Jaggi Exercise- 3(a)

Ques-12. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that

$$u_x^2 + u_y^2 + u_z^2 = 2(x u_x + y u_y + z u_z).$$

Ans Differentiating w.r.t x,

$$\frac{(a^2+u)2x - x^2 u_x}{(a^2+u)^2} + \frac{(-y^2 u_x)}{(b^2+u)^2} + \frac{(-z^2 u_x)}{(c^2+u)^2} = 0$$

$$-\frac{x^2 u_x}{(a^2+u)^2} - \frac{y^2 u_x}{(b^2+u)^2} - \frac{z^2 u_x}{(c^2+u)^2} + \frac{2x}{a^2+u} = 0$$

$$\frac{2x}{a^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] u_x$$

Differentiating w.r.t. y,

$$\frac{-x^2 u_y}{(a^2+u)^2} + \frac{(b^2+u)2y - y^2 u_y}{(b^2+u)^2} + \frac{(-z^2 u_y)}{(c^2+u)^2} = 0$$

$$\Rightarrow \frac{2y}{(b^2+u)} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] u_y$$

Differentiating w.r.t. z,

$$\frac{2z}{c^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] u_z$$

$$\therefore \quad u_x = \left(\frac{2x}{a^2+u}\right) \Big/ \left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right)$$

$$u_y = \left(\frac{2y}{b^2+u}\right) \Big/ \left(\quad '' \quad \right)$$

$$u_z = \left(\frac{2z}{c^2+u}\right) \Big/ \left(\quad '' \quad \right)$$

Then LHS = $u_x^2 + u_y^2 + u_z^2$

$$= \frac{\left(\frac{2x}{a^2+u}\right)^2 + \left(\frac{2y}{b^2+u}\right)^2 + \left(\frac{2z}{c^2+u}\right)^2}{\left[\frac{x^2}{(a^2+u)^2} + \left(\frac{y}{b^2+u}\right)^2 + \left(\frac{z}{c^2+u}\right)^2\right]^2}$$

$$= 4\left[\frac{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}}{\left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}^2}\right] = \frac{4}{\left(\frac{x}{a^2+u}\right)^2 + \left(\frac{y}{b^2+u}\right)^2 + \left(\frac{z}{c^2+u}\right)^2}$$

RHS = $2(xu_x + yu_y + zu_z)$

$$= 2\left[\frac{\frac{2x^2}{a^2+u} + \frac{2y^2}{b^2+u} + \frac{2z^2}{c^2+u}}{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}}\right]$$

$$= 4\left[\frac{\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}}{\left(\frac{x}{a^2+u}\right)^2 + \left(\frac{y}{b^2+u}\right)^2 + \left(\frac{z}{c^2+u}\right)^2}\right]$$

$$= \frac{4(1)}{\left(\frac{x}{a^2+u}\right)^2 + \left(\frac{y}{b^2+u}\right)^2 + \left(\frac{z^2}{c^2+u}\right)^2} \qquad \left(\because \text{ given } \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1\right)$$

= LHS

Hence proved.

Jaggi Exercise-3(b)

Ques-9. Show that the function

$$u(x,y,z,t) = \frac{1}{(2a\sqrt{\pi t})^3} \cdot e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}}$$

where $x_0, y_0, z_0,$ a are constants, satisfies the equation

$$\frac{4a^2}{(4a^2)^2 t^2} \quad \frac{1}{4a^2t^2}$$

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Ans

$$\cancel{\frac{\partial u}{\partial t} = \frac{1}{2a\pi^{3/2}\, t^{3/2}}\, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{[-(x-x_0)^2-(y-y_0)^2-(z-z_0)^2]\,4a^2}{(4a^2t)^2}\right]}$$

$$\cancel{= \frac{1}{2a(\pi t)^{3/2}}\, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{-(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{4a^2t^2}\right]}$$

$$\frac{\partial u}{\partial x} = \frac{1}{(2a)^3(\pi t)^{3/2}}\, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{\cancel{2}(x-x_0)}{{}_2\cancel{4}a^2t}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{(2a)^3(\pi t)^{3/2}}\, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{1}{2a^2t} + \left(\frac{x-x_0}{2a^2t}\right)^2\right]$$

$$\frac{\partial u}{\partial y} = \frac{1}{(2a)^3(\pi t)^{3/2}}\, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left(\frac{y-y_0}{2a^2t}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{(2a)^3(\pi t)^{3/2}}\, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{1}{2a^2t} + \left(\frac{y-y_0}{2a^2t}\right)^2\right]$$

Similarly,

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{(2a)^3(\pi t)^{3/2}}\, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{1}{2a^2t} + \left(\frac{z-z_0}{2a^2t}\right)^2\right]$$

$$\text{RHS} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$= \frac{a^2}{(2a)^3(\pi t)^{3/2}}\, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{3}{2a^2t} + \left(\frac{x-x_0}{2a^2t}\right)^2 + \left(\frac{y-y_0}{2a^2t}\right)^2 + \left(\frac{z-z_0}{2a^2t}\right)^2\right]$$

$$\frac{\partial u}{\partial t} = \frac{1}{(2a)^3(\pi t)^{3/2}} \, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{-(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{(4a^2t)^2} \cdot 4a^2 \right]$$

$$+ \frac{1}{(2a)^3\pi^{3/2}} \left(-\frac{3}{2}\right) t^{-5/2} \, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}}$$

$$\frac{4a^2}{(4a^2)^2 t^2}$$

$$= \frac{a^2}{a^2} \left\{ \frac{1}{(2a)^3(\pi t)^{3/2}} \, e^{\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}} \left[\frac{-(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{4a^2t^2} - \frac{3}{2t} \right] \right\}$$

$$\frac{1}{t^{5/2}} \quad t^{3/2} \cdot t$$

$$= \frac{a^2}{(2a)^3(\pi t)^{3/2}} \, e^{(\quad)} \left[\frac{-(x-x_0)^2-(y-y_0)^2-(z-z_0)^2}{\boxed{4a^2t^2 \cdot a^2}} - \frac{3}{2a^2t} \right]$$

$$(2a^2t)^2$$

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Q-7.

