$\frac{x}{f(-a^2)}$ $f(D)y = x \cdot \sin ax$ $f(-a^2)$ $f(D^2)$ $\frac{x \sin 3x}{f'(D^2+9)}$

2) $= u^2 \; xyz(1+3x^2)(z+x^2)$

3)

Q: If

$u = \sin(xy) + e^{y+z} + \log(x+z) + \tan^{-1}(x+t)$

$\frac{\partial^4 u}{\partial t \cdot \partial x \cdot \partial y \cdot \partial z} = ?$

← This is order of differentiating

$\frac{\partial u}{\partial t} = \frac{1}{1+(x+t)^2}$

~~$\frac{\partial u}{\partial x} = y\cos(xy) + \frac{1}{x+z} + \frac{1}{1+(x+t)^2}$~~

$\frac{\partial^2 u}{\partial t \cdot \partial x} = \frac{-1}{(1+(x+t)^2)^2}(2(x+t))$

$\frac{\partial^3 u}{\partial t \cdot \partial x \cdot \partial y} = 0$.

* Total Differentiation

Consider $z = f(x,y)$ be a function of 2 independ-ent variable of x, y.

then,

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

where $dx, dy \Rightarrow$ general derivative

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \Rightarrow$ partial derivative.

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Case I: If $z = z(x,y) = f(x,y)$ such that $x = \phi(t)$, $y = \psi(t)$

then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Case II: If $z = z(x,y) = f(x,y)$ such that

$x = \phi(u,v)$; $y = \psi(u,v)$

$$\Rightarrow \frac{dz}{du} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\Rightarrow \frac{dz}{dv} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Q. If $u = x^3 + y^3$, $x = a\cos t$ & $y = b\sin t$

find 1) Find $\frac{du}{dt}$ by total derivative & verify.

It is given that

$u = u(x,y) = f(x,y) = x^3 + y^3$ —①

And,

$x = a\cos t = \phi(t)$

$y = b\sin t = \psi(t)$

using $\frac{du}{dt} = \frac{\partial u}{\partial t} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial t} \cdot \frac{dx}{dt}$.

$\frac{du}{dt} = (3x^2)(-a\sin t) + (3y^2)(b\cos t)$

$= (3a^2\cos^2 t)(-a\sin t) + (3b^2\sin^2 t)(b\cos t)$

$$\frac{du}{dt} = -3a^3\cos^2 t \cdot \sin t + 3b^3 \sin^2 t \cdot \cos t. \quad \text{--- (3)}$$

Verification -

$u = a^3\cos^3 t + b^3 \sin^3 t.$

differentiate directly.

Q2) If $w = f(x,y)$; $x = r\cos\theta$ $y = r\sin\theta$.

then P.T.

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2 = \frac{1}{r^2}\cdot\left(\frac{\partial w}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2$$

It is given that ·

$w = f(x,y) = w(x,y)$

$x = r\cos\theta = \phi(r,\theta)$.

$y = r\sin\theta = \psi(r,\theta)$

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta -$$

$$\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y}\cdot r\cos\theta -$$

$$\left(\frac{\partial w}{\partial \theta}\right)^2\frac{1}{r^2} + \left(\frac{\partial w}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2(\sin^2\theta + \cos^2\theta)$$

$$+\left(\frac{\partial f}{\partial x}\right)^2(\sin^2\theta \cos^2\theta)$$

Q. If $u = u(y-z, z-x, x-y)$, is the function of 2 independent variable x and y, then P.T.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Q. If $w = w\left(\left(\frac{y-x}{y\cdot x}\right), \left(\frac{z-x}{z\cdot x}\right)\right)$ is the function of 3 independent variable x, y, z then show that

$$x^2\cdot\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + z^2\cdot\frac{\partial^2 w}{\partial z^2} = 0.$$

* Jacobian.

If $u = u(x,y) \equiv f(x,y)$
$v = v(x,y) \equiv f(x,y)$] u, v are function of 2 independent variable.

then Jacobian for u and v is defined

$$J\left\{\frac{(u,v)}{(x,y)}\right\} \text{ or } \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarly for $u = (x,y,z)$
$v = (x,y,z)$
$w = (x,y,z)$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Q1. If $u = u(x,y)$; $v = v(x,y)$ then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Q2. If $u = u(r,s)$, $v = v(r,s)$;
$r = \phi(x,y)$; $s = \psi(x,y)$

then
$$\frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(f(r,s))}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(f(r,s))}$$

$= 1$ or $\neq 1$ depending on condition

Q3. If $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$ then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$ when u, v, w are non independent variable.

