6

Expansions of Functions and Indeterminate Forms

6·1. The student is already familiar with expansions of elementary functions using Binomial Theorem. In this chapter we shall expand the given function as an infinite convergent series in the form $a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$, known as power series. It is assumed that all the functions dealt here possess finite and continuous derivatives of all orders for the values of variables under consideration and are capable of expansions as power series.

6·2. Maclaurin's Theorem

If a function $f(x)$ can be expanded as an infinite convergent series of positive integral power of x, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^n(0)+\ldots$$

where $f^n(0)$ stands for nth derivative of $f(x)$ at $x=0$.

Proof. Since $f(x)$ is capable of being expanded as an infinite series, let

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots \quad \ldots(1)$$

By successive differentiation, we get

$$f'(x)=a_1+2.a_2x+3.a_3x^2+4a_4x^3+\ldots \quad \ldots(2)$$

$$f''(x)=2.a_2+3.2.a_3x+4.3.a_4x^2+\ldots \quad \ldots(3)$$

$$f'''(x)=3.2.a_3+4.3.2.a_4x+\ldots \quad \ldots(4)$$

Substituting $x=0$, successively in (1), (2), (3) and (4), we get

$f(0)=a_0$ or $a_0=f(0)$

$f'(0)=a_1$ or $a_1=f'(0)$

$f''(0)=2.a_2$ or $a_2=\dfrac{f''(0)}{2!}$

$f'''(0)=3.2.a_3$ or $a_3=\dfrac{f'''(0)}{3!}$ and so on

Substituting the values of a_0, a_1, a_2, a_3, etc. in (1), we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots+\frac{x^n}{n!}f^n(0)+\ldots$$

The series on the R.H.S. is known as Maclaurin's Series.

Note 1. Another useful form of the above series for the function $y=f(x)$ is,

$$f(x)=y=(y)_0+(y_1)_0x+(y_2)_0\frac{x^2}{2!}+\ldots+(y_n)_0\frac{x^n}{n!}+\ldots$$

where $(y_n)_0$ stands for the nth derivative of y at $x=0$.

6·3. Expansion of sin x

Let $f(x)=\sin x$ $\quad\therefore f(0)=0$

$f'(x)=\cos x$ $\quad f'(0)=1$

$f''(x)=-\sin x$ $\quad f''(0)=0$

$f'''(x)=-\cos x$ $\quad f'''(0)=-1$

$f^{iv}(0)=\sin x$ $\quad f^{iv}(0)=0$ and so on.

The values of derivatives at $x=0$ are repeated in cycles of 0, 1, 0, -1.

By Malcaurin's Theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots$$

$$\therefore \quad \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

Aliter

Here $f(x)=\sin x$

$$\therefore \quad f^n(x)=\sin\left(x+\frac{n\pi}{2}\right)$$

Putting $x=0$ on both sides, we have

$$f^n(0)=\sin\frac{n\pi}{2}$$

Substituting $n=0, 1, 2, 3, \ldots$, we get

$f(0)=0$

$'(0)=\sin\frac{\pi}{2}=1$

$f''(0)=\sin\pi=0$

$f'''(0)=\sin\frac{3\pi}{2}=-1$

$f^{iv}(0)=\sin 2\pi=0$ and so on.

Hence by Maclaurin's Theorem, we have

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

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6·4. Expansion of a^x

Let $f(x)=a^x$ $\therefore f(0)=a^0=1$

$f'(x)=a^x \log a$ $f'(0)=\log a$

$f''(x)=a^x (\log a)^2$ $f''(0)=(\log a)^2$

$f'''(x)=a^x (\log a)^3$ $f'''(0)=(\log a)^3$

Proceeding in this manner, we have, $f^n(0)=(\log a)^n$

By Maclaurin's theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots+\frac{x^n}{n\,!}f^n(0)+\ldots$$

$$\therefore\ a^x=1+(x\log a)+\frac{1}{2\,!}(x\log a)^2+\frac{1}{3\,!}(x\log a)^3+\ldots$$

$$+\frac{1}{n\,!}(x\log a)^n+\ldots$$

Note. The expansion of e^x can be obtained by putting $a=e$, in the above result so that $\log a=\log e=1$.

