5

Successive Differentiation

5·1. Introduction

Let $y=f(x)=e^{6x}+5\sin 2x+4x^2-3x+7$, then on differentiating w.r.t. x, we get $y_1=f'(x)=6e^{6x}+10\cos 2x+8x-3$.

Here y_1 or $f'(x)$ is called the first derivative of y or $f(x)$ and it is a function of x. Again differentiating, we have

$$y_2=f''(x)=36e^{6x}-20\sin 2x+8.$$

This is the second derivative of y and it can be further differentiated to give

$y_3=f'''(x)=216\,e^{6x}-40\cos 2x$, which is the third derivative of y. It is denoted by $\dfrac{d^3y}{dx^3}$, D^3y or y'''.

Thus if we differentiate a function y, n-times successively, we will obtain nth derivative of the function y which we denote by

$$y_n,\ f^n(x),\ \frac{d^ny}{dx^n} \text{ or } D^ny.$$

5·2. Standard Results

(*a*) If $\quad y=(ax+b)^m$, then

$$y_1=m.\,a(ax+b)^{m-1}$$

$$y_2=m(m-1).\,a^2(ax+b)^{m-2}$$

$$y_3=m(m-1)(m-2).\,a^3(ax+b)^{m-3}.$$

In general, $y_n=m(m-1)(m-2)\,.\,(m-n+1).\,a^n(ax+b)^{m-n}$

In particular (*i*) $y_n=\dfrac{m\,!}{(m-n)\,!}\,a^n(ax+b)^{m-n}$,

if m is a positive interger$>n$.

(*ii*) $y_n=0$, if m is a positive integer$<n$,

(*iii*) $y_n=n\,!\,a^n$, when $m=n$,

(*iv*) $y=(ax+b)^{-1}$, when $m=-1$,

$$y_n=(-1)(-2)(-3)\,..(-n).\,a^n(ax+b)^{-1-n}$$

$$=(-1)^n (n!) a^n.(ax+b)^{-(n+1)}$$

(*b*) If $y=e^{ax}$, then

$$y_1=ae^{ax},\ y_2=a^2 e^{ax},\ y_3=a^3 e^{ax}.$$

This can be generalised to

$$y_n=a^n e^{ax}.$$

$y=(ax+b)^{-1}$

$(-1)^n n! [a^n][ax+b]^{-(n+1)}$

(*c*) If $y=\log(ax+b)$, then

$$y_1=a/(ax+b)=a(ax+b)^{-1},$$
$$y_2=a^2.(-1)(ax+b)^{-2},$$
$$y_3=a^3(-1)(-2)(ax+b)^{-3},$$
$$y_4=a^4(-1)(-2)(-3)(ax+b)^{-4}$$
$$=a^4(-1)^3\,3\,!\,(ax+b)^{-4}.$$

$$\therefore\quad y_n=a^n(-1)^{n-1}(n-1)\,!\,(ax+b)^{-n}.$$

(*d*) If $y=\sin(ax+b)$, then

$$y_1=a.\cos(ax+b)=a.\sin\left(ax+b+\frac{\pi}{2}\right),$$

on using $\cos\theta=\sin\left(\theta+\dfrac{\pi}{2}\right)$. Again differentiating, we get

$$y_2=a^2.\cos\left(ax+b+\frac{\pi}{2}\right)$$
$$=a^2.\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$
$$=a^2.\sin\left(ax+b+2.\frac{\pi}{2}\right).$$

Similarly, $y_3=a^3.\sin\left(ax+b+3.\dfrac{\pi}{2}\right)$.

Continuing this process, we get

$$y_n=a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

(*e*) If $y=\cos(ax+b)$, then proceeding exactly as above we get $y_n=a^n\cos\left(ax+b+\dfrac{n\pi}{2}\right)$.

