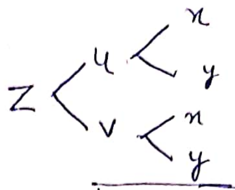
4) $u = xy, \quad v = x/y$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x$$

$$\frac{\partial v}{\partial x} = \frac{1}{y}, \quad \frac{\partial v}{\partial y} = -\frac{x}{y^2}$$

$$uv = x^2$$

$$\frac{u}{v} = \cancel{\frac{xy}{x}}\, xy \quad \text{or} \quad \frac{u}{v} = y^2$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\, y + \frac{\partial z}{\partial v}\frac{1}{y} \qquad \text{(i)}$$

& $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

or $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\, x + \frac{\partial z}{\partial v}\left(-\frac{x}{y^2}\right) \qquad \text{(ii)}$$

Partially Differentiate (i) w.r.t 'x' and (ii) w.r.t 'y' :-

$$\frac{\partial^2 z}{\partial x^2} = y\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial v}{\partial v}\right)\right] + \frac{1}{y}\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial v}{\partial v}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial u}{\partial u}\right)\right]$$

$$\frac{\partial^2 z}{\partial x^2} = y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial v}{\partial x}\right)\right] + \frac{1}{y}\left[\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial u}{\partial x}\right)\right]$$

$$\frac{\partial^2 z}{\partial x^2} = y\left[\frac{\partial^2 z}{\partial u^2}(y) + \frac{\partial^2 z}{\partial v\,\partial u}\left(\frac{1}{y}\right)\right] + \frac{1}{y}\left[\frac{\partial^2 z}{\partial v^2}\left(\frac{1}{y}\right) + \frac{\partial^2 z}{\partial u\,\partial v}(y)\right]$$

$$\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} + \frac{\partial^2 z}{\partial u\,\partial v}$$

or $$\boxed{\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial u^2} + \frac{2\,\partial^2 z}{\partial u\,\partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}} \qquad \text{(iii)}$$

Similarly :-

$$\frac{\partial^2 z}{\partial y^2} = x\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial u}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial v}{\partial v}\right)\right] - \frac{x}{y^2}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial v}{\partial v}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial u}{\partial u}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v}$$

or $$\frac{\partial^2 z}{\partial y^2} = x\left[\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{\partial v}{\partial y}\right)\right] - \frac{x}{y^2}\left[\frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{\partial u}{\partial y}\right)\right] + \frac{2x}{y^3}\left(\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = x\left[x\frac{\partial^2 z}{\partial u^2} + \left(\frac{-x}{y^2}\right)\frac{\partial^2 z}{\partial u\,\partial v}\right] - \frac{x}{y^2}\left[\frac{-x}{y^2}\left(\frac{\partial^2 z}{\partial v^2}\right) + (x)\frac{\partial^2 z}{\partial u\,\partial v}\right] + \frac{2x}{y^3}\left(\frac{\partial z}{\partial v}\right)$$

$$\text{or } \frac{\partial^2 z}{\partial y^2} = x^2\frac{\partial^2 z}{\partial u^2} - \frac{x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2} - \frac{x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{2x}{y^3}\left(\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = x^2\frac{\partial^2 z}{\partial u^2} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{2x}{y^3}\left(\frac{\partial z}{\partial v}\right)$$

$$x = \sqrt{uv}, \quad y = \sqrt{\frac{u}{v}}, \quad \frac{2\sqrt{uv}\, v^{3/2}}{u^{3/2}}, \quad \frac{2v^2}{u}\frac{\partial z}{\partial v}$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = uv\frac{\partial^2 z}{\partial u^2} + \frac{v^3}{u}\frac{\partial^2 z}{\partial v^2} - 2v^2\frac{\partial^2 z}{\partial u\partial v} + \frac{2v^2}{u}\frac{\partial z}{\partial v}} \quad (iv)$$

(iii) can be written as:-

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{u}{v}\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{v}{u}\frac{\partial^2 z}{\partial v^2}} \quad (v)$$

Now, $\boxed{y^2\frac{\partial^2 z}{\partial y^2} = u^2\frac{\partial^2 z}{\partial u^2} + v^2\frac{\partial^2 z}{\partial v^2} - 2uv\frac{\partial^2 z}{\partial u\partial v}} + 2v\frac{\partial z}{\partial v}$ (vi)

& $\boxed{x^2\frac{\partial^2 z}{\partial x^2} = u^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v}(uv) + v^2\frac{\partial^2 z}{\partial v^2}}$ (vii)

Now, $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 0$

$\Downarrow$

$$\Rightarrow \cancel{u^2\frac{\partial^2 z}{\partial u^2}} + 2uv\frac{\partial^2 z}{\partial u\partial v} + \cancel{v^2\frac{\partial^2 z}{\partial v^2}} - \cancel{u^2\frac{\partial^2 z}{\partial u^2}} - \cancel{v^2\frac{\partial^2 z}{\partial v^2}} + 2uv\frac{\partial^2 z}{\partial u\partial v} - 2v\frac{\partial z}{\partial v} = 0$$

$$\Rightarrow \cancel{\boxed{4uv\frac{\partial^2 z}{\partial u\partial v}}}$$

$$\cancel{4}u\cancel{v}\frac{\partial^2 z}{\partial u\partial v} = \cancel{2v}\frac{\partial z}{\partial v}$$

$$\boxed{2u\frac{\partial^2 z}{\partial u\partial v} = \frac{\partial z}{\partial v}}$$

$y = x^2$

$\frac{dy}{dx} = 2x$

$\frac{d^2y}{dx^2} = 2$