$$\therefore\ e^x=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots+\frac{x^n}{n\,!}+\ldots$$

6·5. Expansion of log (1+x)

Let $f(x)=\log(1+x)$ $\therefore f(0)=\log 1=0$

$f'(x)=\dfrac{1}{1+x}$ $f'(0)=1$

$f''(x)=-\dfrac{1}{(1+x)^2}$ $f''(0)=-1$

$f'''(x)=\dfrac{(-1)(-2)}{(1+x)^3}$ $f'''(0)=2\,!$

$f^{iv}(x)=\dfrac{(-1)(-2)(-3)}{(1+x)^4}$ $f^{iv}(0)=-3!$ and so on.

By Maclaurin's Theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\frac{x^4}{4\,!}f^{iv}(0)+\ldots$$

$$\therefore\ \log(1+x)=x+\frac{x^2}{2\,!}(-1)+\frac{x^3}{3\,!}(2\,!)+\frac{x^4}{4\,!}\{-(3!)\}+\ldots$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

Note. Expansion of $\log(1-x)$ can be obtained by replacing x by $(-x)$, in the above result,

$$\therefore\ \log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots$$

Example 1. *Show that*

$$\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots$$

Hence find the value of π.

Sol. Let $y = \sin^{-1} x$ ...(1)

Differentiating both sides w.r.t. x,

$$y_1 = \frac{1}{\sqrt{1-x^2}} \quad \ldots(2)$$

or $y_1^2(1-x^2) = 1$

Differentiating again w.r.t. x, we get

$$\cancel{2}\cancel{y_1}y_2(1-x^2) - \cancel{2}y_1^{\cancel{2}}.x = 0$$

or $y_2(1-x^2) - y_1 x = 0$ ...(3)

Differentiating both sides of (3), n times using Leibnitz's Theorem.

$$[y_{n+2}(1-x^2) + {}^nC_1.y_{n+1}(-2x) + {}^nC_2.y_n(-2)] - [y_{n+1}x + {}^nC_1.y_n.1] = 0$$

or $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$ ...(4)

Putting $x=0$ in (1), (2), (3) and (4) we get

$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0$

and $(y_{n+2})_0 - n^2(y_n)_0 = 0$ ...(5)

or $(y_{n+2})_0 = n^2(y_n)_0$ ...(6)

Substituting $n = 1, 2, 3, 4, 5$, etc. in (6), we get

$(y_3)_0 = 1^2\ (y_1)_0 = 1^2$ $[\because (y_1)_0 = 1]$

$(y_4)_0 = 2^2.(y_2)_0 = 0$ $[\because (y_2)_0 = 0]$

$(y_5)_0 = 3^2.(y_3)_0 = 3^2.1^2$

$(y_6)_0 = 4^2.(y_4)_0 = 0$

$(y_7)_0 = 5^2.(y_5)_0 = 5^2.3^2.1^2.$

By Maclaurin's Theorem,

$$y = (y_0) + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \frac{x^6}{6!}(y_6)_0 + \frac{x^7}{7!}(y_7)_0 + \ldots$$

$$\therefore \quad \sin^{-1} x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}.1^2 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}1^2.3^2 + \frac{x^6}{6!}.0 + \frac{x^7}{7!}.1^2.3^2.5^2 + \ldots$$

or $$\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots$$

Putting $x=\frac{1}{2}$ on both sides, we get

$$\sin^{-1}\frac{1}{2}=\frac{1}{2}+\frac{1}{6}\left(\frac{1}{2}\right)^3+\frac{3}{40}\left(\frac{1}{2}\right)^5+\cdots$$

$$\therefore \quad \frac{\pi}{6}=0{\cdot}5000+0{\cdot}0208+0{\cdot}0023=0{\cdot}5231$$

$$\therefore \quad \pi=3{\cdot}1386=3{\cdot}14 \text{ (approximately).}$$

Example 2. *Expand* $\tan\left(\frac{\pi}{4}+x\right)$ *in ascending powers of* x. *Hence find the value of* $\tan 45°\ 30'$ *to four places of decimals.*