Q4. If $u = u(x,y,z)$, $v = v(x,y,z)$ $w = w(x,y,z)$ and

$u = x+y+z$
$v = xyz$
$w = x^2+y^2+z^2$

then P.T.

$$\frac{\partial(x,y,z)}{\partial(u,v,z)} = \frac{-1}{2} \times \frac{1}{(x-y)(y-z)(z-x)}$$

Q2. If $u = u(x,y,z)$, $v = v(x,y,z)$ and $w = w(x,y,z)$ and $u = x+y+z$ $v = xy+yz+xz$, $w = x^2+y^2+z^2$ then P.T.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

* Jacobian Implicit Function.

Consider u, v are functions of 2 independent variables x and y, if they connected implicitely then

$$f_1(u,v,x,y) = 0 \quad \text{--- (1)}$$

$$f_2(u,v,x,y) = 0 \quad \text{--- (2)}$$

Now by Jacobian property.

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} \\ \text{---} & \text{---} \end{vmatrix}$$

Now differentiating eq^n (1) partially w.r.t x and y

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} = 0.$$

$$\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} = 0.$$

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$$\Rightarrow \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} = -\frac{\partial f_1}{\partial x} \quad \text{— (4)}$$

$$\Rightarrow \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} = -\frac{\partial f_1}{\partial y} \quad \text{— (5)}$$

Similarly for eq^n (2).

$$\frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x} = -\frac{\partial f_2}{\partial x} \quad \text{— (6)}$$

$$\frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x} = -\frac{\partial f_2}{\partial y} \quad \text{— (7)}$$

by eq^n (4), (5), (6) and (7)

$$\frac{\partial(f_1, f_2)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{\partial f_1}{\partial x} & -\frac{\partial f_1}{\partial y} \\ -\frac{\partial f_2}{\partial x} & -\frac{\partial f_2}{\partial y} \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x,y)}$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \frac{(-1)^2 \left\{\frac{\partial(f_1, f_2)}{\partial(x,y)}\right\}}{\left\{\frac{\partial(f_1, f_2)}{\partial(u,v)}\right\}}$$

If $u^2 + v^2 + x^2 - y^2 = 0$ & $xu + yv = 0$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = ?$$

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Q2. If
$$u^2 + v + w = x + y^2 + z^2$$
$$u + v^2 + w = x^2 + y + z^2$$
$$u + v + w^2 = x^2 + y^2 + z.$$

then PT.
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1 - 4(xy + yz + zx) + 16xyz}{1 - 2(u^2 + v^2 + w^2 - 2uvw) + 27uvw}$$

* Taylor's Theorem.

If f is a function of 2 variable x, y. Also we want to write the expansion of taylor series in powers of h and k, where h & k are small constants then.

$$f(x+h, y+k) = f(x,y) + \left(h\left(\frac{\partial f}{\partial x}\right) + k\cdot\frac{\partial f}{\partial y}\right) +$$
$$\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2h\cdot k\cdot\frac{\partial^2 f}{\partial y\cdot\partial x} + k^2\cdot\frac{\partial^2 f}{\partial y^2}\right)$$

$$- - - + -$$

* Relative Extrema (maxima/minima) for function of 2 variable.

If $z = f(x_0, y)$

then

Step 1: find p, q, r, s, t

i.e. $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ & $\frac{\partial p}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial y}$

Step 2.

$p = 0 \longrightarrow x = p_1$

$q = 0 \longrightarrow y = p_2$

$P(p_1, p_2) \rightarrow$ Extreme points

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$$\sin 2x \cdot \frac{1}{D} \cdot \left(\frac{1}{4i} \left(\frac{1}{\left(1 + \frac{D}{4i}\right)} \right) \right) \cdot x^2$$

$$= \sin 2x \cdot \frac{1}{D} \left(\frac{1}{4i} \left(1 - \frac{D}{4i} + \cdots \right) \right) x^2$$

$$= \cdot \sin 2x \cdot \frac{1}{D} \left(\frac{1}{4i} \left(+ \frac{D}{16} \right) x^2 \right)$$

$$= \sin 2x \cdot \left(\frac{1}{4i} \frac{x^3}{3} + \frac{x^2}{16} \right)$$