(*f*) If $y=e^{ax}\sin(bx+c)$, then

$$y_1=ae^{ax}.\sin(bx+c)+e^{ax}.b\cos(bx+c)$$
$$=e^{ax}\{a\sin(bx+c)+b\cos(bx+c)\}.$$

Now let $a=r\cos\phi$ and $b=r\sin\phi$, we get

$$y_1=e^{ax}.r\{\sin(bx+c)\cos\phi+\cos(bx+c)\sin\phi\}$$
$$=re^{ax}\sin(bx+c+\phi)$$

Similarly, again differentiating and simplifying, we have

$$y_2 = r^2 e^{ax} \sin(bx+c+2\phi)$$

and

$$y_3 = r^3 e^{ax} \sin(bx+c+3\phi).$$

If this process is continued n times, we get

$$y_n = r^n e^{ax} \sin(bx+c+n\phi),$$

where

$$r = \sqrt{(a^2+b^2)} \text{ and } \phi = \tan^{-1} \frac{b}{a}.$$

(g) If $y = e^{ax} \cos(bx+c)$, then proceeding as above we have

$$y_n = r^n e^{ax} \cos(bx+c+n\phi).$$

where r and ϕ are given earlier.

Example 1. *Find the nth derivative of* $\dfrac{x^4}{(x-1)(x-2)}$

Sol. Let

$$y = \frac{x^4}{(x-1)(x-2)}$$

$$= x^2+3x+7+\frac{15x-14}{(x-1)(x-2)}$$

Resolving into partial fractions, we have

$$y = x^2+3x+7-\frac{1}{(x-1)}+\frac{16}{(x-2)}$$

Now differentiating n (>2) times and using the result 5·2(a) with $m=-1$, we have

$$y_n = (-1)^n (n!) \left\{ \frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right\}.$$

Example 2. *If $y = \sin^4 x$, find y_n.*

Sol. Here

$$y = \sin^4 x$$

$$= (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2} \right)^2$$

$$= \tfrac{1}{4}(1-2\cos 2x+\cos^2 2x)$$

$$= \tfrac{1}{4}\{1-2\cos 2x+\tfrac{1}{2}(1+\cos 4x)\}$$

$$= \tfrac{1}{8}(3-4\cos 2x+\cos 4x)$$

Now differentiating n times w. r. t. x and using the result 5·2 (e), we have

$$y_n = \tfrac{1}{8} \left\{ -4.2^n \cos\left(2x+\frac{n\pi}{2}\right) + 4^n \cos\left(4x+\frac{n\pi}{2}\right) \right\}$$

$$= 2^{n-1} \left\{ 2^{n-2} \cos\left(4x+\frac{n\pi}{2}\right) - \cos\left(2x+\frac{n\pi}{3}\right) \right\}$$

Example 3. *If $y = e^{3x} \cos x \cos 2x \sin x$, find y_n.*

Sol. Here $y=\frac{1}{2}e^{3x}(2\cos x \sin x)\cos 2x$

$$=\frac{1}{2}e^{3x}\sin 2x\cos 2x$$

$$=\frac{1}{4}e^{3x}\sin 4x$$

Now using the result 5·2 (d), we have

$$y_n=\frac{1}{4}\,.\,5^n e^{3x}\sin\left(4x+n\tan^{-1}\frac{4}{3}\right).$$

Example 4. *Find y_n when $y=\tan^{-1}\left(\frac{x}{a}\right)$.*

Sol. Differentiating y w. r. t. x, we have

$$y_1=\frac{a}{x^2+a^2}=\frac{a}{(x-i\,a)(x+i\,a)}$$

$$=\frac{1}{2i}\left\{\frac{1}{(x-i\,a)}-\frac{1}{(x+i\,a)}\right\}$$

Differentiating $(n-1)$ times, we have

$$y_n=\frac{(-1)^{n-1}\,(n-1)\,!}{2\,i}\left\{\frac{1}{(x-i\,a)^n}-\frac{1}{(x+i\,a)^n}\right\}$$

Now y_n can be simplified by using De Moivre's theorem. Let $x=r\cos\phi$ and $a=r\sin\phi$.

Then $(x-i\,a)^{-n}=(r\cos\phi-i\,r\sin\phi)^{-n}$

$$=r^{-n}(\cos\phi-i\sin\phi)^{-n}=r^{-n}(\cos n\phi+i\sin n\phi).$$

Similarly $(x+i\,a)^{-n}=r^{-n}(\cos n\phi-i\sin n\phi)$.