Sol. Let $f(x)=\tan\left(\frac{\pi}{4}+x\right) \quad \therefore \quad f(0)=\tan\left(\frac{\pi}{4}\right)=1$

$$f'(x)=\sec^2\left(\frac{\pi}{4}+x\right) \quad \therefore \; f'(0)=\sec^2\frac{\pi}{4}=2$$

$$f''(x)=2\sec^2\left(\frac{\pi}{4}+x\right)\tan\left(\frac{\pi}{4}+x\right)$$

$$=2\left\{1+\tan^2\left(\frac{\pi}{4}+x\right)\right\}\tan\left(\frac{\pi}{4}+x\right)$$

$$=2\left\{\tan\left(\frac{\pi}{4}+x\right)+\tan^3\left(\frac{\pi}{4}+x\right)\right\}$$

$$\therefore \quad f''(0)=4$$

$$f'''(x)=2\left\{\sec^2\left(\frac{\pi}{4}+x\right)+3\tan^2\left(\frac{\pi}{4}+x\right)\sec^2\left(\frac{\pi}{4}+x\right)\right\}$$

$$=2\sec^2\left(\frac{\pi}{4}+x\right)\left\{1+3\tan^2\left(\frac{\pi}{4}+x\right)\right\}$$

$$=2\left\{1+\tan^2\left(\frac{\pi}{4}+x\right)\right\}\left\{1+3\tan^2\left(\frac{\pi}{4}+x\right)\right\}$$

$$=2\left\{1+4\tan^2\left(\frac{\pi}{4}+x\right)+3\tan^4\left(\frac{\pi}{4}+x\right)\right\}$$

$$\therefore \quad f'''(0)=16$$

$$f^{iv}(x)=2\left\{8\tan\left(\frac{\pi}{4}+x\right)\sec^2\left(\frac{\pi}{4}+x\right)+12\tan^3\left(\frac{\pi}{4}+x\right)\sec^2\left(\frac{\pi}{4}+x\right)\right\}$$

$$\therefore \quad f^{iv}(0)=80$$

By Maclaurin's Theorem

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\cdots$$

$$\therefore \quad \tan\left(\frac{\pi}{4}+x\right)=1+2x+\frac{x^2}{2\,!}(4)+\frac{x^3}{3\,!}(16)+\frac{x^4}{4\,!}(80)+\ldots$$

$$=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\ldots$$

Putting $\quad x=30'=\dfrac{\pi}{360}$, we get

$$\tan 45^\circ\, 30'=1+2\cdot\frac{\pi}{360}+2\cdot\left(\frac{\pi}{360}\right)^2+\ldots$$

$$=1+0{\cdot}1745+0{\cdot}00015=1{\cdot}0176 \text{ (approximately).}$$

Example 3. *Expand* $\log\,[1-\log\,(1-x)]$ *in powers of* x *by Maclaurin's Theorem upto the term of* x^3 *and deduce the expansion of* $\log\,[1+\log\,(1+x)]$.

Sol. Let $f(x)=\log\,[1-\log\,(1-x)]$

We know, $\log\,(1-x)=-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\ldots$

$$\therefore \qquad f(x)=\log\left[1-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots\right)\right]$$

$$=\log\left[1+\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)\right]$$

Let $x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\ldots=z$

$$\therefore \qquad f(x)=\log\,(1+z)$$

$$=z-\frac{z^2}{2}+\frac{z^3}{3}+\ldots$$

$$=\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)-\frac{1}{2}\left(x+\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)^2$$

$$+\frac{1}{3}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)^3-\ldots$$

$$=x+\frac{x^3}{6}+\ldots$$

$$\therefore \qquad \log\,[1-\log\,(1-x)]=x+\frac{x^3}{6}+\ldots \qquad \ldots(1)$$

Replacing x by $\dfrac{x}{1+x}$ in both sides of (1), we get

$$\log\,[1+\log\,(1+x)]=x(1+x)^{-1}+\tfrac{1}{6}x^3(1+x)^{-3}$$

$$=x(1-x+x^2-\ldots)+\tfrac{1}{6}x^3(1-3x+\ldots)+\ldots$$

$$\therefore\ \log[1+\log(1+x)]=x-x^2+\frac{7x^3}{6}+\ldots$$

Example 4. *Apply Maclaurin's Theorem to obtain the expansion of the function e^{ax} sin bx in an infinite series of powers of x, giving the general term.*