x^2

Ans

u

∂u

∂^2

Jaggi Exercise - 3(e)

Q-7. If $x = r\cos\theta$, $y = r\sin\theta$ show that

$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} - u\right)$

$r\left[r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right.$

$$x^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial x^2} = r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)u.$$

Ans

$$\text{RHS} = r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} - u\right) = r\left(r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial r}\right)$$

$$= r^2\frac{\partial^2 u}{\partial r^2}$$

$u = u(x, y) \qquad x = x(r, \theta) \qquad y = y(r, \theta)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)$$

$$= \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x\partial y}\cdot\frac{\partial y}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial y\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial r}\right]$$

$$= \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\cos\theta + \frac{\partial^2 u}{\partial x\partial y}\sin\theta\right] + \sin\theta\left[\frac{\partial^2 u}{\partial x\partial y}\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin\theta\right]$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}$$

$$\therefore \text{RHS} = r^2\frac{\partial^2 u}{\partial r^2} = r^2\cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + r^2\sin^2\theta\frac{\partial^2 u}{\partial y^2}$$

$$= x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= \text{LHS}$$

Hence proved.

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Jaggi Exercise 3(g)

Ques-6. z is an implicit function of x and y defined by the equation $z^3 - 2xz + y = 0$, which takes the value $z = 1$ for $x = 1$, $y = 1$. Expand the function z in increasing powers of $x-1$ and $y-1$.

Ans We have (using corollary 2 of Taylor's theorem)

$$f(x,y) = f(a,b) + \{(x-a) f_x(a,b) + (y-b) f_y(a,b)\} + \frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\} + \dots \quad (A)$$

Here $a = 1$, $b = 1$ $\quad f(x,y) = z(x,y)$ (Need to find z_x, z_y).

We have $z^3 - 2xz + y = 0$

Differentiating w.r.t. x, $\quad 3z^2 z_x - 2x z_x - 2z = 0 \quad$ — ①

$x=1$ $y=1$ pe $z=1$

$$z_x = \frac{2z}{3z^2 - 2x} \qquad z_x(1,1) = \frac{2z}{3z^2 - 2x} \text{ at } (1,1) = \frac{2}{3-2} = \boxed{2}$$

Differentiating w.r.t y we get, $\quad 3z^2 z_y - 2x z_y + 1 = 0 \quad$ — ③

$$z_y = \frac{-1}{3z^2 - 2x} \qquad \text{At } (1,1) \quad z_y(1,1) = \frac{-1}{3-2} = \boxed{-1}$$

① can be written as, $\quad z_x(3z^2 - 2x) - 2z = 0 \quad$ — ②

Differentiating w.r.t x, $\quad z_x(6z z_x - 2) + z_{xx}(3z^2 - 2x) - 2z_x = 0$

$$6z(z_x)^2 - 2z_x + z_{xx}(3z^2 - 2x) - 2z_x = 0$$

$$\therefore \quad z_{xx} = \frac{4z_x - 6z(z_x)^2}{3z^2 - 2x}$$

-2^{+12}

$$\text{At } (1,1) \quad z_{xx}(1,1) = \frac{4(2) - 6(1)(2)^2}{3(1) - 2(1)} = \frac{8 - 24}{1} = \boxed{-16}$$

Differentiating ② w.r.t. y, $\quad z_x(6z z_y) + z_{xy}(3z^2 - 2x) - 2z_y = 0$

$$\therefore \quad z_{xy} = \frac{2z_y - 6z z_x z_y}{3z^2 - 2x}$$

At $(1,1)$

$$z_{xy} = \frac{2(-1) - 6(1)(2)(-1)}{3(1) - 2(1)} = \boxed{10}$$

③ can be written as $z_y(3z^2-2x)+1=0$

Differentiating w.r.t. y, we get $z_y(6zz_y)+z_{yy}(3z^2-2x)=0$

$$z_{yy} = -\frac{6z(z_y)^2}{3z^2-2x} \qquad \text{At } (1,1) \qquad z_{yy} = \frac{-6(1)(-1)^2}{3(1)-2(1)} = \boxed{-6}$$

Putting values in Ⓐ we get

$$z(x,y) = z(1,1) + \{(x-1)\,z_x(1,1) + (y-1)\,z_y(1,1)\} + \frac{1}{2!}\{(x-1)^2 z_{xx}(1,1)$$
$$+\,2(x-1)(y-1)\,z_{xy}(1,1) + (y-1)^2 z_{yy}(1,1) + \dots\}$$
$$= 1 + (x-1)(2) + (y-1)(-1) + \frac{1}{2!}[(x-1)^2(-16) + 2(x-1)(y-1)(10)$$
$$+ (y-1)^2(-6)] + \dots$$

$$z(x,y) = 1 + 2(x-1) - (y-1) - 8(x-1)^2 + 10(x-1)(y-1) - 3(y-1)^2 + \dots$$

Ans