$$\therefore \quad y_n=\frac{(-1)^{n-1}\,(n-1)\,!}{2\,i}\,r^{-n}\,\{(\cos n\phi+i\sin n\phi)$$

$$-(\cos n\phi-i\sin n\phi)\}$$

$$=(-1)^{n-1}\,(n-1)\,!\,r^{-n}\sin n\phi$$

$$=(-1)^{n-1}\,(n-1)\,!\,a^{-n}\sin^n\phi\sin n\phi$$

for $\qquad a=r\sin\phi \Rightarrow \frac{1}{r}=\frac{1}{a}\sin\phi$

or $\qquad r^{-n}=a^{-n}\sin^n\phi.$

Example 5. *Find the nth derivative of $\frac{1}{(1+x+x^2)}$*

Sol. Let $y=\frac{1}{x^2+x+1}=\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}=\frac{1}{t^2+a^2}$,

where $\qquad t=x+\frac{1}{2}$ and $a=\frac{\sqrt{3}}{2}$

Now applying the method of Example 4, we get

$$y_n=\frac{(-1)^n\,n\,!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\,\theta$$

where $\qquad \tan\theta=\frac{\sqrt{3}}{2x+1}$, $a=\frac{\sqrt{3}}{2}$.

EXERCISE 5 (*a*)

1. If $y=\left(\dfrac{1+x}{1-x}\right)^n$, show that $\dfrac{dy}{dx}=\dfrac{2ny}{(1-x^2)}$ and
$$\frac{d^2y}{dx^2}=\frac{2(n+x)}{(1-x^2)}\cdot\frac{dy}{dx}$$

2. Find $\dfrac{d^2y}{dx^2}$ where $x=a\,(\theta+\sin\theta)$, $y=a(1-\cos\theta)$

3. If $y=a\,e^{-kt}\cos(pt+c)$, show that
$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+n^2y=0, \text{ where } n^2=p^2+k^2.$$

4. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$, prove that
$$p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}$$

5. If $y=\sin kx+\cos kx$, prove that
$$y_n=k^n\,\{1+(-1)^n\sin 2kx\}^{1/2}$$

6. Find the nth differential co-efficients of

(*i*) $\cos 2x\sin 3x$ (*ii*) $3\cos 5x\cos 3x+x^n$

(*iii*) $\cos x\cos 2x\cos 3x$ (*iv*) $\sin^2 x\cos^3 x$

(*v*) $\cos^4 x$ (*vi*) $e^x\sin x\sin 2x$

7. Find the nth derivatives of

(*i*) $\dfrac{2x-1}{(x-2)(x+1)}$ (*ii*) $(1-5x+6x^2)^{-1}$

(*iii*) $\dfrac{4x}{(x-1)^2(x+1)}$ (*iv*) $\tan^{-1}\left\{\left(\dfrac{1+x}{1-x}\right)\right\}$.

8. If $y=\tan^{-1}x$, show that
$$y_n=(n-1)!\cos\left\{ny+(n-1)\frac{\pi}{2}\right\}\cos^n y.$$

9. If $y=e^{x\cos\beta}\cos(x\sin\beta)$, show that
$$y_n=e^{x\cos\beta}\cdot\cos(x\sin\beta+n\beta)$$

10. Prove that the value of the nth derivative of $\dfrac{x^3}{x^3-1}$ for $x=0$, is zero when n is even and $(-n\,!)$ when n is odd and greater than one.

11. If $y=\log\sqrt{\dfrac{2x+1}{x-2}}$, show that
$$y_n=\frac{1}{2}(-1)^{n-1}(n-1)!\left\{\frac{2^n}{(2x+1)^n}-\frac{1}{(x-2)^n}\right\}$$

12. If $y=\cosh 2x$, show that (*i*) $y_n=2^n \sinh 2x$ when n is odd and (*ii*) $y_n=2^n \cosh 2x$ when n is even.

[**Hint.** $y=\cosh 2x=\frac{1}{2}(e^{2x}+e^{-2x})$. Now find y_n.]

13. If $y=e^{ax} \sin bx$, prove that

(*i*) $$y_n=(a \sec \theta)^n e^{ax} \sin \left(bx+n \tan^{-1}\frac{b}{a} \right)$$

(*ii*) $$y_{n+1}-2ay_n+(a^2+b^2)\, y_{n-1}=0$$

14. If $f(x)=e^{-ax} \cos (bx+c)$ show that

$$f^n (x)=(-1)^n (a^2+b^2)^{\frac{n}{2}} e^{-ax} \cos \left(bx+c+n \tan^{-1}\frac{b}{a}\right).$$

Also find f^n (0) when $a=b=1$ and $c=0$.