Sol. Let $f(x)=e^{ax}\sin bx$

then $$f^n(x)=(a^2+b^2)^{n/2}.e^{ax}\sin(bx+n\theta) \quad ...(1)$$

where $$\tan\theta=\frac{b}{a} \quad [\S\ 5{\cdot}2(f)]$$

$$\therefore \quad \sin\theta=\frac{b}{\sqrt{a^2+b^2}} \text{ and } \cos\theta=\frac{a}{\sqrt{a^2+b^2}}.$$

Now $f^n(0)=(a^2+b^2)^{n/2}\sin(n\theta)$ (from 1)

Putting $n=1, 2, 3, ...$ successively, we get

$$f'(0)=(a^2+b^2)^{1/2}.\sin\theta$$
$$=(a^2+b^2)^{1/2}.\frac{b}{\sqrt{a^2+b^2}}=b$$

$$f''(0)=(a^2+b^2).\sin 2\theta$$
$$=(a^2+b^2).2\sin\theta\cos\theta$$
$$=(a^2+b^2).\frac{2ab}{(a^2+b^2)}=2ab.$$

$$f'''(0)=(a^2+b^2)^{3/2}.\sin 3\theta$$
$$=(a^2+b^2)^{3/2}.(3\sin\theta-4\sin^3\theta)$$
$$=(a^2+b^2)^{3/2}.\left[\frac{3b}{\sqrt{a^2+b^2}}-\frac{4b^3}{(a^2+b^2)^{3/2}}\right]$$
$$=b(3a^2-b^2)$$

Also $f(0)=0$

By Maclaurin's theorem,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...$$

$$\therefore \quad e^{ax}\sin bx=bx+abx^2+\frac{b(3a^2-b^2)}{3!}x^3+...$$

6·6. Failure of Maclaurin's Theorem

It should be clearly understood that every function cannot be expanded by Maclaurin's Theorem. This theorem is not applicable in the following cases.

(*i*) The function $f(x)$ or any of its successive derivatives do not exist finitely at $x=0$.

(*ii*) The infinite series obtained by expansion does not converge. For example Maclaurin's Theorem cannot be applied to obtain the expansion of functions like cot x, log x etc.

6·7. Taylor's Theorem

If a function $f(x+h)$ can be expanded as an infinite convergent series of positive integral powers of h, then

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots+\frac{h^n}{n!}f^n(x)+\ldots$$

where $f^n(x)$ stands for the nth derivative of $f(x+h)$ with respect to $(x+h)$, when $(x+h)$ is replaced by x.

Proof. Since $f(x+h)$ is capable of being expanded as an infinite series in powers of h,

$$f(x+h)=a_0+a_1h+a_2h^2+a_3h^3+a_4h^4+\ldots \qquad \ldots(1)$$

Let us find derivative of $f(x+h)$.

Now $\frac{d}{dh}f(x+h)=\frac{d}{d(x+h)}[f(x+h)]\cdot\frac{d}{dh}(x+h)$

$$=f'(x+h)=f'(x+h)$$

Also $\frac{d}{d(x+h)}f(x+h)=f'(x+h)$

Hence differentiation of $f(x+h)$ with respect to $(x+h)$ or h gives the same results.

Differentiating (1) successively, with respect to h, we get

$$f'(x+h)=a_1+2.\,a_2h+3.a_3h^2+4.a_4h^3+\ldots \qquad \ldots(2)$$

$$f''(x+h)=2.a_2+3.2.a_3h+4.3.a_4h^2+\ldots \qquad \ldots(3)$$

$$f'''(x+h)=3.2.a_3+4.3.2\,a_4h+\ldots \qquad \ldots(4)$$

Putting $h=0$, in (1), (2), (3) and (4) etc., we get

$f(x)=a_0$ $\therefore\ a_0=f(x)$

$f'(x)=a_1$ $a_1=f'(x)$

$f''(x)=2a_2$ $a_2=\frac{f''(x)}{2!}$

$f'''(x)=3.2.a_3$ $a_3=\frac{f'''(x)}{3!}$

Substituting these values of a_0, a_1, a_2 and a_3 etc. in (1), we obtain

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots+\frac{h^n}{n!}f^n(x)+\ldots$$

The series on R.H.S. is also known as Taylor's Series.

