CSCI 3190
Introduction to Discrete Mathematics and Algorithms

Extended Exercise 1

ABSTRACT

Keywords

1. PROPOSITIONAL LOGIC

1.1 Propositional Logic

1. Explain, without using a truth table, why $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p, q, and r is true and at least one is false, but is false when all three variables have the same truth value.

1.2 Application

1. Each inhabitant of a remote village always tells the truth or always lies.A villager will give only a "Yes"or a "No"response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit; the other branch leads deep into the jungle. A villager is standing at the fork in the road. What one question can you ask the villager to determine which branch to take?

1.3 Propositional Equivalences

1. Show that $(p \wedge q) \rightarrow p$ is a tautology by using truth tables.
2. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.
3. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology.

1.4 Predicates and Quantifiers

1. Let $P(x)$ denote the statement "$x \leq 4$."What are these truth values?
   (a) $P(0)$
   (b) $P(4)$
   (c) $P(6)$
2. Let $P(x)$ be the statement "x spends more than five hours every weekday in class,"where the domain for x consists of all students. Express each of these quantifications in English.
   (a) $\exists x P(x)$
   (b) $\forall x P(x)$
   (c) $\exists x \neg P(x)$
   (d) $\forall x \neg P(x)$
3. Translate these statements into English, where $C(x)$ is "x is a comedian"and $F(x)$ is "x is funny"and the domain consists of all people.
   (a) $\forall x(C(x) \rightarrow F(x))$
   (b) $\forall x(C(x) \wedge F(x))$
   (c) $\exists x(C(x) \rightarrow F(x))$
   (d) $\exists x(C(x) \wedge F(x))$
4. Let $P(x)$ be the statement "x can speak Cantonese"and let $Q(x)$ be the statement "x knows the computer language C++."Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at CU.
   (a) There is a student at CU who can speak Cantonese and who knows C++.
   (b) There is a student at CU who can speak Cantonese but who doesnt know C++.
   (c) Every student at CU either can speak Cantonese or knows C++.
   (d) No student at CU can speak Cantonese or knows C++.
5. Let $P(x)$ be the statement "$x = x^2$."If the domain consists of the integers, what are these truth values?
   (a) $P(0)$
   (b) $P(1)$
   (c) $P(2)$
   (d) $P(-1)$
   (e) $\exists x P(x)$
   (f) $\forall x P(x)$

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6. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0 and 1. Write out each of these propositions using disjunctions, conjunctions, and negations.

   (a) $\exists x P(x)$
   (b) $\forall x P(x)$
   (c) $\exists x \neg P(x)$
   (d) $\forall x \neg P(x)$
   (e) $\neg \exists x P(x)$
   (f) $\neg \forall x P(x)$

7. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

   (a) Everyone is studying discrete mathematics.
   (b) Everyone is older than 21 years.
   (c) Every two people have the same mother.
   (d) No two different people have the same grandmother.

8. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

   (a) $\forall x (x^2 \geq x)$
   (b) $\forall x (x > 0 \vee x < 0)$
   (c) $\forall (x = 1)$

9. Determine whether $\forall x (P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

10. Show that $\exists x (P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ are logically equivalent.

1.5 Nested Quantifiers

1. Translate these statements into English, where the domain for each variable consists of all real numbers.

   (a) $\forall x \exists y (x < y)$
   (b) $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$
   (c) $\forall x \forall y \exists z (xy = z)$

2. Show that $\forall x P(x) \wedge \exists x Q(x)$ is logically equivalent to $\forall x \exists y (P(x) \wedge Q(y))$, where all quantifiers have the same nonempty domain.

3. Show that $\forall x P(x) \vee \exists x Q(x)$ is equivalent to $\forall x \exists y (P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain.

APPENDIX

A. ANSWER

A.1 Propositional Logic

A.1.1 Propositional Logic

1. The first clause is true if and only if at least one of p, q, and r is true. The second clause is true if and only if at least one of the three variables is false. Therefore the entire statement is true if and only if there is at least one T and one F among the truth values of the variables, in other words, that they dont all have the same truth value.

A.1.2 Application

1. "If I were to ask you whether the right branch leads to the ruins, would you answer yes? "

A.1.3 Propositional Equivalences

1.