15. If $(1-x^2) \tan y=2x$, show that

$$y_n=2(-1)^{n-1} (n-1) ! \sin^n w \sin nw$$

where $\cot w=x$.

16. If $y=x \log\left(\frac{x-1}{x+1} \right)$, prove that

$$y_n=(-1)^n (n-2) ! \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n} \right]$$

17. If $y=x(a^2+x^2)^{-1}$, show that

$$y_n=(-1)^n n ! \, a^{-n-1} \sin^{n+1} \phi \cos (n+1) \phi$$

where $\phi=\tan^{-1}\left(\frac{a}{x} \right)$.

18. If $y=x \log (x+1)$, prove that

$$y_n=\frac{(-1)^{n-2} (n-2) ! \,(x+n)}{(x+1)^n}$$

5·3. Leibnitz 's Theorem

If $y=u.v$, where u and v are functions of x having derivatives of nth order, then

$$y_n=u_n . v+{}^nC_1.u_{n-1} . v_1+{}^nC_2 . u_{n-2} . v_2+\ldots+{}^nC_r\, u_{n-r} . v_r+\ldots+u . v_n.$$

where suffixes in u and v denote the order of differentiation with respect to x and ${}^nC_1$, ${}^nC_2$,...have their usual meanings.

This theorem is proved by Mathematical induction.

Step I. By differentiating $y=u . v$ successively, we get

$$y_1=u_1 v+u . v_1,$$
$$y_2=(u_2 . v+u_1 . v_1)+(u_1 . v_1+u . v_2)$$
$$=u_2 . v+2u_1 . v_1+u . v_2$$
$$=u_2 . v+{}^2C_1\, u_1 v_1+u . v_2$$

$$y_3=(u_3 \,.\, v+u_2 \,.\, v_1)+2(u_2 \,.\, v_1+u_1 \,.\, v_2)+(u_1 \,.\, v_2+u \,.\, v_3)$$
$$=u_3 \,.\, v+3u_2 \,.\, v_1+3u_1 \,.\, v_2+u \,.\, v_3$$
$$=u_3 \,.\, v+{}^3C_1\, u_2 \,.\, v_1+{}^3C_2\, u_1 \,.\, v_2+u \,.\, v_3.$$

Thus the theorem is true for $n=1, 2$ and 3.

Step II. Now we assume that the theorem is true for a particular value of n. Differentiating y_n with respect to x once again, we have

$$y_{n+1}=(u_{n+1} \,.\, v+u_n \,.\, v_1)+({}^nC_1\, u_n \,.\, v_1+{}^nC_1\, u_{n-1} \,.\, v_2)+\ldots\ldots$$
$$+({}^nC_r\, u_{n-r+1} \,.\, v_r+{}^nC_r\, u_{n-r} \,.\, v_{r+1})+\ldots+(u_1 \,.\, v_n+u \,.\, v_{n+1})$$
$$=u_{n+1} \,.\, v+(1+{}^nC_1)\, u_n \,.\, v_1+({}^nC_1+{}^nC_2)\, u_{n-1} \,.\, v_2+\ldots$$
$$+({}^nC_{r-1}+{}^nC_r)\, u_{n-r+1} .\, v_r+\ldots+u \,.\, v_{n+1}$$
$$=u_{n+1} \,.\, v+{}^{n+1}C_1\, u_n \,.\, v_1+{}^{n+1}C_2\, u_{n-1} \,.\, v_2+\ldots$$
$$+{}^{n+1}C_r\, u_{n-r+1} \,.\, v_r+\ldots+u \,.\, v_{n+1},$$

since $({}^nC_{r-1}+{}^nC_r)={}^{n+1}C_r$; ${}^nC_0+{}^nC_1={}^{n+1}C_1$

or $1+{}^nC_1={}^{n+1}C_1$; ${}^nC_1+{}^nC_2={}^{n+1}C_2$ etc.

Thus we see that if the theorem is true for a particular value of n, it is also true for $(n+1)$. But the theorem is true for $n=3$, so it is also true for $n=3+1=4$ and so on. Therefore, it must be true for every positive integral value of n.

Remark 1. If one of the functions out of the product is a polynomial function of x, we will generally take it as v while the function whose nth derivative is easily known, will be taken as u. For example, in case

$$y=(3x^2-7x+4)\, e^{5x},$$

take $u=e^{5x}$ and $v=3x^2-7x+4$.