Note 1. A function $f(x)$ may be expanded in powers of $(x-a)$ by Taylor's Theorem by putting $h=x-a$.

$$\therefore f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$$

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Note 2. Putting $x=0$ and $h=x$ in Taylor's series, we obtain Maclaurin's Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots$$

Thus Maclaurin's Series can be obtained as a particular case of Taylor's Series.

Example 1. *Expand $\log_e (x+h)$ in powers of h by Taylor's Theorem.*

Sol. Let $f(x+h)=\log (x+h)$

$$\therefore \quad f(x)=\log x$$

$$f'(x)=\frac{1}{x}$$

$$f''(x)=-\frac{1}{x^2}=(-1)^1.\frac{1}{x^2}$$

$$f'''(x)=\frac{(-1)(-2)}{x^3}=(-1)^2.\frac{2\,!}{x^3}$$

$$\ldots\ldots\ldots\ldots$$

$$f^n(x)=(-1)^{n-1}.\frac{(n-1)\,!}{x^n}$$

By Taylor's Theorem,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x)+\frac{h^3}{3\,!}f'''(x)+\ldots+\frac{h^n}{n\,!}f^n(x)+\ldots$$

$$\therefore \log (x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}+\ldots+(-1)^{n-1}\frac{h^n}{nx^n}+\ldots$$

Example 2. *Expand $\sin x$ in powers of* $\left(x-\frac{\pi}{2}\right)$.

Sol. Sin x may be written as $\sin\left[\frac{\pi}{2}+\left(x-\frac{\pi}{2}\right)\right]$. Here x is $\frac{\pi}{2}$ and h is $x-\frac{\pi}{2}$. $\left(\frac{\pi}{2}+x-\frac{\pi}{2}\right)$ $f(x)+hf'(x)+\frac{h^2}{2!}f''(x)$ $f(\frac{\pi}{2})+(x-\frac{\pi}{2})f'(\frac{\pi}{2})$

Now $f(x)=\sin x$	$\therefore\ f\left(\frac{\pi}{2}\right)=1$
$f'(x)=\cos x$	$f'\left(\frac{\pi}{2}\right)=0$
$f''(x)=-\sin x$	$f''\left(\frac{\pi}{2}\right)=-1$
$f'''(x)=-\cos x$	$f'''\left(\frac{\pi}{2}\right)=0$
$f^{iv}(x)=\sin x$	$f^{iv}\left(\frac{\pi}{2}\right)=1$

By Taylor's Theorem,

$$f(x)=f\left(\frac{\pi}{2}\right)+\left(\frac{\pi}{2}-x\right)f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right)$$

$$+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}f'''\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}f^{iv}\left(\frac{\pi}{2}\right)+\ldots$$

$$\sin x=1+0-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+0+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}-\ldots$$

or $$\sin x=1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{4}\right)^4}{4!}-\ldots$$

Example 3. *Prove by Taylor's Theorem.*

$$\tan^{-1}(x+h)=\tan^{-1}x+(h.\sin\alpha).\sin\alpha-(h\sin\alpha)^2.\frac{\sin 2\alpha}{2}$$

$$+(h\sin\alpha)^3.\frac{\sin 3\alpha}{3}+\ldots$$

where $\alpha=\cot^{-1}x.$

Sol. Here $f(x+h)=\tan^{-1}(x+h)$

$\therefore$ $f(x)=\tan^{-1}x$

$$f'(x)=\frac{1}{1+x^2}=\frac{1}{1+\cot^2\alpha}$$

$$=\frac{1}{\operatorname{cosec}^2\alpha}=\sin^2\alpha$$

$$f''(x)=-\frac{2x}{(1+x^2)^2}=-\frac{2\cot\alpha}{(1+\cot^2\alpha)^2}$$

$$=-\frac{2\cot\alpha}{\operatorname{cosec}^4\alpha}=-2\cot\alpha\sin^4\alpha$$

$$=-\sin^2\alpha.\sin 2\alpha$$

$$f'''(x)=-\frac{2(1-3x^2)}{(1+x^2)^3}=-\frac{2(1-3\cot^2\alpha)}{(1+\cot^4\alpha)^3}$$

