G_C induced by the required links plus two extra arcs from u to v is a balanced graph. We assume the existence of all of the pairs of opposite (nonrequired) arcs forming the tree T .
- Zigzag inequalities (12) and (14), if there are 4 links defining a zigzag $M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow M^4 \rightarrow M^1$ and,
 - (a) for the even case, the subgraph of G_C induced by the required links is balanced;
 - (b) for the odd case, there are two required edges $e_{13} \in (M^1, M^3)$ and $e_{24} \in (M^2, M^4)$, and the remaining required edges in G_C can be oriented in such a way that, together with the required arcs in G_C , they induce a symmetric graph in which all of the links in $(\mathcal{H} \cup \mathcal{D})_R \setminus \mathcal{F}$ are oriented in the direction of the zigzag and all of the links in \mathcal{F} are oriented in the opposite direction.
- Even-even zigzag inequalities (16) and odd-odd zigzag inequalities (18), under conditions similar to those above for the odd zigzag inequalities.

Given that the MCP and MRPP are special cases of the MGRP, the above inequalities are also facet-inducing for the polyhedra associated with these problems when they are formulated with two variables.

6. Conclusions. Given a “windy” graph (an undirected graph with two costs associated with each edge, representing the costs of traversing it in each direction) and given a subset of “required” edges and a subset of required vertices, the windy general routing problem consists of finding a minimum cost tour traversing each required edge and visiting each required vertex at least once. Since undirected, directed, and mixed graphs can be modeled with a windy graph, and the GRP generalizes the CPP, RPP, and GTSP, all of the routing problems represented in Figure 1 are special cases of the WGRP. Therefore, most of the theoretical and practical results obtained for the WGRP can be applied to these important problems.

In this paper we have presented a polyhedral study of the WGRP. In addition to describing some basic properties of its associated polyhedron $\text{WGRP}(G)$, we have shown that all of its facet-inducing inequalities, except those equivalent to the trivial or

the traversing ones, are *weak* configuration inequalities. This is a generalization of the (strong) configuration property related to the polyhedra associated with the problems in Figure 1. Then each facet-inducing inequality can be represented by means of a configuration graph, and we have proved that, if a given inequality is facet-inducing for the polyhedron defined on this simple configuration graph, it is also facet-inducing for the whole polyhedron $\text{WGRP}(G)$.

We have also described wide families of facet-inducing inequalities. This partial description of $\text{WGRP}(G)$ has been recently used in [9] to develop a branch and cut algorithm capable of solving WGRP instances of large size and other difficult instances of some of the problems shown in Figure 1. Although slightly different versions of most of these inequalities were already known for other routing problems polyhedra, it is in this paper that they have been generalized for the WGRP and proved to be facet-inducing of $\text{WGRP}(G)$.

On the other hand, a partial description of the polyhedra associated with ARPs defined on mixed graphs whose formulation uses two variables per edge has also been obtained. This polyhedral study had not been done before.

A final contribution of this paper is that it also provides a global view of most of the known facet-inducing inequalities for the polyhedra associated with the problems in Figure 1, since versions of facet-inducing inequalities for other problems with formulations using only one variable per link are very similar.

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