Then $\quad u_n=5^n e^{5x},$

$\quad u_{n-1}=5^{n-1} e^{5x},$

$\quad u_{n-2}=5^{n-2} e^{5x}$ etc.

and $\quad v_1=6x-7,\ v_2=6,\ v_3=0=v_4=v_5=..$ etc.

$$\therefore \quad y^n=D^n\{e^{5x}\times(3x^2-7x+4)\}$$
$$=(5^n e^{5x})\times(3x^2-7x+4)+{}^nC_1(5^{n-1}e^{5x})\times(6x-7)+{}^nC_2(5^{n-2}\, e^{5x})\times 6$$
$$=5^{n-2}\, e^{5x}\,\{25(3x^2-7x+4)+5n(6x-7)+3n(n-1)\}$$

Here we get the first three terms only as all the rest vanish.

Remark 2. The Leibnitz's theorem can also be stated as follows.

$$D^n(uv)=D^n\, u \,.\, v+{}^nC_1 \,.\, D^{n-1}\, u \,.\, Dv+{}^nC_2\, D^{n-2}\, u \,.\, D^2v+\ldots$$
$$+{}^nC_r\, D^{n-r}\, u \,.\, D^rv+\ldots+u \,.\, D^nv.$$

Example 1. *If* $y = x^n/(x+1)$,

show that $$y_n = \frac{n!}{(x+1)^{n+1}}.$$

Sol. Let $$u = \frac{1}{(x+1)} = (x+1)^{-1}$$

and $$v = x^n.$$

Then $$u_1 = -(x+1)^{-2},$$
$$u_2 = (-1)^2\, 2!\,(x+1)^{-3}$$
$$u_n = (-1)^n\, n!\,(x+1)^{-(n+1)}$$

and $$v_1 = nx^{n-1},$$
$$v_2 = n(n-1)\,x^{n-2},\ \ldots,\ v_n = n!$$

Now $$y_n = D^n(u \times v)$$
$$= (-1)^n\, n!\,(x+1)^{-(n+1)}.\, x^n + {^nC_1}(-1)^{n-1} \times (n-1)!\,(x+1)^{-n}.\, nx^{n-1} + {^nC_2}\,.\,(-1)^{n-2} \times (n-2)!\,(x+1)^{-(n-1)}\,.\, n(n-1)x^{n-2} + \ldots + \frac{1}{x+1}\,.\, n!$$
$$= \frac{(-1)^n\, n!}{(x+1)^{n+1}}\{x^n - {^nC_1}\, x^{n-1}\,.\,(x+1) + {^nC_2}\, x^{n-2}\,.\,(x+1)^2 - \ldots + (x+1)^n\}$$
$$= \frac{(-1)^n\, n!}{(x+1)^{n+1}}\{[x-(x+1)]^n\}$$
$$= \frac{(-1)^n\, n!}{(x+1)^{n+1}}\{(-1)^n\}$$
$$= n!/(x+1)^{n+1}.$$

Example 2. *If* $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$

prove that $x^2 y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2\, y_n = 0.$

Sol. Simplifying $\cos^{-1}\left(\frac{y}{b}\right) = n \log\left(\frac{x}{n}\right)$

$$\Rightarrow \frac{y}{b} = \cos\left\{n \log\left(\frac{x}{n}\right)\right\}.$$

Differentiating, $$y_1 = -b \sin\left\{n \log\left(\frac{x}{n}\right)\right\} \times \frac{n}{x}$$

$$\Rightarrow y_1\,.\,x = -bn \sin\{n \log(x/n)\}.$$

Differentiating again and simplifying, we have

$$y_2\,.\,x + y_1\,.\,1 = -bn \cos\{n \log(x/n)\}\,.\,n/x$$
$$= -\frac{n^2}{x}\, y$$

or $$y_2 \cdot x^2 + y_1 \cdot x + n^2 y = 0.$$

Now differentiating n times by Leibnitz's theorem, we have

$$(y_{n+2} \cdot x^2 + {}^nC_1\, y_{n+1} \cdot 2x + {}^nC_2\, y_n \cdot 2) + (y_{n+1} \cdot x + {}^nC_1\, y_n \cdot 1) + n^2 y_n = 0$$

or $$\{x^2 y_{n+2} + 2nx\, y_{n+1} + n(n-1) y_n\} + (x\, y_{n+1} + n\, y_n) + n^2\, y_n = 0$$

or $$x^2\, y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2\, y_n = 0.$$

Example 3. *If $y = e^{a \sin^{-1} x}$, prove that*

(i) $(1-x^2)\, y_2 - xy_1 - a^2 y = 0$

(ii) $(1-x^2) y_{n+2} - (2n+1)\, x\, y_{n+1} - (n^2+a^2)\, y_n = 0.$

Hence find the value of y_n when $x = 0$.