$$=-2(\sin^2\alpha-3\cos^2\alpha)\sin^4\alpha$$

$$=-2(4\sin^2\alpha-3)\sin^4\alpha$$

$$=2(3\sin\alpha-4\sin^3\alpha)\sin^3\alpha$$

$$=2.\sin 3\alpha.\sin^3\alpha$$

$(\because 3\sin\alpha-4\sin^3\alpha=\sin 3\alpha)$

By Taylor's Theorem,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots$$

$\therefore \quad \tan^{-1}(x+h)=\tan^{-1}x+(h\sin\alpha)\sin\alpha$

$$-(h\sin\alpha)^2\cdot\frac{\sin 2\alpha}{2}+(h\sin\alpha)^3\cdot\frac{\sin 3\alpha}{3}+\ldots$$

Aliter

$$f(x)=\tan^{-1}x$$

$\therefore \quad f^n(x)=(-1)^{n-1}\cdot(n-1)!\sin^n\alpha\sin n\alpha$

Substituting $n=1, 2, 3,\ldots$, we get

$$f'(x)=\sin\alpha\cdot\sin\alpha \qquad (\because\ 0!=1)$$

$$f''(x)=-\sin^2\alpha\cdot\sin 2\alpha$$

$$f'''(x)=2!\sin^3\alpha\cdot\sin 3\alpha$$

By Taylor's Theorem,

$$\tan^{-1}(x+h)=\tan^{-1}x+(h\sin\alpha)\cdot\sin\alpha$$

$$-(h\sin\alpha)^2\cdot\frac{\sin 2\alpha}{2}+(h\sin\alpha)^3\cdot\frac{\sin 3\alpha}{3}+\ldots$$

Example 4. *Apply Taylor's Theorem to calculate the value of* $f\left(\frac{11}{10}\right)$, *where* $f(x)=x^3+3x^2+15x-10$.

Sol. By Taylor's Theorem, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots$$

Put $x=1$ and $h=\frac{1}{10}$,

$$\therefore \quad f\left(1+\frac{1}{10}\right)=f(1)+\frac{1}{10}f'(1)+\frac{1}{2}\left(\frac{1}{10}\right)^2 f''(1)$$

$$+\frac{1}{6}\cdot\left(\frac{1}{10}\right)^3 f'''(1)+\ldots \qquad \ldots(1)$$

Now

$$f(x)=x^3+3x^2+15x-10 \qquad \therefore\ f(1)=9$$

$$f'(x)=3x^2+6x+15 \qquad f'(1)=24$$

$$f''(x)=6x+6 \qquad f''(1)=12$$

$$f'''(x)=6 \qquad f'''(1)=6$$

All other derivatives of $f(x)$ vanish.

Substituting these values in (1), we get

$$f\left(\frac{11}{10}\right)=9+\frac{1}{10}\cdot 24+\frac{1}{2\cdot 10^2}(12)+\frac{1}{6\cdot 10^3}(6)$$

$$=9+2\cdot4+0\cdot06+0\cdot001$$

$$=11\cdot461$$

Example 5. *Given $\sin 30° = \frac{1}{2}$, use Taylor's Theorem to evaluate $\sin 31°$ correct to four significant figures.* ($\cos 30° = 0{\cdot}8660$)

Sol. Let $f(x+h) = \sin(x+h)$

$\therefore$ $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$ and so on

By Taylor's Theorem, we have

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2}\sin x - \ldots \qquad \ldots(1)$$

Putting $x = \frac{\pi}{6}$ and $h = 1° = \frac{\pi}{180}$ radians in (1), we get

$$\sin 31° = \sin\frac{\pi}{6} + \frac{\pi}{180}\cos\frac{\pi}{6} - \frac{1}{2}\left(\frac{\pi}{180}\right)^2 \cdot \sin\frac{\pi}{6} - \ldots$$

$$= 0{\cdot}5 + 0{\cdot}0175 \times 0{\cdot}866 - \tfrac{1}{2}(0{\cdot}0175)^2 \times 0{\cdot}5 - \ldots$$

$$= 0{\cdot}5 + 0{\cdot}01515 - 0{\cdot}000076 - \ldots$$

$= 0{\cdot}5151$ upto four places of decimal.