Table 1: Truth Table

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

2. It is a tautology.

3. If both q and r are false, $(p \vee q)$ and $(\neg p \vee r)$ is opposite, then $(\neg q \wedge \neg r) \rightarrow \neg(p \vee q) \vee \neg(\neg p \vee r)$ is tautology, so $\neg(q \vee r) \rightarrow \neg((p \vee q) \wedge (\neg p \vee r))$ is tautology, and $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is tautology.

A.1.4 Predicates and Quantifiers

1. (a) T
   (b) T
   (c) F

2. (a) There is a student who spends more than 5 hours every weekday in class.
   (b) Every student spends more than 5 hours every weekday in class.
   (c) There is a student who does not spend more than 5 hours every weekday in class.
   (d) No student spends more than 5 hours every weekday in class.

3. (a) Every comedian is funny.
   (b) Every person is a funny comedian.
   (c) There exists a person such that if she or he is a comedian, then she or he is funny.
   (d) Some comedians are funny.

4. (a) $\exists x (P(x) \wedge Q(x))$

(b) $\exists x(P(x) \wedge \neg Q(x))$

(c) $\forall x(P(x) \vee Q(x))$

(d) $\forall x \neg(P(x) \vee Q(x)$

5. (a) T
   (b) T
   (c) F
   (d) F
   (e) T
   (f) F

6. (a) $P(0) \vee P(1)$
   (b) $P(0) \wedge P(1)$
   (c) $\neg P(0) \vee \neg P(1)$
   (d) $\neg P(0) \wedge \neg P(1)$
   (e) $\neg(P(0) \vee P(1))$
   (f) $\neg(P(0) \wedge P(1))$

7. Many answers are possible.
   (a) All students in CSCI3190; all students in the world
   (b) All members of the Legislative Council; all students in CSCI3190
   (c) Tan Sri Dr Runme Shaw and Sir Run Run Shaw; all members of the Legislative Council
   (d) Sir Donald Tsang Yam-kuen and Leung Chun-ying; all students in the world

8. (a) There is no counterexample.
   (b) $x = 0$
   (c) $x = 2$

9. They are not equivalent. Let $P(x)$ be any propositional function that is sometimes true and sometimes false, and let $Q(x)$ be any propositional function that is always false. Then $\forall x(P(x) \rightarrow Q(x))$ is false but $\forall xP(x) \rightarrow \forall xQ(x)$ is true.

10. Both statements are true precisely when at least one of $P(x)$ and $Q(x)$ is true for at least one value of x in the domain.

A.1.5 Nested Quantifiers

1. (a) For every real number x there exists a real number y such that x is less than y.
   (b) For every real number x and real number y, if x and y are both nonnegative, then their product is nonnegative.
   (c) or every real number x and real number y, there exists a real number z such that $xy = z$.

2. Suppose that $\forall xP(x) \wedge \exists xQ(x)$ is true. Then $P(x)$ is true for all x and there is an element y for which $Q(y)$ is true. Because $P(x) \wedge Q(y)$ is true for all x and there is a y for which $Q(y)$ is true, $\forall x \exists y(P(x) \wedge Q(y))$ is true. Conversely, suppose that the second proposition is true. Let x be an element in the domain. There is a y such that $Q(y)$ is true, so $\exists xQ(x)$ is true. Because $\forall xP(x)$ is also true, it follows that the first proposition is true.

3. Suppose that $\forall xP(x) \vee \exists xQ(x)$ is true. Then either $P(x)$ is true for all x, or there exists a y for which $Q(y)$ is true. In the former case, $P(x) \vee Q(y)$ is true for all x, so $\forall x \exists y(P(x) \vee Q(y))$ is true. In the latter case, $Q(y)$ is true for a particular y, so $P(x) \vee Q(y)$ is true for all x and consequently $\forall x \exists y(P(x) \vee Q(y))$ is true. Conversely, suppose that the second proposition is true. If $P(x)$ is true for all x, then the first proposition is true. If not, $P(x)$ is false for some x, and for this x there must be a y such that $P(x) \vee Q(y)$ is true. Hence, $Q(y)$ must be true, so $\exists yQ(y)$ is true. It follows that the first proposition must hold.