Sol. Differentiating y w.r.t. x, we get

$$y_1 = \frac{ae^{a \sin^{-1} x}}{\sqrt{(1-x^2)}} = \frac{ay}{\sqrt{1-x^2}} \quad \ldots(i)$$

or $$y_1^2(1-x^2) = a^2 y^2$$

Differentiating again, we have

$$2y_1 y_2 \cdot (1-x^2) + y_1^2 \cdot (-2x) = a^2 \cdot 2yy_1$$

or $$y_2 \cdot (1-x^2) - y_1 \cdot x - a^2 y = 0 \quad \ldots(ii)$$

Differentiating (*ii*), n times by Leibnitz theorem, we have

$$\{y_{n+2} \cdot (1-x^2) + {}^nC_1\, y_{n+1} \cdot (-2x) + {}^nC_2 \cdot y_n \cdot (-2)\} - (y_{n+1} \cdot x + {}^nC_1\, y_n \cdot 1) - a^2\, y_n = 0$$

or $$(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} - (n^2+a^2) y_n = 0 \quad \ldots(iii)$$

Now putting $x=0$ in (*i*), (*ii*) and (*iii*), we get

$$(y_1)_0 = a,$$

$$(y_2)_0 = a^2$$

and $$(y_{n+2})_0 = (n^2+a^2)(y_n)_0 \quad \ldots(iv)$$

Taking $n = 1, 3, 5, 7, \ldots (n-2)$ in (*iv*), we get

$$(y_3)_0 = (1^2+a^2)(y_1)_0 = (1+a^2) \cdot a$$

$$(y_5)_0 = (3^2+a^2)(y_3)_0 = (3^2+a^2)(1+a^2) \cdot a$$

$$(y_7)_0 = (5^2+a^2)(y_5)_0 = (5^2+a^2)(3^2+a^2)(1+a^2)a$$

$$\vdots \qquad \vdots$$

$$(y_n)_0 = \{(n-2)^2+a^2\}(y_{n-2})_0$$

$$= \{(n-2)^2+a^2\} \cdot \{(n-4)^2+a^2\}(y_{n-4})_0$$

$$= \{(n-2)^2+a^2\} \cdot \{(n-4)^2+a^2\} \ldots (3^2+a^2)(1+a^2)\, a.$$

This is when n is odd. When n is even, taking $n = 2, 4, 6, \ldots (n-2)$ in (*iv*), we have

$$(y_4)_0 = (2^2+a^2)(y_2)_0 = (2^2+a^2) \cdot a^2$$

$$(y_6)_0=(4^2+a^2)(y_4)_0=(4^2+a^2)\,.\,(2^2+a^2)\,.\,a^2$$
$$(y_8)_0=(6^2+a^2)(y_6)_0=(6^2+a^2)(4^2+a^2)(2^2+a^2)\,.\,a^2$$
$$\vdots \qquad\qquad \vdots$$
$$(y_n)_0=\{(n-2)^2+a^2\}(y_{n-2})_0$$
$$=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}(y_{n-4})_0$$
$$=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots(4^2+a^2)(2^2+a^2)\ a^2,$$

when n is even.