6·8. Expansion by Differentiation and Integration of a known Series

These methods are useful, if the series for a given function is known and it is required to obtain a series for its derivative or integrals. The following examples illustrate the use of these methods.

Example 1. *Using the series for $\sin x$, obtain the series for $\cos x$.*

Sol. We know $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Differentiating both sides with respect to x,

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Example 2. *Find by integration the series for,*

(*i*) $\log_e(1+x)$ (*ii*) $\tan^{-1} x$.

Sol. Now $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

Integrating both sides with respect to x, between limits 0 and x, we get

$$\therefore \quad \frac{d\Delta}{dA} = \frac{1}{2} b^2 \sin C$$

$$\times \left[\frac{\cos A \cdot \sin (A+C) - \cos (A+C) \sin A}{\sin^2 (A+C)} \right]$$

$$= \frac{b^2 \sin^2 C}{2 \sin^2 (A+C)}.$$

Now $$\frac{\delta \Delta}{\Delta} \times 100 = \left(\frac{d\Delta}{dA} \cdot \delta A \right) \times \frac{100}{\Delta}$$

$$= \frac{b^2 \sin^2 C \cdot \delta A}{2 \sin^2 (A+C)} \times \frac{100 \times 2 \sin (A+C)}{b^2 \sin C \sin A}$$

$$= 100\, \delta A \cdot \sin C / \{\sin A \cdot \sin (A+C)\}.$$

EXERCISE 6 (*a*)

1. Apply Maclaurin's Theorem to expand

(*i*) log sec x (*ii*) cos x (*iii*) log $(1+\sin x)$.

Prove the following by Maclaurin's Theorem.

2. $(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots$

3. $e^{x \cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots$

4. $e^{a \sin^{-1} x} = 1 + ax + \frac{(ax)^2}{2\,!} + \frac{a(1^2+a^2)}{3\,!} x^3$

$$+ \frac{a^2(2^2+a^2)}{4\,!} x^4 + \ldots\ldots$$

and hence show that

$$e^{\theta} = 1 + \sin \theta + \frac{\sin^2 \theta}{2\,!} + \frac{2}{3\,!} \sin^3 \theta + \frac{5}{4\,!} \sin^4 \theta + \ldots$$

5. Expand $\sin (m \sin^{-1} x)$ in ascending powers of x.

6. If $y = \sin \log (x^2+2x+1)$, then prove that

$$y = 2x - x^2 - \frac{2}{3} x^3 + \frac{3}{2} x^4 - \frac{5}{3} x^5 + \frac{3}{2} x^6 + \ldots$$

7. Expand $\log_e (x + \sqrt{x^2+1})$ up to first four terms by Maclaurin's theorem; by putting $x = 0{\cdot}75$ in the expansion, calculate the value of $\log_e 2$ to four places of decimals and find the percentage error if any.

8. Expand $\log_e \cos x$ by Maclaurin's theorem as far as the term x^4 and calculate $\log_{10} \cos \pi/12$ up to three places of decimal.

9. Calculate the approximate value of $\sqrt{10}$ to four places of decimal by taking the first four terms of an appropriate expansion.

[**Hint.** Expand $(1+x)^{1/2}$ by Maclaurin's theorem and put $x = 1/9$.]

5. $y_n(0)=0$, when n is even

$y_n(0)=m(1-m^2)(3^2-m^2)\ldots\ldots[(n-2)^2-m^2]$,

when n is odd.

6. $f^n(0)=m^2(2^2+m^2)(4^2+m^2)\ldots\ldots[(n-2)^2+m^2]\,e^{\frac{m\pi}{2}}$ n is even

$=-m(1+m^2)(3^2+m^2)\ \ldots\ [(n-2)^2+m^2]\,e^{\frac{m\pi}{2}}$ n is odd

10. $y_n(0)=(n-2)^2.(n-4)^2\ldots 4^2.2^2.2$, if n is even

$y_n(0)=0$, if n is odd.

11. $f^{n+1}(0)=n^2(n-2)^2(n-4^2)\ldots 4^2.2^2$ when n is even

$=0$, when n is odd.

Exercise 6 (a) (Page 147-149)

1. (i) $\frac{x^2}{2!}+\frac{2x^4}{4!}+\frac{16x^6}{6!}+\ldots$

(ii) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

(iii) $x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{24}-\ldots$

5. $mx+m(1^2-m^2)\cdot\frac{x^3}{3!}+m(1^2-m^2)(3^2-m^2)\cdot\frac{x^5}{5!}+\ldots$

7. $x-\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}+\ldots$; 0·6899 ; 0·46

8. $-\frac{x^2}{2}-\frac{x^4}{12}+\ldots\ldots$; $\bar{1}$·954

9. 3·1629

10. $\log\sin x+h\cot x-\frac{h^2}{2}\operatorname{cosec}^2 x+\frac{h^3}{3}\cot x\operatorname{cosec}^2 x+\ldots$;

$\bar{1}$·36486

11. $\tan x+h\sec^2 x+h^2\sec^2 x\tan x+\frac{h^3}{3}(1+3\tan^2 x).\sec^2 x+\ldots$

12. $\sin^{-1}(h)+x(1-h^2)^{-1/2}+\frac{x^2}{2!}h(1-h^2)^{-3/2}+\ldots$

13. $\log\sin 2+(x-2)\cot 2-\frac{1}{2}(x-2)^2\operatorname{cosec}^2 2+\frac{1}{3}(x-2)^3\operatorname{cosec}^2 2\cot 2+\ldots$

14. $\tan^{-1}\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\cdot\frac{1}{1+\left(\frac{\pi^2}{16}\right)}-\frac{\left(x-\frac{\pi}{4}\right)^2\pi}{4\left(1+\frac{\pi^2}{16}\right)^2}+\ldots$

15. $11+7(x-2)+4(x-2)^2+(x-2)^3$

16. 9·01 (app.)

17. (*i*) 0·50725 (*ii*) 0·02178

21. (*a*) $\frac{\pi rh\,\delta h}{\sqrt{r^2+h^2}}$ (*b*) $\frac{\pi(2r^2+2r\sqrt{r^2+y^2+h^2})}{\sqrt{r^2+h^2}}\delta r.$

22. (*i*) $-5·024$ cu. cm. (*ii*) $-5·024$ sq. cm, 23. 6π km.

24. $4\pm0·008$ 25. 1%

26. 0·71 30. 10%.

Exercise 6 (*b*) (Page 160—161)

1. 16 2. (-1)
3. $\frac{1}{2}$ 4. $(\log a-\log b)$
5. 0 6. 3. 7. 1
8. 0 9. $\frac{1}{2}$ 10. $-\frac{1}{3}$ 11. $-\infty$
12 1 13. a 14. (*i*) e^{-6} (*ii*) ∞. 15. e^a 16. $e^{\frac{1}{6}}$
17. $a=-2$; -1

Exercise 7 (*a*) (Page 165—167)

3. $\frac{i_0 R}{L}e^{-Rt/L}$ 4. $\frac{3}{8\pi}$ cm per minute

5. $\frac{1}{20}$ radian per second. 6. $\frac{90}{\sqrt{34}}$ m per second

8. 120 kg per square cm decreasing.

11. $\frac{8\sqrt{3}}{3}$ m per second 14. 8π km/minute

Exercise 7 (*b*) (Page 175—179)

1. $\frac{4\sqrt{3}}{9}$ 4. $t=\sqrt{(\mu^2+1)}-\mu$

5. $\frac{a}{2}$; $\frac{16}{25\sqrt{5a^4}}$ 6. $\frac{3K}{4}$ when $x=4$.

15. 4·5 km
20. $x=\frac{l}{2}$
27. $\frac{5}{2}$ kr
34. $a=1$,
36. $2-\frac{2}{\sqrt{3}}$
39. π.

1. (*i*)
(*ii*) Y
2. $9x-8$
3. (*a*) (*i*
(*b*) (
4. (*i*) $y=$
(*ii*)
5. Paral
7. $y=2a$
14. (*i*) $\frac{\pi}{2}$
(*ii*)
(*iv*) 4
3. S.T.=
Norn
3. $\sqrt{}$
1. (*i*)