Remark. If $\quad y=f(x)$,

then
$$(y)_0=(y)_{x=0}=f(0),$$
$$(y_1)_0=(y_1)_{x=0}=f'(0),$$
$$(y_2)_0=(y_2)_{x=0}=f''(0),$$
$$(y_3)_0=(y_3)_{x=0}=f'''(0),$$

and in general $\quad (y_n)_0=(y_n)_{x=0}=f^n(0),$

EXERCISE 5 (b)

1. Find the nth derivative of

 (a) $x^3\ e^{3x}$ (b) $x^4 \sin 3x$ (c) $x^n e^x$ (d) $e^x \log x$

2. If $y=a\cos(\log x)+b\sin(\log x)$, show that

 (i) $x^2y_2+xy_1+y=0$

 (ii) $x^2y_{n+2}+(2n+1)x\ y_{n+1}+(n^2+1)y_n=0.$

3. If $y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x$, prove that

 $(x^2-1)\ y_{n+2}+(2n+1)x\ y_{n+1}+(n^2-m^2)\ y_n=0.$

4. If $y=(x^2-1)^n$, prove that

 $(x^2-1)\ y_{n+2}+2x\ y_{n+1}-n(n+1)\ y_n=0.$

5. If $y=\sin(m\sin^{-1}x)$, prove that

 $(1-x^2)\ y_{n+2}-(2n+1)\ x\ y_{n+1}-(n^2-m^2)\ y_n=0.$

 Also find the value of y_n when $x=0$. (*A.M.I.E. 1961, 66, 71, 74*)

6. If $f(x)=e^{m\cos^{-1}x}$, find $f^n(0)$ when n is even.

7. If $f(x)=\{\log(x+\sqrt{1+x^2})\}^2$, show that

 $f^{n+2}(0)=-n^2 f^n(0).$

8. If $x+y=1$, prove that

 $D^n(x^n\ y^n)=n\,!\,[y^n-({}^nC_1)^2\ y^{n-1}\ x+({}^nC_2)^2\ y^{n-2}.\ x^2-\ldots\ldots$
 $\ldots+(-1)^n\ x^n].$

9. If $y=\log(x+\sqrt{x^2+a^2})$, show that

 (i) $(a^2+x^2)y_2+xy_1=0$ (ii) $\lim\limits_{x\to 0}\left(\dfrac{y_{n+2}}{y_n}\right)=-\dfrac{n^2}{a^2}.$

10. If $y=(\sin^{-1} x)^2$, show that

$$(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}-n^2\,y_n=0.$$

Hence find y_n when $x=0$.

11. If $f(x)=\sin^{-1} x/\sqrt{(1-x^2)}$, show that

$$f^{n+1}(0)-n^2 f^{n-1}(0)=0.$$

Hence evaluate $f^n(0)$.

12. If $y=e^{\tan^{-1} x}$, prove that

$$(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0.$$

Hence calculate y_2, y_4, y_5 and y_6 when $x=0$.

13. Show that

$$D^n(x^{n-1}\log x)=\frac{(n-1)\,!}{x}$$

14. Show that

$$(1-x)\,y_{n+1}-(n+\alpha x).\,y_n-n\alpha y_{n-1}=0,$$

where $y=(1-x)^{-\alpha}\,e^{-\alpha x}$.

15. If $x=\cosh\left(\frac{1}{m}\log y\right)$, prove that

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$$

16. If $V_n=\frac{d^n}{dx^n}(x^n\log x)$, show that

$$V_n=nV_{n-1}+(n-1)\,!$$

Hence show that

$$V_n=n\,!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)$$

17. If $y=(\tan^{-1} x)^2$, prove that

(*i*) $(x^2+1)^2y_2+2x(x^2+1)y_1=2$

(*ii*) $(x^2+1)^2y_{n+2}+(4n+2)x(x^2+1)y_{n+1}+2n^2(3x^2+1)y_n$
$+2n(n-1)(2n-1)xy_{n-1}+n(n-1)^2(n-2)y_{n-2}=0.$

18. If $y=e^{\frac{1}{2}x^2}\cos x$, prove that

$$y_{2n+2}(0)-4ny_{2n}(0)+2n(2n-1)y_{2n-2}(0)=0$$

19. If $y=(1+x^2)^{\frac{m}{2}}\sin(m\tan^{-1} x)$, prove that

(*i*) $y_{2n}(0)=0$ (*ii*) $y_{2n+1}(0)=(-1)^n m(m-1)(m-2)\ldots(m-2n).$

20. If $\sin^{-1} y=2\log(x+1)$, prove that

$$(x+1)^2\,y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0.$$

6·1. The
mentary functi
expand the gi
form a_0+a_1x
assumed that
tinuous deriva
consideration

6·2. Maclau

If a fun
series of posi

$f(x)=f($

where $f^n(0)$ s

Proof.
series, let

By suc

f

Substi